

Def (Subset). $\forall x \in X, x \in A$ means $X \subset A$

Def (Set operations). Union, Intersection, $A \times B = \{(a, b) : a \in A, b \in B\}$.

Def (Indexed unions/intersections). $\bigcup_{\alpha \in S} A_\alpha = \{x : \exists \alpha \in S, x \in A_\alpha\}$; $\bigcap_{\alpha \in S} A_\alpha = \{x : \forall \alpha \in S, x \in A_\alpha\}$. For $S = \mathbb{N}$, write $\bigcup_{n=1}^\infty A_n$.

Def (Function). $f : A \rightarrow B$ assigns to each $a \in A$ a unique $f(a) \in B$.

(Injective) $f(x) = f(x') \implies x = x'$.

(Surjective) $\forall y \in B, \exists x \in A : f(x) = y$

(Bijective) Both injective and surjective.

Ex. $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x^2$ not injective $(1, -1)$.

Ex. $f : \mathbb{N} \rightarrow \mathbb{N}, f(x) = x^2$ injective.

Ex. $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = 2x$ not surjective.

Ex. $f : \mathbb{Z} \rightarrow \mathbb{Z}, f(x) = x + 1$ bijective.

Def (Composition). $(g \circ f)(x) = g(f(x))$.

Def (Identity). $\text{id}_A(x) = x$.

Prop Composition of injectives is injective; of surjectives is surjective; of bijjectives is bijective.

Prop $f : A \rightarrow B$ bijective $\iff \exists g : B \rightarrow A$ with $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

Def (Restriction). If $A' \subset A$, then $f|_{A'} : A' \rightarrow B$, $f|_{A'}(x) = f(x)$.

Def (Image). $f(E) = \{f(x) : x \in E\}$.

Def (Inverse image). $f^{-1}(F) = \{x \in A : f(x) \in F\}$.

Ex. $f(x) = x^2, E = [-1, 2]$, then $f(E) = [0, 4]$.

Ex. $f^{-1}([1, 4]) = [-2, -1] \cup [1, 2]$.

Def. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.

Def. S finite if $S = \emptyset$ or $\exists n$ bijection $[n] \rightarrow S$.

Prop $f^{-1}(F \cup G) = f^{-1}(F) \cup f^{-1}(G)$; $f^{-1}(F \cap G) = f^{-1}(F) \cap f^{-1}(G)$; $f(E \cup F) = f(E) \cup f(F)$; $f(E \cap F) \subset f(E) \cap f(F)$.

Ex. $f(x) = x^2, E = \{-1, 1\}, F = \{2\}$: $f(E \cap F) = \emptyset$, but $f(E) \cap f(F) = \{1\} \cap \{4\} = \emptyset$.

Ex. Inclusion can be strict (non-injective f).

Cor If f injective, then $f(E \cap F) = f(E) \cap f(F)$.

Props Subset of finite set finite; finite unions finite; product of finite sets finite.

Def (Field). A set F with $+, \cdot$ satisfies: commutativity, associativity, distributivity, identities $0, 1$, inverses.

Def (Ordered set). (S, \leq) with reflexivity, antisymmetry, transitivity, and comparability.

Def (Maximum/minimum). Max: $y \in A$ with $x \leq y$ for all $x \in A$; Min: $y \in A$ with $y \leq x$ for all $x \in A$.

Lemma 3.20. Every nonempty subset of \mathbb{N} has a minimum (well-ordering).

Cor 3.21. Every nonempty finite subset of an ordered set has min + max.

Def (Upper bound). y is an UB of A if $x \leq y \forall x \in A$.

Def (Supremum). $\sup A =$ least upper bound.

Def. Infimum = greatest lower bound.

Props 3.29–3.30. $\sup A$ unique; if $\max A$ exists then $\sup A = \max A$ (similarly for inf, min).

Thm \mathbb{R} is an ordered field with the LUB property, unique up to isomorphism.

Prop (Archimedean). $x > 0 \implies \forall y \exists n \in \mathbb{N} : nx > y$.

Prop (Density of \mathbb{Q}). If $x < y$, then $\exists q \in \mathbb{Q}$ with $x < q < y$.

Ex (“Hole” in \mathbb{Q}). $A = \{x \in \mathbb{Q} : x > 0, x^2 < 2\}$ is bounded above in \mathbb{Q} but has no LUB in \mathbb{Q} (hence \mathbb{Q} fails LUB property).

Prop (Square roots). For $x \geq 0$ there exists a unique $y \geq 0$ with $y^2 = x$.

Def (Exponentiation). For $\beta = p/q \in \mathbb{Q}, x > 0$: $x^\beta = (\sqrt[q]{x})^p$. For $x > 1, \alpha \in \mathbb{R}$: $x^\alpha = \sup\{x^\beta : \beta \in \mathbb{Q}, \beta \leq \alpha\}$. If $0 < x < 1$, use inf.

Quick sup/inf calculus. $\inf A = -\sup(-A)$; $\sup(A + c) = \sup A + c$; $\inf(A + c) = \inf A + c$; $c \geq 0$: $\sup(cA) = c \sup A$, $\inf(cA) = c \inf A$; $c < 0$: $\sup(cA) = c \inf A$, $\inf(cA) = c \sup A$; $\sup(A \cup B) = \max\{\sup A, \sup B\}$; if A, B nonempty, bdd above: $\sup(A + B) = \sup A + \sup B$ (use $\varepsilon/2$).

Def (Countable). S is at most countable if S is finite or \exists bijection $f : \mathbb{N} \rightarrow S$.

Countability toolkit. Infinite subset of a countable set is countable; countable union of countables is countable; \mathbb{Q} and \mathbb{Q}^n are countable; \mathbb{R} is uncountable (Cantor diagonal).

Def (Inner product, norm, distance). $x \cdot y = \sum x_i y_i$; $\|x\| = \sqrt{x \cdot x}$; $d(x, y) = \|x - y\|$ on \mathbb{R}^n .

Cauchy–Schwarz. $|x \cdot y| \leq \|x\| \|y\|$. **Triangle.** $\|x + y\| \leq \|x\| + \|y\|$; also $||\|x\| - \|y\|| \leq \|x - y\|$.

Def (Metric space). $d : X \times X \rightarrow \mathbb{R}$ with: $d \geq 0$ and $d(x, y) = 0 \iff x = y$; symmetry; triangle inequality.

Def (Balls). $B_\varepsilon(p) = \{q : d(p, q) < \varepsilon\}$ (open ball); $\overline{B}_\varepsilon(p) = \{q : d(p, q) \leq \varepsilon\}$ (closed ball).

Open-ball lemma. Each $B_\varepsilon(p)$ is open: if $q \in B_\varepsilon(p)$, set $\delta = \varepsilon - d(p, q) > 0$, then $B_\delta(q) \subset B_\varepsilon(p)$.

Def (Open/closed, interior). E open if $\forall p \in E \exists \varepsilon > 0$ with $B_\varepsilon(p) \subset E$. Closed if E^c open. Interior $\text{int}(E) =$ largest open subset of E .

Basic open/closed algebra. Arbitrary unions of open sets are open; finite intersections of open sets are open; arbitrary intersections of closed sets are closed; finite unions of closed sets are closed.

Subspace: $E \subset Y \subset X$ open in $Y \iff E = Y \cap G$ for some open $G \subset X$; similarly for closed.

Exs (Open/closed pitfalls). $\bigcap_{n \geq 1} (-1/n, 1/n) = \{0\}$ (not open). $\bigcup_{n \geq 1} [-1/n, 1/n] = [-1, 1]$ (not closed).

Def (Limit points). p is a limit point of E if every $B_\varepsilon(p)$ contains some $q \in E, q \neq p$. E is closed $\iff E$ contains all its limit points.

Limit-point facts. Every neighborhood of a limit point contains infinitely many points of E ; finite sets have no limit points \implies finite sets are closed; $\{1/n : n \in \mathbb{N}\}$ has the unique limit point 0 ; in \mathbb{R} , (a, b) has limit points a, b but omits them.

Def (Closure/boundary). $\overline{E} = E \cup \{\text{limit points of } E\}$; $\partial E = \overline{E} \setminus \text{int}(E)$.

Closure characterizations. \overline{E} is the smallest closed set containing E and equals $\bigcap\{F \supset E : F \text{ closed}\}$; \overline{E} closed; $E = \overline{E}$ iff E closed; if $E \subset F$ and F closed, then $\overline{E} \subset \overline{F}$; in \mathbb{R} : if $E \neq \emptyset$ and bounded above, then $\sup E \in \overline{E}$.

Def (Bounded sets). E is bounded if $\exists M$ with $d(p, q) < M$ for all $p, q \in E$.

Boundedness equivalence. E bounded $\iff \exists x, M$ with $E \subset B_M(x)$. Exs: finite sets are bounded; $\mathbb{Z}^3 \subset \mathbb{R}^3$ unbounded.

One-liners you can reproduce. (1) $\inf A = -\sup(-A)$ (lower bounds \leftrightarrow negative upper bounds). (2) $\sup(A + B) = \sup A + \sup B$ (" \leq ": monotonicity; " \geq ": $\varepsilon/2$ near-sup). (3) Neighborhoods are open: $\delta = \varepsilon - d(p, q)$. (4) Finite sets closed: a limit point forces infinitely many points per ball. (5) Floor trick: if U upper bound of \mathbb{N} , then $\lfloor U \rfloor + 1 > U$.

Fast exemplars. Open not closed: $(0, 1)$; Closed not open: $[0, 1]$; Both: \emptyset, X (discrete metric); Neither: $(0, 1] \cup \{2\}$.
Limit points: $E = \{1/n\}$ has only 0; 1 is isolated
Sup in closure: $E = (0, 1)$ has $\sup E = 1 \in \overline{E}$ but $1 \notin E$.

Canonical counts (know these cold).
(1) \mathbb{N} countable; \mathbb{Z} countable (zig-zag); \mathbb{Q} countable (grid/diagonal).
(2) If A, B countable, then $A \cup B, A \times B$ are countable; countable union of countables is countable.
(3) \mathbb{Q}^n countable; finite strings over a countable alphabet are countable.
(4) $\mathcal{P}(\mathbb{N})$ uncountable (Cantor); hence $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ uncountable; $\mathbb{N}^{\mathbb{N}}$ uncountable.
(5) If F finite with $|F| \geq 2$, then $F^{\mathbb{N}}$ uncountable; if D finite then \mathbb{N}^D countable.

Encodings you can write fast.
(1) Pairs $\leftrightarrow \mathbb{N}$ (Cantor pairing): $\pi(m, n) = \frac{(m+n)(m+n+1)}{2} + n$ is bijection $\mathbb{N}^2 \rightarrow \mathbb{N}$.
(2) Tuples: $\mathbb{N}^k \cong \mathbb{N}$ by iterating π (or prime coding $n \mapsto 2^{a_1} 3^{a_2} \dots p_k^{a_k}$).
(3) Finite sequences (any length): encode as $(\ell, a_1, \dots, a_\ell) \in \mathbb{N}^{\ell+1}$; union over ℓ (countable).
(4) Finite subsets of \mathbb{N} : encode F by $n(F) = \prod_{i \in F} p_i$ (unique) — injects into \mathbb{N} .

Diagonalization template (prove uncountable).
Assume a list f_1, f_2, \dots of all elements of $X^{\mathbb{N}}$ (e.g. $\{0, 1\}^{\mathbb{N}}$). Define g by $g(n) \neq f_n(n)$ (flip $0 \leftrightarrow 1$). Then g differs from f_n at n for every n — contradiction.

function set sizes ex
(1) Finite domain: if F finite then B^F finite ($|B|^{|F|}$), hence countable if B countable.
(2) Finite codomain ≥ 2 : $F^{\mathbb{N}}$ uncountable (reduce from $\{0, 1\}^{\mathbb{N}}$).
(3) Eventually-zero sequences: $E = \{(a_n) : \exists N, a_n = 0 \forall n > N\}$ is countable (finite-sequence union).
(4) All sequences: $\mathbb{N}^{\mathbb{N}}$ uncountable (diagonalize).
(5) Rational vs real sequences: $\mathbb{Q}^{\mathbb{N}}$ uncountable (inject $\{0, 1\}^{\mathbb{N}}$).

Countability of function sets.
(1) Finite domain: if A finite and B countable, then B^A countable (finite power).
(2) Finite codomain ≥ 2 : if A infinite and $|B| \geq 2$ finite, then B^A uncountable (reduce from $\{0, 1\}^{\mathbb{N}}$).
(4) All sequences: $\mathbb{N}^{\mathbb{N}}$ uncountable (diagonalization).
(5) Countable domain + uncountable codomain: B^A uncountable (already $2^{\mathbb{N}}$).
(6) Finite binary strings $\{0, 1\}^*$ countable; infinite binary strings $\{0, 1\}^{\mathbb{N}}$ uncountable.

Micro-Exs (1–3 lines).

- (1) \mathbb{Q} countable: list $\frac{p}{q}$ with $q \geq 1, \gcd(p, q) = 1$; enumerate diagonally, skip repeats/signs.
- (2) $\{0, 1\}^{\mathbb{N}}$ uncountable: diagonal flip of putative list.
- (3) $\text{Fun}(\mathbb{N}, \{0, 1, 2\})$ uncountable: inject $\{0, 1\}^{\mathbb{N}}$ (e.g. append 2's).
- (4) Countable union of countables: enumerate $A_k = \{a_{k,1}, a_{k,2}, \dots\}$, list along diagonals.
- (5) Finite subsets of \mathbb{N} countable: $F_k = \{S \subset \mathbb{N} : |S| = k\} \cong \mathbb{N}^k$, then $\bigcup_k F_k$ is countable.
- (6) Algebraic numbers countable: union over degree/coefficient bounds, each inner set finite.

Creative proof skeletons. Encode & diagonalize; chunk by size (k -tuples / k -element subsets); pairing $\mathbb{N}^k \rightarrow \mathbb{N}$; finite-support \Rightarrow countable; prime-coding injection.

Typical exam prompts & 1-line starts.

1. $\{0, 1\}^{\mathbb{N}}$ uncountable: assume list $(x^{(n)})$, set $y_n = 1 - x_n^{(n)}$.
2. Eventually constant 0/1 sequences countable: $\bigcup_{N,c} \{\text{seqs constant after } N\} \cong \bigcup_N \{0, 1\}^N$.
3. $\mathbb{Q}^{\mathbb{N}}$ uncountable: inject $\{0, 1\}^{\mathbb{N}} \hookrightarrow \mathbb{Q}^{\mathbb{N}}$.
4. Polynomials with integer coeffs countable: coeff tuples $\in \mathbb{Z}^{k+1}$; union over k .
5. $\mathcal{P}(\mathbb{N})$ uncountable: subsets \leftrightarrow characteristic sequences; diagonalize.

Fast pitfalls. "Countable union of *uncountable* sets is countable" — false. "Product of countables is always countable" — only finite products. "Functions from a countable set to a finite set are countable" — false when domain infinite and codomain has ≥ 2 elements.

Extra key Exs.

- (1) On \mathbb{R} , $d(x, y) = |x^2 - y^2|$ is *not* a metric: $d(1, -1) = 0$ though $1 \neq -1$.
- (2) In the discrete metric $d(x, y) = 0$ if $x = y$, else 1, every subset is both open and closed (clopen).
- (3) $E = \{0\} \cup \{1/n : n \in \mathbb{N}\} \cup \{1 + 1/n : n \in \mathbb{N}\}$ has exactly three limit points: 0, 1, 2.

More key Exs.

- (1) On \mathbb{R} , $d(x, y) = \frac{|x - y|}{1 + |x - y|}$ is a *bounded* metric that induces the same open sets as the standard metric.
- (2) In the subspace $Y = [0, 1] \subset \mathbb{R}$, the set $(0, 1]$ is open in Y but not open in \mathbb{R} (relative openness).
- (3) In \mathbb{R}^2 , $E = \{(x, y) : x < y\}$ is open; the distance from $(x, y) \in E$ to the boundary line $y = x$ is $\frac{y - x}{\sqrt{2}}$.

\mathbb{Q} is dense in \mathbb{R} but not closed, since $\sqrt{2} \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$.

Limit vs isolated points.

Limit point: every ball around p meets $E \setminus \{p\}$.
Isolated point: \exists ball around p containing no other point of E .
Ex: $E = \{1/n : n \in \mathbb{N}\}$ has unique limit point 0; each $1/n$ is isolated

Finite sets closed.

Let $F = \{x_1, \dots, x_n\}$ in a metric space. For $p \in F$, the distances $d(p, q)$ ($q \neq p$) are > 0 ; set $\delta = \frac{1}{2} \min\{d(p, q) : q \in F, q \neq p\}$. Then $B_\delta(p) \cap (F \setminus \{p\}) = \emptyset$, so p is not a limit point. Thus F has no limit points and vacuously contains them all $\Rightarrow F$ is closed.

Irrationals in $(0, 1)$ are uncountable.

$(0, 1)$ is uncountable (Cantor); $\mathbb{Q} \cap (0, 1)$ is countable. If the irrationals $I = (0, 1) \setminus \mathbb{Q}$ were countable, then $(0, 1) = I \cup (\mathbb{Q} \cap (0, 1))$ would be a countable union of countables \Rightarrow countable — contradiction.