Math 120 QR

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Contents

Chapter 1

1.1 Day 1 notes

Definition 1.1.1: Distance Formula

Defintion:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Definition 1.1.2: Equation of a sphere

Defintion: An equation of a sphere with center $C(h, k, l)$, and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2$$

In particular, if the center is the origin O, than an equation of the sphere is

$$x^2 + y^2 + z^2$$

1.2 Day 2 Notes

Definition 1.2.1: The lenght/magnitude of a vecotr

In 2D, $\vec{v} = <a, b>$: $|\vec{v}| = \sqrt{a^2 + b^2}$
In 3D, $\vec{v}$, $\vec{v} = <a, b, c>$: $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$
A unit vector is a vector of length 1

Question 1

If $\vec{v}$ is a a vector and a is a scalar, then what is $|a\vec{v}|$

Solution:

$$|a\vec{v}| = |a||\vec{v}|$$

Definition 1.2.2: Vectors in $\mathbf{R}^3$

The standard basis vectors in $\mathbf{R}^3$ are

$$i = <1,0,0>$$
$$j = <0,1,0>$$
$$k = <0,0,1>$$

Question 2

What is special about i,j,k?

Solution:

- Cannot make any of them as a linear combibnation of the other three.

- Any vector $\vec{v} \epsilon \mathbf{R}^3$ can be written uniquily as a linear combibnation of i,j,k

Example 1.2.1:

$$\vec{v} + \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{v} + \vec{u} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$

Definition 1.2.3: Dot product

A dot product between v and w is:
In 2D: $v \cdot w = v_1 w_1 + v_2 w_2$
In 3D: $v \cdot 2 = v_1 w_1 + v_2 w_2 + v_3 w_3$

Geometric defintion:
$|u \cdot w| = |u||w| \cos(\theta)$ where θ is the angle between v and w

Example 1.2.2: Why the 2 defintions are the same

$$v_1 w_1 + v_2 w_2 = |v||w| \cos(\theta) = p|v|$$

$$p = |w| \cos(\theta)$$

$$v_1 = |v| \cos(\theta)$$

$$v_2 = |v| \sin(\theta)$$

$$w_1 = |w| \cos(\theta)$$

$$w_2 = |w| \sin(\theta)$$

$$\text{LHS} = |v||w| \big(\cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta) \big)$$

$$\text{LHS} = |v||w| \sin(\alpha + \beta)$$

$$\text{LHS} = |v||w| \cos(\theta) = \text{RHS}$$

Example 1.2.3: what does the def mean

How much effect of $\vec{w}$ act along $\vec{v}$

Work: $w = \vec{F} \cdot \vec{S}$

Question 3

Find a relation between $|v|$ and $v \cdot v$

Solution: $|v|^2 = v \cdot v$

Question 4

v, w of fixed lenghts when is $v \cdot w$ largest?

Solution: v paralell: $\theta = 0$

$\cos(\theta) = 1$

Example 1.2.4: Projections

Given $\vec{v} \neq \vec{o}$

The project of w on v is proj $w = \left(\dfrac{\vec{w} \cdot \vec{v}}{|v|} \right) \dfrac{\vec{v}}{|v|}$

Directon of v is $\dfrac{\vec{v}}{|v|}$

Dor project of w with direction is:

$$\vec{w} \cdot \frac{\vec{v}}{|v|}$$

Question 5

TRUE or False:

u, v, w: vectors $(u \cdot v)w = u(v \cdot w)$

Solution: false

Question 6

TRUE or FALSE:

$v - w| = |v| - |w|$ if $v \parallel w$

Solution: false

Question 7

When is this ideal square sum happening?

$$|v + w|^2 = |v|^2 + |w|^2$$

Solution: when $v \perp w$

Definition 1.2.4: Cross Product

The cross product of two vectors v and w, $v * w$ is a vector u defined by $u \perp v$ and $u \perp w$.
Direction of u is given by the right hand rule
Magnitude: $|u|$ = Area of the parallelogram spanned by v and w.

1.3 Day 2 Reading notes

Definition 1.3.1: Vector Addition

If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

Definition 1.3.2: Scalar Multiplication

If c is a scalar and **v** is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c = 0$ or $\mathbf{v} = 0$, then $c\mathbf{v} = 0$

Example 1.3.1:

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with represenation $\overrightarrow{AB}$ is:

$$a = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 1.3.2:

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarily, for three demensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Note:-

Properties of vectors: If **a**, **b**, and **c** are vectors in V_n and c and d are scalars than

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $a + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + 0 = \mathbf{a}$
- $\mathbf{a} + \mathbf{a} + \text{-}\mathbf{a} = 0$

- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$

- $(c + d)a = c\mathbf{a} + d\mathbf{a}$

- $(cd)\mathbf{a} = c(d\mathbf{a})$

- $l\mathbf{a} = \mathbf{a}$

1.4 Day 3 Reading notes

Example 1.4.1:

$\in$

1.5 Day 3 Class notes

Example 1.5.1:

Use the geometric def of cross product to calculate:
$i \times (i + j)$
$(i + j) \times (i - j)$

Definition 1.5.1: Second definition of dot product

Arithmetic Definition:

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = |a||b|\sin(\theta)$$

$$\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} i - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} j - \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 k \end{bmatrix}$$

$$= (a_2 b_3 - a_3 b_2)i - (a_1 b_3 - a_3 b_1)j + (a_1 b_2 - a_2 b_1)k$$

$$|a \times b|^2 = (a_2 b_3 - a_3 b_2) - (a_1 b_3 - a_3 b_1) + (a_1 b_2 - a_2 b_1)$$

$$|a \times b|^2 = (a_2 b_3 - a_3 b_2) - (a_1 b_3 - a_3 b_1) + (a_1 b_2 - a_2 b_1)$$

Question 8

Calculate the corss product of $v = \langle -1, 3, 4 \rangle$ and $w = \langle 2, 1, -2 \rangle$

Solution:

$$v \times w = \begin{bmatrix} i & j & k \\ -1 & 4 & 4 \\ 2 & 1 & -2 \end{bmatrix} = 10i + 6j - 7k$$

Question 9

Calculate the area of the triangle with vertices P(1,0,1), Q (-3, 4, 0) and R(2,1,0)

Solution:

$$\overrightarrow{RP} = \langle -1, -1, 1 \rangle$$

$$\overrightarrow{RQ} = \langle 5, 0, 0 \rangle$$

$$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{bmatrix} i & j & k \\ -1 & -1 & 1 \\ -5 & 0 & 0 \end{bmatrix} = -5j - 5k$$

$$\text{area of } \delta PQR = \frac{1}{2}|\overrightarrow{RP} \times \overrightarrow{RQ}| = \frac{1}{2}\sqrt{5^2 + 5^2} = \frac{5\sqrt{2}}{2}$$

Question 10

What is special about: P(1,4,2), 1(3,3,5), R(9,1,11), and S(5, 11, 9)

Solution: If $(\overrightarrow{AC} \times \overrightarrow{AB} \parallel (\overrightarrow{AD} \times \overrightarrow{AC}))$ then A, B, C, D are on the same plane

Question 11

1. If $u \times w = u \times v$ then $w = v$
2. If $u \cdot v = 0$ and $u \times v = \vec{0}$ then either $u = \vec{0}$ or $v = \vec{0}$

Solution:
1. False

$$u \times (u + w) = u \times w$$

2. True
The reason is because the dot product is $|u||v| \cos(\theta)$ and the cross product is $|u||v| \sin(\theta)$

Question 12

Find the distance of a point p to the line passing through A and B

Solution:

$$= \frac{1}{2}\text{dist}(P, AB) \cdot |AB|$$

$$\text{dist}(P, AB) = \frac{|\overrightarrow{AB} \times \overrightarrow{AP}|}{|\overrightarrow{AB}}$$

Note:-

Line in 2D equation: $ax + by = c$
Line in 3D: Equation of a line passing through (x_0, y_0, z_0) and parallel to $\vec{v} = \langle a, b, c \rangle$ is

- $x = x_0 + at$

- $y = y_0 + bt$

- $z = z_0 + zt$

Equation of a plane passing through (x_0, y_0, z_0) and orthogonal to $\vec{n} = \langle a, b, c \rangle$ is $(r - r_0) \cdot n = 0$

$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$ax + by + cz + d = 0$

Question 13

What is the normal of the plane $3x + z + 2 = 0$?

Solution:

$$\vec{n} = \langle 3, 0, 1 \rangle$$

Question 14

Is the line $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$ parallel to the plane $x - 2y + z = 7$

1.6   Integration Review

Example 1.6.1:

$$\int \frac{x}{\sqrt{1 - x^2}} dx$$

$$v = 1 - x^2$$

$$dv = -2x\,dx$$

$$\int \frac{-\frac{1}{2}}{\sqrt{v}} dv$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{v}}$$

Example 1.6.2:

$$(3x^2 5)e^{x^3 + 5x} dx$$

$$u = x^3 + 5x$$

$$du = 3x^2 5\,dx$$

$$= \int e^u du$$

$$= e^{x^3 + 5x} c$$

Example 1.6.3:

$$\cos^5(2t)dt$$

$$u = 2t$$

$$du = 2dt$$

$$= \frac{1}{2}\cos^3(u)du = \frac{1}{2}\cos^4(u) + \cos(u)du$$

$$= \frac{1}{2}\int (1 - \sin^2(u))^2 \cdot \cos(u)du$$

$$v = \sin(u)$$

$$dv = \cos(u)du$$

Example 1.6.4:

$$xe^{2x}dx$$

$$x' = 1$$

$$\int e^{2x}dx = \frac{1}{2}e^{2x} + c$$

$$\int vdv = uv + \int vdu$$

$$v = x \rightarrow dv = 1dx$$

$$dv = e^{2x}dx \rightarrow v = \frac{1}{2}e^{2x}$$

$$\int = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$$

Example 1.6.5:

$$\int t\sin(2t)dt$$

$$t' = 1$$

$$\int \sin(2t)dt = -\frac{1}{2}\cos(2t)$$

$$\int vdv = uv + \int vdv$$

$$v = t \rightarrow dv = 1dt$$

$$dv = \sin(2t)dx \rightarrow v = -\frac{1}{2}\cos(2t)$$

$$\int = -\frac{1}{2}\cos(2t) - \int \frac{1}{2}e^{2x}dx$$

1.7 Day 4 Class Notes

Example 1.7.1: Warm up 1

True or False
If Q_1R are not on P and $\vec{n} \cdot \overrightarrow{QR} = 0$ than Q_1R are on the same side of (P). True: Since the dot product of the normal and $\overrightarrow{QR}$ are 0 than it is parallel to the plane

Example 1.7.2: Warm up 2

What is the shape (in 3D) of

$$\text{a) } x^2 + y^2 = 1 \quad \text{b) } x^2 + y^2 + z^2 = 4$$

a) Cylinder
b) Sphere

1.8 Day 5 Class Notes

Definition 1.8.1: Tangent Line

iven a cruve $r(t)$, and a point $r(t_0)$ Equation of the tangent line passing through $r(t_0)$ is

$$l(t) = r(t_0) + t \cdot r'(t_0)$$

$$r(t) = \langle \cos(t), \sin(t), 0 \rangle \quad \text{for } t \in [0, 2\pi]$$

Note:-

Differential Rules:

1. $(u \pm v)' = u' \pm v'$

2. $cu' = (cu)'$

3. $(u \cdot v)' = u' \cdot v + u \cdot v'$

4. $(u \times v)' = u' \times v + u \times v'$

5. $(u_0 f)' = u'(f)f'$

Definition 1.8.2

If a curve is parameterized by $r(t)$ for $\leqslant t \leqslant b$ then its (arc)length is

$$L = \int_a^b |r'(t)| dt$$

$$\langle -2\cos(t), 0, -2\sin(t) \rangle$$

$$\int_{-\pi}^{\frac{\pi}{2}} |\langle -2\sin(t), -4, 2\cos(t) \rangle$$

1.9 Day 7 Notes

Example 1.9.1 (Warm Up)

Calculate first and second derivatives of $f(x,y) = 2x^2 - 3xy^2$ and $f(x,y) = \sin(x-y) + \sin(x+y)$

Chain rule compute $\frac{d}{dt}$ in 2 ways (chain rule, and subtitution)

$$f(x,y) = 2x^2 - 3xy^2$$
$$f'(x) = 3x^2 - 3y^2$$
$$f'(y) = -6xy$$
$$f''(x) = 6x$$
$$f''(y) = -6y$$

1.10 Day 8 Notes

Definition 1.10.1: Gradient

Gradient of $f(x,y)$

$$\nabla f = \langle f_x, f_y \rangle$$

RMK:

Given a direction (unit) $\vec{u}$, we have

$$D_{\vec{u}} f(x_0, y_0) = \vec{u} \cdot \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$$

$$= \vec{u} \cdot \vec{\nabla} f(x_0, y_0)$$

Example 1.10.1:

Given (x_0, y_0) what is the direction that f increases/decreases teh most?

f increases the most in the direction $\frac{\nabla f(x_0, y_0)}{|\nabla f|}$

f decreases the most in the direction $\frac{-\nabla f}{|\nabla f|}(x_0, y_0)$

1.11 Day 9 notes

Example 1.11.1:

$$f(x,y) = xy - x^2 y - xy^2$$
$$\nabla f(x,y) = \vec{0}$$
$$\langle y - 2x - y^2, x - x^2 - 2xy \rangle = 0$$
$$y - 2xy - y^2 = 0 \quad x - x^2 - 2xy = 0$$
$$y(1 - 2x - y) = 0 \quad x(1 - x - 2y) = 0$$

Example 1.11.2:

$$f(x, y) = x^2 - y^2$$

Critical points