Math 120 Cheat Sheet

Directional Derivatives and Gradient Vector

Directional Derivative:

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot \mathbf{u}$$

where $\mathbf{u}$ is a unit vector.
Gradient Vector:

$$\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y)\rangle$$

Properties: - ∇f points in the direction of maximum increase of f.
- ∇f is perpendicular to level curves (surfaces) of f.
Maximum Rate of Change:

$$\text{Max Rate} = |\nabla f(x,y)|$$

Maximum and Minimum Values

Second Derivative Test:
Compute $D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
- If $D > 0$ and $f_{xx}(a,b) > 0$, then local minimum at (a,b).
- If $D > 0$ and $f_{xx}(a,b) < 0$, then local maximum at (a,b).
- If $D < 0$, saddle point at (a,b).
- If $D = 0$, test is inconclusive.

Critical Points: Solve $f_x = 0$ and $f_y = 0$.

Lagrange Multipliers

To find extrema of $f(x,y,z)$ subject to constraint $g(x,y,z) = 0$:

$$\nabla f = \lambda \nabla g$$

Solve:

$$f_x = \lambda g_x,\quad f_y = \lambda g_y,\quad f_z = \lambda g_z$$
$$g(x,y,z) = 0$$

For Two Variables:

$$f_x = \lambda g_x,\quad f_y = \lambda g_y,\quad g(x,y) = 0$$

Double Integrals over Rectangles

Definition:

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

where $R = [a,b]\times[c,d]$.
Fubini's Theorem: If f is continuous on R:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Double Integrals over General Regions

Type I Region (Vertical):

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Type II Region (Horizontal):

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

Double Integrals in Polar Coordinates

Transformation:

$$x = r\cos\theta,\quad y = r\sin\theta$$

Jacobian:

$$dA = r\,dr\,d\theta$$

Integral:

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Vector Fields

Definition:
In $\mathbb{R}^2$:

$$\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$$

Gradient Field: $\mathbf{F} = \nabla f$
Conservative Field: If $\mathbf{F} = \nabla f$, then $\mathbf{F}$ is conservative.
Curl in $\mathbb{R}^2$:

$$\text{curl}\ \mathbf{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Line Integrals

Scalar Function:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\,|\mathbf{r}'(t)|\,dt$$

Vector Field:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy$$

$$= \int_a^b [P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)]\,dt$$

Fundamental Theorem for Line Integrals

If $\mathbf{F} = \nabla f$ is conservative, then for any curve C from A to B:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(B) - f(A)$$

Conservative Field Test: - In $\mathbb{R}^2$, if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in a simply connected domain, then $\mathbf{F}$ is conservative.

Green's Theorem

For a positively oriented, piecewise smooth, simple closed curve C enclosing region D:

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Area Using Green's Theorem:

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Trigonometric Identities

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Double Angle Formulas:

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Sum and Difference Formulas:

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

Half-Angle Formulas:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

Product to Sum:

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

Common Derivatives and Integrals

Derivatives:

$$\frac{d}{dx}[e^{ax}] = ae^{ax}$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\frac{d}{dx}[\sin ax] = a\cos ax$$

$$\frac{d}{dx}[\cos ax] = -a\sin ax$$

$$\frac{d}{dx}[\tan ax] = a\sec^2 ax$$

Integrals:

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

Integration Techniques: - Substitution: Let $u = g(x)$, then $\int f(g(x))g'(x)\,dx = \int f(u)\,du$.
- Integration by Parts: $\int u\,dv = uv - \int v\,du$.

Jacobian Determinant

For a transformation from (x, y) to (u, v):

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Use in Integration:

$$\iint_D f(x,y)\,dA = \iint_{D'} f(x(u,v), y(u,v))\,|J|\,du\,dv$$

Conservative Vector Fields

Tests for Conservativeness in $\mathbb{R}^2$: - $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
Finding Potential Function f: - Integrate P with respect to x, then Q with respect to y, and combine results. **Zero Curl Condition: -** For $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, if $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is conservative (in simply connected domains).

Common Coordinate Transformations

Polar to Cartesian:

$$x = r\cos\theta, \quad y = r\sin\theta$$

Cartesian to Polar:

$$r = \sqrt{x^2+y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Cylindrical Coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Spherical Coordinates:

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

Derivative Rules

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$