Math 120 QR

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Contents

Chapter 1

1.1 12.1 Notes (Three Dimensional Coodinate Systems)

Definition 1.1.1: Distance Formula

Defintion:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Definition 1.1.2: Equation of a sphere

Defintion: An equation of a sphere with center $C(h, k, l)$, and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2$$

In particular, if the center is the origin O, than an equation of the sphere is

$$x^2 + y^2 + z^2$$

1.2 12.2 Notes (Vectors)

Definition 1.2.1: Vector Addition

If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the **sum** $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

Definition 1.2.2: Scalar Multiplication

If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c = 0$ or $\mathbf{v} = 0$, then c$\mathbf{v} = 0$

Example 1.2.1:

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with represenation $\overrightarrow{AB}$ is:

$$a = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 1.2.2:

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarily, for three demensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Note:-

Properties of vectors: If **a**, **b**, and **c** are vectors in V_n and c and d are scalars than

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $a + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + 0 = \mathbf{a}$
- $\mathbf{a} + \mathbf{a} + \mathbf{-a} = 0$
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
- $(c + d)a = c\mathbf{a} + d\mathbf{a}$
- $(cd)\mathbf{a} = c(d\mathbf{a})$
- $l\mathbf{a} = \mathbf{a}$

1.3 12.3 Notes (Dot Product)

Definition 1.3.1: Dot Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties of the Dot Product: If $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

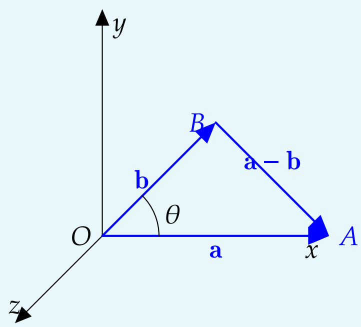
1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

Definition 1.3.2: Geometric Definition of the Dot Product

If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

Proof:



$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos\theta$$

Corollary: If θ is the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Note:-

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if an only if $\mathbf{a} \cdot \mathbf{b} = 0$

Example 1.3.1 (Direction Angles and Cosines)

The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ (in the interval $[0, \pi]$) that $\mathbf{a}$ makes with the positive x-, y-, and z-axes, respectively .

The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector $\mathbf{a}$. Using Corollary 6 with $\mathbf{b}$ replaced by $\mathbf{i}$, we obtain:

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|} \tag{1}$$

Similarly, we also have:

$$\cos\beta = \frac{a_2}{|\mathbf{a}|} \quad \text{and} \quad \cos\gamma = \frac{a_3}{|\mathbf{a}|} \tag{2}$$

By squaring the expressions in Equations 8 and 9 and adding, we see that:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{3}$$

We can also use Equations 8 and 9 to write:

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle = \langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma\rangle = |\mathbf{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$

Therefore,

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle \tag{4}$$

which says that the direction cosines of $\mathbf{a}$ are the components of the unit vector in the direction of $\mathbf{a}$.

Definition 1.3.3: Projections

The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This is denoted by $\mathrm{comp}_{\mathbf{a}}\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leqslant \pi$. The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of $\mathbf{a}$ and $\mathbf{b}$ can be interpreted as the length of $\mathbf{a}$ times the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of $\mathbf{b}$ along $\mathbf{a}$ can be computed by taking the dot product of $\mathbf{b}$ with the unit vector in the direction of $\mathbf{a}$. We summarize these ideas as follows.

Scalar projection of b onto a: $\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of b onto a: $\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

1.4 12.4 Notes (Cross Product)

Definition 1.4.1: Cross Product

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, suppose that a nonzero vector $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$, and so:

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \tag{1}$$
$$b_1c_1 + b_2c_2 + b_3c_3 = 0 \tag{2}$$

To eliminate c_3, we multiply (1) by b_3 and (2) by a_3 and subtract:

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0 \tag{3}$$

Equation (3) has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So, a solution of (3) is:

$$c_1 = a_2b_3 - a_3b_2$$
$$c_2 = a_3b_1 - a_1b_3$$

Substituting these values into (1) and (2), we then get:

$$c_3 = a_1b_2 - a_2b_1$$

This means that a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ is:

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ and is denoted by $\mathbf{a} \times \mathbf{b}$.

Definition 1.4.2: Cross Product of two vectors

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

Note:-

Determinant of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Note:-

Determinant of order 3:
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Definition 1.4.3: Second definition of cross product

Arithmetic Definition:

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = |a||b|\sin(\theta)$$

$$\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} i - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} j + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 k \end{bmatrix}$$

$$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$

Example 1.4.1: Proof that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1a_3b_2 - a_2a_1b_3 + a_2a_3b_1 + a_3a_1b_2 - a_3a_2b_1 \\ &= 0 \end{aligned}$$

Definition 1.4.4: sin definition of cross product

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$$

Proof:

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \quad \text{(by Theorem 12.3.3)} \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta \end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geqslant 0$ when $0 \leqslant \theta \leqslant \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

Note:-

Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = 0$$

Example 1.4.2: Geometric interpretation of $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$

If **a** and **b** are represented by directed line segments with the same inital point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $\mathbf{b}\sin(\theta)$ and area

$$A = |\mathbf{a}|(|\mathbf{b}|\sin\theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product:
The length of the cross product of $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**

Note:-

If we apply the following theorem:
The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**, and

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

to the standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ using $\theta = \frac{\pi}{2}$, we obtain

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

Note:-

If **a**, **b**, and **c** are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Example 1.4.3: Proof of property 5 of cross products

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then:

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

Definition 1.4.5: Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the *scalar triple product* of the vectors **a**, **b**, and **c**.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**. The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between **a** and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Therefore, the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus, we have proved the following formula: The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Note:-

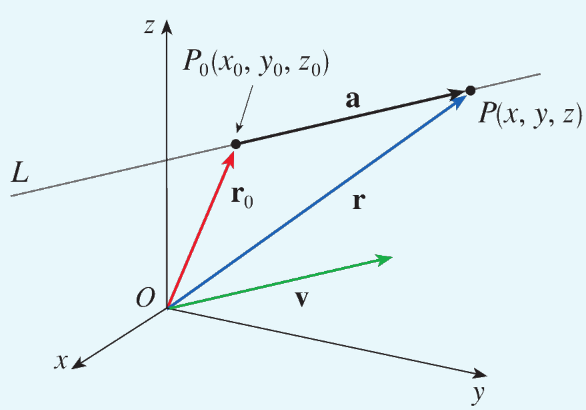
If we use the formula in $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ and discover that the volume of the parallelepiped determined by a, b, and c is 0, then the vectors must lie in the same plane; that is, they are coplanar

1.5 12.5 Notes (Equations of Lines and Planes)

Definition 1.5.1: Hi

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and a direction for L, which is conveniently described by a vector **v** parallel to the line. Let $P(x, y, z)$ be an arbitrary point on L and let $\mathbf{r_0}$ and **r** be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If **a** is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives

$$\mathbf{r} = \mathbf{r_0} + \mathbf{a}.$$



Since **a** and **v** are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$ Thus

$$r = r_0 + t\mathbf{v}$$

Note:-

If the vector $\mathbf{v}$ that gives the direction of the line L is written in component form as

$$\mathbf{v} = \langle a, b, c \rangle,$$

then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$. We can also write $\mathbf{r} = \langle x, y, z \rangle$ and

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle,$$

so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle.$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

Example 1.5.1: Line example

Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

Parametric equations are

$$x = 5 + t \quad y = 1 + 4t \quad z = 3 - 2t$$

Note:-

In general, if a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line L, then the numbers a, b, and c are called *direction numbers* of L. Since any vector parallel to $\mathbf{v}$ could also be used, we see that any three numbers proportional to a, b, and c could also be used as a set of direction numbers for L.

Another way of describing a line L is to eliminate the parameter t from Equations 2. If none of a, b, or c is 0, we can solve each of these equations for t:

$$t = \frac{x - x_0}{a} \quad t = \frac{y - y_0}{b} \quad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called symetric equations of L

Definition 1.5.2: Line segment

The line segment from r_0 to r_1 is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r_0} + tr_1 \quad 0 \leqslant t \leqslant 1$$

Definition 1.5.3: Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r_0}$ and $\mathbf{r}$ be the position vectors of P_0 and P. Then the vector $\mathbf{r} - \mathbf{r_0}$ is represented by $\overrightarrow{P_0P}$. The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane. In particular, $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r_0}$ and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0 \tag{1.1}$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0} \tag{1.2}$$

These can be reffered to as the **vector equation of the plane**
To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c \rangle, \mathbf{r} = \langle x, y, x \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. then the vector equation becomes:

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Expanding the left side of this equation gives the following:
A **scalar equation of the plane** through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

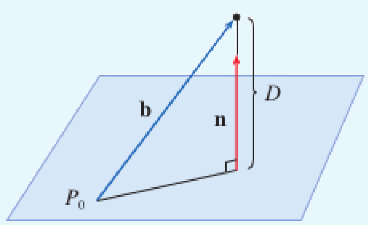
by colecting terms can be rewritten as:

$$ax + by + cz + d = 0$$

Definition 1.5.4: Distance of a plane

In order to find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$, we let $P_0(x_0, y_0, z_0)$ be any point in the given plane and $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$



From Figure, you can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$. Thus,

$$\begin{aligned} D = |\text{comp}_{\mathbf{n}}\mathbf{b}| &= \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

1.6 12.6 Reading Notes (Cylinders and Quadric Surfaces)

Definition 1.6.1: Cylinder

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

Definition 1.6.2: Quadric Surfaces

A Quadric Surface is the graph of a second-degree equation in three variables x, y, and z. The most general such equation is

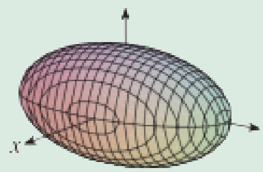
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0$$

where A, B, C, $\ldots$, J are constants, but by translation and rotation it can be brought into one of the two *standard forms*

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

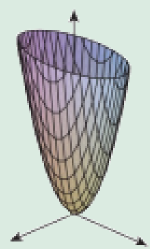
Example 1.6.1: Graphs of Quadric Surfaces PT 1

Ellipsoid:



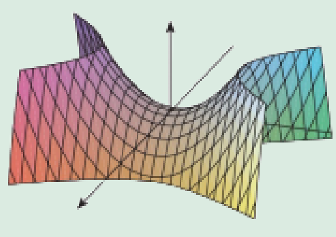
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.



$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

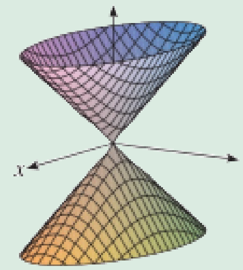
Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.



$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

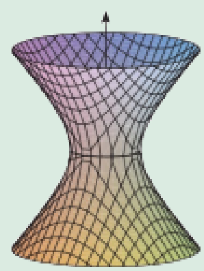
Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.

Example 1.6.2: Quadric Surfaces Pt 2



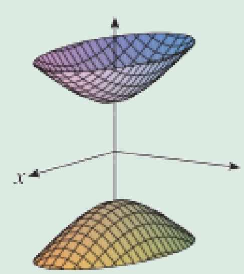
$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.



$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Chapter 2

2.1 13.1 Reading Notes(Vector Functions and Space Curves)

Definition 2.1.1: Vector Value Functions

A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions r whose values are three-dimensional vectors. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f , g, and h are real-valued functions called the **component functions** of r and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

Definition 2.1.2: Limit of Vectors

The **limit** of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions as follows. If $(\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle)$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

provided the limits of the component functions exist.

Definition 2.1.3: Space Curves

here is a close connection between continuous vector functions and space curves. Suppose that f, g, and h are continuous real-valued functions on an interval I. Then the set C of all points (x, y, z) in space, where

$$x = f(t) \quad y = g(t) \quad z = h(t)$$

and (t) varies throughout the interval I, is called a **space curve**. The equations in are called **parametric equations of** C and t is called a **parameter**.

2.2 13.2 Notes (Derivatives and Integrals of Vector Functions)

Definition 2.2.1: Derivatives

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

Definition 2.2.2: Derivatives of vectors pt 2

he following theorem gives us a convenient method for computing the derivative of a vector function $\mathbf{r}$: just differentiate each component of $\mathbf{r}$. **Theorem** If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Example 2.2.1: Proof of Definition 2.2.2

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)] \\
&= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)\rangle - \langle f(t), g(t), h(t)\rangle] \\
&= \lim_{\Delta t\to 0} \left\langle \frac{f(t+\Delta t)-f(t)}{\Delta t}, \frac{g(t+\Delta t)-g(t)}{\Delta t}, \frac{h(t+\Delta t)-h(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t\to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}, \lim_{\Delta t\to 0} \frac{g(t+\Delta t)-g(t)}{\Delta t}, \lim_{\Delta t\to 0} \frac{h(t+\Delta t)-h(t)}{\Delta t} \right\rangle \\
&= \langle f'(t), g'(t), h'(t)\rangle
\end{aligned}$$

A unit vector that has the same direction as the tangent vector is called the **unit tangent vector T** and is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Definition 2.2.3: Differentiation Rules

Proof: **Theorem** Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Note:-

We use Formula 4 to prove the following theorem. **Theorem** If $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Definition 2.2.4: Interation of Vectors

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of $\mathbf{r}$ in terms of the integrals of its component functions f, g, and h as follows. (We use the notation of Chapter 5.)

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{n\to\infty} \sum_{j=1}^{n} \mathbf{r}(t_j^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\,\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\,\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\,\Delta t \right) \mathbf{k} \right]$$

and so

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt \right) \mathbf{i} + \left(\int_a^b g(t)\,dt \right) \mathbf{j} + \left(\int_a^b h(t)\,dt \right) \mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function. We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$. We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (antiderivatives).

2.3 13.3 Notes (Arc Length and Curvature)

Definition 2.3.1: Length of a space curve

Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \leqslant t \leqslant b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous. If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \tag{2.1}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{2.2}$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

$$L = \int_a^b |\mathbf{r}'(t)|\, dt \tag{2.3}$$

because, for plane curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

A single curve C can be respresented by more than one vector function. For instance the twisted cube

$$r_1(t) = \langle t, t^2, t^3\rangle \quad 1 \leqslant t \leqslant 2 \tag{2.4}$$

could also be represented by the function

$$r_2(u) = \langle e^u, e^{2u}, e^{3u}\rangle \quad o \leqslant u \leqslant \ln 2 \tag{2.5}$$

where the connection between the parameters t and u is given by $t = e^u$ We say that equations 2.4 and 2.5 are parameterizations of the curve C. If we were to use Equation 2.3 to compute the length of C using Equations 2.4 and 2.5, we would get the same answer. This is because arc length is a geometric property of the curve and hence is independent of the parametrization that is used.

Definition 2.3.2: Arc Length Function

Now we supose that the curve C is a cruve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \leqslant t \leqslant b$$

where **r**' is continuous and C is transvered exactly once as t increases from a to b. We define its **arc length functions** by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du \tag{2.6}$$

Thus $s(t)$ is the length of part C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. If we differentiate both sides of equation 2.6 using part 1 of the Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = |\mathbf{r}(t)| \tag{2.7}$$

It is often useful to **paramterize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system of a particular parametrization.

Definition 2.3.3: Curvature

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if **r**' is continuous and **r**'$(t) \neq 0$ on I. A curve is called smooth if it has a smooth parameterization. A smooth corner has no cusp or sharp corner; when the tangent vector turns it does so continuously.

If C is a smooth curve defined by the vector **r**, recall that the unit tangent vector **T**(t) is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The curvature of a curve is

$$k = |\frac{d\mathbf{T}}{ds}| \tag{2.8}$$

where **T** is the unit tangent vector

The curvature is easier to compute if it is easier to compute if it is expressed in terms of the paramteter t instead of s, so we use the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \Rightarrow k = \left|\frac{d\mathbf{T}}{s}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|$$

but $ds/dt = |\mathbf{r}'(t|)$ from equation 2.7

$$k(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \tag{2.9}$$

The curvature of the curve given by the vector function **r** is

$$k(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}^n(t)|}{|\mathbf{r}'(t)|^3} \tag{2.10}$$

Note:-

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = 0$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f'(x)\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \tag{2.11}$$

Note:-

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t, we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$ by Theorem 13.2.4, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that, typically, $\mathbf{T}'(t)$ is itself not a unit vector. But at any point where $\kappa \neq 0$ we can define the *principal unit normal vector* $\mathbf{N}(t)$ (or simply *unit normal*) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. The vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

Note:-

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Definition 2.3.4: Torision

The **torsion** of a curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Torsion is easier to compute if it is expressed in terms of the parameter t instead of s, so we use the Chain Rule to write

$$\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\frac{ds}{dt} \quad \text{so} \quad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{\mathbf{B}'(t)}{|\mathbf{r}'(t)|}$$

$$\tau(t) = \frac{-\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|}$$

Theorem The torsion of the curve given by the vector function $\mathbf{r}$ is

$$\tau(t) = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

2.4 13.4 Notes (Motion in Space: Velocity and Acceleration)

Chapter 3

3.1 14.1 Functions of Several Variables

Definition 3.1.1: Functions of Two Variables

Defintion: A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the *domain* of f and its *range* is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

Definition 3.1.2: Graph of a Function of Two Variables

Defintion If f is a function of two variables with domain D, then the *graph* of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Definition 3.1.3: Level Curves and Contour Maps

Defintion: The *level curves* of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Definition 3.1.4: Functions of Three Variables

Defintion: **A function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature...

3.2 14.2 Limits and Continuity

Definition 3.2.1: Limit of Two Variable Functions

Defintion: Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the *limit of* $f(x, y)$ *as* (x, y) *approaches* (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$.

3.3 14.3 Partial Derivatives

Definition 3.3.1: Partial Derivatives

Defintion Partial Derivative with respect to x

$$f_x(a,b) = g'(a) \quad \text{where} \quad g(x) = f(x,b)$$

Defintion Partial Derivative with respect to y

$$f_y(a,b) = h'(a) \quad \text{where} \quad h(x) = f(a,y)$$

Note:-

If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Note:-

Rule for Finding Partial Derivatives of $z = f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

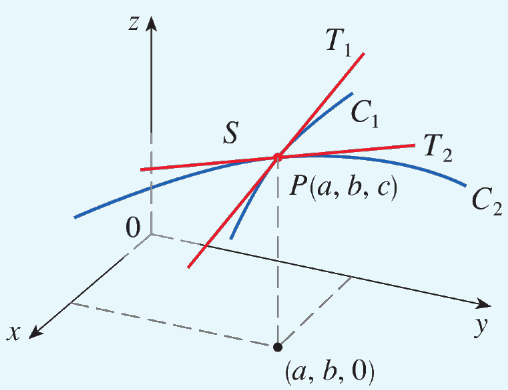
Definition 3.3.2: interpretation of Partial Derivatives

To understand partial derivatives geometrically, think of the equation $z = f(x,y)$ as representing a surface S (the graph of f). If $f(a,b) = c$, then the point $P(a,b,c)$ lies on this surface.

By fixing $y = b$, we focus on the curve C_1 where the vertical plane $y = b$ intersects S. Similarly, fixing $x = a$ gives us the curve C_2, which is where the vertical plane $x = a$ intersects S. Both curves C_1 and C_2 pass through the point P.

The curve C_1 is the graph of the function $g(x) = f(x,b)$, and the slope of its tangent at P is $f_x(a,b)$. The curve C_2 is the graph of $G(y) = f(a,y)$, and the slope of its tangent at P is $f_y(a,b)$.

Thus, the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ represent the slopes of the tangent lines at P along these curves.



Definition 3.3.3: Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the *second partial derivatives* of f. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\frac{\partial^2 f}{\partial y \partial x}$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Definition 3.3.4: Clairut's Theorem

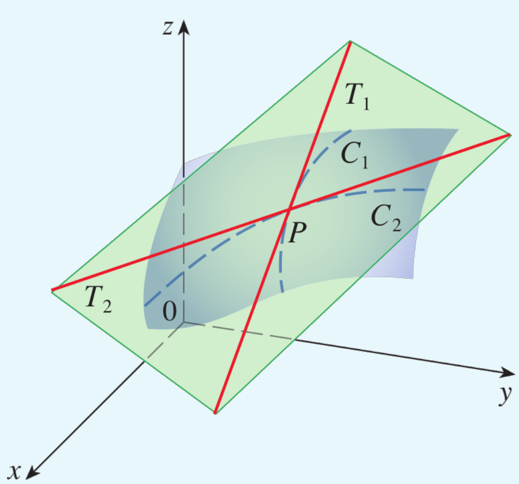
Defintion: Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

3.4 14.4 Tangent Planes and Linear Approximation

Definition 3.4.1: Tangent Planes

Let's consider a surface S given by the equation $z = f(x, y)$, where f has continuous first derivatives. Let $P(x_0, y_0, z_0)$ be a point on the surface. Two curves, C_1 and C_2, are formed by slicing the surface with vertical planes $y = y_0$ and $x = x_0$. These curves pass through the point P. The tangent lines to C_1 and C_2 at P are denoted T_1 and T_2. The **tangent plane** to the surface at P is the plane that contains both tangent lines T_1 and T_2.



Definition 3.4.2: Equation of a tangent plan

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{3.1}$$

Definition 3.4.3: Linear Approximations

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \tag{3.2}$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy such that ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

3.5 14.5 The Chain Rule

Definition 3.5.1: Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{3.3}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial z}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{3.4}$$

Definition 3.5.2: Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \tag{3.5}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{3.6}$$

Definition 3.5.3: Chain Rule (General Case)

Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i} \tag{3.7}$$

for each $i = 1, 2, \ldots, m$.

Definition 3.5.4: Implicit Differentiation

Defintion:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial Y}} = -\frac{F_x}{F_y} \tag{3.8}$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \tag{3.9}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z} \tag{3.10}$$

3.6 14.6 Directional Derivatives and the Gradient Vector

Definition 3.6.1: Directional Derivative

Defintion: The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \tag{3.11}$$

if this limit exists.

Theorem 3.6.1

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b \tag{3.12}$$

Definition 3.6.2: The Gradient Vector

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \tag{3.13}$$

Note:-

Equation 3.12 can be rewritten as

$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u} \tag{3.14}$$

Definition 3.6.3: Gradient of Three Variable Functions

Defintion:

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{3.15}$$

Theorem 3.6.2 Maximizing Directional Derivative

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Definition 3.6.4: Tangent Planes to Level Surfaces

Consider a surface S defined by $F(x, y, z) = k$, where F is a function of three variables. Let $P(x_0, y_0, z_0)$ be a point on S and C be a curve on S that passes through P. The curve is given by a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ such that $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since C lies on S, the equation $F(x(t), y(t), z(t)) = k$ must hold.

By using the Chain Rule to differentiate both sides of this equation, we get:

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

This can be written as a dot product:

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

which means that the gradient ∇F is perpendicular to the tangent vector $\mathbf{r}'(t)$ at P.
At $t = t_0$, the gradient at P, $\nabla F(x_0, y_0, z_0)$, is normal to the tangent plane at P. The equation of the tangent plane is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{3.16}$$

Note:-

Properties of the Gradient Vecotr
Let f be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x}) \neq 0$.

- The directional derivative of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.
- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that maximum rate of change is $|\nabla f(\mathbf{x})|$.
- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.

3.7 14.7 Maxium and Minimum Values