Math 2550

PSET 3

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Contents

Ex 3.1

Question

For each of the following, determine whether X with the distance function d is a metric space, and prove your answer.

1. $X = \mathbb{R}$, $d(x, y) = |x^2 - y^2|$
2. $X = \mathbb{R}$, $d(x, y) = |x - 2y|$
3. $X = \mathbb{R}$, $d(x, y) = \frac{|x-y|}{1+|x-y|}$

Proof

1. We check the 4 conditions. First, $d(p, q) = |p^2 - q^2| > 0$ which isn't always true for example $d(1, -1) = 0$ so this condition does not hold and so this is not a metric space.

2. We check the 4 conditions
First, $d(x, y) = |x - 2y| > 0$ always so we satisfy the nonnegativity condition. Second, $d(p, p) = 0$ is not satisfied because $|p - 2p| = 0$ is only true when $p = 0$ since this fails this is not a metric space.

3. We check the 4 conditions. First, $d(x, y) = \frac{|x-y|}{1+|x-y|} \geq 0$ always. Second, $d(x, x) = \frac{0}{1+0} = 0$, and if $d(x, y) = 0$ then $|x - y| = 0$ so $x = y$. Third, $d(x, y) = \frac{|x-y|}{1+|x-y|} = \frac{|y-x|}{1+|y-x|} = d(y, x)$. Fourth, we check the triangle inequality. Let $a = |x - y|$, $b = |y - z|$, $c = |x - z|$. Then $c \leq a + b$ and the function $f(t) = \frac{t}{1+t}$ is increasing. So

$$d(x, z) = f(c) \leq f(a + b).$$

We need $f(a + b) \leq f(a) + f(b)$.

$$f(a) + f(b) - f(a + b) = 1 - \frac{1}{1 + a} - \frac{1}{1 + b} + \frac{1}{1 + a + b}.$$

Since $(1 + a + b) \leq (1 + a)(1 + b)$, we have $\frac{1}{1+a+b} \geq \frac{1}{(1+a)(1+b)}$, so the RHS is $\geq \frac{ab}{(1+a)(1+sob)} \geq 0$. So $f(a + b) \leq f(a) + f(b)$.
All conditions met this is a metric space

Ex 3.2

Question

Let X be any set, let and $d : X \times X \longrightarrow \mathbb{R}$ be the discrete metric, defined by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$$

for all $x, y \in X$.

1. Prove that, with this distance function, X is a metric space.
2. For any $x \in X$, what is $N_\epsilon(x)$ when $\epsilon = \frac{1}{2}$, 1, and 2?
3. Which subsets of X are open? Which are closed?

Proof

1. need to check the four conditions. (i) $d(p,q) \geq 0$ since $d(p,q)$ is either 0 or 1.
(ii) $d(p,q) = 0$ iff $p = q$ by definition.
(iii) $d(p,q) = d(q,p)$ because $p \neq q$ is the same as $q \neq p$.
(iv) $d(p,q) \leq d(p,r) + d(r,q)$. if $p = q$ then $d(p,q) = 0$ and the inequality is true. if $p \neq q$ then $d(p,q) = 1$. if $r = p$ then RHS $= 0 + 1 = 1$. if $r = q$ then RHS $= 1 + 0 = 1$. if r is different from both then RHS $= 1 + 1 = 2$. in all cases the inequality holds. so this is a metric space.

2. for open neighborhoods $N_\epsilon(x)$. since $d(x,y) \in \{0,1\}$ we check cases. if $\epsilon = \frac{1}{2}$ then only $y = x$ works so $N_{1/2}(x) = \{x\}$. if $\epsilon = 1$ then again only $y = x$ works so $N_1(x) = \{x\}$. if $\epsilon = 2$ then both 0 and 1 are less than 2 so $N_2(x) = X$.

3. let $A \subset X$ and $x \in A$. then $N_{1/2}(x) = \{x\} \subset A$ so A is open. but then A^c has the same property so A^c is also open. so every subset of X is both open and closed.

Ex 3.3

Question

Show that the subset of $\mathbb{R}^2$ given by

$$E = \{(x,y) \in \mathbb{R}^2 \mid x < y\}$$

is open.

Proof

Let $(x,y) \in E$. WTS that it is an interior point. Since $x < y$ we have $y - x > 0$. Let $y - x = \lambda$ and set $\epsilon = \frac{\lambda}{3}$. Now WTS $N_\epsilon(x,y) \subset E$.
So take another point $(a,b) = q \in \mathbb{R}^2$ with $d(p,q) = \sqrt{(x-a)^2 + (y-b)^2} < \epsilon$. Then $(x-a)^2 \leq (x-a)^2 + (y-b)^2 < \epsilon^2$, so $|x-a| < \epsilon$, and similarly $|y-b| < \epsilon$.
Then

$$b - a = (y-x) + (b-y) - (a-x).$$

Using triangle inequality,

$$b - a \geq (y-x) - |b-y| - |a-x| > \lambda - \epsilon - \epsilon = \lambda - 2\epsilon = \tfrac{\lambda}{3} > 0.$$

So $b > a$, meaning $(a,b) \in E$.
So $N_\epsilon(x,y) \subset E$. Since (x,y) was arbitrary, every point of E is interior, so E is open.

Ex 3.4

Question

Construct a bounded set of real numbers with exactly three limit points (using the standard metric on $\mathbb{R}$). (You need not prove carefully what the limit points are; it is sufficient to give the set and state what are the limit points.)

Proof

$$A = \left\{\tfrac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{1 + \tfrac{1}{n} : n \in \mathbb{N}\right\} \cup \left\{2 + \tfrac{1}{n} : n \in \mathbb{N}\right\}.$$

$A \subset [0,3]$, so it's bounded. The exactly three limit points are $0, 1, 2$.

Ex 3.5

Question

Let E be a subset of a metric space. Define the *interior* of E, denoted E°, to be the set of all interior points of E.

1. Prove that E° is always open.
2. Prove that E is open if and only if $E^\circ = E$.
3. Prove that, if G is an open subset of E, then $G \subset E^\circ$.
4. Prove that the complement of E° is the closure of the complement of E.
5. Do E and $\overline{E}$ always have the same interiors?
6. Do E and E° always have the same closures?

Proof

1. WTS E° is open. Take $x \in E^\circ$. By definition $\exists \epsilon > 0$ with $N_\epsilon(x) \subset E$. For any $y \in N_\epsilon(x)$ pick $\delta := \epsilon - d(x,y) > 0$. Then $N_\delta(y) \subset N_\epsilon(x) \subset E$, so y is also interior to E. so each point of E° has a neighborhood inn E°, so E° is open.
2. WTS: E open $\Leftrightarrow E^\circ = E$. ($\Longrightarrow$) If E is open and $x \in E$, then some neighborhood around x sits in E, so $x \in E^\circ$. Hence $E \subset E^\circ$, and always $E^\circ \subset E$, so $E^\circ = E$. ($\Longleftarrow$) If $E^\circ = E$, then by (1) E is open.
3. Let $G \subset E$ be open. Take $x \in G$. Since G is open, $\exists \epsilon > 0$ with $N_\epsilon(x) \subset G \subset E$. So x is interior to E, i.e. $x \in E^\circ$. Hence $G \subset E^\circ$.
4. Claim: $(E^\circ)^c = \overline{E^c}$. ($\subset$) If $x \notin E^\circ$, then no neighborhood around x sits inside E. Equivalently, every neighborhood around x meets E^c, i.e. $x \in \overline{E^c}$. ($\supset$) If $x \in \overline{E^c}$, every neighborhood around x meets E^c, so no neighborhood is contained in E; hence $x \notin E^\circ$. So equality holds.
5. Do E and $\overline{E}$ always have the same interiors? No. Example in $\mathbb{R}$: let $E = \mathbb{Q}$. Then $E^\circ = \varnothing$, but $\overline{E} = \mathbb{R}$ so $(\overline{E})^\circ = \mathbb{R}$.
6. Do E and E° always have the same closures? No. Same space, take $E = \mathbb{Q}$. Then $E^\circ = \varnothing$ so $\overline{E^\circ} = \varnothing$, while $\overline{E} = \mathbb{R}$.