# Math 2550

## PSET 3

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# Contents

## Ex 3.1

**Question**

Prove that $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

**Proof**

Suppose for contradiction that $\mathbb{N}$ is bounded above in $\mathbb{R}$. Then there is some $x' \in \mathbb{R}$ such that $n \leq x'$ for all $n \in \mathbb{N}$. We can take the floor of $x'$, $k = \lfloor x' \rfloor$. We need to take the floor of the upper bound as we know our upper bound would be in $\mathbb{R}$ but it is not guranteed to be in the set $\mathbb{N}$ but its floor is. Then $k \in \mathbb{N}$ and $k \leq x'$. But $k+1$ is also in $\mathbb{N}$ and $k+1 > x'$. This contradicts $x'$ being an upper bound. So $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

## Ex 3.2

**Question**

In class we proved: if $I_n \subset \mathbb{R}$ is a closed interval for each $n \in \mathbb{N}$, and $I_{n+1} \subset I_n$ for all $n$, then $\bigcap_{n=1}^{\infty} I_n$ is nonempty. What if the $I_n$ were open intervals instead? Then is $\bigcap_{n=1}^{\infty} I_n$ still nonempty? Prove it or give a counterexample.

**Proof**

The result does not always hold for open intervals. Take $I_n = (0, 1/n)$ for $n \in \mathbb{N}$. These are all open intervals and they are nested, since if $n < m$ then $1/m < 1/n$ and so $(0, 1/m) \subset (0, 1/n)$. So $I_{n+1} \subset I_n$.

Now look at the intersection $\bigcap_{n=1}^{\infty} I_n$. 0 cannot be in any $I_m$ because our interval is open on 0. Next, take any $x > 0$ for $x$ to be in $I_n$ we need $0 < x < 1/n$. But as $n$ grows, $1/n$ gets smaller and smaller. Since $1/n \longrightarrow 0$ as n increases. We can always pick an $n$ so large that $1/n$ becomes less then our orignal $x$. Once this happens, the condition $x < 1/n$ fails, so $x \notin (0, 1/n)$.
So $x$ is not part of that subset and outside the interval and therefore is not in the intersection. You can do this for all $x > 0$, no positive number can lie in all of the $I_n$. So for open intervals the intersection can be empty.

## Ex 3.3

**Question**

A real number $x$ is called *algebraic* if there exist $n \in \mathbb{N}$ and $a_0, \ldots, a_n \in \mathbb{Z}$, with $a_0 \neq 0$, such that

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n = 0.$$

1. Prove that $\sqrt{5}$ and $\sqrt{2+\sqrt{3}}$ are algebraic.
2. Prove that the set of all algebraic real numbers is countable. (You may use without proof the fact that a polynomial of degree $n$ has at most $n$ roots.)
3. Prove that there exist real numbers which are not algebraic.

**Proof**

1. Let $y = \sqrt{5}$. To check if $y$ is algebraic we need to see if it is the root of a polynomial with integer coefficients. We can square both sides

$$y^2 = 5 \Rightarrow y^2 - 5 = 0.$$

This is now a polynomial equation with integer coefficients. The polynomial is $x^2 - 5$. Our leading coefficient is 1, this means we satisfy the requirement of $a_0 \neq 0$. Our $y$ being eqyal to 5 is a solution to the polynomial which means that $\sqrt{5}$ is algebraic.

Let $z = \sqrt{2 + \sqrt{3}}$. To check if $z$ is algebraic we want to see if it is the root of a polynomial with integer coefficients. We can square both sides

$$z^2 = 2 + \sqrt{3} \Rightarrow \sqrt{3} = z^2 - 2.$$

We need to square again to remove the radical

$$(z^2 - 2)^2 = 3 \Rightarrow z^4 - 4z^2 + 4 = 3 \Rightarrow z^4 - 4z^2 + 1 = 0.$$

The polynomial is $x^4 - 4x^2 + 1$ has integer coefficients and the leading coefficient is 1, so it satisfies the requirement of $a_0 \neq 0$. z would be a solution to the polynomial so $\sqrt{2 + \sqrt{3}}$ is algebraic.

2. WTS that the set of all algebraic real numbers is countable. Every algebraic number is a root of some polynomial with integer coefficients.

We proved in class that the set $\mathbb{Z}$ is countable as you can make a bijection between it and $\mathbb{N}$, and any finite product of countable sets is also countable. So the set of all $(a_0, a_1, \ldots, a_n)$ with $a_i \in \mathbb{Z}$ is countable. We also proved in class that the union of countable sets is counatiable, the set of all polynomials with integer coefficients is a union of countable sets, so it is countable.

We also know that a polynomial of degree $n$ has at most $n$ roots. So each polynomial gives only finitely many algebraic numbers. Taking the union over all polynomials, we get the set of all algebraic numbers. This is a countable union of finite sets, which as proved in class means its counatiable. So the set of all algebraic real numbers is countable.

3. WTS that there exist real numbers which are not algebraic. We already showed that the set of algebraic real numbers is countable.

We also proved in class that the set of all real numbers $\mathbb{R}$ is uncountable. So if every real number were algebraic, then $\mathbb{R}$ would also have to be countable. But this is a contradiction since $\mathbb{R}$ is uncountable.

Therefore there must be real numbers that are not algebraic.

## Ex 3.4

**Question**

Suppose $a, b \in \mathbb{R}$ with $a < b$. Prove that there are uncountably many irrational numbers in the interval $(a, b)$.

**Proof**

Assume for contradiction the irrationals in $(a, b)$ are countable, so we can list them as $x_1, x_2, x_3, \ldots$. We could then build intervals $I_n = [a_n, b_n]$ where we start with $I_1 \subset (a, b)$ that does not contain $x_1$. Given an $I_n$ we can then pick $I_{n+1} \subset I_n$ that does not contain $x_{n+1}$.

So we get a chain of nested intervals

$$I_1 \supset I_2 \supset I_3 \supset \ldots$$

and their lengths $b_n - a_n$ shrink to 0. The sequence of left endpoints $(a_n)$ is increasing and bounded above, while the sequence of right endpoints $(b_n)$ is decreasing and bounded below. So $a_n \longrightarrow L$ and $b_n \longrightarrow L$ for some $L \in \mathbb{R}$.

$L$ lies in every $I_n$, and since each $I_n \subset (a, b)$ we know $L \in (a, b)$. But we also made sure that at step $n$ that $x_n \notin I_n$, but $L \in I_n$. So $L \neq x_n$ for all $n$. That means $L$ is an irrational in $(a, b)$ that was not in our original list which is a contradiction so the irrationals in $(a, b)$ are uncountable.

## Ex 3.5

**Question**

Are the following sets finite, countable or uncountable? Prove your answers.

1. The set of all finite subsets of $\mathbb{N}$.
2. The set of all subsets of $\mathbb{N}$.
3. The set of all functions $f : \mathbb{Q} \to \mathbb{Q}$.

**Proof**

1. WTS that the set of all finite subsets of $\mathbb{N}$ is countable. We can split the set up by the size of the subsets. For each $k \in \mathbb{N}$, let $A_k$ be the set of subsets of $\mathbb{N}$ that have exactly $k$ elements. Then every finite subset must appear in one of these sets, so the collection of all finite subsets is

$$\bigcup_{k=0}^{\infty} A_k.$$

In class we proved that $\mathbb{N}$ is countable because we can build a bijection $f : \mathbb{N} \to \mathbb{N}$ with $f(n) = n$. We also proved that the cartesian product of countable sets is countable, so $\mathbb{N}^k$ is countable. Each $A_k$ can be injected into $\mathbb{N}^k$ where a $k$-element subset corresponds to a $k$-tuple so each $A_k$ is countable.
Finally, since the set of all finite subsets of $\mathbb{N}$ is a countable union of countable sets, it is countable.

2. WTS that the set of all subsets of $\mathbb{N}$ is uncountable. Suppose for contradiction that it was countable. Then we could list all subsets like

$$S_1, S_2, S_3, \ldots$$

where each $S_n \subseteq \mathbb{N}$.
Now we construct a new subset $T$ of $\mathbb{N}$ that is not equal to any $S_n$. Define $T$ so that for each $n \in \mathbb{N}$, we include $n$ in $T$ if and only if $n \notin S_n$. This means that $T$ differs from $S_1$ at element 1, it differs from $S_2$ at element 2, and in general it differs from $S_n$ at element $n$.
So $T \neq S_n$ for all $n$. But $T$ is still a subset of $\mathbb{N}$. This is a contradiction because we assumed our list contained all subsets.
Therefore the set of all subsets of $\mathbb{N}$ cannot be countable. So it is uncountable.

3. WTS that the set of all functions $f : \mathbb{Q} \to \mathbb{Q}$ is uncountable. In class we proved that $\mathbb{Q}$ is countable. So there is a bijection $b : \mathbb{N} \to \mathbb{Q}$. Using $b$ we can convert any function $f : \mathbb{Q} \to \mathbb{Q}$ into a function $h : \mathbb{N} \to \mathbb{N}$ by

$$h(n) = b^{-1}(f(b(n))).$$

It is a bijection so WTS that the set of all functions $\mathbb{N} \to \mathbb{N}$ is uncountable.
Assume for contradiction that the set of all functions $\mathbb{N} \to \mathbb{N}$ is countable. Then we can list them as $f_1, f_2, f_3, \ldots$. Define a new function $g : \mathbb{N} \to \mathbb{N}$ by

$$g(n) = f_n(n) + 1.$$

For each $n$, we have $g(n) \neq f_n(n)$, so $g \neq f_n$ for every $n$ on the list. This contradicts the assumption that our list contained all functions $\mathbb{N} \to \mathbb{N}$. So the set of all functions $\mathbb{N} \to \mathbb{N}$ is uncountable. and since it was a bijection the set of all functions $\mathbb{Q} \to \mathbb{Q}$ is also uncountable.