

Math 120

PSet 10

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Contents

Chapter 1

Page 2

1.1 PSet 10

2

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Question 1

Find the surface area of the part of the paraboloid $x = 3y^2 + 3z^2$ satisfying $x \leq 3$.

Solution:

$$y = r \cos \theta, \quad z = r \sin \theta, \quad x = 3r^2$$

$$A = \iint_{\text{Domain}} \left\| \frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} \right\| dr d\theta$$

$$\mathbf{S}(r, \theta) = (3r^2, r \cos \theta, r \sin \theta)$$

$$\frac{\partial \mathbf{S}}{\partial r} = (6r, \cos \theta, \sin \theta), \quad \frac{\partial \mathbf{S}}{\partial \theta} = (0, -r \sin \theta, r \cos \theta)$$

$$\frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} = (r, -6r^2 \cos \theta, -6r^2 \sin \theta)$$

$$\left\| \frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} \right\| = r \sqrt{1 + 36r^2}$$

$$A = \int_0^{2\pi} \int_0^1 r \sqrt{1 + 36r^2} dr d\theta$$

$$u = 1 + 36r^2, \quad du = 72r dr, \quad r dr = \frac{du}{72}$$

$$\int_{r=0}^{r=1} r \sqrt{1 + 36r^2} dr = \frac{1}{72} \int_{u=1}^{u=37} \sqrt{u} du = \frac{1}{72} \left[\frac{2}{3} u^{3/2} \right]_{u=1}^{u=37}$$

$$\frac{1}{72} \left[\frac{2}{3} u^{3/2} \right]_{u=1}^{u=37} = \frac{1}{108} (37^{3/2} - 1)$$

$$A = \frac{\pi}{54} (37^{3/2} - 1)$$

$$37^{3/2} = \sqrt{37^3} = 37\sqrt{37}$$

$$A = \frac{\pi}{54} (37\sqrt{37} - 1)$$

Question 2

In this problem, we'll find the surface area of the part of the cylinder $x^2 + y^2 = 9$ that lies between the planes $x + y + z = -6$ and $x + y + z = 5$.

- Find a parametrization $\vec{r}_1(u)$ of the curve C_1 of intersection of the plane $x + y + z = 5$ with the cylinder $x^2 + y^2 = 9$. Also find a parametrization $\vec{r}_2(u)$ of the curve C_2 of intersection of the plane $x + y + z = -6$ with the cylinder.
- Write down a parametrization $\vec{s}(u, v)$ of the part of the cylinder that lies between the two planes. The curves C_1 and C_2 should be two grid curves of the parametrization, and the bounds on the parameters should be of the form $a \leq v \leq b$ and $c \leq u \leq d$ for constants a , b , c , and d .
- Use the parametrization you found to calculate the surface area of the part of the cylinder that lies between the two planes.
- Does your answer from (c) make sense? Give an intuitive geometric reason that the surface area you found is the same as the surface area of a cylinder of radius 3 and height 11.

Solution:

(a)

$$\begin{aligned} x &= 3 \cos u, & y &= 3 \sin u \\ z &= 5 - x - y = 5 - (3 \cos u + 3 \sin u) \\ \vec{r}_1(u) &= (3 \cos u, 3 \sin u, 5 - 3 \cos u - 3 \sin u) \\ z &= -6 - x - y = -6 - (3 \cos u + 3 \sin u) \\ \vec{r}_2(u) &= (3 \cos u, 3 \sin u, -6 - 3 \cos u - 3 \sin u) \end{aligned}$$

(b)

$$\begin{aligned} \vec{s}(u, v) &= (3 \cos u, 3 \sin u, v - 3 \cos u - 3 \sin u) \\ 0 &\leq u \leq 2\pi, & -6 &\leq v \leq 5 \end{aligned}$$

(c)

$$\begin{aligned} \frac{\partial \vec{s}}{\partial u} &= (-3 \sin u, 3 \cos u, 3 \sin u - 3 \cos u) \\ \frac{\partial \vec{s}}{\partial v} &= (0, 0, 1) \\ \frac{\partial \vec{s}}{\partial u} \times \frac{\partial \vec{s}}{\partial v} &= (3 \cos u, 3 \sin u, 0) \\ \left\| \frac{\partial \vec{s}}{\partial u} \times \frac{\partial \vec{s}}{\partial v} \right\| &= \sqrt{(3 \cos u)^2 + (3 \sin u)^2} = 3 \\ A &= \int_{v=-6}^5 \int_{u=0}^{2\pi} 3 \, du \, dv = 3 \cdot \left(\int_{u=0}^{2\pi} du \right) \cdot \left(\int_{v=-6}^5 dv \right) = 3 \cdot 2\pi \cdot 11 = 66\pi \end{aligned}$$

(d)

$$A = 2\pi r h = 2\pi \times 3 \times 11 = 66\pi$$

Question 3

Evaluate the surface integral

$$\iint_S (x^2 + y^2 + z^2) dS$$

where S is the surface of the solid cylinder defined by the inequalities $x^2 + z^2 \leq 1$ and $0 \leq y \leq 5$. Note that S consists of a hollow cylinder and two disks.

Solution:

$$I = \iint_S (x^2 + y^2 + z^2) dS$$

$$x^2 + z^2 \leq 1, \quad 0 \leq y \leq 5$$

$$S = S_{\text{cyl}} \cup S_{\text{top}} \cup S_{\text{bot}}$$

$$\text{For } S_{\text{cyl}}, \quad x = \cos \theta, \quad z = \sin \theta, \quad y = y, \quad \theta \in [0, 2\pi), \quad y \in [0, 5]$$

$$\vec{r}(\theta, y) = (\cos \theta, y, \sin \theta)$$

$$\vec{r}_\theta = (-\sin \theta, 0, \cos \theta), \quad \vec{r}_y = (0, 1, 0)$$

$$\vec{r}_\theta \times \vec{r}_y = (-\cos \theta, 0, -\sin \theta), \quad |\vec{r}_\theta \times \vec{r}_y| = 1$$

$$dS_{\text{cyl}} = d\theta dy$$

$$x^2 + y^2 + z^2 = \cos^2 \theta + y^2 + \sin^2 \theta = 1 + y^2$$

$$I_{\text{cyl}} = \int_0^{2\pi} \int_0^5 (1 + y^2) dy d\theta$$

$$\int_0^5 (1 + y^2) dy = \int_0^5 1 dy + \int_0^5 y^2 dy = [y]_0^5 + \left[\frac{y^3}{3} \right]_0^5 = 5 + \frac{125}{3} = \frac{140}{3}$$

$$I_{\text{cyl}} = \int_0^{2\pi} \frac{140}{3} d\theta = \frac{140}{3} \cdot 2\pi = \frac{280\pi}{3}$$

$$\text{For } S_{\text{top}}, \quad x = r \cos \theta, \quad z = r \sin \theta, \quad y = 5, \quad r \in [0, 1], \quad \theta \in [0, 2\pi)$$

$$dS_{\text{top}} = r dr d\theta$$

$$x^2 + y^2 + z^2 = r^2 + 25$$

$$I_{\text{top}} = \int_0^{2\pi} \int_0^1 (r^2 + 25)r dr d\theta$$

$$\int_0^1 (r^2 + 25)r dr = \int_0^1 (r^3 + 25r) dr = \left[\frac{r^4}{4} + \frac{25r^2}{2} \right]_0^1 = \frac{1}{4} + \frac{25}{2} = \frac{51}{4}$$

$$I_{\text{top}} = \int_0^{2\pi} \frac{51}{4} d\theta = \frac{51}{4} \cdot 2\pi = \frac{51\pi}{2}$$

For S_{bot} , $x = r \cos \theta$, $z = r \sin \theta$, $y = 0$

$$x^2 + y^2 + z^2 = r^2$$

$$dS_{\text{bot}} = r \, dr \, d\theta$$

$$I_{\text{bot}} = \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta$$

$$\int_0^1 r^3 \, dr = \left[\frac{r^4}{4} \right]_0^1 = \frac{1}{4}$$

$$I_{\text{bot}} = \int_0^{2\pi} \frac{1}{4} \, d\theta = \frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$$

$$I = I_{\text{cyl}} + I_{\text{top}} + I_{\text{bot}} = \frac{280\pi}{3} + \frac{51\pi}{2} + \frac{\pi}{2}$$

$$I = \frac{560\pi}{6} + \frac{153\pi}{6} + \frac{3\pi}{6} = \frac{716\pi}{6} = \frac{358\pi}{3}$$

$$\boxed{\frac{358\pi}{3}}$$

Question 4

Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2 + y^2}$, $1 \leq z \leq 4$, if its density function is $\rho(x, y, z) = z + 2$.

Solution:

$$z = \sqrt{x^2 + y^2}, \quad 1 \leq z \leq 4$$

$$\rho(x, y, z) = z + 2$$

$$\begin{cases} x = z \cos \theta \\ y = z \sin \theta \\ z = z \end{cases}$$

$$\mathbf{r}(\theta, z) = (z \cos \theta, z \sin \theta, z)$$

$$\mathbf{r}_\theta = (-z \sin \theta, z \cos \theta, 0)$$

$$\mathbf{r}_z = (\cos \theta, \sin \theta, 1)$$

$$\mathbf{N} = \mathbf{r}_\theta \times \mathbf{r}_z = (z \cos \theta, z \sin \theta, -z)$$

$$|\mathbf{N}| = \sqrt{(z \cos \theta)^2 + (z \sin \theta)^2 + (-z)^2} = z\sqrt{2}$$

$$dS = z\sqrt{2} \, d\theta \, dz$$

$$\begin{aligned}
 M &= \int_{\theta=0}^{2\pi} \int_{z=1}^4 \rho(z) dS \\
 M &= \int_{z=1}^4 \int_{\theta=0}^{2\pi} (z+2)z\sqrt{2} d\theta dz \\
 &\quad \int_{\theta=0}^{2\pi} d\theta = 2\pi \\
 M &= 2\pi\sqrt{2} \int_{z=1}^4 (z+2)z dz \\
 M &= 2\pi\sqrt{2} \int_{z=1}^4 (z^2 + 2z) dz \\
 &\quad \int (z^2 + 2z) dz = \frac{1}{3}z^3 + z^2 \\
 &\quad \left[\frac{1}{3}(4)^3 + (4)^2 \right] - \left[\frac{1}{3}(1)^3 + (1)^2 \right] \\
 &\quad \left(\frac{64}{3} + 16 \right) - \left(\frac{1}{3} + 1 \right) \\
 &\quad 36 \\
 M &= 2\pi\sqrt{2} \times 36
 \end{aligned}$$

$$M = 72\pi\sqrt{2}$$

Question 5

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}, \quad \text{where } \vec{F} = \langle x, y, 2z \rangle$$

and S is the part of the paraboloid $z = 4 - x^2 - y^2$, oriented downwards, that lies above the unit square $[0, 1] \times [0, 1]$.

Solution:

$$\begin{aligned}
 &\iint_S \vec{F} \cdot d\vec{S}, \quad \vec{F} = \langle x, y, 2z \rangle \\
 &\quad z = 4 - x^2 - y^2 \\
 &\quad \vec{n} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1 \right) \\
 &\quad \frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = -2y \\
 &\quad \vec{n} = (-2x, -2y, -1) \\
 &\quad d\vec{S} = \vec{n} dx dy = (-2x, -2y, -1) dx dy \\
 &\quad \vec{F} = (x, y, 2z) = (x, y, 2(4 - x^2 - y^2))
 \end{aligned}$$

$$\vec{F} \cdot d\vec{S} = (x, y, 2z) \cdot (-2x, -2y, -1) dx dy$$

$$\vec{F} \cdot d\vec{S} = [-2x^2 - 2y^2 - 2(4 - x^2 - y^2)] dx dy$$

$$\vec{F} \cdot d\vec{S} = [-2x^2 - 2y^2 - 8 + 2x^2 + 2y^2] dx dy = -8 dx dy$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-8) dx dy = -8 \iint_D dx dy$$

$$\iint_S \vec{F} \cdot d\vec{S} = -8 \times 1 = -8$$

$$\iint_S \vec{F} \cdot d\vec{S} = -8$$

Question 6

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}, \quad \text{where } \vec{F} = \langle -z, x, y \rangle$$

and S is the part of the unit sphere $x^2 + y^2 + z^2 = 1$ in the first octant, oriented upwards.

Solution:

$$\vec{F} = \langle -z, x, y \rangle$$

$$x = \sin \phi \cos \theta, \quad y = \sin \phi \sin \theta, \quad z = \cos \phi$$

$$0 \leq \phi \leq \frac{\pi}{2}, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$\vec{r}_\phi = \frac{\partial \vec{r}}{\partial \phi} = \begin{pmatrix} \cos \phi \cos \theta \\ \cos \phi \sin \theta \\ -\sin \phi \end{pmatrix}, \quad \vec{r}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \begin{pmatrix} -\sin \phi \sin \theta \\ \sin \phi \cos \theta \\ 0 \end{pmatrix}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\ -\sin \phi \sin \theta & \sin \phi \cos \theta & 0 \end{vmatrix}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{pmatrix} \sin^2 \phi \cos \theta \\ \sin^2 \phi \sin \theta \\ \cos \phi \sin \phi \end{pmatrix}$$

$$d\vec{S} = \begin{pmatrix} \sin^2 \phi \cos \theta \\ \sin^2 \phi \sin \theta \\ \cos \phi \sin \phi \end{pmatrix} d\phi d\theta$$

$$\vec{F} = \langle -z, x, y \rangle = \langle -\cos \phi, \sin \phi \cos \theta, \sin \phi \sin \theta \rangle$$

$$\vec{F} \cdot d\vec{S} = (-\cos \phi) (\sin^2 \phi \cos \theta) + (\sin \phi \cos \theta) (\sin^2 \phi \sin \theta) + (\sin \phi \sin \theta) (\cos \phi \sin \phi)$$

$$\vec{F} \cdot d\vec{S} = \cos \phi \sin^2 \phi (-\cos \theta + \sin \theta) + \sin^3 \phi \cos \theta \sin \theta$$

$$\int_0^{\frac{\pi}{2}} (-\cos \theta + \sin \theta) d\theta = 0, \quad \int_0^{\frac{\pi}{2}} \sin 2\theta d\theta = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^3 \phi \, d\phi = \frac{4}{3}$$

$$\iint_S \vec{F} \cdot d\vec{S} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$$

$$\boxed{\frac{2}{3}}$$

Question 7

Find the flux of $\vec{F}(x, y, z) = z\hat{i} + y\hat{j} + x\hat{k}$ across the helicoid

$$\vec{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v \leq 2\pi,$$

oriented upward.

Question 8

Let S be the part of the elliptical cylinder $y^2 + 4z^2 = 4$ that lies above the xy -plane and between the planes $x = -2$ and $x = 2$. Let S have the upward orientation; that is, let S be oriented so that the normal vectors have positive z -component.

- (a) Find a parameterization of S .
- (b) Does your parameterization match the given orientation of S ? Explain.
- (c) Let \vec{F} be the vector field

$$\vec{F}(x, y, z) = e^{x^2 y^2 z^2} \hat{i} + x^2 y \hat{j} + z^2 e^{x/5} \hat{k}.$$

Find the flux of \vec{F} across the oriented surface S .