Math 120

PSet 7

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Contents

Chapter 1

1.1 PSet 7

Question 1

Evaluate the scalar line integral

$$\int_C (3x + y)\, ds,$$

where C is the line segment from $(-1, 3)$ to $(4, 2)$.

Solution:

$$\int_C (3x + y) ds$$

$$(-1, 3) \quad (4, 2)$$

$$f(t) = (1-, 3) + t((4, 2) - (-1, 3))$$

$$f(t) = (-1, 3) + t(5, -1) = \langle -1 + 5t, 3 - t\rangle$$

$$x = -1 + 5t \quad y = 3 - t \quad t \,\epsilon\, [0, 1]$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\frac{dx}{dt} = 5 \quad \frac{dy}{dt} = -1$$

$$ds = \sqrt{5^2 + (-1)^2}\, dt = \sqrt{26}\, dt$$

$$3x + y \Rightarrow 3(-1 + 5t) + (3 - t) \Rightarrow -3 + 15t + 3 - t = 14t$$

$$\int_0^1 14t\sqrt{26}\, dt \Rightarrow \sqrt{26}\int_0^1 14t\, dt$$

$$7\sqrt{26}t^2\Big|_0^1 = 7\sqrt{26}(1)^2 - 7\sqrt{27}(0)^2 = 7\sqrt{26}$$

Question 2

In this problem we will sketch part of the argument that a scalar line integral $\int_C f\,ds$ is independent of the parameterization of C that we choose to compute the integral. Suppose $\vec{r}_1(t)$, $a \leqslant t \leqslant b$, and $\vec{r}_2(t)$, $c \leqslant t \leqslant d$, are two smooth parameterizations of the same smooth curve C. Assuming that both parameterizations are in the same direction it can be shown that $\vec{r}_2(t) = \vec{r}_1(w(t))$, for some increasing function $w(t)$ satisfying $w(c) = a$ and $w(d) = b$. If this is the case, show that

$$\int_a^b f(\vec{r}_1(t))\left|\vec{r}_1'(t)\right|\,dt = \int_c^d f(\vec{r}_2(t))\left|\vec{r}_2'(t)\right|\,dt$$

for any continuous function f.

Solution:

$$\int_c f\,ds$$

$$\vec{r}_1(t) \quad a \leqslant t \leqslant b$$

$$\vec{r}_2(t) \quad c \leqslant t \leqslant d$$

$$\vec{r}_2(r) = \vec{r}_1(w(t)) \quad w(c) = a \quad w(d) = b$$

$$\int_a^b f(\vec{r}_1(t))|\vec{r}_1{}'(t)\,dt = \int_c^d f(\vec{r}_2(t))|\vec{r}_2{}'(t)\,dt$$

$$\vec{r}_2{}'(t) = \frac{d}{dt}\vec{r}_2(t) = \frac{d}{dt}\vec{r}_1(w(t)) = \vec{r}_1{}'(w(t))w'(t)$$

$$|\vec{r}_2{}'(t)| = |\vec{r}_1{}'(w(t))| \cdot |w'(t)|$$

$$\int_c^d f(\vec{r}_2(t))|\vec{r}_2{}'(t)|\,dt = \int_a^b f(\vec{r}_1(t))|\vec{r}_1{}'(w(t))| \cdot |w'(t)|\,dt$$

w maps $[c, d]$ to $[a, b]$, when $t = c\,, s = a$, and when $t = d\,, s = b$

$$\int_a^b f(\vec{r}_1)(w(t))|\vec{r}_1{}'(w(t))| \cdot |w(t)|\,dt = \int_a^b f(\vec{r}_1(s))|\vec{r}_1{}'(s)|\,ds$$

$$\int_a^b f(\vec{r}_1{}'(t))|\vec{r}_1{}'(t)|\,dt = \int_a^b f(r_1(s))|\vec{r}_1{}'(s)|\,ds = \int_c^d f(\vec{r}_2(t))|\vec{r}_2{}'(t)|\,dt$$

$\therefore$ the scalar line integral is independent of the parameterization and the equality holds true for any continuous function f

Question 3

Sketch the vector field $\vec{F}(x, y) = xy\,\hat{\imath} + \frac{1}{2}\,\hat{\jmath}$.

Question 4

Given the contour diagram for a function f shown below, in which dark colors correspond to low values of f and light colors correspond to high values of f, sketch the gradient vector field $\vec{F} = \nabla f$.

Question 5

A thin wire has the shape of the curve C parameterized by $x = \cos t$, $y = \sin t$, $z = t$, $0 \leqslant t \leqslant 4\pi$, where x, y, and z are measured in centimeters. The linear density of the wire is given by $\rho(x, y, z) = x^2 z$ grams per centimeter. Find the mass of the wire.

Solution:

Question 6

Let $\vec{F}$ be the vector field shown below, and let C be the unit circle, oriented clockwise. Is the vector line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

positive, negative, or zero? Explain your reasoning.

Question 7

Evaluate the line integral

$$\int_C \sin x \, dx + \cos y \, dy$$

where C consists of the top half of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$ and the line segment from $(-1, 0)$ to $(-2, 3)$. (Remember that when you see an integral that looks like

$$\int_C P(x, y) \, dx + \int_C Q(x, y) \, dy$$

it is a shorthand notation for

$$\int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r}$$

where $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$. The analogous thing is true in three dimensions.)

Question 8

Compute the line integral of the vector field

$$\vec{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \hat{\imath} + \frac{y}{\sqrt{x^2 + y^2}} \hat{\jmath}$$

along the parabola $x = 1 + y^2$ from $(2, -1)$ to $(2, 1)$.

Question 9

Evaluate the line integral of the vector field

$$\vec{F}(x, y, z) = (x + y)\hat{\imath} + (y - z)\hat{\jmath} + z^2 \hat{k}$$

along the path parameterized by

$$\vec{r}(t) = t^2 \hat{\imath} + t^3 \hat{\jmath} + t^2 \hat{k}, \quad 0 \leqslant t \leqslant 1.$$

Question 10

For each of the following vector fields $\vec{F}$ and curves C, find a function f such that $\vec{F} = \nabla f$ and use this function to evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

along the given directed curve C.

1. $\vec{F}(x, y) = \langle x^2, y^2 \rangle$, C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.
2. $\vec{F}(x, y, z) = \langle e^y, xe^y, (z + 1)e^z \rangle$, $C : \vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leqslant t \leqslant 1$.

Question 11

Clairaut's Theorem implies that if the vector field $\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$ is conservative and $P, Q,$ and R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

1. Use the statement above to show that the vector line integral

$$\int_C x\,dx + 2x\,dy + xz\,dz$$

   is not independent of path.

2. Find two directed curves C_1 and C_2 that start at the same point and end at the same point, such that

$$\int_{C_1} x\,dx + 2x\,dy + xz\,dz \neq \int_{C_2} x\,dx + 2x\,dy + xz\,dz.$$

Question 12

The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\vec{r} = \langle x, y, z\rangle$ is

$$\vec{F}(\vec{r}) = K\frac{\vec{r}}{|\vec{r}|^3},$$

where K is a constant. Find the work done on the particle as it moves along the straight line from $(0, 3, 0)$ to $(1, 3, 2)$ in two ways:

1. Parameterize the line segment, and compute

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\,dt$$

   directly.

2. Although $\vec{F}$ is not defined at the origin, it turns out that $\vec{F}$ is conservative on its domain. Find a potential function f, and use the Fundamental Theorem of Line Integrals to compute the work done on the particle.