

Math 120

PSet 4

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Question 1

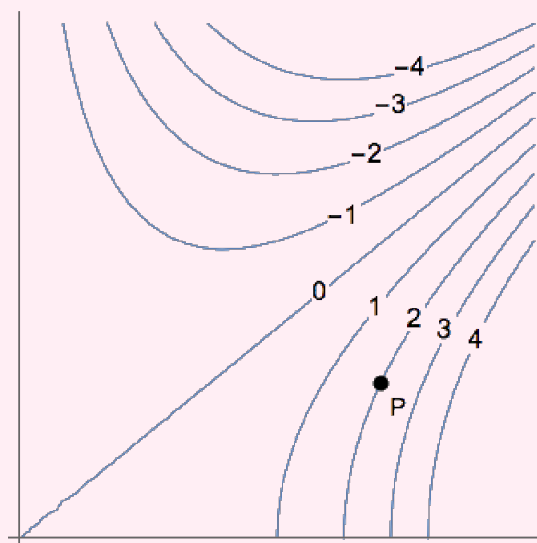
By finding both $f_{xy} = (f_x)_y$ and $f_{yx} = (f_y)_x$, verify that Clairaut's Theorem holds for the function $f(x, y) = y \arctan(xy)$.

Solution:

$$\begin{aligned}f_x &= \frac{\partial}{\partial x} y \arctan(xy) = y \cdot \frac{1}{1 + (xy)^2} \cdot y = \frac{y^2}{1 + (xy)^2} \\ \frac{\partial}{\partial y} \frac{y^2}{1 + (xy)^2} &= \frac{(2y)(1 + (xy)^2) - (y^2)(2x^2y)}{(1 + (xy)^2)^2} \\ \frac{(2y)(1 + (xy)^2) - (y^2)(2x^2y)}{(1 + (xy)^2)^2} &= \frac{2y}{(1 + (xy)^2)} \\ f_y &= \frac{\partial}{\partial y} y \arctan(xy) = \arctan(xy) + \frac{xy}{1 + (xy)^2} \\ \frac{\partial}{\partial x} \arctan(xy) + \frac{xy}{1 + (xy)^2} &= \frac{y}{(1 + (xy)^2)} + \frac{y(1 + (xy)^2) - (2y^2x)(xy)}{(1 + (xy)^2)^2} = \frac{2y}{(1 + (xy)^2)}\end{aligned}$$

Question 2

Level curves are shown below for a function f



- Determine the signs of f_x , f_y , f_{xx} and f_{yy} at the point P . Explain your reasoning. *You should assume that the undrawn level curves are nicely and evenly distributed between the ones drawn.*
- Mark a point on the contour plot where $f_x = 0$. (You can either mark the point on a screenshot and insert the picture in your homework file, or just make a rough copy of the contour plot by hand.)

Solution:

a)

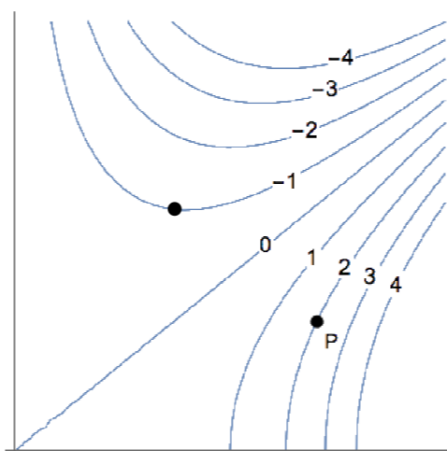
$f_x > 0$ as you move to the right of P f increases.

$f_y < 0$ as you go up in the y direction at point P there is a decrease in the value of f .

$f_{xx} > 0$ The level curves in the x -direction appear to be getting closer together as we move to the right, so it means that f_x is increasing.

$f_{yy} > 0$ the difference in the level curves is decreasing as you move up so it means that f_y is increasing

b)



Question 3

Use the Chain Rule to find the indicated partial derivatives.

- (a) Compute $\frac{dz}{dt}$ if $z = \tan(y/x)$, $x = e^t$, and $y = 1 - e^{-t}$.
- (b) Compute $\frac{\partial M}{\partial u}$ and $\frac{\partial M}{\partial v}$ at $u = 3$ and $v = -1$ if $M = xe^{y-z^2}$, $x = 2uv$, $y = u - v$, and $z = u + v$.

Solution:

a)

$$\begin{aligned}\frac{dz}{dt} &= \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt} \\ \frac{dz}{dx} &= z_x = \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{(0)x - (1)(y)}{x^2}\right) \Rightarrow \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{-y}{x^2}\right) \\ \frac{dz}{dy} &= z_y = \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{(1)x - (0)(y)}{x^2}\right) \Rightarrow \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{1}{x}\right) \\ \frac{dx}{dt} &= e^t \quad \frac{dy}{dt} = e^{-t} \\ \frac{dz}{dt} &= \sec\left(\frac{1 - e^{-t}}{e^t}\right)^2 \cdot \left(\frac{-(1 - e^{-t})}{e^{2t}}\right) \cdot e^t + \sec\left(\frac{1 - e^{-t}}{e^t}\right)^2 \cdot \left(\frac{1}{e^t}\right) \cdot e^{-t}\end{aligned}$$

b)

$$\begin{aligned}\frac{\partial M}{\partial u} &= \frac{\partial M}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial M}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial M}{\partial z} \cdot \frac{dz}{dt} \\ M_x &= e^{y-z^2} \quad \frac{dx}{du} = 2v \quad \frac{dx}{dv} = 2u \\ M_y &= xe^{y-z^2} \quad \frac{dy}{du} = 1 \quad \frac{dy}{dv} = -1 \\ M_z &= -2zxe^{y-z^2} \quad \frac{dz}{du} = 1 \quad \frac{dz}{dv} = 1 \\ \frac{\partial M}{\partial u} &= e^{y-z^2}(2v) + (xe^{y-z^2})(1) + (-2zxe^{y-z^2})(1) \\ x &= 2uv \Rightarrow 2(3)(-1) = -6 \\ y &= u - v \Rightarrow (3) - (-1) = 4 \\ z &= u + v \Rightarrow 3 + (-1) = 2 \\ \frac{\partial M}{\partial u} &= e^{4-2^2}(2(-1)) + ((-6)e^{4-2^2})(1) + (-2(2)(-6)e^{4-2^2})(1) = 1(-2) + (-6)(1) + (24)(1) = 16 \\ \frac{\partial M}{\partial v} &= \frac{\partial M}{\partial x} \cdot \frac{dx}{dv} + \frac{\partial M}{\partial y} \cdot \frac{dy}{dv} + \frac{\partial M}{\partial z} \cdot \frac{dz}{dv} \\ \frac{\partial M}{\partial v} &= e^{y-z^2}(2u) + (xe^{y-z^2})(-1) + (-2zxe^{y-z^2})(1) \\ \frac{\partial M}{\partial v} &= e^{4-2^2}(2(3)) + ((-6)e^{4-2^2})(-1) + (-2(2)(-6)e^{4-2^2})(1) = 1(6) + (-6)(1) + (24)(1) = 36\end{aligned}$$

Question 4

Use implicit differentiation to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the surface given by $z^3 + 3xyz + x^2 + y^2 = 0$ at the point $(1, -2, 1)$.

Solution:

$$\begin{aligned}\frac{\partial z}{\partial x} (z^3 + 3xyz + x^2 + y^2) &= 3z^2 \frac{\partial z}{\partial x} + 3yz + 3xy \frac{\partial z}{\partial x} + 2x = 0 \\ 3z^2 \frac{\partial z}{\partial x} + 3yz + 3xy \frac{\partial z}{\partial x} + 2x &= 0 \Rightarrow 3z^2 \frac{\partial z}{\partial x} + 3xy \frac{\partial z}{\partial x} = -3yz - 2x \\ 3z^2 \frac{\partial z}{\partial x} + 3xy \frac{\partial z}{\partial x} &= -3yz - 2x \Rightarrow \frac{\partial z}{\partial x} (3z^2 + 3xy) = -3yz - 2x \\ \frac{\partial z}{\partial x} (3z^2 + 3xy) &= -3yz - 2x \Rightarrow \frac{\partial z}{\partial x} = \frac{-3yz - 2x}{3z^2 + 3xy} \\ \frac{\partial z}{\partial x} &= \frac{-3(-2)(1) - 2(1)}{3(1)^2 + 3(1)(-2)} = \frac{-4}{3} \\ \frac{\partial z}{\partial y} (z^3 + 3xyz + x^2 + y^2) &= 3z^2 \frac{\partial z}{\partial y} + 3xz + 3xy \frac{\partial z}{\partial y} + 2y = 0 \\ 3z^2 \frac{\partial z}{\partial y} + 3xz + 3xy \frac{\partial z}{\partial y} + 2y &= 0 \Rightarrow 3z^2 \frac{\partial z}{\partial y} + 3xy \frac{\partial z}{\partial y} = -3xz - 2y \\ 3z^2 \frac{\partial z}{\partial y} + 3xy \frac{\partial z}{\partial y} &= -3xz - 2y \Rightarrow \frac{\partial z}{\partial y} (3z^2 + 3xy) = -3xz - 2y \\ \frac{\partial z}{\partial y} (3z^2 + 3xy) &= -3xz - 2y \Rightarrow \frac{\partial z}{\partial y} = \frac{-3xz - 2y}{3z^2 + 3xy} \\ \frac{\partial z}{\partial y} &= \frac{-3(1)(1) - 2(-2)}{3(1)^2 + 3(1)(-2)} = -\frac{1}{3}\end{aligned}$$

Question 5

[(5.) (Stewart problem 14.5.36)] Wheat production W in a given year depends on the average temperature T and annual rainfall R . Scientists estimate that the average temperature is rising at a rate of $0.15^\circ\text{C}/\text{year}$ and rainfall is decreasing at a rate of $0.1 \text{ cm}/\text{year}$. They also estimate that at current production levels, $\partial W/\partial T = -2$ and $\partial W/\partial R = 8$.

- What is the significance of the signs of these partial derivatives?
- Estimate the current rate of change of wheat production $\frac{dW}{dt}$.

Solution:

a) $\frac{\partial W}{\partial T} = -2$ means that for every drop in 1°C there is a decrease of 2 units in wheat production. If the sign were positive it would mean that there would be an increase in wheat production as a result of the decrease of temperature.
 $\frac{\partial W}{\partial R} = 8$ means for every increase in 1 cm rainfall there is an increase of 8 units in wheat production. If the sign were negative instead it would mean that an increase in rainfall brought down wheat production. b)

$$\frac{dW}{dt} = \frac{\partial W}{\partial T} \cdot \frac{dT}{dt} + \frac{\partial W}{\partial R} \cdot \frac{dR}{dt} = -2(0.15) + 8(-0.1) = -1.1$$