

Math 120

PSet 11

Nov 21 2024

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Question 1

Find the surface area of the part of the paraboloid $x = 3y^2 + 3z^2$ satisfying $x \leq 3$.

Solution:

$$y = r \cos \theta, \quad z = r \sin \theta, \quad x = 3r^2$$

$$A = \iint_{\text{Domain}} \left\| \frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} \right\| dr d\theta$$

$$\mathbf{S}(r, \theta) = (3r^2, r \cos \theta, r \sin \theta)$$

$$\frac{\partial \mathbf{S}}{\partial r} = (6r, \cos \theta, \sin \theta), \quad \frac{\partial \mathbf{S}}{\partial \theta} = (0, -r \sin \theta, r \cos \theta)$$

$$\frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} = (r, -6r^2 \cos \theta, -6r^2 \sin \theta)$$

$$\left\| \frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} \right\| = r \sqrt{1 + 36r^2}$$

$$A = \int_0^{2\pi} \int_0^1 r \sqrt{1 + 36r^2} dr d\theta$$

$$u = 1 + 36r^2, \quad du = 72r dr, \quad r dr = \frac{du}{72}$$

$$\int_{r=0}^{r=1} r \sqrt{1 + 36r^2} dr = \frac{1}{72} \int_{u=1}^{u=37} \sqrt{u} du = \frac{1}{72} \left[\frac{2}{3} u^{3/2} \right]_{u=1}^{u=37}$$

$$\frac{1}{72} \left[\frac{2}{3} u^{3/2} \right]_{u=1}^{u=37} = \frac{1}{108} (37^{3/2} - 1)$$

$$A = \frac{\pi}{54} (37^{3/2} - 1)$$

$$37^{3/2} = \sqrt{37^3} = 37\sqrt{37}$$

$$A = \frac{\pi}{54} (37\sqrt{37} - 1)$$

Question 2

In this problem, we'll find the surface area of the part of the cylinder $x^2 + y^2 = 9$ that lies between the planes $x + y + z = -6$ and $x + y + z = 5$.

- Find a parametrization $\vec{r}_1(u)$ of the curve C_1 of intersection of the plane $x + y + z = 5$ with the cylinder $x^2 + y^2 = 9$. Also find a parametrization $\vec{r}_2(u)$ of the curve C_2 of intersection of the plane $x + y + z = -6$ with the cylinder.
- Write down a parametrization $\vec{s}(u, v)$ of the part of the cylinder that lies between the two planes. The curves C_1 and C_2 should be two grid curves of the parametrization, and the bounds on the parameters should be of the form $a \leq v \leq b$ and $c \leq u \leq d$ for constants a , b , c , and d .
- Use the parametrization you found to calculate the surface area of the part of the cylinder that lies between the two planes.
- Does your answer from (c) make sense? Give an intuitive geometric reason that the surface area you found is the same as the surface area of a cylinder of radius 3 and height 11.

Solution:

(a)

$$\begin{aligned} x &= 3 \cos u, & y &= 3 \sin u \\ z &= 5 - x - y = 5 - (3 \cos u + 3 \sin u) \\ \vec{r}_1(u) &= (3 \cos u, 3 \sin u, 5 - 3 \cos u - 3 \sin u) \\ z &= -6 - x - y = -6 - (3 \cos u + 3 \sin u) \\ \vec{r}_2(u) &= (3 \cos u, 3 \sin u, -6 - 3 \cos u - 3 \sin u) \end{aligned}$$

(b)

$$\begin{aligned} \vec{s}(u, v) &= (3 \cos u, 3 \sin u, v - 3 \cos u - 3 \sin u) \\ 0 &\leq u \leq 2\pi, & -6 &\leq v \leq 5 \end{aligned}$$

(c)

$$\begin{aligned} \frac{\partial \vec{s}}{\partial u} &= (-3 \sin u, 3 \cos u, 3 \sin u - 3 \cos u) \\ \frac{\partial \vec{s}}{\partial v} &= (0, 0, 1) \\ \frac{\partial \vec{s}}{\partial u} \times \frac{\partial \vec{s}}{\partial v} &= (3 \cos u, 3 \sin u, 0) \\ \left\| \frac{\partial \vec{s}}{\partial u} \times \frac{\partial \vec{s}}{\partial v} \right\| &= \sqrt{(3 \cos u)^2 + (3 \sin u)^2} = 3 \\ A &= \int_{v=-6}^5 \int_{u=0}^{2\pi} 3 \, du \, dv = 3 \cdot \left(\int_{u=0}^{2\pi} du \right) \cdot \left(\int_{v=-6}^5 dv \right) = 3 \cdot 2\pi \cdot 11 = 66\pi \end{aligned}$$

(d)

$$A = 2\pi r h = 2\pi \times 3 \times 11 = 66\pi$$

Question 3

Evaluate the surface integral

$$\iint_S (x^2 + y^2 + z^2) dS$$

where S is the surface of the solid cylinder defined by the inequalities $x^2 + z^2 \leq 1$ and $0 \leq y \leq 5$. Note that S consists of a hollow cylinder and two disks.

Solution:

$$I = \iint_S (x^2 + y^2 + z^2) dS$$

$$x^2 + z^2 \leq 1, \quad 0 \leq y \leq 5$$

$$S = S_{\text{cyl}} \cup S_{\text{top}} \cup S_{\text{bot}}$$

$$\text{For } S_{\text{cyl}}, \quad x = \cos \theta, \quad z = \sin \theta, \quad y = y, \quad \theta \in [0, 2\pi), \quad y \in [0, 5]$$

$$\vec{r}(\theta, y) = (\cos \theta, y, \sin \theta)$$

$$\vec{r}_\theta = (-\sin \theta, 0, \cos \theta), \quad \vec{r}_y = (0, 1, 0)$$

$$\vec{r}_\theta \times \vec{r}_y = (-\cos \theta, 0, -\sin \theta), \quad |\vec{r}_\theta \times \vec{r}_y| = 1$$

$$dS_{\text{cyl}} = d\theta dy$$

$$x^2 + y^2 + z^2 = \cos^2 \theta + y^2 + \sin^2 \theta = 1 + y^2$$

$$I_{\text{cyl}} = \int_0^{2\pi} \int_0^5 (1 + y^2) dy d\theta$$

$$\int_0^5 (1 + y^2) dy = \int_0^5 1 dy + \int_0^5 y^2 dy = [y]_0^5 + \left[\frac{y^3}{3} \right]_0^5 = 5 + \frac{125}{3} = \frac{140}{3}$$

$$I_{\text{cyl}} = \int_0^{2\pi} \frac{140}{3} d\theta = \frac{140}{3} \cdot 2\pi = \frac{280\pi}{3}$$

$$\text{For } S_{\text{top}}, \quad x = r \cos \theta, \quad z = r \sin \theta, \quad y = 5, \quad r \in [0, 1], \quad \theta \in [0, 2\pi)$$

$$dS_{\text{top}} = r dr d\theta$$

$$x^2 + y^2 + z^2 = r^2 + 25$$

$$I_{\text{top}} = \int_0^{2\pi} \int_0^1 (r^2 + 25)r dr d\theta$$

$$\int_0^1 (r^2 + 25)r dr = \int_0^1 (r^3 + 25r) dr = \left[\frac{r^4}{4} + \frac{25r^2}{2} \right]_0^1 = \frac{1}{4} + \frac{25}{2} = \frac{51}{4}$$

$$I_{\text{top}} = \int_0^{2\pi} \frac{51}{4} d\theta = \frac{51}{4} \cdot 2\pi = \frac{51\pi}{2}$$

For S_{bot} , $x = r \cos \theta$, $z = r \sin \theta$, $y = 0$

$$x^2 + y^2 + z^2 = r^2$$

$$dS_{\text{bot}} = r \, dr \, d\theta$$

$$I_{\text{bot}} = \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta$$

$$\int_0^1 r^3 \, dr = \left[\frac{r^4}{4} \right]_0^1 = \frac{1}{4}$$

$$I_{\text{bot}} = \int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$

$$I = I_{\text{cyl}} + I_{\text{top}} + I_{\text{bot}} = \frac{280\pi}{3} + \frac{51\pi}{2} + \frac{\pi}{2}$$

$$I = \frac{560\pi}{6} + \frac{153\pi}{6} + \frac{3\pi}{6} = \frac{716\pi}{6} = \frac{358\pi}{3}$$

$$\boxed{\frac{358\pi}{3}}$$

Question 4

Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$, if its density function is $\rho(x, y, z) = z + 2$.

Solution:

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 4$$

$$\rho(x, y, z) = z + 2$$

$$\begin{cases} x = z \cos \theta \\ y = z \sin \theta \\ z = z \end{cases}$$

$$\mathbf{r}(\theta, z) = (z \cos \theta, z \sin \theta, z)$$

$$\mathbf{r}_\theta = (-z \sin \theta, z \cos \theta, 0)$$

$$\mathbf{r}_z = (\cos \theta, \sin \theta, 1)$$

$$\mathbf{N} = \mathbf{r}_\theta \times \mathbf{r}_z = (z \cos \theta, z \sin \theta, -z)$$

$$|\mathbf{N}| = \sqrt{(z \cos \theta)^2 + (z \sin \theta)^2 + (-z)^2} = z\sqrt{2}$$

$$dS = z\sqrt{2} \, d\theta \, dz$$

$$\begin{aligned}
 M &= \int_{\theta=0}^{2\pi} \int_{z=1}^4 \rho(z) dS \\
 M &= \int_{z=1}^4 \int_{\theta=0}^{2\pi} (z+2)z\sqrt{2} d\theta dz \\
 &\quad \int_{\theta=0}^{2\pi} d\theta = 2\pi \\
 M &= 2\pi\sqrt{2} \int_{z=1}^4 (z+2)z dz \\
 M &= 2\pi\sqrt{2} \int_{z=1}^4 (z^2 + 2z) dz \\
 &\quad \int (z^2 + 2z) dz = \frac{1}{3}z^3 + z^2 \\
 &\quad \left[\frac{1}{3}(4)^3 + (4)^2 \right] - \left[\frac{1}{3}(1)^3 + (1)^2 \right] \\
 &\quad \left(\frac{64}{3} + 16 \right) - \left(\frac{1}{3} + 1 \right) \\
 &\quad 36
 \end{aligned}$$

$$M = 2\pi\sqrt{2} \times 36$$

$$M = 72\pi\sqrt{2}$$

Question 5

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}, \quad \text{where } \vec{F} = \langle x, y, 2z \rangle$$

and S is the part of the paraboloid $z = 4 - x^2 - y^2$, oriented downwards, that lies above the unit square $[0, 1] \times [0, 1]$.

Solution:

$$\begin{aligned}
 &\iint_S \vec{F} \cdot d\vec{S}, \quad \vec{F} = \langle x, y, 2z \rangle \\
 &\quad z = 4 - x^2 - y^2 \\
 &\quad \vec{n} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right) \\
 &\quad \frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = -2y \\
 &\quad \vec{n} = (-2x, -2y, -1) \\
 &\quad d\vec{S} = \vec{n} dx dy = (-2x, -2y, -1) dx dy \\
 &\quad \vec{F} = (x, y, 2z) = (x, y, 2(4 - x^2 - y^2))
 \end{aligned}$$

$$\vec{F} \cdot d\vec{S} = (x, y, 2z) \cdot (-2x, -2y, -1) dx dy$$

$$\vec{F} \cdot d\vec{S} = [-2x^2 - 2y^2 - 2(4 - x^2 - y^2)] dx dy$$

$$\vec{F} \cdot d\vec{S} = [-2x^2 - 2y^2 - 8 + 2x^2 + 2y^2] dx dy = -8 dx dy$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-8) dx dy = -8 \iint_D dx dy$$

$$\iint_S \vec{F} \cdot d\vec{S} = -8 \times 1 = -8$$

$$\iint_S \vec{F} \cdot d\vec{S} = -8$$

Question 6

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}, \quad \text{where } \vec{F} = \langle -z, x, y \rangle$$

and S is the part of the unit sphere $x^2 + y^2 + z^2 = 1$ in the first octant, oriented upwards.

Solution:

$$\vec{F} = \langle -z, x, y \rangle$$

$$x = \sin \phi \cos \theta, \quad y = \sin \phi \sin \theta, \quad z = \cos \phi$$

$$0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\vec{r}_\phi = \frac{\partial \vec{r}}{\partial \phi} = \begin{pmatrix} \cos \phi \cos \theta \\ \cos \phi \sin \theta \\ -\sin \phi \end{pmatrix}, \quad \vec{r}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \begin{pmatrix} -\sin \phi \sin \theta \\ \sin \phi \cos \theta \\ 0 \end{pmatrix}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{vmatrix}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{pmatrix} \sin^2 \phi \cos \theta \\ \sin^2 \phi \sin \theta \\ \cos \phi \sin \phi \end{pmatrix}$$

$$d\vec{S} = \begin{pmatrix} \sin^2 \phi \cos \theta \\ \sin^2 \phi \sin \theta \\ \cos \phi \sin \phi \end{pmatrix} d\phi d\theta$$

$$\vec{F} = \langle -z, x, y \rangle = \langle -\cos \phi, \sin \phi \cos \theta, \sin \phi \sin \theta \rangle$$

$$\vec{F} \cdot d\vec{S} = (-\cos \phi) (\sin^2 \phi \cos \theta) + (\sin \phi \cos \theta) (\sin^2 \phi \sin \theta) + (\sin \phi \sin \theta) (\cos \phi \sin \phi)$$

$$\vec{F} \cdot d\vec{S} = \cos \phi \sin^2 \phi (-\cos \theta + \sin \theta) + \sin^3 \phi \cos \theta \sin \theta$$

$$\int_0^{\frac{\pi}{2}} (-\cos \theta + \sin \theta) d\theta = 0, \quad \int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^3 \phi \, d\phi = \frac{4}{3}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$$

$$\boxed{\frac{2}{3}}$$

Question 7

Find the flux of $\vec{F}(x, y, z) = z\hat{i} + y\hat{j} + x\hat{k}$ across the helicoid

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi,$$

oriented upward.

Solution:

$$\begin{aligned} \vec{r}(u, v) &= \langle u \cos v, u \sin v, v \rangle \\ \vec{r}_u &= \langle \cos v, \sin v, 0 \rangle, \quad \vec{r}_v = \langle -u \sin v, u \cos v, 1 \rangle \\ \vec{n} = \vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u \rangle \\ \vec{F}(x, y, z) &= z\hat{i} + y\hat{j} + x\hat{k}, \quad \vec{F}(\vec{r}(u, v)) = \langle v, u \sin v, u \cos v \rangle \\ \vec{F} \cdot \vec{n} &= v \sin v - u \sin v \cos v + u^2 \cos v \\ \Phi &= \iint_D (\vec{F} \cdot \vec{n}) \, du \, dv = \int_0^{2\pi} \int_0^1 (v \sin v - u \sin v \cos v + u^2 \cos v) \, du \, dv \\ \int_0^1 v \sin v \, du &= v \sin v, \quad \int_0^1 u \sin v \cos v \, du = \frac{1}{2} \sin v \cos v, \quad \int_0^1 u^2 \cos v \, du = \frac{1}{3} \cos v \\ \Phi &= \int_0^{2\pi} \left(v \sin v - \frac{1}{2} \sin v \cos v + \frac{1}{3} \cos v \right) dv \\ \sin v \cos v &= \frac{1}{2} \sin 2v \\ \Phi &= \int_0^{2\pi} \left(v \sin v - \frac{1}{4} \sin 2v + \frac{1}{3} \cos v \right) dv \\ \int_0^{2\pi} v \sin v \, dv &= -2\pi, \quad \int_0^{2\pi} \sin 2v \, dv = 0, \quad \int_0^{2\pi} \cos v \, dv = 0 \\ \Phi &= -2\pi - \frac{1}{4}(0) + \frac{1}{3}(0) \\ \Phi &= -2\pi \end{aligned}$$

Question 8

Let S be the part of the elliptical cylinder $y^2 + 4z^2 = 4$ that lies above the xy -plane and between the planes $x = -2$ and $x = 2$. Let S have the upward orientation; that is, let S be oriented so that the normal vectors have positive z -component.

- Find a parameterization of S .
- Does your parameterization match the given orientation of S ? Explain.

(c) Let \vec{F} be the vector field

$$\vec{F}(x, y, z) = e^{x^2 y^2 z^2} \hat{i} + x^2 y \hat{j} + z^2 e^{x/5} \hat{k}.$$

Find the flux of \vec{F} across the oriented surface S .

Solution:

$$(a) \quad \vec{r}(x, \theta) = \langle x, 2 \sin \theta, \cos \theta \rangle, \quad -2 \leq x \leq 2, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$(b) \quad \vec{r}_x = \langle 1, 0, 0 \rangle, \quad \vec{r}_\theta = \langle 0, 2 \cos \theta, -\sin \theta \rangle, \quad \vec{N} = \vec{r}_x \times \vec{r}_\theta = \langle 0, \sin \theta, 2 \cos \theta \rangle$$

$$(c) \quad F_x = e^{4x^2 \sin^2 \theta \cos^2 \theta}, \quad F_y = 2x^2 \sin \theta, \quad F_z = \cos^2 \theta e^{x/5}$$

$$\vec{F} \cdot \vec{N} = 2x^2 \sin^2 \theta + 2 \cos^3 \theta e^{x/5}$$

$$\Phi = \int_{x=-2}^2 \int_{\theta=0}^{\frac{\pi}{2}} (2x^2 \sin^2 \theta + 2 \cos^3 \theta e^{x/5}) d\theta dx$$

$$\int_{\theta=0}^{\frac{\pi}{2}} \sin^2 \theta d\theta = \frac{\pi}{4}, \quad \int_{\theta=0}^{\frac{\pi}{2}} \cos^3 \theta d\theta = \frac{2}{3}$$

$$\Phi = \int_{x=-2}^2 \left(2x^2 \cdot \frac{\pi}{4} + 2 \cdot \frac{2}{3} e^{x/5} \right) dx$$

$$\int_{x=-2}^2 x^2 dx = \frac{16}{3}, \quad \int_{x=-2}^2 e^{x/5} dx = 5 \left(e^{2/5} - e^{-2/5} \right)$$

$$\Phi = \frac{\pi}{2} \cdot \frac{16}{3} + \frac{4}{3} \cdot 5 \left(e^{2/5} - e^{-2/5} \right)$$

$$\Phi = \frac{8\pi}{3} + \frac{20}{3} \left(e^{2/5} - e^{-2/5} \right)$$

Question 9

For each of the following parameterizations $\vec{r}(u, v)$, and vector fields $\vec{F}(x, y, z)$:

- Describe the surface S that is parameterized by $\vec{r}(u, v)$.
- Describe in words the positive orientation of S given by the family of unit normal vectors $\vec{n} = (\vec{r}_u \times \vec{r}_v) / |\vec{r}_u \times \vec{r}_v|$. That is, give a description of which way the normal vectors point for this orientation.
- The flux of \vec{F} through S is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) du dv.$$

Explain without any calculation whether the flux of \vec{F} through S is positive, negative, or zero; or explain why you don't have enough information to do so.

- $\vec{r}(u, v) = \langle u, v, \sqrt{1 - u^2 - v^2} \rangle$ where $u^2 + v^2 \leq 1$. The vector field is $\vec{F}(x, y, z) = \langle -x, -y, -z \rangle$.
- $\vec{r}(u, v) = \langle u \cos v, u \sin v, -u^2 \rangle$ where $0 \leq v \leq 2\pi$ and $0 \leq u$. The vector field is $\vec{F}(x, y, z) = \langle 0, 0, 1 \rangle$.
- $\vec{r}(u, v) = \langle \cos u \sin v, \sin u \sin v, \cos v \rangle$ where $0 \leq v \leq \pi$ and $0 \leq u \leq \pi$. The vector field is $\vec{F}(x, y, z) = \langle 0, 1, z \rangle$.

- (d) $\vec{r}(u, v) = \langle \cos u, v, \sin u \rangle$ where $0 \leq u \leq 2\pi$ and $-\infty < v < \infty$. The vector field is $\vec{F}(x, y, z) = \langle 0, e^x, 0 \rangle$.

Solution:

(a)

Parameterization: $\vec{r}(u, v) = \langle u, v, \sqrt{1 - u^2 - v^2} \rangle$, where $u^2 + v^2 \leq 1$.

Vector Field: $\vec{F}(x, y, z) = \langle -x, -y, -z \rangle$.

- i. **Description of Surface S:** The parameterization maps the unit disk $u^2 + v^2 \leq 1$ in the uv -plane to the upper hemisphere of the unit sphere centered at the origin, as:

$$x = u, \quad y = v, \quad z = \sqrt{1 - u^2 - v^2}.$$

This satisfies $x^2 + y^2 + z^2 = 1$ with $z \geq 0$.

- ii. **Positive Orientation:** The normal vectors \vec{n} point outward from the sphere. The cross product $\vec{r}_u \times \vec{r}_v$ aligns with the position vector, confirming the outward orientation.
- iii. **Flux through S:** The vector field $\vec{F}(x, y, z) = \langle -x, -y, -z \rangle$ points inward toward the origin. Since \vec{F} is opposite to the outward normals, the dot product $\vec{F} \cdot \vec{n} < 0$, so the flux is **negative**.

—

(b)

Parameterization: $\vec{r}(u, v) = \langle u \cos v, u \sin v, -u^2 \rangle$, where $0 \leq v \leq 2\pi$ and $0 \leq u$.

Vector Field: $\vec{F}(x, y, z) = \langle 0, 0, 1 \rangle$.

- i. **Description of Surface S:** The parameterization describes the paraboloid:

$$z = -(x^2 + y^2),$$

which opens downward for $u \geq 0$.

- ii. **Positive Orientation:** The normal vectors point upward in the positive z -direction. The computation of $\vec{r}_u \times \vec{r}_v$ confirms that the z -component is positive.
- iii. **Flux through S:** The vector field $\vec{F} = \langle 0, 0, 1 \rangle$ points upward, aligning with the normals. Therefore, $\vec{F} \cdot \vec{n} > 0$, and the flux is **positive**.

—

(c)

Parameterization: $\vec{r}(u, v) = \langle \cos u \sin v, \sin u \sin v, \cos v \rangle$, where $0 \leq v \leq \pi$ and $0 \leq u \leq \pi$.

Vector Field: $\vec{F}(x, y, z) = \langle 0, 1, z \rangle$.

- i. **Description of Surface S:** The parameterization covers the entire unit sphere centered at the origin, as:

$$x^2 + y^2 + z^2 = 1.$$

- ii. **Positive Orientation:** The normal vectors \vec{n} point inward toward the center of the sphere, as shown by the cross product $\vec{r}_u \times \vec{r}_v \propto -\vec{r}(u, v)$.
- iii. **Flux through S:** The vector field \vec{F} has components that depend on z . Without explicit calculation, it is unclear whether the contributions from the positive and negative flux cancel, so the sign of the flux cannot be determined.

—

(d)

Parameterization: $\vec{r}(u, v) = \langle \cos u, v, \sin u \rangle$, where $0 \leq u \leq 2\pi$ and $-\infty < v < \infty$.

Vector Field: $\vec{F}(x, y, z) = \langle 0, e^x, 0 \rangle$.

i. **Description of Surface S:** The parameterization describes an infinite circular cylinder of radius 1 centered along the y -axis.

ii. **Positive Orientation:** The normal vectors \vec{n} point radially inward toward the y -axis:

$$\vec{n} \propto \langle -\cos u, 0, -\sin u \rangle.$$

iii. **Flux through S:** The vector field $\vec{F} = \langle 0, e^x, 0 \rangle$ has only a y -component, while the normals have zero y -component. Therefore:

$$\vec{F} \cdot \vec{n} = 0,$$

and the flux is **zero**.

Fourier's Law of Heat Conduction states that if the temperature at a point is given by $u(x, y, z)$, then the heat flow is given by the vector field $\vec{F} = -K\nabla u$, where K is the thermal conductivity of the material through which the heat is flowing. The rate of heat flow across a surface S is then given by the vector surface integral

$$\iint_S \vec{F} \cdot d\vec{S} = -K \iint_S \nabla u \cdot d\vec{S}.$$

Question 10

Suppose the temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Show that the rate of heat flow across a sphere S of radius a with center at the center of the ball and oriented outwards is independent of the radius a .

Solution:

$$u(x, y, z) = \frac{c}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}.$$

$$\frac{\partial u}{\partial x} = -\frac{cx}{r^3}, \quad \frac{\partial u}{\partial y} = -\frac{cy}{r^3}, \quad \frac{\partial u}{\partial z} = -\frac{cz}{r^3}.$$

$$\nabla u = -\frac{c}{r^3} \langle x, y, z \rangle.$$

$$\vec{F} = -K\nabla u = \frac{Kc}{r^3} \langle x, y, z \rangle.$$

$$\vec{n} = \frac{\langle x, y, z \rangle}{a}, \quad d\vec{S} = \vec{n} dS = \frac{\langle x, y, z \rangle}{a} dS.$$

$$\vec{F} \cdot d\vec{S} = \left(\frac{Kc}{a^3} \langle x, y, z \rangle \right) \cdot \left(\frac{\langle x, y, z \rangle}{a} \right) dS = \frac{Kc}{a^4} (x^2 + y^2 + z^2) dS.$$

$$x^2 + y^2 + z^2 = a^2 \implies \vec{F} \cdot d\vec{S} = \frac{Kc}{a^4} a^2 dS = \frac{Kc}{a^2} dS.$$

$$\iint_S \vec{F} \cdot d\vec{S} = \frac{Kc}{a^2} \iint_S dS = \frac{Kc}{a^2} (4\pi a^2) = 4\pi Kc.$$

Question 11

Use Stokes's Theorem to evaluate $\iint_S \text{curl } \vec{F} \cdot d\vec{S}$.

- (a) $\vec{F}(x, y, z) = x^2 y^3 z \hat{i} + \sin(xyz) \hat{j} + xyz \hat{k}$,
 S is the part of the cone $y^2 = x^2 + z^2$ that lies between the planes $y = 0$ and $y = 3$, oriented in the direction of the positive y -axis.
- (b) $\vec{F}(x, y, z) = e^x \cos z \hat{i} + x^2 z \hat{j} + xy \hat{k}$,
 S is the hemisphere $x = \sqrt{1 - y^2 - z^2}$, oriented in the direction of the positive x -axis.

Solution:

a)

$$\vec{F}(x, y, z) = x^2 y^3 z \hat{i} + \sin(xyz) \hat{j} + xyz \hat{k}$$

$$C : \begin{cases} x = 3 \cos \theta \\ y = 3 \\ z = 3 \sin \theta \\ \theta \in [0, 2\pi] \end{cases}$$

$$dx = -3 \sin \theta d\theta, \quad dy = 0, \quad dz = 3 \cos \theta d\theta$$

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

$$F_x = x^2 y^3 z = (3 \cos \theta)^2 \cdot 3^3 \cdot (3 \sin \theta) = 243 \cos^2 \theta \sin \theta$$

$$F_y = \sin(xyz) = \sin(9 \cos \theta \sin \theta)$$

$$F_z = xyz = (3 \cos \theta)(3)(3 \sin \theta) = 9 \cos \theta \sin \theta$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [F_x dx + F_z dz]$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [-729 \cos^2 \theta \sin^2 \theta + 27 \cos^2 \theta \sin \theta] d\theta$$

$$I = \int_0^{2\pi} [-729 \cos^2 \theta \sin^2 \theta + 27 \cos^2 \theta \sin \theta] d\theta$$

$$= -729 \int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta + 27 \int_0^{2\pi} \cos^2 \theta \sin \theta d\theta$$

$$= -729 \left(\frac{\pi}{4} \right) + 0$$

$$= -\frac{729\pi}{4}$$

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = -\frac{729\pi}{4}$$

b)

$$\vec{F}(x, y, z) = e^x \cos z \hat{i} + x^2 z \hat{j} + xy \hat{k}$$

$$C : \begin{cases} x = 0 \\ y = \cos \theta \\ z = \sin \theta \\ \theta \in [0, 2\pi] \end{cases}$$

$$\vec{F} = \cos z \, \hat{i}$$

$$d\vec{r} = dy \, \hat{j} + dz \, \hat{k} = (-\sin \theta \, \hat{j} + \cos \theta \, \hat{k}) \, d\theta$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \cos(\sin \theta) \cdot 0 \, d\theta = 0$$

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = 0$$