

Math 120

PSet 10

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Question 1

The vector field \vec{F} is shown below in the xy -plane and looks the same in all other horizontal planes. (In other words, \vec{F} is independent of z and its z -component is 0.)

- (a) Is $\text{div } \vec{F}$ positive, negative, or zero? Explain.
- (b) Determine whether $\text{curl } \vec{F} = \vec{0}$. If not, in which direction does $\text{curl } \vec{F}$ point?

Solution:

- (a) The divergence of \vec{F} is negative because the vectors converge toward the origin, indicating a net inward flux. This suggests material is "flowing in" rather than spreading out.
- (b) The curl of \vec{F} is zero, as there is no rotational pattern; the vectors are purely radial and do not exhibit any circular motion.

Question 2

Find the curl and divergence of the given vector field.

1. $\vec{F}(x, y, z) = \langle x^2yz, xy^2z, xyz^2 \rangle$
2. $\vec{F}(x, y, z) = e^{xy} \sin z \hat{j} + y \arctan(x/z) \hat{k}$

Solution:

$$P = x^2yx \quad Q = xy^2z \quad R = xyz^2$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\text{div } \vec{F} = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\frac{\partial P}{\partial x} = 2xyz,$$

$$\frac{\partial Q}{\partial y} = 2xyz,$$

$$\frac{\partial R}{\partial z} = 2xyz.$$

$$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2xyz + 2xyz + 2xyz = 6xyz.$$

$$\left(\text{curl } \vec{F} \right)_x = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = xz^2 - xy^2.$$

$$\left(\operatorname{curl} \vec{F}\right)_y = \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = x^2 y - y z^2.$$

$$\left(\operatorname{curl} \vec{F}\right)_z = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 z - x^2 z.$$

$$\operatorname{curl} \vec{F} = (xz^2 - xy^2) \hat{i} + (x^2 y - yz^2) \hat{j} + (y^2 z - x^2 z) \hat{k}.$$

b)

$$\vec{F}(x, y, z) = e^{xy} \sin z \hat{j} + y \arctan\left(\frac{x}{z}\right) \hat{k}$$

$$P = 0,$$

$$Q = e^{xy} \sin z,$$

$$R = y \arctan\left(\frac{x}{z}\right).$$

$$\frac{\partial P}{\partial x} = 0,$$

$$\frac{\partial Q}{\partial y} = x e^{xy} \sin z,$$

$$\frac{\partial R}{\partial z} = -\frac{xy}{x^2 + z^2}.$$

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + x e^{xy} \sin z - \frac{xy}{x^2 + z^2}.$$

$$\begin{aligned} \left(\operatorname{curl} \vec{F}\right)_x &= \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ &= \arctan\left(\frac{x}{z}\right) - e^{xy} \cos z. \end{aligned}$$

$$\begin{aligned} \left(\operatorname{curl} \vec{F}\right)_y &= \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ &= 0 - \left(y \cdot \frac{z}{x^2 + z^2}\right) \\ &= -\frac{yz}{x^2 + z^2}. \end{aligned}$$

$$\begin{aligned} \left(\operatorname{curl} \vec{F}\right)_z &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\ &= y e^{xy} \sin z - 0 \\ &= y e^{xy} \sin z. \end{aligned}$$

$$\operatorname{curl} \vec{F} = \left(\arctan\left(\frac{x}{z}\right) - e^{xy} \cos z\right) \hat{i} - \left(\frac{yz}{x^2 + z^2}\right) \hat{j} + (y e^{xy} \sin z) \hat{k}.$$

Question 3

- (a) Consider the surface given by $\sin(x - y) - x - y + z = 0$. Find a parametrization of the part of the surface that has $|x| \leq 3$ and $|y| \leq 3$. Be sure to include the bounds for your parameters.
- (b) Look at the parametrization that you found in part (a). It should be of the form $\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$. Explain why it is not possible to find a parametrization of the surface $x^2 - y + z^2 = 1$ that is of the form $\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$.
- (c) Find a parametrization of $x^2 - y + z^2 = 1$ that is of the form $\vec{r}(u, v) = \langle u, f(u, v), v \rangle$.
- (d) Consider the surface $x^2 - y^2 + \frac{z^2}{4} = 1$. Explain why it is not possible to find a parametrization of this surface that is of the form $\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$, or of the form $\vec{r}(u, v) = \langle f(u, v), u, v \rangle$.
- (e) Find a parametrization of the surface $x^2 - y^2 + \frac{z^2}{4} = 1$ of the form

$$\vec{r}(u, v) = \langle f(v) \cos u, v, g(v) \sin u \rangle,$$

where $0 \leq u \leq 2\pi$ and $-\infty < v < \infty$.

Solution:

Part (a):

$$\sin(x - y) - x - y + z = 0$$

$$|x| \leq 3 \quad \text{and} \quad |y| \leq 3$$

$$z = x + y - \sin(x - y)$$

$$\vec{r}(u, v) = \langle u, v, u + v - \sin(u - v) \rangle$$

$$-3 \leq u \leq 3 \quad \text{and} \quad -3 \leq v \leq 3$$

Part (b):

$$\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$$

$$z^2 = 1 - x^2 + y$$

$$z = \pm \sqrt{1 - x^2 + y}$$

However, the expression under the square root, $1 - x^2 + y$, can be negative for certain values of x and y , which means z is not defined over the entire domain of x and y . Additionally, z has two possible values (positive and negative square roots) for each (x, y) , so z is not a single-valued function of x and y . Therefore, it's impossible to express z as a function $f(u, v)$ over the entire surface, making such a parametrization infeasible.

Part (c):

$$x^2 - y + z^2 = 1$$

$$y = x^2 + z^2 - 1$$

$$\vec{r}(u, v) = \langle u, u^2 + v^2 - 1, v \rangle$$

Part (d):

For the surface $x^2 - y^2 + \frac{z^2}{4} = 1$, attempting to parametrize it as $\vec{r}(u, v) = \langle u, v, f(u, v) \rangle$ or $\vec{r}(u, v) = \langle f(u, v), u, v \rangle$ is not feasible because:

- **First Form (z in terms of x and y):** Solving for z:

$$z^2 = 4(1 - x^2 + y^2)$$

The right side depends on both x and y in a way that z cannot be uniquely expressed as a function of x and y over the entire surface, especially when the expression under the square root is negative.

- **Second Form (x in terms of y and z):** Solving for x :

$$x^2 = y^2 - \frac{z^2}{4} + 1$$

Similar to the first form, x cannot be uniquely expressed as a function of y and z over the entire surface.

In both cases, the variables cannot be separated into a single-valued function necessary for the parametrization.

Part (e):

$$\vec{r}(u, v) = \langle f(v) \cos u, v, g(v) \sin u \rangle$$

$$[f(v) \cos u]^2 - v^2 + \frac{[g(v) \sin u]^2}{4} = 1$$

$$f(v)^2 \cos^2 u - v^2 + \frac{g(v)^2 \sin^2 u}{4} = 1$$

$$f(v)^2 = \frac{g(v)^2}{4}$$

$$f(v)^2 + \frac{g(v)^2}{4} = 1 + v^2$$

$$g(v)^2 = 4f(v)^2$$

$$f(v)^2 + f(v)^2 = 1 + v^2 \implies 2f(v)^2 = 1 + v^2$$

$$f(v) = \sqrt{\frac{1 + v^2}{2}}$$

$$g(v) = \sqrt{2(1 + v^2)}$$

$$\vec{r}(u, v) = \left\langle \sqrt{\frac{1 + v^2}{2}} \cos u, v, \sqrt{2(1 + v^2)} \sin u \right\rangle$$

$$0 \leq u \leq 2\pi, \quad -\infty < v < \infty$$

Question 4

Identify and sketch the surface with the given parameterization.

(a) $\vec{r}(u, v) = (2 \sin u) \hat{j} + (3 \cos u) \hat{i} + v \hat{k}, \quad 0 \leq u \leq \pi, \quad -2 \leq v \leq 2$

(b) $\vec{r}(u, v) = \langle u \sin v, u^2, u \cos v \rangle, \quad 0 \leq u \leq 3, \quad 0 \leq v \leq 2\pi$

(a) Given parameterization:

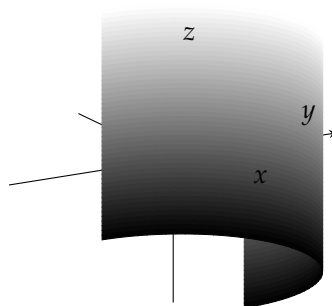
$$\vec{r}(u, v) = (2 \sin u) \hat{j} + (3 \cos u) \hat{i} + v \hat{k}, \quad 0 \leq u \leq \pi, \quad -2 \leq v \leq 2$$

$$x(u, v) = 3 \cos u, \quad y(u, v) = 2 \sin u, \quad z(u, v) = v$$

$$\cos u = \frac{x}{3}, \quad \sin u = \frac{y}{2}$$

$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$$

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$



(b) Given parameterization:

$$\vec{r}(u, v) = \langle u \sin v, u^2, u \cos v \rangle, \quad 0 \leq u \leq 3, \quad 0 \leq v \leq 2\pi$$

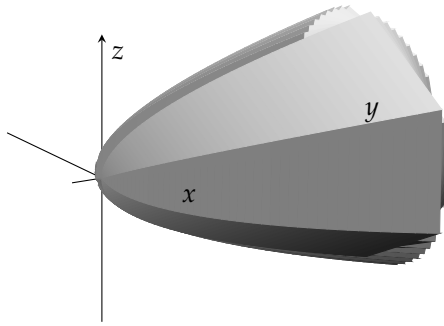
$$x(u, v) = u \sin v, \quad y(u, v) = u^2, \quad z(u, v) = u \cos v$$

$$\sin v = \frac{x}{u}, \quad \cos v = \frac{z}{u}$$

$$\left(\frac{x}{u}\right)^2 + \left(\frac{z}{u}\right)^2 = 1$$

$$x^2 + z^2 = u^2$$

$$x^2 + z^2 = y$$



Question 5

Consider the surface S described by $x - 4y^2 - z^2 + 3 = 0$.

- Find a parametrization of S of the form $\vec{r}_1(u, v) = \langle f(u, v), u, v \rangle$. Give the domain for your parametrization.
- Find a parametrization of S of the form $\vec{r}_2(u, v) = \langle v, f(v) \cos u, g(v) \sin u \rangle$. Give the domain for your parametrization.
- How must we restrict the parameters (u, v) in part (a) if we only want the part of S that lies in front of the yz -plane, i.e., where $x \geq 0$?
- How must we restrict the parameters (u, v) in part (b) if we only want the part of S that lies in front of the yz -plane?

(a)

$$\begin{aligned}
 x &= 4y^2 + z^2 - 3. \\
 \vec{r}_1(u, v) &= \langle 4u^2 + v^2 - 3, u, v \rangle. \\
 (u, v) &\in \mathbb{R}^2.
 \end{aligned}$$

(b)

$$\begin{aligned}
 \vec{r}_2(u, v) &= \langle v, f(v) \cos u, g(v) \sin u \rangle \\
 x &= v \quad y = f(v) \cos u \quad z = g(v) \sin u \\
 v - 4[f(v) \cos u]^2 - [g(v) \sin u]^2 + 3 &= 0. \\
 v - 4f(v)^2 \cos^2 u - g(v)^2 \sin^2 u + 3 &= 0. \\
 4f(v)^2 &= k \quad \text{and} \quad g(v)^2 = k \\
 k &= v + 3 \\
 f(v) &= \frac{1}{2}\sqrt{v+3}, \quad g(v) = \sqrt{v+3}. \\
 \vec{r}_2(u, v) &= \left\langle v, \frac{1}{2}\sqrt{v+3} \cos u, \sqrt{v+3} \sin u \right\rangle \\
 v &\geq -3, \quad u \in \mathbb{R}
 \end{aligned}$$

(c)

$$\begin{aligned}
 x &= 4u^2 + v^2 - 3 \geq 0 \\
 4u^2 + v^2 &\geq 3
 \end{aligned}$$

(d)

$$v \geq 0.$$

$$v \geq 0, \quad u \in \mathbb{R}$$

Question 6

Find parametric equations for each of the following surfaces.

- (a) The part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.
- (b) The surface obtained by rotating the curve $x = 4y^2 - y^4$, $-2 \leq y \leq 2$ about the y -axis.
- (c) The ellipsoid $\frac{x^2}{4} + 4y^2 + \frac{z^2}{9} = 1$.

Part (a)

$$x = r \cos \theta, \quad y = r \sin \theta,$$

$$r \in [0, 1] \quad \text{and} \quad \theta \in [0, 2\pi)$$

$$z = r \cos \theta + 3.$$

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \\ z = r \cos \theta + 3, \end{cases}$$

$$0 \leq r \leq 1 \quad \text{and} \quad 0 \leq \theta < 2\pi$$

Part (b)

$$\begin{cases} x = [4y^2 - y^4] \cos \theta, \\ z = [4y^2 - y^4] \sin \theta, \\ y = y, \end{cases}$$

$$-2 \leq y \leq 2 \quad \text{and} \quad 0 \leq \theta < 2\pi$$

$$\begin{cases} x = (4y^2 - y^4) \cos \theta, \\ y = y, \\ z = (4y^2 - y^4) \sin \theta, \end{cases}$$

$$-2 \leq y \leq 2 \quad \text{and} \quad 0 \leq \theta < 2\pi$$

Part (c)

Solution:

$$X = \frac{x}{2}, \quad Y = 2y, \quad Z = \frac{z}{3}$$

$$X^2 + Y^2 + Z^2 = 1$$

$$\begin{cases} X = \sin \phi \cos \theta, \\ Y = \sin \phi \sin \theta, \\ Z = \cos \phi, \end{cases}$$

$$\phi \in [0, \pi] \quad \text{and} \quad \theta \in [0, 2\pi)$$

$$\begin{cases} x = 2X = 2 \sin \phi \cos \theta, \\ y = \frac{Y}{2} = \frac{1}{2} \sin \phi \sin \theta, \\ z = 3Z = 3 \cos \phi. \end{cases}$$

$$\begin{cases} x = 2 \sin \phi \cos \theta, \\ y = \frac{1}{2} \sin \phi \sin \theta, \\ z = 3 \cos \phi, \end{cases}$$

$$0 \leq \phi \leq \pi \quad \text{and} \quad 0 \leq \theta < 2\pi$$

Question 7

Find the tangent plane to the parametric surface $\vec{r}(u, v) = \langle u \sin v, u^2, u \cos v \rangle$ at the point where $u = 1$ and $v = \frac{\pi}{3}$. Write the plane both in the vector form $\vec{r}(u, v) = \vec{r}_0 + u\vec{a} + v\vec{b}$ and in the form $ax + by + cz = d$.

Solution:

$$\vec{r}_0 = \vec{r}(1, \frac{\pi}{3}) = \left\langle 1 \cdot \sin \frac{\pi}{3}, 1^2, 1 \cdot \cos \frac{\pi}{3} \right\rangle = \left\langle \frac{\sqrt{3}}{2}, 1, \frac{1}{2} \right\rangle.$$

$$\vec{r}_u(u, v) = \langle \sin v, 2u, \cos v \rangle$$

$$\vec{r}_v(u, v) = \langle u \cos v, 0, -u \sin v \rangle$$

$$\vec{r}_u(1, \frac{\pi}{3}) = \left\langle \sin \frac{\pi}{3}, 2, \cos \frac{\pi}{3} \right\rangle = \left\langle \frac{\sqrt{3}}{2}, 2, \frac{1}{2} \right\rangle,$$

$$\vec{r}_v(1, \frac{\pi}{3}) = \left\langle 1 \cdot \cos \frac{\pi}{3}, 0, -1 \cdot \sin \frac{\pi}{3} \right\rangle = \left\langle \frac{1}{2}, 0, -\frac{\sqrt{3}}{2} \right\rangle.$$

$$\vec{r}(s, t) = \vec{r}_0 + s \vec{r}_u(1, \frac{\pi}{3}) + t \vec{r}_v(1, \frac{\pi}{3})$$

$$\vec{r}(s, t) = \left\langle \frac{\sqrt{3}}{2}, 1, \frac{1}{2} \right\rangle + s \left\langle \frac{\sqrt{3}}{2}, 2, \frac{1}{2} \right\rangle + t \left\langle \frac{1}{2}, 0, -\frac{\sqrt{3}}{2} \right\rangle.$$

$$\vec{n} = \vec{r}_u(1, \frac{\pi}{3}) \times \vec{r}_v(1, \frac{\pi}{3}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\sqrt{3}}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \end{vmatrix}.$$

$$\vec{n} = (-\sqrt{3}, 1, -1)$$

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0.$$

$$-\sqrt{3}(x - \frac{\sqrt{3}}{2}) + 1(y - 1) - 1(z - \frac{1}{2}) = 0$$

$$-\sqrt{3}x + y - z + (\frac{3}{2} - 1 + \frac{1}{2}) = 0,$$

$$-\sqrt{3}x + y - z + 1 = 0.$$

$$\sqrt{3}x - y + z = 1$$

$$\sqrt{3}x - y + z = 1$$

$$\vec{r}(s, t) = \left\langle \frac{\sqrt{3}}{2}, 1, \frac{1}{2} \right\rangle + s \left\langle \frac{\sqrt{3}}{2}, 2, \frac{1}{2} \right\rangle + t \left\langle \frac{1}{2}, 0, -\frac{\sqrt{3}}{2} \right\rangle.$$

$$\sqrt{3}x - y + z = 1$$