Math 120

Final Review

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Parametric Surfaces

Definition 0.0.2: Parametric Surface

A *parametric surface* is a surface in three-dimensional space $\mathbb{R}^3$ defined by a vector-valued function $\mathbf{r}(u,v)$, which depends on two parameters u and v. The function is expressed as:

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k},$$

where $x(u,v)$, $y(u,v)$, and $z(u,v)$ are the component functions of $\mathbf{r}$, representing the x-, y-, and z-coordinates of the surface, respectively. These functions are defined over a region D in the uv-plane. The set of all points $(x,y,z) \in \mathbb{R}^3$ that satisfy:

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v),$$

as (u,v) varies over D, forms the parametric surface S.

Parametric Equations

Definition 0.0.3: Parametric Equations

For a parametric surface the *parametric equations* are equations that describe the coordinates (x,y,z) of points on the surface as functions of two independent parameters u and v. For a parametric surface S, these equations are given by:

$$x = x(u,v), \quad y = y(u,v), \quad z = z(u,v),$$

where $x(u,v)$, $y(u,v)$, and $z(u,v)$ are the component functions of a vector-valued function $\mathbf{r}(u,v)$. These equations define the spatial coordinates of the surface for every pair of parameters (u,v) in a specified domain D in the uv-plane.

Grid Curves

Definition 0.0.4: Grid Curves

On a parametric surface s *grid curves* are families of curves defined by the vector function $\mathbf{r}(u, v)$. They are obtained by fixing one parameter and varying the other:
1. **Curves with $u = u_0$:** When u is held constant, the parametric surface reduces to a curve:

$$\mathbf{r}(u_0, v) = \langle x(u_0, v), y(u_0, v), z(u_0, v)\rangle,$$

which traces a curve C_1 on the surface as v varies.
2. **Curves with $v = v_0$:** When v is held constant, the parametric surface reduces to a curve:

$$\mathbf{r}(u, v_0) = \langle x(u, v_0), y(u, v_0), z(u, v_0)\rangle,$$

which traces a curve C_2 on the surface as u varies.
These two families of curves correspond to horizontal and vertical lines in the uv-plane and form a grid-like structure when plotted on the surface.

Spherical Coordinates

Surfaces of Revolution

Definition 0.0.5: Surfaces of Revolution

A **surface of revolution** is generated by rotating a curve C, defined parametrically or as a function, about a fixed axis in three-dimensional space. The parametric equations of the surface can be expressed as:

$$\begin{aligned} x &= u, \\ y &= r(u)\cos\theta, \\ z &= r(u)\sin\theta, \end{aligned}$$

where:

- u is a parameter describing the curve C,
- $r(u)$ is the radial distance of the curve from the axis of rotation,
- $\theta \in [0, 2\pi]$ is the angle of rotation.

The domain of the parameters u and θ depends on the curve and the extent of rotation.

Tangent Planes

Definition 0.0.6: Tangent Planes

The **tangent plane** to a parametric surface S at a point $P_0(u_0, v_0)$ is the plane that best approximates S near P_0.
If S is defined by a vector-valued function:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

then the tangent plane at P_0 is determined by the two tangent vectors at P_0:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k},$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

The tangent plane at P_0 is spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$. A normal vector to the plane is given by:

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v.$$

The equation of the tangent plane can be expressed in the point-normal form:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}(u_0, v_0)) = 0,$$

where $\mathbf{r}(u_0, v_0)$ is the position vector of P_0.
For the tangent plane to exist, the cross product $\mathbf{r}_u \times \mathbf{r}_v$ must be nonzero, ensuring that S is smooth at P_0.

Surface Area for a Parametric Surface

Definition 0.0.7: Surface Area for a Parametric Surface

The **surface area** of a smooth parametric surface S, defined by the vector-valued function:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D,$$

where D is the parameter domain, is given by the integral:

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA,$$

where:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

The cross product $\mathbf{r}_u \times \mathbf{r}_v$ represents a vector orthogonal to the tangent plane at each point on the surface, and its magnitude $|\mathbf{r}_u \times \mathbf{r}_v|$ gives the infinitesimal area of a parallelogram spanned by the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. Integrating this quantity over the parameter domain D yields the total surface area of S.

Surface Area of the Graph of a Function

Definition 0.0.8: Surface Area of the Graph of a Function

The **surface area** of the graph of a function $z = f(x, y)$, where $f(x, y)$ has continuous partial derivatives, over a region D in the xy-plane is given by:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA.$$

Explanation

- The parametric representation of the surface is:

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}.$$

- The tangent vectors are:

$$\mathbf{r}_x = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}, \quad \mathbf{r}_y = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}.$$

- The magnitude of the cross product of the tangent vectors is:

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

Integrating this quantity over the region D in the xy-plane gives the total surface area of the graph of $f(x, y)$.

16.7 Surface Integral

Parametric Surfaces

Definition 0.0.9: Surface Integral for Parametric Surfaces

The **surface integral** of a scalar function $f(x, y, z)$ over a parametric surface S, defined by the vector equation:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D,$$

is given by:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\; dA.$$

Explanation

- The parameter domain D is divided into subrectangles with dimensions Δu and Δv, and each corresponding surface patch is approximated as a parallelogram in the tangent plane.
- The area of a surface patch is approximated as:

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u \Delta v.$$

- The surface integral is defined as the limit of a Riemann sum:

$$\iint_S f(x, y, z)\, dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}.$$

Key Components

- $\mathbf{r}_u$ and $\mathbf{r}_v$ are the partial derivatives of $\mathbf{r}(u, v)$ with respect to u and v, respectively:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}.$$

- $\mathbf{r}_u \times \mathbf{r}_v$ gives a vector normal to the surface at each point, and $|\mathbf{r}_u \times \mathbf{r}_v|$ represents the infinitesimal surface area element.

This integral evaluates the contribution of $f(x, y, z)$ across the entire surface S.

Graphs of Functions

Definition 0.0.10: Surafe Integrals for Graphs of Functions

The **surface integral** of a scalar function $f(x, y, z)$ over the graph of a function $z = g(x, y)$, where $g(x, y)$ has continuous partial derivatives, is given by:

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA.$$

Explanation

- The graph of the function $z = g(x, y)$ can be regarded as a parametric surface with:

$$x = x, \quad y = y, \quad z = g(x, y).$$

- The tangent vectors to this surface are:

$$\mathbf{r}_x = \mathbf{i} + \frac{\partial g}{\partial x}\mathbf{k}, \quad \mathbf{r}_y = \mathbf{j} + \frac{\partial g}{\partial y}\mathbf{k}.$$

- The cross product of the tangent vectors is:

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{j} - \frac{\partial g}{\partial y}\mathbf{i} + \mathbf{k}.$$

- The magnitude of the cross product is:

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}.$$

By integrating this quantity over the region D in the xy-plane, we account for the contributions of $f(x, y, z)$ over the entire surface S.

Oriented Surfaces

Definition 0.0.11: Oriented Surfaces

An **oriented surface** is an orientable (two-sided) surface S where it is possible to define a continuous, unit normal vector $\mathbf{n}$ at every point (x, y, z) on the surface, except possibly at boundary points.

Key Properties

- **Two Possible Orientations**: For any orientable surface, there are two choices for the unit normal vector:
  - $\mathbf{n}_1$, the chosen unit normal vector.
  - $\mathbf{n}_2 = -\mathbf{n}_1$, the opposite orientation.
- A surface is called **orientable** if it is possible to assign $\mathbf{n}$ continuously over the entire surface S.
- A classic example of a non-orientable surface is the Möbius strip, which has only one side and no consistent orientation.

Explanation

An oriented surface requires the existence of a consistent way to assign a "positive" or "negative" side across all points on the surface. The orientation is provided by the chosen direction of the normal vector $\mathbf{n}$, which varies smoothly across the surface.

Surface Integrals of Vector Fields; Flux

Definition 0.0.12: Flux

The **surface integral of a vector field** (also called the **flux**) over an oriented surface S with a unit normal vector $\mathbf{n}$ is defined as:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS,$$

where:

- $\mathbf{F}$ is a continuous vector field defined on S,
- $\mathbf{n}$ is the unit normal vector to S,
- dS represents the infinitesimal surface area element.

Physical Interpretation

This integral measures the total "flow" of $\mathbf{F}$ through the surface S in the direction of the normal vector $\mathbf{n}$. For example:

- If $\mathbf{F} = \rho\mathbf{v}$, where $\rho(x, y, z)$ is the density of a fluid and $\mathbf{v}(x, y, z)$ is the velocity field, the flux represents the mass flow rate of the fluid through S.

Parametric Form

If the surface S is parameterized by $\mathbf{r}(u, v)$, with tangent vectors:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v},$$

then the flux integral can be written as:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA,$$

where D is the parameter domain.

Notes

- The orientation of S determines the sign of the flux.
- If the opposite orientation is used, the flux is multiplied by -1.

16.8 Stokes' Theorem

16.9