Math 120

PSet 2

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Contents

Chapter 1

1.1 PSet 2

Question 1

Consider the line L_1 given by $x + 2y = 7$ and the line L_2 given by $5x - y = 2$.

1. There are two unit vectors that are parallel to L_1. What are they?
2. There are two unit vectors that are perpendicular to L_1. What are they?
3. Find the acute angle between the lines L_1 and L_2. First find an exact expression and then approximate to the nearest degree.

Solution:
a)

$$L_1 = x + 2y = 7$$

$$y = -\frac{1}{2}x + \frac{7}{2}$$

$$v_1 = (1, m_1) = (1, -\frac{1}{2})$$

$$|v_1| = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$u_1 = \frac{1}{\frac{\sqrt{5}}{2}}\left(1, -\frac{1}{2}\right) = \frac{2}{\sqrt{5}}\left(1, -\frac{1}{2}\right) = \left(\frac{\sqrt{2}}{5}, -\frac{1}{\sqrt{5}}\right)$$

$$-u_1 = \left(-\frac{\sqrt{2}}{5}, \frac{1}{\sqrt{5}}\right)$$

b)

$$\text{slope of line perpendicular to } L_1 : -\frac{1}{-\frac{1}{2}} = 2$$

$$\langle 1, 2 \rangle$$

$$|n| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$u_1 = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$u_2 = -u_1 = \left\langle -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \right\rangle$$

c)

$$\vec{v_1} = \langle 2, -1 \rangle$$

$$y_2 = 5x - 2$$

$$\vec{v_2} = \langle 1, 5 \rangle$$
$$\cos(\theta) = \frac{v_1 \cdot v_2}{|v_1||v_2|}$$
$$\cos(\theta) = \frac{2-5}{\sqrt{5}\sqrt{26}} = -\frac{3}{\sqrt{130}}$$
$$\theta = 180 - \arccos\left(-\frac{3}{\sqrt{130}}\right) \approx 75$$

Question 2

Find all values of x such that the angle between the vectors $\langle 1, -1, 0 \rangle$ and $\langle 2, x, 1 \rangle$ is $\frac{\pi}{3}$.

Solution:

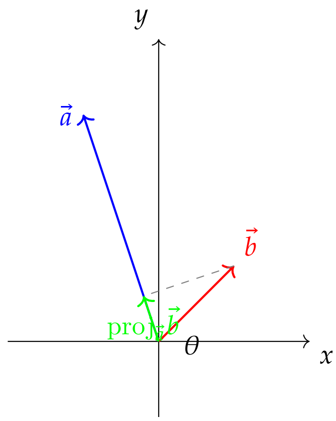
$$v_1 = \langle 1, -1, 0 \rangle$$
$$v_2 = \langle 2, x, 1, \rangle$$
$$\cos(\frac{\pi}{2}) = \frac{1}{2}$$
$$\frac{1}{2} = \frac{v_1 \cdot v_2}{|v_1||v_2|} = \frac{2-x}{(\sqrt{2})\sqrt{5+x^2}}$$
$$4 - 2x = \sqrt{10 + 2x^2}$$
$$10 + 2x^2 = 16 - 16x + 4x^2$$
$$-2x^2 + 16x - 6 = 0$$
$$x^2 - 8x + 3 = 0$$
$$x = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(3)}}{2 \cdot 1}$$
$$x = \frac{8 \pm \sqrt{52}}{2}$$
$$\frac{2 - \left(4 + \sqrt{13}\right)}{2\sqrt{5 + (4 + \sqrt{13})^2}} = -0.5$$
$$\frac{2 - \left(4 - \sqrt{13}\right)}{2\sqrt{5 + (4 - \sqrt{13})^2}} = 0.5$$
$$x = 4 - \sqrt{13}$$

Question 3

Find the scalar and vector projections of $\vec{b} = \hat{\imath} + \hat{\jmath}$ onto $\vec{a} = -\hat{\imath} + 3\hat{\jmath}$, and illustrate your answers with a sketch.

Solution:
Scalar Projection:

$$\vec{b} = \hat{\imath} + \hat{\jmath}$$
$$\vec{a} = \hat{\imath} + 3\hat{\jmath}$$
$$\text{comp}_a\mathbf{b} = \frac{a \cdot b}{|a|}$$
$$\frac{a \cdot b}{|a|} = \frac{2}{\sqrt{10}} = \frac{2\sqrt{10}}{10} = \frac{\sqrt{10}}{5}$$



Vector Projection:

$$\text{proj}_a\mathbf{b} = \left(\frac{a \cdot b}{|a|^2}\right) a$$

$$\left(\frac{a \cdot b}{|a|^2}\right) a = \frac{2}{\sqrt{10}^2} a = \frac{2}{10} a = \frac{1}{5} a$$

$$\frac{1}{5} a = \frac{1}{5}(-\hat{\imath}, 3\hat{\jmath}) = \langle -\frac{1}{5}\hat{\imath}, \frac{3}{5}\hat{\jmath} \rangle$$

Question 4

Find two vectors of length 2 that are orthogonal to both $\vec{v} = \langle 2, 4, 4 \rangle$ and $\vec{w} = \langle 1, -1, -3 \rangle$.

Solution:

$$v \times w = \langle 4(-3) - 4(-1), 4(1) - 2(-3), 2(-1) - 4(1) \rangle = \langle -8, 10, -6 \rangle$$

$$|u| = \sqrt{(-8)^2 + 10^2 + (-6)^2} = \sqrt{200}$$

$$2 = x \cdot \sqrt{200}$$

$$x = \frac{2}{\sqrt{200}} = \frac{1}{5\sqrt{2}}$$

$$u_1 = \frac{1}{5\sqrt{2}} \langle -8, 10, -6 \rangle$$

$$u_2 = -\frac{1}{5\sqrt{2}} \langle -8, 10, -6 \rangle$$

Question 5

Let $\vec{a} = \langle 3, 1, 0 \rangle$. Find all vectors $\vec{b} = \langle b_1, b_2, b_3 \rangle$ such that $\vec{a} \times \vec{b}$ is parallel to the z-axis and pointing in the positive z direction. Illustrate with a sketch, in which all vectors are drawn as position vectors, i.e., with the tail at the origin.

Solution:

$$a \times b = \langle 0, 0, c \rangle$$

$$\langle 3, 1, 0 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle 0, 0, c \rangle$$

$$\langle 3, 1, 0 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle 1(b_3) - 0(b_2), 0(b_1) - 3(b_3), 3(b_2) - 1(b_1) \rangle$$

$$\langle 3, 1, 0 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle b_3, -3b_3, 3b_2 - b_1 \rangle$$

$$3b_2 - b_1 > 0$$

$$b_3 = 0$$

$$-3b_3 = 0$$

All vectors in the form of $\langle b_1, b_2, 0 \rangle$ where $3b_2 - b_1 > 0$

Question 6

Consider the four points in $\mathbb{R}^3$, $K(1,2,3)$, $L(1,3,6)$, $M(3,8,6)$, and $N(3,7,3)$.

1. Show that the vectors $\overrightarrow{KL}$, $\overrightarrow{KM}$, and $\overrightarrow{KN}$ are coplanar. Explain why this means that K, L, M, and N all lie in the same plane.
2. From part (a), we know that K, L, M, and N are the vertices of a quadrilateral. Explain how you can tell that this quadrilateral is actually a parallelogram.
3. Find the area of the parallelogram with vertices K, L, M, and N.

Solution: a)

$$\overrightarrow{KL} = \langle 1-1, 3-2, 6-3\rangle = \langle 0,1,3\rangle$$
$$\overrightarrow{KM} = \langle 3-1, 8-2, 6-3\rangle = \langle 2,6,3\rangle$$
$$\overrightarrow{KN} = \langle 3-1, 7-2, 3-3\rangle = \langle 2,5,0\rangle$$
$$\overrightarrow{KL}\cdot(\overrightarrow{KM}\times\overrightarrow{KN}) = \overrightarrow{KL}\times\langle 6(0)-3(5), 3(2)-2(0), 2(5)-6(2)\rangle$$
$$\overrightarrow{KL}\cdot(\overrightarrow{KM}\times\overrightarrow{KN}) = \overrightarrow{KL}\times\langle -15,6,-2\rangle$$
$$\overrightarrow{KL}\cdot(\overrightarrow{KM}\times\overrightarrow{KN}) = \overrightarrow{KL}\times\langle -15,6,-2\rangle = 0(15)+6(1)+3(-2) = 0$$
$$|\overrightarrow{KL}\cdot(\overrightarrow{KM}\times\overrightarrow{KN})| = 0$$

They are coplanar because the volume determined by the vectors is 0, therefore they must lie on the same plane.
b).

$$\overrightarrow{KL} = \langle 0,1,3\rangle$$
$$\overrightarrow{MN} = \langle 3-3, 7-8, 3-6\rangle = \langle 0,-1,-3\rangle$$
$$\overrightarrow{KL} = -\overrightarrow{MN}$$

Since $\overrightarrow{KL} = -\overrightarrow{MN}$ these two sides are parallel.

$$\overrightarrow{KL} = \langle 2,6,3\rangle$$
$$\overrightarrow{LN} = \langle 3-1, 7-3, 3-6\rangle = \langle 2,4,-3\rangle$$

Although $\overrightarrow{KM} = -\overrightarrow{LN}$ are not negatives of each other or equal in magnitude they form the other side of the parallelogram
c)

$$\overrightarrow{KL}\times\overrightarrow{KM} = \langle 1(3)-3(6), 3(2)-0(3), 0(6)-(1)(2)\rangle = \langle -15,6,-2\rangle$$
$$\sqrt{(-15)^2+6^2+(-2)^2} = \sqrt{265}$$

Question 7

Find the vector equation and parametric equations for the line through the point $(1,2,-2)$ parallel to the line $x = t-2$, $y = -2t+1$, $z = 3$.

Solution:

$$x = t-2 \quad y = -2t+1 \quad z = 3$$
$$\vec{d} = \langle 1,-2,0\rangle$$
$$\vec{r}(t) = \vec{r}_0 + t\vec{d}$$
$$\vec{r}(t) = \langle 1,2,-2\rangle + t\langle 1,-2,0\rangle$$
$$x(t) = 1+t \quad y(t) = 2-2t \quad z(t) = -2$$

Vector Equation: $\vec{r}(t) = \langle 1,2,-2\rangle + t\langle 1,-2,0\rangle$
Parametric Equation: $x(t) = 1+t$, $y(t) = 2-2t$, $z(t) = -2$

Question 8

Consider the lines $L_1 : x = t + 3,\ y = 2t - 1,\ z = -t$, and $L_2 : x = t - 1,\ y = t - 4,\ z = -t + 4$. Determine whether the L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

Solution:

$$d_1 = \langle 1, 2, -1\rangle \quad \langle d_2 = 1, 1, -1\rangle$$

$$\frac{1}{1} \neq \frac{2}{1} \neq \frac{-1}{-1}$$

$$t_1 + 3 = t_2 - 1 \quad 2t_1 - 1 = t_2 - 4 \quad -t_1 = -t_2 + 4$$

$$t_1 + 3 = t_2 - 1 \Rightarrow t_1 - t_2 = -4$$

$$-t_1 = -t_2 + 4 \Rightarrow t_1 = t_2 - 4$$

$$(t_2 - 4) - t_2 = -4 \Rightarrow -4 = -4$$

$$2t_1 - 1 = t_2 - 4$$

$$2(t_2 - 4) - 1 = t_2 - 4$$

$$2t_2 - 9 = t_2 - 4$$

$$t_2 = 5$$

$$t_1 = t_2 - 4 = 5 - 4 = 1$$

$$x_1 = 1 + 3 = 4 \quad y_1 = 2(1) - 1 = 1 \quad z_1 = -1$$

Point of Intersection: (4, 1, - 1)

Question 9

Consider the planes $x + y + 2z = 4$ and $2x - y - 2z = 1$.

1. Find a vector equation for the line of intersection of the planes.
2. Find the angle between the planes. First find an exact expression and then approximate to the nearest degree.

Solution:

a)

$$n_1 = \langle 1, 1, 2\rangle \quad n_2 = \langle 2, -1, -2\rangle$$

$$d = n_1 \times n_2 = \langle 1(-2) - 2(-1), 2(2) - 1(-2), 1(-1) - 1(2)\rangle = \langle 0, 6, -3\rangle$$

$$(x + y + 2z) + (2x - y - 2z) = 4 + 1$$

$$3x = 5 \Rightarrow x = \frac{5}{3}$$

$$\frac{5}{3} + y + 2z = 4 \Rightarrow \frac{7}{3}$$

$$y + 2z = \frac{7}{3}$$

$$2\left(\frac{5}{3}\right) - y - 2z = 1$$

$$-y - 2z = -\frac{7}{3}$$

$$y + 2z = \frac{7}{3}$$

$$y + 0 = \frac{7}{3}$$

$$\left(\frac{5}{3}, \frac{7}{3}, 0\right)$$

$$\vec{r}(t) = \left(\frac{5}{3}, \frac{7}{3}, 0\right) + t\langle 0, 6, -3\rangle$$

b)

$$n_1 = \langle 1, 1, 2\rangle \quad n_2 = \langle 2, -1, -2\rangle$$

$$\cos(\theta) = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{1(2) + 1(-1) + 2(-2)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{-3}{\sqrt{6}\sqrt{9}} = -\frac{1}{\sqrt{6}}$$

$$\theta = 180 - \arccos\left(-\frac{1}{\sqrt{6}}\right) \approx 180 - 114 = 66$$

Question 10

Let P be the plane $x + y + 2z = 1$ and let A be the point $(1, 1, 1)$.

(a) Find an equation of the plane through point A parallel to plane P.

(b) Find a vector equation for the line through the point A which is perpendicular to the plane P. Call this line L.

(c) Find the point of intersection of the line L (from part (b)) and the plane P.

(d) Find the point on the plane P closest to the point A, and then find the shortest distance from the point A to the plane P.

Solution:

a)

$$P : x + y + 2z = 1 \quad n_1 = \langle 1, 1, 2\rangle$$

$$n_2 = t\langle 1, 1, 2\rangle$$

$$P_2 : tx + ty + 2tz = d$$

$$t1 + t1 + 2t(1) = d \Rightarrow t + t + 2t = d \Rightarrow d = 4t$$

$$tx + ty + 2tz = 4t \Rightarrow x + y + 2z = 4$$

b)

$$n = (1, 1, 2)$$

$$r(t) = (1, 1, 1) + t(1, 1, 2)$$

c)

$$P : x + y + 2z = 1$$

$$L : x = 1 + t \quad y = 1 + t \quad z = 1 + 2t$$

$$(1 + t) + (1 + t) + 2(1 + 2t) = 1$$

$$2 + 2t + 2 + 4t = 1 \Rightarrow 6t + 4 = 1$$

$$6t = -3 \Rightarrow t = -\frac{1}{2}$$

$$x : -\frac{1}{2} + 1 \quad y : -\frac{1}{2} + 1 \quad z : 1 + 2\left(-\frac{1}{2}\right)$$

Point: $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

d) Point: $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ because it is when the line through point A is perpendicular to the plane P

$$d = \sqrt{\left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)^2 + (1 - 0)^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}$$