3D Coordinate Systems

Distance: $|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
Sphere: $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$

Vectors

Properties: $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
Normalize vector : $\frac{\vec{n}}{|n|}$

Dot Product

Formula: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
Angle: $\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
Orthogonality: $\mathbf{a} \cdot \mathbf{b} = 0$
Projections:

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Positive Dot Product: The angle θ between the vectors is acute. The vectors point in the same general direction.
Zero Dot Product: The angle θ between the vectors is right. The vectors are perpendicular (orthogonal) to each other.
Negative Dot Product: The angle θ between the vectors is obtuse. The vectors point in opposite general directions.

Cross Product

Formula:

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

Magnitude: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$
Triple Product: $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$
If **Triple Product** $= 0$ than they are coplanar.

Lines and Planes

Line: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$
Plane: $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$
$ax + by + cz + d = 0$
Distance: '

$$D = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}}$$

Curve Length: $s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} \, du$

Equation for Planes containing three points

$$\vec{PQ} \times \vec{PR}$$

Normal Vector of a Plane

$$ax + by + cz + d = 0 \quad \text{with normal vector} \quad \vec{n} = \langle a, b, c \rangle$$

Plane containing L_1 and parallel to L_2

$$\vec{n} = v_1 \times v_2 \Rightarrow \vec{n} \cdot (\langle x, y, z \rangle - v_1(t))$$

Vector Value Functions

Form:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

Limit:

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

Derivative: $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$
Unit Tangent Vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
Orthogonality: If $|\mathbf{r}(t)| = c$, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Definite Integral: $\int_a^b \mathbf{r}(t) \, dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$

Length: $L = \int_a^b |\mathbf{r}'(t)| \, dt$

Curvature

Form: $s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} \, du$

Motion in space

Form: $\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$

Partial Derivatives

Form: $f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$
Defintion:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Notation:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rules
To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.
Higher Derivatives:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Clairaut's Theorem $f_{xy}(a, b) = f_{yx}(a, b)$

Chain Rule
Example:

$$\frac{d}{dt}\left(x^2(t)y(t)\right) = \langle 2x(t)y(t), x^2(t)\rangle \cdot \langle x'(t), y'(t)\rangle$$

Case 1:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Case 2:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

General Form:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Implicit Differentiation
Form:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Three variables:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Directional Derivative and Gradient Vector
Directional Derivative:

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$$

Gradient For 2 variables

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Gradient for 3 Variables

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Properties of Gradient Vecotr
$\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.

Tangent Planes and Surfaces
Form:

$$F_x(x_0,y_0,z_0)(x-x_0) + F_y(x_0,y_0,z_0)(y-y_0) + F_z(x_0,y_0,z_0)(z-z_0) = 0$$

Derivative Rules

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$