# Math 120

## PSet 9

Nov 5 2024

# Contents

# Chapter 1

## 1.1 PSet 9

**Question 1**

Let $\vec{F}(x,y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$, and let $C$ be the triangle from $(0,0)$ to $(2,6)$ to $(2,0)$ to $(0,0)$. Use Green's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$. (Check the orientation of the curve before applying the theorem.)

***Solution:***

$$\vec{F}(x,y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2y \sin x) = 2x + 2y \cos x$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(y^2 \cos x) = 2y \cos x$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = [2x + 2y \cos x] - [2y \cos x] = 2x$$

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_D 2x\, dx\, dy$$

$$\int_{y=0}^{y=3x} 2x\, dy = 2x(3x - 0) = 6x^2$$

$$-\int_{x=0}^{2} 6x^2\, dx = -6\int_0^2 x^2\, dx = -6\left[\frac{x^3}{3}\right]_0^2 = -6\left(\frac{8}{3}\right) = -16$$

$$\int_C \vec{F} \cdot d\vec{r} = -16$$

**Question 2**

Let $P(x, y) = x - x^2y^3$ and $Q(x, y) = xy^2$, and let $C$ be the circle $x^2 + y^2 = 4$, oriented counterclockwise.

(a) Compute $\int_C \vec{F} \cdot d\vec{r}$ directly, by parameterizing $C$ and finding the line integral.

(b) Compute $\int_C \vec{F} \cdot d\vec{r}$ using Green's Theorem.

***Solution:***

a)

$$x(t) = 2\cos t, \quad y(t) = 2\sin t, \quad t \in [0, 2\pi]$$

$$dx = -2\sin t\, dt, \quad dy = 2\cos t\, dt$$

$$\begin{aligned} P &= x - x^2y^3 = 2\cos t - (2\cos t)^2(2\sin t)^3 \\ &= 2\cos t - 4\cos^2 t \cdot 8\sin^3 t = 2\cos t - 32\cos^2 t\sin^3 t \end{aligned}$$

$$Q = xy^2 = (2\cos t)(2\sin t)^2 = 2\cos t \cdot 4\sin^2 t = 8\cos t\sin^2 t$$

$$\begin{aligned} P\,dx &= \left(2\cos t - 32\cos^2 t\sin^3 t\right)(-2\sin t\, dt) \\ &= -4\cos t\sin t\, dt + 64\cos^2 t\sin^4 t\, dt \end{aligned}$$

$$Q\,dy = \left(8\cos t\sin^2 t\right)(2\cos t\, dt) = 16\cos^2 t\sin^2 t\, dt$$

$$P\,dx + Q\,dy = \left[-4\cos t\sin t + 64\cos^2 t\sin^4 t + 16\cos^2 t\sin^2 t\right] dt$$

$$P\,dx + Q\,dy = \left[-4\cos t\sin t + 16\cos^2 t\sin^2 t(1 + 4\sin^2 t)\right] dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \left[-4\cos t\sin t + 16\cos^2 t\sin^2 t(1 + 4\sin^2 t)\right] dt$$

$$I = I_1 + I_2 + I_3$$

$$I_1 = \int_0^{2\pi} -4\cos t\sin t\, dt = -2\int_0^{2\pi} \sin 2t\, dt = 0$$

$$I_2 = 16\int_0^{2\pi} \cos^2 t\sin^2 t\, dt$$

$$I_3 = 64\int_0^{2\pi} \cos^2 t\sin^4 t\, dt$$

$$\cos^2 t\sin^2 t = \frac{1}{4}\sin^2 2t = \frac{1}{8}(1 - \cos 4t)$$

$$I_2 = 16 \cdot \frac{1}{8}\int_0^{2\pi} (1 - \cos 4t)\, dt = 2\left[t - \frac{\sin 4t}{4}\right]_0^{2\pi} = 4\pi$$

$$\sin^4 t = \left(\sin^2 t\right)^2 = \left(\frac{1 - \cos 2t}{2}\right)^2 = \frac{1 - 2\cos 2t + \cos^2 2t}{4}$$

$$\cos^2 t = \frac{1 + \cos 2t}{2}$$

$$\cos^2 t\sin^4 t = \frac{1 + \cos 2t}{2} \cdot \frac{1 - 2\cos 2t + \cos^2 2t}{4} = \frac{(1 + \cos 2t)(1 - 2\cos 2t + \cos^2 2t)}{8}$$

$$\begin{aligned} (1 + \cos 2t)(1 - 2\cos 2t + \cos^2 2t) &= (1)(1 - 2\cos 2t + \cos^2 2t) + \cos 2t(1 - 2\cos 2t + \cos^2 2t) \\ &= 1 - 2\cos 2t + \cos^2 2t + \cos 2t - 2\cos^2 2t + \cos^3 2t \\ &= 1 - \cos 2t - \cos^2 2t + \cos^3 2t \end{aligned}$$

$$\cos^2 t\sin^4 t = \frac{1 - \cos 2t - \cos^2 2t + \cos^3 2t}{8}$$

$$I_3 = 64\int_0^{2\pi} \cos^2 t \sin^4 t \, dt = 8\int_0^{2\pi} \left(1 - \cos 2t - \cos^2 2t + \cos^3 2t\right) dt$$

$$\int_0^{2\pi} 1 \, dt = 2\pi$$

$$\int_0^{2\pi} \cos 2t \, dt = 0$$

$$\int_0^{2\pi} \cos^2 2t \, dt = \int_0^{2\pi} \frac{1 + \cos 4t}{2} \, dt = \pi$$

$$\cos^3 2t = \frac{3\cos 2t + \cos 6t}{4} \implies \int_0^{2\pi} \cos^3 2t \, dt = 0$$

$$I_3 = 8\,(2\pi - 0 - \pi + 0) = 8 \cdot \pi = 8\pi$$

$$\int_C \vec{F} \cdot d\vec{r} = I_1 + I_2 + I_3 = 0 + 4\pi + 8\pi = \boxed{12\pi}$$

b)

$$\int_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy$$

$$\frac{\partial Q}{\partial x} = y^2$$

$$\frac{\partial P}{\partial y} = -3x^2y^2$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + 3x^2y^2 = y^2(1 + 3x^2)$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (r\sin\theta)^2\left(1 + 3r^2\cos^2\theta\right) = r^2\sin^2\theta\left(1 + 3r^2\cos^2\theta\right)$$

$$\int_0^{2\pi}\int_0^2 r^2\sin^2\theta\left(1 + 3r^2\cos^2\theta\right) r\, dr\, d\theta = \int_0^{2\pi} \sin^2\theta\left(\int_0^2 r^3(1 + 3r^2\cos^2\theta)\, dr\right) d\theta$$

$$I = \int_0^{2\pi} \sin^2\theta\left(\int_0^2 r^3\, dr + 3\cos^2\theta\int_0^2 r^5\, dr\right) d\theta$$

$$\int_0^2 r^3\, dr = \left[\frac{r^4}{4}\right]_0^2 = \frac{16}{4} = 4$$

$$\int_0^2 r^5\, dr = \left[\frac{r^6}{6}\right]_0^2 = \frac{64}{6} = \frac{32}{3}$$

$$I = \int_0^{2\pi} \sin^2\theta\left(4 + 3\cos^2\theta \cdot \frac{32}{3}\right) d\theta = \int_0^{2\pi} \sin^2\theta\left(4 + 32\cos^2\theta\right) d\theta$$

$$I = \int_0^{2\pi} \left(4\sin^2\theta + 32\sin^2\theta\cos^2\theta\right) d\theta$$

$$I_1 = 4\int_0^{2\pi} \sin^2\theta\, d\theta = 4 \cdot \pi$$

$$I_2 = 32\int_0^{2\pi} \sin^2\theta\cos^2\theta\, d\theta = 32 \cdot \frac{1}{8}\int_0^{2\pi} (1 - \cos 4\theta)\, d\theta = 4\,(2\pi - 0) = 8\pi$$

$$I = I_1 + I_2 = 4\pi + 8\pi = \boxed{12\pi}$$

**Question 3**

Use Green's Theorem to find the area enclosed by the parametric curve $\vec{r}(t) = \langle \sin t, \sin 2t \rangle$, $0 \leqslant t \leqslant \pi$.

***Solution:***

$$A = \frac{1}{2}_C x\, dy - y\, dx \Rightarrow \frac{1}{2}\int_C \left( x\frac{dy}{dy} - y\frac{dx}{dt} \right) dt$$

$$x = \sin t \quad y = \sin 2t \quad \frac{dx}{dt} = \cos t \quad \frac{dy}{dt} = -2\cos t$$

$$\sin 2t = 2\sin t\cos t \quad \cos 2t = \cos^2 t - \sin^2 t$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = 2\sin t\left(\cos^2 t - \sin^2 t\right) - 2\sin t\cos t(\cos t)$$

$$= 2\sin t\left(\cos^2 t - \sin^2 t - \cos^2 t\right) = -2\sin^3 t$$

$$A = \frac{1}{2} - 2\sin^3 t\, dt = -\int \sin^3 t\, dt$$

$$\sin^3 t = \sin t\left(1 - \cos^2 t\right) = \sin t - \sin t\cos^2 t$$

$$\int_0^\pi \sin t\, dt - \int_0^\pi \sin t\cos^2 t$$

$$\int_0^\pi \sin t\, dt \Rightarrow -\cos t|_0^\pi = 2$$

$$\int_0^\pi \sin t\cos^2 t$$

$$u = \cos t \quad du = -\sin t\, dt$$

$$-\int_0^\pi u^2 \Rightarrow \left.\frac{u^3}{3}\right|_0^\pi$$

$$-\left.\frac{\cos^3 t}{3}\right|_0^\pi = \frac{2}{3}$$

$$A = 2 - \frac{2}{3} = \frac{4}{3}$$

**Question 4**

Consider the vector field $\vec{F} = -\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$.

(a) Show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ at every point in the domain of $\vec{F}$.

(b) Let $C$ be the short arc of the circle $x^2 + y^2 = 2$ from $(1, 1)$ to $(-1, 1)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ directly, by parameterizing the curve and computing $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$.

(c) Integrate $P(x, y) = -\frac{y}{x^2+y^2}$ with respect to $x$, and check that the partial derivative of the result with respect to $y$ is $Q(x, y) = \frac{x}{x^2+y^2}$. You have now found a function $f$ such that $\nabla f = \vec{F}$.

What is the domain of this function $f$? Is it the same as the domain of $\vec{F}$?

(d) Use your answer to part (c) and the Fundamental Theorem of Line Integrals to check your answer to part (b).

(e) Now let $C$ be the circle of radius $R$ centered at the origin, oriented counterclockwise. Compute $\oint_C \vec{F} \cdot d\vec{r}$. Explain why your answer doesn't contradict the statement that the integral of a conservative vector field around any closed curve must be zero. Hint: Look carefully at the domain of the potential function $f$ you found in part (b).

***Solution:*** a)

$$P(x,y) = -\frac{y}{x^2+y^2}, \quad Q(x,y) = \frac{x}{x^2+y^2}$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{-\left[(x^2+y^2) - 2y^2\right]}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

b)

$$x = \sqrt{2}\cos\theta, \quad y = \sqrt{2}\sin\theta \quad \theta \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

$$dx = -\sqrt{2}\sin\theta\, d\theta, \quad dy = \sqrt{2}\cos\theta\, d\theta$$

$$\vec{F} = -\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}.$$

$$\vec{F} = -\frac{\sqrt{2}\sin\theta}{2}\hat{\imath} + \frac{\sqrt{2}\cos\theta}{2}\hat{\jmath} = \left\langle -\frac{\sin\theta}{\sqrt{2}}, \frac{\cos\theta}{\sqrt{2}}\right\rangle.$$

$$\vec{F}\cdot d\vec{r} = \left\langle -\frac{\sin\theta}{\sqrt{2}}, \frac{\cos\theta}{\sqrt{2}}\right\rangle \cdot \left\langle -\sqrt{2}\sin\theta, \sqrt{2}\cos\theta\right\rangle.$$

$$\vec{F}\cdot d\vec{r} = d\theta\left(-\frac{\sin\theta}{\sqrt{2}}\right)(-\sqrt{2}\sin\theta) + d\theta\left(\frac{\cos\theta}{\sqrt{2}}\right)(\sqrt{2}\cos\theta).$$

$$= \sin^2\theta\, d\theta + \cos^2\theta\, d\theta = 1 \Rightarrow d\theta.$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{\pi/4}^{3\pi/4} d\theta = \frac{\pi}{2}$$

c)

$$f(x,y) = \int P(x,y)\,dx = -\int \frac{y}{x^2+y^2}\,dx = -\arctan\left(\frac{x}{y}\right) + h(y)$$

$$\frac{\partial f}{\partial y} = -\frac{\partial}{\partial y}\left[\arctan\left(\frac{x}{y}\right)\right] + h'(y) = \frac{x}{x^2+y^2} + h'(y)$$

$$h'(y) = 0 \Rightarrow f(x,y) = -\arctan\left(\frac{x}{y}\right) + C$$

d)

$$\int_C \vec{F}\cdot d\vec{r} = f(-1,1) - f(1,1) = \left(-\arctan\left(\frac{-1}{1}\right)\right) - \left(-\arctan\left(\frac{1}{1}\right)\right) = \frac{\pi}{2}$$

e)

$$x = R\cos\theta, \quad y = R\sin\theta \quad 0 \leqslant \theta \leqslant 2\pi$$
$$\vec{F}\cdot d\vec{r} = \sin^2\theta\, d\theta + \cos^2\theta\, d\theta = d\theta$$
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} d\theta = 2\pi$$

**Question 5**

Again consider the vector field $\vec{F} = -\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$. Let $C_1$ be any closed curve going counterclockwise around the origin, such as the orange curve below. Let $C_2$ be a circle, centered around the origin, with radius less than the shortest distance between $C_1$ and the origin. (This condition guarantees that the two curves don't intersect.) Let $D$ be the region between the two curves.

(a) Explain why Green's Theorem applies on the region $D$.

(b) The boundary of $D$ is the union of the two curves $C_1$ and $-C_2$, where by $-C_2$ we mean the inside circle oriented clockwise. Since $\int_{-C_2}\vec{F}\cdot d\vec{r} = -\int_{C_2}\vec{F}\cdot d\vec{r}$, Green's Theorem implies that

$$\int_{C_1}\vec{F}\cdot d\vec{r} - \int_{C_2}\vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\, dA.$$

Use the results of Problem # 4 above to determine the value of $\int_{C_1}\vec{F}\cdot d\vec{r}$.

***Solution:***

a)

Green's Theorem connects a line integral around a closed curve $C$ to a double integral over the region $D$ enclosed by $C$, given a vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ with continuous partial derivatives in the area around $D$ and $C$. Here, the vector field $\vec{F} = \left(-\frac{y}{x^2+y^2}\right)\hat{\imath} + \left(\frac{x}{x^2+y^2}\right)\hat{\jmath}$ is not defined at the origin due to division by zero, creating a singularity. However, the region $D$ is bounded by two curves, $C_1$ and $C_2$, where $C_2$ is a smaller circle within $C_1$ that avoids the origin. This exclusion of the origin ensures $\vec{F}$ remains continuously differentiable in $D$, satisfying Green's Theorem's conditions. Therefore, Green's Theorem is applicable to $D$.

b)

$$\int_{C_1}\vec{F}\cdot d\vec{r} - \oint_{C_2}\vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 0\, dA = 0.$$
$$\int_{C_1}\vec{F}\cdot d\vec{r} - \int_{C_2}\vec{F}\cdot d\vec{r} = 0 \implies \int_{C_1}\vec{F}\cdot d\vec{r} = \int_{C_2}\vec{F}\cdot d\vec{r}.$$
$$\int_C \vec{F}\cdot d\vec{r} = 2\pi$$
$$\int_{C_2}\vec{F}\cdot d\vec{r} = 2\pi$$
$$\int_{C_1}\vec{F}\cdot d\vec{r} = 2\pi$$

## Question 6

Let $\vec{F} = \langle 2y - x^2, 4x + ye^{\cos y}\rangle$, and let $C$ be the curve $y = x^2 - 9$, $-3 \leqslant x \leqslant 3$, oriented from left to right.

(a) Parameterize the curve $C$, and write the vector line integral $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$. Do not try to compute this integral directly!

(b) Let $C^*$ be the line segment along the $x$-axis from $(3,0)$ to $(-3,0)$. Compute $\int_{C^*} \vec{F} \cdot d\vec{r}$.

(c) Let $D$ be the region bounded by the parabola $y = x^2 - 9$ and the $x$-axis. Compute $\iint_D (Q_x - P_y)\,dA$.

(d) Use your answers to (b) and (c) to compute $\int_C \vec{F} \cdot d\vec{r}$.

***Solution:***

a)

$$\vec{r}(t) = \langle t, t^2 - 9\rangle, \quad t \in [-3, 3].$$

$$\vec{r}'(t) = \langle 1, 2t\rangle.$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{-3}^{3} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt.$$

b)

$$\vec{r}(t) = \langle t, 0\rangle, \quad t \in [3, -3].$$

$$\vec{r}'(t) = \langle 1, 0\rangle.$$

$$\vec{F}(\vec{r}(t)) = \langle -t^2, 4t\rangle.$$

$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = -t^2.$$

$$\int_{C^*} \vec{F} \cdot d\vec{r} = -\int_{3}^{-3} t^2\,dt = \int_{-3}^{3} t^2\,dt = \left[\frac{t^3}{3}\right]_{-3}^{3} = 18.$$

$$\int_{C^*} \vec{F} \cdot d\vec{r} = 18.$$

c)

$$P = 2y - x^2 \implies P_y = 2, \quad Q = 4x + ye^{\cos y} \implies Q_x = 4.$$

$$Q_x - P_y = 4 - 2 = 2.$$

$$\text{Area} = \int_{-3}^{3} [0 - (x^2 - 9)]\,dx = \int_{-3}^{3} (9 - x^2)\,dx = 36.$$

$$\iint_D (Q_x - P_y)\,dA = 2 \times 36 = 72.$$

$$\iint_D (Q_x - P_y)\,dA = 72.$$

d)

$$\int_C \vec{F} \cdot d\vec{r} + \int_{C^*} \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA.$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA - \int_{C^*} \vec{F} \cdot d\vec{r} = 72 - 18 = 54.$$

$$\int_C \vec{F} \cdot d\vec{r} = 54.$$