Math 2550

PSET 2

September 11 2025

Contents

Ex 1.1

Question

Let F be any ordered field (for example $F = \mathbb{Q}$ or $F = \mathbb{R}$). Suppose $A \subset F$ has a supremum. Suppose $\epsilon \in F$ with $\epsilon > 0$. Prove that there exists some $x \in A$ such that $x > (\sup A) - \epsilon$.

Proof

A having a supremum means that there is some $y \in F$ that is a least upper bound for F such that for any upper bound $y' \in F$, $y \leq y'$. Yet for all $x \in A$, $x \leq y$.
Let us suppose that the statement is false. That would mean that $\forall x \in A$, $x \leq (\sup A) - \epsilon$, wehre $\epsilon \geq 0$
Letting $\sup A = y$ this would mean that there exists some uppwer bound $y - \epsilon < y$, since $\epsilon > 0$. Which contradicts the statement that y is a upper bound of A such that for all other upper bounds, y' $y \leq y'$
Meaning the statement is true.

Ex 1.2, Rudin 1.5

Question

Let $A \subset \mathbb{R}$ be nonempty and bounded below. Define

$$-A = \{-x \mid x \in A\}. \tag{1}$$

Prove that $\inf A = -\sup(-A)$.

Proof

The infimum of A is a number $y \in \mathbb{R}$ such that y is a lower bound for A and for any other lower bound y' of A we have $y' \leq y$. The supremum of a set B is a number $z \in \mathbb{R}$ such that z is an upper bound for B and for any other upper bound z' of B we have $z \leq z'$.
$-A$ is defined by multiplying each element of A by -1. In $\mathbb{R}$, if $x < x'$ and $c < 0$ then $xc > x'c$, so the order is reversed when we pass from A to $-A$. That means every lower bound of A becomes an upper bound of $-A$, and conversely. In particular, the greatest lower bound of A corresponds to the least upper bound of $-A$ under this reversal.
So let $y = \inf A$. Then for all $x \in A$ we have $y \leq x$, which is equivalent to $-x \leq -y$ for all $x \in A$. This shows $-y$ is an upper bound of $-A$.
Now suppose u is any other upper bound of $-A$. Then for all $x \in A$ we have $-x \leq u$, which means $x \geq -u$. Thus $-u$ is a lower bound of A, and since y is the greatest lower bound of A we have $-u \leq y$. Rearranging gives $u \geq -y$. Therefore $-y$ is the least upper bound of $-A$, so $\sup(-A) = -y$.
So

$$\inf A = -\sup(-A).$$

Ex 1.3

Question

Suppose $A, B \subset \mathbb{R}$ are both nonempty and bounded above. Define the set $A + B$ by

$$A + B = \{a + b \mid a \in A, b \in B\}. \tag{2}$$

Prove that $\sup(A + B) = \sup A + \sup B$.

Proof

The supremum of a set is a number x in the set such that x is a lower bound for the set and any lower bound x' for the set has $x \leq x'$. $\mathbb{R}$ is an ordered field meaning that if follows the order axiom that means that if $a < b$ and $c < d$ then $a + b < c + d$. A and B are both subsets of the ordered field R meaning that they also have an ordering. So that if a, a' are in $\mathbb{R}$ and b, b' are in $\mathbb{R}$ with $a < a'$ and $b < b'$ it follows that $a + b < a' + b'$ by the order axiom of fields so for all $x \in \mathbb{R}$ there exists some least upper bound q such that $\forall x \in A$, $x \leq q$ and the same applies for B. Following the order axiom that means that $q + p \geq a + b$ for all $a + b \in A, B$

Ex 1.4

Question

In each of the following, S is an ordered set, and $A \subset S$. Answer the following in each case, and prove your answers:

- Is A bounded above?
- Does A have a maximum element, and if so, what is it?
- Does A have a supremum in S, and if so, what is it?

1. $S = \mathbb{Z}$, $A = \{2, 3\}$.
2. $S = \mathbb{Q}$, $A = \{-\frac{2n}{5} \mid n \in \mathbb{N}\}$.
3. $S = \mathbb{Q}$, $A = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$.
4. $S = \mathbb{Q}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.
5. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x \leq 1\}$.
6. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.

Proof

1. $S = \mathbb{Z}$, $A = \{2, 3\}$
   (a) It is bounded above, it has 2 elements $2, 3$ and is a subset of the ordered set $\mathbb{Z}$ this means that it inheris the ordering. With $2 < 3$. So A is bounded above by 3
   (b) Yes, 3.
   (c) Yes 3.
2. $S = \mathbb{Q}$, $A = \{-\frac{2n}{5} \mid n \in \mathbb{N}\}$.
   (a) no
   (b) no
   (c) no
3. $S = \mathbb{Q}$, $A = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$.
   (a) Yes
   (b) Yes, 1
   (c) Yes, 1
4. $S = \mathbb{Q}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.

(a) Yes
(b) Yes
(c) Yes, 1

5. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x \leq 1\}$.
   (a) Yes
   (b) Yes, 1
   (c) Yes, 1
6. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.
   (a) Yes
   (b) No
   (c) Yes, 1

Question

Suppose F is an ordered field and $x, y, z \in F$.

1. Prove that if $x > 0$, then $x^{-1} > 0$.
2. Prove that if $x > 0$, then $y > z$ if and only if $xy > xz$.
3. Recall the field $\mathbb{F}_3 = \{0, 1, 2\}$ which we discussed in class. Prove that there does not exist an order on $\mathbb{F}_3$ such that it is an ordered field.

Proof

1. Let F be an ordered field and let $x > 0$. Because F is a field, there exists $x^{-1} \in F$ with $x \cdot x^{-1} = 1$, and $x^{-1} \neq 0$. Assume for contradiction that $x^{-1} \leq 0$. Since $x^{-1} \neq 0$, we have $x^{-1} < 0$. In an ordered field, multiplying an inequality by a negative number reverses the inequality; hence from $x > 0$ we obtain

$$x \cdot x^{-1} \; < \; 0 \quad \Longrightarrow \quad 1 < 0.$$

But in any ordered field every nonzero square is positive, so $1 = 1^2 > 0$, a contradiction. Therefore $x^{-1} > 0$.

2. Let $x > 0$. First suppose for contradiction that $y < z$ but $xy > xz$. Since $x^{-1} > 0$ by part (1), multiplying both sides of $xy > xz$ by x^{-1} preserves the inequality:

$$xy > xz \;\; \Rightarrow \;\; x^{-1}(xy) > x^{-1}(xz) \;\; \Rightarrow \;\; (x \cdot x^{-1})y > (x \cdot x^{-1})z \;\; \Rightarrow \;\; y > z,$$

contradicting $y < z$.
For the other direction, suppose for contradiction that $y > z$ but $xy < xz$. Multiplying both sides by $x^{-1} > 0$ preserves the inequality:

$$xy < xz \;\; \Rightarrow \;\; x^{-1}(xy) < x^{-1}(xz) \;\; \Rightarrow \;\; (x \cdot x^{-1})y < (x \cdot x^{-1})z \;\; \Rightarrow \;\; y < z,$$

contradicting $y > z$.
Hence $y > z \iff xy > xz$.

Question

This exercise is a reminder that the "size" of infinite sets can be a bit counterintuitive.

1. Suppose $a, b \in \mathbb{R}$, with $a < b$. Prove that there is a bijection from (a, b) to $(0, 1)$.

2. Prove that there is a bijection from $(0, 1)$ to $(1, \infty)$.
3. Suppose $a, b \in \mathbb{R}$. Prove that there is a bijection from (a, ∞) to (b, ∞).

Proof

1. Let $(0, 1)$ be a subset of $\mathbb{R}$ that inherits its conventional ordering this means that $0 < 1$. A bijection
   There is a bijection if there is both a surjection a injection