

Math 120

PSet 7

Oct 24 2024

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Chapter 1

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Question 1

Calculate the given iterated integrals.

$$1. \int_0^1 \int_0^1 x\sqrt{1+4y} dy dx$$

$$2. \int_0^1 \int_1^2 \frac{xe^x}{y} dy dx$$

Solution:

1)

$$\int_0^1 \int_0^1 x\sqrt{1+4y} dy dx$$

$$\int_0^1 x\sqrt{1+4y} dy$$

$$1+4y = t \quad r = dt$$

$$x \int_0^1 \frac{1}{4}\sqrt{t} dt$$

$$\frac{1}{4}x \int_0^1 \sqrt{t} dt$$

$$\frac{1}{4}x \cdot \frac{2t\sqrt{t}}{3} \Big|_0^1$$

$$\frac{1}{4}x \cdot \frac{2(1+4y)\sqrt{1+4y}}{3} \Big|_0^1$$

$$\frac{x\sqrt{1+4y}(1+4y)}{6} \Big|_0^1$$

$$\frac{x\sqrt{1+4}(1+4)}{6} - \frac{x\sqrt{11}}{6}$$

$$\frac{5x\sqrt{5}}{6} - \frac{x}{6}$$

$$\int_0^1 \frac{5x\sqrt{5}}{6} - \frac{x}{6} dx$$

$$\frac{1}{6} \int_0^1 5\sqrt{5}x - x dx$$

$$\begin{aligned}
& \frac{1}{6} \left(\int_0^1 5\sqrt{5}x dx - \int_0^1 x dx \right) \\
& \int_0^1 5\sqrt{5}x dx \Rightarrow \frac{5\sqrt{5}x^2}{2} \Big|_0^1 \\
& \frac{5\sqrt{5}(1)^2}{2} - 0 = \frac{5\sqrt{5}}{2} \\
& \int_0^1 x dx \Rightarrow \frac{x^2}{2} \Big|_0^1 \\
& \frac{1}{2} - 0 = \frac{1}{2} \\
& \frac{1}{6} \left(\frac{5\sqrt{5}}{2} - \frac{1}{2} \right) = \frac{5\sqrt{5} - 1}{12}
\end{aligned}$$

2)

$$\begin{aligned}
& \int_0^1 \int_1^2 \frac{xe^x}{y} dy dx \\
& xe^x \int_1^2 \frac{1}{y} dy \\
& xe^x \ln(y) \Big|_1^2 \Rightarrow xe^x \ln(2) - xe^x \ln(1) = xe^x \ln(2) \\
& \ln(2) \int_0^1 xe^x dx \\
& \ln(2) (xe^x - e^x) \Big|_0^1 \\
& (\ln(2)e - \ln(2)e) - (\ln(2)(0) - \ln(2)e^0) = 0 - (-\ln(2)(1)) = \ln(2)
\end{aligned}$$

Question 2

- (a) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx.$$

- (b) Explain why

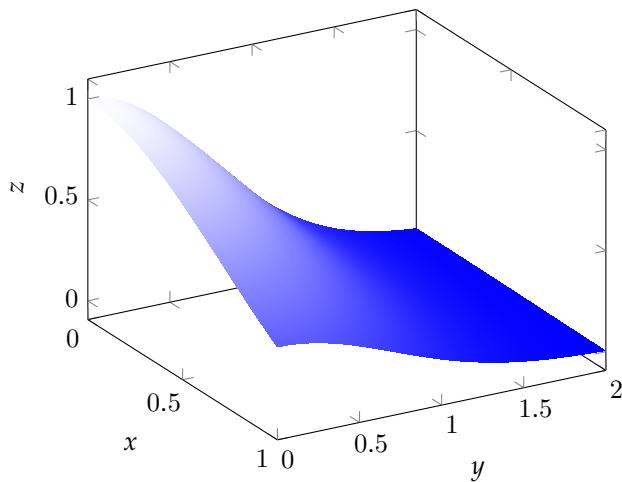
$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx = \int_0^1 e^{-x^2} dx \cdot \int_0^2 e^{-y^2} dy.$$

- (c) Use Desmos to compute

$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx.$$

(Desmos will give a numerical approximation, but this is fine. In fact, there is no way to compute the antiderivatives necessary to get an exact answer.)

Solution:



b)

It is because $e^{-x^2} = e^{x^2} \cdot e^{-y^2}$ and the bounds of y are independent of x , so that allows e^{-x^2} to be treated as a constant when integrating with respect to y and vice versa.

c)

$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx \approx 0.6588$$

Question 3

- (a) Find the average value of the function $f(x, y) = \sin x \cos y$ on the rectangle $R = [0, \pi] \times [-\pi/2, \pi/2]$.

- (b) Use symmetry to find the average value of $f(x, y) = \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$ on the region $R = [2\pi, 4\pi] \times [2\pi, 6\pi]$. Please explain your answer carefully.

Solution: a)

$$f(x, y) = \sin x \cos y$$

$$R = [0, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) dA$$

$$A(R) = (\pi - 0) \times \left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = \pi^2$$

$$\frac{1}{\pi^2} \int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos y \, dy \, dx$$

$$\sin x \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos y \, dy$$

$$(\sin x) \sin y \Big|_{\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$(\sin x) \sin \left(\frac{\pi}{2}\right) - (\sin x) \sin \left(-\frac{\pi}{2}\right) = 2 \sin x$$

$$\int_0^\pi 2 \sin x \, dx$$

$$-2 \cos x \Big|_0^\pi$$

$$-\cos \pi - (-2) \cos(0) = 4$$

$$\frac{1}{\pi^2} \cdot 4 = \frac{4}{\pi^2}$$

b)

$$f(x, y) = \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$$

$$R = [2\pi, 4\pi] \times [2\pi, 6\pi]$$

$$f_{avg} = \iint_R f(x, y) \, dA$$

$$A(R) = [4\pi - 2\pi] \times [6\pi - 2\pi] = 8\pi^2$$

$$\int_{2\pi}^{4\pi} \int_{2\pi}^{6\pi} \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3 \, dy \, dx$$

$$\iint_R f(x, y) \, dA - \iint_R \frac{4 \sin y}{e^{x^2}} \, dA - \iint_R \frac{\cos x}{\ln y} \, dA + \iint_R 3 \, dA$$

$$\int_{2\pi}^{6\pi} 4 \sin y \, dy = -4 [\cos y]_{2\pi}^{6\pi} = -4(6 \cos \pi - \cos 2\pi) = -4(1 - 1) = 0$$

$$\iint_R f(x, y) \frac{4 \sin y}{e^{x^2}} \, dA = \int_{2\pi}^{4\pi} \frac{1}{e^{x^2}} \, dx \times 0 = 0$$

$$\int_{2\pi}^{4\pi} \cos x \, dx = \sin x \Big|_{2\pi}^{4\pi} = \sin 4\pi - \sin 2\pi = 0 - 0 = 0$$

$$\iint_R \frac{\cos x}{\ln y} \, dA = \int_{2\pi}^{6\pi} \frac{1}{\ln y} \times 0 = 0$$

$$\iint_R 3 \, dA = 3 \times A(R) = 3 \times 8\pi^2 = 24\pi^2$$

$$\frac{24\pi^2}{8\pi^2} = 3$$

Question 4

In each part, draw the region D , and evaluate the integral.

1. $\iint_D \frac{y}{x^5+1} dA$, where D is the region $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.
2. $\iint_D x^3 dA$, where $D = \{(x, y) \mid 1 \leq x \leq e, 0 \leq y \leq \ln x\}$.

Solution: 1.

$$\iint_D \frac{y}{x^5+1} dA \quad D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\}$$

$$\int_0^1 \int_0^{x^2} \frac{y}{x^5+1} dy dx$$

$$\frac{1}{x^5+1} \int_0^{x^2} y dy$$

$$\frac{y^2}{2} \Big|_0^{x^2} \Rightarrow \frac{(x^2)^2}{2} - \frac{0}{2} = \frac{x^4}{2}$$

$$\int_0^1 \frac{1}{x^5+1} \times \frac{x^4}{2} dx$$

$$x^5 + 1 = t \quad dt = 5x^4 dx$$

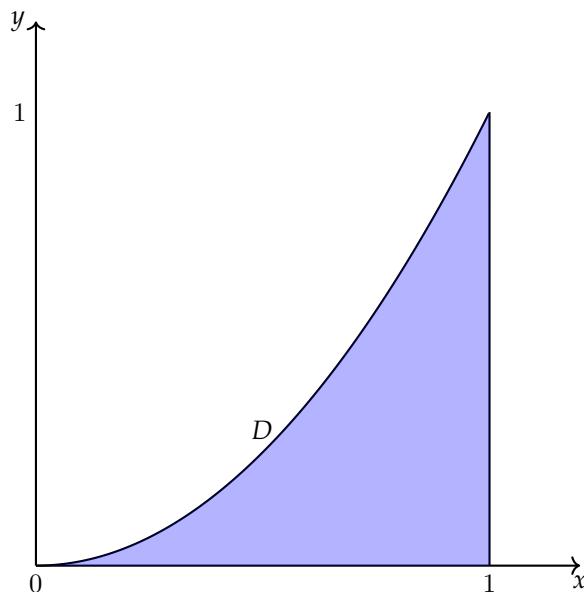
$$\frac{1}{10} \int_0^1 \frac{1}{t} dt$$

$$\frac{1}{10} \ln|t| \Big|_0^1$$

$$\frac{1}{10} |x^5 + 1| \Big|_0^1$$

$$\frac{1}{10} \ln(1^5 + 1) - \frac{1}{10} \ln(1)$$

$$\frac{1}{10} \ln(2) - \frac{1}{10} \ln(1) = \frac{1}{10} \ln(2)$$



2.

$$\iint_D x^3 \, dA \quad D = \{(x, y) \mid 1 \leq x \leq e, 0 \leq y \leq \ln x\}$$

$$\int_1^e \int_0^{\ln x} x^3 \, dy \, dx$$

$$x^3 \int_0^{\ln x} 1 \, dy$$

$$(x^3) y \Big|_0^{\ln x}$$

$$x^3 \ln x - 0$$

$$uv - \int v \, du$$

$$u = \ln x \quad du = \frac{1}{x} \, dx$$

$$v = \frac{x^4}{4} \quad x^3 \, dx$$

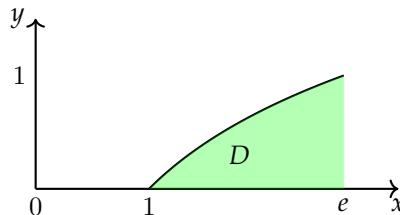
$$\frac{\ln x \cdot x^4}{4} - \int \frac{x^3}{4} \, dx$$

$$\left[\frac{\ln x^4 \cdot x^4}{4} - \frac{x^4}{16} \right]_1^e$$

$$\left(\frac{\ln e \cdot e^4}{4} - \frac{e^4}{16} \right) - \left(\frac{\ln 1 \cdot 1^4}{4} - \frac{1^4}{16} \right)$$

$$\left(\frac{\ln e \cdot e^4}{4} \right) - \left(0 - \frac{1}{16} \right)$$

$$\left(\frac{e^4}{4} - \frac{e^4}{16} \right) + \frac{1}{16}$$

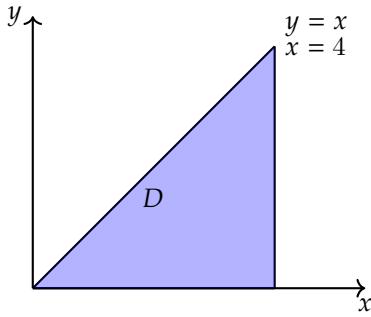


Question 5

Draw the region D . Set up the iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$\iint_D x^2 e^{-xy} \, dA \quad \text{where } D \text{ is bounded by } y = x, x = 4, \text{ and } y = 0.$$

Solution:



$$\begin{aligned}
 & \int_0^4 \int_0^x x^2 e^{-yx} dy dx \\
 & \int_0^4 \int_y^4 x^2 e^{-yx} dx dy \\
 & x^2 \int_y^4 e^{-yx} dy \\
 & (x^2) \frac{e^{-yx}}{x} \Big|_0^x = (x^2) \frac{e^{-yx}}{x} - (x^2) \frac{e^{-0 \cdot x}}{x} \\
 & xe^{-x^2} - x^2 \cdot \frac{1}{x} \Rightarrow -xe^{-x^2} + x \\
 & \int_0^4 -xe^{-x^2} + x dx \Rightarrow \int_0^4 -xe^{-x^2} dx \int_0^4 x dx \\
 & -x^2 = t \quad -2x = dt \\
 & \int_0^4 \frac{1}{2} e^t dt \Rightarrow \frac{1}{2} \int_0^4 e^t dt \\
 & \frac{1}{2} e^t \Big|_0^4 \Rightarrow \frac{1}{2} \Big|_0^4 \\
 & \frac{1}{2} e^{-4^2} - \frac{1}{2} e^{-0^2} \Rightarrow \frac{1}{2} e^{-16} - \frac{1}{2} \\
 & \int_0^4 x dx \\
 & \frac{x^2}{2} \Big|_0^4 \\
 & \frac{4^2}{2} - \frac{0^2}{2} = \frac{16}{2} = 8 \\
 & \int_0^4 -xe^{-x^2} + x dx = \frac{1}{2} e^{-16} + \frac{15}{2}
 \end{aligned}$$

Question 6

- (a) Find the volume of the solid in the first octant enclosed by the parabolic cylinder $y = 1 - x^2$ and the planes $z = 2 - y$ and $z = y$.

(b) Sketch the solid whose volume is given by the iterated integral

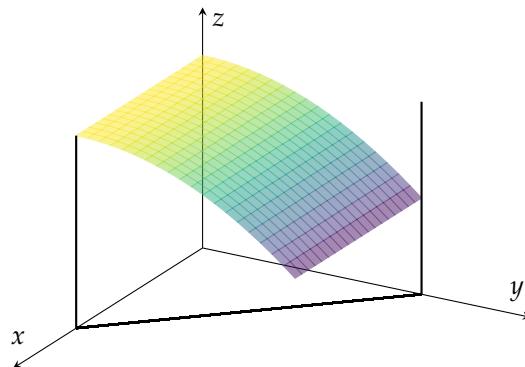
$$\int_0^1 \int_0^{1-x} (2 - y^2) dy dx.$$

Solution:

a)

$$\begin{aligned}
 y &= -x^2 & z &= 2 - y & z &= y \\
 x, y, z &\geq 0 \\
 2 - y &= y \Rightarrow 2y = 2 \Rightarrow y = 1 \Rightarrow 0 \leq y \leq 1 - x^2 \\
 \text{height} &= (2 - y) - y \Rightarrow 2 - 2y \\
 V &= \int_0^1 \int_0^{1-x^2} 2 - 2y dy dx \\
 &\quad \int_0^{1-x^2} 2 - 2y dy \\
 &\quad 2y - y^2 \Big|_0^{1-x^2} \\
 &\quad 2(1 - x^2) - (1 - x^2) - 2(0) - (0)^2 \\
 &\quad 2 - 2x^2 - 1 + 2x^2 - x^4 \\
 &\quad 1 - x^4 \\
 &\quad \int_0^1 1 - x^4 dx \\
 &\quad x - \frac{x^5}{5} \Big|_0^1 \\
 &\quad 1 - \frac{1^5}{5} - 0 - \frac{0^5}{5} \\
 &\quad 1 - \frac{1}{5} = \frac{4}{5}
 \end{aligned}$$

b)



Question 7

Sketch the region of integration and change the order of integration.

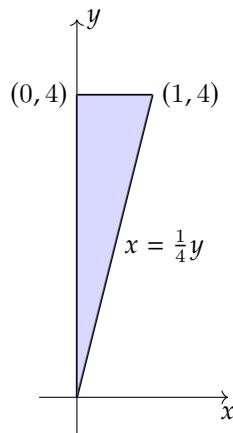
$$1. \int_0^1 \int_{4x}^4 f(x, y) dy dx$$

$$2. \int_0^3 \int_0^{\sqrt{9-y}} f(x, y) dx dy$$

$$3. \int_0^4 \int_0^{\ln 2x} f(x, y) dy dx$$

Solution:

a)



$$\int_0^1 \int_{4x}^4 f(x, y) dy dx$$

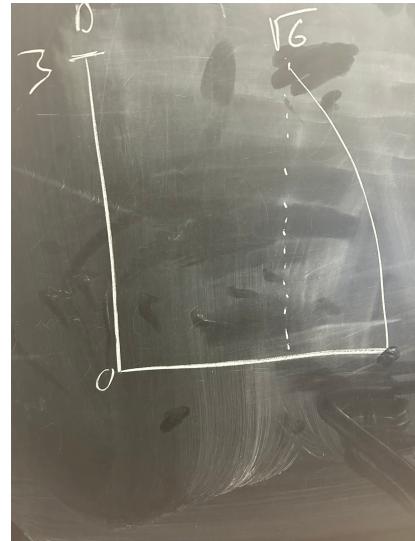
$$\iint_D f(x, y) dA$$

$$D = \{(x, y) | 0 \leq x \leq 1, 4x \leq y \leq 4\}$$

$$D = \{(x, y) | 0 \leq y \leq 4, 0 \leq x \leq \frac{1}{4}y\}$$

$$\iint_D f(x, y) dA = \int_0^4 \int_0^{\frac{1}{4}y} f(x, y) dx dy$$

b)



$$\int_0^3 \int_0^{\sqrt{9-y}} f(x, y) dx dy$$

$$\iint_D f(x, y) dA$$

$$D = \{(x, y) | 0 \leq x \leq \sqrt{9-y}, 0 \leq y \leq 3\}$$

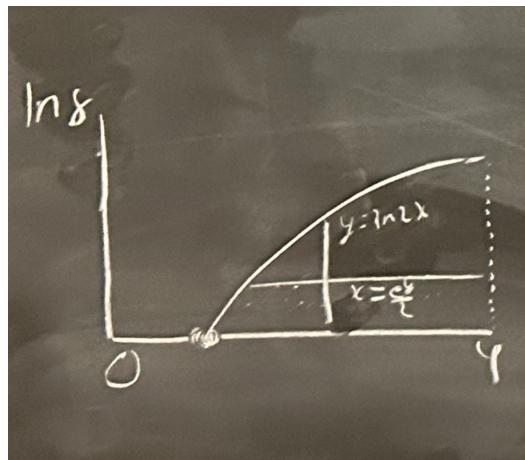
$$x = \sqrt{9-y} \quad x^2 - 9 = -y \quad -x^2 + 9 = y$$

$$-x^2 + 9 = 3 \quad -x^2 = -6 \Rightarrow x^2 = 6 \Rightarrow x = \sqrt{6}$$

$$D = \{(x, y) | 0 \leq y \leq -x^2 + 9, \sqrt{6} \leq x \leq 3\}$$

$$\int_{\sqrt{6}}^3 \int_0^{-x^2+9} f(x, y) dy dx$$

c)



$$\int_0^4 \int_0^{\ln 2x} f(x, y) dy dx$$

$$\iint_D f(x, y) dA \quad D = \{0 \leq x \leq 4, 0 \leq y \leq \ln x\}$$

$$y = \ln 2x \Rightarrow y = \ln 2(4) \Rightarrow y = \ln 8 \Rightarrow 0 = \ln 2x \Rightarrow \ln 1 = \ln 2x \Rightarrow x = \frac{1}{2}$$

$$\iint_D f(x, y) dA \quad D = \{(x, y) | 0 \leq y \leq \ln 8, \frac{1}{2} \leq x \leq \frac{e^y}{2}\}$$

$$\int_0^{\ln 8} \int_{\frac{1}{2}}^{\frac{e^y}{2}} f(x, y) dx dy$$

Question 8

Evaluate the integral

$$\int_0^1 \int_x^1 e^{\frac{x}{y}} dy dx$$

by reversing the order of integration.

Solution:

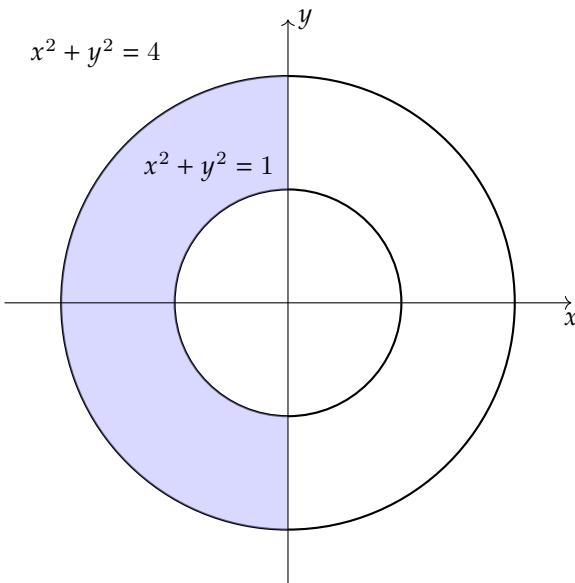
$$\begin{aligned} & \int_0^1 \int_0^y e^{\frac{x}{y}} dx dy \\ & \quad ye^{\frac{x}{y}} \Big|_0^y \\ & \int_0^1 ye - y dy \\ & \quad \frac{ey^2}{2} - \frac{y^2}{2} \Big|_0^1 \\ & \quad \frac{e-1}{2} \end{aligned}$$

Question 9

Evaluate the given integral by converting to polar coordinates. Be sure to draw the region of integration in each part.

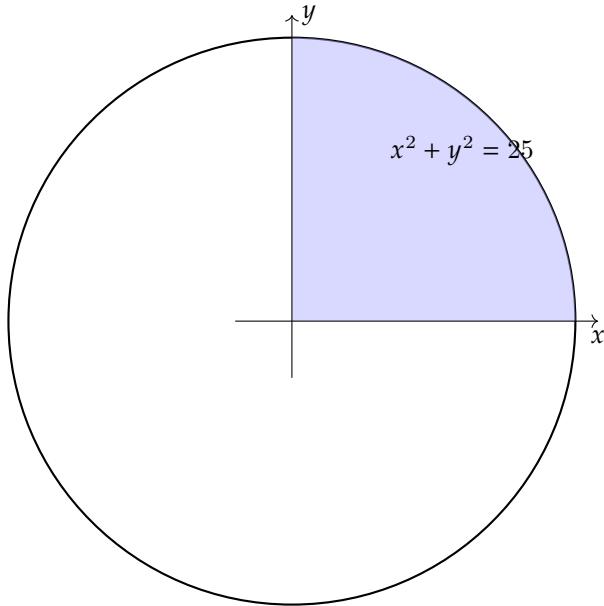
1. $\iint_R (x + y) dA$, where R is the region that lies to the left of the y -axis between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
2. $\iint_R ye^x dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$.

Solution: a)



$$\begin{aligned}
& x \leq 0 \quad x + y^2 = 1 \quad x^2 + y^2 = 4 \\
& x = r \cos \theta \quad y = r \sin \theta \quad dA = r dr d\theta \\
& R : 1 \leq r \leq 2 \\
& x + y = r \cos \theta + r \sin \theta \Rightarrow r(\cos \theta \sin \theta) \\
& \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^2 r(\cos \theta \sin \theta) r dr d\theta \\
& \cos \theta + \sin \theta \int_1^2 r^2 dr \\
& \left. \frac{r^3}{3} \right|_1^2 \\
& \frac{2^3}{3} - \frac{1}{3} = \frac{7}{3} \\
& \frac{7}{3} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos \theta + \sin \theta d\theta \\
& \left. \frac{7}{3} \sin \theta \right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \left. \frac{7}{3} \cos \theta \right|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
& \frac{7}{3}(-1) - \frac{7}{3}(1) = -\frac{14}{3} \\
& -\frac{7}{3} \cos \frac{3\pi}{2} + \frac{7}{3} \cos \frac{\pi}{2} \\
& -\frac{7}{3}(0) + \frac{7}{3} = 0 \\
& -\frac{14}{3}
\end{aligned}$$

b)



$$\iint_R ye^x dA$$

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = 25$$

$$\text{1st quadrant} \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$ye^x \Rightarrow r \sin \theta e^{r \cos \theta}$$

$$dA = r dr d\theta$$

$$\int_0^{\frac{\pi}{2}} \int_0^5 r \sin \theta e^{r \cos \theta}$$

$$I(\theta) = \int_0^5 r^2 \sin \theta e^{r \cos \theta} dr$$

$$\frac{\partial}{\partial \theta} e^{r \cos \theta} = -r \sin \theta e^{r \cos \theta}$$

$$r \sin \theta e^{r \cos \theta} = -\frac{\partial}{\partial \theta} e^{r \cos \theta}$$

$$I = \int_0^{\frac{\pi}{2}} \int_0^5 -r \frac{\partial}{\partial \theta} e^{r \cos \theta} dr d\theta$$

$$I = - \int_0^5 r \left(\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \theta} e^{r \cos \theta} d\theta \right) dr$$

$$\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \theta} e^{r \cos \theta} d\theta = e^{r \cos \frac{\pi}{2}} \Big|_0^{\frac{\pi}{2}}$$

$$e^{r \cos \frac{\pi}{2}} - e^{r \cos 0} = e^{r \cdot 0} - e^{r \cdot 1} = 1 - e^r$$

$$I = \int_0^5 re^r dr - \int_0^3 rdr$$

$$u = r \quad du = dr \quad v = e^r \quad dv = e^r dr$$

$$\int_0^5 re^r dr = re^r - \int_0^5 e^r dr = re^r - e^r + K$$

$$\int_0^5 re^r dr = [re^r - e^r]_0^5 = (5e^5 - e^5) - (0 - e^0) = 4e^5 + 1$$

$$\int_0^5 r dr = \frac{1}{2} r^2 \Big|_0^5 = \frac{1}{2} (25 - 0) = \frac{25}{2}$$

$$I = (4e^5 + 1) - \frac{25}{2} = 4e^5 - 11.5$$

Question 10

Use polar coordinates to find the volume of the given solid.

- (a) Inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.
- (b) Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

Solution:

a)

$$\begin{aligned}
 & x^2 + y^2 + z^2 = 4 \quad x^2 + y^2 = 1 \\
 & x = r \cos \theta \quad y = r \sin \theta \quad 0 \leq \theta \leq 2\pi \\
 & r^2 + z^2 = 4 \quad x^2 + y^2 = 1 \Rightarrow r^2 = 1 \Rightarrow r = 1 \\
 & r^2 + z^2 = 4 \Rightarrow r^2 = 4 - z^2 \Rightarrow r = \sqrt{4 - z^2} \\
 & V = \iint_D [z_{\text{upper}} - z_{\text{lower}}] \\
 & V = \int_0^{2\pi} \int_1^2 r \sqrt{4 - r^2 - (-\sqrt{4 - r^2})} r dr d\theta \\
 & V = 2 \int_0^{2\pi} \int_1^2 \sqrt{4 - r^2} dr d\theta \\
 & 4 - r^2 = t \quad -2r = dt \\
 & - \int_1^2 \frac{1}{2} \sqrt{t} dt \Rightarrow -\frac{1}{2} \int_1^2 \sqrt{t} dt = -\frac{1}{2} \cdot \frac{2t\sqrt{t}}{3} \Big|_1^2 \\
 & -\frac{1}{2} \cdot \frac{2(4 - r^2)\sqrt{4 - r^2}}{3} \Big|_1^2 \left(-\frac{1}{2} \cdot \frac{2(4 - 2^2)\sqrt{4 - 2^2}}{3} \right) - \left(-\frac{1}{2} \cdot \frac{2(4 - 1^2)\sqrt{4 - 1^2}}{3} \right) \\
 & V = 2 \left(\int_0^{2\pi} d\theta \right) \left(\int_1^2 r \sqrt{4 - r^2} dr \right) \\
 & \int_0^{2\pi} d\theta = 2\pi \quad V = 2 \cdot 2\pi \cdot \int_1^2 r \sqrt{4 - r^2} dr = 4\pi \int_1^2 r \sqrt{4 - r^2} dr \\
 & - \left(-\frac{1}{2} \cdot \frac{2(3)\sqrt{3}}{3} \right) \Rightarrow \left(\frac{1}{2}\sqrt{3} \right) \Rightarrow -(-\sqrt{3}) \\
 & 4\pi\sqrt{3}
 \end{aligned}$$

b)

$$\begin{aligned}
 z &= 3x^2 + 3y^2 = 3(x^2 + y^2) = 3(x^2 + y^2) = 3r^2 \\
 z &= 4 - x^2 - y^2 = 4 - r^2 \\
 3r^2 &= 4 - r^2 \\
 4r^2 &= 4 \Rightarrow r = 1 \\
 \int_0^{2\pi} \int_0^1 (4 - r^2 - 3r^2) r dr d\theta & \\
 \int_0^{2\pi} \int_0^1 (4 - 4r^2) r dr d\theta & \\
 2r - r^4 \Big|_0^1 d\theta & \Rightarrow (2(1)^2 - 1) - 0 \\
 \int_0^{2\pi} 1 d\theta & \\
 \theta \Big|_0^{2\pi} & = 2\pi
 \end{aligned}$$

Question 11

Evaluate the iterated integral

$$\int_0^b \int_{-\sqrt{b^2-y^2}}^0 x^2 y \, dx \, dy$$

by converting to polar coordinates.

Solution:

$$\begin{aligned}
& \int_0^b \int_{-\sqrt{b^2-y^2}}^0 x^2 y \, dx \, dy \\
& y = 0 \quad \text{to} \quad y = b \\
& x = -\sqrt{b^2 - y^2} \text{ to} \quad \text{to } x = 0 \\
& \text{left half of } x^2 + y^2 = b^2 \\
& x = r \cos \theta \quad y = r \sin \theta \quad 0 \leq r \leq b \quad \frac{\pi}{2} \leq \theta \leq \pi \\
& x^y = (r \cos \theta)^2 (r \sin \theta) = r^3 \cos \theta \sin \theta \\
& \int_{\frac{\pi}{2}}^{\pi} \int_0^b (r^3 \cos^2 \theta \sin \theta) r \, dr \, d\theta \\
& \int_{\frac{\pi}{2}}^{\pi} \int_0^b r^4 \cos^2 \theta \sin \theta \, d\theta \, dr \\
& \cos^2 \theta \sin \theta \int_0^b r^4 \, dr \\
& \cos^2 \theta \sin \theta \frac{r^5}{5} \Big|_0^b \\
& \cos^2 \theta \sin \theta \frac{b^5}{5} - 0 \\
& \int_{\frac{\pi}{2}}^{\pi} \frac{b^5}{5} \cos^2 \theta \sin \theta \, d\theta \\
& \frac{b^5}{5} \int_{\frac{\pi}{2}}^{\pi} \cos^2 \theta \sin \theta \, d\theta \\
& \frac{b^5}{5} \frac{\cos^3 \theta}{3} \Big|_{\frac{\pi}{2}}^{\pi} \\
& \left(\frac{b^5}{5} \right) \left(\frac{\cos^3 \frac{\pi}{2}}{3} \right) - \left(\frac{b^5}{5} \right) \left(\frac{\cos^3 \pi}{3} \right) \\
& 0 - \frac{b^5}{5} \cdot \frac{-1}{3} \\
& \frac{b^5}{15}
\end{aligned}$$

Question 12

Let D be the disk with center at the origin and radius a .

(a) Use your intuition: what do you expect is the average distance from points on the disk to the origin?

- less than $a/2$
- $a/2$
- between $a/2$ and a
- more than a

Give an intuitive explanation of your answer.

(b) What is the average distance from points in the disk to the origin?

Solution: The area of the disk should be greater on the interval of $\left[\frac{a}{2}, a\right]$ than from $\left[0, \frac{a}{2}\right]$ which means there are more points on the interval of $\left[\frac{a}{2}, a\right]$ meaning the average distance is on this interval.
b)

$$\begin{aligned} D &= \frac{1}{A} \iint_A d \, da \\ D &= \frac{1}{A} \int_0^{2\pi} \int_0^a r \cdot r \, dr \, d\theta \\ \frac{r^3}{3} \Big|_0^a &\Rightarrow \frac{a^3}{3} - 0 = \frac{a^3}{3} \\ A &= a^2\pi \\ \frac{1}{a^2\pi} \int_0^{2\pi} \frac{a^3}{3} \, d\theta & \\ \frac{1}{a^2\pi} \left(\frac{a^3}{3}\right) \Big|_0^{2\pi} & \\ \frac{2a}{3} & \end{aligned}$$