Irrationals in $(0,1)$ are uncountable.
$(0,1)$ is uncountable (Cantor); $\mathbb{Q}\cap(0,1)$ is countable. If the irrationals $I=(0,1)\setminus\mathbb{Q}$ were countable, then $(0,1)=I\cup(\mathbb{Q}\cap(0,1))$ would be a countable union of countables $\Rightarrow$ countable — contradiction.

Def (Distance to a set). $d(x,A)=\inf\{d(x,a):a\in A\}$; $d(x,A)=0 \iff x\in\overline{A}$.

Metric counterexamples. (1) $d(x,y)=|x^2-y^2|$ *fails identity of indiscernibles*: $d(1,-1)=0$ but $1\neq -1$.
(2) $d(x,y)=|x-2y|$ *fails symmetry*: $d(0,1)=2$ but $d(1,0)=1$.

Finite codomain ≥ 2 $\rightarrow$ uncountable. If D infinite countable, $|C|\geq 2$ finite, then C^D uncountable. Reason: inject $\{0,1\}^{\mathbb{N}}$ (already uncountable).

Counterexample style. "Functions $\mathbb{N}\to\{0,1\}$ are countable." False — diagonalization. "Functions $\mathbb{N}\to\{0\}$ are uncountable." False — only 1 function.

Encoding trick. Any function $f:\mathbb{N}\to A$ can be seen as an infinite sequence $(f(1),f(2),\dots)$. So $A^{\mathbb{N}}$ is the set of A-valued sequences.

Dense corollary. If A is uncountable, then $A^{\mathbb{N}}$ is also uncountable (since already true for 2-element subset).

Other common T/F. - $\mathbb{N}^{\mathbb{N}}$ is countable. (False, diag argument.) - $\mathbb{Q}^{\mathbb{N}}$ is countable. (False, inject $\{0,1\}^{\mathbb{N}}$.) - $\mathbb{N}^{\mathbb{Q}}$ is uncountable. (True, domain countable, codomain ≥ 2.) - If D finite, A countable $\Rightarrow A^D$ countable. (True.)

Finite-support sequences countable. $E=\{(a_n):\exists N,\ a_n=0\ \forall n>N\}$. Then $E=\bigcup_N A^{N+1}$ (finite product of countables) $\Rightarrow$ countable.

Polynomials $\rightarrow$ algebraic numbers countable. Coeff tuples $\in\mathbb{Z}^{k+1}$ countable, union over k countable. Each polynomial has finitely many roots. Countable union of finite sets $\Rightarrow$ algebraics countable.

Power set argument. $|\mathcal{P}(\mathbb{N})|=2^{\aleph_0}$ uncountable. Each subset $\leftrightarrow$ characteristic function $\mathbb{N}\to\{0,1\}$. So $2^{\mathbb{N}}$ uncountable. Generalizes: for $|A|\geq 2$, $A^{\mathbb{N}}$ uncountable.

$d(x,A)=0 \iff x\in\overline{A}$ (metric space).
($\Rightarrow$) Suppose $d(x,A)=0$. For each n pick $a_n\in A$ with $d(x,a_n)<\frac{1}{n}$ (by def. of infimum). Then $a_n\to x$, hence every ball around x meets A; thus $x\in\overline{A}$.
($\Leftarrow$) If $x\in\overline{A}$, then for each n there exists $a_n\in A$ with $d(x,a_n)<\frac{1}{n}$, so $\inf\{d(x,a):a\in A\}=0$. *Conclusion:* $d(x,A)=0$ iff $x\in\overline{A}$.

Finite codomain $\geq 2\Rightarrow C^D$ uncountable (countable D).
Let $D\cong\mathbb{N}$ and $|C|\geq 2$. Choose distinct $c_0,c_1\in C$. Map each binary sequence $b\in\{0,1\}^{\mathbb{N}}$ to $f_b:\mathbb{N}\to C$ by $f_b(n)=c_{b(n)}$. This is injective, so $|\{0,1\}^{\mathbb{N}}|\leq|C^{\mathbb{N}}|$. Since $\{0,1\}^{\mathbb{N}}$ is uncountable (Cantor diagonal), C^D is uncountable.

Finite-support sequences are countable.
Let A be countable and $E=\{(a_n):\exists N\ \forall n>N,\ a_n=0\}$. Then $E=\bigcup_{N\geq 0}A^{N+1}$ (the tail is forced zeros). Each A^{N+1} is a finite product of countables, hence countable; a countable union of countables is countable. Therefore E is countable.

Algebraic numbers are countable.
Fix a degree $k\geq 0$. Each polynomial $p(x)=a_kx^k+\cdots+a_0$ with $a_i\in\mathbb{Z}$ corresponds to a vector $(a_0,\dots,a_k)\in\mathbb{Z}^{k+1}$; $\mathbb{Z}^{k+1}$ is countable. Every nonzero degree-k polynomial has at most k roots, i.e. finitely many. Hence the set of roots of all degree-k polynomials is a countable union of finite sets $\Rightarrow$ countable. Take the union over k to conclude: algebraic numbers form a countable set.

Dense corollary: if A uncountable then $A^{\mathbb{N}}$ uncountable.
Any uncountable A contains a 2-element subset $\{a_0,a_1\}$. The injection $\{0,1\}^{\mathbb{N}}\hookrightarrow A^{\mathbb{N}}$ given by $b\mapsto(a_{b(n)})_{n\in\mathbb{N}}$ shows $A^{\mathbb{N}}$ is uncountable.

Finite domain + countable codomain $\Rightarrow$ countable.
If D is finite, C countable, then C^D is in bijection with a finite product of C's: $C^D\cong C^{|D|}$. Finite products of countables are countable; hence C^D is countable. *Template use:* Replace D by any finite set (e.g. $\{1,2,3\}$) and C by $\mathbb{N}$ or $\mathbb{Q}$.

Countable domain + finite codomain with ≥ 2 elements $\Rightarrow$ uncountable.
Let $D\cong\mathbb{N}$ and pick distinct $a,b\in C$ (with $|C|\geq 2$). Inject $\{0,1\}^{\mathbb{N}}\hookrightarrow C^D$ via $b\mapsto f_b$ with $f_b(n)=a$ if $b(n)=0$, $f_b(n)=b$ if $b(n)=1$. Since $\{0,1\}^{\mathbb{N}}$ is uncountable (Cantor), so is C^D. *Template use:* Any time the domain is countably infinite and the codomain has at least 2 elements.

Diagonalization template (prove $A^{\mathbb{N}}$ uncountable when $|A|\geq 2$).
Assume $f_1,f_2,\dots$ list all $A^{\mathbb{N}}$. Choose distinct $a_0,a_1\in A$. Define g by $g(n)=a_0$ if $f_n(n)\neq a_0$, else $g(n)=a_1$. Then $g\neq f_n$ for every n (differs at n). Contradiction. *Template use:* Works verbatim for binary/ternary/finite alphabets.

Eventually-zero (finite-support) sequences are countable.
Let A be countable and $E=\{(a_n):\exists N\,\forall n>N,\ a_n=0\}$. Then $E=\bigcup_{N\geq 0}A^{N+1}$, a countable union of countables $\Rightarrow$ countable. *Template use:* Any "finitely non-default" objects (finite support, finite exceptions).

Eventually constant sequences over a finite alphabet are countable.
Fix finite C. For N,c, let $S_{N,c}=\{(x_n):x_n=c\ \forall n\geq N\}\cong C^N$ (finite). Then $\bigcup_{N,c}S_{N,c}$ is a countable union of finite sets $\Rightarrow$ countable.

When C^D is countable vs. uncountable (quick classifier).
Countable: (i) D finite, C countable; (ii) D finite, C finite.
Uncountable: (i) D infinite countable, $|C|\geq 2$ finite; (ii) D infinite countable, C uncountable; (iii) D uncountable and C has ≥ 2 elements. *Proof ideas:* finite products, injections from $\{0,1\}^{\mathbb{N}}$, or direct diagonalization.

Apply in 1–2 lines.
(1) $\{1,2,3\}\to\mathbb{N}$: finite domain + countable codomain $\Rightarrow$ countable (or prime-coding).
(3) $\mathbb{Q}^{\mathbb{N}}$: inject $\{0,1\}^{\mathbb{N}}\hookrightarrow\mathbb{Q}^{\mathbb{N}}$ via two rationals $\Rightarrow$ uncountable.
(3) $\mathbb{Q}^{\mathbb{N}}$: inject $\{0,1\}^{\mathbb{N}}\hookrightarrow\mathbb{Q}^{\mathbb{N}}$ via two rationals $\Rightarrow$ uncountable.
(4) $\mathbb{N}^{\mathbb{Q}}$: domain countable? No, codomain $\mathbb{N}$ (≥ 2) and domain *uncountable* $\Rightarrow$ uncountable (even larger).
(5) Functions $\mathbb{N}\to\{0\}$: only one function $\Rightarrow$ finite (countable).