Math 2550

PSET 2

September 11 2025

Contents

Ex 2.1

Question

Let F be any ordered field (for example $F = \mathbb{Q}$ or $F = \mathbb{R}$). Suppose $A \subset F$ has a supremum. Suppose $\epsilon \in F$ with $\epsilon > 0$. Prove that there exists some $x \in A$ such that $x > (\sup A) - \epsilon$.

Proof

A having a supremum means that there is some $y \in F$ that is a least upper bound for A such that for any upper bound $y' \in F$, $y \leq y'$. Yet for all $x \in A$, $x \leq y$.

Let us suppose that the statement is false. That would mean that $\forall x \in A$, $x \leq (\sup A) - \epsilon$, where $\epsilon > 0$.

Letting $\sup A = y$, this would mean that $y - \epsilon$ is an upper bound of A, since for all $x \in A$, $x \leq y - \epsilon$. But because $\epsilon > 0$, we have $y - \epsilon < y$.

This contradicts the statement that y is a least upper bound of A, such that for all other upper bounds y', $y \leq y'$. So there must exist form $x \in A$ such that $s > (\sup A) - \epsilon$

Ex 2.2, Rudin 1.5

Question

Let $A \subset \mathbb{R}$ be nonempty and bounded below. Define

$$-A = \{-x \ \mid x \in A\}\,. \tag{1}$$

Prove that $\inf A = -\sup(-A)$.

Proof

The infimum of A is a number $y \in \mathbb{R}$ such that y is a lower bound for A and for any other lower bound y' of A we have $y' \leq y$. The supremum of a set B is a number $z \in \mathbb{R}$ such that z is an upper bound for B and for any other upper bound z' of B we have $z \leq z'$.

$-A$ is defined by multiplying each element of A by -1. Suppose $x, x' \in A$ with $x \leq x'$. Multiplying both sides by -1 yields $-x \geq -x'$. So the original ordering of A is reversed in -A. Smaller elements of A correspond to larger elements of $-A$, and larger elements of A correspond to smaller elements of $-A$.

In particular, if y is a lower bound of A, i.e. $y \leq x$ for all $x \in A$, then multiplying by -1 gives $-x \leq -y$ for all $x \in A$, so $-y$ is an upper bound of $-A$. If u is an upper bound of $-A$, i.e. $-x \leq u$ for all $x \in A$, then $x \geq -u$ for all $x \in A$, so $-u$ is a lower bound of A.

So let $y = \inf A$. Then for all $x \in A$ we have $y \leq x$, which is equivalent to $-x \leq -y$ for all $x \in A$. So $-y$ is an upper bound of $-A$.

Now suppose u is any other upper bound of $-A$. Then for all $x \in A$ we have $-x \leq u$, which means $x \geq -u$. So $-u$ is a lower bound of A, and since y is the greatest lower bound of A we have $-u \leq y$. Rearranging gives $u \geq -y$. Therefore $-y$ is the least upper bound of $-A$, so $\sup(-A) = -y$.

$$\inf A = -\sup(-A).$$

Ex 2.3

Question

Suppose $A, B \subset \mathbb{R}$ are both nonempty and bounded above. Define the set $A + B$ by

$$A + B = \{a + b \mid a \in A, b \in B\}\,. \tag{2}$$

Prove that $\sup(A + B) = \sup A + \sup B$.

Proof

The supremum of a set S is a number $y \in \mathbb{R}$ such that y is an upper bound for S and for any other upper bound y' of S we have $y \le y'$.

Let $q = \sup A$ and $p = \sup B$. Then for all $a \in A$, $a \le q$, and for all $b \in B$, $b \le p$. Adding these gives $a + b \le q + p$ for all $a \in A$, $b \in B$. So $q + p$ is an upper bound of $A + B$.

WTS that $q + p$ is a least upper bound. Let $\epsilon > 0$. By definition of the supremum, for every $\epsilon > 0$ there exists $a \in A$ with $q - \frac{\epsilon}{2} < a \le q$. Similarly, since $p = \sup B$, there exists $b \in B$ with $p - \frac{\epsilon}{2} < b \le p$. Then

$$a + b > (q - \tfrac{\epsilon}{2}) + (p - \tfrac{\epsilon}{2}) = q + p - \epsilon.$$

So for every $\epsilon > 0$, there is an element $a + b \in A + B$ greater than $q + p - \epsilon$. So $q + p$ is the least upper bound of $A + B$.

Therefore

$$\sup(A + B) = \sup A + \sup B.$$

Ex 2.4

Question

In each of the following, S is an ordered set, and $A \subset S$. Answer the following in each case, and prove your answers:

- Is A bounded above?
- Does A have a maximum element, and if so, what is it?
- Does A have a supremum in S, and if so, what is it?

1. $S = \mathbb{Z}$, $A = \{2, 3\}$.
2. $S = \mathbb{Q}$, $A = \{-\frac{2n}{5} \mid n \in \mathbb{N}\}$.
3. $S = \mathbb{Q}$, $A = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$.
4. $S = \mathbb{Q}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.
5. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x \le 1\}$.
6. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.

Proof

1. $S = \mathbb{Z}$, $A = \{2, 3\}$.

(a) A is bounded above since 3 is an upper bound, and any $u \geq 3$ also works.

(b) A has a maximum. The largest element is 3.

(c) The supremum in $\mathbb{Z}$ is 3, because the maximum is also the least upper bound.

2. $S = \mathbb{Q}$, $A = \{-\frac{2n}{5} \mid n \in \mathbb{N}\} = \{-\frac{2}{5}, -\frac{4}{5}, -\frac{6}{5}, \dots\}$.

   (a) A is bounded above. The element $-\frac{2}{5}$ is an upper bound.

   (b) A has a maximum. The largest element is $-\frac{2}{5}$.

   (c) The supremum in $\mathbb{Q}$ is $-\frac{2}{5}$, since the maximum is the least upper bound.

3. $S = \mathbb{Q}$, $A = \{-\frac{1}{n} \mid n \in \mathbb{N}\} = \{-1, -\frac{1}{2}, -\frac{1}{3}, \dots\}$.

   (a) A is bounded above, because every element is ≤ 0.

   (b) A does not have a maximum. The elements increase toward 0, but $0 \notin A$.

   (c) A has a supremum, which is 0. First, 0 is an upper bound. If $y < 0$, choose $n > \frac{-1}{y}$. Then $-\frac{1}{n} > y$, so y is not an upper bound. So 0 is the least upper bound.

4. $S = \mathbb{Q}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.

   (a) A is bounded above, because 1 is an upper bound.

   (b) A has a maximum. The largest element is 1.

   (c) The supremum is 1, since the maximum is always the supremum.

5. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x \leq 1\}$.

   (a) A is bounded above, because 1 is an upper bound.

   (b) A has a maximum. The largest element is 1.

   (c) The supremum in $\mathbb{Q}$ is 1, since it is the maximum.

6. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.

   (a) A is bounded above, because 1 is an upper bound.

   (b) A does not have a maximum. For any $x < 1$ in A, the rational $x' = \frac{x+1}{2}$ is still in A and larger than x.

   (c) A has a supremum, which is 1. First, 1 is an upper bound. If $y < 1$, pick n large and set $x = 1 - \frac{1}{n}$. Then $x \in A$ and $x > y$, so y is not an upper bound. So 1 is the least upper bound.

Ex 2.5

Question

Suppose F is an ordered field and $x, y, z \in F$.

1. Prove that if $x > 0$, then $x^{-1} > 0$.
2. Prove that if $x > 0$, then $y > z$ if and only if $xy > xz$.
3. Recall the field $\mathbb{F}_3 = \{0, 1, 2\}$ which we discussed in class. Prove that there does not exist an order on $\mathbb{F}_3$ such that it is an ordered field.

Proof

1. Let F be an ordered field and let $x > 0$. Because F is a field, there exists $x^{-1} \in F$ with $x \cdot x^{-1} = 1$, and $x^{-1} \neq 0$. Assume for contradiction that $x^{-1} \leq 0$. Since $x^{-1} \neq 0$, we have $x^{-1} < 0$. In an ordered field,

multiplying an inequality by a negative number reverses the inequality; hence from $x > 0$ we obtain

$$x \cdot x^{-1} < 0 \quad \Longrightarrow \quad 1 < 0.$$

But in any ordered field every nonzero square is positive, so $1 = 1^2 > 0$, a contradiction. Therefore $x^{-1} > 0$.

2. Let $x > 0$. First suppose for contradiction that $y < z$ but $xy > xz$. Since $x^{-1} > 0$ by part (1), multiplying both sides of $xy > xz$ by x^{-1} preserves the inequality:

$$xy > xz \quad \Rightarrow \quad x^{-1}(xy) > x^{-1}(xz) \quad \Rightarrow \quad (x \cdot x^{-1})y > (x \cdot x^{-1})z \quad \Rightarrow \quad y > z,$$

contradicting $y < z$.
For the other direction, suppose for contradiction that $y > z$ but $xy < xz$. Multiplying both sides by $x^{-1} > 0$ preserves the inequality:

$$xy < xz \quad \Rightarrow \quad x^{-1}(xy) < x^{-1}(xz) \quad \Rightarrow \quad (x \cdot x^{-1})y < (x \cdot x^{-1})z \quad \Rightarrow \quad y < z,$$

contradicting $y > z$.
Hence $y > z \iff xy > xz$.

3. Suppose for contradiction that there is an order on $\mathbb{F}_3 = \{0, 1, 2\}$ making it an ordered field. By definition, an ordered field is a field with an order satisfying two axioms. First, for all $x, y, z \in F$ with $y < z$, we must have $x + y < x + z$. Second, for all $x, y \in F$ with $x > 0$ and $y > 0$, we must have $xy > 0$.

Since $\mathbb{F}_3 = \{0, 1, 2\}$ has three elements, any total order must be a linear arrangement of these three numbers. There are $3! = 6$ possible such orderings. We show that each leads to a contradiction.

Case 1: $0 < 1 < 2$. From $1 < 2$, by axiom (1) with $x = 1$, we get $2 < 0$ (since $2 + 1 = 0$ in $\mathbb{F}_3$), contradicting $0 < \cdots < 2$.

Case 2: $0 < 2 < 1$. From $0 < 2$, by axiom (1) with $x = 1$, we get $1 < 0$, contradicting $0 < \cdots < 1$.

Case 3: $1 < 0 < 2$. From $1 < 0$, by axiom (1) with $x = 1$, we get $2 < 1$, contradicting $1 < \cdots < 2$.

Case 4: $1 < 2 < 0$. From $2 < 0$, by axiom (1) with $x = 1$, we get $0 < 1$, contradicting $1 < \cdots < 0$.

Case 5: $2 < 0 < 1$. From $0 < 1$, by axiom (1) with $x = 1$, we get $1 < 2$, contradicting $2 < \cdots < 1$.

Case 6: $2 < 1 < 0$. From $2 < 1$, by axiom (1) with $x = 1$, we get $0 < 2$, contradicting $2 < \cdots < 0$.

In every possible ordering of $\mathbb{F}_3$ assuming such an order leads to a contradiction with the property that for all $x, y, z \in F$ with $y < z$, we must have $x + y < x + z$. Therefore no order exists on $\mathbb{F}_3$ that would make it an ordered field.

Ex 2.6

Question

This exercise is a reminder that the "size" of infinite sets can be a bit counterintuitive.

1. Suppose $a, b \in \mathbb{R}$, with $a < b$. Prove that there is a bijection from (a, b) to $(0, 1)$.
2. Prove that there is a bijection from $(0, 1)$ to $(1, \infty)$.
3. Suppose $a, b \in \mathbb{R}$. Prove that there is a bijection from (a, ∞) to (b, ∞).

Proof

1. Suppose $a, b \in \mathbb{R}$ with $a < b$. We want a bijection from (a, b) to $(0, 1)$. I want to send a to 0 and b to 1, so

$$f(x) = \frac{x - a}{b - a}.$$

Since $a < x < b$, we get $0 < f(x) < 1$, so f maps into $(0, 1)$. If $f(x_1) = f(x_2)$ then $\frac{x_1 - a}{b - a} = \frac{x_2 - a}{b - a}$, which gives $x_1 = x_2$, so f is injective. For surjectivity, if $y \in (0, 1)$ then $x = a + y(b - a)$ lies in (a, b) and satisfies $f(x) = y$. So f is a bijection.

2. We want a bijection from $(0, 1)$ to $(1, \infty)$. Let

$$g(x) = \frac{1}{x}.$$

If $0 < x < 1$ then $1 < 1/x < \infty$, so $g(x) \in (1, \infty)$. If $g(x_1) = g(x_2)$ then $1/x_1 = 1/x_2$, so $x_1 = x_2$, so this function is injective. For surjectivity, given $y > 1$ we take $x = 1/y$, which lies in $(0, 1)$ and satisfies $g(x) = y$. So g is a bijection.

3. Suppose $a, b \in \mathbb{R}$. WTS there is a bijection from (a, ∞) to (b, ∞). Both sets are unbounded intervals, and WTS if we can list out all the elements of one interval as a sequence, then we can relabel that list to get the other interval.
Consider (a, ∞). For each natural number n we can define

$$x_n = a + n.$$

This gives a sequence (x_n) in (a, ∞), and in fact every point of (a, ∞) lies between some consecutive terms of this sequence. Similarly, we can build a sequence in (b, ∞) by

$$y_n = b + n.$$

Again, every point of (b, ∞) lies between some consecutive y_n's.
The map $h : (a, \infty) \to (b, \infty)$ defined by

$$h(x) = x + (b - a)$$

lines these sequences up. We have $h(x_n) = y_n$ for every n. If $x > a$, then $h(x) > b$, so the map goes into (b, ∞). If $h(x_1) = h(x_2)$, then $x_1 + (b - a) = x_2 + (b - a)$ so $x_1 = x_2$, proving injectivity. And given any $y > b$, we can write $y = h(x)$ with $x = y - (b - a) > a$, so the map is also surjective.
Thus h is a bijection from (a, ∞) to (b, ∞).