

Math 120

PSet 6

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Question 1

Find all the (local) maximum and minimum values and saddle points of the function.

(a) $f(x, y) = xy + \frac{1}{x} + \frac{1}{y}$

(b) $f(x, y) = e^y(x^2 - y^2)$

Solution:

a)

$$f_x = y - \frac{1}{x^2} \quad f_y = x - \frac{1}{y^2}$$

$$y = \frac{1}{x^2} \quad x = \frac{1}{y^2}$$

$$\frac{1}{x^2} = 0 \quad \frac{1}{y^2} = 0$$

$$x = \pm 1 \quad y = 1 \quad x \geq 0 \quad y = \frac{1}{x^2} \quad x = 1$$

$$f_{xx} = \frac{2x}{x^3} \quad f_{yy} = \frac{2y}{y^3} \quad f_{xy} = 0$$

$$f_{xx}(1, 1) = \frac{2(1)}{2^3} = 2 \quad f_{yy}(1, 1) = \frac{2(1)}{2^3} = 2$$

$$f_{xx}(1, 1)f_{yy}(1, 1) - 0^2 = 4$$

(1,1) is a local min because $D > 0$ and $f_{xx} > 0$ b)

$$f_x = 2xe^y \quad f_y = x^2e^y - e^yy^2 - 2ye^y$$

$$2xe^y = 0 \Rightarrow x = 0$$

$$x^2e^y - e^yy^2 - 2ye^y \Rightarrow 0^2e^y - e^yy^2 - 2ye^y \Rightarrow -e^yy^2 - 2ye^y$$

$$-e^yy^2 - 2ye^y \Rightarrow 2ye^y = y^2e^y \Rightarrow y = 0 \quad y = -2$$

$$f_{xx} = 2e^y \quad f_{yy} = x^2e^y - 2ye^y - e^yy^2 - 2e^y - 2ye^y$$

$$D = 2e^0 (0^2e^0 - 2(0)e^0 - e^0(0)^2 - 2e^0 - 2(0)e^0) = (2)(-2) = -4$$

$$D = 2e^{-2} (0^2e^{-2} - 2(-2)e^{-2} - e^{-2}(-2)^2 - 2e^{-2} - 2(-2)e^{-2}) = \frac{16}{e^4}$$

(0,0) is a saddle point because $D < 0$ and (0,-2) is a local min because $D > 0$ and $f_{xx} > 0$

Question 2

Find the absolute maximum and minimum values of the function

$$f(x, y) = x + y - xy$$

on the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$.

Solution:

$$f_x = 1 - y = 0 \Rightarrow y = 1$$

$$f_y = 1 - x = 0 \Rightarrow x = 1$$

$$f_{xx} = 0 \quad f_{yy} = 0 \quad f_{xy} = -1$$

$$(0, 0) \text{ to } (0, 2)$$

$$(0, y) = y \quad [0, 2]$$

$$(0, 2) \text{ to } (4, 0)$$

$$y = 2 - \frac{1}{2}x$$

$$f(x, 2 - \frac{1}{2}x) = x + (2 - \frac{1}{2}x) - x(2 - \frac{1}{2}x) \Rightarrow x + 2 - \frac{1}{2}x - 2x + \frac{1}{2}x^2 \Rightarrow \frac{1}{2}x^2 - \frac{3}{2}x + 2$$

$$\frac{d}{dx} \left[\frac{1}{2}x^2 - \frac{3}{2}x + 2 \right] = x - \frac{3}{2}$$

$$x = \frac{3}{2} \quad y = 2 - \left(\frac{3}{2} \right) = \frac{5}{4}$$

$$f\left(\frac{3}{2}, \frac{5}{4}\right) = \frac{7}{8}$$

$$(0, 0) \text{ to } (4, 0)$$

$$y = 0$$

$$f(x, 0) = x$$

$$f(0, 0) = 0$$

$$f(4, 0) = 4$$

Abs Max at $(4, 0)$ where $f(x, y) = 4$ Abs Min $(0, 0)$ where $f(x, y) = 0$

Question 3

Find the absolute maximum and minimum values of the function

$$f(x, y) = xy^2$$

on the region $x^2 + y^2 \leq 3$, $x \geq 0$, $y \geq 0$.

Solution:

$$f(x, y) = xy^2$$

$$f_x = y^2 = 0 \Rightarrow y = 0$$

$$f_y = 2xy = 0 \Rightarrow y = 0 \quad x = 0$$

$$\nabla f = (y^2, 2xy) \quad \nabla g = (2x, 2y)$$

$$y^2 = 2x\lambda$$

$$2xy = 2y\lambda$$

$$\frac{y}{2x} = \frac{x}{y}$$

$$y^2 = 2x^2 \Rightarrow y = \pm\sqrt{2} \quad x^2 > 1$$

$$(1, \sqrt{2})$$

$$f(0,0) = 0(0)^2 = 0$$

$$f(1, \sqrt{2}) = 1\sqrt{2}^2 = 2$$

$$f(\sqrt{3}, 0) = \sqrt{3}(0)^2 = 0$$

Absolute max $(1, \sqrt{2}) = 2$

Absolute min $(0, 0)$ and $(\sqrt{3}, 0) = 0$

Question 4

Find the maximum and minimum values of the function $f(x, y) = x + 4y$ subject to the constraint

$$\sqrt{x} + \sqrt{y} = 3.$$

Solution:

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$\nabla f(x, y) = (1, 4)$$

$$\nabla g = \left(\frac{\partial}{\partial x} \sqrt{x} + \sqrt{y} - 3, \frac{\partial}{\partial y} \sqrt{x} + \sqrt{y} - 3 \right)$$

$$\nabla g = \left(\frac{1}{2\sqrt{x}}, \frac{1}{2\sqrt{y}} \right)$$

$$1 = \lambda \frac{1}{2\sqrt{x}}$$

$$4 = \lambda \frac{1}{2\sqrt{y}}$$

$$\lambda = 2\sqrt{x}$$

$$4 = 2\sqrt{x} \left(\frac{1}{2\sqrt{y}} \right) = \frac{\sqrt{x}}{\sqrt{y}}$$

$$\sqrt{16y} + \sqrt{y} = 3 \Rightarrow 4\sqrt{y} + \sqrt{y} = 3 \Rightarrow 5\sqrt{y} = 3$$

$$\sqrt{y} = \frac{3}{5} \Rightarrow y = \frac{9}{25}$$

$$x = 16 \cdot \frac{9}{25} = \frac{144}{25}$$

$$f\left(\frac{144}{25}, \frac{9}{25}\right) = \frac{180}{25}$$

$$x = 0 \quad y = 9$$

$$x = 9 \quad y = 0$$

$$f(0, 9) = 36 \quad f(9, 0) = 0$$

Absolute max at $(0, 9) = 36$

Absolute min at $\left(\frac{144}{25}, \frac{9}{25}\right) = \frac{180}{25}$

Question 5

Consider the function $f(x, y) = e^{xy}$ and the constraint $x^3 + y^3 = 16$.

- Use Lagrange multipliers to find the coordinates (x, y) of any points on the constraint where the function f could attain a maximum or minimum.
- For each point you found in part (a), is the point a maximum, a minimum, both or neither? Explain your answer carefully. What are the minimum and maximum values of f on the constraint? Please explain your answers carefully.
- The Extreme Value Theorem, which we covered last week, guarantees that under the right circumstances, we are guaranteed to find absolute minima and maxima for a function f on a certain constraint. Please explain why parts (a) and (b) don't violate the Extreme Value Theorem.

Solution:

a)

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$(ye^{yx} = \lambda(3x^2, 3y^2))$$

$$ye^{yx} = \lambda 3x^2$$

$$xe^{xy} = \lambda 3y^2$$

$$\frac{ye^{yx}}{xe^{xy}} = \frac{\lambda 3x^2}{\lambda 3y^2}$$

$$\frac{y}{x} = \frac{x^2}{y^2}$$

$$y^3 = x^3$$

$$y = x$$

$$x^3 + x^3 = 16 \Rightarrow 2x^3 = 16 \Rightarrow x = 2$$

point is (2,2)

b)

(2,2) is a maximum value because no other critical point exists on the constraint. It will evaluate to $e^{(2)(2)}$ which is e^4 .

c)

We get an absolute maximum, but not an absolute minimum, meaning this region is neither closed nor bounded, so the Extreme Value Theorem doesn't apply here.

Question 6

Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y, z) = x^2y^2z^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$.

Solution:

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

$$\nabla f = (2xy^2z^2, 2yx^2z^2, 2zx^2y^2)$$

$$\nabla g = (2x, 2y, 2z)$$

$$2xy^2z^2 = \lambda 2x$$

$$2yx^2z^2 = \lambda 2y$$

$$2zx^2y^2 = \lambda 2z$$

$$y^2z^2 = \lambda$$

$$\begin{aligned}
z^2 x^2 &= \lambda \\
x^2 y^2 &= \lambda \\
x^2 y^2 &= z^2 x^2 = y^2 z^2 \\
x &= y = z \\
x^2 + y^2 + z^2 &= 1 \\
3x^2 = 1 &\Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm \frac{1}{\sqrt{3}}
\end{aligned}$$

$$y = x = z = \frac{1}{\sqrt{3}}$$

$$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{1}{27}$$

$$f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) = \frac{1}{27}$$

$$\begin{aligned}
&\text{max: } \frac{1}{27} \\
&\text{min: } 0
\end{aligned}$$

Question 7

Use Lagrange multipliers to find the maximum and minimum values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $x^4 + y^4 + z^4 = 1$.

Solution:

$$\begin{aligned}
\nabla f(x, y, z) &= (2x, 2y, 2z) \\
\nabla g(x, y, z) &= (4x^3, 4y^3, 4z^3) \\
\nabla f &= \lambda \nabla g \\
2x &= \lambda 4x^3 \quad 2y = \lambda 4y^3 \quad 2z = \lambda 4z^3 \\
z &= \lambda 4x^2 \quad z = \lambda 4y^2 \quad z = \lambda 4z^3 \\
\lambda &= \frac{1}{2x^2} \quad \lambda = \frac{1}{2y^2} \quad \lambda = \frac{1}{2z^2} \\
x^4 + y^4 + z^4 &= 1 \quad x^4 = y^4 = z^4 = t \\
3t &= 1 \quad t = \frac{1}{3} \\
z^4 &= \frac{1}{3} \quad z^2 = \frac{1}{\sqrt{3}}
\end{aligned}$$

$$f(x, y, z) = 3 \cdot \frac{1}{\sqrt{3}} = \sqrt{3}$$

Case 1: One variable is 0

$$y^2 + z^2 = 1 \quad 2y^4 = 1$$

$$y^2 = \frac{1}{\sqrt{2}}$$

$$f(x, y, z) = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$$

Case 1: Two variables are 0

$$z^4 = 1$$

$$z^2 = 1$$

$$f(x, y, z) = z^2 = 1$$

$$\begin{aligned}
&\text{Min } 1 \\
&\text{Max } \sqrt{3}
\end{aligned}$$

Question 8

Find the absolute minimum and maximum values of the function $f(x, y) = x^2 - (y - 2)^2$ on the region

$$D = \{x^2 + y^2 \leq 9 \text{ and } y \geq 0\},$$

and the points at which those extrema occur.

Solution:

$$\nabla f(x, y) = (2x - 2y + 4)$$

$$\nabla g(x, y) = 2x, 2y$$

$$\nabla f(x, y) = \lambda \nabla g(x, y)$$

$$2x = \lambda 2x$$

$$-2y + 4 = \lambda 2y$$

$$x^2 + y^2 = 9$$

$$2x = 0 \quad -2y + 4 = 0 \Rightarrow y = 2$$

$$f(0, 2) = 0^2 - (2 - 2)^2 = 0$$

$$\text{Case: } x = 0$$

$$0^2 + y^2 = 9 \Rightarrow y = \pm 3 \quad y \geq 0 \quad y = 3$$

$$f(0, 3) = 0^2 - (3 - 2)^2 = -1$$

$$\text{Case: } x \neq 0$$

$$1 = \lambda$$

$$-2y + 4 = 2y \Rightarrow 4 = 4y \Rightarrow y = 1$$

$$f(x, 1) = x^2 + 1^2 = 9 \Rightarrow x = \pm 2\sqrt{2}$$

$$f(0, 2) = 0 \quad f(0, 3) = -1 \quad f(\pm 2\sqrt{2}, 1) = 7 \quad f(0, 0) = -4$$

Max: 7 at (0,2)

Min: -4 at (0,0)