## Directional Derivatives and Gradient Vector

**Directional Derivative:**

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y) \cdot \mathbf{u}$$

where $\mathbf{u}$ is a unit vector.
**Gradient Vector:**

$$\nabla f(x,y) = \langle f_x, f_y \rangle$$

**Properties:** - $\nabla f$ points in the direction of max increase of $f$.
- $\nabla f$ is perpendicular to level curves of $f$.
**Max Rate of Change:**

$$\text{Max Rate} = |\nabla f(x,y)|$$

## Maximum and Minimum Values

**Second Derivative Test:**
Compute $D = f_{xx}f_{yy} - (f_{xy})^2$.
- If $D > 0$ and $f_{xx} > 0$, local min at $(a,b)$.
- If $D > 0$ and $f_{xx} < 0$, local max at $(a,b)$.
- If $D < 0$, saddle point at $(a,b)$.
- If $D = 0$, test inconclusive.

**Critical Points:** Solve $f_x = 0$, $f_y = 0$.

## Lagrange Multipliers

**Purpose:** Find extrema of $f(x,y)$ subject to $g(x,y) = 0$.
**Method:** 1. $\nabla f = \lambda \nabla g$
$f_x = \lambda g_x, \quad f_y = \lambda g_y$
2. Include $g(x,y) = 0$.
3. Solve for $x, y, \lambda$.
4. Evaluate $f$ at solutions.

## Double Integrals over Rectangles

**Definition:**

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

where $R = [a,b] \times [c,d]$.
**Fubini's Theorem:** If $f$ is continuous:

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

## Average Value of a Function

**Average Value over $R$:**

$$f_{\text{avg}} = \frac{1}{(b-a)(d-c)} \iint_R f(x,y)\,dA$$

## Double Integrals over General Regions

**Type I Region (Vertical):**

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D f\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f\,dy\,dx$$

**Type II Region (Horizontal):**

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$\iint_D f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f\,dx\,dy$$

## Double Integrals in Polar Coordinates

**When to Convert:** - Circular regions or integrands with $x^2 + y^2$.
- When $f(x,y)$ is easier to integrate in polar form.
**Transformation:**

$$x = r\cos\theta,\ y = r\sin\theta$$
$$dA = r\,dr\,d\theta$$

**Integral:**

$$\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

**Tips:** - Adjust limits of $r$ and $\theta$ to match $D$.
- Common for circles, sectors, annuli.

## Vector Fields

**Definition:** $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$
**Gradient Field:** $\mathbf{F} = \nabla f$
**Conservative Field:** $\mathbf{F} = \nabla f$.
**Curl in $\mathbb{R}^2$:**

$$\text{curl } \mathbf{F} = Q_x - P_y$$

## Line Integrals

**When to Use:** - To compute work done by a force field along a path.
- To integrate a scalar function over a curve (mass, length).
**Types of Line Integrals:** - **Scalar Line Integral** (with respect to arc length): $\int_C f\,ds$
- **Vector Line Integral** (work): $\int_C \mathbf{F} \cdot d\mathbf{r}$
**How to Compute:** 1. Parameterize $C$ by $\mathbf{r}(t)$, $t \in [a,b]$.
2. Compute $\mathbf{r}'(t)$ and $|\mathbf{r}'(t)|$ if necessary.
3. Substitute into the integral: - Scalar: $\int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt$
- Vector: $\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt$
**When to Convert to Polar Coordinates:** - When $C$ is a circle or curve naturally described in polar coordinates.
- When integrand involves $x^2 + y^2$ or trigonometric functions.
**Converting to Polar Coordinates:** - Use $x = r\cos\theta$, $y = r\sin\theta$.
- Express $\mathbf{F}$ and $d\mathbf{r}$ in terms of $r$ and $\theta$.
**Tips:** - Choose the simplest parameterization possible.
- For circles: $x = a\cos t,\ y = a\sin t,\ t \in [0, 2\pi]$.
- For straight lines, use linear parameterizations.
**Applications:** - Calculating work, circulation, or flux.
- Finding mass of a wire with variable density.

## Fundamental Theorem for Line Integrals

If $\mathbf{F} = \nabla f$, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

**Conservative Field Test:** - If $P_y = Q_x$, then $\mathbf{F}$ is conservative.

## Green's Theorem

**When to Use:** - To convert a difficult line integral into a double integral (or vice versa).
- When dealing with circulation or flux over a closed curve $C$ in the plane.
- $C$ must be a positively oriented (counter-clockwise) simple closed curve.
**Statement:**

$$\oint_C P\,dx + Q\,dy = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

**Applications:** - Calculating area: Area $= \frac{1}{2} \oint_C x\,dy - y\,dx$
- Computing work done by a force field around a closed path.
**How to Apply:** 1. Verify conditions (closed curve, positive orientation).
2. Identify $P(x,y)$ and $Q(x,y)$.
3. Compute $Q_x - P_y$.
4. Evaluate $\iint_D (Q_x - P_y)dA$.
**Tips:** - Simplify the integrand before integrating.
- Choose the order of integration based on $D$.
- For circular regions, consider polar coordinates.

## Example: Evaluating a Line Integral Using Green's Theorem

**Problem:** Let $\vec{F}(x,y) = \left(x^2 + y^2,\ \frac{1}{3}x^3 + 2xy + x\right)$. Compute $\int_C \vec{F} \cdot d\vec{r}$ along the semicircle $C$ defined by $x^2 + y^2 = 16$ for $y \ge 0$.
**Solution:** 1. **Close the Curve:** - Add the interval from $x = 4$ to $x = -4$ along $y = 0$ to form a closed curve $C'$. 2. **Apply Green's Theorem:**

$$\int_{C'} \vec{F} \cdot d\vec{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy$$

- Compute $\frac{\partial Q}{\partial x} = x^2 + 2y + 1$, $\frac{\partial P}{\partial y} = 2y$. - Integrand: $x^2 + 1$. 3. **Compute the Double Integral:**

$$\iint_D (x^2 + 1)\,dxdy = \int_0^\pi \int_0^4 (r^2\cos^2\theta + 1)\,r\,drd\theta$$

- Evaluate:

$$\int_0^\pi \left( \frac{64}{4}\cos^2\theta + 8 \right) d\theta = 40\pi$$

4. **Compute Integral over Straight Segment:** - Parametrize: $x = t$, $y = 0$, $t \in [4, -4]$. - $\vec{F} \cdot d\vec{r} = x^2 dx$. - Integral: $\int_4^{-4} x^2 dx = -\frac{128}{3}$. 5. **Find Integral along** $C$:

$$\int_C \vec{F} \cdot d\vec{r} = \int_{C'} \vec{F} \cdot d\vec{r} - \left( -\frac{128}{3} \right) = 40\pi + \frac{128}{3}$$

**Answer:** $\int_C \vec{F} \cdot d\vec{r} = 40\pi + \dfrac{128}{3}$

## Trigonometric Identities

**Pythagorean:**

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$

**Double Angle:**

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

**Sum and Difference:**

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

## Common Derivatives and Integrals

**Derivatives:**

$$\frac{d}{dx}e^{ax} = ae^{ax}$$
$$\frac{d}{dx}\ln x = \frac{1}{x}$$
$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$$
$$\frac{d}{dx}\sin ax = a\cos ax$$
$$\frac{d}{dx}\cos ax = -a\sin ax$$
$$\frac{d}{dx}\tan ax = a\sec^2 ax$$

**Integrals:**

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

**Techniques:** - **Substitution:** Let $u = g(x)$.
- **Integration by Parts:** $\int u\,dv = uv - \int v\,du$.

## Jacobian Determinant

Transformation from $(x, y)$ to $(u, v)$:

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u$$

**Use in Integration:**

$$\iint_D f(x,y)\,dA = \iint_{D'} f(x(u,v), y(u,v))|J|\,du\,dv$$

## Conservative Vector Fields

**Tests:** - If $P_y = Q_x$, $\mathbf{F}$ is conservative.
**Finding Potential $f$:** 1. Integrate $P$ w.r.t $x$ to get $f$.
2. Differentiate $f$ w.r.t $y$, compare with $Q$.
3. Adjust $f$ as needed.

## Coordinate Transformations

**Polar to Cartesian:**

$$x = r\cos\theta,\ y = r\sin\theta$$

**Cartesian to Polar:**

$$r = \sqrt{x^2 + y^2},\ \theta = \arctan\left(\frac{y}{x}\right)$$

**Cylindrical:**

$$x = r\cos\theta,\ y = r\sin\theta,\ z = z$$

**Spherical:**

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

## Derivative Rules

$$\frac{d}{dx}c = 0$$
$$\frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}[cf(x)] = cf'(x)$$
$$\frac{d}{dx}[f \pm g] = f' \pm g'$$
$$\frac{d}{dx}[fg] = f'g + fg'$$
$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$
$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

## Example: Computing Area Between Circles

**Problem:** Compute the area of the region $R$ with $y \geq 0$ outside $C_2$ and inside $C_1$, where:

$$C_1 : (x-1)^2 + y^2 = 1, \quad C_2 : x^2 + y^2 = 2$$

**Solution Steps:** 1. **Express the curves in polar coordinates:** - For $C_1$:

$$(x-1)^2 + y^2 = 1$$

Substitute $x = r\cos\theta$, $y = r\sin\theta$:

$$(r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1$$

Simplify:

$$r^2 - 2r\cos\theta = 0$$

So $r = 0$ or $r = 2\cos\theta$. Since $r = 0$ is trivial, $C_1$ corresponds to $r = 2\cos\theta$.
- For $C_2$:

$$x^2 + y^2 = 2$$

In polar coordinates:

$$r^2 = 2$$

So $r = \sqrt{2}$.
2. **Determine the limits of integration:** - Find the angle $\theta$ where the curves intersect:

$$r = \sqrt{2} = 2\cos\theta$$
$$\cos\theta = \frac{\sqrt{2}}{2}$$
$$\theta = \frac{\pi}{4}$$

- Therefore, $\theta$ ranges from 0 to $\frac{\pi}{4}$.
3. **Set up the double integral in polar coordinates:**

$$A = \int_{\theta=0}^{\frac{\pi}{4}} \int_{r=\sqrt{2}}^{r=2\cos\theta} r\,dr\,d\theta$$

4. **Compute the integral:** - Integrate with respect to $r$:

$$\int_{r=\sqrt{2}}^{r=2\cos\theta} r\,dr = \left[\frac{1}{2}r^2\right]_{r=\sqrt{2}}^{r=2\cos\theta} = \frac{1}{2}\left((2\cos\theta)^2 - (\sqrt{2})^2\right) = \frac{1}{2}\left(4\cos^2\theta - 2\right)$$

- Integrate with respect to $\theta$:

$$A = \int_0^{\frac{\pi}{4}} \left(2\cos^2\theta - 1\right) d\theta$$

5. **Simplify and evaluate the integral:** - Use the identity $\cos 2\theta = 2\cos^2\theta - 1$:

$$2\cos^2\theta - 1 = \cos 2\theta$$

- Therefore:

$$A = \int_0^{\frac{\pi}{4}} \cos 2\theta\,d\theta = \left[\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin 0\right) = \frac{1}{2}(1 - 0) = \frac{1}{2}$$

6. **Final Answer:** - The area $A = \frac{1}{2}$ square units.

## Example: Evaluating a Line Integral

**Problem:** Evaluate the line integral $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle 4xy^2 + 9x^2, 3e^y + 4x^2y\rangle$ and $C$ is the part of the parabola $4y = x^2$ from $(2, 1)$ to $(-2, 1)$.
**Solution:** 1. **Verify if the Vector Field is Conservative:** - Let $P = 4xy^2 + 9x^2$ and $Q = 3e^y + 4x^2y$. - Compute $\frac{\partial P}{\partial y} = 8xy$ and $\frac{\partial Q}{\partial x} = 8xy$.
- Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, $\vec{F}$ is conservative.
2. **Find the Potential Function** $f(x, y)$**:** - $f_x = 4xy^2 + 9x^2 \implies f(x, y) = \int(4xy^2 + 9x^2)\,dx = 2x^2y^2 + 3x^3 + g(y)$. - Differentiate with respect to $y$: $f_y = 4x^2y + g'(y)$. - Set equal to $Q$: $4x^2y + g'(y) = 3e^y + 4x^2y \implies g'(y) = 3e^y$. - Integrate $g'(y)$: $g(y) = 3e^y$. - Potential function: $f(x, y) = 2x^2y^2 + 3x^3 + 3e^y$.
3. **Apply the Fundamental Theorem for Line Integrals:**

$$\int_C \vec{F} \cdot d\vec{r} = f(-2, 1) - f(2, 1).$$

- Compute $f(2, 1) = 2(2)^2(1)^2 + 3(2)^3 + 3e^1 = 8 + 24 + 3e$. - Compute $f(-2, 1) = 2(-2)^2(1)^2 + 3(-2)^3 + 3e^1 = 8 - 24 + 3e$. - Result:

$$\int_C \vec{F} \cdot d\vec{r} = (8 - 24 + 3e) - (8 + 24 + 3e) = -48.$$

4. **Path Independence Verification:** - Choose the line segment $C'$ from $(2, 1)$ to $(-2, 1)$ and parametrize by $\vec{r}(t) = \langle -t, 1\rangle$ with $-2 \leq t \leq 2$. - $d\vec{r} = \langle -1, 0\rangle dt$ and $\vec{F}(\vec{r}(t)) = \langle 4t + 9t^2, 3e + 4t^2\rangle$. - $\vec{F} \cdot d\vec{r} = -4t - 9t^2$.
5. **Evaluate the Integral Directly:**

$$\int_{-2}^{2}(-4t - 9t^2)\,dt = \left[-2t^2 - 3t^3\right]_{-2}^{2} = -48.$$

**Answer:** $\int_C \vec{F} \cdot d\vec{r} = -48$.