Directional Derivatives and Gradient Vector

Directional Derivative:

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

where $\mathbf{u}$ is a unit vector.
Gradient Vector:

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$

Properties: - ∇f points in the direction of max increase of f.
- ∇f is perpendicular to level curves of f.
Max Rate of Change:

$$\text{Max Rate} = |\nabla f(x,y)|$$

Maximum and Minimum Values

Second Derivative Test:
Compute $D = f_{xx}f_{yy} - (f_{xy})^2$.
- If $D > 0$ and $f_{xx} > 0$, local min at (a,b).
- If $D > 0$ and $f_{xx} < 0$, local max at (a,b).
- If $D < 0$, saddle point at (a,b).
- If $D = 0$, test inconclusive.

Critical Points: Solve $f_x = 0,\ f_y = 0$.

Lagrange Multipliers

Purpose: Find extrema of $f(x,y)$ subject to $g(x,y) = 0$.
Method: 1. $\nabla f = \lambda \nabla g$
$f_x = \lambda g_x, \quad f_y = \lambda g_y$
2. Include $g(x,y) = 0$.
3. Solve for x, y, λ.
4. Evaluate f at solutions.

Double Integrals over Rectangles

Definition:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

where $R = [a,b] \times [c,d]$.
Fubini's Theorem: If f is continuous:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Average Value of a Function

Average Value over R:

$$f_{\text{avg}} = \frac{1}{(b-a)(d-c)} \iint_R f(x,y)\,dA$$

Double Integrals over General Regions

Type I Region (Vertical):

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D f\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f\,dy\,dx$$

Type II Region (Horizontal):

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$\iint_D f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f\,dx\,dy$$

Double Integrals in Polar Coordinates

When to Convert: - Circular regions or integrands with $x^2 + y^2$.
- When $f(x,y)$ is easier to integrate in polar form.
Transformation:

$$x = r\cos\theta,\ \ y = r\sin\theta$$
$$dA = r\,dr\,d\theta$$

Integral:

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

Tips: - Adjust limits of r and θ to match D.
- Common for circles, sectors, annuli.

Vector Fields

Definition: $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$
Gradient Field: $\mathbf{F} = \nabla f$
Conservative Field: $\mathbf{F} = \nabla f$.
Curl in $\mathbb{R}^2$:

$$\text{curl } \mathbf{F} = Q_x - P_y$$

Fundamental Theorem for Line Integrals

If $\mathbf{F} = \nabla f$, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

Conservative Field Test: - If $P_y = Q_x$, then $\mathbf{F}$ is conservative.

Line Integrals

When to Use: - To compute work done by a force field along a path.
- To integrate a scalar function over a curve (mass, length).
Types of Line Integrals: - **Scalar Line Integral** (with respect to arc length): $\int_C f\,ds$
- **Vector Line Integral** (work): $\int_C \mathbf{F} \cdot d\mathbf{r}$
How to Compute: 1. Parameterize C by $\mathbf{r}(t)$, $t \in [a,b]$.
2. Compute $\mathbf{r}'(t)$ and $|\mathbf{r}'(t)|$ if necessary.
3. Substitute into the integral: - Scalar: $\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Vector: $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$
When to Convert to Polar Coordinates: - When C is a circle or curve naturally described in polar coordinates.
- When integrand involves $x^2 + y^2$ or trigonometric functions.
Converting to Polar Coordinates: - Use $x = r\cos\theta$, $y = r\sin\theta$.
- Express $\mathbf{F}$ and $d\mathbf{r}$ in terms of r and θ.
Tips: - Choose the simplest parameterization possible.
- For circles: $x = a\cos t,\ \ y = a\sin t,\ t \in [0, 2\pi]$.
- For straight lines, use linear parameterizations.
Applications: - Calculating work, circulation, or flux.
- Finding mass of a wire with variable density.

Green's Theorem

When to Use: - To convert a difficult line integral into a double integral (or vice versa).
- When dealing with circulation or flux over a closed curve C in the plane.
- C must be a positively oriented (counter-clockwise) simple closed curve.
Statement:

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Applications: - Calculating area: Area $= \frac{1}{2}\oint_C x\,dy - y\,dx$
- Computing work done by a force field around a closed path.
How to Apply: 1. Verify conditions (closed curve, positive orientation).
2. Identify $P(x,y)$ and $Q(x,y)$.
3. Compute $Q_x - P_y$.
4. Evaluate $\iint_D (Q_x - P_y)dA$.
Tips: - Simplify the integrand before integrating.
- Choose the order of integration based on D.
- For circular regions, consider polar coordinates.

Example: Evaluating a Line Integral Using Green's Theorem

Problem: Let $\vec{F}(x,y) = \left\langle x^2y + y^2,\ \frac{1}{3}x^3 + 2xy + x \right\rangle$. Compute $\int_C \vec{F} \cdot d\vec{r}$ along the semicircle C defined by $x^2 + y^2 = 16$ for $y \ge 0$.
Solution Steps: 1. **Close the Curve:** - Since C is not closed, add the line segment L along the x-axis from $(4,0)$ to $(-4,0)$ to form a closed curve C'.
2. **Apply Green's Theorem:**

$$\oint_{C'} \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

- Identify $P(x,y) = x^2y + y^2$, $Q(x,y) = \frac{1}{3}x^3 + 2xy + x$. - Compute $\frac{\partial Q}{\partial x} = x^2 + 2y + 1$. - Compute $\frac{\partial P}{\partial y} = x^2 + 2y$. - The integrand simplifies to 1:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

3. **Compute the Double Integral:** - Since the integrand is 1, the double integral equals the area of D. - Area of the upper half-circle of radius 4:

$$\text{Area} = \frac{1}{2}\pi(4)^2 = 8\pi$$

- Therefore:

$$\oint_{C'} \vec{F} \cdot d\vec{r} = 8\pi$$

4. **Compute the Line Integral over L:** - Along L: $y = 0$, $dy = 0$, $d\vec{r} = \langle dx, 0 \rangle$. - Evaluate $\vec{F}$ on L:

$$\vec{F}(x,0) = \left\langle 0,\ \frac{1}{3}x^3 + x \right\rangle$$

- Compute $\vec{F} \cdot d\vec{r}$:

$$\vec{F} \cdot d\vec{r} = 0 \cdot dx + \left(\frac{1}{3}x^3 + x \right) \cdot 0 = 0$$

- Thus:

$$\int_L \vec{F} \cdot d\vec{r} = 0$$

5. **Compute the Original Line Integral:** - Since $\oint_{C'} = \int_C + \int_L$:

$$\int_C \vec{F} \cdot d\vec{r} = \oint_{C'} \vec{F} \cdot d\vec{r} - \int_L \vec{F} \cdot d\vec{r} = 8\pi - 0 = 8\pi$$

Answer: $\int_C \vec{F} \cdot d\vec{r} = 8\pi$

Trigonometric Identities

Pythagorean:

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$

Double Angle:

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

Sum and Difference:

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

Common Derivatives and Integrals

Derivatives:

$$\frac{d}{dx}e^{ax} = ae^{ax} \quad \frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \quad \frac{d}{dx}\sin ax = a\cos ax$$
$$\frac{d}{dx}\cos ax = -a\sin ax \quad \frac{d}{dx}\tan ax = a\sec^2 ax$$

Integrals:

$$\int e^{ax}dx = \frac{1}{a}e^{ax} + C \quad \int \frac{1}{x}dx = \ln|x| + C$$
$$\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C \quad \int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C \quad \int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

Techniques: - **Substitution:** Let $u = g(x)$.
- **Integration by Parts:** $\int u\,dv = uv - \int v\,du$.

Jacobian Determinant

Transformation from (x, y) to (u, v):

$$J = \left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left|\begin{smallmatrix} x_u & x_v \\ y_u & y_v \end{smallmatrix}\right| = x_u y_v - x_v y_u$$

Use in Integration:

$$\iint_D f(x, y)\,dA = \iint_{D'} f(x(u, v), y(u, v))|J|\,du\,dv$$

Conservative Vector Fields

Tests: - If $P_y = Q_x$, $\mathbf{F}$ is conservative.
Finding Potential f**:** 1. Integrate P w.r.t x to get f.
2. Differentiate f w.r.t y, compare with Q.
3. Adjust f as needed.

Coordinate Transformations

Polar to Cartesian:

$$x = r\cos\theta,\ y = r\sin\theta$$

Cartesian to Polar:

$$r = \sqrt{x^2 + y^2},\ \theta = \arctan\left(\frac{y}{x}\right)$$

Cylindrical:

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

Spherical:

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

Derivative Rules

$$\frac{d}{dx}c = 0 \quad \frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}[cf(x)] = cf'(x) \quad \frac{d}{dx}[f \pm g] = f' \pm g'$$
$$\frac{d}{dx}[fg] = f'g + fg' \quad \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$
$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Example: Computing Area Between Circles

Problem: Compute the area of the region R with $y \geq 0$ outside C_2 and inside C_1, where:

$$C_1 : (x-1)^2 + y^2 = 1, \quad C_2 : x^2 + y^2 = 2$$

Solution Steps: 1. **Express the curves in polar coordinates:** - For C_1:

$$(x-1)^2 + y^2 = 1$$

Substitute $x = r\cos\theta$, $y = r\sin\theta$:

$$(r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1$$

Simplify:

$$r^2 - 2r\cos\theta = 0$$

So $r = 0$ or $r = 2\cos\theta$. Since $r = 0$ is trivial, C_1 corresponds to $r = 2\cos\theta$.
- For C_2:

$$x^2 + y^2 = 2$$

In polar coordinates:

$$r^2 = 2$$

So $r = \sqrt{2}$.
2. **Determine the limits of integration:** - Find the angle θ where the curves intersect:

$$r = \sqrt{2} = 2\cos\theta$$
$$\cos\theta = \frac{\sqrt{2}}{2}$$
$$\theta = \frac{\pi}{4}$$

- Therefore, θ ranges from 0 to $\frac{\pi}{4}$.
3. **Set up the double integral in polar coordinates:**

$$A = \int_{\theta=0}^{\frac{\pi}{4}}\int_{r=\sqrt{2}}^{r=2\cos\theta} r\,dr\,d\theta$$

4. **Compute the integral:** - Integrate with respect to r:

$$\int_{r=\sqrt{2}}^{r=2\cos\theta} r\,dr = \left[\frac{1}{2}r^2\right]_{r=\sqrt{2}}^{r=2\cos\theta} = \frac{1}{2}\left((2\cos\theta)^2 - (\sqrt{2})^2\right) = \frac{1}{2}\left(4\cos^2\theta - 2\right)$$

- Integrate with respect to θ:

$$A = \int_0^{\frac{\pi}{4}}\left(2\cos^2\theta - 1\right)d\theta$$

5. **Simplify and evaluate the integral:** - Use the identity $\cos 2\theta = 2\cos^2\theta - 1$:

$$2\cos^2\theta - 1 = \cos 2\theta$$

- Therefore:

$$A = \int_0^{\frac{\pi}{4}}\cos 2\theta\,d\theta = \left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{2}(1 - 0) = \frac{1}{2}$$

6. **Final Answer:** - The area $A = \frac{1}{2}$ square units.

Example: Evaluating a Line Integral

Problem: Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle 4xy^2 + 9x^2, 3e^y + 4x^2y\rangle$ and C is the part of the parabola $4y = x^2$ from $(2, 1)$ to $(-2, 1)$.
Solution: 1. **Verify if the Vector Field is Conservative:** - Let $P = 4xy^2 + 9x^2$ and $Q = 3e^y + 4x^2y$. - Compute $\frac{\partial P}{\partial y} = 8xy$ and $\frac{\partial Q}{\partial x} = 8xy$. - Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\vec{F}$ is conservative.
2. **Find the Potential Function** $f(x, y)$**:** - $f_x = 4xy^2 + 9x^2 \implies f(x, y) = \int(4xy^2 + 9x^2)\,dx = 2x^2y^2 + 3x^3 + g(y)$. - Differentiate with respect to y: $f_y = 4x^2y + g'(y)$. - Set equal to Q: $4x^2y + g'(y) = 3e^y + 4x^2y \implies g'(y) = 3e^y$. - Integrate $g'(y)$: $g(y) = 3e^y$. - Potential function: $f(x, y) = 2x^2y^2 + 3x^3 + 3e^y$.
3. **Apply the Fundamental Theorem for Line Integrals:**

$$\int_C \vec{F}\cdot d\vec{r} = f(-2, 1) - f(2, 1).$$

- Compute $f(2, 1) = 2(2)^2(1)^2 + 3(2)^3 + 3e^1 = 8 + 24 + 3e$. - Compute $f(-2, 1) = 2(-2)^2(1)^2 + 3(-2)^3 + 3e^1 = 8 - 24 + 3e$. - Result:

$$\int_C \vec{F}\cdot d\vec{r} = (8 - 24 + 3e) - (8 + 24 + 3e) = -48.$$

4. **Path Independence Verification:** - Choose the line segment C' from $(2, 1)$ to $(-2, 1)$ and parametrize by $\vec{r}(t) = \langle -t, 1\rangle$ with $-2 \leq t \leq 2$. - $d\vec{r} = \langle -1, 0\rangle dt$ and $\vec{F}(\vec{r}(t)) = \langle 4t + 9t^2, 3e + 4t^2\rangle$. - $\vec{F}\cdot d\vec{r} = -4t - 9t^2$.
5. **Evaluate the Integral Directly:**

$$\int_{-2}^{2}(-4t - 9t^2)\,dt = \left[-2t^2 - 3t^3\right]_{-2}^{2} = -48.$$

Answer: $\int_C \vec{F}\cdot d\vec{r} = -48$.