Math 120

PSet 7

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Contents

Chapter 1

1.1 PSet 7

Question 1

Evaluate the scalar line integral

$$\int_C (3x + y)\, ds,$$

where C is the line segment from $(-1, 3)$ to $(4, 2)$.

Solution:

$$\int_C (3x + y) ds$$

$$(-1, 3) \quad (4, 2)$$

$$f(t) = (1-, 3) + t((4, 2) - (-1, 3))$$

$$f(t) = (-1, 3) + t(5, -1) = \langle -1 + 5t, 3 - t\rangle$$

$$x = -1 + 5t \quad y = 3 - t \quad t \in [0, 1]$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = 5 \quad \frac{dy}{dt} = -1$$

$$ds = \sqrt{5^2 + (-1)^2}\, dt = \sqrt{26}\, dt$$

$$3x + y \Rightarrow 3(-1 + 5t) + (3 - t) \Rightarrow -3 + 15t + 3 - t = 14t$$

$$\int_0^1 14t\sqrt{26}\, dt \Rightarrow \sqrt{26}\int_0^1 14t\, dt$$

$$7\sqrt{26}t^2\Big|_0^1 = 7\sqrt{26}(1)^2 - 7\sqrt{27}(0)^2 = 7\sqrt{26}$$

Question 2

In this problem we will sketch part of the argument that a scalar line integral $\int_C f\,ds$ is independent of the parameterization of C that we choose to compute the integral. Suppose $\vec{r}_1(t)$, $a \leqslant t \leqslant b$, and $\vec{r}_2(t)$, $c \leqslant t \leqslant d$, are two smooth parameterizations of the same smooth curve C. Assuming that both parameterizations are in the same direction it can be shown that $\vec{r}_2(t) = \vec{r}_1(w(t))$, for some increasing function $w(t)$ satisfying $w(c) = a$ and $w(d) = b$. If this is the case, show that

$$\int_a^b f(\vec{r}_1(t))\left|\vec{r}_1'(t)\right|\,dt = \int_c^d f(\vec{r}_2(t))\left|\vec{r}_2'(t)\right|\,dt$$

for any continuous function f.

Solution:

$$\int\limits_c f\,ds$$

$$\vec{r_1}(t) \quad a \leqslant t \leqslant b$$

$$\vec{r_2}(t) \quad c \leqslant t \leqslant d$$

$$\vec{r_2}(r) = \vec{r_1}(w(t)) \quad w(c) = a \quad w(d) = b$$

$$\int_a^b f(\vec{r_1}(t))|\vec{r_1}'(t)\,dt = \int_c^d f(\vec{r_2}(t))|\vec{r_2}'(t)\,dt$$

$$\vec{r_2}'(t) = \frac{d}{dt}\vec{r_2}(t) = \frac{d}{dt}\vec{r_1}(w(t)) = \vec{r_1}'(w(t))w'(t)$$

$$|\vec{r_2}'(t)| = |\vec{r_1}'(w(t))| \cdot |w'(t)|$$

$$\int_c^d f(\vec{r_2}(t))|\vec{r_2}'(t)|\,dt = \int_a^b f(\vec{r_1}(t))|\vec{r_1}'(w(t))| \cdot |w'(t)|\,dt$$

w maps $[c, d]$ to $[a, b]$, when $t = c\,, s = a$, and when $t = d\,, s = b$

$$\int_a^b f(\vec{r_1})(w(t))|\vec{r_1}'(w(t))| \cdot |w(t)|\,dt = \int_a^b f(\vec{r_1}(s))|\vec{r_1}'(s)|\,ds$$

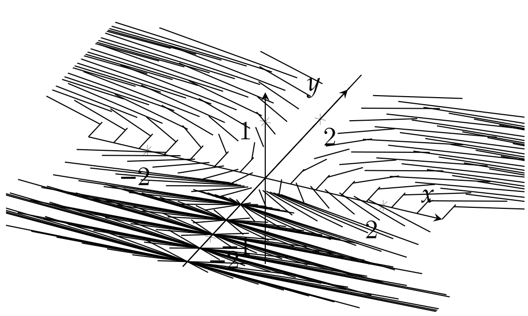
$$\int_a^b f(\vec{r_1}'(t))|\vec{r_1}'(t)|\,dt = \int_a^b f(r_1(s))|\vec{r_1}'(s)|\,ds = \int_c^d f(\vec{r_2}(t))|\vec{r_2}'(t)|\,dt$$

$\therefore$ the scalar line integral is independent of the parameterization and the equality holds true for any continuous function f

Question 3

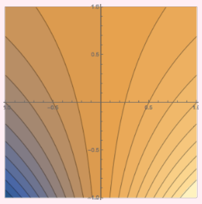
Sketch the vector field $\vec{F}(x, y) = xy\,\hat{\imath} + \frac{1}{2}\,\hat{\jmath}$.

Solution:



Question 4

Given the contour diagram for a function f shown below, in which dark colors correspond to low values of f and light colors correspond to high values of f, sketch the gradient vector field $\vec{F} = \nabla f$.



Question 5

A thin wire has the shape of the curve C parameterized by $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 4\pi$, where $x, y,$ and z are measured in centimeters. The linear density of the wire is given by $\rho(x, y, z) = x^2 z$ grams per centimeter. Find the mass of the wire.

Solution:

$$x = \cos t \quad y = \sin t \quad z = t$$

$$0 \leqslant t \leqslant 4\pi$$

$$\rho(x, y, x) = x^2 z \frac{\text{grams}}{\text{cm}}$$

$$\text{Mass: } = \int_C \rho(x, y, z) ds$$

$$\frac{dx}{dt} = -\sin t \quad \frac{dy}{dt} = \cos t \quad \frac{dz}{dt} = 1$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$ds = \sqrt{(-\sin(t))^2 + (\cos(t))^2 + (1)^2}\, dt$$

$$ds = \sqrt{1 + 1}\, dt = \sqrt{2}\, dt$$

$$\rho(x, y, z) = x^2 z \Rightarrow \cos^2 t \cdot t \Rightarrow t \cos^2 t$$

$$\text{Mass: } \int_0^{4\pi} \rho(t) ds = \int_0^{4\pi} t \cos^2 t \sqrt{2}\, dt$$

$$\sqrt{2} \int_0^{4} t \cos^2 t dt \quad \cos^2 t = \frac{1 + \cos 2t}{2}$$

$$\sqrt{2} \int_0^{4\pi} t \left(\frac{1 + \cos 2t}{2}\right) dt \Rightarrow \frac{\sqrt{2}}{2} \int_0^{4\pi} t(1 + \cos 2t)\, dt$$

$$\frac{\sqrt{2}}{2} \int_0^{4\pi} t\, dt + \frac{\sqrt{2}}{2} \int_0^{4\pi} t \cos 2t\, dt$$

$$\int_0^{4\pi} t\, dt = \left.\frac{t^2}{2}\right|_0^{4\pi} \Rightarrow \frac{\sqrt{2}}{2} \frac{16\pi^2}{2} - \frac{\sqrt{2}}{2} \frac{0}{2} = 4\sqrt{2}\pi^2$$

$$u = t \quad du = dt$$

$$v = \frac{1}{2} \sin 2t \quad dv = \cos 2t$$

$$\int t\cos 2t\,dt = t\cdot\frac{1}{2}\sin 2t - \int\frac{1}{2}\sin 2t dt = \frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t + k$$

$$\left[\frac{1}{2}t\sin 2t + \frac{1}{4}\cos 2t\right]_0^{4\pi} = \left(\frac{1}{2}\cdot 4\pi\cdot 0 + \frac{1}{4}\cdot 1\right) - \left(0 + \frac{1}{4}\cdot 1\right) = 0$$

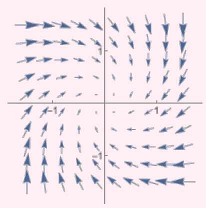
$$4\sqrt{2}\pi^2 + 0 = 4\sqrt{2}\pi^2$$

Question 6

Let $\vec{F}$ be the vector field shown below, and let C be the unit circle, oriented clockwise. Is the vector line integral

$$\int_C \vec{F}\cdot d\vec{r}$$

positive, negative, or zero? Explain your reasoning.



Solution:
In the first quadrant (top right), the vectors point in the counterclockwise direction. In the second quadrant (top left), the vectors still circulate in a way consistent with a counterclockwise motion. In the third and fourth quadrants, the vectors similarly maintain this pattern, suggesting the field is rotating counterclockwise overall. Since the problem states the circle C is clockwise oriented, the direction of the field is opposite to the direction of the curve's traversal.
The line integral $\int_C \vec{F}\,d\vec{r}$ measures the component of the vector field $\vec{F}$ that aligns with the direction of traversal around the curve C. Here, the vector field rotates counterclockwise, but the traversal of CC is clockwise. Since the field vectors mostly oppose the direction of movement along the curve, the dot product $\vec{F}\cdot d\vec{r}$ will tend to be negative along the path.

Question 7

Evaluate the line integral

$$\int_C \sin x\,dx + \cos y\,dy$$

where C consists of the top half of the circle $x^2+y^2=1$ from $(1,0)$ to $(-1,0)$ and the line segment from $(-1,0)$ to $(-2,3)$. (Remember that when you see an integral that looks like

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

it is a shorthand notation for

$$\int_C \vec{F}(\vec{r}(t))\cdot d\vec{r}$$

where $\vec{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$. The analogous thing is true in three dimensions.)

Solution:

$$x^2+y^2=1 \quad x=\cos t \quad y=\sin t \quad t\in[0,\pi]$$

$$x(t) = (1-t)(-1)+t(-2) \quad y(t) = (1-t)(0)+t(3) \quad t\in[0,1]$$

$$\vec{F}(x,y) = \langle \sin x, \cos y\rangle$$

$$\int_C \sin x\,dx + \cos y\,dy$$

$$x(t) = \cos t, \quad y(t) = \sin t, \quad t \in [0, \pi]$$

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t$$

$$\int_{C_1} \sin x\,dx + \cos y\,dy = \int_0^\pi [\sin(\cos t)(-\sin t) + \cos(\sin t)\cos t]\,dt$$

$$\int_0^\pi \sin(\cos t)(-\sin t)\,dt + \int_0^\pi \cos(\sin t)\cos t\,dt$$

$$f(x, y) = -\cos x + \sin y$$

At $(1, 0)$:

$$f(1, 0) = -\cos(1) + \sin(0) = -\cos(1)$$

At $(-1, 0)$:

$$f(-1, 0) = -\cos(1)$$

$$\int_{C_1} \sin x\,dx + \cos y\,dy = f(-1, 0) - f(1, 0) = 0$$

$$x(t) = -1 - t, \quad y(t) = 3t, \quad t \in [0, 1]$$

$$dx = -1\,dt, \quad dy = 3\,dt$$

$$\int_{C_2} \sin x\,dx + \cos y\,dy = \int_0^1 [\sin(-1-t)(-1) + \cos(3t)(3)]\,dt$$

Using $\sin(-1-t) = -\sin(1+t)$, the integral becomes:

$$\int_0^1 \sin(1+t)\,dt + 3\int_0^1 \cos(3t)\,dt$$

$$\int_0^1 \sin(1+t)\,dt = -\cos(1+t)\Big|_0^1 = -\cos(2) + \cos(1)$$

$$3\int_0^1 \cos(3t)\,dt = 3\left(\frac{1}{3}\sin(3t)\Big|_0^1\right) = \sin(3)$$

$$\int_{C_2} \sin x\,dx + \cos y\,dy = \cos(1) - \cos(2) + \sin(3)$$

$$\int_C \sin x\,dx + \cos y\,dy = \int_{C_1} \sin x\,dx + \cos y\,dy + \int_{C_2} \sin x\,dx + \cos y\,dy$$

Since $\int_{C_1} \sin x\,dx + \cos y\,dy = 0$, the total integral is:

$$\int_C \sin x\,dx + \cos y\,dy = \cos(1) - \cos(2) + \sin(3)$$

Question 8

Compute the line integral of the vector field

$$\vec{F}(x,y) = \frac{x}{\sqrt{x^2+y^2}}\hat{\imath} + \frac{y}{\sqrt{x^2+y^2}}\hat{\jmath}$$

along the parabola $x = 1 + y^2$ from $(2,-1)$ to $(2,1)$.

Solution:

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \left[F_1(x(t),y(t))\frac{dx}{dt} + F_2(x(t),y(t))\frac{dy}{dt}\right]dt$$

$$y = t \quad t \in [-1,1] \quad x(t) = 1 + t^2$$

$$r(t) = \langle x(t), y(t)\rangle = \langle 1+t^2, t\rangle \quad t \in [-1,1]$$

$$\frac{dx}{dt} = 2t \quad \frac{dy}{dt} = 2$$

$$\vec{F}(x,y) \Rightarrow \vec{F}(1+t^2,t) = \frac{1+t^2}{\sqrt{(1+t^2)^2+t^2}}\hat{\imath} + \frac{t}{\sqrt{(1+t^2)^2+t^2}}\hat{\jmath}$$

$$\left[\frac{1+t^2}{\sqrt{(1+t^2)^2+t^2}} \times 2t\right] + \left[\frac{t}{\sqrt{(1+t^2)^2+t^2}} \times 1\right]$$

$$\int_a^b \left[\frac{3t+2t^3}{\sqrt{(1+t^2)^2+t^2}}\right]dt$$

$$\int_a^b \frac{3t+2t^3}{\sqrt{1+3t^2+t^4}}dt$$

$$d\vec{r} = (2t\hat{\imath}, 2\hat{\jmath})$$

$$\vec{F}\cdot d\vec{r} = \frac{3t+2t^3}{\sqrt{1+3t^2+t^4}}$$

$$d\frac{d}{dt}\sqrt{1+3t^2+t^4} = \frac{3t+t^2}{\sqrt{t^4+3t^2+1}}$$

$$\vec{F}\cdot d\vec{r} = d\left(\sqrt{t^4+3t^2+1}\right)$$

$$\int_a^b \vec{F}d\vec{r} = \left[t^4+3t^2+1\right]_{-1}^{1} = \sqrt{1^4+3(1)^2+1} - \sqrt{1+3(-1)^2+(-1)^4} = 0$$

Question 9

Evaluate the line integral of the vector field

$$\vec{F}(x,y,z) = (x+y)\hat{\imath} + (y-z)\hat{\jmath} + z^2\hat{k}$$

along the path parameterized by

$$\vec{r}(t) = t^2\hat{\imath} + t^3\hat{\jmath} + t^2\hat{k}, \quad 0 \leqslant t \leqslant 1.$$

Solution:

$$\vec{F}(x,y,x) = (x+y)\hat{\imath} + (y-z)\hat{\jmath} + z^2\hat{k}$$

$$\vec{r}(t) = t^2\hat{\imath} + t^3\hat{\jmath} + t^2\hat{k}, \quad 0 \leqslant t \leqslant 1$$

$$\frac{d\vec{r}}{dt} = 2t\,\hat{\imath} + 3t^2\hat{\jmath} + 2t\,\hat{k}$$

$$\vec{F}(t) = \left[t^2 + t^3\right]\,\hat{\imath} + \left[t^3 - t^2\right]\,\hat{\jmath} + \left[t^4\right]\,\hat{k}$$

$$\vec{F} \cdot \frac{dr}{dt} = \left[\left(t^2 + t^3\right)(2t)\right] + \left[\left(t^2 - t^3\right)\left(3t^2\right)\right] + \left[\left(t^4\right)(2t)\right]$$

$$\vec{F}\frac{d\vec{r}}{dt} = 2t^3 - t^4 + 5t^5$$

$$\int_0^1 \left[2t^3 - t^4 + 5t^5\right] dt = \left[\frac{1}{2}t^{-}\frac{1}{5}t^5 + \frac{5t^6}{6}\right]_0^1$$

$$\left(\frac{1}{2} - \frac{1}{5} + \frac{5}{6}\right) - 0 = \frac{17}{15}$$

Question 10

For each of the following vector fields $\vec{F}$ and curves C, find a function f such that $\vec{F} = \nabla f$ and use this function to evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

along the given directed curve C.

1. $\vec{F}(x, y) = \langle x^2, y^2 \rangle$, C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.
2. $\vec{F}(x, y, z) = \langle e^y, xe^y, (z+1)e^z \rangle$, $C : \vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leqslant t \leqslant 1$.

Solution:

$$F = \nabla f$$

$$\int_C F \cdot d\vec{r} = f(\text{end point}) - f(\text{start point})$$

Problem 1

$$\frac{\partial f}{\partial x} = x^2 \quad \frac{\partial f}{\partial y} = y^2$$

$$f(x, y) = \int x^2\,dx = \frac{1}{3}x^2 + g(y)$$

$$\frac{\partial g}{\partial y} = g'(y) = y^2$$

$$g(y) = \int_y^2 dy = \frac{1}{3}y^3$$

$$f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$$

$$\int F\,d\vec{r} = f(2, 8) - f(-1, 2) = 171$$

Problem 2

$$\vec{F}(x, y, z) = \langle e^y, xe^y, (z+1)e^z \rangle \quad \vec{r}(t) = \left(t, t^2, t^3\right)$$

$$\frac{\partial F}{x} = e^x \quad \frac{\partial F}{\partial y} = xe^y \quad \frac{\partial F}{\partial z} = (z+1)e^z$$

$$f(x, y, z) = \int e^y\,dx = xe^y + \rho(y, z)$$

$$\frac{\partial f}{\partial y} = xe^y + \frac{\partial \rho}{\partial y} \quad xe^y + \frac{\partial \rho}{\partial y} \quad \frac{\partial \rho}{\partial y} = 0$$

$$\rho(x, y, z) = \phi(z)$$

$$\frac{\partial f}{\partial z} = (z+1)e^z$$

$$\rho(z) = \int (z+1)e^z \, dz$$

$$u = z + 1 \quad du = 1\, dz$$

$$v = e^z \, dv = e^z \, dz$$

$$\int (z+1)e^z \, dz = (z+1)e^z - \int e^z \, dz$$

$$(z+1)e^z - e^z \Rightarrow ze^z + e^z - e^z = ze^z$$

$$f(x, y, z) = xe^y + ze^z$$

$$f(1, 1, 1) = (1)e^1 + (1)e^1 = ze$$

$$f(0, 0, 0) = (0)e^0 + +(0)e^0 = 0$$

$$\int_C \vec{F} \cdot d\vec{r} = f(1, 1, 1) - f(0, 0, 0) = 2e$$

Question 11

Clairaut's Theorem implies that if the vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ is conservative and P, Q, and R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

1. Use the statement above to show that the vector line integral

   $$\int_C x\, dx + 2x\, dy + xz\, dz$$

   is not independent of path.

2. Find two directed curves C_1 and C_2 that start at the same point and end at the same point, such that

   $$\int_{C_1} x\, dx + 2x\, dy + xz\, dz \neq \int_{C_2} x\, dx + 2x\, dy + xz\, dz.$$

Question 12

The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\vec{r} = \langle x, y, z \rangle$ is

$$\vec{F}(\vec{r}) = K \frac{\vec{r}}{|\vec{r}|^3},$$

where K is a constant. Find the work done on the particle as it moves along the straight line from $(0, 3, 0)$ to $(1, 3, 2)$ in two ways:

1. Parameterize the line segment, and compute

   $$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

   directly.

2. Although $\vec{F}$ is not defined at the origin, it turns out that $\vec{F}$ is conservative on its domain. Find a potential function f, and use the Fundamental Theorem of Line Integrals to compute the work done on the particle.

Solution:
a)

$$\vec{r}(t) = (t, 3, 2t), \quad t \in [0, 1]$$
$$\vec{r}'(t) = \langle 1, 0, 2 \rangle$$
$$|\vec{r}(t)| = \sqrt{5t^2 + 9}$$
$$\vec{F}(\vec{r}(t)) = K\frac{\langle t, 3, 2t \rangle}{(5t^2 + 9)^{3/2}}$$
$$\vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) = K\frac{5t}{(5t^2 + 9)^{3/2}}$$
$$W = \int_0^1 K\frac{5t}{(5t^2 + 9)^{3/2}}\, dt$$

Substitution: $u = 5t^2 + 9$, $du = 10t\, dt$

$$W = K\int_9^{14} \frac{du}{2u^{3/2}}$$
$$W = \frac{K}{2}\int_9^{14} u^{-3/2}\, du$$
$$W = -K\left[u^{-1/2}\right]_9^{14} = K\left(\frac{1}{3} - \frac{1}{\sqrt{14}}\right)$$

b)

$$f(\vec{r}) = -\frac{K}{|\vec{r}|}$$
$$W = f(\vec{r}_B) - f(\vec{r}_A)$$
$$|\vec{r}_A| = 3, \quad |\vec{r}_B| = \sqrt{14}$$
$$W = K\left(\frac{1}{3} - \frac{1}{\sqrt{14}}\right)$$