Math 120

PSet 7

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Chapter 1

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Question 1

Evaluate the scalar line integral

$$\int_C (3x + y)\, ds,$$

where C is the line segment from $(-1, 3)$ to $(4, 2)$.

Solution:

$$\int_C (3x + y) ds$$

$$(-1, 3) \quad (4, 2)$$

$$f(t) = (1-, 3) + t((4, 2) - (-1, 3))$$

$$f(t) = (-1, 3) + t(5, -1) = \langle -1 + 5t, 3 - t \rangle$$

$$x = -1 + 5t \quad y = 3 - t \quad t \in [0, 1]$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = 5 \quad \frac{dy}{dt} = -1$$

$$ds = \sqrt{5^2 + (-1)^2}\, dt = \sqrt{26}\, dt$$

$$3x + y \Rightarrow 3(-1 + 5t) + (3 - t) \Rightarrow -3 + 15t + 3 - t = 14t$$

$$\int_0^1 14t\sqrt{26}\, dt \Rightarrow \sqrt{26} \int_0^1 14t\, dt$$

$$7\sqrt{26}t^2 \Big|_0^1 = 7\sqrt{26}(1)^2 - 7\sqrt{27}(0)^2 = 7\sqrt{26}$$

Question 2

In this problem we will sketch part of the argument that a scalar line integral $\int_C f\,ds$ is independent of the parameterization of C that we choose to compute the integral. Suppose $\vec{r}_1(t)$, $a \leqslant t \leqslant b$, and $\vec{r}_2(t)$, $c \leqslant t \leqslant d$, are two smooth parameterizations of the same smooth curve C. Assuming that both parameterizations are in the same direction it can be shown that $\vec{r}_2(t) = \vec{r}_1(w(t))$, for some increasing function $w(t)$ satisfying $w(c) = a$ and $w(d) = b$. If this is the case, show that

$$\int_a^b f(\vec{r}_1(t))\left|\vec{r}_1'(t)\right|\,dt = \int_c^d f(\vec{r}_2(t))\left|\vec{r}_2'(t)\right|\,dt$$

for any continuous function f.

Solution:

$$\int\limits_c f\,ds$$

$$\vec{r}_1(t) \quad a \leqslant t \leqslant b$$

$$\vec{r}_2(t) \quad c \leqslant t \leqslant d$$

$$\vec{r}_2(r) = \vec{r}_1(w(t)) \quad w(c) = a \quad w(d) = b$$

$$\int_a^b f(\vec{r}_1(t))|\vec{r}_1{}'(t)\,dt = \int_c^d f(\vec{r}_2(t))|\vec{r}_2{}'(t)\,dt$$

$$\vec{r}_2{}'(t) = \frac{d}{dt}\vec{r}_2(t) = \frac{d}{dt}\vec{r}_1(w(t)) = \vec{r}_1{}'(w(t))w'(t)$$

$$|\vec{r}_2{}'(t)| = |\vec{r}_1{}'(w(t))| \cdot |w'(t)|$$

$$\int_c^d f(\vec{r}_2(t))|\vec{r}_2{}'(t)|\,dt = \int_a^b f(\vec{r}_1(t))|\vec{r}_1{}'(w(t))| \cdot |w'(t)|\,dt$$

w maps $[c, d]$ to $[a, b]$, when $t = c\,, s = a$, and when $t = d\,, s = b$

$$\int_a^b f(\vec{r}_1)(w(t))|\vec{r}_1{}'(w(t))| \cdot |w(t)|\,dt = \int_a^b f(\vec{r}_1(s))|\vec{r}_1{}'(s)|\,ds$$

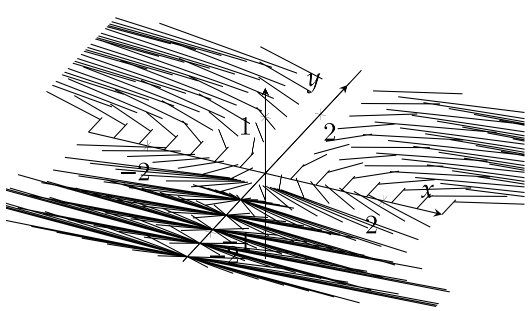
$$\int_a^b f(\vec{r}_1{}'(t))|\vec{r}_1{}'(t)|\,dt = \int_a^b f(r_1(s))|\vec{r}_1{}'(s)|\,ds = \int_c^d f(\vec{r}_2(t))|\vec{r}_2{}'(t)|\,dt$$

$\therefore$ the scalar line integral is independent of the parameterization and the equality holds true for any continuous function f

Question 3

Sketch the vector field $\vec{F}(x, y) = xy\,\hat{\imath} + \frac{1}{2}\,\hat{\jmath}$.

Solution:



Question 4

Given the contour diagram for a function f shown below, in which dark colors correspond to low values of f and light colors correspond to high values of f, sketch the gradient vector field $\vec{F} = \nabla f$.

Question 5

A thin wire has the shape of the curve C parameterized by $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 4\pi$, where x, y, and z are measured in centimeters. The linear density of the wire is given by $\rho(x, y, z) = x^2 z$ grams per centimeter. Find the mass of the wire.

Solution:

$$x = \cos t \quad y = \sin t \quad z = t$$

$$0 \leqslant t \leqslant 4\pi$$

$$\rho(x, y, x) = x^2 z \frac{\text{grams}}{\text{cm}}$$

$$\text{Mass: } = \int_C \rho(x, y, z) ds$$

$$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t \quad \frac{dz}{dt} = 1$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$ds = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (1)^2}\, dt$$

$$ds = \sqrt{1+1}\, dt = \sqrt{2}\, dt$$

$$\rho(x, y, z) = x^2 z \Rightarrow \cos^2 t \cdot t \Rightarrow t \cos^2 t$$

$$\text{Mass: } \int_0^{4\pi} \rho(t) ds = \int_0^{4\pi} t \cos^2 t \sqrt{2}\, dt$$

$$\sqrt{2} \int_0^4 t \cos^2 t dt \qquad \cos^2 t = \frac{1 + \cos 2t}{2}$$

$$\sqrt{2} \int_0^{4\pi} t \left(\frac{1 + \cos 2t}{2}\right) dt \Rightarrow \frac{\sqrt{2}}{2} \int_0^{4\pi} t(1 + \cos 2t)\, dt$$

$$\frac{\sqrt{2}}{2} \int_0^{4\pi} t\, dt + \frac{\sqrt{2}}{2} \int_0^{4\pi} t \cos 2t\, dt$$

$$\int_0^{4\pi} t\, dt = \left.\frac{t^2}{2}\right|_0^{4\pi} \Rightarrow \frac{\sqrt{2}}{2} \frac{16\pi^2}{2} - \frac{\sqrt{2}}{2} \frac{0}{2} = 4\sqrt{2}\pi^2$$

$$u = t \quad du = dt$$

$$v = \frac{1}{2} \sin 2t \quad dv = \cos 2t$$

$$\int t \cos 2t\, dt = t \cdot \frac{1}{2} \sin 2t - \int \frac{1}{2} \sin 2t dt = \frac{1}{2} t \sin 2t + \frac{1}{4} \cos 2t + k$$

$$\left[\frac{1}{2} t \sin 2t + \frac{1}{4} \cos 2t\right]_0^{4\pi} = \left(\frac{1}{2} \cdot 4\pi \cdot 0 + \frac{1}{4} \cdot 1\right) - \left(0 + \frac{1}{4} \cdot 1\right) = 0$$

$$4\sqrt{2}\pi^2 + 0 = 4\sqrt{2}\pi^2$$

Question 6

Let $\vec{F}$ be the vector field shown below, and let C be the unit circle, oriented clockwise. Is the vector line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

positive, negative, or zero? Explain your reasoning.

Question 7

Evaluate the line integral

$$\int_C \sin x \, dx + \cos y \, dy$$

where C consists of the top half of the circle $x^2 + y^2 = 1$ from $(1,0)$ to $(-1,0)$ and the line segment from $(-1,0)$ to $(-2,3)$. (Remember that when you see an integral that looks like

$$\int_C P(x,y)\, dx + \int_C Q(x,y)\, dy$$

it is a shorthand notation for

$$\int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r}$$

where $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$. The analogous thing is true in three dimensions.)

Solution:

$$x^2 + y^2 = 1 \quad x = \cos t \quad y = \sin t \quad t \in [0,\pi]$$

$$x(t) = (1-t)(-1) + t(-2) \quad y(t) = (1-t)(0) + t(3) \quad t \in [0,1]$$

$$\vec{F}(x,y) = \langle \sin x, \cos y \rangle$$

$$\int\limits_C \sin x \, dx + \cos y \, dy$$

$$x(t) = \cos t, \quad y(t) = \sin t, \quad t \in [0,\pi]$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t$$

$$\int\limits_{C_1} \sin x \, dx + \cos y \, dy = \int_0^\pi [\sin(\cos t)(-\sin t) + \cos(\sin t)\cos t]\, dt$$

$$\int_0^\pi \sin(\cos t)(-\sin t)\, dt + \int_0^\pi \cos(\sin t)\cos t \, dt$$

$$f(x,y) = -\cos x + \sin y$$

At $(1,0)$:

$$f(1,0) = -\cos(1) + \sin(0) = -\cos(1)$$

At $(-1,0)$:

$$f(-1,0) = -\cos(1)$$

$$\int_{C_1} \sin x\, dx + \cos y\, dy = f(-1,0) - f(1,0) = 0$$

$$x(t) = -1 - t, \quad y(t) = 3t, \quad t \in [0,1]$$

$$dx = -1\, dt, \quad dy = 3\, dt$$

$$\int_{C_2} \sin x\, dx + \cos y\, dy = \int_0^1 [\sin(-1-t)(-1) + \cos(3t)(3)]\, dt$$

Using $\sin(-1-t) = -\sin(1+t)$, the integral becomes:

$$\int_0^1 \sin(1+t)\, dt + 3\int_0^1 \cos(3t)\, dt$$

$$\int_0^1 \sin(1+t)\, dt = -\cos(1+t)\Big|_0^1 = -\cos(2) + \cos(1)$$

$$3\int_0^1 \cos(3t)\, dt = 3\left(\frac{1}{3}\sin(3t)\Big|_0^1\right) = \sin(3)$$

$$\int_{C_2} \sin x\, dx + \cos y\, dy = \cos(1) - \cos(2) + \sin(3)$$

$$\int_{C} \sin x\, dx + \cos y\, dy = \int_{C_1} \sin x\, dx + \cos y\, dy + \int_{C_2} \sin x\, dx + \cos y\, dy$$

Since $\int_{C_1} \sin x\, dx + \cos y\, dy = 0$, the total integral is:

$$\int_{C} \sin x\, dx + \cos y\, dy = \cos(1) - \cos(2) + \sin(3)$$

Question 8

Compute the line integral of the vector field

$$\vec{F}(x,y) = \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath}$$

along the parabola $x = 1 + y^2$ from $(2,-1)$ to $(2,1)$.

Solution:

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left[F_1(x(t),y(t))\frac{dx}{dt} + F_2(x(t),y(t))\frac{dy}{dt}\right] dt$$

$$y = t \quad t \in [-1,1] \quad x(t) = 1 + t^2$$

$$r(t) = \langle x(t), y(t)\rangle = \langle 1+t^2, t\rangle \quad t \in [-1,1]$$

$$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 2$$

$$\vec{F}(x,y) \Rightarrow \vec{F}(1+t^2,t) = \frac{1+t^2}{\sqrt{(1+t^2)^2+t^2}}\hat{\imath} + \frac{t}{\sqrt{(1+t^2)^2+t^2}}\hat{\jmath}$$

$$\left[\frac{1+t^2}{\sqrt{(1+t^2)^2+t^2}}\times 2t\right]+\left[\frac{t}{\sqrt{(1+t^2)^2+t^2}}\times 1\right]$$

$$\int_a^b \left[\frac{3t+2t^3}{\sqrt{(1+t^2)^2+t^2}}\right] dt$$

$$\int_a^b \frac{3t+2t^3}{\sqrt{1+3t^2+t^4}} dt$$

$$d\vec{r} = (2t\hat{\imath}, 2\hat{\jmath})$$

$$\vec{F}\cdot d\vec{r} = \frac{3t+2t^3}{\sqrt{1+3t^2+t^4}}$$

$$d\frac{d}{dt}\sqrt{1+3t^2+t^4} = \frac{3t+t^2}{\sqrt{t^4+3t^2+1}}$$

$$\vec{F}\cdot d\vec{r} = d\left(\sqrt{t^4+3t^2+1}\right)$$

$$\int_a^b \vec{F}d\vec{r} = \left[t^4+3t^2+1\right]_{-1}^{1} = \sqrt{1^4+3(1)^2+1} - \sqrt{1+3(-1)^2+(-1)^4} = 0$$

Question 9

Evaluate the line integral of the vector field

$$\vec{F}(x,y,z) = (x+y)\hat{\imath} + (y-z)\hat{\jmath} + z^2\hat{k}$$

along the path parameterized by

$$\vec{r}(t) = t^2\hat{\imath} + t^3\hat{\jmath} + t^2\hat{k}, \quad 0 \leqslant t \leqslant 1.$$

Solution:

$$\vec{F}(x,y,x) = (x+y)\hat{\imath} + (y-z)\hat{\jmath} + z^2\hat{k}$$

$$\vec{r}(t) = t^2\hat{\imath} + t^3\hat{\jmath} + t^2\hat{k}, \quad 0 \leqslant t \leqslant 1$$

$$\frac{d\vec{r}}{dt} = 2t\,\hat{\imath} + 3t^2\hat{\jmath} + 2t\,\hat{k}$$

$$\vec{F}(t) = \left[t^2+t^3\right]\,\hat{\imath} + \left[t^3-t^2\right]\,\hat{\jmath} + \left[t^4\right]\,\hat{k}$$

$$\vec{F}\cdot\frac{dr}{dt} = \left[\left(t^2+t^3\right)(2t)\right] + \left[\left(t^2-t^3\right)\left(3t^2\right)\right] + \left[\left(t^4\right)(2t)\right]$$

$$\vec{F}\frac{d\vec{r}}{dt} = 2t^3 - t^4 + 5t^5$$

$$\int_0^1 \left[2t^3 - t^4 + 5t^5\right] dt = \left[\frac{1}{2}t^{-}\frac{1}{5}t^5 + \frac{5t^6}{6}\right]_0^1$$

$$\left(\frac{1}{2} - \frac{1}{5} + \frac{5}{6}\right) - 0 = \frac{17}{15}$$

Question 10

For each of the following vector fields $\vec{F}$ and curves C, find a function f such that $\vec{F} = \nabla f$ and use this function to evaluate

$$\int_C \vec{F}\cdot d\vec{r}$$

along the given directed curve C.

1. $\vec{F}(x, y) = \langle x^2, y^2 \rangle$, C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.
2. $\vec{F}(x, y, z) = \langle e^y, xe^y, (z+1)e^z \rangle$, $C : \vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leqslant t \leqslant 1$.

Solution:

$$F = \nabla f$$

$$\int_C F \cdot d\vec{r} = f(\text{end point}) - f(\text{start point})$$

Problem 1

$$\frac{\partial f}{\partial x} = x^2 \quad \frac{\partial f}{\partial y} = y^2$$

$$f(x, y) = \int x^2 \, dx = \frac{1}{3}x^2 + g(y)$$

$$\frac{\partial g}{\partial y} = g'(y) = y^2$$

$$g(y) = \int_y^2 dy = \frac{1}{3}y^3$$

$$f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$$

$$\int F \, d\vec{r} = f(2, 8) - f(-1, 2) = 171$$

Problem 2

$$\vec{F}(x, y, z) = \langle e^y, xe^y, (z+1)e^z \rangle \quad \vec{r}(t) = \left(t, t^2, t^3\right)$$

$$\frac{\partial F}{x} = e^x \quad \frac{\partial F}{\partial y} = xe^y \quad \frac{\partial F}{\partial z} = (z+1)e^z$$

$$f(x, y, z) = \int e^y \, dx = xe^y + \rho(y, z)$$

$$\frac{\partial f}{\partial y} = xe^y + \frac{\partial \rho}{\partial y} \quad xe^y + \frac{\partial \rho}{\partial y} \quad \frac{\partial \rho}{\partial y} = 0$$

$$\rho(x, y, z) = \phi(z)$$

$$\frac{\partial f}{\partial z} = (z+1)e^z$$

$$\rho(z) = \int (z+1)e^z \, dz$$

$$u = z + 1 \quad du = 1 \, dz$$

$$v = e^z \, dv = e^z \, dz$$

$$\int (z+1)e^z \, dz = (z+1)e^z - \int e^z \, dz$$

$$(z+1)e^z - e^z \Rightarrow ze^z + e^z - e^z = ze^z$$

$$f(x, y, z) = xe^y + ze^z$$

$$f(1, 1, 1) = (1)e^1 + (1)e^1 = ze$$

$$f(0, 0, 0) = (0)e^0 + +(0)e^0 = 0$$

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 1, 1) - f(0, 0, 0) = 2e$$

Question 11

Clairaut's Theorem implies that if the vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ is conservative and P, Q, and R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

1. Use the statement above to show that the vector line integral

$$\int_C x\,dx + 2x\,dy + xz\,dz$$

is not independent of path.

2. Find two directed curves C_1 and C_2 that start at the same point and end at the same point, such that

$$\int_{C_1} x\,dx + 2x\,dy + xz\,dz \neq \int_{C_2} x\,dx + 2x\,dy + xz\,dz.$$

Question 12

The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\vec{r} = \langle x, y, z\rangle$ is

$$\vec{F}(\vec{r}) = K\frac{\vec{r}}{|\vec{r}|^3},$$

where K is a constant. Find the work done on the particle as it moves along the straight line from $(0, 3, 0)$ to $(1, 3, 2)$ in two ways:

1. Parameterize the line segment, and compute

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

directly.

2. Although $\vec{F}$ is not defined at the origin, it turns out that $\vec{F}$ is conservative on its domain. Find a potential function f, and use the Fundamental Theorem of Line Integrals to compute the work done on the particle.

Solution:

a)

$$\vec{r}(t) = (t, 3, 2t), \quad t \in [0, 1]$$

$$\vec{r}'(t) = \langle 1, 0, 2\rangle$$

$$|\vec{r}(t)| = \sqrt{5t^2 + 9}$$

$$\vec{F}(\vec{r}(t)) = K\frac{\langle t, 3, 2t\rangle}{(5t^2 + 9)^{3/2}}$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = K\frac{5t}{(5t^2 + 9)^{3/2}}$$

$$W = \int_0^1 K\frac{5t}{(5t^2 + 9)^{3/2}}\,dt$$

Substitution: $u = 5t^2 + 9$, $du = 10t\,dt$

$$W = K\int_9^{14} \frac{du}{2u^{3/2}}$$

$$W = \frac{K}{2}\int_{9}^{14} u^{-3/2}\,du$$

$$W = -K\left[u^{-1/2}\right]_{9}^{14} = K\left(\frac{1}{3} - \frac{1}{\sqrt{14}}\right)$$

b)

$$f(\vec{r}) = -\frac{K}{|\vec{r}|}$$

$$W = f(\vec{r}_B) - f(\vec{r}_A)$$

$$|\vec{r}_A| = 3, \quad |\vec{r}_B| = \sqrt{14}$$

$$W = K\left(\frac{1}{3} - \frac{1}{\sqrt{14}}\right)$$