

Math 120 QR

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Chapter 1

1.1 Day 1 notes

Definition 1.1.1: Distance Formula

Defintion:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



Definition 1.1.2: Equation of a sphere

Defintion: An equation of a sphere with center $C(h, k, l)$, and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2$$

In particular, if the center is the origin O , than an equation of the sphere is

$$x^2 + y^2 + z^2$$



1.2 Day 2 Notes

Definition 1.2.1: The lenght/magnitude of a vecotr

In 2D, $\vec{v} = \langle a, b \rangle$: $|\vec{v}| = \sqrt{a^2 + b^2}$

In 3D, $\vec{v} = \langle a, b, c \rangle$: $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$

A unit vector is a vector of length 1



Question 1

If \vec{v} is a vector and a is a scalar, then what is $|a\vec{v}|$

Solution:

$$|a\vec{v}| = |a||\vec{v}|$$

Definition 1.2.2: Vectors in \mathbf{R}^3

The standard basis vectors in \mathbf{R}^3 are

$$i = \langle 1, 0, 0 \rangle$$

$$j = \langle 0, 1, 0 \rangle$$

$$k = \langle 0, 0, 1 \rangle$$



Question 2

What is special about i, j, k ?

Solution:

- Cannot make any of them as a linear combination of the other three.
- Any vector $\vec{v} \in \mathbf{R}^3$ can be written uniquely as a linear combination of i, j, k

Example 1.2.1:

$$\vec{v} + \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\vec{v} + \vec{u} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$



Definition 1.2.3: Dot product

A dot product between v and w is:

In 2D: $v \cdot w = v_1 w_1 + v_2 w_2$

In 3D: $v \cdot w = v_1 w_1 + v_2 w_2 + v_3 w_3$

Geometric definition:

$|u \cdot w| = |u||w| \cos(\theta)$ where θ is the angle between v and w



Example 1.2.2: Why the 2 definitions are the same

$$v_1 w_1 + v_2 w_2 = |v||w| \cos(\theta) = p|v|$$

$$p = |w| \cos(\theta)$$

$$v_1 = |v| \cos(\theta)$$

$$v_2 = |v| \sin(\theta)$$

$$w_1 = |w| \cos(\theta)$$

$$w_2 = |w| \sin(\theta)$$

$$\text{LHS} = |v||w| (\cos(\alpha) \cos(\beta) + \sin(\alpha) \sin(\beta))$$

$$\text{LHS} = |v||w| \sin(\alpha + \beta)$$

$$\text{LHS} = |v||w| \cos(\theta) = \text{RHS}$$



Example 1.2.3: what does the def mean

How much effect of \vec{w} act along \vec{v}

Work: $w = \vec{F} \cdot \vec{S}$



Question 3

Find a relation between $|v|$ and $v \cdot v$

Solution: $|v|^2 = v \cdot v$

Question 4

v, w of fixed lengths when is $v \cdot w$ largest?

Solution: v parallel: $\theta = 0$
 $\cos(\theta) = 1$

Example 1.2.4: Projections

Given $\vec{v} \neq \vec{0}$

The project of w on v is $\text{proj } w = \left(\frac{\vec{w} \cdot \vec{v}}{|\vec{v}|} \right) \frac{\vec{v}}{|\vec{v}|}$

Direction of v is $\frac{\vec{v}}{|\vec{v}|}$

Dir project of w with direction is:

$$\vec{w} \cdot \frac{\vec{v}}{|\vec{v}|}$$



Question 5

TRUE or False:

u, v, w : vectors $(u \cdot v)w = u(v \cdot w)$

Solution: false

Question 6

TRUE or FALSE:

$|v - w| = |v| - |w|$ if $v \parallel w$

Solution: false

Question 7

When is this ideal square sum happening?

$$|v + w|^2 = |v|^2 + |w|^2$$

Solution: when $v \perp w$

Definition 1.2.4: Cross Product

The cross product of two vectors v and w , $v \times w$ is a vector u defined by $u \perp v$ and $u \perp w$.

Direction of u is given by the right hand rule

Magnitude: $|u| = \text{Area of the parallelogram spanned by } v \text{ and } w$.



1.3 Day 2 Reading notes

Definition 1.3.1: Vector Addition

If \mathbf{u} and \mathbf{v} are vectors positioned so the initial point of \mathbf{v} is at the terminal point of \mathbf{u} , then the **sum** $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of \mathbf{u} to the terminal point of \mathbf{v} .



Definition 1.3.2: Scalar Multiplication

If c is a scalar and \mathbf{v} is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of \mathbf{v} and whose direction is the same as \mathbf{v} if $c > 0$ and is opposite to \mathbf{v} if $c < 0$ or $\mathbf{v} = \mathbf{0}$, then $c\mathbf{v} = \mathbf{0}$.



Example 1.3.1:

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector \mathbf{a} with representation \overrightarrow{AB} is:

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$



Example 1.3.2:

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$



Note:-

Properties of vectors: If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors in V_n and c and d are scalars then

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\mathbf{a} + \mathbf{a} + -\mathbf{a} = \mathbf{0}$

- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
- $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
- $(cd)\mathbf{a} = c(d\mathbf{a})$
- $l\mathbf{a} = \mathbf{a}$



1.4 Day 3 Reading notes

Example 1.4.1:

∈



1.5 Day 3 Class notes

Example 1.5.1:

Use the geometric def of cross product to calculate:

$$\mathbf{i} \times (\mathbf{i} + \mathbf{j})$$

$$(\mathbf{i} + \mathbf{j}) \times (\mathbf{i} - \mathbf{j})$$



Definition 1.5.1: Second definition of dot product

Arithmetic Definition:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = |\mathbf{a}||\mathbf{b}| \sin(\theta)$$

$$\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \mathbf{i} - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} \mathbf{j} - \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \mathbf{k} \\ = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$|\mathbf{a} \times \mathbf{b}|^2 = (a_2b_3 - a_3b_2)^2 - (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2$$

$$|\mathbf{a} \times \mathbf{b}|^2 = (a_2b_3 - a_3b_2)^2 - (a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2$$



Question 8

Calculate the cross product of $\mathbf{v} = \langle -1, 3, 4 \rangle$ and $\mathbf{w} = \langle 2, 1, -2 \rangle$

Solution:

$$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & 4 \\ 2 & 1 & -2 \end{bmatrix} = 10\mathbf{i} + 6\mathbf{j} - 7\mathbf{k}$$

Question 9

Calculate the area of the triangle with vertices P(1,0,1), Q (-3, 4, 0) and R(2,1,0)

Solution:

$$\overrightarrow{RP} = \langle -1, -1, 1 \rangle$$

$$\overrightarrow{RQ} = \langle 5, 0, 0 \rangle$$

$$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{bmatrix} i & j & k \\ -1 & -1 & 1 \\ -5 & 0 & 0 \end{bmatrix} = -5j - 5k$$

$$\text{area of } \delta PQR = \frac{1}{2} |\overrightarrow{RP} \times \overrightarrow{RQ}| = \frac{1}{2} \sqrt{5^2 + 5^2} = \frac{5\sqrt{2}}{2}$$

Question 10

What is special about: P(1,4,2), 1(3,3,5), R(9,1,11), and S(5, 11, 9)

Solution: If $(\overrightarrow{AC} \times \overrightarrow{AB} \parallel (\overrightarrow{AD} \times \overrightarrow{AC}))$ then A, B, C, D are on the same plane

Question 11

1. If $u \times w = u \times v$ then $w = v$
2. If $u \cdot v = 0$ and $u \times v = \vec{0}$ then either $u = \vec{0}$ or $v = \vec{0}$

Solution:

1. False

$$u \times (u + w) = u \times w$$

2. True

The reason is because the dot product is $|u||v| \cos(\theta)$ and the cross product is $|u||v| \sin(\theta)$

Question 12

Find the distance of a point p to the line passing through A and B

Solution:

$$= \frac{1}{2} \text{dist}(P, AB) \cdot |AB|$$

$$\text{dist}(P, AB) = \frac{|\overrightarrow{AB} \times \overrightarrow{AP}|}{|\overrightarrow{AB}|}$$

Note:-

Line in 2D equation: $ax + by = c$

Line in 3D: Equation of a line passing through (x_0, y_0, z_0) and parallel to $\vec{v} = \langle a, b, c \rangle$ is

- $x = x_0 + at$
- $y = y_0 + bt$
- $z = z_0 + ct$



Note:-

Equation of a plane passing through (x_0, y_0, z_0) and orthogonal to $\vec{n} = \langle a, b, c \rangle$ is $(r - r_0) \cdot n = 0$
 $\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$
 $ax + by + cz + d = 0$

**Question 13**

What is the normal of the plane $3x + z + 2 = 0$?

Solution:

$$\vec{n} = \langle 3, 0, 1 \rangle$$

Question 14

Is the line $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$ parallel to the plane $x - 2y + z = 7$?

1.6 Integration Review

Example 1.6.1:

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} dx \\ v = 1 - x^2 \\ dv = -2x dx \\ \int \frac{-\frac{1}{2}}{\sqrt{v}} dv \\ = \frac{1}{2} \int \frac{1}{\sqrt{v}} \end{aligned}$$

**Example 1.6.2:**

$$\begin{aligned} (3x^2 5) e^{x^3+5x} dx \\ u = x^3 + 5x \\ du = 3x^2 5 dx \\ = \int e^u du \\ = e^{x^3+5x} c \end{aligned}$$



Example 1.6.3:

$$\begin{aligned} & \cos^5(2t)dt \\ & u = 2t \\ & du = 2dt \\ & = \frac{1}{2} \cos^3(u)du = \frac{1}{2} \cos^4(u) + \cos(u)du \\ & = \frac{1}{2} \int (1 - \sin^2(u))^2 \cdot \cos(u)du \\ & v = \sin(u) \\ & dv = \cos(u)du \end{aligned}$$



Example 1.6.4:

$$\begin{aligned} & xe^{2x}dx \\ & x' = 1 \\ & \int e^{2x}dx = \frac{1}{2}e^{2x} + c \\ & \int vdv = uv + \int vdu \\ & v = x \rightarrow dv = 1dx \\ & dv = e^{2x}dx \rightarrow v = \frac{1}{2}e^{2x} \\ & \int = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx \end{aligned}$$



Example 1.6.5:

$$\begin{aligned} & \int t \sin(2t)dt \\ & t' = 1 \\ & \int \sin(2t)dt = -\frac{1}{2} \cos(2t) \\ & \int vdv = uv + \int vdv \\ & v = t \rightarrow dv = 1dt \\ & dv = \sin(2t)dx \rightarrow v = -\frac{1}{2} \cos(2t) \\ & \int = -\frac{1}{2} \cos(2t) - \int \frac{1}{2}e^{2x}dx \end{aligned}$$



1.7 Day 4 Class Notes

Example 1.7.1: Warm up 1

True or False

If Q_1R are not on P and $\vec{n} \cdot \overrightarrow{QR} = 0$ then Q_1R are on the same side of (P). True: Since the dot product of the normal and \overrightarrow{QR} are 0 then it is parallel to the plane



Example 1.7.2: Warm up 2

What is the shape (in 3D) of

$$\text{a) } x^2 + y^2 = 1 \quad \text{b) } x^2 + y^2 + z^2 = 4$$

- a) Cylinder
- b) Sphere



1.8 Day 5 Class Notes

Definition 1.8.1: Tangent Line

Given a curve $r(t)$, and a point $r(t_0)$ Equation of the tangent line passing through $r(t_0)$ is

$$l(t) = r(t_0) + t \cdot r'(t_0)$$

$$r(t) = \langle \cos(t), \sin(t), 0 \rangle \quad \text{for } t \in [0, 2\pi]$$



Note:-

Differential Rules:

1. $(u \pm v)' = u' \pm v'$
2. $cu' = (cu)'$
3. $(u \cdot v)' = u' \cdot v + u \cdot v'$
4. $(u \times v)' = u' \times v + u \times v'$
5. $(u_0 f)' = u'(f)f'$



Definition 1.8.2

If a curve is parameterized by $r(t)$ for $a \leq t \leq b$ then its (arc)length is

$$L = \int_a^b |r'(t)| dt$$



$$\langle -2 \cos(t), 0, -2 \sin(t) \rangle$$

$$\int_{-\pi}^{\frac{\pi}{2}} |\langle -2 \sin(t), -4, 2 \cos(t) \rangle| dt$$

1.9 Day 7 Notes

Example 1.9.1 (Warm Up)

Calculate first and second derivatives of $f(x, y) = 2x^2 - 3xy^2$ and $f(x, y) = \sin(x - y) + \sin(x + y)$

Chain rule compute $\frac{d}{dt}$ in 2 ways (chain rule, and substitution)

$$f(x, y) = 2x^2 - 3xy^2$$

$$f'(x) = 3x^2 - 3y^2$$

$$f'(y) = -6xy$$

$$f''(x) = 6x$$

$$f''(y) = -6y$$

1.10 Day 8 Notes

Definition 1.10.1: Gradient

Gradient of $f(x, y)$

$$\nabla f = \langle f_x, f_y \rangle$$

RMK:

Given a direction (unit) \vec{u} , we have

$$D_{\vec{u}}f(x_0, y_0) = \vec{u} \cdot \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$$

$$= \vec{u} \cdot \vec{\nabla} f(x_0, y_0)$$



Example 1.10.1:

Given (x_0, y_0) what is the direction that f increases/decreases the most?

f increases the most in the direction $\frac{\nabla f(x_0, y_0)}{|\nabla f|}$

f decreases the most in the direction $\frac{-\nabla f}{|\nabla f|}(x_0, y_0)$



1.11 Day 9 notes

Example 1.11.1:

$$f(x, y) = xy - x^2y - xy^2$$

$$\nabla f(x, y) = \vec{0}$$

$$\langle y - 2x - y^2, x - x^2 - 2xy \rangle = 0$$

$$y - 2xy - y^2 = 0 \quad x - x^2 - 2xy = 0$$

$$y(1 - 2x - y) = 0 \quad x(1 - x - 2y) = 0$$



Example 1.11.2:

Critical points

$$f(x, y) = x^2 - y^2$$



1.12 Day 12 Notes