Irrationals in $(0,1)$ are uncountable. $(0,1)$ is uncountable (Cantor); $\mathbb{Q}\cap(0,1)$ is countable. If the irrationals $I=(0,1)\setminus\mathbb{Q}$ were countable, then $(0,1)=I\cup(\mathbb{Q}\cap(0,1))$ would be a countable union of countables $\Rightarrow$ countable — contradiction.

Def (Distance to a set). $d(x,A)=\inf\{d(x,a):a\in A\}$; $d(x,A)=0 \iff x\in\overline{A}$.

Metric counterexamples. (1) $d(x,y)=|x^2-y^2|$ *fails identity of indiscernibles*: $d(1,-1)=0$ but $1\neq -1$.
(2) $d(x,y)=|x-2y|$ *fails symmetry*: $d(0,1)=2$ but $d(1,0)=1$.

(1) $\inf A=-\sup(-A)$ (one-liner). Lower bounds of $A\leftrightarrow$ upper bounds of $-A$ via $x\mapsto -x$. Least $\iff$ greatest.

(2) $\sup(A+B)=\sup A+\sup B$ ($\varepsilon/2$ template). Let $a^*=\sup A$, $b^*=\sup B$. Then $a+b\le a^*+b^*$ so $\sup(A+B)\le a^*+b^*$. For $\varepsilon>0$ pick $a>a^*-\varepsilon/2$, $b>b^*-\varepsilon/2$, then $a+b>a^*+b^*-\varepsilon$. Conclude equality.

(3) $\mathbb{N}$ not bounded above in $\mathbb{R}$ (floor trick). If U bounds $\mathbb{N}$, then $n=\lfloor U\rfloor\in\mathbb{N}$ and $n+1>U$ contradicts boundedness.

(4) Nested open intervals can intersect trivially. $U_n=(0,1/n)$ are open, $U_{n+1}\subset U_n$, but $\bigcap_n U_n=\varnothing$. (Any $x>0$ fails $x<1/n$ for n large.)

(5) $\mathbb{Q}$ lacks LUB property (classic hole). $A=\{q\in\mathbb{Q}: q>0,\ q^2<2\}$ is bounded in $\mathbb{Q}$ but has no $\sup_{\mathbb{Q}}$ (would be $\sqrt{2}\notin\mathbb{Q}$).

(6) Irrationals in $(0,1)$ are uncountable (nested-intervals proof). Assume a list of irrationals $\{\alpha_n\}$. Build closed $I_1\supset I_2\supset\cdots$ with $I_{n+1}\subset I_n$ and $\alpha_n\notin I_n$, lengths $\to 0$. Unique $x\in\bigcap I_n\subset(0,1)$ is irrational and not on the list. Contradiction.

(7) Finite sets are closed (no limit points). For $F=\{x_1,\dots,x_k\}$ and $p=x_i$, set $\delta=\frac12\min_{j\neq i}d(x_i,x_j)>0$. Then $B_\delta(p)\cap(F\setminus\{p\})=\varnothing$. Thus F has no limit points $\Rightarrow$ closed.

(8) $\sup E\in\overline{E}$ (when $E\neq\varnothing$ bounded above in $\mathbb{R}$). Let $s=\sup E$. For $\varepsilon>0$ pick $x\in E$ with $s-\varepsilon<x\le s$. Then $B_\varepsilon(s)$ meets E. Hence $s\in\overline{E}$.

(9) Discrete metric: every set is clopen. If $d(x,y)=\mathbf{1}_{x\neq y}$, then for $E\subset X$ and $x\in E$, $B_{1/2}(x)=\{x\}\subset E$ so E open. Complement is also open.

(10) Quick metric checks (failures). (1) $d(x,y)=|x^2-y^2|$ fails $d(x,y)=0\Rightarrow x=y$ since $d(1,-1)=0$. (2) $d(x,y)=|x-2y|$ fails symmetry: $d(0,1)=2$ but $d(1,0)=1$.

(11) Bounded metric inducing usual opens (equivalence idea). $\rho(x,y)=\frac{|x-y|}{1+|x-y|}$ on $\mathbb{R}$: monotone in $|x-y|$, so for $r>0$, $B_r^\rho(x)=\{y:\rho(x,y)<r\}=\{y:|x-y|<\frac{r}{1-r}\}$, an ordinary open ball. Hence same open sets.

(12) Subspace openness/closedness ($\Leftarrow\Rightarrow$). If $E=Y\cap G$ with G open in X, then for $y\in E$ there is ε with $B_\varepsilon^X(y)\subset G$; hence $B_\varepsilon^Y(y)=Y\cap B_\varepsilon^X(y)\subset E$ so E is open in Y. The converse: if E open in Y, set $G=\bigcup_{y\in E}B_\varepsilon^X(y)$ from witnessing balls; then $E=Y\cap G$. Dual statement for closed sets.

(13) Interior/closure dualities. $(E^\circ)^c=\overline{E^c}$ and $(\overline{E})^c=(E^c)^\circ$. (Take complements and use de Morgan with openness/closedness.)

(14) Sequential closedness (metric spaces). F is closed $\iff$ whenever $x_n\in F$ and $x_n\to x$, then $x\in F$. ($\Rightarrow$) Complements of closed sets are open $\Rightarrow$ some ball misses F if $x\notin F$. ($\Leftarrow$) If $x\notin F$ and F not open at x, build $x_n\in F\cap B_{1/n}(x)\to x$, contradiction.

(15) Closure/interior algebra (useful equalities). $\overline{A\cup B}=\overline{A}\cup\overline{B}$ and $(A\cap B)^\circ=A^\circ\cap B^\circ$. *Proofs:* Each inclusion is easy: use monotonicity and openness/closedness definitions.

(16) When equalities fail (good counterexamples). $\overline{A\cap B}\subsetneq\overline{A}\cap\overline{B}$: take $A=\mathbb{Q}$, $B=\mathbb{R}\setminus\mathbb{Q}$ in $\mathbb{R}$. LHS $=\varnothing$, RHS $=\mathbb{R}$.
$(A\cup B)^\circ\supsetneq A^\circ\cup B^\circ$ can fail: in $\mathbb{R}$, let $A=(0,1)$ and $B=(1,2)$. LHS $=\varnothing$, RHS $=(0,1)\cup(1,2)$? (Instead: take A dense open, B dense open disjoint? Use rationals/irrationals by intervals—safe rule is only $\subseteq$ holds.)

(17) Every nbhd of a limit point contains infinitely many points of E. If some nbhd of p had only finitely many points of E, remove them via $\delta=\frac12\min d(p,q)$ over those points; then $B_\delta(p)$ misses $E\setminus\{p\}$, contradicting p limit point.

(18) Bounded $\Leftrightarrow$ contained in some ball. ($\Rightarrow$) If $\sup\{d(x,y):x,y\in E\}=M<\infty$, then for any fixed $x_0\in E$, $d(x,x_0)\le M$ so $E\subset B_M(x_0)$. ($\Leftarrow$) Trivial.

(19) Openness via distance-to-boundary (clean trick). In $\mathbb{R}^2$, $E=\{(x,y):x<y\}$. Boundary is the line $y=x$. For $(x,y)\in E$, distance to line is $\frac{y-x}{\sqrt2}>0$, so the open ball of that radius stays in E; hence E is open.

(20) Interior is open; E open $\Leftrightarrow$ $E^\circ=E$. $E^\circ=\bigcup\{B_\varepsilon(x)\subset E\}$ is a union of opens $\Rightarrow$ open. If E open, then each $x\in E$ has a ball inside E, so $x\in E^\circ$.

(21) Creative diagonalization (functions). If $\{f_n:\mathbb{N}\to\mathbb{N}\}$ is listed, define $g(n)=f_n(n)+1$. Then $g\neq f_n$ for all n; hence the set of functions $\mathbb{N}\to\mathbb{N}$ is uncountable. Use bijections $\mathbb{Q}\leftrightarrow\mathbb{N}$ to lift to $\mathrm{Fun}(\mathbb{Q},\mathbb{Q})$.

(22) Finite-support sequences are countable. $E=\{(a_n):\exists N\ \forall n>N,\ a_n=0\}=\bigcup_{N\ge0}\mathbb{Z}^{N+1}$ (or countable alphabet) $\Rightarrow$ countable union of countables.

(23) Bijections between intervals (fast formulas). Affine: $(0,1)\to(a,b)$ by $x\mapsto a+(b-a)x$. Reciprocal: $(0,1)\to(1,\infty)$ by $x\mapsto 1/x$. Compositions yield many cardinality equivalences.

(24) Sequence test for openness (contrapositive style). G not open $\Rightarrow\exists x\in G$ such that every $B_{1/n}(x)$ meets G^c. Pick $y_n\in G^c\cap B_{1/n}(x)$, then $y_n\to x\in G$. Useful: builds counterexamples by sequences.

Key rule: If A has at least two elements and domain is infinite countable (e.g. $\mathbb{N}$), then $A^{\mathbb{N}}$ is uncountable.

Diag proof: Suppose $f_1,f_2,\dots$ lists all $f:\mathbb{N}\to A$. Pick $a_0\neq a_1\in A$. Define $g(n)=a_0$ if $f_n(n)\neq a_0$, else a_1. Then $g\neq f_n$ for all n. Contradiction $\Rightarrow$ uncountable.

Finite domain $\to$ countable. If D finite and C countable, then C^D countable ($|C|^{|D|}$ finite power of countable).