

Math 120 QR

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Chapter 1

1.1 12.1 Notes (Three Dimensional Coordinate Systems)

Definition 1.1.1: Distance Formula

Defintion:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Definition 1.1.2: Equation of a sphere

Defintion: An equation of a sphere with center $C(h, k, l)$, and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2$$

In particular, if the center is the origin O , than an equation of the sphere is

$$x^2 + y^2 + z^2$$



1.2 12.2 Notes (Vectors)

Definition 1.2.1: Vector Addition

If \mathbf{u} and \mathbf{v} are vectors positioned so the initial point of \mathbf{v} is at the terminal point of \mathbf{u} , then the **sum** $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of \mathbf{u} to the terminal point of \mathbf{v} .



Definition 1.2.2: Scalar Multiplication

If c is a scalar and \mathbf{v} is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of \mathbf{v} and whose direction is the same as \mathbf{v} if $c > 0$ and is opposite to \mathbf{v} if $c < 0$. If $c = 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.



Example 1.2.1:

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector \mathbf{a} with representation \overrightarrow{AB} is:

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**Example 1.2.2:**

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

**Note:-**

Properties of vectors: If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors in V_n and c and d are scalars then

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\mathbf{a} + \mathbf{a} + -\mathbf{a} = \mathbf{0}$
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
- $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
- $(cd)\mathbf{a} = c(d\mathbf{a})$
- $l\mathbf{a} = \mathbf{a}$



1.3 12.3 Notes (Dot Product)

Definition 1.3.1: Dot Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of \mathbf{a} and \mathbf{b} is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties of the Dot Product: If \mathbf{a}, \mathbf{b} , and \mathbf{c} are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

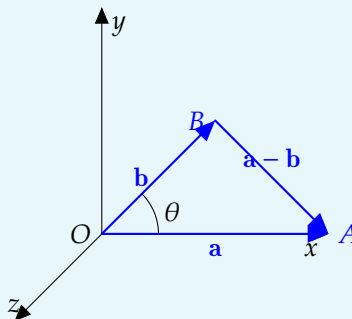


Definition 1.3.2: Geometric Definition of the Dot Product

If θ is the angle between vectors \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

Proof:



$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta$$

Corollary: If θ is the angle between nonzero vectors \mathbf{a} and \mathbf{b} , then

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$



Note:-

Two vectors \mathbf{a} and \mathbf{b} are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$



Example 1.3.1 (Direction Angles and Cosines)

The **direction angles** of a nonzero vector \mathbf{a} are the angles α , β , and γ (in the interval $[0, \pi]$) that \mathbf{a} makes with the positive x -, y -, and z -axes, respectively.

The cosines of these direction angles, $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, are called the **direction cosines** of the vector **a**. Using Corollary 6 with **b** replaced by **i**, we obtain:

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|} \quad (1)$$

Similarly, we also have:

$$\cos \beta = \frac{a_2}{|\mathbf{a}|} \quad \text{and} \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|} \quad (2)$$

By squaring the expressions in Equations 8 and 9 and adding, we see that:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad (3)$$

We can also use Equations 8 and 9 to write:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle = |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

Therefore,

$$\frac{1}{|\mathbf{a}|} \mathbf{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \quad (4)$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

Definition 1.3.3: Projections

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}| \cos \theta$, where θ is the angle between **a** and **b**. This is denoted by $\text{comp}_{\mathbf{a}} \mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leq \pi$. The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = |\mathbf{a}|(|\mathbf{b}| \cos \theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**. We summarize these ideas as follows.

Scalar projection of b onto a: $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

Vector projection of b onto a: $\text{proj}_{\mathbf{a}} \mathbf{b} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$



1.4 12.4 Notes (Cross Product)

Definition 1.4.1: Cross Product

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, suppose that a nonzero vector $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is perpendicular to both \mathbf{a} and \mathbf{b} . Then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$, and so:

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \quad (1)$$

$$b_1c_1 + b_2c_2 + b_3c_3 = 0 \quad (2)$$

To eliminate c_3 , we multiply (1) by b_3 and (2) by a_3 and subtract:

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0 \quad (3)$$

Equation (3) has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So, a solution of (3) is:

$$c_1 = a_2b_3 - a_3b_2$$

$$c_2 = a_3b_1 - a_1b_3$$

Substituting these values into (1) and (2), we then get:

$$c_3 = a_1b_2 - a_2b_1$$


This means that a vector perpendicular to both \mathbf{a} and \mathbf{b} is:

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the **cross product** of \mathbf{a} and \mathbf{b} and is denoted by $\mathbf{a} \times \mathbf{b}$. 


Definition 1.4.2: Cross Product of two vectors

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then the **cross product** of \mathbf{a} and \mathbf{b} is:

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$



Note:-

Determinant of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$


Note:-

Determinant of order 3:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$


Definition 1.4.3: Second definition of cross product

Arithmetic Definition:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = |\mathbf{a}||\mathbf{b}| \sin(\theta) \\ &= \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} i - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} j + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} k \\ &= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k \end{aligned}$$

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b}



Example 1.4.1: Proof that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b}

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1a_3b_2 - a_2a_1b_3 + a_2a_3b_1 + a_3a_1b_2 - a_3a_2b_1 \\ &= 0 \end{aligned}$$



Definition 1.4.4: sin definition of cross product

If θ is the angle between \mathbf{a} and \mathbf{b} (so $0 \leq \theta \leq \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$$

Proof:

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2 \theta \quad (\text{by Theorem 12.3.3}) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2 \theta) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2 \theta \end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2 \theta} = \sin \theta$ because $\sin \theta \geq 0$ when $0 \leq \theta \leq \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$



Note:-

Two nonzero vectors \mathbf{a} and \mathbf{b} are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = 0$$

**Example 1.4.2: Geometric interpretation of $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$**

If \mathbf{a} and \mathbf{b} are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $|\mathbf{b}| \sin(\theta)$ and area

$$A = |\mathbf{a}|(|\mathbf{b}| \sin \theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product:

The length of the cross product of $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by \mathbf{a} and \mathbf{b}

**Note:-**

If we apply the following theorem:

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both \mathbf{a} and \mathbf{b} , and

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

to the standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ using $\theta = \frac{\pi}{2}$, we obtain

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

**Note:-**

If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

**Example 1.4.3: Proof of property 5 of cross products**

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then:

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$



Definition 1.4.5: Triple Products

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ that occurs in Property 5 is called the *scalar triple product* of the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} .

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} . The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between \mathbf{a} and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}| \cos \theta$. (We must use $|\cos \theta|$ instead of $\cos \theta$ in case $\theta > \pi/2$.) Therefore, the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}| |\mathbf{a}| \cos \theta = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus, we have proved the following formula: The volume of the parallelepiped determined by the vectors \mathbf{a} , \mathbf{b} , and \mathbf{c} is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$



Note:-

If we use the formula in $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ and discover that the volume of the parallelepiped determined by \mathbf{a} , \mathbf{b} , and \mathbf{c} is 0, then the vectors must lie in the same plane; that is, they are coplanar.

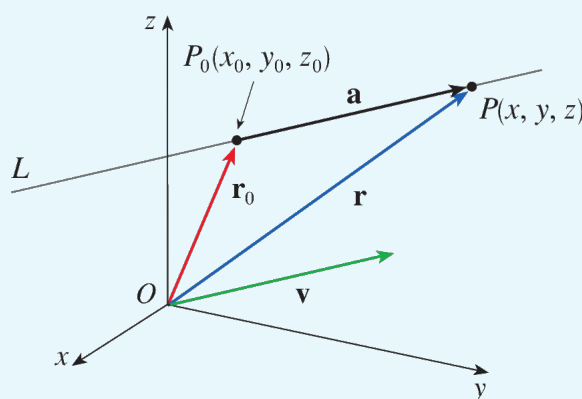


12.5 Notes (Equations of Lines and Planes)

Definition 1.5.1: Hi

Likewise, a line L in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and a direction for L , which is conveniently described by a vector \mathbf{v} parallel to the line. Let $P(x, y, z)$ be an arbitrary point on L and let \mathbf{r}_0 and \mathbf{r} be the position vectors of P_0 and P (that is, they have representations $\overrightarrow{OP_0}$ and \overrightarrow{OP}). If \mathbf{a} is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{a}.$$



Since \mathbf{a} and \mathbf{v} are parallel vectors, there is a scalar t such that $\mathbf{a} = t\mathbf{v}$. Thus

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$



Note:-

If the vector \mathbf{v} that gives the direction of the line L is written in component form as

$$\mathbf{v} = \langle a, b, c \rangle,$$

then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$. We can also write $\mathbf{r} = \langle x, y, z \rangle$ and

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle,$$

so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle.$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

**Example 1.5.1: Line example**

Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

Parametric equations are

$$x = 5 + t \quad y = 1 + 4t \quad z = 3 - 2t$$

**Note:-**

In general, if a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line L , then the numbers a , b , and c are called *direction numbers* of L . Since any vector parallel to \mathbf{v} could also be used, we see that any three numbers proportional to a , b , and c could also be used as a set of direction numbers for L .

Another way of describing a line L is to eliminate the parameter t from Equations 2. If none of a , b , or c is 0, we can solve each of these equations for t :

$$t = \frac{x - x_0}{a} \quad t = \frac{y - y_0}{b} \quad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called symmetric equations of L

**Definition 1.5.2: Line segment**

The line segment from \mathbf{r}_0 to \mathbf{r}_1 is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$



Definition 1.5.3: Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector \mathbf{n} that is orthogonal to the plane. This orthogonal vector \mathbf{n} is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane, and let \mathbf{r}_0 and \mathbf{r} be the position vectors of P_0 and P . Then the vector $\mathbf{r} - \mathbf{r}_0$ is represented by $\overrightarrow{P_0P}$. The normal vector \mathbf{n} is orthogonal to every vector in the given plane. In particular, \mathbf{n} is orthogonal to $\mathbf{r} - \mathbf{r}_0$ and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad (1.1)$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 \quad (1.2)$$

These can be referred to as the **vector equation of the plane**

To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c \rangle$, $\mathbf{r} = \langle x, y, z \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$. then the vector equation becomes:

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Expanding the left side of this equation gives the following:

A **scalar equation of the plane** through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

by collecting terms can be rewritten as:

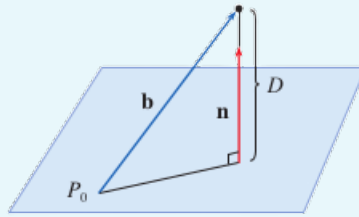
$$ax + by + cz + d = 0$$



Definition 1.5.4: Distance of a plane

In order to find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$, we let $P_0(x_0, y_0, z_0)$ be any point in the given plane and \mathbf{b} be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$



From Figure, you can see that the distance D from P_1 to the plane is equal to the absolute value of the scalar projection of \mathbf{b} onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$. Thus,

$$\begin{aligned} D &= |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$



1.6 12.6 Reading Notes (Cylinders and Quadric Surfaces)

Definition 1.6.1: Cylinder

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.



Definition 1.6.2: Quadric Surfaces

A Quadric Surface is the graph of a second-degree equation in three variables x , y , and z . The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0$$

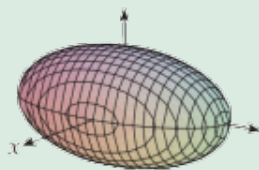
where A, B, C, \dots, J are constants, but by translation and rotation it can be brought into one of the two *standard forms*

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$



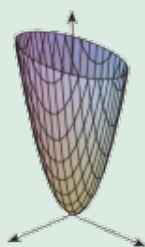
Example 1.6.1: Graphs of Quadric Surfaces PT 1

Ellipsoid:



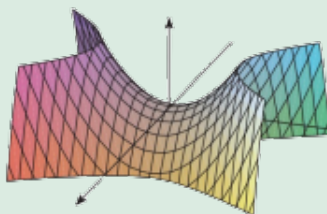
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.



$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.

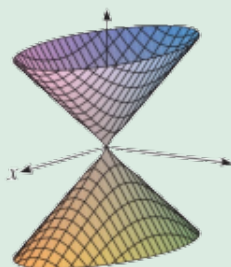


$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.

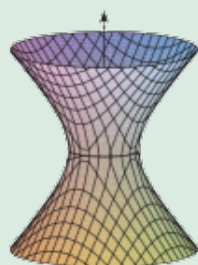


Example 1.6.2: Quadric Surfaces Pt 2



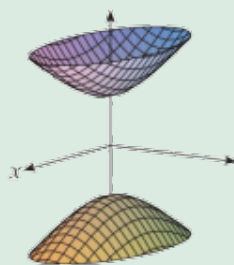
$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.



$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.



Chapter 2

2.1 13.1 Reading Notes (Vector Functions and Space Curves)

Definition 2.1.1: Vector Value Functions

A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions \mathbf{r} whose values are three-dimensional vectors. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then f , g , and h are real-valued functions called the **component functions** of \mathbf{r} and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

Definition 2.1.2: Limit of Vectors

The **limit** of a vector function \mathbf{r} is defined by taking the limits of its component functions as follows. If $(\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle)$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Definition 2.1.3: Space Curves

here is a close connection between continuous vector functions and space curves. Suppose that f , g , and h are continuous real-valued functions on an interval I . Then the set C of all points (x, y, z) in space, where

$$x = f(t) \quad y = g(t) \quad z = h(t)$$

and (t) varies throughout the interval I , is called a **space curve**. The equations in are called **parametric equations** of C and t is called a **parameter**.

2.2 13.2 Notes (Derivatives and Integrals of Vector Functions)

Definition 2.2.1: Derivatives

The derivative \mathbf{r}' of a vector function \mathbf{r} is defined in much the same way as for real-valued functions:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

Definition 2.2.2: Derivatives of vectors pt 2

The following theorem gives us a convenient method for computing the derivative of a vector function \mathbf{r} : just differentiate each component of \mathbf{r} . **Theorem** If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$



Example 2.2.1: Proof of Definition 2.2.2

$$\begin{aligned} \mathbf{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\mathbf{r}(t + \Delta t) - \mathbf{r}(t)] \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\langle f(t + \Delta t), g(t + \Delta t), h(t + \Delta t) \rangle - \langle f(t), g(t), h(t) \rangle] \\ &= \lim_{\Delta t \rightarrow 0} \left\langle \frac{f(t + \Delta t) - f(t)}{\Delta t}, \frac{g(t + \Delta t) - g(t)}{\Delta t}, \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\ &= \left\langle \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\ &= \langle f'(t), g'(t), h'(t) \rangle \end{aligned}$$

A unit vector that has the same direction as the tangent vector is called the **unit tangent vector** \mathbf{T} and is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$



Definition 2.2.3: Differentiation Rules

Proof: **Theorem** Suppose \mathbf{u} and \mathbf{v} are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

(Chain Rule)

Note:-

We use Formula 4 to prove the following theorem. **Theorem** If $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t .



Definition 2.2.4: Iteration of Vectors

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of \mathbf{r} in terms of the integrals of its component functions f , g , and h as follows. (We use the notation of Chapter 5.)

$$\begin{aligned}\int_a^b \mathbf{r}(t) dt &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \mathbf{r}(t_j^*) \Delta t \\ &= \lim_{n \rightarrow \infty} \left[\left(\sum_{i=1}^n f(t_i^*) \Delta t \right) \mathbf{i} + \left(\sum_{i=1}^n g(t_i^*) \Delta t \right) \mathbf{j} + \left(\sum_{i=1}^n h(t_i^*) \Delta t \right) \mathbf{k} \right]\end{aligned}$$

and so

$$\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function. We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(t) \Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where \mathbf{R} is an antiderivative of \mathbf{r} , that is, $\mathbf{R}'(t) = \mathbf{r}(t)$. We use the notation $\int \mathbf{r}(t) dt$ for indefinite integrals (antiderivatives).



2.3 13.3 Notes (Arc Length and Curvature)

Definition 2.3.1: Length of a space curve

Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f' , g' , and h' are continuous. If the curve is traversed exactly once as t increases from a to b , then it can be shown that its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \quad (2.1)$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \quad (2.2)$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

$$L = \int_a^b |\mathbf{r}'(t)| dt \quad (2.3)$$

because, for plane curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

A single curve C can be represented by more than one vector function. For instance the twisted cube

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \quad 1 \leq t \leq 2 \quad (2.4)$$

could also be represented by the function

$$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \quad 0 \leq u \leq \ln 2 \quad (2.5)$$

where the connection between the parameters t and u is given by $t = e^u$. We say that equations 2.4 and 2.5 are parameterizations of the curve C . If we were to use Equation 2.3 to compute the length of C using Equations 2.4 and 2.5, we would get the same answer. This is because arc length is a geometric property of the curve and hence is independent of the parametrization that is used.



Definition 2.3.2: Arc Length Function

Now we suppose that the curve C is a curve given by a vector function


$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \leq t \leq b$$

where \mathbf{r}' is continuous and C is traversed exactly once as t increases from a to b . We define its **arc length functions** by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du \quad (2.6)$$

Thus $s(t)$ is the length of part C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. If we differentiate both sides of equation 2.6 using part 1 of the Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)| \quad (2.7)$$

It is often useful to **parameterize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system of a particular parametrization. 

Definition 2.3.3: Curvature

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval I if \mathbf{r}' is continuous and $\mathbf{r}'(t) \neq 0$ on I . A curve is called smooth if it has a smooth parameterization. A smooth corner has no cusp or sharp corner; when the tangent vector turns it does so continuously.

If C is a smooth curve defined by the vector \mathbf{r} , recall that the unit tangent vector $\mathbf{T}(t)$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The curvature of a curve is

$$k = \left| \frac{d\mathbf{T}}{ds} \right| \quad (2.8)$$

where \mathbf{T} is the unit tangent vector


The curvature is easier to compute if it is expressed in terms of the parameter t instead of s , so we use the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} \Rightarrow k = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right|$$

but $ds/dt = |\mathbf{r}'(t)|$ from equation 2.7

$$k(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \quad (2.9)$$

The curvature of the curve given by the vector function \mathbf{r} is

$$k(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad (2.10) $$

Note:-

For the special case of a plane curve with equation $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = 0$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f'(x)\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \quad (2.11)$$

Note:-

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t , we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$ by Theorem 13.2.4, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that, typically, $\mathbf{T}'(t)$ is itself not a unit vector. But at any point where $\kappa \neq 0$ we can define the *principal unit normal vector* $\mathbf{N}(t)$ (or simply *unit normal*) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. The vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

Note:-

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} & \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} & \mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\ \kappa &= \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \end{aligned}$$

Definition 2.3.4: Torision

The **torsion** of a curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Torsion is easier to compute if it is expressed in terms of the parameter t instead of s , so we use the Chain Rule to write

$$\begin{aligned} \frac{d\mathbf{B}}{dt} &= \frac{d\mathbf{B}}{ds} \frac{ds}{dt} \quad \text{so} \quad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{\mathbf{B}'(t)}{|\mathbf{r}'(t)|} \\ \tau(t) &= \frac{-\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|} \end{aligned}$$

Theorem The torsion of the curve given by the vector function \mathbf{r} is

$$\tau(t) = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

2.4 13.4 Notes (Motion in Space: Velocity and Acceleration)

Chapter 3

3.1 14.1 Functions of Several Variables

Definition 3.1.1: Functions of Two Variables

Definition: A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the *domain* of f and its *range* is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.



Definition 3.1.2: Graph of a Function of Two Variables

Definition If f is a function of two variables with domain D , then the *graph* of f is the set of all points (x, y, z) in \mathbb{R}^3 such that $z = f(x, y)$ and (x, y) is in D .



Definition 3.1.3: Level Curves and Contour Maps

Definition: The *level curves* of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).



Definition 3.1.4: Functions of Three Variables

Definition: A **function of three variables**, f , is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature...



3.2 14.2 Limits and Continuity

Definition 3.2.1: Limit of Two Variable Functions

Definition: Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b) . Then we say that the *limit of $f(x, y)$ as (x, y) approaches (a, b)* is L and we write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$.



3.3 14.3 Partial Derivatives

Definition 3.3.1: Partial Derivatives

Definition Partial Derivative with respect to x

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

Definition Partial Derivative with respect to y

$$f_y(a, b) = h'(a) \quad \text{where} \quad h(x) = f(a, y)$$



Note:-

If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$



Note:-

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x , regard y as a constant and differentiate $f(x, y)$ with respect to x .
2. To find f_y , regard x as a constant and differentiate $f(x, y)$ with respect to y .



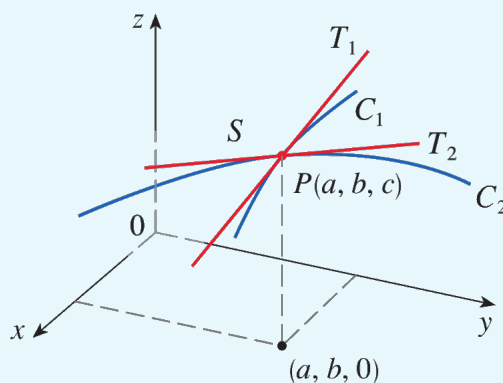
Definition 3.3.2: interpretation of Partial Derivatives

To understand partial derivatives geometrically, think of the equation $z = f(x, y)$ as representing a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on this surface.

By fixing $y = b$, we focus on the curve C_1 where the vertical plane $y = b$ intersects S . Similarly, fixing $x = a$ gives us the curve C_2 , which is where the vertical plane $x = a$ intersects S . Both curves C_1 and C_2 pass through the point P .

The curve C_1 is the graph of the function $g(x) = f(x, b)$, and the slope of its tangent at P is $f_x(a, b)$. The curve C_2 is the graph of $G(y) = f(a, y)$, and the slope of its tangent at P is $f_y(a, b)$.

Thus, the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ represent the slopes of the tangent lines at P along these curves.



Definition 3.3.3: Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the *second partial derivatives* of f . If $z = f(x, y)$, we use the following notation:

$$\begin{aligned}(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Thus the notation f_{xy} (or $\frac{\partial^2 f}{\partial y \partial x}$) means that we first differentiate with respect to x and then with respect to y , whereas in computing f_{yx} the order is reversed.



Definition 3.3.4: Clairut's Theorem

Defintion: Suppose f is defined on a disk D that contains the point (a, b) . If the functions f_{xy} and f_{yx} are both continuous on D , then

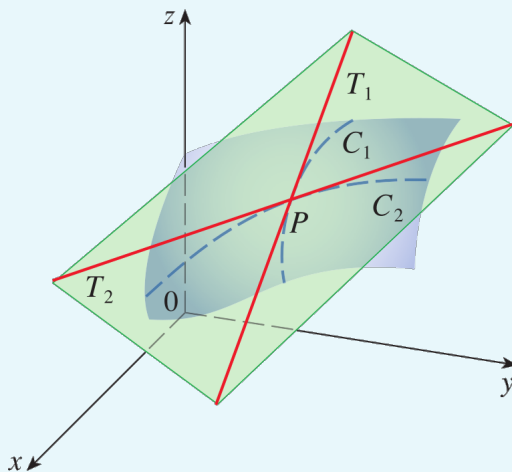
$$f_{xy}(a, b) = f_{yx}(a, b)$$



3.4 14.4 Tangent Planes and Linear Approximation

Definition 3.4.1: Tangent Planes

Let's consider a surface S given by the equation $z = f(x, y)$, where f has continuous first derivatives. Let $P(x_0, y_0, z_0)$ be a point on the surface. Two curves, C_1 and C_2 , are formed by slicing the surface with vertical planes $y = y_0$ and $x = x_0$. These curves pass through the point P . The tangent lines to C_1 and C_2 at P are denoted T_1 and T_2 . The **tangent plane** to the surface at P is the plane that contains both tangent lines T_1 and T_2 .



Definition 3.4.2: Equation of a tangent plan

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad (3.1)$$

Definition 3.4.3: Linear Approximations

If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \quad (3.2)$$

where ϵ_1 and ϵ_2 are functions of Δx and Δy such that ϵ_1 and $\epsilon_2 \rightarrow 0$ as $(\Delta x, \Delta y) \rightarrow (0, 0)$.

3.5 14.5 The Chain Rule

Definition 3.5.1: Chain Rule (Case 1)

Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \quad (3.3)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \quad (3.4)$$

Definition 3.5.2: Chain Rule (Case 2)

Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t . Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad (3.5)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \quad (3.6)$$

Definition 3.5.3: Chain Rule (General Case)

Suppose that u is a differentiable function of the n variables x_1, x_2, \dots, x_n and each x_j is a differentiable function of the m variables t_1, t_2, \dots, t_m . Then u is a function of t_1, t_2, \dots, t_m and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i} \quad (3.7)$$

for each $i = 1, 2, \dots, m$.

Definition 3.5.4: Implicit Differentiation

Defintion:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \quad (3.8)$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \quad (3.9)$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z} \quad (3.10)$$



3.6 14.6 Directional Derivatives and the Gradient Vector

Definition 3.6.1: Directional Derivative

Defintion: The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \quad (3.11)$$

if this limit exists.



Theorem 3.6.1

If f is a differentiable function of x and y , then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b \quad (3.12)$$

Definition 3.6.2: The Gradient Vector

If f is a function of two variables x and y , then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \quad (3.13)$$



Note:-

Equation 3.12 can be rewritten as

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u} \quad (3.14)$$



Definition 3.6.3: Gradient of Three Variable Functions

Defintion:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \quad (3.15)$$



Theorem 3.6.2 Maximizing Directional Derivative

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when \mathbf{u} has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Definition 3.6.4: Tangent Planes to Level Surfaces

Consider a surface S defined by $F(x, y, z) = k$, where F is a function of three variables. Let $P(x_0, y_0, z_0)$ be a point on S and C be a curve on S that passes through P . The curve is given by a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ such that $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S , the equation $F(x(t), y(t), z(t)) = k$ must hold.

By using the Chain Rule to differentiate both sides of this equation, we get:

$$\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0$$

This can be written as a dot product:

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

which means that the gradient ∇F is perpendicular to the tangent vector $\mathbf{r}'(t)$ at P .

At $t = t_0$, the gradient at P , $\nabla F(x_0, y_0, z_0)$, is normal to the tangent plane at P . The equation of the tangent plane is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \quad (3.16)$$

Note:-

Properties of the Gradient Vector

Let f be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x}) \neq 0$.

- The directional derivative of f at \mathbf{x} in the direction of a unit vector \mathbf{u} is given by $D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.
- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at \mathbf{x} , and that maximum rate of change is $|\nabla f(\mathbf{x})|$.
- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through \mathbf{x} .



3.7 14.7 Maximum and Minimum Values

Definition 3.7.1: Local Min and Max

Definition: A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) . [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b) .] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) , then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.



Theorem 3.7.1 Critical Point

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Definition 3.7.2: Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b) , and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \quad (3.17)$$

- (a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
- (b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
- (c) If $D < 0$, then (a, b) is a saddle point of f .



Definition 3.7.3: Absolute Maximums and Absolute Minimums

Let (a, b) be a point in the domain D of a function f of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of f on D if $f(a, b) \geq f(x, y)$ for all (x, y) in D .
- **absolute minimum** value of f on D if $f(a, b) \leq f(x, y)$ for all (x, y) in D .



Definition 3.7.4: Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in \mathbb{R}^2 , then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D .



Note:-

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D :

1. Find the values of f at the critical points of f in D .
2. Find the extreme values of f on the boundary of D .
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.



3.8 14.8 Lagrange Multipliers

Definition 3.8.1: Geometric Explanation Lagrange Multipliers (One Constraint)

The geometric basis of Lagrange's method for two variables involves finding the extreme values of $f(x, y)$ under the constraint $g(x, y) = k$. This means finding the extreme values of f along the level curve defined by $g(x, y) = k$. To maximize $f(x, y)$ subject to this constraint, we look for the largest value of f where its level curve touches the constraint curve at one point. At this point, the gradients of f and g are parallel, which gives the relationship $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ .

This reasoning also applies to functions of three variables. To find extreme values of $f(x, y, z)$ under the constraint $g(x, y, z) = k$, the point (x, y, z) must lie on the level surface defined by $g(x, y, z) = k$. The gradients of f and g are again parallel at the point where the maximum value of f is reached.

To make this precise, consider a curve C on the surface where $g(x, y, z) = k$. The function f has an extreme value at the point $P(x_0, y_0, z_0)$. If we parameterize the curve as $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, then f has an extreme value at t_0 when $h'(t_0) = 0$, where $h(t) = f(\mathbf{r}(t))$. Using the Chain Rule, we find that the gradients are again parallel.



Definition 3.8.2: Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]:

1. Find all values of x , y , z , and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f ; the smallest is the minimum value of f .



Definition 3.8.3: Lagrange Multipliers Two Constraints

To find the maximum and minimum values of $f(x, y, z)$ subject to two constraints, $g(x, y, z) = k$ and $h(x, y, z) = c$, we look for extreme values when (x, y, z) lies on the curve formed by the intersection of the level surfaces of g and h .

At the point where f has an extreme value, the gradient of f , ∇f , is orthogonal to the curve. The gradients of g and h are also orthogonal to this curve, which means ∇f must lie in the plane formed by ∇g and ∇h .

Thus, there exist two numbers, λ and μ , called Lagrange multipliers, such that $\nabla f(x_0, y_0, z_0)$ is a linear combination of $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$.

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

(3.18)



Chapter 4

Multiple Integrals

4.1 15.1 Double Integral Over Rectangles