

## 3D Coordinate Systems

**Distance:**  $|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$   
**Sphere:**  $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$

## Vectors

**Properties:**  $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$   
 $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

## Dot Product

**Formula:**  $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

**Angle:**  $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|}$

**Orthogonality:**  $\mathbf{a} \cdot \mathbf{b} = 0$

**Projections:**  $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

$\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$

## Cross Product

**Formula:**  $\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$

**Magnitude:**  $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \sin \theta$

**Triple Product:**  $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$

## Lines and Planes

**Line:**  $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$

**Plane:**  $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$

$ax + by + cz + d = 0$

**Distance:**

$$D = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

**Curve Length:**  $s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$

## Vector Value Functions

**Form:**  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

**Limit:**  $\lim_{t \rightarrow a} \mathbf{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle$

**Derivative:**  $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$

**Unit Tangent Vector**  $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

**Orthogonality:** If  $|\mathbf{r}(t)| = c$ , then  $\mathbf{r}'(t)$  is orthogonal to  $\mathbf{r}(t)$ .

**Definite Integral:**  $\int_a^b \mathbf{r}(t) dt = \mathbf{R}(t) \Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$

**Length:**  $L = \int_a^b |\mathbf{r}'(t)| dt$

## Curvature

**Form:**  $s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du$

## Motion in space

**Form:**  $\mathbf{v}(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$

## Partial Derivatives

**Form:**  $f_x(a, b) = g'(a)$  where  $g(x) = f(x, b)$

**Definition:**

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h}$$

**Notation:**

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

**Rules**

To find  $f_x$ , regard  $y$  as a constant and differentiate  $f(x, y)$  with respect to  $x$ .

To find  $f_y$ , regard  $x$  as a constant and differentiate  $f(x, y)$  with respect to  $y$ .

**Higher Derivatives:**

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

**Clairaut's Theorem**  $f_{xy}(a, b) = f_{yx}(a, b)$

## Chain Rule

**Case 1:**

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

**Case 2:**

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

**General Form:**

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}$$

## Implicit Differentiation

**Form:**

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

**Three variables:**

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Directional Derivative and Gradient Vector

Directional Derivative:

$$D_{\mathbf{u}}f(x,y)=f_x(x,y)a+f_y(x,y)b$$

Gradient For 2 variables

$$\nabla f(x,y)=\langle f_x(x,y),f_y(x,y)\rangle=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}$$

Gradient for 3 Variables

$$\nabla f=\langle f_x,f_y,f_z\rangle=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$$

Properties of Gradient Vecotr

$\nabla f(\mathbf{x})$  is perpendicular to the level curve or level surface of  $f$  through  $\mathbf{x}$ .

Tange Planes and Surfaces

Form:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$