

# Math 120 QR

Alex Hernandez Juarez

Fall 2024

# Contents

## Chapter 1

	Page
1.1 12.1 Notes (Three Dimensional Coodinate Systems)	2
1.2 12.2 Notes (Vectors)	2
1.3 12.3 Notes (Dot Product)	4
1.4 12.4 Notes (Cross Product)	6
1.5 12.5 Notes (Equations of Lines and Planes)	9
1.6 12.6	12

# Chapter 1

## 1.1 12.1 Notes (Three Dimensional Coordinate Systems)

### Definition 1.1.1: Distance Formula

Defintion:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



### Definition 1.1.2: Equation of a sphere

Defintion: An equation of a sphere with center  $C(h, k, l)$ , and radius  $r$  is

$$(x - h)^2 + (y - k)^2 + (z - l)^2$$

In particular, if the center is the origin  $O$ , than an equation of the sphere is

$$x^2 + y^2 + z^2$$



## 1.2 12.2 Notes (Vectors)

### Definition 1.2.1: Vector Addition

If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors positioned so the initial point of  $\mathbf{v}$  is at the terminal point of  $\mathbf{u}$ , then the **sum**  $\mathbf{u} + \mathbf{v}$  is the vector from the initial point of  $\mathbf{u}$  to the terminal point of  $\mathbf{v}$ .



### Definition 1.2.2: Scalar Multiplication

If  $c$  is a scalar and  $\mathbf{v}$  is a vector, then the **scalar multiple**  $c\mathbf{v}$  is the vector whose length is  $|c|$  times the length of  $\mathbf{v}$  and whose direction is the same as  $\mathbf{v}$  if  $c > 0$  and is opposite to  $\mathbf{v}$  if  $c < 0$  or  $\mathbf{v} = \mathbf{0}$ , then  $c\mathbf{v} = \mathbf{0}$ .



**Example 1.2.1:**

Given the points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$ , the vector  $\mathbf{a}$  with representation  $\overrightarrow{AB}$  is:

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**Example 1.2.2:**

If  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$ , then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

**Note:-**

Properties of vectors: If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_n$  and  $c$  and  $d$  are scalars then

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\mathbf{a} + \mathbf{a} + -\mathbf{a} = \mathbf{0}$
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
- $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
- $(cd)\mathbf{a} = c(d\mathbf{a})$
- $l\mathbf{a} = \mathbf{a}$



## 1.3 12.3 Notes (Dot Product)

### Definition 1.3.1: Dot Product

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then the **dot product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the number  $\mathbf{a} \cdot \mathbf{b}$  given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties of the Dot Product: If  $\mathbf{a}, \mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_3$  and  $c$  is a scalar, then

1.  $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2.  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3.  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4.  $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5.  $\mathbf{0} \cdot \mathbf{a} = 0$

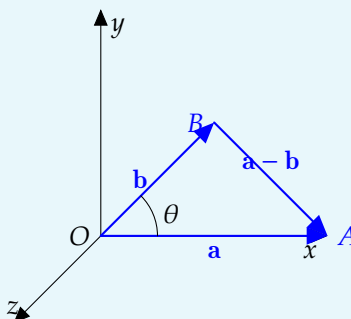


### Definition 1.3.2: Geometric Definition of the Dot Product

If  $\theta$  is the angle between vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

Proof:



$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta$$

Corollary: If  $\theta$  is the angle between nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$



### Note:-

Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal if and only if  $\mathbf{a} \cdot \mathbf{b} = 0$



### Example 1.3.1 (Direction Angles and Cosines)

The **direction angles** of a nonzero vector  $\mathbf{a}$  are the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  (in the interval  $[0, \pi]$ ) that  $\mathbf{a}$  makes with the positive  $x$ -,  $y$ -, and  $z$ -axes, respectively.

The cosines of these direction angles,  $\cos \alpha$ ,  $\cos \beta$ , and  $\cos \gamma$ , are called the **direction cosines** of the vector **a**. Using Corollary 6 with **b** replaced by **i**, we obtain:

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|} \quad (1)$$

Similarly, we also have:

$$\cos \beta = \frac{a_2}{|\mathbf{a}|} \quad \text{and} \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|} \quad (2)$$

By squaring the expressions in Equations 8 and 9 and adding, we see that:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad (3)$$

We can also use Equations 8 and 9 to write:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle = |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

Therefore,

$$\frac{1}{|\mathbf{a}|} \mathbf{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \quad (4)$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

### Definition 1.3.3: Projections

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number  $|\mathbf{b}| \cos \theta$ , where  $\theta$  is the angle between **a** and **b**. This is denoted by  $\text{comp}_{\mathbf{a}} \mathbf{b}$ . Observe that it is negative if  $\pi/2 < \theta \leq \pi$ . The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = |\mathbf{a}|(|\mathbf{b}| \cos \theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**. We summarize these ideas as follows.

**Scalar projection of b onto a:**  $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

**Vector projection of b onto a:**  $\text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$



## 1.4 12.4 Notes (Cross Product)

### Definition 1.4.1: Cross Product

Given two nonzero vectors  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , suppose that a nonzero vector  $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$  is perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ . Then  $\mathbf{a} \cdot \mathbf{c} = 0$  and  $\mathbf{b} \cdot \mathbf{c} = 0$ , and so:

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \quad (1)$$

$$b_1c_1 + b_2c_2 + b_3c_3 = 0 \quad (2)$$

To eliminate  $c_3$ , we multiply (1) by  $b_3$  and (2) by  $a_3$  and subtract:

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0 \quad (3)$$

Equation (3) has the form  $pc_1 + qc_2 = 0$ , for which an obvious solution is  $c_1 = q$  and  $c_2 = -p$ . So, a solution of (3) is:

$$c_1 = a_2b_3 - a_3b_2$$

$$c_2 = a_3b_1 - a_1b_3$$

Substituting these values into (1) and (2), we then get:

$$c_3 = a_1b_2 - a_2b_1$$


This means that a vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$  is:

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the **cross product** of  $\mathbf{a}$  and  $\mathbf{b}$  and is denoted by  $\mathbf{a} \times \mathbf{b}$ . 


### Definition 1.4.2: Cross Product of two vectors

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  then the **cross product** of  $\mathbf{a}$  and  $\mathbf{b}$  is:

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$



#### Note:-

Determinant of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$


#### Note:-

Determinant of order 3:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$


### Definition 1.4.3: Second definition of cross product

Arithmetic Definition:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = |\mathbf{a}||\mathbf{b}| \sin(\theta) \\ &= \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} i - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} j + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} k \\ &= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k \end{aligned}$$

The vector  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$



### Example 1.4.1: Proof that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1a_3b_2 - a_2a_1b_3 + a_2a_3b_1 + a_3a_1b_2 - a_3a_2b_1 \\ &= 0 \end{aligned}$$



### Definition 1.4.4: sin definition of cross product

If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$  (so  $0 \leq \theta \leq \pi$ ), then the length of the cross product  $\mathbf{a} \times \mathbf{b}$  is given by:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$$

Proof:

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2 \theta \quad (\text{by Theorem 12.3.3}) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2 \theta) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2 \theta \end{aligned}$$

Taking square roots and observing that  $\sqrt{\sin^2 \theta} = \sin \theta$  because  $\sin \theta \geq 0$  when  $0 \leq \theta \leq \pi$ , we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$





**Note:-**

Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = \mathbf{0}$$

**Example 1.4.2: Geometric interpretation of  $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$** 

If  $\mathbf{a}$  and  $\mathbf{b}$  are represented by directed line segments with the same initial point, then they determine a parallelogram with base  $|\mathbf{a}|$ , altitude  $|\mathbf{b}| \sin(\theta)$  and area

$$A = |\mathbf{a}|(|\mathbf{b}| \sin \theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product:

The length of the cross product of  $\mathbf{a} \times \mathbf{b}$  is equal to the area of the parallelogram determined by  $\mathbf{a}$  and  $\mathbf{b}$

**Note:-**

If we apply the following theorem:

The vector  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ , and

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

to the standard basis vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  using  $\theta = \frac{\pi}{2}$ , we obtain

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

**Note:-**

If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors and  $c$  is a scalar, then

1.  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2.  $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3.  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4.  $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5.  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6.  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

**Example 1.4.3: Proof of property 5 of cross products**

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ ,  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , and  $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ , then:

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$



### Definition 1.4.5: Triple Products

The product  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$  that occurs in Property 5 is called the *scalar triple product* of the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ .

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ . The area of the base parallelogram is  $A = |\mathbf{b} \times \mathbf{c}|$ . If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b} \times \mathbf{c}$ , then the height  $h$  of the parallelepiped is  $h = |\mathbf{a}| \cos \theta$ . (We must use  $|\cos \theta|$  instead of  $\cos \theta$  in case  $\theta > \pi/2$ .) Therefore, the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}| |\mathbf{a}| \cos \theta = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus, we have proved the following formula: The volume of the parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$



### Note:-

If we use the formula in  $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$  and discover that the volume of the parallelepiped determined by  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is 0, then the vectors must lie in the same plane; that is, they are coplanar.

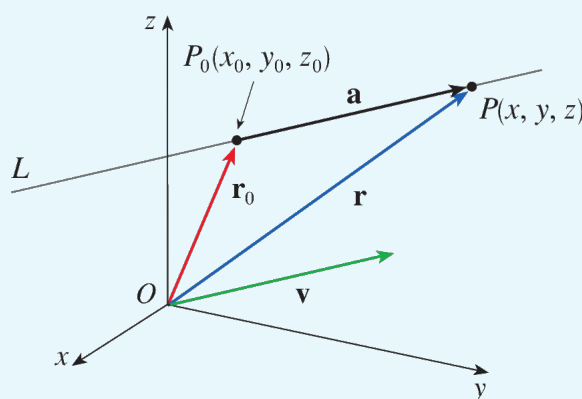


## 12.5 Notes (Equations of Lines and Planes)

### Definition 1.5.1: Hi

Likewise, a line  $L$  in three-dimensional space is determined when we know a point  $P_0(x_0, y_0, z_0)$  on  $L$  and a direction for  $L$ , which is conveniently described by a vector  $\mathbf{v}$  parallel to the line. Let  $P(x, y, z)$  be an arbitrary point on  $L$  and let  $\mathbf{r}_0$  and  $\mathbf{r}$  be the position vectors of  $P_0$  and  $P$  (that is, they have representations  $\overrightarrow{OP_0}$  and  $\overrightarrow{OP}$ ). If  $\mathbf{a}$  is the vector with representation  $\overrightarrow{P_0P}$ , as in Figure 1, then the Triangle Law for vector addition gives

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{a}.$$



Since  $\mathbf{a}$  and  $\mathbf{v}$  are parallel vectors, there is a scalar  $t$  such that  $\mathbf{a} = t\mathbf{v}$ . Thus

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$



**Note:-**

If the vector  $\mathbf{v}$  that gives the direction of the line  $L$  is written in component form as

$$\mathbf{v} = \langle a, b, c \rangle,$$

then we have  $t\mathbf{v} = \langle ta, tb, tc \rangle$ . We can also write  $\mathbf{r} = \langle x, y, z \rangle$  and

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle,$$

so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle.$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

**Example 1.5.1: Line example**

Find a vector equation and parametric equations for the line that passes through the point  $(5, 1, 3)$  and is parallel to the vector  $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ . Here  $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$  and  $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ , so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

Parametric equations are

$$x = 5 + t \quad y = 1 + 4t \quad z = 3 - 2t$$

**Note:-**

In general, if a vector  $\mathbf{v} = \langle a, b, c \rangle$  is used to describe the direction of a line  $L$ , then the numbers  $a$ ,  $b$ , and  $c$  are called *direction numbers* of  $L$ . Since any vector parallel to  $\mathbf{v}$  could also be used, we see that any three numbers proportional to  $a$ ,  $b$ , and  $c$  could also be used as a set of direction numbers for  $L$ .

Another way of describing a line  $L$  is to eliminate the parameter  $t$  from Equations 2. If none of  $a$ ,  $b$ , or  $c$  is 0, we can solve each of these equations for  $t$ :

$$t = \frac{x - x_0}{a} \quad t = \frac{y - y_0}{b} \quad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called symmetric equations of  $L$

**Definition 1.5.2: Line segment**

The line segment from  $\mathbf{r}_0$  to  $\mathbf{r}_1$  is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$



### Definition 1.5.3: Planes

A plane in space is determined by a point  $P_0(x_0, y_0, z_0)$  in the plane and a vector  $\mathbf{n}$  that is orthogonal to the plane. This orthogonal vector  $\mathbf{n}$  is called a **normal vector**. Let  $P(x, y, z)$  be an arbitrary point in the plane, and let  $\mathbf{r}_0$  and  $\mathbf{r}$  be the position vectors of  $P_0$  and  $P$ . Then the vector  $\mathbf{r} - \mathbf{r}_0$  is represented by  $\overrightarrow{P_0P}$ . The normal vector  $\mathbf{n}$  is orthogonal to every vector in the given plane. In particular,  $\mathbf{n}$  is orthogonal to  $\mathbf{r} - \mathbf{r}_0$  and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad (1.1)$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 \quad (1.2)$$

These can be referred to as the **vector equation of the plane**

To obtain a scalar equation for the plane, we write  $\mathbf{n} = \langle a, b, c \rangle$ ,  $\mathbf{r} = \langle x, y, z \rangle$ , and  $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ . then the vector equation becomes:

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Expanding the left side of this equation gives the following:

A **scalar equation of the plane** through the point  $P_0(x_0, y_0, z_0)$  with normal vector  $\mathbf{n} = \langle a, b, c \rangle$  is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

by collecting terms can be rewritten as:

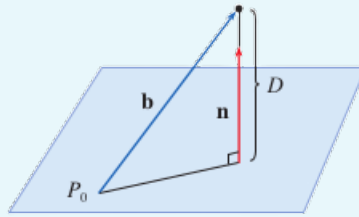
$$ax + by + cz + d = 0$$



### Definition 1.5.4: Distance of a plane

In order to find a formula for the distance  $D$  from a point  $P_1(x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$ , we let  $P_0(x_0, y_0, z_0)$  be any point in the given plane and  $\mathbf{b}$  be the vector corresponding to  $\overrightarrow{P_0P_1}$ . Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$



From Figure, you can see that the distance  $D$  from  $P_1$  to the plane is equal to the absolute value of the scalar projection of  $\mathbf{b}$  onto the normal vector  $\mathbf{n} = \langle a, b, c \rangle$ . Thus,

$$\begin{aligned} D &= |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$



## 1.6 12.6

### Definition 1.6.1: Cylinder

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.



### Definition 1.6.2: Quadric Surfaces

A Quadric Surface is the graph of a second-degree equation in three variables  $x$ ,  $y$ , and  $z$ . The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0$$

where  $A, B, C, \dots, J$  are constants, but by translation and rotation it can be brought into one of the two *standard forms*

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

