Math 120

PSet 10

Nov 14 2024

Contents

Chapter 1

1.1 PSet 10

Question 1

The vector field $\vec{F}$ is shown below in the xy-plane and looks the same in all other horizontal planes. (In other words, $\vec{F}$ is independent of z and its z-component is 0.)

(a) Is $\operatorname{div}\vec{F}$ positive, negative, or zero? Explain.

(b) Determine whether $\operatorname{curl}\vec{F} = \vec{0}$. If not, in which direction does $\operatorname{curl}\vec{F}$ point?

Solution:

(a) The divergence of $\vec{F}$ is negative because the vectors converge toward the origin, indicating a net inward flux. This suggests material is "flowing in" rather than spreading out.

(b) The curl of $\vec{F}$ is zero, as there is no rotational pattern; the vectors are purely radial and do not exhibit any circular motion.

Question 2

Find the curl and divergence of the given vector field.

1. $\vec{F}(x,y,z) = \langle x^2yz, xy^2z, xyz^2\rangle$
2. $\vec{F}(x,y,z) = e^{xy}\sin z\,\hat{\jmath} + y\arctan(x/z)\,\hat{k}$

Solution:

$$P = x^2yx \quad Q = xy^2z \quad R = xyz^2$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\operatorname{div}\vec{F} = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\frac{\partial P}{\partial x} = 2xyz,$$

$$\frac{\partial Q}{\partial y} = 2xyz,$$

$$\frac{\partial R}{\partial z} = 2xyz.$$

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2xyz + 2xyz + 2xyz = 6xyz.$$

$$\left(\operatorname{curl}\vec{F}\right)_x = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = xz^2 - xy^2.$$

$$\left(\operatorname{curl}\vec{F}\right)_y = \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = x^2y - yz^2.$$

$$\left(\operatorname{curl}\vec{F}\right)_z = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2z - x^2z.$$

$$\operatorname{curl}\vec{F} = \left(xz^2 - xy^2\right)\hat{\imath} + \left(x^2y - yz^2\right)\hat{\jmath} + \left(y^2z - x^2z\right)\hat{k}.$$

b)

$$\vec{F}(x,y,z) = e^{xy}\sin z\,\hat{\jmath} + y\arctan\left(\frac{x}{z}\right)\hat{k}$$

$$\begin{aligned} P &= 0, \\ Q &= e^{xy}\sin z, \\ R &= y\arctan\left(\frac{x}{z}\right). \end{aligned}$$

$$\begin{aligned} \frac{\partial P}{\partial x} &= 0, \\ \frac{\partial Q}{\partial y} &= xe^{xy}\sin z, \\ \frac{\partial R}{\partial z} &= -\frac{xy}{x^2+z^2}. \end{aligned}$$

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + xe^{xy}\sin z - \frac{xy}{x^2+z^2}.$$

$$\begin{aligned} \left(\operatorname{curl}\vec{F}\right)_x &= \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ &= \arctan\left(\frac{x}{z}\right) - e^{xy}\cos z. \end{aligned}$$

$$\begin{aligned} \left(\operatorname{curl}\vec{F}\right)_y &= \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ &= 0 - \left(y \cdot \frac{z}{x^2+z^2}\right) \\ &= -\frac{yz}{x^2+z^2}. \end{aligned}$$

$$\begin{aligned} \left(\operatorname{curl}\vec{F}\right)_z &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\ &= ye^{xy}\sin z - 0 \\ &= ye^{xy}\sin z. \end{aligned}$$

$$\operatorname{curl}\vec{F} = \left(\arctan\left(\frac{x}{z}\right) - e^{xy}\cos z\right)\hat{\imath} - \left(\frac{yz}{x^2+z^2}\right)\hat{\jmath} + \left(ye^{xy}\sin z\right)\hat{k}.$$

Question 3

(a) Consider the surface given by $\sin(x-y) - x - y + z = 0$. Find a parametrization of the part of the surface that has $|x| \leqslant 3$ and $|y| \leqslant 3$. Be sure to include the bounds for your parameters.

(b) Look at the parametrization that you found in part (a). It should be of the form $\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$. Explain why it is not possible to find a parametrization of the surface $x^2 - y + z^2 = 1$ that is of the form $\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$.

(c) Find a parametrization of $x^2 - y + z^2 = 1$ that is of the form $\vec{r}(u,v) = \langle u, f(u,v), v\rangle$.

(d) Consider the surface $x^2 - y^2 + \frac{z^2}{4} = 1$. Explain why it is not possible to find a parametrization of this surface that is of the form $\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$, or of the form $\vec{r}(u,v) = \langle f(u,v), u, v\rangle$.

(e) Find a parametrization of the surface $x^2 - y^2 + \frac{z^2}{4} = 1$ of the form

$$\vec{r}(u,v) = \langle f(v)\cos u, v, g(v)\sin u\rangle,$$

where $0 \leqslant u \leqslant 2\pi$ and $-\infty < v < \infty$.

Solution:

Part (a):

$$\sin(x-y) - x - y + z = 0$$
$$|x| \leqslant 3 \quad \text{and} \quad |y| \leqslant 3$$

.

$$z = x + y - \sin(x-y)$$
$$\vec{r}(u,v) = \langle u, v, u + v - \sin(u-v)\rangle$$

$$-3 \leqslant u \leqslant 3 \quad \text{and} \quad -3 \leqslant v \leqslant 3$$

Part (b):

$$\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$$
$$z^2 = 1 - x^2 + y$$
$$z = \pm\sqrt{1 - x^2 + y}$$

However, the expression under the square root, $1 - x^2 + y$, can be negative for certain values of x and y, which means z is not defined over the entire domain of x and y. Additionally, z has two possible values (positive and negative square roots) for each (x, y), so z is not a single-valued function of x and y. Therefore, it's impossible to express z as a function $f(u,v)$ over the entire surface, making such a parametrization infeasible.

Part (c):

$$x^2 - y + z^2 = 1$$
$$y = x^2 + z^2 - 1$$
$$\vec{r}(u,v) = \langle u, u^2 + v^2 - 1, v\rangle$$

Part (d):

For the surface $x^2 - y^2 + \frac{z^2}{4} = 1$, attempting to parametrize it as $\vec{r}(u, v) = \langle u, v, f(u, v)\rangle$ or $\vec{r}(u, v) = \langle f(u, v), u, v\rangle$ is not feasible because:

- **First Form (z in terms of x and y):** Solving for z:

$$z^2 = 4(1 - x^2 + y^2)$$

  The right side depends on both x and y in a way that z cannot be uniquely expressed as a function of x and y over the entire surface, especially when the expression under the square root is negative.

- **Second Form (x in terms of y and z):** Solving for x:

$$x^2 = y^2 - \frac{z^2}{4} + 1$$

  Similar to the first form, x cannot be uniquely expressed as a function of y and z over the entire surface.

  In both cases, the variables cannot be separated into a single-valued function necessary for the parametrization.

Part (e):

$$\vec{r}(u, v) = \langle f(v)\cos u, v, g(v)\sin u\rangle$$

$$\left[f(v)\cos u\right]^2 - v^2 + \frac{\left[g(v)\sin u\right]^2}{4} = 1$$

$$f(v)^2\cos^2 u - v^2 + \frac{g(v)^2\sin^2 u}{4} = 1$$

$$f(v)^2 = \frac{g(v)^2}{4}$$

$$f(v)^2 + \frac{g(v)^2}{4} = 1 + v^2$$

$$g(v)^2 = 4f(v)^2$$

$$f(v)^2 + f(v)^2 = 1 + v^2 \implies 2f(v)^2 = 1 + v^2$$

$$f(v) = \sqrt{\frac{1 + v^2}{2}}$$

$$g(v) = \sqrt{2(1 + v^2)}$$

$$\vec{r}(u, v) = \left\langle \sqrt{\frac{1 + v^2}{2}}\cos u,\ v,\ \sqrt{2(1 + v^2)}\sin u \right\rangle$$

$$0 \leqslant u \leqslant 2\pi, \quad -\infty < v < \infty$$

Question 4

Identify and sketch the surface with the given parameterization.

(a) $\vec{r}(u,v) = (2\sin u)\,\hat{\jmath} + (3\cos u)\,\hat{\imath} + v\,\hat{k}, \quad 0 \leqslant u \leqslant \pi, \quad -2 \leqslant v \leqslant 2$

(b) $\vec{r}(u,v) = \langle u\sin v, u^2, u\cos v\rangle, \quad 0 \leqslant u \leqslant 3, \quad 0 \leqslant v \leqslant 2\pi$

(a) Given parameterization:

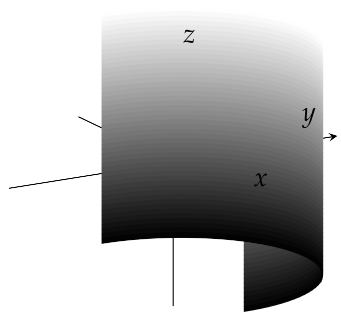
$$\vec{r}(u,v) = (2\sin u)\,\hat{\jmath} + (3\cos u)\,\hat{\imath} + v\,\hat{k}, \quad 0 \leqslant u \leqslant \pi, \quad -2 \leqslant v \leqslant 2$$

$$x(u,v) = 3\cos u, \quad y(u,v) = 2\sin u, \quad z(u,v) = v$$

$$\cos u = \frac{x}{3}, \quad \sin u = \frac{y}{2}$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$



(b) Given parameterization:

$$\vec{r}(u,v) = \langle u\sin v, u^2, u\cos v\rangle, \quad 0 \leqslant u \leqslant 3, \quad 0 \leqslant v \leqslant 2\pi$$

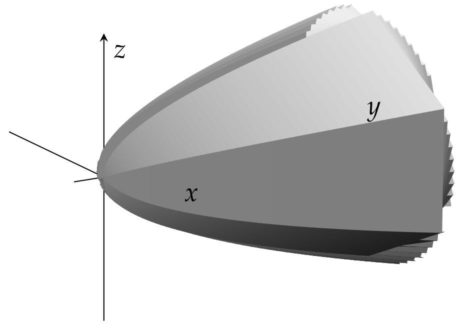
$$x(u,v) = u\sin v, \quad y(u,v) = u^2, \quad z(u,v) = u\cos v$$

$$\sin v = \frac{x}{u}, \quad \cos v = \frac{z}{u}$$

$$\left(\frac{x}{u}\right)^2 + \left(\frac{z}{u}\right)^2 = 1$$

$$x^2 + z^2 = u^2$$

$$x^2 + z^2 = y$$



Question 5

Consider the surface S described by $x - 4y^2 - z^2 + 3 = 0$.

(a) Find a parametrization of S of the form $\vec{r}_1(u,v) = \langle f(u,v), u, v\rangle$. Give the domain for your parametrization.

(b) Find a parametrization of S of the form $\vec{r}_2(u,v) = \langle v, f(v)\cos u, g(v)\sin u\rangle$. Give the domain for your parametrization.

(c) How must we restrict the parameters (u,v) in part (a) if we only want the part of S that lies in front of the yz-plane, i.e., where $x \geqslant 0$?

(d) How must we restrict the parameters (u,v) in part (b) if we only want the part of S that lies in front of the yz-plane?

(a)

$$x = 4y^2 + z^2 - 3.$$
$$\vec{r}_1(u,v) = \langle 4u^2 + v^2 - 3, u, v\rangle.$$
$$(u,v) \in \mathbb{R}^2.$$

(b)

$$\vec{r}_2(u,v) = \langle v, f(v)\cos u, g(v)\sin u\rangle$$
$$x = v \quad y = f(v)\cos u \quad z = g(v)\sin u$$
$$v - 4\left[f(v)\cos u\right]^2 - \left[g(v)\sin u\right]^2 + 3 = 0.$$
$$v - 4f(v)^2\cos^2 u - g(v)^2\sin^2 u + 3 = 0.$$
$$4f(v)^2 = k \quad \text{and} \quad g(v)^2 = k$$
$$k = v + 3$$
$$f(v) = \frac{1}{2}\sqrt{v+3}, \quad g(v) = \sqrt{v+3}.$$
$$\vec{r}_2(u,v) = \left\langle v, \frac{1}{2}\sqrt{v+3}\cos u, \sqrt{v+3}\sin u\right\rangle$$
$$v \geqslant -3, \quad u \in \mathbb{R}$$

(c)

$$x = 4u^2 + v^2 - 3 \geqslant 0$$
$$4u^2 + v^2 \geqslant 3$$

(d)

$$v \geqslant 0.$$

$$v \geqslant 0, \quad u \in \mathbb{R}$$

Question 6

Find parametric equations for each of the following surfaces.

(a) The part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

(b) The surface obtained by rotating the curve $x = 4y^2 - y^4$, $-2 \leqslant y \leqslant 2$ about the y-axis.

(c) The ellipsoid $\frac{x^2}{4} + 4y^2 + \frac{z^2}{9} = 1$.

Part (a)

$$x = r\cos\theta, \quad y = r\sin\theta,$$
$$r \in [0,1] \quad \text{and} \theta \in [0, 2\pi)$$

$$z = r\cos\theta + 3.$$

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \\ z = r\cos\theta + 3, \end{cases}$$
$$0 \leqslant r \leqslant 1 \quad \text{and} \quad 0 \leqslant \theta < 2\pi$$

.

Part (b)

$$\begin{cases} x = [4y^2 - y^4]\cos\theta, \\ z = [4y^2 - y^4]\sin\theta, \\ y = y, \end{cases}$$
$$-2 \leqslant y \leqslant 2 \quad \text{and} \quad 0 \leqslant \theta < 2\pi$$
$$\begin{cases} x = \left(4y^2 - y^4\right)\cos\theta, \\ y = y, \\ z = \left(4y^2 - y^4\right)\sin\theta, \end{cases}$$
$$-2 \leqslant y \leqslant 2 \quad \text{and} \quad 0 \leqslant \theta < 2\pi$$

.

Part (c)

Solution:

$$X = \frac{x}{2}, \quad Y = 2y, \quad Z = \frac{z}{3}$$
$$X^2 + Y^2 + Z^2 = 1$$

$$\begin{cases} X = \sin\phi\cos\theta, \\ Y = \sin\phi\sin\theta, \\ Z = \cos\phi, \end{cases}$$
$$\phi \in [0, \pi] \quad \text{and} \quad \theta \in [0, 2\pi)$$

$$\begin{cases} x = 2X = 2\sin\phi\cos\theta, \\ y = \dfrac{Y}{2} = \dfrac{1}{2}\sin\phi\sin\theta, \\ z = 3Z = 3\cos\phi. \end{cases}$$

$$\begin{cases} x = 2\sin\phi\cos\theta, \\ y = \dfrac{1}{2}\sin\phi\sin\theta, \\ z = 3\cos\phi, \end{cases}$$

$$0 \leqslant \phi \leqslant \pi \quad \text{and} \quad 0 \leqslant \theta < 2\pi$$

Question 7

Find the tangent plane to the parametric surface $\vec{r}(u,v) = \langle u\sin v, u^2, u\cos v\rangle$ at the point where $u = 1$ and $v = \frac{\pi}{3}$. Write the plane both in the vector form $\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$ and in the form $ax + by + cz = d$.

Solution:

$$\vec{r}_0 = \vec{r}(1, \tfrac{\pi}{3}) = \left\langle 1\cdot\sin\tfrac{\pi}{3},\ 1^2,\ 1\cdot\cos\tfrac{\pi}{3}\right\rangle = \left\langle \tfrac{\sqrt{3}}{2},\ 1,\ \tfrac{1}{2}\right\rangle.$$

$$\vec{r}_u(u,v) = \langle \sin v,\ 2u,\ \cos v\rangle$$

$$\vec{r}_v(u,v) = \langle u\cos v,\ 0,\ -u\sin v\rangle$$

$$\vec{r}_u(1, \tfrac{\pi}{3}) = \left\langle \sin\tfrac{\pi}{3},\ 2,\ \cos\tfrac{\pi}{3}\right\rangle = \left\langle \tfrac{\sqrt{3}}{2},\ 2,\ \tfrac{1}{2}\right\rangle,$$

$$\vec{r}_v(1, \tfrac{\pi}{3}) = \left\langle 1\cdot\cos\tfrac{\pi}{3},\ 0,\ -1\cdot\sin\tfrac{\pi}{3}\right\rangle = \left\langle \tfrac{1}{2},\ 0,\ -\tfrac{\sqrt{3}}{2}\right\rangle.$$

$$\vec{r}(s,t) = \vec{r}_0 + s\,\vec{r}_u(1, \tfrac{\pi}{3}) + t\,\vec{r}_v(1, \tfrac{\pi}{3})$$

$$\vec{r}(s,t) = \left\langle \tfrac{\sqrt{3}}{2},\ 1,\ \tfrac{1}{2}\right\rangle + s\left\langle \tfrac{\sqrt{3}}{2},\ 2,\ \tfrac{1}{2}\right\rangle + t\left\langle \tfrac{1}{2},\ 0,\ -\tfrac{\sqrt{3}}{2}\right\rangle.$$

$$\vec{n} = \vec{r}_u(1, \tfrac{\pi}{3}) \times \vec{r}_v(1, \tfrac{\pi}{3}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{3}}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{vmatrix}.$$

$$\vec{n} = \left(-\sqrt{3},\ 1,\ -1\right)$$

$$\vec{n}\cdot(\vec{r} - \vec{r}_0) = 0.$$

$$-\sqrt{3}(x - \tfrac{\sqrt{3}}{2}) + 1(y - 1) - 1(z - \tfrac{1}{2}) = 0$$

$$-\sqrt{3}x + y - z + \left(\tfrac{3}{2} - 1 + \tfrac{1}{2}\right) = 0,$$

$$-\sqrt{3}x + y - z + 1 = 0.$$

$$\sqrt{3}x - y + z = 1$$

$$\sqrt{3}\,x \ - \ y \ + \ z \ = \ 1$$

$$\vec{r}(s,t) = \left\langle \tfrac{\sqrt{3}}{2},\ 1,\ \tfrac{1}{2}\right\rangle + s\left\langle \tfrac{\sqrt{3}}{2},\ 2,\ \tfrac{1}{2}\right\rangle + t\left\langle \tfrac{1}{2},\ 0,\ -\tfrac{\sqrt{3}}{2}\right\rangle.$$

$$\sqrt{3}\,x \ - \ y \ + \ z \ = \ 1$$