

Math 120

PSet 8

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Contents

Chapter 1

Page 2

1.1 PSet 8

2

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Question 1

Let $\vec{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$, and let C be the triangle from $(0, 0)$ to $(2, 6)$ to $(2, 0)$ to $(0, 0)$. Use Green's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$. (Check the orientation of the curve before applying the theorem.)

Solution:

$$\begin{aligned}\vec{F}(x, y) &= \langle y^2 \cos x, x^2 + 2y \sin x \rangle \\ \int_C \vec{F} \cdot d\vec{r} &= - \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x} (x^2 + 2y \sin x) = 2x + 2y \cos x \\ \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y} (y^2 \cos x) = 2y \cos x \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= [2x + 2y \cos x] - [2y \cos x] = 2x \\ \int_C \vec{F} \cdot d\vec{r} &= - \iint_D 2x dx dy \\ \int_{y=0}^{y=3x} 2x dy &= 2x(3x - 0) = 6x^2 \\ - \int_{x=0}^2 6x^2 dx &= -6 \int_0^2 x^2 dx = -6 \left[\frac{x^3}{3} \right]_0^2 = -6 \left(\frac{8}{3} \right) = -16 \\ \int_C \vec{F} \cdot d\vec{r} &= -16\end{aligned}$$

Question 2

Let $P(x, y) = -x^2 y^3$ and $Q(x, y) = xy^2$, and let C be the circle $x^2 + y^2 = 4$, oriented counterclockwise.

- (a) Compute $\int_C \vec{F} \cdot d\vec{r}$ directly, by parameterizing C and finding the line integral.
- (b) Compute $\int_C \vec{F} \cdot d\vec{r}$ using Green's Theorem.

Question 3

Use Green's Theorem to find the area enclosed by the parametric curve $\vec{r}(t) = \langle \sin t, \sin 2t \rangle$, $0 \leq t \leq \pi$.

Question 4

Consider the vector field $\vec{F} = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$.

- Show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ at every point in the domain of \vec{F} .
- Let C be the short arc of the circle $x^2 + y^2 = 2$ from $(1, 1)$ to $(-1, 1)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ directly, by parameterizing the curve and computing $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$.
- Integrate $P(x, y) = -\frac{y}{x^2+y^2}$ with respect to x , and check that the partial derivative of the result with respect to y is $Q(x, y) = \frac{x}{x^2+y^2}$. You have now found a function f such that $\nabla f = \vec{F}$.

What is the domain of this function f ? Is it the same as the domain of \vec{F} ?

- Use your answer to part (c) and the Fundamental Theorem of Line Integrals to check your answer to part (b).
- Now let C be the circle of radius R centered at the origin, oriented counterclockwise. Compute $\oint_C \vec{F} \cdot d\vec{r}$. Explain why your answer doesn't contradict the statement that the integral of a conservative vector field around any closed curve must be zero. Hint: Look carefully at the domain of the potential function f you found in part (b).

Question 5

Again consider the vector field $\vec{F} = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$. Let C_1 be any closed curve going counterclockwise around the origin, such as the orange curve below. Let C_2 be a circle, centered around the origin, with radius less than the shortest distance between C_1 and the origin. (This condition guarantees that the two curves don't intersect.) Let D be the region between the two curves.

- Explain why Green's Theorem applies on the region D .
- The boundary of D is the union of the two curves C_1 and $-C_2$, where by $-C_2$ we mean the inside circle oriented clockwise. Since $\int_{-C_2} \vec{F} \cdot d\vec{r} = -\int_{C_2} \vec{F} \cdot d\vec{r}$, Green's Theorem implies that

$$\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y) dA.$$

Use the results of Problem # 4 above to determine the value of $\int_{C_1} \vec{F} \cdot d\vec{r}$.

Question 6

Let $\vec{F} = \langle 2y - x^2, 4x + ye^{\cos y} \rangle$, and let C be the curve $y = x^2 - 9$, $-3 \leq x \leq 3$, oriented from left to right.

- Parameterize the curve C , and write the vector line integral $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$. Do not try to compute this integral directly!
- Let C^* be the line segment along the x -axis from $(3, 0)$ to $(-3, 0)$. Compute $\int_{C^*} \vec{F} \cdot d\vec{r}$.
- Let D be the region bounded by the parabola $y = x^2 - 9$ and the x -axis. Compute $\iint_D (Q_x - P_y) dA$.
- Use your answers to (b) and (c) to compute $\int_C \vec{F} \cdot d\vec{r}$.