

Math 120

PSet 11

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# Contents

Chapter 1

Page 2

1.1 PSet 10

2

# Chapter 1

## 1.1 PSet 10

### Question 1

Find the surface area of the part of the paraboloid  $x = 3y^2 + 3z^2$  satisfying  $x \leq 3$ .

*Solution:*

$$y = r \cos \theta, \quad z = r \sin \theta, \quad x = 3r^2$$

$$A = \iint_{\text{Domain}} \left\| \frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} \right\| dr d\theta$$

$$\mathbf{S}(r, \theta) = (3r^2, r \cos \theta, r \sin \theta)$$

$$\frac{\partial \mathbf{S}}{\partial r} = (6r, \cos \theta, \sin \theta), \quad \frac{\partial \mathbf{S}}{\partial \theta} = (0, -r \sin \theta, r \cos \theta)$$

$$\frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} = (r, -6r^2 \cos \theta, -6r^2 \sin \theta)$$

$$\left\| \frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} \right\| = r \sqrt{1 + 36r^2}$$

$$A = \int_0^{2\pi} \int_0^1 r \sqrt{1 + 36r^2} dr d\theta$$

$$u = 1 + 36r^2, \quad du = 72r dr, \quad r dr = \frac{du}{72}$$

$$\int_{r=0}^{r=1} r \sqrt{1 + 36r^2} dr = \frac{1}{72} \int_{u=1}^{u=37} \sqrt{u} du = \frac{1}{72} \left[ \frac{2}{3} u^{3/2} \right]_{u=1}^{u=37}$$

$$\frac{1}{72} \left[ \frac{2}{3} u^{3/2} \right]_{u=1}^{u=37} = \frac{1}{108} (37^{3/2} - 1)$$

$$A = \frac{\pi}{54} (37^{3/2} - 1)$$

$$37^{3/2} = \sqrt{37^3} = 37\sqrt{37}$$

$$A = \frac{\pi}{54} (37\sqrt{37} - 1)$$

## Question 2

In this problem, we'll find the surface area of the part of the cylinder  $x^2 + y^2 = 9$  that lies between the planes  $x + y + z = -6$  and  $x + y + z = 5$ .

- Find a parametrization  $\vec{r}_1(u)$  of the curve  $C_1$  of intersection of the plane  $x + y + z = 5$  with the cylinder  $x^2 + y^2 = 9$ . Also find a parametrization  $\vec{r}_2(u)$  of the curve  $C_2$  of intersection of the plane  $x + y + z = -6$  with the cylinder.
- Write down a parametrization  $\vec{s}(u, v)$  of the part of the cylinder that lies between the two planes. The curves  $C_1$  and  $C_2$  should be two grid curves of the parametrization, and the bounds on the parameters should be of the form  $a \leq v \leq b$  and  $c \leq u \leq d$  for constants  $a$ ,  $b$ ,  $c$ , and  $d$ .
- Use the parametrization you found to calculate the surface area of the part of the cylinder that lies between the two planes.
- Does your answer from (c) make sense? Give an intuitive geometric reason that the surface area you found is the same as the surface area of a cylinder of radius 3 and height 11.

**Solution:**

(a)

$$\begin{aligned} x &= 3 \cos u, & y &= 3 \sin u \\ z &= 5 - x - y = 5 - (3 \cos u + 3 \sin u) \\ \vec{r}_1(u) &= (3 \cos u, 3 \sin u, 5 - 3 \cos u - 3 \sin u) \\ z &= -6 - x - y = -6 - (3 \cos u + 3 \sin u) \\ \vec{r}_2(u) &= (3 \cos u, 3 \sin u, -6 - 3 \cos u - 3 \sin u) \end{aligned}$$

(b)

$$\begin{aligned} \vec{s}(u, v) &= (3 \cos u, 3 \sin u, v - 3 \cos u - 3 \sin u) \\ 0 &\leq u \leq 2\pi, & -6 &\leq v \leq 5 \end{aligned}$$

(c)

$$\begin{aligned} \frac{\partial \vec{s}}{\partial u} &= (-3 \sin u, 3 \cos u, 3 \sin u - 3 \cos u) \\ \frac{\partial \vec{s}}{\partial v} &= (0, 0, 1) \\ \frac{\partial \vec{s}}{\partial u} \times \frac{\partial \vec{s}}{\partial v} &= (3 \cos u, 3 \sin u, 0) \\ \left\| \frac{\partial \vec{s}}{\partial u} \times \frac{\partial \vec{s}}{\partial v} \right\| &= \sqrt{(3 \cos u)^2 + (3 \sin u)^2} = 3 \\ A &= \int_{v=-6}^5 \int_{u=0}^{2\pi} 3 \, du \, dv = 3 \cdot \left( \int_{u=0}^{2\pi} du \right) \cdot \left( \int_{v=-6}^5 dv \right) = 3 \cdot 2\pi \cdot 11 = 66\pi \end{aligned}$$

(d)

$$A = 2\pi r h = 2\pi \times 3 \times 11 = 66\pi$$

### Question 3

Evaluate the surface integral

$$\iint_S (x^2 + y^2 + z^2) dS$$

where  $S$  is the surface of the solid cylinder defined by the inequalities  $x^2 + z^2 \leq 1$  and  $0 \leq y \leq 5$ . Note that  $S$  consists of a hollow cylinder and two disks.

**Solution:**

$$I = \iint_S (x^2 + y^2 + z^2) dS$$

$$x^2 + z^2 \leq 1, \quad 0 \leq y \leq 5$$

$$S = S_{\text{cyl}} \cup S_{\text{top}} \cup S_{\text{bot}}$$

$$\text{For } S_{\text{cyl}}, \quad x = \cos \theta, \quad z = \sin \theta, \quad y = y, \quad \theta \in [0, 2\pi), \quad y \in [0, 5]$$

$$\vec{r}(\theta, y) = (\cos \theta, y, \sin \theta)$$

$$\vec{r}_\theta = (-\sin \theta, 0, \cos \theta), \quad \vec{r}_y = (0, 1, 0)$$

$$\vec{r}_\theta \times \vec{r}_y = (-\cos \theta, 0, -\sin \theta), \quad |\vec{r}_\theta \times \vec{r}_y| = 1$$

$$dS_{\text{cyl}} = d\theta dy$$

$$x^2 + y^2 + z^2 = \cos^2 \theta + y^2 + \sin^2 \theta = 1 + y^2$$

$$I_{\text{cyl}} = \int_0^{2\pi} \int_0^5 (1 + y^2) dy d\theta$$

$$\int_0^5 (1 + y^2) dy = \int_0^5 1 dy + \int_0^5 y^2 dy = [y]_0^5 + \left[ \frac{y^3}{3} \right]_0^5 = 5 + \frac{125}{3} = \frac{140}{3}$$

$$I_{\text{cyl}} = \int_0^{2\pi} \frac{140}{3} d\theta = \frac{140}{3} \cdot 2\pi = \frac{280\pi}{3}$$

$$\text{For } S_{\text{top}}, \quad x = r \cos \theta, \quad z = r \sin \theta, \quad y = 5, \quad r \in [0, 1], \quad \theta \in [0, 2\pi)$$

$$dS_{\text{top}} = r dr d\theta$$

$$x^2 + y^2 + z^2 = r^2 + 25$$

$$I_{\text{top}} = \int_0^{2\pi} \int_0^1 (r^2 + 25)r dr d\theta$$

$$\int_0^1 (r^2 + 25)r dr = \int_0^1 (r^3 + 25r) dr = \left[ \frac{r^4}{4} + \frac{25r^2}{2} \right]_0^1 = \frac{1}{4} + \frac{25}{2} = \frac{51}{4}$$

$$I_{\text{top}} = \int_0^{2\pi} \frac{51}{4} d\theta = \frac{51}{4} \cdot 2\pi = \frac{51\pi}{2}$$

For  $S_{\text{bot}}$ ,  $x = r \cos \theta$ ,  $z = r \sin \theta$ ,  $y = 0$

$$x^2 + y^2 + z^2 = r^2$$

$$dS_{\text{bot}} = r \, dr \, d\theta$$

$$I_{\text{bot}} = \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta$$

$$\int_0^1 r^3 \, dr = \left[ \frac{r^4}{4} \right]_0^1 = \frac{1}{4}$$

$$I_{\text{bot}} = \int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$

$$I = I_{\text{cyl}} + I_{\text{top}} + I_{\text{bot}} = \frac{280\pi}{3} + \frac{51\pi}{2} + \frac{\pi}{2}$$

$$I = \frac{560\pi}{6} + \frac{153\pi}{6} + \frac{3\pi}{6} = \frac{716\pi}{6} = \frac{358\pi}{3}$$

$$\boxed{\frac{358\pi}{3}}$$

#### Question 4

Find the mass of a thin funnel in the shape of a cone  $z = \sqrt{x^2 + y^2}$ ,  $1 \leq z \leq 4$ , if its density function is  $\rho(x, y, z) = z + 2$ .

**Solution:**

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 4$$

$$\rho(x, y, z) = z + 2$$

$$\begin{cases} x = z \cos \theta \\ y = z \sin \theta \\ z = z \end{cases}$$

$$\mathbf{r}(\theta, z) = (z \cos \theta, z \sin \theta, z)$$

$$\mathbf{r}_\theta = (-z \sin \theta, z \cos \theta, 0)$$

$$\mathbf{r}_z = (\cos \theta, \sin \theta, 1)$$

$$\mathbf{N} = \mathbf{r}_\theta \times \mathbf{r}_z = (z \cos \theta, z \sin \theta, -z)$$

$$|\mathbf{N}| = \sqrt{(z \cos \theta)^2 + (z \sin \theta)^2 + (-z)^2} = z\sqrt{2}$$

$$dS = z\sqrt{2} \, d\theta \, dz$$

$$\begin{aligned}
M &= \int_{\theta=0}^{2\pi} \int_{z=1}^4 \rho(z) dS \\
M &= \int_{z=1}^4 \int_{\theta=0}^{2\pi} (z+2)z\sqrt{2} d\theta dz \\
&\int_{\theta=0}^{2\pi} d\theta = 2\pi \\
M &= 2\pi\sqrt{2} \int_{z=1}^4 (z+2)z dz \\
M &= 2\pi\sqrt{2} \int_{z=1}^4 (z^2 + 2z) dz \\
&\int (z^2 + 2z) dz = \frac{1}{3}z^3 + z^2 \\
&\left[ \frac{1}{3}(4)^3 + (4)^2 \right] - \left[ \frac{1}{3}(1)^3 + (1)^2 \right] \\
&\left( \frac{64}{3} + 16 \right) - \left( \frac{1}{3} + 1 \right) \\
&36
\end{aligned}$$

$$M = 2\pi\sqrt{2} \times 36$$

$$M = 72\pi\sqrt{2}$$

#### Question 5

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}, \quad \text{where } \vec{F} = \langle x, y, 2z \rangle$$

and  $S$  is the part of the paraboloid  $z = 4 - x^2 - y^2$ , oriented downwards, that lies above the unit square  $[0, 1] \times [0, 1]$ .

**Solution:**

$$\begin{aligned}
&\iint_S \vec{F} \cdot d\vec{S}, \quad \vec{F} = \langle x, y, 2z \rangle \\
&z = 4 - x^2 - y^2 \\
&\vec{n} = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right) \\
&\frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = -2y \\
&\vec{n} = (-2x, -2y, -1) \\
&d\vec{S} = \vec{n} dx dy = (-2x, -2y, -1) dx dy \\
&\vec{F} = (x, y, 2z) = (x, y, 2(4 - x^2 - y^2))
\end{aligned}$$

$$\vec{F} \cdot d\vec{S} = (x, y, 2z) \cdot (-2x, -2y, -1) dx dy$$

$$\vec{F} \cdot d\vec{S} = [-2x^2 - 2y^2 - 2(4 - x^2 - y^2)] dx dy$$

$$\vec{F} \cdot d\vec{S} = [-2x^2 - 2y^2 - 8 + 2x^2 + 2y^2] dx dy = -8 dx dy$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-8) dx dy = -8 \iint_D dx dy$$

$$\iint_S \vec{F} \cdot d\vec{S} = -8 \times 1 = -8$$

$$\iint_S \vec{F} \cdot d\vec{S} = -8$$

### Question 6

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}, \quad \text{where } \vec{F} = \langle -z, x, y \rangle$$

and  $S$  is the part of the unit sphere  $x^2 + y^2 + z^2 = 1$  in the first octant, oriented upwards.

**Solution:**

$$\text{Divergence theorem: } \iint_{S_{\text{closed}}} \vec{F} \cdot d\vec{S} = \iiint_V \nabla \cdot \vec{F} dV.$$

$$\nabla \cdot \vec{F} = \frac{\partial(-z)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial y}{\partial z} = 0.$$

$$\iint_{S_{\text{closed}}} \vec{F} \cdot d\vec{S} = 0.$$

$$\iint_S \vec{F} \cdot d\vec{S} = - \left( \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} + \iint_{S_3} \vec{F} \cdot d\vec{S} \right).$$

$$\iint_{S_1} \vec{F} \cdot d\vec{S}, \quad \vec{n}_1 = \langle -1, 0, 0 \rangle, \quad \vec{F} = \langle -z, 0, y \rangle.$$

$$\vec{F} \cdot \vec{n}_1 = (-z)(-1) = z.$$

$$\iint_{S_1} z dy dz, \quad y^2 + z^2 \leq 1, \quad y, z \geq 0.$$

Polar coordinates:  $y = r \cos \theta$ ,  $z = r \sin \theta$ ,  $r \in [0, 1]$ ,  $\theta \in [0, \frac{\pi}{2}]$ .

$$\iint_{S_1} z dy dz = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \sin \theta dr d\theta = \int_0^{\frac{\pi}{2}} \sin \theta d\theta \int_0^1 r^2 dr.$$

$$\int_0^{\frac{\pi}{2}} \sin \theta d\theta = 1, \quad \int_0^1 r^2 dr = \frac{1}{3}.$$

$$\iint_{S_1} z dy dz = \frac{1}{3}$$

$$\iint_{S_2} \vec{F} \cdot d\vec{S}, \quad \vec{n}_2 = \langle 0, -1, 0 \rangle, \quad \vec{F} = \langle -z, x, 0 \rangle.$$

$$\vec{F} \cdot \vec{n}_2 = x(-1) = -x.$$



$$\begin{aligned}
& - \iint_{S_2} x \, dx \, dz, \quad x^2 + z^2 \leq 1, \quad x, z \geq 0. \\
& \iint_{S_2} x \, dx \, dz = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cos \theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \cos \theta \, d\theta \int_0^1 r^2 \, dr. \\
& \int_0^{\frac{\pi}{2}} \cos \theta \, d\theta = 1, \quad \int_0^1 r^2 \, dr = \frac{1}{3}. \\
& \iint_{S_2} x \, dx \, dz = \frac{1}{3}, \quad - \iint_{S_2} x \, dx \, dz = -\frac{1}{3}. \\
& \iint_{S_3} \vec{F} \cdot d\vec{S}, \quad \vec{n}_3 = \langle 0, 0, -1 \rangle, \quad \vec{F} = \langle -z, x, y \rangle. \\
& \vec{F} \cdot \vec{n}_3 = y(-1) = -y. \\
& - \iint_{S_3} y \, dx \, dy, \quad x^2 + y^2 \leq 1, \quad x, y \geq 0. \\
& \iint_{S_3} y \, dx \, dy = \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \sin \theta \, dr \, d\theta = \int_0^{\frac{\pi}{2}} \sin \theta \, d\theta \int_0^1 r^2 \, dr. \\
& \int_0^{\frac{\pi}{2}} \sin \theta \, d\theta = 1, \quad \int_0^1 r^2 \, dr = \frac{1}{3}. \\
& \iint_{S_3} y \, dx \, dy = \frac{1}{3}, \quad - \iint_{S_3} y \, dx \, dy = -\frac{1}{3}. \\
& \iint_S \vec{F} \cdot d\vec{S} = -\left(\frac{1}{3} - \frac{1}{3} - \frac{1}{3}\right) = \frac{1}{3}.
\end{aligned}$$

### Question 7

Find the flux of  $\vec{F}(x, y, z) = z\hat{i} + y\hat{j} + x\hat{k}$  across the helicoid

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi,$$

oriented upward.

**Solution:**

$$\begin{aligned}
\vec{r}(u, v) &= \langle u \cos v, u \sin v, v \rangle \\
\vec{r}_u &= \langle \cos v, \sin v, 0 \rangle, \quad \vec{r}_v = \langle -u \sin v, u \cos v, 1 \rangle \\
\vec{n} = \vec{r}_u \times \vec{r}_v &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u \rangle \\
\vec{F}(x, y, z) &= z\hat{i} + y\hat{j} + x\hat{k}, \quad \vec{F}(\vec{r}(u, v)) = \langle v, u \sin v, u \cos v \rangle \\
\vec{F} \cdot \vec{n} &= v \sin v - u \sin v \cos v + u^2 \cos v \\
\Phi &= \iint_D (\vec{F} \cdot \vec{n}) \, du \, dv = \int_0^{2\pi} \int_0^1 (v \sin v - u \sin v \cos v + u^2 \cos v) \, du \, dv \\
\int_0^1 v \sin v \, du &= v \sin v, \quad \int_0^1 u \sin v \cos v \, du = \frac{1}{2} \sin v \cos v, \quad \int_0^1 u^2 \cos v \, du = \frac{1}{3} \cos v \\
\Phi &= \int_0^{2\pi} \left( v \sin v - \frac{1}{2} \sin v \cos v + \frac{1}{3} \cos v \right) \, dv \\
\sin v \cos v &= \frac{1}{2} \sin 2v
\end{aligned}$$

$$\begin{aligned}\Phi &= \int_0^{2\pi} \left( v \sin v - \frac{1}{4} \sin 2v + \frac{1}{3} \cos v \right) dv \\ \int_0^{2\pi} v \sin v \, dv &= -2\pi, \quad \int_0^{2\pi} \sin 2v \, dv = 0, \quad \int_0^{2\pi} \cos v \, dv = 0 \\ \Phi &= -2\pi - \frac{1}{4}(0) + \frac{1}{3}(0) \\ \Phi &= -2\pi\end{aligned}$$

### Question 8

Let  $S$  be the part of the elliptical cylinder  $y^2 + 4z^2 = 4$  that lies above the  $xy$ -plane and between the planes  $x = -2$  and  $x = 2$ . Let  $S$  have the upward orientation; that is, let  $S$  be oriented so that the normal vectors have positive  $z$ -component.

- (a) Find a parameterization of  $S$ .
- (b) Does your parameterization match the given orientation of  $S$ ? Explain.
- (c) Let  $\vec{F}$  be the vector field

$$\vec{F}(x, y, z) = e^{x^2 y^2 z^2} \hat{i} + x^2 y \hat{j} + z^2 e^{x/5} \hat{k}.$$

Find the flux of  $\vec{F}$  across the oriented surface  $S$ .

**Solution:**

(a)

$$\vec{r}(x, \theta) = \langle x, 2 \cos \theta, \sin \theta \rangle, \quad x \in [-2, 2], \quad \theta \in [0, \pi].$$

(b)

$$\begin{aligned}\vec{r}_x &= \frac{\partial \vec{r}}{\partial x} = \langle 1, 0, 0 \rangle, \quad \vec{r}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \langle 0, -2 \sin \theta, \cos \theta \rangle. \\ \vec{n} = \vec{r}_\theta \times \vec{r}_x &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 \sin \theta & \cos \theta \\ 1 & 0 & 0 \end{vmatrix} = \langle 0, \cos \theta, 2 \sin \theta \rangle. \\ n_z &= 2 \sin \theta \geq 0 \quad \text{for } \theta \in [0, \pi].\end{aligned}$$

(c)

$$\begin{aligned}\text{Flux} &= \iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x, \theta)) \cdot (\vec{r}_\theta \times \vec{r}_x) \, d\theta \, dx. \\ \vec{F}(x, y, z) &= \langle e^{x^2 y^2 z^2}, x^2 y, z^2 e^{x/5} \rangle. \\ \vec{F}(\vec{r}(x, \theta)) &= \langle e^{4x^2 \cos^2 \theta \sin^2 \theta}, 2x^2 \cos \theta, \sin^2 \theta e^{x/5} \rangle. \\ \vec{F} \cdot \vec{n} &= 2x^2 \cos^2 \theta + 2 \sin^3 \theta e^{x/5}. \\ \text{Flux} &= \int_{x=-2}^2 \int_{\theta=0}^{\pi} (2x^2 \cos^2 \theta + 2 \sin^3 \theta e^{x/5}) \, d\theta \, dx. \\ \int_0^{\pi} \cos^2 \theta \, d\theta &= \frac{\pi}{2}, \quad \int_0^{\pi} \sin^3 \theta \, d\theta = \frac{4}{3}. \\ \text{Flux} &= \int_{x=-2}^2 \left[ 2x^2 \cdot \frac{\pi}{2} + 2 \cdot \frac{4}{3} \cdot e^{x/5} \right] dx. \\ \int_{x=-2}^2 x^2 dx &= \left[ \frac{x^3}{3} \right]_{-2}^2 = \frac{8}{3} - \left( -\frac{8}{3} \right) = \frac{16}{3}. \\ \int_{x=-2}^2 e^{x/5} dx &= 5 \left[ e^{x/5} \right]_{-2}^2 = 5 \left( e^{2/5} - e^{-2/5} \right). \\ \text{Flux} &= \frac{16\pi}{3} + \frac{40}{3} \left( e^{2/5} - e^{-2/5} \right).\end{aligned}$$

### Question 9

For each of the following parameterizations  $\vec{r}(u, v)$ , and vector fields  $\vec{F}(x, y, z)$ :

- Describe the surface  $S$  that is parameterized by  $\vec{r}(u, v)$ .
- Describe in words the positive orientation of  $S$  given by the family of unit normal vectors  $\vec{n} = (\vec{r}_u \times \vec{r}_v) / |\vec{r}_u \times \vec{r}_v|$ . That is, give a description of which way the normal vectors point for this orientation.
- The flux of  $\vec{F}$  through  $S$  is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} \, dS = \iint_S \vec{F} \cdot (\vec{r}_u \times \vec{r}_v) \, du \, dv.$$

Explain without any calculation whether the flux of  $\vec{F}$  through  $S$  is positive, negative, or zero; or explain why you don't have enough information to do so.

- $\vec{r}(u, v) = \langle u, v, \sqrt{1 - u^2 - v^2} \rangle$  where  $u^2 + v^2 \leq 1$ . The vector field is  $\vec{F}(x, y, z) = \langle -x, -y, -z \rangle$ .
- $\vec{r}(u, v) = \langle u \cos v, u \sin v, -u^2 \rangle$  where  $0 \leq v \leq 2\pi$  and  $0 \leq u$ . The vector field is  $\vec{F}(x, y, z) = \langle 0, 0, 1 \rangle$ .
- $\vec{r}(u, v) = \langle \cos u \sin v, \sin u \sin v, \cos v \rangle$  where  $0 \leq v \leq \pi$  and  $0 \leq u \leq \pi$ . The vector field is  $\vec{F}(x, y, z) = \langle 0, 1, z \rangle$ .
- $\vec{r}(u, v) = \langle \cos u, v, \sin u \rangle$  where  $0 \leq u \leq 2\pi$  and  $-\infty < v < \infty$ . The vector field is  $\vec{F}(x, y, z) = \langle 0, e^x, 0 \rangle$ .

**Solution:**

(a)

**Parameterization:**  $\vec{r}(u, v) = \langle u, v, \sqrt{1 - u^2 - v^2} \rangle$ , where  $u^2 + v^2 \leq 1$ .

**Vector Field:**  $\vec{F}(x, y, z) = \langle -x, -y, -z \rangle$ .

- Description of Surface  $S$ :** The parameterization maps the unit disk  $u^2 + v^2 \leq 1$  in the  $uv$ -plane to the upper hemisphere of the unit sphere centered at the origin, as:

$$x = u, \quad y = v, \quad z = \sqrt{1 - u^2 - v^2}.$$

This satisfies  $x^2 + y^2 + z^2 = 1$  with  $z \geq 0$ .

- Positive Orientation:** The normal vectors  $\vec{n}$  point outward from the sphere. The cross product  $\vec{r}_u \times \vec{r}_v$  aligns with the position vector, confirming the outward orientation.
- Flux through  $S$ :** The vector field  $\vec{F}(x, y, z) = \langle -x, -y, -z \rangle$  points inward toward the origin. Since  $\vec{F}$  is opposite to the outward normals, the dot product  $\vec{F} \cdot \vec{n} < 0$ , so the flux is **negative**.

—

(b)

**Parameterization:**  $\vec{r}(u, v) = \langle u \cos v, u \sin v, -u^2 \rangle$ , where  $0 \leq v \leq 2\pi$  and  $0 \leq u$ .

**Vector Field:**  $\vec{F}(x, y, z) = \langle 0, 0, 1 \rangle$ .

- Description of Surface  $S$ :** The parameterization describes the paraboloid:

$$z = -(x^2 + y^2),$$

which opens downward for  $u \geq 0$ .

- ii. **Positive Orientation:** The normal vectors point upward in the positive  $z$ -direction. The computation of  $\vec{r}_u \times \vec{r}_v$  confirms that the  $z$ -component is positive.
- iii. **Flux through  $S$ :** The vector field  $\vec{F} = \langle 0, 0, 1 \rangle$  points upward, aligning with the normals. Therefore,  $\vec{F} \cdot \vec{n} > 0$ , and the flux is **positive**.

—

(c)

**Parameterization:**  $\vec{r}(u, v) = \langle \cos u \sin v, \sin u \sin v, \cos v \rangle$ , where  $0 \leq v \leq \pi$  and  $0 \leq u \leq \pi$ .

**Vector Field:**  $\vec{F}(x, y, z) = \langle 0, 1, z \rangle$ .

- i. **Description of Surface  $S$ :** The parameterization covers the entire unit sphere centered at the origin, as:

$$x^2 + y^2 + z^2 = 1.$$

- ii. **Positive Orientation:** The normal vectors  $\vec{n}$  point inward toward the center of the sphere, as shown by the cross product  $\vec{r}_u \times \vec{r}_v \propto -\vec{r}(u, v)$ .
- iii. **Flux through  $S$ :** The vector field  $\vec{F}$  has components that depend on  $z$ . Without explicit calculation, it is unclear whether the contributions from the positive and negative flux cancel, so the sign of the flux cannot be determined.

—

(d)

**Parameterization:**  $\vec{r}(u, v) = \langle \cos u, v, \sin u \rangle$ , where  $0 \leq u \leq 2\pi$  and  $-\infty < v < \infty$ .

**Vector Field:**  $\vec{F}(x, y, z) = \langle 0, e^x, 0 \rangle$ .

- i. **Description of Surface  $S$ :** The parameterization describes an infinite circular cylinder of radius 1 centered along the  $y$ -axis.
- ii. **Positive Orientation:** The normal vectors  $\vec{n}$  point radially inward toward the  $y$ -axis:

$$\vec{n} \propto \langle -\cos u, 0, -\sin u \rangle.$$

- iii. **Flux through  $S$ :** The vector field  $\vec{F} = \langle 0, e^x, 0 \rangle$  has only a  $y$ -component, while the normals have zero  $y$ -component. Therefore:

$$\vec{F} \cdot \vec{n} = 0,$$

and the flux is **zero**.

Fourier's Law of Heat Conduction states that if the temperature at a point is given by  $u(x, y, z)$ , then the heat flow is given by the vector field  $\vec{F} = -K\nabla u$ , where  $K$  is the thermal conductivity of the material through which the heat is flowing. The rate of heat flow across a surface  $S$  is then given by the vector surface integral

$$\iint_S \vec{F} \cdot d\vec{S} = -K \iint_S \nabla u \cdot d\vec{S}.$$

#### Question 10

Suppose the temperature at a point in a ball with conductivity  $K$  is inversely proportional to the distance from the center of the ball. Show that the rate of heat flow across a sphere  $S$  of radius  $a$  with center at the center of the ball and oriented outwards is independent of the radius  $a$ .

**Solution:**

$$u(x, y, z) = \frac{c}{r}, \quad r = \sqrt{x^2 + y^2 + z^2}.$$

$$\frac{\partial u}{\partial x} = -\frac{cx}{r^3}, \quad \frac{\partial u}{\partial y} = -\frac{cy}{r^3}, \quad \frac{\partial u}{\partial z} = -\frac{cz}{r^3}.$$

$$\nabla u = -\frac{c}{r^3} \langle x, y, z \rangle.$$

$$\vec{F} = -K\nabla u = \frac{Kc}{r^3} \langle x, y, z \rangle.$$

$$\vec{n} = \frac{\langle x, y, z \rangle}{a}, \quad d\vec{S} = \vec{n} dS = \frac{\langle x, y, z \rangle}{a} dS.$$

$$\vec{F} \cdot d\vec{S} = \left( \frac{Kc}{a^3} \langle x, y, z \rangle \right) \cdot \left( \frac{\langle x, y, z \rangle}{a} \right) dS = \frac{Kc}{a^4} (x^2 + y^2 + z^2) dS.$$

$$x^2 + y^2 + z^2 = a^2 \implies \vec{F} \cdot d\vec{S} = \frac{Kc}{a^4} a^2 dS = \frac{Kc}{a^2} dS.$$

$$\iint_S \vec{F} \cdot d\vec{S} = \frac{Kc}{a^2} \iint_S dS = \frac{Kc}{a^2} (4\pi a^2) = 4\pi Kc.$$

### Question 11

Use Stokes's Theorem to evaluate  $\iint_S \text{curl } \vec{F} \cdot d\vec{S}$ .

- (a)  $\vec{F}(x, y, z) = x^2 y^3 z \hat{i} + \sin(xyz) \hat{j} + xyz \hat{k}$ ,  
 $S$  is the part of the cone  $y^2 = x^2 + z^2$  that lies between the planes  $y = 0$  and  $y = 3$ , oriented in the direction of the positive  $y$ -axis.
- (b)  $\vec{F}(x, y, z) = e^x \cos z \hat{i} + x^2 z \hat{j} + xy \hat{k}$ ,  
 $S$  is the hemisphere  $x = \sqrt{1 - y^2 - z^2}$ , oriented in the direction of the positive  $x$ -axis.

**Solution:**

a)

$$\vec{F}(x, y, z) = x^2 y^3 z \hat{i} + \sin(xyz) \hat{j} + xyz \hat{k}$$

$$C : \begin{cases} x = 3 \cos \theta \\ y = 3 \\ z = 3 \sin \theta \\ \theta \in [0, 2\pi] \end{cases}$$

$$dx = -3 \sin \theta d\theta, \quad dy = 0, \quad dz = 3 \cos \theta d\theta$$

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

$$F_x = x^2 y^3 z = (3 \cos \theta)^2 \cdot 3^3 \cdot (3 \sin \theta) = 243 \cos^2 \theta \sin \theta$$

$$F_y = \sin(xyz) = \sin(9 \cos \theta \sin \theta)$$

$$F_z = xyz = (3 \cos \theta)(3)(3 \sin \theta) = 9 \cos \theta \sin \theta$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [F_x dx + F_z dz]$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [-729 \cos^2 \theta \sin^2 \theta + 27 \cos^2 \theta \sin \theta] d\theta$$

$$I = \int_0^{2\pi} [-729 \cos^2 \theta \sin^2 \theta + 27 \cos^2 \theta \sin \theta] d\theta$$

$$= -729 \int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta + 27 \int_0^{2\pi} \cos^2 \theta \sin \theta d\theta$$

$$= -729 \left( \frac{\pi}{4} \right) + 0$$

$$= -\frac{729\pi}{4}$$

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = -\frac{729\pi}{4}$$

b)

$$\vec{F}(x, y, z) = e^x \cos z \hat{i} + x^2 z \hat{j} + xy \hat{k}$$

$$C : \begin{cases} x = 0 \\ y = \cos \theta \\ z = \sin \theta \\ \theta \in [0, 2\pi] \end{cases}$$

$$\vec{F} = \cos z \, \hat{i}$$

$$d\vec{r} = dy \, \hat{j} + dz \, \hat{k} = (-\sin \theta \, \hat{j} + \cos \theta \, \hat{k}) \, d\theta$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \cos(\sin \theta) \cdot 0 \, d\theta = 0$$

$$\iint_S \text{curl } \vec{F} \cdot d\vec{S} = 0$$