Definition 2.3 (Subset). If A, B are sets and all $x \in A$ also satisfy $x \in B$, then $A \subset B$. If $A \subset B$ and $B \subset A$, then $A = B$.

Definition 2.5 (Set operations). $A \cup B = \{x : x \in A \vee x \in B\}$; $A \cap B = \{x : x \in A \wedge x \in B\}$; $A \times B = \{(a,b) : a \in A, b \in B\}$.

Definition 2.6 (Indexed unions/intersections). $\bigcup_{\alpha \in S} A_\alpha = \{x : \exists \alpha \in S,\ x \in A_\alpha\}$; $\bigcap_{\alpha \in S} A_\alpha = \{x : \forall \alpha \in S,\ x \in A_\alpha\}$. For $S = \mathbb{N}$, write $\bigcup_{n=1}^{\infty} A_n$.

Definition 2.7 (Function). $f : A \to B$ assigns to each $a \in A$ a unique $f(a) \in B$. Domain $= A$, codomain $= B$. Two functions equal iff $f(a) = g(a)$ for all $a \in A$.

Definition 2.8 (Injective). $f(x) = f(x') \implies x = x'$. **Definition 2.11 (Surjective).** $\forall y \in B, \exists x \in A : f(x) = y$. **Definition 2.14 (Bijective).** Both injective and surjective.

Example 2.9. $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = x^2$ not injective $(1, -1)$.
Example 2.10. $f : \mathbb{N} \to \mathbb{N},\ f(x) = x^2$ injective.
Example 2.12. $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = 2x$ not surjective.
Example 2.13. $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = x + 1$ bijective.

Definition 2.15 (Composition). $(g \circ f)(x) = g(f(x))$. **Definition 2.16 (Identity).** $\mathrm{id}_A(x) = x$.

Proposition 2.17. Composition of injectives is injective; of surjectives is surjective; of bijectives is bijective.
Proposition 2.18. $f : A \to B$ bijective $\iff \exists g : B \to A$ with $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$.

Definition 2.19 (Restriction). If $A' \subset A$, then $f|_{A'} : A' \to B$, $f|_{A'}(x) = f(x)$. **Definition 2.20 (Image).** $f(E) = \{f(x) : x \in E\}$. **Definition 2.21 (Inverse image).** $f^{-1}(F) = \{x \in A : f(x) \in F\}$.

Example 2.22. $f(x) = x^2$, $E = [-1, 2]$, then $f(E) = [0, 4]$. **Example 2.23.** $f^{-1}([1, 4]) = [-2, -1] \cup [1, 2]$.

Definition 2.24. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. **Definition 2.25.** S finite if $S = \varnothing$ or $\exists n$ bijection $[n] \to S$.

Proposition 2.26. $f^{-1}(F \cup G) = f^{-1}(F) \cup f^{-1}(G)$; $f^{-1}(F \cap G) = f^{-1}(F) \cap f^{-1}(G)$; $f(E \cup F) = f(E) \cup f(F)$; $f(E \cap F) \subset f(E) \cap f(F)$.

Example 2.27. $f(x) = x^2$, $E = \{-1, 1\}$, $F = \{2\}$: $f(E \cap F) = \varnothing$, but $f(E) \cap f(F) = \{1\} \cap \{4\} = \varnothing$.
Example 2.28. Inclusion can be strict (non-injective f).

Corollary 2.29. If f injective, then $f(E \cap F) = f(E) \cap f(F)$. **Props 2.30–2.33.** Subset of finite set finite; finite unions finite; product of finite sets finite.

Definition 3.1 (Field). A set F with $+, \cdot$ satisfies: commutativity, associativity, distributivity, identities $0, 1$, inverses.

Definition 3.9 (Ordered set). $(S, \leq)$ with reflexivity, antisymmetry, transitivity, and comparability.

Definition 3.13 (Maximum/minimum). Max: $y \in A$ with $x \leq y$ for all $x \in A$; Min: $y \in A$ with $y \leq x$ for all $x \in A$.

Lemma 3.20. Every nonempty subset of $\mathbb{N}$ has a minimum (well-ordering). **Cor 3.21.** Every nonempty finite subset of an ordered set has min + max.

Definition 3.22 (Upper bound). y is an UB of A if $x \leq y\ \forall x \in A$. **Definition 3.26 (Supremum).** $\sup A$ = least upper bound. **Definition 3.31–3.32.** Infimum = greatest lower bound.

Props 3.29–3.30. $\sup A$ unique; if $\max A$ exists then $\sup A = \max A$ (similarly for $\inf, \min$).

Theorem 3.43. $\mathbb{R}$ is an ordered field with the LUB property, unique up to isomorphism. **Prop 3.44 (Archimedean).** $x > 0 \Rightarrow \forall y\ \exists n \in \mathbb{N} : nx > y$. **Prop 3.45 (Density of $\mathbb{Q}$).** If $x < y$, then $\exists q \in \mathbb{Q}$ with $x < q < y$.

3A ("Hole" in $\mathbb{Q}$). $A = \{x \in \mathbb{Q} : x > 0,\ x^2 < 2\}$ is bounded above in $\mathbb{Q}$ but has no LUB in $\mathbb{Q}$ (hence $\mathbb{Q}$ fails LUB property).

Proposition 3.46 (Square roots). For $x \geq 0$ there exists a unique $y \geq 0$ with $y^2 = x$.

Definition 3.48–3.49 (Exponentiation). For $\beta = p/q \in \mathbb{Q}$, $x > 0$: $x^\beta = (\sqrt[q]{x})^p$. For $x > 1$, $\alpha \in \mathbb{R}$: $x^\alpha = \sup\{x^\beta : \beta \in \mathbb{Q}, \beta \leq \alpha\}$. If $0 < x < 1$, use inf.

Quick sup/inf calculus. $\inf A = -\sup(-A)$; $\sup(A + c) = \sup A + c$; $\inf(A + c) = \inf A + c$; $c \geq 0 :\ \sup(cA) = c \sup A$, $\inf(cA) = c \inf A$; $c < 0 :\ \sup(cA) = c \inf A$, $\inf(cA) = c \sup A$; $\sup(A \cup B) = \max\{\sup A, \sup B\}$; if A, B nonempty, bdd above: $\sup(A + B) = \sup A + \sup B$ (use $\varepsilon/2$).

Definition 4.1 (Countable). S is at most countable if S is finite or $\exists$ bijection $f : \mathbb{N} \to S$.

Countability toolkit. Infinite subset of a countable set is countable; countable union of countables is countable; $\mathbb{Q}$ and $\mathbb{Q}^n$ are countable; $\mathbb{R}$ is uncountable (Cantor diagonal).

Def 5.1 (Inner product, norm, distance). $x \cdot y = \sum x_i y_i$; $\|x\| = \sqrt{x \cdot x}$; $d(x, y) = \|x - y\|$ on $\mathbb{R}^n$.

Cauchy–Schwarz. $|x \cdot y| \leq \|x\|\,\|y\|$. **Triangle.** $\|x + y\| \leq \|x\| + \|y\|$; also $|\|x\| - \|y\|| \leq \|x - y\|$.

Def 5.5 (Metric space). $d : X \times X \to \mathbb{R}$ with: $d \geq 0$ and $d(x, y) = 0 \iff x = y$; symmetry; triangle inequality.

Def 5.9 (Balls). $B_\varepsilon(p) = \{q : d(p, q) < \varepsilon\}$ (open ball); $\overline{B}_\varepsilon(p) = \{q : d(p, q) \leq \varepsilon\}$ (closed ball).

Open-ball lemma. Each $B_\varepsilon(p)$ is open: if $q \in B_\varepsilon(p)$, set $\delta = \varepsilon - d(p, q) > 0$, then $B_\delta(q) \subset B_\varepsilon(p)$.

Def 5.11 (Open/closed, interior). E open if $\forall p \in E\ \exists \varepsilon > 0$ with $B_\varepsilon(p) \subset E$. Closed if E^c open. Interior $\mathrm{int}(E)$ = largest open subset of E.

Basic open/closed algebra. Arbitrary unions of open sets are open; finite intersections of open sets are open; arbitrary intersections of closed sets are closed; finite unions of closed sets are closed.
Subspace: $E \subset Y \subset X$ open in $Y \iff E = Y \cap G$ for some open $G \subset X$; similarly for closed.

Open/closed pitfalls. $\bigcap_{n \geq 1}(-1/n, 1/n) = \{0\}$ (not open). $\bigcup_{n \geq 1}[-1/n, 1/n] = (-1, 1]$ (not closed).

Def 5.20 (Limit points). p is a limit point of E if every $B_\varepsilon(p)$ contains some $q \in E$, $q \neq p$. E is closed $\iff E$ contains all its limit points.

Limit-point facts. Every neighborhood of a limit point contains infinitely many points of E; finite sets have no limit points $\Rightarrow$ finite sets are closed; $\{1/n : n \in \mathbb{N}\}$ has the unique limit point 0; in $\mathbb{R}$, (a, b) has limit points a, b but omits them.

Def 5.35 (Closure/boundary). $\overline{E} = E \cup \{\text{limit points of } E\}$; $\partial E = \overline{E} \setminus \text{int}(E)$.

Closure characterizations. $\overline{E}$ is the smallest closed set containing E and equals $\bigcap\{F \supset E : F \text{ closed}\}$; $\overline{E}$ closed; $E = \overline{E}$ iff E closed; if $E \subset F$ and F closed, then $\overline{E} \subset F$; in $\mathbb{R}$: if $E \neq \varnothing$ and bounded above, then $\sup E \in \overline{E}$.

Def 5.42 (Bounded sets). E is bounded if $\exists M$ with $d(p, q) < M$ for all $p, q \in E$.

Boundedness equivalence. E bounded $\iff \exists x, M$ with $E \subset B_M(x)$. Examples: finite sets are bounded; $\mathbb{Z}^3 \subset \mathbb{R}^3$ unbounded.

One-liners you can reproduce. (1) $\inf A = -\sup(-A)$ (lower bounds $\leftrightarrow$ negative upper bounds). (2) $\sup(A + B) = \sup A + \sup B$ ("$\le$": monotonicity; "$\ge$": $\varepsilon/2$ near-sups). (3) Neighborhoods are open: $\delta = \varepsilon - d(p, q)$. (4) Finite sets closed: a limit point forces infinitely many points per ball. (5) Floor trick: if U upper bound of $\mathbb{N}$, then $\lfloor U \rfloor + 1 > U$.

Fast exemplars. Open not closed: $(0, 1)$; Closed not open: $[0, 1]$; Both: $\varnothing$, X (discrete metric); Neither: $(0, 1] \cup \{2\}$.
Limit points: $E = \{1/n\}$ has only 0; 1 is isolated (take $(0.9, 1.1)$).
Sup in closure: $E = (0, 1)$ has $\sup E = 1 \in \overline{E}$ but $1 \notin E$.

Canonical counts (know these cold).
(1) $\mathbb{N}$ countable; $\mathbb{Z}$ countable (zig-zag); $\mathbb{Q}$ countable (grid/diagonal).
(2) If A, B countable, then $A \cup B$, $A \times B$ are countable; countable union of countables is countable.
(3) $\mathbb{Q}^n$ countable; finite strings over a countable alphabet are countable.
(4) $\mathcal{P}(\mathbb{N})$ uncountable (Cantor); hence $2^{\mathbb{N}} = \{0, 1\}^{\mathbb{N}}$ uncountable; $\mathbb{N}^{\mathbb{N}}$ uncountable.
(5) If F finite with $|F| \ge 2$, then $F^{\mathbb{N}}$ uncountable; if D finite then $\mathbb{N}^D$ countable.

Encodings you can write fast.
(1) Pairs $\leftrightarrow$ $\mathbb{N}$ (Cantor pairing): $\pi(m, n) = \frac{(m+n)(m+n+1)}{2} + n$ is bijection $\mathbb{N}^2 \to \mathbb{N}$.
(2) Tuples: $\mathbb{N}^k \cong \mathbb{N}$ by iterating π (or prime coding $n \mapsto 2^{a_1} 3^{a_2} \cdots p_k^{a_k}$).
(3) Finite sequences (any length): encode as $(\ell, a_1, \ldots, a_\ell) \in \mathbb{N}^{\ell+1}$; union over ℓ (countable).
(4) Finite subsets of $\mathbb{N}$: encode F by $n(F) = \prod_{i \in F} p_i$ (unique) — injects into $\mathbb{N}$.

Diagonalization template (prove uncountable). Assume a list $f_1, f_2, \ldots$ of all elements of $X^{\mathbb{N}}$ (e.g. $\{0, 1\}^{\mathbb{N}}$). Define g by $g(n) \neq f_n(n)$ (flip $0 \leftrightarrow 1$). Then g differs from f_n at n for every n — contradiction.

At-a-glance: function set sizes.
(1) Finite domain: if F finite then B^F finite ($|B|^{|F|}$), hence countable if B countable.
(2) Finite codomain ≥ 2: $F^{\mathbb{N}}$ uncountable (reduce from $\{0, 1\}^{\mathbb{N}}$).
(3) Eventually-zero sequences: $E = \{(a_n) : \exists N,\ a_n = 0\ \forall n > N\}$ is countable (finite-sequence union).
(4) All sequences: $\mathbb{N}^{\mathbb{N}}$ uncountable (diagonalize).
(5) Rational vs real sequences: $\mathbb{Q}^{\mathbb{N}}$ uncountable (inject $\{0, 1\}^{\mathbb{N}}$).

Micro-examples (1–3 lines).
(1) $\mathbb{Q}$ countable: list $\frac{p}{q}$ with $q \ge 1$, $\gcd(p, q) = 1$; enumerate diagonally, skip repeats/signs.
(2) $\{0, 1\}^{\mathbb{N}}$ uncountable: diagonal flip of putative list.
(3) $\text{Fun}(\mathbb{N}, \{0, 1, 2\})$ uncountable: inject $\{0, 1\}^{\mathbb{N}}$ (e.g. append 2's).
(4) Countable union of countables: enumerate $A_k = \{a_{k,1}, a_{k,2}, \ldots\}$, list along diagonals.
(5) Finite subsets of $\mathbb{N}$ countable: $F_k = \{S \subset \mathbb{N} : |S| = k\} \cong \mathbb{N}^k$, then $\bigcup_k F_k$ is countable.
(6) Algebraic numbers countable: union over degree/coeff bounds, each inner set finite.

Creative proof skeletons. Encode & diagonalize; chunk by size (k-tuples / k-element subsets); pairing $\mathbb{N}^k \to \mathbb{N}$; finite-support $\Rightarrow$ countable; prime-coding injection.

Typical exam prompts & 1-line starts.
1. $\{0, 1\}^{\mathbb{N}}$ uncountable: assume list $(x^{(n)})$, set $y_n = 1 - x_n^{(n)}$.
2. Eventually constant 0/1 sequences countable: $\bigcup_{N,c}\{\text{seqs constant after } N\} \cong \bigcup_N \{0, 1\}^N$.
3. $\mathbb{Q}^{\mathbb{N}}$ uncountable: inject $\{0, 1\}^{\mathbb{N}} \hookrightarrow \mathbb{Q}^{\mathbb{N}}$.
4. Polynomials with integer coeffs countable: coeff tuples $\in \mathbb{Z}^{k+1}$; union over k.
5. $\mathcal{P}(\mathbb{N})$ uncountable: subsets $\leftrightarrow$ characteristic sequences; diagonalize.

Fast pitfalls. "Countable union of *uncountable* sets is countable" — false. "Product of countables is always countable" — only finite products. "Functions from a countable set to a finite set are countable" — false when domain infinite and codomain has ≥ 2 elements.

Extra key examples.
(1) On $\mathbb{R}$, $d(x, y) = |x^2 - y^2|$ is *not* a metric: $d(1, -1) = 0$ though $1 \neq -1$.
(2) In the discrete metric $d(x, y) = 0$ if $x = y$, else 1, every subset is both open and closed (clopen).
(3) $E = \{0\} \cup \{1/n : n \in \mathbb{N}\} \cup \{1 + 1/n : n \in \mathbb{N}\}$ has exactly three limit points: $0, 1, 2$.

More key examples.
(1) On $\mathbb{R}$, $d(x, y) = \dfrac{|x - y|}{1 + |x - y|}$ is a *bounded* metric that induces the same open sets as the standard metric.
(2) In the subspace $Y = [0, 1] \subset \mathbb{R}$, the set $(0, 1]$ is open in Y but not open in $\mathbb{R}$ (relative openness).
(3) In $\mathbb{R}^2$, $E = \{(x, y) : x < y\}$ is open; the distance from $(x, y) \in E$ to the boundary line $y = x$ is $\dfrac{y - x}{\sqrt{2}}$.

$\mathbb{Q}$ is dense in $\mathbb{R}$ but not closed, since $\sqrt{2} \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$.

Open vs Closed (quick test).
Open $\iff$ every point has a small ball inside the set.
Closed $\iff$ contains all limit points (equiv. complement open).
Interior point: has some ball fully inside E.
Boundary point: every ball meets both E and E^c.
Closure: $E \cup \{\text{limit points of } E\}$.

Limit vs isolated points.
Limit point: every ball around p meets $E \setminus \{p\}$.
Isolated point: $\exists$ ball around p containing no other point of E.
Example: $E = \{1/n : n \in \mathbb{N}\}$ has unique limit point 0; each $1/n$ is isolated (e.g. $(0.9, 1.1)$ shows 1 is not a limit point).