Math 120

PSet 4

Sep 22 2024

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Chapter 1

1.1 PSet 4

Question 1

By finding both $f_{xy} = (f_x)_y$ and $f_{yx} = (f_y)_x$, verify that Clairaut's Theorem holds for the function $f(x, y) = y \arctan(xy)$.

Solution:

$$f_x = \frac{\partial}{\partial x} y \arctan(xy) = y \cdot \frac{1}{1 + (xy)^2} \cdot y = \frac{y^2}{1 + (xy)^2}$$

$$\frac{\partial}{\partial y} \frac{y^2}{1 + (xy)^2} = \frac{(2y)\left(1 + (xy)^2\right) - \left(y^2\right)\left(2x^2 y\right)}{\left(1 + \left(xy\right)^2\right)^2}$$

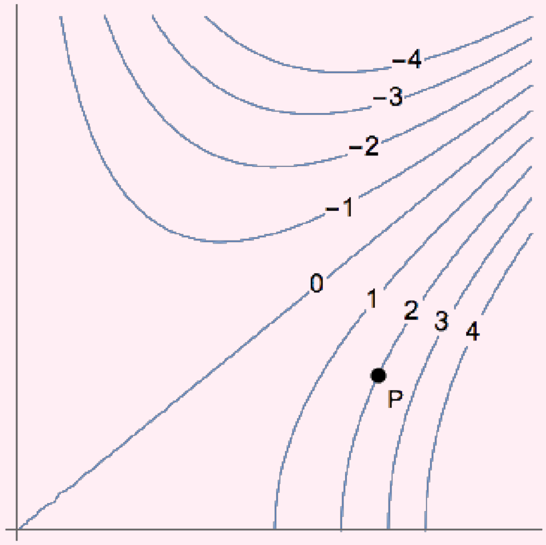
$$\frac{(2y)\left(1 + (xy)^2\right) - \left(y^2\right)\left(2x^2 y\right)}{\left(1 + \left(xy\right)^2\right)} = \frac{2y}{\left(1 + (xy)^2\right)}$$

$$f_y = \frac{\partial}{\partial y} y \arctan(xy) = \arctan(xy) + \frac{xy}{1 + (xy)^2}$$

$$\frac{\partial}{\partial x} \arctan(xy) + \frac{xy}{1 + (xy)^2} = \frac{y}{\left(1 + (xy)^2\right)} + \frac{y\left(1 + (xy)^2\right) - \left(2y^2 x\right)(xy)}{\left(1 + \left(xy\right)^2\right)^2} = \frac{2y}{\left(1 + (xy)^2\right)}$$

Question 2

Level curves are shown below for a function f



(a) Determine the signs of f_x, f_y, f_{xx} and f_{yy} at the point P. Explain your reasoning. *You should assume that the undrawn level curves are nicely and evenly distributed between the ones drawn.*

(b) Mark a point on the contour plot where $f_x = 0$. (You can either mark the point on a screenshot and insert the picture in your homework file, or just make a rough copy of the contour plot by hand.)

Solution:

a)

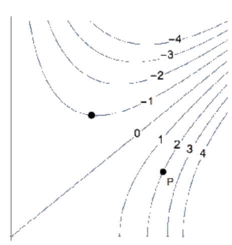
$f_x > 0$ as you move to the right of P f increases .

$f_y < 0$ as you go up in the y direction at point p there is a decrease in the value of f.

$f_{xx} > 0$ The level curves in the x-direction appear to be getting closer together as we move to the right, so it means that f_x is increasing.

$f_{yy} > 0$ the difference in teh level cirves is decreasing as you move up so it means that f_y is increasing

b)



Question 3

Use the Chain Rule to find the indicated partial derivatives.

(a) Compute $\frac{dz}{dt}$ if $z = \tan(y/x)$, $x = e^t$, and $y = 1 - e^{-t}$.

(b) Compute $\frac{\partial M}{\partial u}$ and $\frac{\partial M}{\partial v}$ at $u = 3$ and $v = -1$ if $M = xe^{y-z^2}$, $x = 2uv$, $y = u - v$, and $z = u + v$.

Solution:

a)

$$\frac{dz}{dt} = \frac{dz}{dx} \cdot \frac{dx}{dt} + \frac{dz}{dy} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dx} = z_x = \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{(0)x - (1)(y)}{x^2}\right) \Rightarrow \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{-y}{x^2}\right)$$

$$\frac{dz}{dy} = z_y = \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{(1)x - (0)(y)}{x^2}\right) \Rightarrow \sec\left(\frac{y}{x}\right)^2 \cdot \left(\frac{1}{x}\right)$$

$$\frac{dx}{dt} = e^t \quad \frac{dy}{dt} = e^{-t}$$

$$\frac{dz}{dt} = \sec\left(\frac{1-e^{-t}}{e^t}\right)^2 \cdot \left(\frac{-(1-e^{-t})}{e^{2t}}\right) \cdot e^t + \sec\left(\frac{1-e^{-t}}{e^t}\right)^2 \cdot \left(\frac{1}{e^t}\right) \cdot e^{-t}$$

b)

$$\frac{\partial M}{\partial u} = \frac{\partial M}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial M}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial M}{\partial z} \cdot \frac{dz}{dt}$$

$$M_x = e^{y-z^2} \quad \frac{dx}{du} = 2v \quad \frac{dx}{dv} = 2u$$

$$M_y = xe^{y-z^2} \quad \frac{dy}{du} = 1 \quad \frac{dy}{dv} = -1$$

$$M_z = -2zxe^{y-z^2} \quad \frac{dz}{du} = 1 \quad \frac{dz}{dt} = 1$$

$$\frac{\partial M}{\partial u} = e^{y-z^2}(2v) + \left(xe^{y-z^2}\right)(1) + \left(-2zxe^{y-z^2}\right)(1)$$

$$x = 2uv \Rightarrow 2(3)(-1) = -6$$

$$y = u - v \Rightarrow (3) - -(1) = 4$$

$$z = u + v \Rightarrow 3 + (-1) = 2$$

$$\frac{\partial M}{\partial u} = e^{4-2^2}(2(-1)) + \left((-6)e^{4-2^2}\right)(1) + \left(-2(2)(-6)e^{4-2^2}\right)(1) = 1(-2) + (-6)(1) + (24)(1) = 16$$

$$\frac{\partial M}{\partial v} = \frac{\partial M}{\partial x} \cdot \frac{dx}{dv} + \frac{\partial M}{\partial y} \cdot \frac{dy}{dv} + \frac{\partial M}{\partial z} \cdot \frac{dz}{dv}$$

$$\frac{\partial M}{\partial v} = e^{y-z^2}(2u) + \left(xe^{y-z^2}\right)(-1) + \left(-2zxe^{y-z^2}\right)(1)$$

$$\frac{\partial M}{\partial v} = e^{4-2^2}(2(3)) + \left((-6)e^{4-2^2}\right)(-1) + \left(-2(2)(-6)e^{4-2^2}\right)(1) = 1(6) + (-6)(1) + (24)(1) = 36$$

Question 4

Use implicit differentiation to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the surface given by $z^3 + 3xyz + x^2 + y^2 = 0$ at the point $(1, -2, 1)$.

Solution:

$$\frac{\partial z}{\partial x}\left(z^3+3xyz+x^2+y^2\right)=3z^2\frac{\partial z}{\partial x}+3yz+3xy\frac{\partial z}{\partial x}+2x=0$$

$$3z^2\frac{\partial z}{\partial x}+3yz+3xy\frac{\partial z}{\partial x}+2x=0 \Rightarrow 3z^2\frac{\partial z}{\partial x}+3xy\frac{\partial z}{\partial x}=-3yz-2x$$

$$3x^2\frac{\partial z}{\partial x}+3xy\frac{\partial z}{\partial x}=-3yz-2x \Rightarrow \frac{\partial z}{\partial x}\left(3z^2+3xy\right)=-3yz-2x$$

$$\frac{\partial z}{\partial x}\left(3z^2+3xy\right)=-3yz-2x \Rightarrow \frac{\partial z}{\partial x}=\frac{-3yz-2x}{3z^2+3xy}$$

$$\frac{\partial z}{\partial x}=\frac{-3(-2)(1)-2(1)}{3(1)^2+3(1)(-2)}=\frac{-4}{3}$$

$$\frac{\partial z}{\partial y}\left(z^3+3xyz+x^2+y^2\right)=3z^2\frac{\partial z}{\partial y}+3xz+3xy\frac{\partial z}{\partial y}+2y=0$$

$$3z^2\frac{\partial z}{\partial y}+3xz+3xy\frac{\partial z}{\partial y}+2y=0 \Rightarrow 3z^2\frac{\partial z}{\partial y}+3xy\frac{\partial z}{\partial x}=-3xz-2y$$

$$3z^2\frac{\partial z}{\partial y}+3xy\frac{\partial z}{\partial y}=-3xz-2y \Rightarrow \frac{\partial z}{\partial y}\left(3z^2+3xy\right)=-3xz-2y$$

$$\frac{\partial z}{\partial y}\left(3z^2+3xy\right)=-3yz-2y \Rightarrow \frac{\partial z}{\partial x}=\frac{-3xz-2y}{3z^2+3xy}$$

$$\frac{\partial z}{\partial y}=\frac{-3(1)(1)-2(-2)}{3(1)^2+3(1)(-2)}=-\frac{1}{3}$$

Question 5

[(5.) (Stewart problem 14.5.36)] Wheat production W in a given year depends on the average temperature T and annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.15°C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that at current production levels, $\partial W/\partial T=-2$ and $\partial W/\partial R=8$.

(a) What is the significance of the signs of these partial derivatives?

(b) Estimate the current rate of change of wheat production $\frac{dW}{dt}$.

Solution:

a)
$\frac{\partial W}{\partial T}=-2$ means that for every drop in 1 °C there is a decrease if Wheat production of 2 units.
If the sign were positive it would mean that there would an increase in wheat production as a result in the decrease of temperature.
$\frac{\partial W}{\partial R}$ means for every increase in 1 cm rainfall there is an increases of 8 units in wheat production.
If the sign were negative instead it would mean that increase in rainfall brought down wheat production. b)

$$\frac{dW}{dt}=\frac{\partial W}{\partial T}\cdot\frac{dT}{dt}+\frac{\partial W}{\partial R}+\frac{dR}{dt}=-2(.15)+8(-.1)=1.1$$