

Math 120

PSet 2

Sep 12 2024

Contents

Chapter 1

Page 2

1.1 PSet 2

2

Chapter 1

1.1 PSet 2

Question 1

Consider the line L_1 given by $x + 2y = 7$ and the line L_2 given by $5x - y = 2$.

1. There are two unit vectors that are parallel to L_1 . What are they?
2. There are two unit vectors that are perpendicular to L_1 . What are they?
3. Find the acute angle between the lines L_1 and L_2 . First find an exact expression and then approximate to the nearest degree.

Solution:

a)

$$L_1 = x + 2y = 7$$

$$y = -\frac{1}{2}x + \frac{7}{2}$$

$$v_1 = (1, m_1) = (1, -\frac{1}{2})$$

$$|v_1| = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$u_1 = \frac{1}{\frac{\sqrt{5}}{2}} \left(1, -\frac{1}{2}\right) = \frac{2}{\sqrt{5}} \left(1, -\frac{1}{2}\right) = \left(\frac{\sqrt{2}}{5}, -\frac{1}{\sqrt{5}}\right)$$

$$-u_1 = \left(-\frac{\sqrt{2}}{5}, \frac{1}{\sqrt{5}}\right)$$

b)

$$\text{slope of line perpendicular to } L_1 : -\frac{1}{-\frac{1}{2}} = 2$$

$$\langle 1, 2 \rangle$$

$$|n| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$u_1 = \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}$$

$$u_2 = -u_1 = -\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}$$

c)

$$\vec{v}_1 = \langle 1, -2 \rangle$$

$$\vec{v}_2 = \langle 5, 1 \rangle$$

$$\begin{aligned}\cos(\theta) &= \frac{v_1 \cdot v_2}{|v_1||v_2|} \\ \cos(\theta) &= \frac{-2+5}{\sqrt{5}\sqrt{26}} = \frac{3}{\sqrt{130}} \\ \theta &= \arccos\left(\frac{3}{\sqrt{130}}\right)\end{aligned}$$

Question 2

Find all values of x such that the angle between the vectors $\langle 1, -1, 0 \rangle$ and $\langle 2, x, 1 \rangle$ is $\frac{\pi}{3}$.

Solution:

$$\begin{aligned}v_1 &= \langle 1, -1, 0 \rangle \\ v_2 &= \langle 2, x, 1 \rangle \\ \cos\left(\frac{\pi}{3}\right) &= \frac{1}{2} \\ \frac{1}{2} &= \frac{v_1 \cdot v_2}{|v_1||v_2|} = \frac{2-x}{(\sqrt{2})\sqrt{5+x^2}} \\ 4-2x &= \sqrt{10+2x^2} \\ 10+2x^2 &= 16-16x+4x^2 \\ -2x^2+16x-6 &= 0 \\ x^2-8x+3 &= 0 \\ x &= \frac{8 \pm \sqrt{(-8)^2 - 4(1)(3)}}{2 \cdot 1} \\ x &= \frac{8 \pm \sqrt{52}}{2} \\ x &= 4 - \sqrt{13}\end{aligned}$$

Question 3

Find the scalar and vector projections of $\vec{b} = \hat{i} + \hat{j}$ onto $\vec{a} = -\hat{i} + 3\hat{j}$, and illustrate your answers with a sketch.

Solution:

Scalar Projection:

$$\begin{aligned}\vec{b} &= \hat{i} + \hat{j} \\ \vec{a} &= -\hat{i} + 3\hat{j} \\ \text{comp}_a \mathbf{b} &= \frac{a \cdot b}{|a|} \\ \frac{a \cdot b}{|a|} &= \frac{2}{\sqrt{10}} = \frac{2\sqrt{10}}{10} = \frac{\sqrt{10}}{5}\end{aligned}$$

Vector Projection:

$$\begin{aligned}\text{proj}_a \mathbf{b} &= \left(\frac{a \cdot b}{|a|^2}\right) a \\ \left(\frac{a \cdot b}{|a|^2}\right) a &= \frac{2}{\sqrt{10}^2} a = \frac{2}{10} a = \frac{1}{5} a \\ \frac{1}{5} a &= \frac{1}{5}(-\hat{i}, 3\hat{j}) = \left\langle -\frac{1}{5}\hat{i}, \frac{3}{5}\hat{j} \right\rangle\end{aligned}$$

Question 4

Find two vectors of length 2 that are orthogonal to both $\vec{v} = \langle 2, 4, 4 \rangle$ and $\vec{w} = \langle 1, -1, -3 \rangle$.

Solution:

$$\vec{v} \times \vec{w} = \langle 4(-3) - 4(-1), 4(1) - 2(-3), 2(-1) - 4(1) \rangle = \langle -8, 10, -6 \rangle$$

$$|u| = \sqrt{(-8)^2 + 10^2 + (-6)^2} = \sqrt{200}$$

$$2 = x \cdot \sqrt{200}$$

$$x = \frac{2}{\sqrt{200}} = \frac{1}{5\sqrt{2}}$$

$$u_1 = \frac{1}{5\sqrt{2}} \langle -8, 10, -6 \rangle$$

$$u_2 = -\frac{1}{5\sqrt{2}} \langle -8, 10, -6 \rangle$$

Question 5

Let $\vec{a} = \langle 3, 1, 0 \rangle$. Find all vectors $\vec{b} = \langle b_1, b_2, b_3 \rangle$ such that $\vec{a} \times \vec{b}$ is parallel to the z-axis and pointing in the positive z direction. Illustrate with a sketch, in which all vectors are drawn as position vectors, i.e., with the tail at the origin.

Solution:

$$\vec{a} \times \vec{b} = \langle 0, 0, c \rangle$$

$$\langle 3, 1, 0 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle 0, 0, c \rangle$$

$$\langle 3, 1, 0 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle 1(b_3) - 0(b_2), 0(b_1) - 3(b_3), 3(b_2) - 1(b_1) \rangle$$

$$\langle 3, 1, 0 \rangle \times \langle b_1, b_2, b_3 \rangle = \langle b_3, -3b_3, 3b_2 - b_1 \rangle$$

$$3b_2 - b_1 > 0$$

$$b_3 = 0$$

$$-3b_3 = 0$$

All vectors in the form of $\langle b_1, b_2, 0 \rangle$ where $3b_2 - b_1 > 0$

Question 6

Consider the four points in \mathbb{R}^3 , $K(1, 2, 3)$, $L(1, 3, 6)$, $M(3, 8, 6)$, and $N(3, 7, 3)$.

1. Show that the vectors \vec{KL} , \vec{KM} , and \vec{KN} are coplanar. Explain why this means that K , L , M , and N all lie in the same plane.
2. From part (a), we know that K , L , M , and N are the vertices of a quadrilateral. Explain how you can tell that this quadrilateral is actually a parallelogram.
3. Find the area of the parallelogram with vertices K , L , M , and N .

Solution: a)

$$\vec{KL} = \langle 1 - 1, 3 - 2, 6 - 3 \rangle = \langle 0, 1, 3 \rangle$$

$$\vec{KM} = \langle 3 - 1, 8 - 2, 6 - 3 \rangle = \langle 2, 6, 3 \rangle$$

$$\vec{KN} = \langle 3 - 1, 7 - 2, 3 - 3 \rangle = \langle 2, 5, 0 \rangle$$

$$\vec{KL} \cdot (\vec{KM} \times \vec{KN}) = \vec{KL} \cdot \langle 6(0) - 3(5), 3(2) - 2(0), 2(5) - 6(2) \rangle$$

$$\begin{aligned}\vec{KL} \cdot (\vec{KM} \times \vec{KN}) &= \vec{KL} \times \langle -15, 6, -2 \rangle \\ \vec{KL} \cdot (\vec{KM} \times \vec{KN}) &= \vec{KL} \times \langle -15, 6, -2 \rangle = 0(15) + 6(1) + 3(-2) = 0 \\ |\vec{KL} \cdot (\vec{KM} \times \vec{KN})| &= 0\end{aligned}$$

They are coplanar because the volume determined by the vectors is 0, therefore they must lie on the same plane.
b).

$$\begin{aligned}\vec{KL} &= \langle 0, 1, 3 \rangle \\ \vec{MN} &= \langle 3 - 3, 7 - 8, 3 - 6 \rangle = \langle 0, -1, -3 \rangle \\ \vec{KL} &= -\vec{MN}\end{aligned}$$

Since $\vec{KL} = -\vec{MN}$ these two sides are parallel.

$$\begin{aligned}\vec{KL} &= \langle 2, 6, 3 \rangle \\ \vec{LN} &= \langle 3 - 1, 7 - 3, 3 - 6 \rangle = \langle 2, 4, -3 \rangle\end{aligned}$$

Although $\vec{KM} = -\vec{LN}$ are not negatives of each other or equal in magnitude they form the other side of the parallelogram

c)

$$\begin{aligned}\vec{KL} \times \vec{KM} &= \langle 1(3) - 3(6), 3(2) - 0(3), 0(6) - (1)(2) \rangle = \langle -15, 6, -2 \rangle \\ \sqrt{(-15)^2 + 6^2 + (-2)^2} &= \sqrt{265}\end{aligned}$$

Question 7

Find the vector equation and parametric equations for the line through the point $(1, 2, -2)$ parallel to the line $x = t - 2$, $y = -2t + 1$, $z = 3$.

Solution:

$$\begin{aligned}x &= t - 2 \quad y = -2t + 1 \quad z = 3 \\ \vec{d} &= \langle 1, -2, 0 \rangle \\ \vec{r}(t) &= \vec{r}_0 + t\vec{d} \\ \vec{r}(t) &= \langle 1, 2, -2 \rangle + t\langle 1, -2, 0 \rangle \\ x(t) &= 1 + t \quad y(t) = 2 - 2t \quad z(t) = -2\end{aligned}$$

Vector Equation: $\vec{r}(t) = \langle 1, 2, -2 \rangle + t\langle 1, -2, 0 \rangle$

Parametric Equation: $x(t) = 1 + t$, $y(t) = 2 - 2t$, $z(t) = -2$

Question 8

Consider the lines $L_1 : x = t + 3$, $y = 2t - 1$, $z = -t$, and $L_2 : x = t - 1$, $y = t - 4$, $z = -t + 4$. Determine whether the L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

Solution:

$$\begin{aligned}d_1 &= \langle 1, 2, -1 \rangle \quad d_2 = \langle 1, 1, -1 \rangle \\ \frac{1}{1} &\neq \frac{2}{1} \neq \frac{-1}{-1} \\ t_1 + 3 &= t_2 - 1 \quad 2t_1 - 1 = t_2 - 4 \quad -t_1 = -t_2 + 4 \\ t_1 + 3 &= t_2 - 1 \Rightarrow t_1 - t_2 = -4 \\ -t_1 &= -t_2 + 4 \Rightarrow t_1 = t_2 - 4 \\ (t_2 - 4) - t_2 &= -4 \Rightarrow -4 = -4\end{aligned}$$

$$2t_1 - 1 = t_2 - 4$$

$$2(t_2 - 4) - 1 = t_2 - 4$$

$$2t_2 - 9 = t_2 - 4$$

$$t_2 = 5$$

$$t_1 = t_2 - 4 = 5 - 4 = 1$$

$$x_1 = 1 + 3 = 4 \quad y_1 = 2(1) - 1 = 1 \quad z_1 = -1$$

Point of Intersection: (4, 1, -1)

Question 9

Consider the planes $x + y + 2z = 4$ and $2x - y - 2z = 1$.

1. Find a vector equation for the line of intersection of the planes.
2. Find the angle between the planes. First find an exact expression and then approximate to the nearest degree.

Solution:

a)

$$n_1 = \langle 1, 1, 2 \rangle \quad n_2 = \langle 2, -1, -2 \rangle$$

$$d = n_1 \times n_2 = \langle 1(-2) - 2(-1), 2(2) - 1(-2), 1(-1) - 1(2) \rangle = \langle 0, 6, -3 \rangle$$

$$(x + y + 2z) + (2x - y - 2z) = 4 + 1$$

$$3x = 5 \Rightarrow x = \frac{5}{3}$$

$$\frac{5}{3} + y + 2z = 4 = \frac{7}{3}$$

$$y + 2z = \frac{7}{3}$$

$$2\left(\frac{5}{3}\right) - y - 2z = 1$$

$$-y - 2z = -\frac{7}{3}$$

$$y + 2z = \frac{7}{3}$$

$$y + 0 = \frac{7}{3}$$

$$\left(\frac{5}{3}, \frac{7}{3}, 0\right)$$

$$\vec{r}(t) = \left(\frac{5}{3}, \frac{7}{3}, 0\right) + t\langle 0, 6, -3 \rangle$$

b)

$$n_1 = \langle 1, 1, 2 \rangle \quad n_2 = \langle 2, -1, -2 \rangle$$

$$\cos(\theta) = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{1(2) + 1(-1) + 2(-2)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{-3}{\sqrt{6}\sqrt{9}} = -\frac{1}{\sqrt{6}}$$

$$\theta = \arccos\left(-\frac{1}{\sqrt{6}}\right) \approx 114$$

Question 10

Let P be the plane $x + y + 2z = 1$ and let A be the point $(1, 1, 1)$.

- Find an equation of the plane through point A parallel to plane P .
- Find a vector equation for the line through the point A which is perpendicular to the plane P . Call this line L .
- Find the point of intersection of the line L (from part (b)) and the plane P .
- Find the point on the plane P closest to the point A , and then find the shortest distance from the point A to the plane P .

Solution:

a)

$$P : x + y + 2z = 1 \quad n_1 = \langle 1, 1, 2 \rangle$$

$$n_2 = t\langle 1, 1, 2 \rangle$$

$$P_2 : tx + ty + 2tz = d$$

$$t1 + t1 + 2t(1) = d \Rightarrow t + t + 2t = d \Rightarrow d = 4t$$

$$tx + ty + 2tz = 4t \Rightarrow x + y + 2z = 4$$

b)

$$n = (1, 1, 2)$$

$$r(t) = (1, 1, 1) + t(1, 1, 2)$$

c)

$$P : x + y + 2z = 1$$

$$L : x = 1 + t \quad y = 1 + t \quad z = 1 + 2t$$

$$(1 + t) + (1 + t) + 2(1 + 2t) = 1$$

$$2 + 2t + 2 + 4t = 1 \Rightarrow 6t + 4 = 1$$

$$6t = -3 \Rightarrow t = -\frac{1}{2}$$

$$x : -\frac{1}{2} + 1 \quad y : -\frac{1}{2} + 1 \quad z : 1 + 2\left(-\frac{1}{2}\right)$$

Point: $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$

d) Point: $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ because it is when the line through point A is perpendicular to the plane P

$$d = \sqrt{\left(1 - \frac{1}{2}\right)^2 + \left(1 - \frac{1}{2}\right)^2 + (1 - 0)^2} = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{\frac{3}{2}}$$