Math 120

PSet 7

Oct 24 2024

Contents

Chapter 1

1.1 PSet 7

Question 1

Calculate the given iterated integrals.

1. $\int_0^1 \int_0^1 x\sqrt{1+4y}\,dy\,dx$
2. $\int_0^1 \int_1^2 \frac{xe^x}{y}\,dy\,dx$

Solution:
1)

$$\int_0^1 \int_0^1 x\sqrt{1+4y}\,dy\,dx$$

$$\int_0^1 x\sqrt{1+4y}\,dy$$

$$1+4y = t \quad r = dt$$

$$x\int_0^1 \frac{1}{4}\sqrt{t}dt$$

$$\frac{1}{4}x\int_0^1 \sqrt{t}dt$$

$$\frac{1}{4}x \cdot \frac{2t\sqrt{t}}{3}\Bigg|_0^1$$

$$\frac{1}{4}x \cdot \frac{2(1+4y)\sqrt{1+4y}}{3}\Bigg|_0^1$$

$$\frac{x\sqrt{1+4y}(1+4y)}{6}\Bigg|_0^1$$

$$\frac{x\sqrt{1+4}(1+4)}{6} - \frac{x\sqrt{1}1}{6}$$

$$\frac{5x\sqrt{5}}{6} - \frac{x}{6}$$

$$\int_0^1 \frac{5x\sqrt{5}}{6} - \frac{x}{6}dx$$

$$\frac{1}{6}\int_0^1 5\sqrt{5}x - x\,dx$$

$$\frac{1}{6}\left(\int_0^1 5\sqrt{5}x dx - \int_o^1 x\,dx\right)$$

$$\int_0^1 5\sqrt{5}x\,dx \Rightarrow \left.\frac{5\sqrt{5}x^2}{2}\right|_0^1$$

$$\frac{5\sqrt{5}(1)^2}{2} - 0 = \frac{5\sqrt{5}}{2}$$

$$\int_0^1 x\,dx \Rightarrow \left.\frac{x^2}{2}\right|_0^1$$

$$\frac{1}{2} - 0 = \frac{1}{2}$$

$$\frac{1}{6}\left(\frac{5\sqrt{5}}{2} - \frac{1}{2}\right) = \frac{5\sqrt{5}-1}{12}$$

2)

$$\int_0^1 \int_1^2 \frac{xe^x}{y}\,dy\,dx$$

$$xe^x \int_1^2 \frac{1}{y}\,dy$$

$$xe^x \ln(y)\Big|_1^2 \Rightarrow xe^x\ln(2) - xe^x\ln(1) = xe^x\ln(2)$$

$$\ln(2)\int_0^1 xe^x\,dx$$

$$\ln(2)\,(xe^x - e^x)|_0^1$$

$$(\ln(2)e - \ln(2)e) - \left(\ln(2)(0) - \ln(2)e^0\right) = 0 - (-\ln(2)(1)) = \ln(2)$$

Question 2

(a) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^2 e^{-x^2-y^2}\,dy\,dx.$$

(b) Explain why

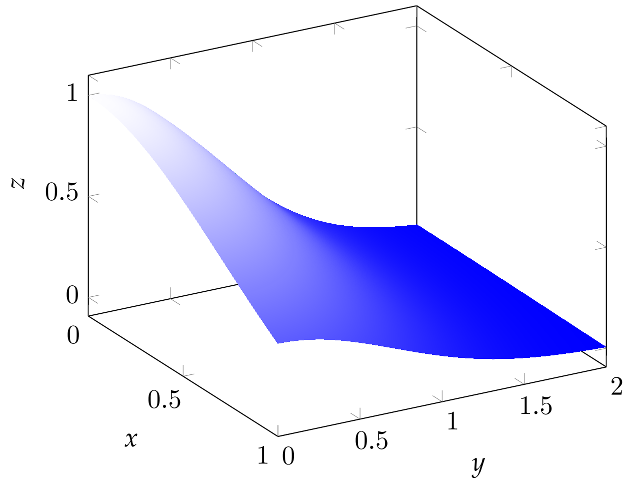
$$\int_0^1 \int_0^2 e^{-x^2-y^2}\,dy\,dx = \int_0^1 e^{-x^2}\,dx \cdot \int_0^2 e^{-y^2}\,dy.$$

(c) Use Desmos to compute

$$\int_0^1 \int_0^2 e^{-x^2-y^2}\,dy\,dx.$$

(Desmos will give a numerical approximation, but this is fine. In fact, there is no way to compute the antiderivatives necessary to get an exact answer.)

Solution:



b)

It is because $e^{-x^2y^2 = e^{x^2} \cdot e^{y^2}}$ and the bounds of y are independent of x, so that allows e^{-x^2} to be treated as a constant when integrating with respect to y and vice versa.

c)

$$\int_0^1 \int_0^2 e^{-x^2-y^2}\, dy\, dx \approx 0.6588$$

Question 3

(a) Find the average value of the function $f(x,y) = \sin x \cos y$ on the rectangle $R = [0,\pi] \times [-\pi/2, \pi/2]$.

(b) Use symmetry to find the average value of $f(x,y) = \frac{4\sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$ on the region $R = [2\pi, 4\pi] \times [2\pi, 6\pi]$. Please explain your answer carefully.

Solution: a)

$$f(x,y) = \sin x \cos y$$

$$R = [0,\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$f_{avg} = \frac{1}{A(R)} \iint\limits_R f(x,y) dA$$

$$A(R) = (\pi - 0) \times (\frac{\pi}{2} - -\frac{\pi}{2}) = \pi^2$$

$$\frac{1}{\pi^2} \int_0^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos y\, dy\, dx$$

$$\sin x \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos y\, dy$$

$$(\sin x) \sin y \Big|_{\frac{pi}{2}}^{\frac{\pi}{2}}$$

$$(\sin x) \sin\left(\frac{\pi}{2}\right) - (\sin x) \sin\left(\frac{-\pi}{2}\right) = 2 \sin x$$

$$\int_0^{\pi} 2 \sin x dx$$

$$-2 \cos x|_0^{\pi}$$

$$-\cos \pi - (-2)\cos(0) = 4$$

$$\frac{1}{\pi^2} \cdot 4 = \frac{4}{\pi^2}$$

b)

$$f(x,y) = \frac{4\sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$$

$$R = [2\pi, 4\pi] \times [2\pi, 6\pi]$$

$$f_{avg} = \iint\limits_R f(x,y) dA$$

$$A(R) = [4\pi - 2\pi] \times [6\pi - 2\pi] = 8\pi^2$$

$$\int_{2\pi}^{4\pi} \int_{2\pi}^{6\pi} \frac{4\sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3\, dy\, dx$$

$$\iint\limits_R f(x,y) dA - \iint\limits_R \frac{4\sin y}{e^{x^2}} dA - \iint\limits_R \frac{\cos x}{\ln y} dA + \iint\limits_R 3dA$$

$$\int_{2\pi}^{6\pi} 4\sin y\, dy = -4\left[\cos y\right]_{2\pi}^{6\pi} = -4(6\cos\pi - \cos 2\pi) = -4(1-1) = 0$$

$$\iint\limits_R f(x,y) \frac{4\sin y}{e^{x^2}} dA = \int_{2\pi}^{4\pi} \frac{1}{e^{x^2}} dx \times 0 = 0$$

$$\int_{2\pi}^{4\pi} \cos x dx = \sin x|_{2\pi}^{4\pi} = sin 4\pi - \sin 2\pi = 0 - 0 = 0$$

$$\iint\limits_R \frac{\cos x}{\ln y} dA = \int_{2\pi}^{6\pi} \frac{1}{\ln y} \times 0 = 0$$

$$\iint\limits_R 3dA = 3 \times A(R) = 3 \times 8\pi^2 = 24\pi^2$$

$$\frac{24\pi^2}{8\pi^2} = 3$$

Question 4

In each part, draw the region D, and evaluate the integral.

1. $\iint_D \frac{y}{x^5+1}\, dA$, where D is the region $D = \{(x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x^2\}$.
2. $\iint_D x^3\, dA$, where $D = \{(x,y) \mid 1 \leqslant x \leqslant e,\ 0 \leqslant y \leqslant \ln x\}$.

Solution: 1.

$$\iint_D \frac{y}{x^5+1}\, dA \quad D = \{(x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x^2\}$$

$$\int_0^1 \int_0^{x^2} \frac{y}{x^5+1} dy\, dx$$

$$\frac{1}{x^5+1} \int_0^{x^2} y dy$$

$$\frac{y^2}{2}\Big|_0^{x^2} \Rightarrow \frac{(x^2)^2}{2} - \frac{0}{2} = \frac{x^4}{2}$$

$$\int_0^1 \frac{1}{x^5+1} \times \frac{x^4}{2} dx$$

$$x^5 + 1 = t \quad dt = 5x^4 dx$$

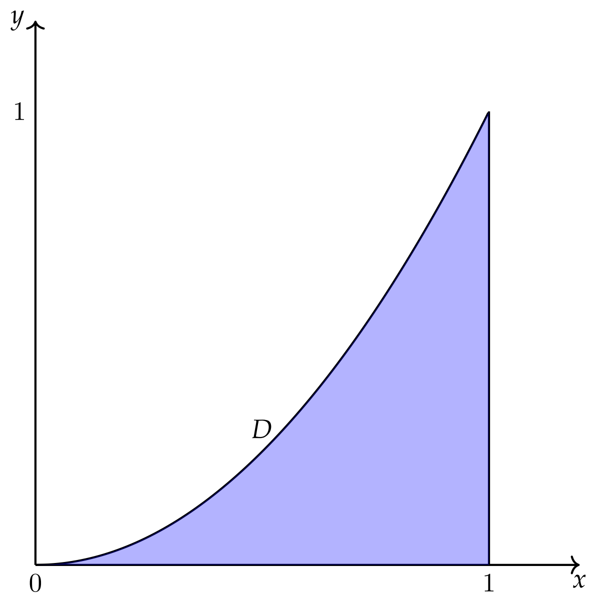
$$\frac{1}{10}\int_0^1 \frac{1}{t} dt$$

$$\frac{1}{10}\ln|t|\Big|_0^1$$

$$\frac{1}{10}|x^5 + 1|\Big|_0^1$$

$$\frac{1}{10}\ln(1^5 + 1) - \frac{1}{10}\ln(1)$$

$$\frac{1}{10}\ln(2) - \frac{1}{10}\ln(1) = \frac{1}{10}\ln(2)$$



2.

$$\iint_D x^3\, dA \quad D = \{(x,y) \mid 1 \leqslant x \leqslant e,\ 0 \leqslant y \leqslant \ln x\}$$

$$\int_1^e \int_0^{\ln x} x^3\, dy\, dx$$

$$x^3 \int_0^{\ln x} 1 dy$$

$$\left(x^3\right) y\Big|_0^{\ln x}$$

$$x^3 \ln x - 0$$

$$uv - \int v\, du$$

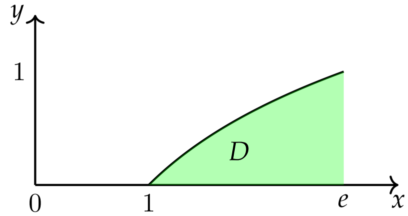
$$u = \ln x \quad du = \frac{1}{x}\, dx$$

$$v = \frac{x^4}{4} \quad x^3 dx$$

$$\frac{\ln x \cdot x^4}{4} - \int \frac{x^3}{4}\, dx$$

$$\left[\frac{\ln x^4 \cdot x^4}{4} - \frac{x^4}{16}\right]_1^e$$

$$\left(\frac{\ln e \cdot e^4}{4} - \frac{e^4}{16}\right) - \left(\frac{\ln 1 \cdot 1^4}{4} - \frac{1^4}{16}\right)$$
$$\left(\frac{\ln e \cdot e^4}{4}\right) - \left(0 - \frac{1}{16}\right)$$
$$\left(\frac{e^4}{4} - \frac{e^4}{16}\right) + \frac{1}{16}$$

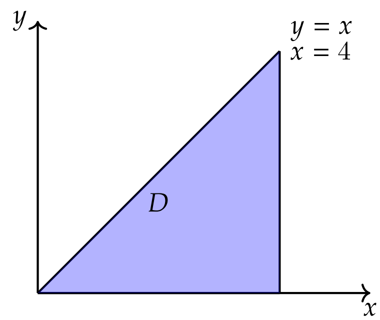


Question 5

Draw the region D. Set up the iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$\iint_D x^2 e^{-xy}\, dA \quad \text{where } D \text{ is bounded by } y = x,\ x = 4, \text{ and } y = 0.$$

Solution:



$$\int_0^4 \int_0^x x^2 e^{-yx}\, dy\, dx$$
$$\int_0^4 \int_y^4 x^2 e^{-yx}\, dx\, dy$$
$$x^2 \int_y^4 e^{-yx} dy$$
$$\left(x^2\right) \frac{e^{-yx}}{x}\bigg|_0^x \left(x^2\right) \frac{e^{-yx}}{x} - \left(x^2\right) \frac{e^{-0\cdot x}}{x}$$
$$xe^{-x^2} - x^2 \cdot \frac{1}{x} \Rightarrow -xe^{-x^2} + x$$
$$\int_0^4 -xe^{-x^2} + x\, dx \Rightarrow \int_0^4 -xe^{-x^2}\, dx \int_0^4 x\, dx$$
$$-x^2 = t \quad -2x = dt$$

$$\int_0^4 \frac{1}{2} e^t\, dt \Rightarrow \frac{1}{2} \int_0^4 e^t\, dt$$

$$\frac{1}{2}e^{t}\Big|_{0}^{4} \Rightarrow \frac{1}{2}^{x^2}\Big|_{0}^{4}$$

$$\frac{1}{2}e^{-4^2} - \frac{1}{2}e^{-0^2} \Rightarrow \frac{1}{2}e^{-16} - \frac{1}{2}$$

$$\int_0^4 x\,dx$$

$$\frac{x^2}{2}\Big|_0^4$$

$$\frac{4^2}{2} - \frac{0^2}{2} = \frac{16}{2} = 8$$

$$\int_0^4 -xe^{-x^2} + x\,dx = \frac{1}{2}e^{-16} + \frac{15}{2}$$

Question 6

(a) Find the volume of the solid in the first octant enclosed by the parabolic cylinder $y = 1 - x^2$ and the planes $z = 2 - y$ and $z = y$.

(b) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^{1-x} (2 - y^2)\,dy\,dx.$$

Solution:
a)

$$y = -x^2 \quad z = 2 - y \quad z = y$$

$$x, y, z \geqslant 0$$

$$2 - y = y \Rightarrow 2y = 2 \Rightarrow y = 1 \Rightarrow 0 \leqslant y \leqslant 1 - x^2$$

$$\text{height } = (2 - y) - y \Rightarrow 2 - 2y$$

$$V = \int_0^1 \int_0^{1-x^2} 2 - 2y\,dy\,dx$$

$$\int_0^{1-x^2} 2 - 2y\,dy$$

$$2y - y^2\Big|_0^{1-x^2}$$

$$2\left(1 - x^2\right) - \left(1 - x^2\right) - 2(0) - (0)^2$$

$$2 - 2x^2 - 1 + 2x^2 - x^4$$

$$1 - x^4$$

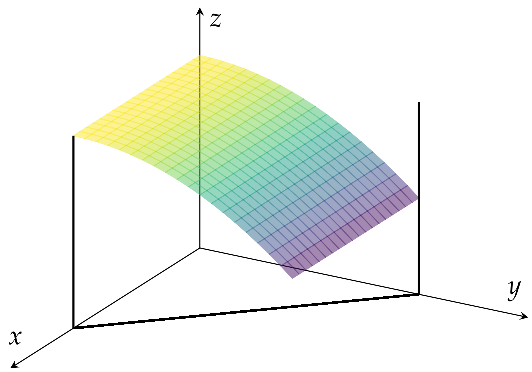
$$\int_0^1 1 - x^4 dx$$

$$x - \frac{x^5}{5}\Big|_0^1$$

$$1 - \frac{1^5}{5} - 0 - \frac{0^5}{5}$$

$$1 - \frac{1}{5} = \frac{4}{5}$$

b)



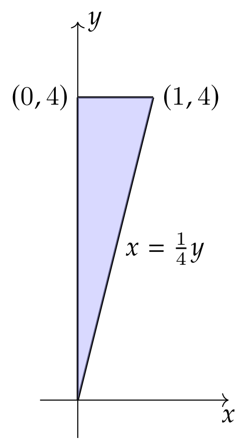
Question 7

Sketch the region of integration and change the order of integration.

1. $\int_0^1 \int_{4x}^4 f(x,y)\,dy\,dx$
2. $\int_0^3 \int_0^{\sqrt{9-y}} f(x,y)\,dx\,dy$
3. $\int_0^4 \int_0^{\ln 2x} f(x,y)\,dy\,dx$

Solution:

a)



$$\int_0^1 \int_{4x}^4 f(x,y)\,dy\,dx$$

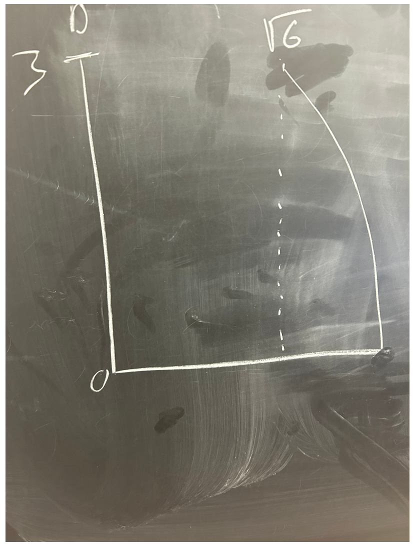
$$\iint\limits_D f(x,y)dA$$

$$D = \{(x,y)|0 \leqslant x \leqslant 1, 4x \leqslant y \leqslant 4\}$$

$$D = \{(x,y)|0 \leqslant y \leqslant 4, 0 \leqslant x \leqslant \frac{1}{4}y\}$$

$$\iint_D f(x,y)dA = \int_0^4 \int_0^{\frac{1}{4}y} f(x,y)\,dx\,dy$$

b)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