Math 120

Final Review

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Parametric Surfaces

Definition 0.0.2: Parametric Surface

A *parametric surface* is a surface in three-dimensional space $\mathbb{R}^3$ defined by a vector-valued function $\mathbf{r}(u, v)$, which depends on two parameters u and v. The function is expressed as:

$$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k},$$

where $x(u, v)$, $y(u, v)$, and $z(u, v)$ are the component functions of $\mathbf{r}$, representing the x-, y-, and z-coordinates of the surface, respectively. These functions are defined over a region D in the uv-plane. The set of all points $(x, y, z) \in \mathbb{R}^3$ that satisfy:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

as (u, v) varies over D, forms the parametric surface S.

Parametric Equations

Definition 0.0.3: Parametric Equations

For a parametric surface the *parametric equations* are equations that describe the coordinates (x, y, z) of points on the surface as functions of two independent parameters u and v. For a parametric surface S, these equations are given by:

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

where $x(u, v)$, $y(u, v)$, and $z(u, v)$ are the component functions of a vector-valued function $\mathbf{r}(u, v)$. These equations define the spatial coordinates of the surface for every pair of parameters (u, v) in a specified domain D in the uv-plane.

Grid Curves

Definition 0.0.4: Grid Curves

On a parametric surface s *grid curves* are families of curves defined by the vector function $\mathbf{r}(u, v)$. They are obtained by fixing one parameter and varying the other:
1. **Curves with $u = u_0$:** When u is held constant, the parametric surface reduces to a curve:

$$\mathbf{r}(u_0, v) = \langle x(u_0, v), y(u_0, v), z(u_0, v)\rangle,$$

which traces a curve C_1 on the surface as v varies.
2. **Curves with $v = v_0$:** When v is held constant, the parametric surface reduces to a curve:

$$\mathbf{r}(u, v_0) = \langle x(u, v_0), y(u, v_0), z(u, v_0)\rangle,$$

which traces a curve C_2 on the surface as u varies.
These two families of curves correspond to horizontal and vertical lines in the uv-plane and form a grid-like structure when plotted on the surface.

Spherical Coordinates

Surfaces of Revolution

Definition 0.0.5: Surfaces of Revolution

A **surface of revolution** is generated by rotating a curve C, defined parametrically or as a function, about a fixed axis in three-dimensional space. The parametric equations of the surface can be expressed as:

$$\begin{aligned} x &= u, \\ y &= r(u)\cos\theta, \\ z &= r(u)\sin\theta, \end{aligned}$$

where:

- u is a parameter describing the curve C,
- $r(u)$ is the radial distance of the curve from the axis of rotation,
- $\theta \in [0, 2\pi]$ is the angle of rotation.

The domain of the parameters u and θ depends on the curve and the extent of rotation.

Tangent Planes

Definition 0.0.6: Tangent Planes

The **tangent plane** to a parametric surface S at a point $P_0(u_0, v_0)$ is the plane that best approximates S near P_0.
If S is defined by a vector-valued function:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

then the tangent plane at P_0 is determined by the two tangent vectors at P_0:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k},$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

The tangent plane at P_0 is spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$. A normal vector to the plane is given by:

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v.$$

The equation of the tangent plane can be expressed in the point-normal form:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}(u_0, v_0)) = 0,$$

where $\mathbf{r}(u_0, v_0)$ is the position vector of P_0.
For the tangent plane to exist, the cross product $\mathbf{r}_u \times \mathbf{r}_v$ must be nonzero, ensuring that S is smooth at P_0.

Surface Area for a Parametric Surface

Definition 0.0.7: Surface Area for a Parametric Surface

The **surface area** of a smooth parametric surface S, defined by the vector-valued function:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D,$$

where D is the parameter domain, is given by the integral:

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA,$$

where:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

The cross product $\mathbf{r}_u \times \mathbf{r}_v$ represents a vector orthogonal to the tangent plane at each point on the surface, and its magnitude $|\mathbf{r}_u \times \mathbf{r}_v|$ gives the infinitesimal area of a parallelogram spanned by the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. Integrating this quantity over the parameter domain D yields the total surface area of S.

Surface Area of the Graph of a Function

Definition 0.0.8: Surface Area of the Graph of a Function

The **surface area** of the graph of a function $z = f(x, y)$, where $f(x, y)$ has continuous partial derivatives, over a region D in the xy-plane is given by:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA.$$

Explanation

- The parametric representation of the surface is:

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}.$$

- The tangent vectors are:

$$\mathbf{r}_x = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}, \quad \mathbf{r}_y = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}.$$

- The magnitude of the cross product of the tangent vectors is:

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

Integrating this quantity over the region D in the xy-plane gives the total surface area of the graph of $f(x, y)$.

16.7 Surface Integral

Parametric Surfaces

Definition 0.0.9: Surface Integral for Parametric Surfaces

The **surface integral** of a scalar function $f(x,y,z)$ over a parametric surface S, defined by the vector equation:

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v) \in D,$$

is given by:

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))\,|\mathbf{r}_u \times \mathbf{r}_v|\ dA.$$

Explanation

- The parameter domain D is divided into subrectangles with dimensions Δu and Δv, and each corresponding surface patch is approximated as a parallelogram in the tangent plane.
- The area of a surface patch is approximated as:

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u \Delta v.$$

- The surface integral is defined as the limit of a Riemann sum:

$$\iint_S f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}.$$

Key Components

- $\mathbf{r}_u$ and $\mathbf{r}_v$ are the partial derivatives of $\mathbf{r}(u,v)$ with respect to u and v, respectively:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}.$$

- $\mathbf{r}_u \times \mathbf{r}_v$ gives a vector normal to the surface at each point, and $|\mathbf{r}_u \times \mathbf{r}_v|$ represents the infinitesimal surface area element.

This integral evaluates the contribution of $f(x,y,z)$ across the entire surface S.

Graphs of Functions

Definition 0.0.10: Surafe Integrals for Graphs of Functions

The **surface integral** of a scalar function $f(x, y, z)$ over the graph of a function $z = g(x, y)$, where $g(x, y)$ has continuous partial derivatives, is given by:

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA.$$

Explanation

- The graph of the function $z = g(x, y)$ can be regarded as a parametric surface with:

$$x = x, \quad y = y, \quad z = g(x, y).$$

- The tangent vectors to this surface are:

$$\mathbf{r}_x = \mathbf{i} + \frac{\partial g}{\partial x}\mathbf{k}, \quad \mathbf{r}_y = \mathbf{j} + \frac{\partial g}{\partial y}\mathbf{k}.$$

- The cross product of the tangent vectors is:

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{j} - \frac{\partial g}{\partial y}\mathbf{i} + \mathbf{k}.$$

- The magnitude of the cross product is:

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}.$$

By integrating this quantity over the region D in the xy-plane, we account for the contributions of $f(x, y, z)$ over the entire surface S.

Oriented Surfaces

Definition 0.0.11: Oriented Surfaces

An **oriented surface** is an orientable (two-sided) surface S where it is possible to define a continuous, unit normal vector $\mathbf{n}$ at every point (x, y, z) on the surface, except possibly at boundary points.

Key Properties

- **Two Possible Orientations**: For any orientable surface, there are two choices for the unit normal vector:
  - $\mathbf{n}_1$, the chosen unit normal vector.
  - $\mathbf{n}_2 = -\mathbf{n}_1$, the opposite orientation.
- A surface is called **orientable** if it is possible to assign $\mathbf{n}$ continuously over the entire surface S.
- A classic example of a non-orientable surface is the Möbius strip, which has only one side and no consistent orientation.

Explanation

An oriented surface requires the existence of a consistent way to assign a ”positive” or ”negative” side across all points on the surface. The orientation is provided by the chosen direction of the normal vector $\mathbf{n}$, which varies smoothly across the surface.

Surface Integrals of Vector Fields; Flux

Definition 0.0.12: Flux

he **surface integral of a vector field** (also called the **flux**) over an oriented surface S with a unit normal vector $\mathbf{n}$ is defined as:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where:

- $\mathbf{F}$ is a continuous vector field defined on S,
- $\mathbf{n}$ is the unit normal vector to S,
- dS represents the infinitesimal surface area element.

Special Case: Surface Defined by $z = g(x, y)$

If the surface S is defined by the graph $z = g(x, y)$, and $\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, the surface integral can be expressed as:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA,$$

where D is the projection of the surface onto the xy-plane.
This formula assumes the upward orientation of S. For a downward orientation, the integral is multiplied by -1.

Parametric Form

If the surface S is parameterized by $\mathbf{r}(u, v)$, with tangent vectors:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v},$$

then the flux integral can be written as:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA,$$

where D is the parameter domain.

Physical Interpretation

The flux integral measures the total flow of the vector field $\mathbf{F}$ across the surface S, representing quantities like mass flow rate, electric flux, or fluid flow through S.

16.8 Stokes' Theorem

Definition 0.0.13: Stokes' Theorem

Stokes' Theorem relates the surface integral of the curl of a vector field over an oriented surface S to the line integral of the vector field along the boundary curve C of S. Mathematically, it is expressed as:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where:

- S is a piecewise-smooth, oriented surface with unit normal vector $\mathbf{n}$,
- C is the positively oriented, closed boundary curve of S,
- $\mathbf{F}$ is a vector field with continuous partial derivatives,
- $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$,
- $d\mathbf{S} = \mathbf{n}\, dS$ is the oriented surface element,
- $d\mathbf{r}$ is the infinitesimal vector along C.

Key Notes

- **Positive Orientation:** The orientation of C is determined by the right-hand rule: when you walk along C with your head pointing in the direction of $\mathbf{n}$, the surface S remains on your left.
- **Special Case:** If S lies flat in the xy-plane and $\mathbf{n} = \mathbf{k}$, Stokes' Theorem reduces to Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$$

Stokes' Theorem provides a fundamental relationship between the circulation of $\mathbf{F}$ along C and the total rotational effects (curl) of $\mathbf{F}$ over the surface S.

Note:-

Stokes' Theorem allows us to compute a surface integral simply by knowing the values of $\mathbf{F}$ on the boundary curve C. This means that if we have another oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral. In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then:

$$\int_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

16.9