

# Math 120

## PSet 4

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# Chapter 1

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### Question 1

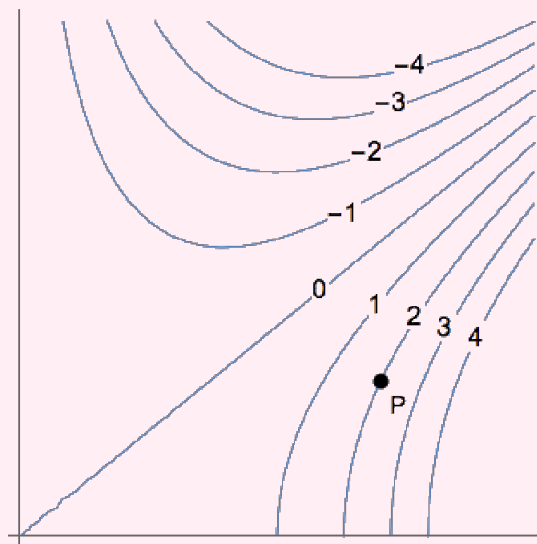
By finding both  $f_{xy} = (f_x)_y$  and  $f_{yx} = (f_y)_x$ , verify that Clairaut's Theorem holds for the function  $f(x, y) = y \arctan(xy)$ .

**Solution:**

$$\begin{aligned}f_x &= \frac{\partial}{\partial x} y \arctan(xy) = y \cdot \frac{1}{1 + (xy)^2} \cdot y = \frac{y^2}{1 + (xy)^2} \\ \frac{\partial}{\partial y} \frac{y^2}{1 + (xy)^2} &= \frac{(2y)(1 + (xy)^2) - (y^2)(2x^2y)}{(1 + (xy)^2)^2} \\ \frac{(2y)(1 + (xy)^2) - (y^2)(2x^2y)}{(1 + (xy)^2)^2} &= \frac{2y}{(1 + (xy)^2)^2} \\ f_y &= \frac{\partial}{\partial y} y \arctan(xy) = \arctan(xy) + \frac{xy}{1 + (xy)^2} \\ \frac{\partial}{\partial x} \arctan(xy) + \frac{xy}{1 + (xy)^2} &= \frac{y}{(1 + (xy)^2)} + \frac{y(1 + (xy)^2) - (2y^2x)(xy)}{(1 + (xy)^2)^2} = \frac{2y}{(1 + (xy)^2)^2}\end{aligned}$$

## Question 2

Level curves are shown below for a function  $f$



- Determine the signs of  $f_x$ ,  $f_y$ ,  $f_{xx}$  and  $f_{yy}$  at the point  $P$ . Explain your reasoning. *You should assume that the undrawn level curves are nicely and evenly distributed between the ones drawn.*
- Mark a point on the contour plot where  $f_x = 0$ . (You can either mark the point on a screenshot and insert the picture in your homework file, or just make a rough copy of the contour plot by hand.)

### Solution:

a)

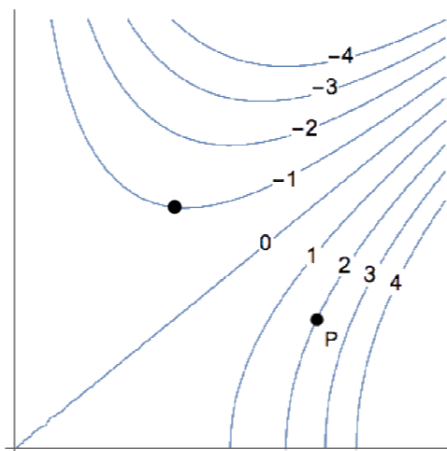
$f_x > 0$  as you move to the right of  $P$   $f$  increases.

$f_y < 0$  as you go up in the  $y$  direction at point  $P$  there is a decrease in the value of  $f$ .

$f_{xx} > 0$  The level curves in the  $x$ -direction appear to be getting closer together as we move to the right, so it means that  $f_x$  is increasing.

$f_{yy} > 0$  the difference in the level curves is decreasing as you move up so it means that  $f_y$  is increasing

b)



### Question 3

Use the Chain Rule to find the indicated partial derivatives.

(a) Compute  $\frac{dz}{dt}$  if  $z = \tan(y/x)$ ,  $x = e^t$ , and  $y = 1 - e^{-t}$ .

(b) Compute  $\frac{\partial M}{\partial u}$  and  $\frac{\partial M}{\partial v}$  at  $u = 3$  and  $v = -1$  if  $M = xe^{y-z^2}$ ,  $x = 2uv$ ,  $y = u - v$ , and  $z = u + v$ .

**Solution:**

a)

$$\begin{aligned}\frac{dz}{dt} &= \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt} \\ \frac{dz}{dx} &= z_x = \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{(0)x - (1)(y)}{x^2}\right) \Rightarrow \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{-y}{x^2}\right) \\ \frac{dz}{dy} &= z_y = \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{(1)x - (0)(y)}{x^2}\right) \Rightarrow \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{1}{x}\right) \\ \frac{dx}{dt} &= e^t \quad \frac{dy}{dt} = e^{-t} \\ \frac{dz}{dt} &= \sec\left(\frac{1 - e^{-t}}{e^t}\right)^2 \cdot \left(\frac{-(1 - e^{-t})}{e^{2t}}\right) \cdot e^t + \sec\left(\frac{1 - e^{-t}}{e^t}\right)^2 \cdot \left(\frac{1}{e^t}\right) \cdot e^{-t}\end{aligned}$$

b)

$$\begin{aligned}\frac{\partial M}{\partial u} &= \frac{\partial M}{\partial x} \cdot \frac{dx}{du} + \frac{\partial M}{\partial y} \cdot \frac{dy}{du} + \frac{\partial M}{\partial z} \cdot \frac{dz}{du} \\ M_x &= e^{y-z^2} \quad \frac{dx}{du} = 2v \quad \frac{dx}{dv} = 2u \\ M_y &= xe^{y-z^2} \quad \frac{dy}{du} = 1 \quad \frac{dy}{dv} = -1 \\ M_z &= -2zxe^{y-z^2} \quad \frac{dz}{du} = 1 \quad \frac{dz}{dv} = 1 \\ \frac{\partial M}{\partial u} &= e^{y-z^2}(2v) + (xe^{y-z^2})(1) + (-2zxe^{y-z^2})(1) \\ x &= 2uv \Rightarrow 2(3)(-1) = -6 \\ y &= u - v \Rightarrow (3) - (-1) = 4 \\ z &= u + v \Rightarrow 3 + (-1) = 2 \\ \frac{\partial M}{\partial u} &= e^{4-2^2}(2(-1)) + ((-6)e^{4-2^2})(1) + (-2(2)(-6)e^{4-2^2})(1) = 1(-2) + (-6)(1) + (24)(1) = 16 \\ \frac{\partial M}{\partial v} &= \frac{\partial M}{\partial x} \cdot \frac{dx}{dv} + \frac{\partial M}{\partial y} \cdot \frac{dy}{dv} + \frac{\partial M}{\partial z} \cdot \frac{dz}{dv} \\ \frac{\partial M}{\partial v} &= e^{y-z^2}(2u) + (xe^{y-z^2})(-1) + (-2zxe^{y-z^2})(1) \\ \frac{\partial M}{\partial v} &= e^{4-2^2}(2(3)) + ((-6)e^{4-2^2})(-1) + (-2(2)(-6)e^{4-2^2})(1) = 1(6) + (-6)(1) + (24)(1) = 36\end{aligned}$$

#### Question 4

Use implicit differentiation to compute  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  for the surface given by  $z^3 + 3xyz + x^2 + y^2 = 0$  at the point  $(1, -2, 1)$ .

**Solution:**

$$\begin{aligned}\frac{\partial z}{\partial x} (z^3 + 3xyz + x^2 + y^2) &= 3z^2 \frac{\partial z}{\partial x} + 3yz + 3xy \frac{\partial z}{\partial x} + 2x = 0 \\ 3z^2 \frac{\partial z}{\partial x} + 3yz + 3xy \frac{\partial z}{\partial x} + 2x &= 0 \Rightarrow 3z^2 \frac{\partial z}{\partial x} + 3xy \frac{\partial z}{\partial x} = -3yz - 2x \\ 3z^2 \frac{\partial z}{\partial x} + 3xy \frac{\partial z}{\partial x} &= -3yz - 2x \Rightarrow \frac{\partial z}{\partial x} (3z^2 + 3xy) = -3yz - 2x \\ \frac{\partial z}{\partial x} (3z^2 + 3xy) &= -3yz - 2x \Rightarrow \frac{\partial z}{\partial x} = \frac{-3yz - 2x}{3z^2 + 3xy} \\ \frac{\partial z}{\partial x} &= \frac{-3(-2)(1) - 2(1)}{3(1)^2 + 3(1)(-2)} = \frac{-4}{3} \\ \frac{\partial z}{\partial y} (z^3 + 3xyz + x^2 + y^2) &= 3z^2 \frac{\partial z}{\partial y} + 3xz + 3xy \frac{\partial z}{\partial y} + 2y = 0 \\ 3z^2 \frac{\partial z}{\partial y} + 3xz + 3xy \frac{\partial z}{\partial y} + 2y &= 0 \Rightarrow 3z^2 \frac{\partial z}{\partial y} + 3xy \frac{\partial z}{\partial y} = -3xz - 2y \\ 3z^2 \frac{\partial z}{\partial y} + 3xy \frac{\partial z}{\partial y} &= -3xz - 2y \Rightarrow \frac{\partial z}{\partial y} (3z^2 + 3xy) = -3xz - 2y \\ \frac{\partial z}{\partial y} (3z^2 + 3xy) &= -3xz - 2y \Rightarrow \frac{\partial z}{\partial y} = \frac{-3xz - 2y}{3z^2 + 3xy} \\ \frac{\partial z}{\partial y} &= \frac{-3(1)(1) - 2(-2)}{3(1)^2 + 3(1)(-2)} = \frac{1}{-3}\end{aligned}$$

#### Question 5

[(5.) (Stewart problem 14.5.36)] Wheat production  $W$  in a given year depends on the average temperature  $T$  and annual rainfall  $R$ . Scientists estimate that the average temperature is rising at a rate of  $0.15^\circ\text{C}/\text{year}$  and rainfall is decreasing at a rate of  $0.1 \text{ cm}/\text{year}$ . They also estimate that at current production levels,  $\partial W/\partial T = -2$  and  $\partial W/\partial R = 8$ .

(a) What is the significance of the signs of these partial derivatives?

(b) Estimate the current rate of change of wheat production  $\frac{dW}{dt}$ .

**Solution:**

a)

$\frac{\partial W}{\partial T} = -2$  means that for every drop in  $1^\circ\text{C}$  there is a decrease of 2 units in wheat production.

If the sign were positive it would mean that there would be an increase in wheat production as a result of the decrease of temperature.

$\frac{\partial W}{\partial R} = 8$  means for every increase in 1 cm rainfall there is an increase of 8 units in wheat production.

If the sign were negative instead it would mean that an increase in rainfall brought down wheat production. b)

$$\frac{dW}{dt} = \frac{\partial W}{\partial T} \cdot \frac{dT}{dt} + \frac{\partial W}{\partial R} \cdot \frac{dR}{dt} = -2(0.15) + 8(-0.1) = -1.1$$