Math 120

PSet 3

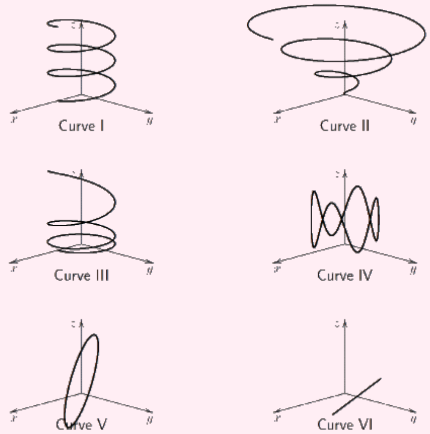
Sep 19 2024

Contents

Chapter 1

1.1 PSet 3

Question 1



Find the curve parameterized by each vector-valued function. Justify your answers

(a) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

(b) $\vec{r}(t) = t\langle \cos t, \sin t, t \rangle$

(c) $\vec{r}(t) = \langle \cos t, \sin t, t^3 \rangle$

(d) $\vec{r}(t) = \langle \cos(t^3), \sin(t^3), t^3 \rangle$

(e) $\vec{r}(u) = \langle \cos u, \sin u, 1 + \sin(4u) \rangle$

(f) $\vec{r}(u) = \langle \cos u, \sin u, 1 + 4\sin(u) \rangle$

(g) $\vec{r}(t) = \langle 2\cos t, 1 + 4\cos t, 3\cos t \rangle$

Solution:
Equation a should make a helix with fixed radi lengths so it goes wiht curve I.
Equation b is similar to equation a but would have increasing ring sizes as t increases so it goes with curve II.
Equation c should be III because the radi is constant but it grows much faster.
Equation d is similar to equation a except the helix would get rings faster. It would still go with curve I.
Equation e would go with curve IV because the x and y portions should form circles but due to $1 + \sin(4u)$ there should also be oscillation in the z axis.
Equation f would go with curve V because it has oscillating height with periods that match the y-axis.
Equation g would go with curve VI because all components are proportional to $\cos t$ which suggests a straight line.

Question 2

Find a vector function that represents the curve of intersection of the plane $z = -2$ and the sphere $x^2 + (y-1)^2 + (z+1)^2 = 9$.

Solution:

$$x^2 + (y-1)^2 + ((-2)+1)^2 = 9$$
$$x^2 + (y-1)^2 = 8$$
$$r = 2\sqrt{2}$$
$$x(t) = 2\sqrt{2}\cos(t)$$
$$y - 1 = 2\sqrt{2}\sin(t) \Rightarrow y = 2\sqrt{t}\sin(t) + 1$$
$$\vec{r}(t) = \langle 2\sqrt{2}\cos(t), 2\sqrt{2}\sin(t) + 1, -2\rangle$$

Question 3

Consider the vector-valued function $\vec{r}_1(t) = \langle 2\sin t, -3\cos t, 0\rangle$, $0 \leqslant t \leqslant 2\pi$.

(a) Sketch the plane curve given by $\vec{r}_1(t)$.

(b) Compute and draw on your sketch from part (a) the position vector $\vec{r}_1\left(\frac{2\pi}{3}\right)$ and the tangent vector $\vec{r}_1'\left(\frac{2\pi}{3}\right)$.

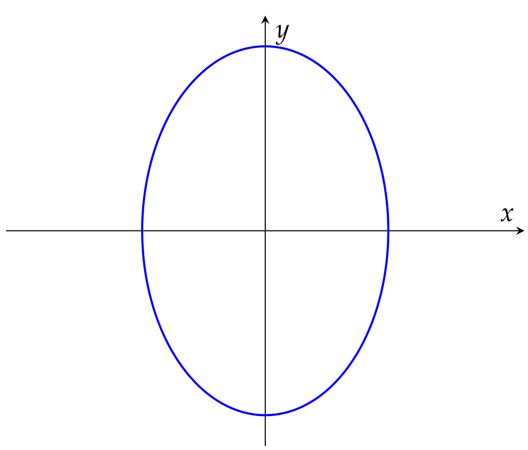
(c) The vector-valued function $\vec{r}_2(t) = \langle 2\cos(3t), -3\sin(3t)\rangle$ parameterizes the same curve. Find the smallest $t^* > 0$ such that $\vec{r}_2(t^*) = \vec{r}_1\left(\frac{2\pi}{3}\right)$, and compute $\vec{r}_2'(t^*)$. Explain how and why $\vec{r}_2'(t^*)$ differs from the tangent vector $\vec{r}_1'\left(\frac{2\pi}{3}\right)$ you computed in part (b).

Solution:

a)

$$x = 2\sin t \quad y = -3\cos t$$
$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{-3}\right)^2 = \sin^2 t + \cos^2 t = 1$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$



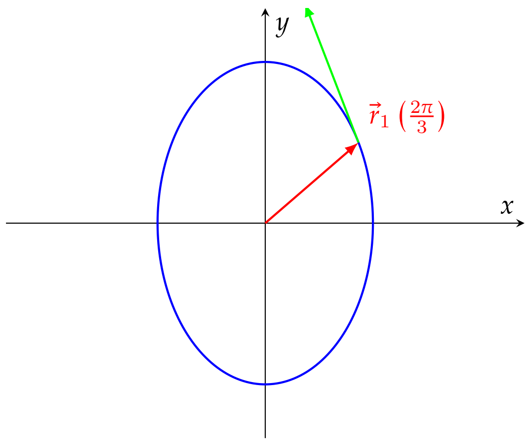
b)

$$\vec{r}_1\left(\frac{2\pi}{3}\right) = \langle 2\sin\left(\frac{2\pi}{3}\right), -3\cos\left(\frac{2\pi}{3}\right)\rangle = \left\langle\sqrt{3}, \frac{3}{2}\right\rangle$$
$$\vec{r}_1'(t) = \langle 2\cos(t), 3\sin(t)\rangle$$

$$\vec{r}_1'\left(\frac{2\pi}{3}\right) = \langle 2\cos\left(\frac{2\pi}{3}\right), 3\sin\left(\frac{2\pi}{3}\right)\rangle$$

$$\vec{r}_1'\left(\frac{2\pi}{3}\right) = \langle 2\left(-\frac{1}{2}\right), 3\left(\frac{\sqrt{3}}{2}\right)\rangle$$

$$\vec{r}_1'\left(\frac{2\pi}{3}\right) = \langle -1, \frac{3\sqrt{3}}{2}\rangle$$



c)

$$r_2(t*) = \langle 2\cos(3t), -2\sin(3t*)\rangle = \langle \sqrt{3}, \frac{3}{2}\rangle$$

$$2\cos(3t*) = \sqrt{3} \quad -3\sin(3t*) = \frac{3}{2}$$

$$\cos(3t*) = \frac{\sqrt{3}}{2} \Rightarrow 3t* = \frac{\pi}{6}, \frac{11\pi}{6}$$

$$t* = \frac{\pi}{18}, \frac{11\pi}{18}$$

$$-3\sin(3t*) = \frac{3}{2} \Rightarrow \sin(3t*) = -\frac{1}{2}$$

$$3t* = \frac{-7\pi}{6}, \frac{11\pi}{6}$$

$$t* = \frac{7\pi}{18}, \frac{11\pi}{18}$$

$$t* = \frac{11\pi}{18}$$

$$r_2'(t) = \langle -6\sin(3t), -9\cos(3t)\rangle$$

$$r_2'\left(\frac{11\pi}{18}\right) = \langle 3, \frac{-9\sqrt{3}}{2}\rangle$$

Tangent vectors differ because $r_2(t)$ traces the same curve more rapidly which affects the magnitude and direction of the tangent vector.

Question 4

Find parametric equations for the tangent line to the curve parameterized by

$$x = 2t + 1, \quad y = e^{t^2-4}, \quad z = \ln(1 + t^2)$$

at the point $(5, 1, \ln 5)$.

Solution:

$$x(t) = 2(t) + 1 \quad y(t) = e^{t^2-4} \quad z(t) = \ln(1 + t)^2$$

$$x'(t) = 2 \quad y'(t) = 2te^{t^2-4} \quad z'(t) = \frac{2t}{1 + t^2}$$

$$5 = 2t + 1 \Rightarrow 4 = 2t \Rightarrow t = 2$$

$$x'(2) = 2 \quad y'(2) = 4e^{2^2-4} = 4 \quad z'(t) = \frac{4}{5}$$

$$x : 5 + 2t \quad y : 1 + 4t \quad z : \ln(5) + \frac{4}{5}t$$

Question 5

(a) Evaluate the integral $\int \left(\tan t\,\hat{\imath} + \sin^2 t\,\hat{\jmath} + \sec^2 t\,\tan t\,\hat{k}\right) dt$.

(b) Suppose a particle is at the point $(-2, 1, 4)$ at time $t = 0$, and moves according to the velocity function $\vec{v}(t) = \tan t\,\hat{\imath} + \sin^2 t\,\hat{\jmath} + \sec^2 t\,\tan t\,\hat{k}$. Find the particle's position at time $t = \frac{\pi}{4}$.

Solution:

a)

$$\int \left(\tan t\,\hat{\imath} + \sin^2 t\,\hat{\jmath} + \sec^2 t\,\tan t\,\hat{k}\right) dt = \int \tan t\,\hat{\imath}dt + \int \sin^2 t\,\hat{\jmath}dt + \int \sec^2 t \tan t\,\hat{k}dt$$

$$\int \tan t\,\hat{\imath}dt = \hat{\imath}\int \frac{\sin t}{\cos t}dt$$

$$x = \cos(t)$$

$$\hat{\imath}\int \tan t dt = \hat{\imath}\int -\frac{1}{x}dx = -\ln|x| + k$$

$$(-\ln|x| + k)\,\hat{\imath} = (-\ln|\cos(t)| + a)\,\hat{\imath}$$

$$\hat{\jmath}\int \sin^2 t dt = \hat{\jmath} = \hat{\jmath}\int \frac{1 - 2\cos\theta}{2}dt$$

$$\hat{\jmath}\int \frac{1 - 2\cos t}{2}dt \Rightarrow \hat{\jmath}\frac{1}{2}\int 1 - 2\cos t dt$$

$$\hat{\jmath}\frac{1}{2}\int 1 - 2\cos t dt = \hat{\jmath}\frac{1}{2}t - \hat{\jmath}\frac{1}{2}\int \cos 2t dt$$

$$\hat{\jmath}\frac{1}{2}t - \hat{\jmath}\int \cos t dt = \left(\frac{1}{2}t - \frac{\sin(2t)}{4} + b\right)\hat{\jmath}$$

$$\hat{k}\int \sec^2 t \tan t dt$$

$$\tan t = u \quad \sec^2 dt = du$$

$$\int u du = \frac{u^2}{2} + c \Rightarrow \left(\frac{\tan^2 t}{2} + c\right)\hat{k}$$

$$\int \left(\tan t\,\hat{\imath} + \sin^2 t\,\hat{\jmath} + \sec^2 t\ \tan t\,\hat{k}\right)\,dt = (-\ln|\cos(x)| + a)\,\hat{\imath} + \left(\frac{1}{2}t - \frac{\sin(2t)}{4} + b\right)\hat{\jmath} + \left(\frac{\tan^2 t}{2} + c\right)\hat{k}$$

b)

$$(-\ln|\cos(0)| + a)\,,\left(\frac{1}{2}t - \frac{\sin(2(0))}{4} + b\right),\left(\frac{\tan^2(0)}{2} + c\right) = (-2,1,4)$$

$$-\ln(1) + a, 0 - \frac{0}{4} + b, \frac{0}{2} + c = (-2,1,4)$$

$$a = -2 \quad b = 1 \quad c = 4$$

$$\left(-\ln|\cos\left(\frac{\pi}{4}\right)| - 2\right),\left(\frac{1}{2}t - \frac{\sin(2\left(\frac{\pi}{4}\right))}{4} + 1\right),\left(\frac{\tan^2\left(\frac{\pi}{4}\right)}{2} + 4\right) = \left(-\ln\left(\frac{\sqrt{2}}{2}\right) - 2, \frac{\pi}{8} + \frac{3}{4}, \frac{9}{2}\right)$$

Question 6

Consider the curve parameterized by $\vec{r}(t) = \langle e^{2t}, e^{-2t}, \sqrt{8t}\rangle$, $0 \leqslant t \leqslant 1$.

(a) Sketch the projections of $\vec{r}(t)$ in the xy-, zx-, and yz-planes.

(b) Find the length of the curve. *Hint:* To integrate, you will need to write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$ as a perfect square.

Solution:

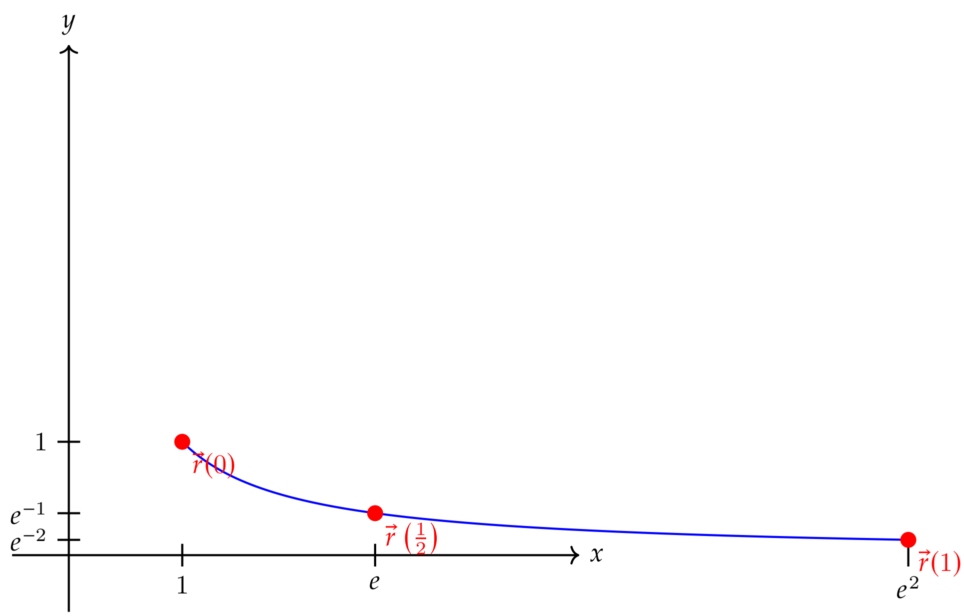
a)

$$t = 0 \quad \vec{r}(0) = \langle 1, 1, 0\rangle$$

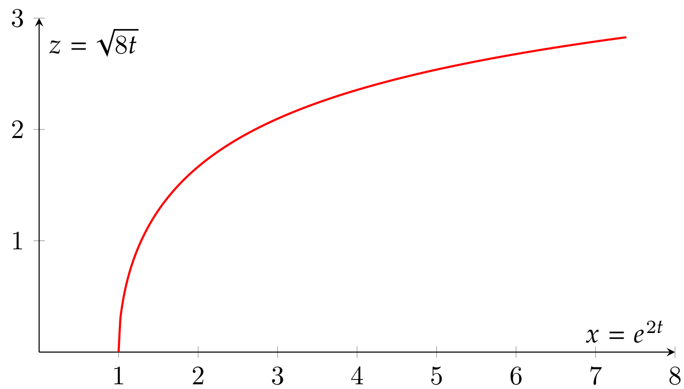
$$t = \frac{1}{2} \quad \vec{r}\left(\frac{1}{2}\right) = \langle e, e^{-1}, \sqrt{2}\rangle$$

$$t = 1, \vec{r}(1) = \langle e^2, e^{-2}, 2\sqrt{2}\rangle$$

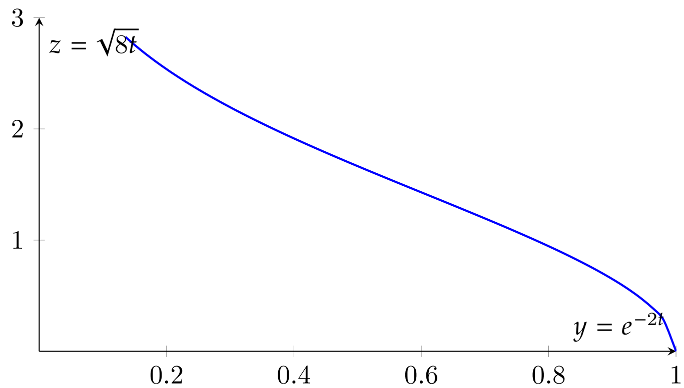
xy



xz



yz



b)

$$L = \int_0^1 ||\vec{r}(t)||dt$$

$$\vec{r}(t) = \langle e^{2t}, e^{-2t}, \sqrt{8}t\rangle$$

$$\vec{r}'(t) = \left\langle \frac{d}{dt}\left(e^{2t}\right), \frac{d}{dt}\left(e^{-2t}\right) \frac{d}{dt}\left(\sqrt{8}t\right)\right\rangle$$

$$\vec{r}'(t) = \left\langle 2e^{2t}, -2e^{-2t}, \sqrt{8}\right\rangle$$

$$L = |\vec{r}'(t)| = \int_0^1 \sqrt{(2e^{2t})^2 + (-2e^{-2t})^2 + \left(\sqrt{8}\right)^2}$$

$$\left(2e^{2t}\right)^2 + \left(-2e^{-2t}\right)^2 + \left(\sqrt{8}\right)^2 = \left(2e^{2t}\right)^2 + \left(-2e^{-2t}\right)^2 + 8$$

$$\left(2e^{2t}\right)^2 + \left(-2e^{-2t}\right)^2 + 8 = \left(2e^{2t} + 2e^{-2t}\right)^2$$

$$L = \int_0^1 \sqrt{(2e^{2t} + 2e^{-2t})^2} \Rightarrow \int_0^1 \left(2e^{2t} + 2e^{-2t}\right)$$

$$\cosh = \frac{e^t + e^{-t}}{2}$$

$$2e^{2t} + 2e^{-2t} = 4\cosh(2t)$$

$$L = \int_0^1 4\cosh(2t)dt \Rightarrow 2\sinh(2t)\Big|_0^1 \rightarrow 2\sinh(2(1)) - 2\sinh(2(0))$$

$$L = 2\sinh(2) - 2\sinh(0)$$

Question 7

Let C be the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $2x + y + z = 4$.

(a) Find a parameterization of C.

(b) Write down an integral for the length of C.

(c) Find the length accurate to five decimal places by using Desmos: https://www.desmos.com/calculator. (Click on the keyboard icon, then "functions", then "Misc", to find the integral symbol.)

Solution:

a)

$$x^2 + y^2 = 4 \quad \mathrm{r} = 2$$
$$x(t) = 2\cos(t) \quad y(t) = 2\sin(t)$$
$$2(2\cos(t)) + 2\sin(t) + z = 4 \Longrightarrow z = 4 - 4\cos(t) - 2\sin(t)$$
$$x^2 + y^2 = 4 \quad \mathrm{r} = 2 \quad z(t) = 4 - 4\cos(t) - 2\sin(t)$$

b)

$$L = \int_a^b \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2 + \left(\frac{d}{dt}z(t)\right)^2}$$
$$\vec{r}'(t) = \langle -2\sin t, 2\cos t, 4\sin t - 2\cos t\rangle$$
$$L = \int_a^b \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (4\sin t - 2\cos t)^2}$$
$$L = \int_a^b \sqrt{4\sin^2 t + 4\cos^2 t + 16\sin^2 t - 16\sin t\cos t + 4\cos^2 t}$$
$$L = \int_a^b \sqrt{20\sin^2 t + 8\cos^2 t - 16sint\cos t}$$

c)

$$\approx 22.64159$$

Question 8

Find the velocity and position vectors of a particle that has acceleration given by

$$\vec{a}(t) = 2\hat{\imath} + 6t\hat{\jmath} + 12t^2\hat{k},$$

and initial velocity and position given by

$$\vec{v}(0) = \hat{\imath} \quad \text{and} \quad \vec{r}(0) = \hat{\jmath} - \hat{k}.$$

Solution:

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$$
$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$$
$$\vec{a}(t) = 2\hat{\imath} + 6t\hat{\jmath} + 12t^2\hat{k}$$
$$\vec{v}(t) = \int \vec{a}(t)dt = \int 2\hat{\imath} + 6t\hat{\jmath} + 12t^2\hat{k}dt$$
$$\int 2\hat{\imath} + 6t\hat{\jmath} + 12t^2\hat{k} = (2t + a)\hat{\imath} + \left(3t^2 + b\right)\hat{\jmath} + \left(4t^3 + c\right)\hat{k}$$

$$\vec{v}(0) = (2(0)+a)\hat{\imath}, \left(3(0)^2+b\right)\hat{\jmath}, \left(4(0)^{3+c}\right)\hat{k} = \langle i, 0, 0\rangle$$
$$\vec{v}(t) = (2t+1)\hat{\imath} + \left(3t^2\right)\hat{\jmath} + \left(4t^3\right)\hat{k}$$

$$\vec{r}(t) = \int \vec{v}(t)dt = \int (2t+1)\hat{\imath} + \left(3t^2\right)\hat{\jmath} + \left(4t^3\right)\hat{k}dt$$
$$\int (2t+1)\hat{\imath} + \left(3t^2\right)\hat{\jmath} + \left(4t^3\right)\hat{k}dt = \left(t^2+t+a_2\right)\hat{\imath} + \left(t^3+b_2\right)\hat{\jmath} + \left(t^4+c_2\right)\hat{k}$$
$$\vec{r}(0) = \left((0)^2+(0)+a_2\right)\hat{\imath} + \left((0)^3+b_2\right)\hat{\jmath} + \left((0)^4+c_2\right)\hat{k} = (a_2)\hat{\imath} + (b_1)\hat{\jmath} + (c_2)\hat{k}$$
$$(a_2)\hat{\imath} + (b_1)\hat{\jmath} + (c_2)\hat{k} = \langle 0, \hat{\jmath}, -\hat{k}\rangle$$
$$a_2 = 0 \quad b_2 = 1 \quad c_2 = -1$$
$$\vec{r}(t) = \left(t^2+t\right)\hat{\imath} + \left(t^3+1\right)\hat{\jmath} + \left(t^4-1\right)\hat{k}$$

Question 9

Consider the function $f(x,y) = \frac{\sqrt{y-3x}}{\ln(4-x^2-y^2)}$.

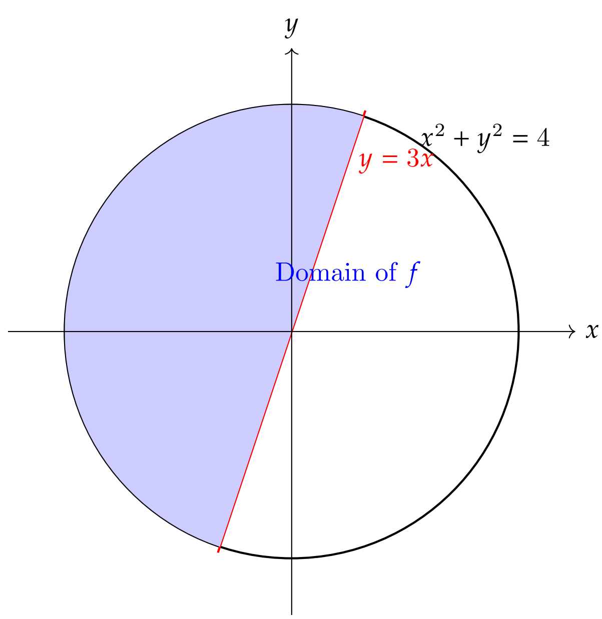
(a) Find and sketch the domain of f.

(b) On your sketch from part (a), mark where $f(x,y) = 0$, and indicate the region(s) where $f(x,y)$ is positive and negative.

Solution:

a)

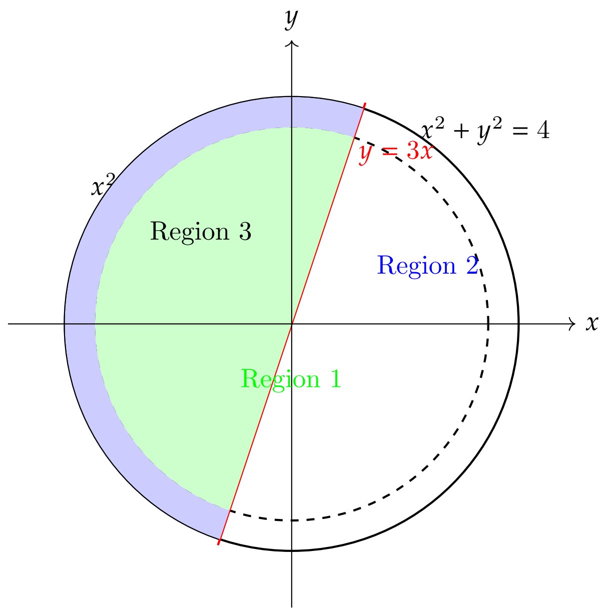
$$(x,y) = \frac{\sqrt{y-3x}}{\ln(4-x^2-y^2)}$$
$$y \geqslant 3x$$
$$4 - x^2 - y^2 > 0$$
$$\text{domain:} \quad x^2+y^2 < 4 \quad \text{and} \quad y \geqslant 3x$$



b)

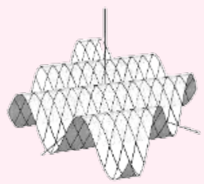
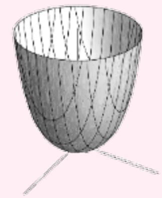
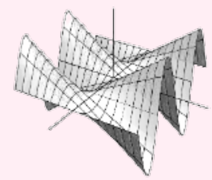
$$\text{Positive:}\quad y > 3x \quad \text{and} \quad x^2 + y^2 < 4$$

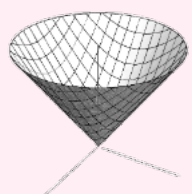
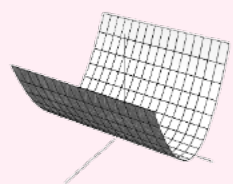
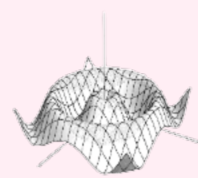
$$\text{Negative:}\quad y > 3x \quad \text{and} \quad 0 < 4 - x^2 - y^2 < 1$$



Question 10

Here are several surfaces.

Match each function with its graph. Justify your answers.

(a) $f(x, y) = x^2$

(b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $f(x, y) = e^{x^2+y^2} - 1$

(d) $f(x, y) = y \sin x$

(e) $f(x, y) = \sin(x + y)$

(f) $f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$

Solution:
Equation a goes with Surface V bceause it should be parabolic along the x-axis and independent of y.
Equation b goes with surface IV because the value of z should increase linearly with radical distance which should make a cone like shape
Equation c goes with surface II z increases exponentially as x and y increase which should make for something cone like but that grows faster which should make a steep smooth rise
Equation d goes with surface III because the function should oscilate in the x-direction while increasing due to y values so
Equation e goes with surface I because it depends on the sum of x and y and would have a constant phase along lines x + y equals a constant. So there should be a diagonal wave pattern.
Equation f goes with surface VI because it should still have waves that depend on distance from origin, meaning there should be ripples.

Question 11

Draw a contour map of the function $f(x, y) = x^2e^{-y}$ showing several level curves.

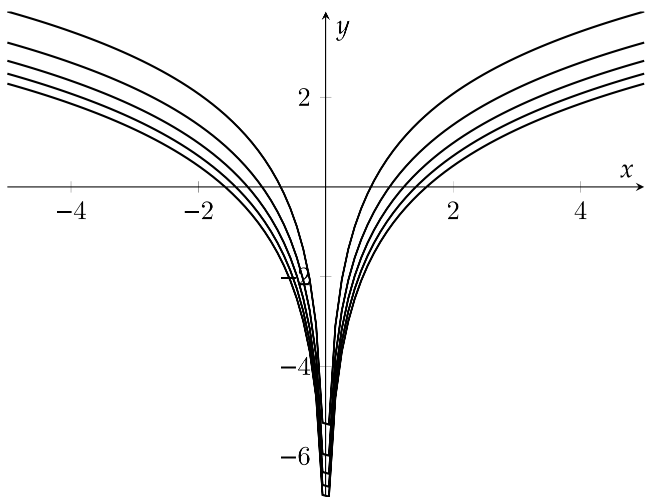
Solution:

$$f(x, y) = x^2e^{-y}$$
$$x^2e^{-y} = k$$
$$e^{-y} = \frac{k}{x^2}$$
$$\ln\left(e^{-y}\right) = \ln\left(\frac{k}{x^2}\right)$$
$$-y = \ln(k) - \ln(x^2) \Rightarrow y = -\ln(k) + 2\ln(x)$$

$x^2 > 0$ for all x in the orignal function defintion so the equation is actually

$$-y = \ln(k) - \ln(x^2) \Rightarrow y = -\ln(k) + 2\ln(|x|)$$

$$y = -\ln(k) + 2\ln(|x|)$$



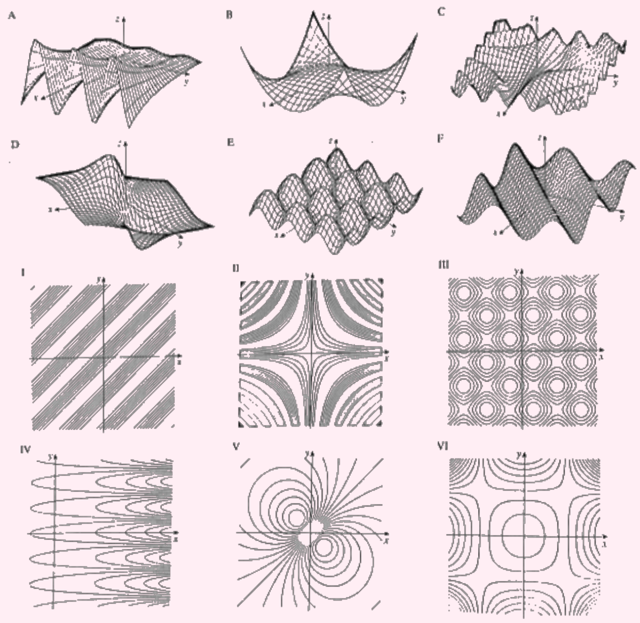
Question 12

Match the function with its graph (labeled A-F below) and with its contour map (labeled I-VI). Give reasons for your choices.

(a) $z = e^x \cos y$

(b) $z = \sin x - \sin y$

(c) $z = \frac{x-y}{1+x^2+y^2}$



Solution:

a)

e^x is an exponential function in the x-direction, meaning that as x increases the value of z grow rapidly.
$\cos(y)$ means that there are oscillations in the y-direction causing wave-like behavior along the y-axis.
Graph A shows an exponential rise in the x-direction with some oscillations in the y-direction. Countour IV because of the oscillations and because it is what graph A would look like from the top.

b) $\sin x - \sin y$ would have oscillations along the x-direction and y-direction. These oscillations would be of the same size as there are fixed values that this equation can result in.
The graph is E for this reason. Contour is III because it is a top view of the graph E and the circles are the same size which is a trait you would expect.

c)

numerator of $x - y$ suggests a linear slope or difference between x and y, so one side will be positive and the other negative.
The denominator makes the effect of the numerator decrease as x and y increase since it outgrows them. So the graph of this function will have a positive peak and a negative peak near the origin and then it should level out on the sides.
This is why the graph is D. It is contour v because of the increasing size of the ring-like shapes as you move away from the origin which is a trait of graph d.