Math 120

PSet 6

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Contents

Chapter 1

1.1 PSet 6

Question 1

Find all the (local) maximum and minimum values and saddle points of the function.

(a) $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

(b) $f(x, y) = e^y(x^2 - y^2)$

Solution:
a)

$$f_x = y - \frac{1}{x^2} \quad f_y = x - \frac{1}{y^2}$$

$$y = \frac{1}{x^2} \quad x = \frac{1}{y^2}$$

$$\frac{1}{x^2} = 0 \quad \frac{1}{y^2} = 0$$

$$x = \pm 1 \quad y = 1 \quad x \geqslant 0 \quad y = \frac{1}{x^2} \quad x = 1$$

$$f_{xx} = \frac{2x}{x^3} \quad f_{yy} = \frac{2y}{y^3} \quad f_{xy} = 0$$

$$f_{xx}(1, 1) = \frac{2(1)}{2^3} = 2 \quad f_{yy}(1, 1) = \frac{2(1)}{2^3} = 2$$

$$f_{xx}(1, 1) f_{yy}(1, 1) - 0^2 = 4$$

(1,1) is a local min because $D > 0$ and $f_{xx} > 0$ b)

$$f_x = 2x^{ey} \quad f_y = x^2 e^y - e^y y^2 - 2ye^y$$

$$2x^{ey} = 0 \Rightarrow x = 0$$

$$x^2 e^y - e^y y^2 - 2ye^y \Rightarrow 0^2 e^y - e^y y^2 - 2ye^y \Rightarrow -e^y y^2 - 2ye^y$$

$$-e^y y^2 - 2ye^y \Rightarrow 2ye^y = y^2 e^y \Rightarrow y = 0 \quad y = -2$$

$$f_{xx} = 2e^y \quad f_{yy} = x^2 e^y - 2ye^y - e^y y^2 - 2e^y - 2ye^y$$

$$D = 2e^0 \left(0^2 e^0 - 2(0)e^0 - e^0(0)^2 - 2e^0 - 2(0)e^0\right) = (2)(-2) = -4$$

$$D = 2e^- \left(0^2 e^{-2} - 2(-2)e^{-2} - e^{-2}(-2)^2 - 2e^{-2} - 2(-2)e^{-2}\right) = \frac{16}{e^4}$$

(0,0) is a saddle point becase $D < 0$ and (0,-2) is a lacal min because $D > 0$ and $f_{xx} > 0$

Question 2

Find the absolute maximum and minimum values of the function

$$f(x, y) = x + y - xy$$

on the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$.

Solution:

$$f_x = 1 - y = 0 \Rightarrow y = 1$$
$$f_y = 1 - x = 0 \Rightarrow x = 1$$
$$f_{xx} = 0 \quad f_{yy} = 0 \quad f_{xy} = -1$$
$$\text{(0,0) to (0,2)}$$
$$(0, y) = y \quad [0, 2]$$
$$\text{(0,2) to (4,0)}$$
$$y = 2 - \frac{1}{2}x$$
$$f(x, 2 - \frac{1}{2}x) = x + (2 - \frac{1}{2}x) - x(2 - \frac{1}{2}x) \Rightarrow x + 2 - \frac{1}{2}x - 2x + \frac{1}{2}x^2 \Rightarrow \frac{1}{2}x^2 - \frac{3}{2}x + 2$$
$$\frac{d}{dx}\left[\frac{1}{2}x^2 - \frac{3}{2}x + 2\right] = x - \frac{3}{2}$$
$$x = \frac{3}{2} \quad y = 2 - \left(\frac{3}{2}\right) = \frac{5}{4}$$
$$f\left(\frac{3}{2}, \frac{5}{4}\right) = \frac{7}{8}$$
$$\text{(0,0) to (4,0)}$$
$$y = 0$$
$$f(x, 0) = x$$
$$f(0, 0) = 0$$
$$f(4, 0) = 4$$

Abs Max at (4,0) where $f(x, y) = 4$ Abs Min (0,0) where $f(x, y) = 0$

Question 3

Find the absolute maximum and minimum values of the function

$$f(x, y) = xy^2$$

on the region $x^2 + y^2 \leqslant 3$, $x \geqslant 0$, $y \geqslant 0$.

Solution:

$$f(x, y) = xy^2$$
$$f_x = y^2 = 0 \Rightarrow y = 0$$
$$f_y = 2xy = 0 \Rightarrow y = 0 \quad x = 0$$
$$\nabla f = (y^2, 2xy) \quad \nabla g = (2x, 2y)$$
$$y^2 = 2x\lambda$$
$$2xy = 2y\lambda$$

$$\frac{y}{2x} = \frac{x}{y}$$

$$y^2 = 2x^2 \Rightarrow y = \pm\sqrt{2} \quad x^2 > 1$$

$$\left(1, \sqrt{2}\right)$$

$$f(0,0) = 0(0)^2 = 0$$

$$f\left(1, \sqrt{2}\right) = 1\sqrt{2}^2 = 2$$

$$f\left(\sqrt{3}, 0\right) = \sqrt{3}(0)^2 = 0$$

Absolute max $\left(1, \sqrt{2}\right) = 2$

Absolute min $(0,0)$ and $\left(\sqrt{3}, 0\right) = 0$

Question 4

Find the maximum and minimum values of the function $f(x,y) = x + 4y$ subject to the constraint

$$\sqrt{x} + \sqrt{y} = 3.$$

Solution:

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$

$$\nabla f(x,y) = (1,4)$$

$$\nabla g = \left(\frac{\partial}{\partial x}\sqrt{x} + \sqrt{y} - 3, \frac{\partial}{\partial y}\sqrt{x} + \sqrt{y} - 3\right)$$

$$\nabla g = \left(\frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}}\right)$$

$$1 = \lambda\frac{1}{2\sqrt{x}}$$

$$4 = \lambda\frac{1}{2\sqrt{y}}$$

$$\lambda = 2\sqrt{x}$$

$$4 = 2\sqrt{x}\left(\frac{1}{2\sqrt{y}}\right) = \frac{\sqrt{x}}{\sqrt{y}}$$

$$\sqrt{16y} + \sqrt{y} = 3 \Rightarrow 4\sqrt{y} + \sqrt{y} = 3 \Rightarrow 5\sqrt{3} = 3$$

$$\sqrt{y} = \frac{3}{5} \Rightarrow y = \frac{9}{25}$$

$$x = 16 \cdot \frac{9}{25} = \frac{144}{25}$$

$$f\left(\frac{144}{25}, \frac{9}{25}\right) = \frac{180}{25}$$

$$x = 0 \quad y = 9$$

$$x = 9 \quad y = 0$$

$$f(0,9) = 36 \quad f(9,0) = 36$$

Absolute max at $(0,9) = 36$
Absolute min at $\left(\frac{144}{25}, \frac{9}{25}\right) = \frac{180}{25}$

Question 5

Consider the function $f(x, y) = e^{xy}$ and the constraint $x^3 + y^3 = 16$.

(a) Use Lagrange multipliers to find the coordinates (x, y) of any points on the constraint where the function f could attain a maximum or minimum.

(b) For each point you found in part (a), is the point a maximum, a minimum, both or neither? Explain your answer carefully. What are the minimum and maximum values of f on the constraint? Please explain your answers carefully.

(c) The Extreme Value Theorem, which we covered last week, guarantees that under the right circumstances, we are guaranteed to find absolute minima and maxima for a function f on a certain constraint. Please explain why parts (a) and (b) don't violate the Extreme Value Theorem.

Solution:
a)

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$(ye^{yx} = \lambda(3x^2, 3y^2))$$
$$ye^{yx} = \lambda 3x^2$$
$$xe^{xy} = \lambda 3y^2$$
$$\frac{ye^{yx}}{xe^{xy}} = \frac{\lambda 3x^2}{\lambda 3y^2}$$
$$\frac{y}{x} = \frac{x^2}{y^2}$$
$$y^3 = x^3$$
$$y = x \quad y = -x$$
$$x^3 + x^3 = 16 \Rightarrow 2x^3 = 16 \Rightarrow x = 2$$
$$x^3 + (-x)^3 = 16 \Rightarrow 0 = 16$$

point is (2,2)
b)

Question 6

Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = x^2y^2z^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Solution:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$$
$$\nabla f = \left(2xy^2z^2, 2yx^2z^2, 2zx^2y^2\right)$$
$$\nabla g = \left(2x, 2y, 2z\right)$$
$$2xy^2z^2 = \lambda 2x$$
$$2yx^2z^2 = \lambda 2y$$
$$2zx^2y^2 = \lambda 2z$$
$$y^2z^2 = \lambda$$
$$z^2x^2 = \lambda$$
$$x^2y^2 = \lambda$$

$$x^2y^2 = z^2x^2 = y^2z^2$$
$$x = y = z$$
$$x^2 + y^2 + z^2 = 1$$
$$3x^2 = 1 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$
$$y = x = z = \frac{1}{\sqrt{3}}$$
$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{27}$$

max: $\frac{1}{27}$
min: 0

Question 7

Use Lagrange multipliers to find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x^4 + y^4 + z^4 = 1$.

Solution:

$$\nabla f(x, y, z) = (2x, 2y, 2z)$$
$$\nabla g(x, y, z) = \left(4x^3, 4y^3, 4z^3\right)$$
$$\nabla f = \lambda \nabla g$$
$$2x = \lambda 4x^3 \quad zy = \lambda 4y^3 \quad xy = \lambda 4z^3$$
$$z = \lambda 4x^2 \quad z = \lambda 4y^3 \quad z = \lambda 4z^3$$
$$\lambda = \frac{1}{2x^2} \quad \lambda = \frac{1}{2y^2} \quad \lambda = \frac{1}{2z^2}$$
$$x^4 + y^4 + z^4 = 1 \quad x^4 = y^4 = z^4 = t$$
$$3t = 1 \quad t = \frac{1}{3}$$
$$z^4 = \frac{1}{3} \quad z^2 = \frac{1}{\sqrt{3}}$$
$$f(x, y, z) = 3 \cdot \frac{1}{\sqrt{3}} = \sqrt{3}$$
$$\text{Case 1: One variabe is 0}$$
$$y^2 + z^2 = 1 \quad 2y^4 = 1$$
$$y^2 = \frac{1}{\sqrt{2}}$$
$$f(x, y, z) = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$$
$$\text{Case 1: Two variabes are 0}$$
$$z^4 = 1$$
$$z^2 = 1$$
$$f(x, y, z) = z^2 = 1$$

Min 1
Max $\sqrt{3}$

Question 8

Find the absolute minimum and maximum values of the function $f(x, y) = x^2 - (y-2)^2$ on the region

$$D = \{x^2 + y^2 \leqslant 9 \text{ and } y \geqslant 0\},$$

and the points at which those extrema occur.

Solution:

$$\nabla f(x, y) = (2x - 2y + 4)$$
$$\nabla g(x, y) = 2x, 2y$$
$$\nabla f(x, y) = \lambda \nabla g(x, y)$$
$$2x = \lambda 2x$$
$$-2y + 4 = \lambda 2y$$
$$x^2 + y^2 = 9$$
$$2x = 0 \quad -2y + 4 = 0 \Rightarrow y = 2$$
$$f(0, 2) = 0^2 - (2-2)^2 = 0$$
$$\text{Case: } x = 0$$
$$0^2 + y^2 = 9 \Rightarrow y = \pm \quad y \geqslant 0 \quad y = 3$$
$$f(0, 2) = 0^2 - (3-2)^2 = -1$$
$$\text{Case: } x \neq 0$$
$$1 = \lambda$$
$$-2y + 4 = 2y \Rightarrow 4 = 4y \Rightarrow y = 1$$
$$f(x, 1) = x^2 + 1^2 = 9 \Rightarrow x = \pm 2\sqrt{2}$$
$$f(0, 2) = 0 \quad f(0, 3) = -1 \quad f(\pm 2\sqrt{2}, 1) = 7$$

Max: 7 at (0,2)
Min: -1 at (0,3)