# Math 120

## PSet 7

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# Contents

# Chapter 1

## 1.1 PSet 7

**Question 1**

Calculate the given iterated integrals.

1. $\int_0^1 \int_0^1 x\sqrt{1+4y}\,dy\,dx$

2. $\int_0^1 \int_1^2 \frac{xe^x}{y}\,dy\,dx$

***Solution:***
1)

$$\int_0^1 \int_0^1 x\sqrt{1+4y}\,dy\,dx$$

$$\int_0^1 x\sqrt{1+4y}\,dy$$

$$1+4y=t \quad r=dt$$

$$x\int_0^1 \frac{1}{4}\sqrt{t}dt$$

$$\frac{1}{4}x\int_0^1 \sqrt{t}dt$$

$$\frac{1}{4}x\cdot\frac{2t\sqrt{t}}{3}\bigg|_0^1$$

$$\frac{1}{4}x\cdot\frac{2(1+4y)\sqrt{1+4y}}{3}\Bigg|_0^1$$

$$\frac{x\sqrt{1+4y}(1+4y)}{6}\Bigg|_0^1$$

$$\frac{x\sqrt{1+4}(1+4)}{6}-\frac{x\sqrt{1}1}{6}$$

$$\frac{5x\sqrt{5}}{6}-\frac{x}{6}$$

$$\int_0^1 \frac{5x\sqrt{5}}{6}-\frac{x}{6}dx$$

$$\frac{1}{6}\int_0^1 5\sqrt{5}x-x\,dx$$

$$\frac{1}{6}\left(\int_0^1 5\sqrt{5}x dx - \int_o^1 x\, dx\right)$$

$$\int_0^1 5\sqrt{5}x\, dx \Rightarrow \left.\frac{5\sqrt{5}x^2}{2}\right|_0^1$$

$$\frac{5\sqrt{5}(1)^2}{2} - 0 = \frac{5\sqrt{5}}{2}$$

$$\int_0^1 x\, dx \Rightarrow \left.\frac{x^2}{2}\right|_0^1$$

$$\frac{1}{2} - 0 = \frac{1}{2}$$

$$\frac{1}{6}\left(\frac{5\sqrt{5}}{2} - \frac{1}{2}\right) = \frac{5\sqrt{5}-1}{12}$$

2)

$$\int_0^1 \int_1^2 \frac{xe^x}{y}\, dy\, dx$$

$$xe^x \int_1^2 \frac{1}{y}\, dy$$

$$xe^x \ln(y)\Big|_1^2 \Rightarrow xe^x \ln(2) - xe^x \ln(1) = xe^x \ln(2)$$

$$\ln(2) \int_0^1 xe^x\, dx$$

$$\ln(2)\ (xe^x - e^x)|_0^1$$

$$(\ln(2)e - \ln(2)e) - \left(\ln(2)(0) - \ln(2)e^0\right) = 0 - (-\ln(2)(1)) = \ln(2)$$

**Question 2**

(a) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^2 e^{-x^2-y^2}\, dy\, dx.$$

(b) Explain why

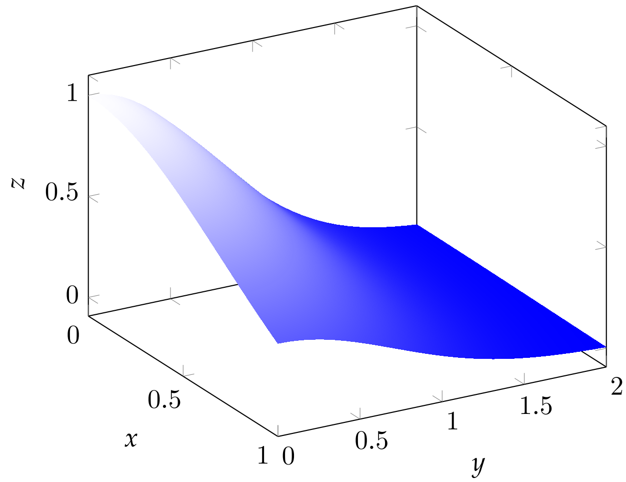
$$\int_0^1 \int_0^2 e^{-x^2-y^2}\, dy\, dx = \int_0^1 e^{-x^2}\, dx \cdot \int_0^2 e^{-y^2}\, dy.$$

(c) Use Desmos to compute

$$\int_0^1 \int_0^2 e^{-x^2-y^2}\, dy\, dx.$$

(Desmos will give a numerical approximation, but this is fine. In fact, there is no way to compute the antiderivatives necessary to get an exact answer.)

***Solution:***




**Question 3**

(a) Find the average value of the function $f(x, y) = \sin x \cos y$ on the rectangle $R = [0, \pi] \times [-\pi/2, \pi/2]$.

(b) Use symmetry to find the average value of $f(x, y) = \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$ on the region $R = [2\pi, 4\pi] \times [2\pi, 6\pi]$. Please explain your answer carefully.

***Solution:*** a)

$$f(x, y) = \sin x \cos y$$

$$R = [0, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) dA$$

$$A(R) = (\pi - 0) \times (\frac{\pi}{2} - -\frac{\pi}{2}) = \pi^2$$

$$\frac{1}{\pi^2} \int_0^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos y \, dy \, dx$$

$$\sin x \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos y \, dy$$

$$(\sin x) \sin y \Big|_{\frac{pi}{2}}^{\frac{\pi}{2}}$$

$$(\sin x) \sin\left(\frac{\pi}{2}\right) - (\sin x) \sin\left(\frac{-\pi}{2}\right) = 2 \sin x$$

$$\int_0^{\pi} 2 \sin x dx$$

$$-2 \cos x|_0^{\pi}$$

$$-\cos \pi - (-2) \cos(0) = 4$$

$$\frac{1}{\pi^2} \cdot 4 = \frac{4}{\pi^2}$$

b)

$$f(x, y) = \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$$

$$R = [2\pi, 4\pi] \times [2\pi, 6\pi]$$

$$f_{avg} = \iint\limits_R f(x,y)dA$$

$$A(R) = [4\pi - 2\pi] \times [6\pi - 2\pi] = 8\pi^2$$

$$\int_{2\pi}^{4\pi} \int_{2\pi}^{6\pi} \frac{4\sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3\,dy\,dx$$

$$\iint\limits_R f(x,y)dA - \iint\limits_R \frac{4\sin y}{e^{x^2}}dA - \iint\limits_R \frac{\cos x}{\ln y}dA + \iint\limits_R 3dA$$

$$\int_{2\pi}^{6\pi} 4\sin y\,dy = -4\left[\cos y\right]_{2\pi}^{6\pi} = -4(6\cos\pi - \cos 2\pi) = -4(1-1) = 0$$

$$\iint\limits_R f(x,y)\frac{4\sin y}{e^{x^2}}dA = \int_{2\pi}^{4\pi} \frac{1}{e^{x^2}}dx \times 0 = 0$$

$$\int_{2\pi}^{4\pi} \cos x dx = \sin x|_{2\pi}^{4\pi} = sin4\pi - \sin 2\pi = 0 - 0 = 0$$

$$\iint\limits_R \frac{\cos x}{\ln y}dA = \int_{2\pi}^{6\pi} \frac{1}{\ln y} \times 0 = 0$$

$$\iint\limits_R 3dA = 3 \times A(R) = 3 \times 8\pi^2 = 24\pi^2$$

$$\frac{24\pi^2}{8\pi^2} = 3$$

**Question 4**

In each part, draw the region $D$, and evaluate the integral.

1. $\iint_D \frac{y}{x^5+1}\,dA$, where $D$ is the region $D = \{(x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x^2\}$.
2. $\iint_D x^3\,dA$, where $D = \{(x,y) \mid 1 \leqslant x \leqslant e,\ 0 \leqslant y \leqslant \ln x\}$.

***Solution:*** 1.

$$\iint_D \frac{y}{x^5+1}\,dA \quad D = \{(x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x^2\}$$

$$\int_0^1 \int_0^{x^2} \frac{y}{x^5+1}\,dy\,dx$$

$$\frac{1}{x^5+1}\int_0^{x^2} ydy$$

$$\left.\frac{y^2}{2}\right|_0^{x^2} \Rightarrow \frac{\left(x^2\right)^2}{2} - \frac{0}{2} = \frac{x^4}{2}$$

$$\int_0^1 \frac{1}{x^5+1} \times \frac{x^4}{2}dx$$

$$x^5 + 1 = t \quad dt = 5x^4dx$$

$$\frac{1}{10}\int_0^1 \frac{1}{t}dt$$

$$\frac{1}{10}\ln|t|\Big|_0^1$$

$$\frac{1}{10}\left|x^5+1\right|\Big|_0^1$$

$$\frac{1}{10}\ln(1^5+1)-\frac{1}{10}\ln(1)$$

$$\frac{1}{10}\ln(2)-\frac{1}{10}\ln(1)=\frac{1}{10}\ln(2)$$

2.

$$\iint_D x^3\,dA \quad D=\{(x,y)\mid 1\leqslant x\leqslant e,\ 0\leqslant y\leqslant \ln x\}$$

$$\int_1^e\int_0^{\ln x} x^3\,dy\,dx$$

$$x^3\int_0^{\ln x} 1dy$$

$$\left(x^3\right)y\Big|_0^{\ln x}$$

$$x^3\ln x-0$$

$$uv-\int v\,du$$

$$u=\ln x \quad du=\frac{1}{x}\,dx$$

$$v=\frac{x^4}{4} \quad x^3dx$$

$$\frac{\ln x\cdot x^4}{4}-\int\frac{x^3}{4}\,dx$$

$$\left[\frac{\ln x^4\cdot x^4}{4}-\frac{x^4}{16}\right]_1^e$$

$$\left(\frac{\ln e\cdot e^4}{4}-\frac{e^4}{16}\right)-\left(\frac{\ln 1\cdot 1^4}{4}-\frac{1^4}{16}\right)$$

$$\left(\frac{\ln e\cdot e^4}{4}\right)-\left(0-\frac{1}{16}\right)$$

$$\left(\frac{e^4}{4}-\frac{e^4}{16}\right)+\frac{1}{16}$$

**Question 5**

Draw the region $D$. Set up the iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$\iint_D x^2e^{-xy}\,dA \quad \text{where } D \text{ is bounded by } y=x,\ x=4, \text{ and } y=0.$$

***Solution:***

**Question 6**

(a) Find the volume of the solid in the first octant enclosed by the parabolic cylinder $y=1-x^2$ and the planes $z=2-y$ and $z=y$.

(b) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^{1-x} (2 - y^2)\, dy\, dx.$$

**Question 7**

Sketch the region of integration and change the order of integration.

1. $\int_0^1 \int_{4x}^4 f(x, y)\, dy\, dx$
2. $\int_0^3 \int_{\sqrt{9-y}}^3 f(x, y)\, dx\, dy$
3. $\int_0^4 \int_0^{\ln 2x} f(x, y)\, dy\, dx$

**Question 8**

Evaluate the integral

$$\int_0^1 \int_x^1 \frac{e^x}{y}\, dy\, dx$$

by reversing the order of integration.

**Question 9**

Evaluate the given integral by converting to polar coordinates. Be sure to draw the region of integration in each part.

1. $\iint_R (x + y)\, dA$, where $R$ is the region that lies to the left of the $y$-axis between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
2. $\iint_R ye^x\, dA$, where $R$ is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$.

**Question 10**

Use polar coordinates to find the volume of the given solid.

(a) Inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.

(b) Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

**Question 11**

Evaluate the iterated integral

$$\int_0^b \int_{-\sqrt{b^2-y^2}}^0 x^2 y\, dx\, dy$$

by converting to polar coordinates.

**Question 12**

Let $D$ be the disk with center at the origin and radius $a$.

(a) Use your intuition: what do you expect is the average distance from points on the disk to the origin?

- less than $a/2$
- $a/2$
- between $a/2$ and $a$
- more than $a$

Give an intuitive explanation of your answer.

(b) What is the average distance from points in the disk to the origin?