Math 2550

PSET 2

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Contents

Ex 1.1

Question

Let F be any ordered field (for example $F = \mathbb{Q}$ or $F = \mathbb{R}$). Suppose $A \subset F$ has a supremum. Suppose $\epsilon \in F$ with $\epsilon > 0$. Prove that there exists some $x \in A$ such that $x > (\sup A) - \epsilon$.

Proof

A having a supremum means that there is some $y \in F$ that is a least upper bound for F such that for any upper bound $y' \in F$, $y \leq y'$. Yet for all $x \in A$, $x \leq y$.
Let us suppose that the statement is false. That would mean that $\forall x \in A$, $x \leq (\sup A) - \epsilon$, wehre $\epsilon \geq 0$
Letting $\sup A = y$ this would mean that there exists some uppwer bound $y - \epsilon < y$, since $\epsilon > 0$. Which contradicts the statement that y is a upper bound of A such that for all other upper bounds, y' $y \leq y'$
Meaning the statement is true.

Ex 1.2, Rudin 1.5

Question

Let $A \subset \mathbb{R}$ be nonempty and bounded below. Define

$$-A = \{-x \mid x \in A\}. \tag{1}$$

Prove that $\inf A = -\sup(-A)$.

Proof

The infemum of A is a number $y \in A$ such that y is a lower bound for A and any lower bound y' for A has $y' \leq y$. The supremum of A is a number $z \in -A$ such that z is a lower bound for $-A$ and any lower bound z' for A has $z \leq z'$. $\mathbb{R}$ is an ordered field and set meaning that the subset A of $\mathbb{R}$ inherits that ordering. So for $x, x' \in A$, $x < x'$ if $x < x'$ in $\mathbb{R}$ but since $-A$ is defined by multiplying all values of $x \in A$ by -1 and if $x < x'$ and $c < 0$ where c is -1 then $xc > x'(c)$. This means that the ordering of A is flipped in $-A$. So what would have been our infimum of A is now the supremum of A. Yet the sign is flipped again in the equation when we take the negative of the supremum of $-A$ making it once again the infimum of A

Ex 1.3

Question

Suppose $A, B \subset \mathbb{R}$ are both nonempty and bounded above. Define the set $A + B$ by

$$A + B = \{a + b \mid a \in A, b \in B\}. \tag{2}$$

Prove that $\sup(A + B) = \sup A + \sup B$.

Proof

The supremum of a set is a number x in the set such that x is a lower bound for the set and any lower bound x' for the set has $x \leq x'$. $\mathbb{R}$ is an ordered field meaning that if follows the order axiom that means that if $a < b$ and $c < d$ then $a + b < c + d$. A and B are both subsets of the ordered field R meaning that they also have an ordering. So that if a, a' are in $\mathbb{R}$ and b, b' are in $\mathbb{R}$ with $a < a'$ and $b < b'$ it follows that $a + b < a' + b'$ by the order axiom of fields so for all $x \in \mathbb{R}$ there exists some least upper bound q such that $\forall x \in A$, $x \leq q$ and the same applies for B. Following the order axiom that means that $q + p \geq a + b$

for all $a + b \in A, B$

Ex 1.4

Question

In each of the following, S is an ordered set, and $A \subset S$. Answer the following in each case, and prove your answers:

- Is A bounded above?
- Does A have a maximum element, and if so, what is it?
- Does A have a supremum in S, and if so, what is it?

1. $S = \mathbb{Z}$, $A = \{2, 3\}$.
2. $S = \mathbb{Q}$, $A = \{-\frac{2n}{5} \mid n \in \mathbb{N}\}$.
3. $S = \mathbb{Q}$, $A = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$.
4. $S = \mathbb{Q}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.
5. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x \leq 1\}$.
6. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.

Proof

1. $S = \mathbb{Z}$, $A = \{2, 3\}$
   (a) It is bounded above, it has 2 elements $2, 3$ and is a subset of the ordered set $\mathbb{Z}$ this means that it inheris the ordering. With $2 < 3$. So A is bounded above by 3
   (b) Yes, 3.
   (c) Yes 3.
2. $S = \mathbb{Q}$, $A = \{-\frac{2n}{5} \mid n \in \mathbb{N}\}$.
   (a) no
   (b) no
   (c) no
3. $S = \mathbb{Q}$, $A = \{-\frac{1}{n} \mid n \in \mathbb{N}\}$.
   (a) Yes
   (b) Yes, 1
   (c) Yes, 1
4. $S = \mathbb{Q}$, $A = \{\frac{1}{n} \mid n \in \mathbb{N}\}$.
   (a) Yes
   (b) Yes
   (c) Yes, 1
5. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x \leq 1\}$.
   (a) Yes

(b) Yes, 1

(c) Yes, 1

6. $S = \mathbb{Q}$, $A = \{x \in \mathbb{Q} \mid 0 < x < 1\}$.

(a) Yes

(b) No

(c) Yes, 1

Question

Suppose F is an ordered field and $x, y, z \in F$.

1. Prove that if $x > 0$, then $x^{-1} > 0$.
2. Prove that if $x > 0$, then $y > z$ if and only if $xy > xz$.
3. Recall the field $\mathbb{F}_3 = \{0, 1, 2\}$ which we discussed in class. Prove that there does not exist an order on $\mathbb{F}_3$ such that it is an ordered field.

Proof

1. F is an ordered field and fields have the property that $\forall x \in F \exists$ some $x' \in F$ such that $x \cdot x^{-1} = 1$ We know that $x > 0$ and that x' is its multiplicative inverse. Lets suppose for sake of contradiction that $x^{-1} < 0$. This means that x is a positive integer and x' is a negative integer. The sign rules of multiplication state that a negative times a positive leads to a negative but if x^{-1} is a multiplicative inverse of x then x times x^{-1} is a negative number which contradicts the statement that $x \cdot x^{-1} = 1$

2. Let us suppose that $x > 0$ and for sake of contradiction that $y < z$ however $xy > yz$. Since F is an ordered field that means that there is some $x^{-1} \forall x \in F$ such that $x \cdot x^{-1} = 1$. This along with the prperty that if $x, y \in F$ then $xy \in F$ means that we can multiply both sides of the inequality $xy > yz$ by x^{-1}

$$xy > yx \Rightarrow x \cdot x^{-1} \cdot y > x \cdot x^{-1} \cdot z \Rightarrow (x \cdot x^{-1})y > (x \cdot x^{-1})z$$

$$\rightarrow (1)y > (1)z \Rightarrow y > z$$

Which contradicts the statement that $y < z$.
Now lets do the other side let us suppose for sake of contradiction that $xy < xz$ however $y > z$. Since F is an ordered field that means that there is some $x^{-1} \forall x \in F$ such that $x \cdot x^{-1} = 1$. This along with the prperty that if $x, y \in F$ then $xy \in F$ means that we can multiply both sides of the inequality $xy > yz$ by x^{-1}

Question

This exercise is a reminder that the "size" of infinite sets can be a bit counterintuitive.

1. Suppose $a, b \in \mathbb{R}$, with $a < b$. Prove that there is a bijection from (a, b) to $(0, 1)$.
2. Prove that there is a bijection from $(0, 1)$ to $(1, \infty)$.
3. Suppose $a, b \in \mathbb{R}$. Prove that there is a bijection from (a, ∞) to (b, ∞).