Math 120

PSet 2

Sep 12 2024

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Chapter 1

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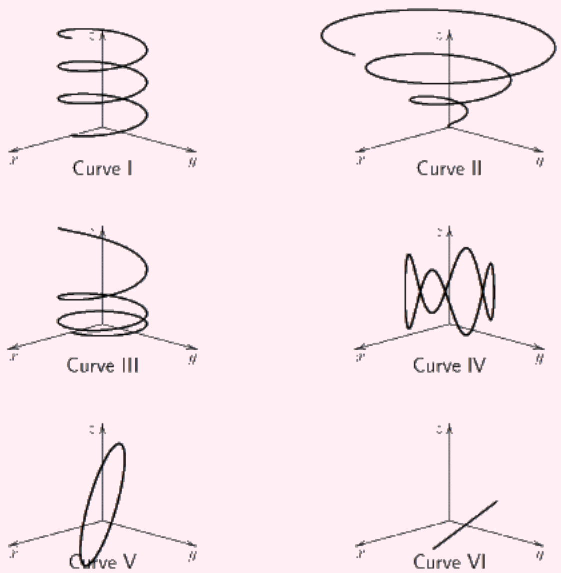
Question 1



(a) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

(b) $\vec{r}(t) = t\langle \cos t, \sin t, t \rangle$

(c) $\vec{r}(t) = \langle \cos t, \sin t, t^3 \rangle$

(d) $\vec{r}(t) = \langle \cos(t^3), \sin(t^3), t^3 \rangle$

(e) $\vec{r}(u) = \langle \cos u, \sin u, 1 + \sin(4u) \rangle$

(f) $\vec{r}(u) = \langle \cos u, \sin u, 1 + 4\sin(u) \rangle$

(g) $\vec{r}(t) = \langle 2\cos t, 1 + 4\cos t, 3\cos t \rangle$

Solution:

Question 2

2. Find a vector function that represents the curve of intersection of the plane $z = -2$ and the sphere $x^2 + (y-1)^2 + (z+1)^2 = 9$.

Solution:

$$x^2 + (y-1)^2 + ((-2)+1)^2 = 9$$
$$x^2 + (y-1)^2 = 8$$
$$r = 2\sqrt{2}$$
$$x(t) = 2\sqrt{2}\cos(t)$$
$$y - 1 = 2\sqrt{2}\sin(t) \Rightarrow y = 2\sqrt{t}\sin(t) + 1$$
$$\vec{r}(t) = \langle 2\sqrt{2}\cos(t), 2\sqrt{2}\sin(t) + 1, -2\rangle$$

Question 3

Consider the vector-valued function $\vec{r}_1(t) = \langle 2\sin t, -3\cos t, 0\rangle$, $0 \leqslant t \leqslant 2\pi$.

(a) Sketch the plane curve given by $\vec{r}_1(t)$.

(b) Compute and draw on your sketch from part (a) the position vector $\vec{r}_1\left(\frac{2\pi}{3}\right)$ and the tangent vector $\vec{r}_1'\left(\frac{2\pi}{3}\right)$.

(c) The vector-valued function $\vec{r}_2(t) = \langle 2\cos(3t), -3\sin(3t)\rangle$ parameterizes the same curve. Find the smallest $t^* > 0$ such that $\vec{r}_2(t^*) = \vec{r}_1\left(\frac{2\pi}{3}\right)$, and compute $\vec{r}_2'(t^*)$. Explain how and why $\vec{r}_2'(t^*)$ differs from the tangent vector $\vec{r}_1'\left(\frac{2\pi}{3}\right)$ you computed in part (b).

Solution:

Question 4

Find parametric equations for the tangent line to the curve parameterized by

$$x = 2t + 1, \quad y = e^{t^2-4}, \quad z = \ln(1 + t^2)$$

at the point $(5, 1, \ln 5)$.

Solution:

$$x(t) = 2(t) + 1 \quad y(t) = e^{t^2-4} \quad z(t) = \ln(1+t)^2$$
$$x'(t) = 2 \quad y'(t) = 2te^{t^2} \quad z'(t) = \frac{2t}{1+t^2}$$
$$5 = 2t + 1 \Rightarrow 4 = 2t \Rightarrow t = 2$$
$$x'(2) = 2 \quad y'(2) = 4e^{4t} \quad z'(t) = \frac{4}{5}$$

Question 5

(a) Evaluate the integral $\int \left(\tan t\,\hat{i} + \sin^2 t\,\hat{j} + \sec^2 t\ \tan t\,\hat{k}\right) dt$.

(b) Suppose a particle is at the point $(-2, 1, 4)$ at time $t = 0$, and moves according to the velocity function $\vec{v}(t) = \tan t\,\hat{i} + \sin^2 t\,\hat{j} + \sec^2 t\ \tan t\,\hat{k}$. Find the particle's position at time $t = \frac{\pi}{4}$.

Solution:

Question 6

Consider the curve parameterized by $\vec{r}(t) = \langle e^{2t}, e^{-2t}, \sqrt{8}t\rangle$, $0 \leqslant t \leqslant 1$.

(a) Sketch the projections of $\vec{r}(t)$ in the xy-, zx-, and yz-planes.

(b) Find the length of the curve. *Hint:* To integrate, you will need to write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$ as a perfect square.

Solution:

a)

$$x(t) = e^{2t} \quad y(t) = e^{-2t}$$
$$x \cdot y = e^{2t} \cdot e^{-2y} = 1$$
$$x(t) = e^{2t} \quad z(t) = \sqrt{8}t$$

b)

$$L = \int_0^1 ||\vec{r}(t)||dt$$
$$\vec{r}(t) = \langle e^{2t}, e^{-2t}, \sqrt{8}t\rangle$$
$$\vec{r}'(t) = \left\langle \frac{d}{dt}\left(e^{2t}\right), \frac{d}{dt}\left(e^{-2t}\right) \frac{d}{dt}\left(\sqrt{8}t\right)\right\rangle$$
$$\vec{r}'(t) = \left\langle 2e^{2t}, -2e^{-2t}, \sqrt{8}\right\rangle$$
$$L = |\vec{r}'(t)| = \int_0^1 \sqrt{(2e^{2t})^2 + (2e^{-2t})^2 + \left(\sqrt{8}\right)^2}$$
$$\left(2e^{2t}\right)^2 + \left(2e^{-2t}\right)^2 + \left(\sqrt{8}\right)^2 = \left(2e^{2t}\right)^2 + \left(2e^{-2t}\right)^2 + 8$$
$$\left(2e^{2t}\right)^2 + \left(2e^{-2t}\right)^2 + 8 = \left(2e^{2t} + 2e^{-2t}\right)^2$$
$$L = \int_0^1 \sqrt{(2e^{2t} + 2e^{-2t})^2} \Rightarrow \int_0^1 \left(2e^{2t} + 2e^{-2t}\right)$$
$$2e^{2t} + 2e^{-2t} = 4\cosh(2t)$$
$$L = \int_0^1 4\cosh(2t)dt \Rightarrow 2\sinh(2t)\Big|_0^1 \rightarrow 2\sinh(2(1)) - 2\sinh(2(0))$$
$$L = 2\sinh(2) - 2\sinh(0)$$

Question 7

Let C be the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $2x + y + z = 4$.

(a) Find a parameterization of C.

(b) Write down an integral for the length of C.

(c) Find the length accurate to five decimal places by using Desmos: https://www.desmos.com/calculator. (Click on the keyboard icon, then "functions", then "Misc", to find the integral symbol.)

Solution:
a)

$$x^2 + y^2 = 4 \quad \text{r} = 2$$
$$x(t) = 2\cos(t) \quad y(t) = 2\sin(t)$$
$$2(2cos(t)) + 2\sin(t) + z = 4 \Rightarrow z = 4 - 4\cos(t) - 2\sin(t)$$
$$x^2 + y^2 = 4 \quad \text{r} = 2 \quad z(t) = 4 - 4\cos(t) - 2\sin(t)$$
$$L = \int_a^b \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2 + \left(\frac{d}{dt}z(t)\right)^2}$$
$$\vec{r}'(t) = \langle\rangle$$

Question 8

Find the velocity and position vectors of a particle that has acceleration given by

$$\vec{a}(t) = 2\hat{i} + 6t\hat{j} + 12t^2\hat{k},$$

and initial velocity and position given by

$$\vec{v}(0) = \hat{i} \quad \text{and} \quad \vec{r}(0) = \hat{j} - \hat{k}.$$

Solution:

Question 9

Consider the function $f(x, y) = \frac{\sqrt{y}-3x}{\ln(4-x^2-y^2)}$.

(a) Find and sketch the domain of f.

(b) On your sketch from part (a), mark where $f(x, y) = 0$, and indicate the region(s) where $f(x, y)$ is positive and negative.

Solution:

Question 10

Here are several surfaces.
Match each function with its graph. Justify your answers.

(a) $f(x, y) = x^2$

(b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $f(x, y) = e^{x^2+y^2} - 1$

(d) $f(x, y) = y \sin x$

(e) $f(x, y) = \sin(x + y)$

(f) $f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$

Solution:

Question 11

Draw a contour map of the function $f(x, y) = x^2 e^{-y}$ showing several level curves.

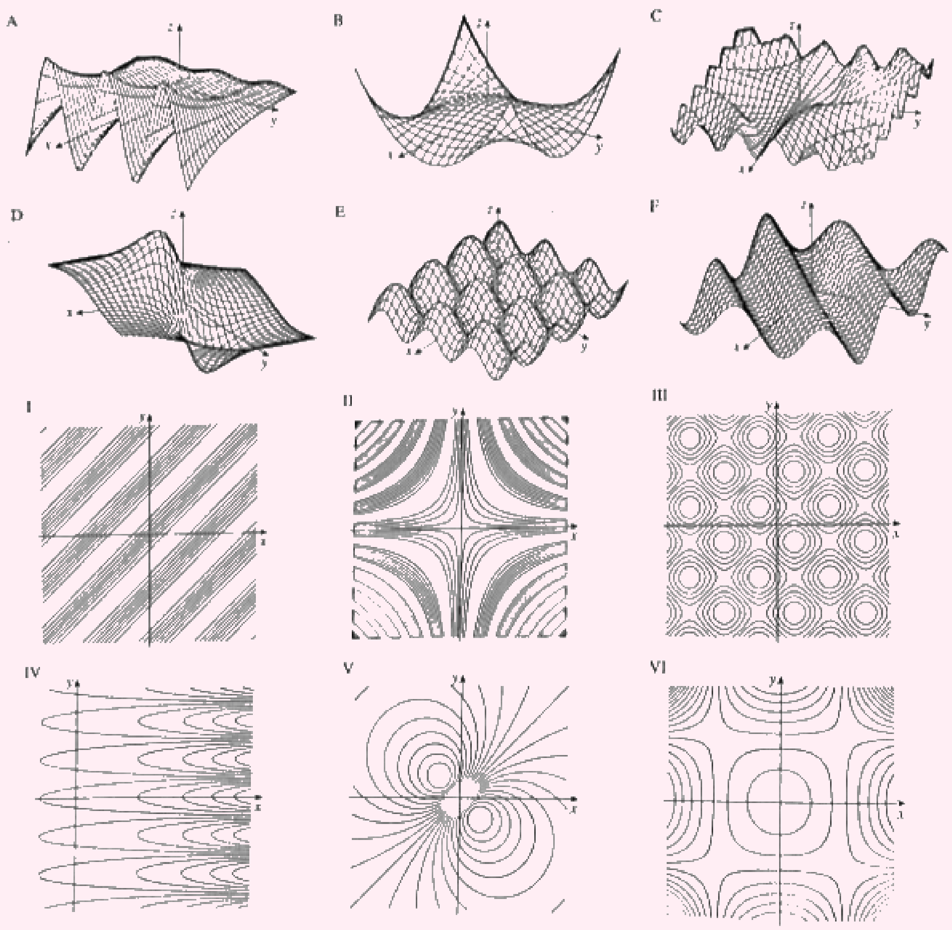
Question 12

Match the function with its graph (labeled A-F below) and with its contour map (labeled I-VI). Give reasons for your choices.

(a) $z = e^x \cos y$

(b) $z = \sin x - \sin y$

(c) $z = \frac{x-y}{1+x^2+y^2}$



Solution:

a)

e^x is an exponential function in the x-direction, meaning that as x increases the value of z grow rapidly.
$\cos(y)$ means that there are oscillations in the y-direction causing wave-like behavior along the y-axis.
Graph A shows an exponential rise in the x-direction with some oscillations in the y-direction.

b) $\sin x - \sin y$ would have oscillations along the x-direction and y-direction. These oscillations would be of the same size as there are fixed values that this equation can result in.
The graph is E for this reason.

c)

numerator of $x - y$ suggests a linear slope or difference between x and y, so one side will be positive and the other negative.
The denominator makes the effect of the numerator decrease as x and y increase since it outgrows them. So the graph of this function will have a positive peak and a negative peak near the origin and then it should level out on the sides.
This is why the graph is D.