Math 120 QR

Alex Hernandez Juarez

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Contents

Chapter 1

1.1 Day 1 notes

Definition 1.1.1: Distance Formula

Defintion:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Definition 1.1.2: Equation of a sphere

Defintion: An equation of a sphere with center $C(h, k, l)$, and radius r is

$$(x - h)^2 + (y - k)^2 + (z - l)^2$$

In particular, if the center is the origin O, than an equation of the sphere is

$$x^2 + y^2 + z^2$$

1.2 Day 2 Notes

Definition 1.2.1: The lenght/magnitude of a vecotr

In 2D, $\vec{v} =< a, b >$: $|\vec{v}| = \sqrt{a^2 + b^2}$
In 3D, $\vec{v}$, $\vec{v} =< a, b, c >$: $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$
A unit vector is a vector of length 1

Question 1

If $\vec{v}$ is a a vector and a is a scalar, then what is $|a\vec{v}|$

Solution:

$$|a\vec{v}| = |a||\vec{v}|$$

Definition 1.2.2: Vectors in $\mathbf{R}^3$

The standard basis vectors in $\mathbf{R}^3$ are

$$i =< 1, 0, 0 >$$

$$j =< 0, 1, 0 >$$

$$k =< 0, 0, 1 >$$

Question 2

What is special about i,j,k?

Solution:

- Cannot make any of them as a linear combibnation of the other three.
- Any vector $\vec{v} \epsilon \mathbf{R}^3$ can be written uniquily as a linear combibnation of i,j,k

Example 1.2.1:

$\vec{v} + \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

$\vec{v} + \vec{u} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$

Definition 1.2.3: Dot product

A dot product between v and w is:
In 2D: $v \cdot w = v_1 w_1 + v_2 w_2$
In 3D: $v \cdot 2 = v_1 w_1 + v_2 w_2 + v_3 w_3$

Geometric defintion:
$|u \cdot w| = |u||w| \cos(\theta)$ where θ is the angle between v and w

Example 1.2.2: Why the 2 defintions are the same

$$v_1 w_1 + v_2 w_2 = |v||w| \cos(\theta) = p|v|$$

$$p = |w| \cos(\theta)$$

$$v_1 = |v| \cos(\theta)$$

$$v_2 = |v| \sin(\theta)$$

$$w_1 = |w| \cos(\theta)$$

$$w_2 = |w| \sin(\theta)$$

$$\text{LHS} = |v||w| \left(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)\right)$$

$$\text{LHS} = |v||w| \sin(\alpha + \beta)$$

$$\text{LHS} = |v||w| \cos(\theta) = \text{RHS}$$

Example 1.2.3: what does the def mean

How much effect of $\vec{w}$ act along $\vec{v}$
Work: $w = \vec{F} \cdot \vec{S}$

Question 3

Find a relation between $|v|$ and $v \cdot v$

Solution: $|v|^2 = v \cdot v$

Question 4

v, w of fixed lenghts when is $v \cdot w$ largest?

Solution: v paralell: $\theta = 0$
$\cos(\theta) = 1$

Example 1.2.4: Projections

Given $\vec{v} \neq \vec{o}$
The project of w on v is proj $w = \left(\frac{\vec{w}\cdot\vec{v}}{|v|}\right)\frac{\vec{v}}{|v|}$
Directon of v is $\frac{\vec{v}}{|v|}$
Dor project of w with direction is:

$$\vec{w} \cdot \frac{\vec{v}}{|v|}$$

Question 5

TRUE or False:
u, v, w: vectors $(u \cdot v)w = u(v \cdot w)$

Solution: false

Question 6

TRUE or FALSE:
$v - w| = |v| - |w|$ if $v \parallel w$

Solution: false

Question 7

When is this ideal square sum happening?

$$|v + w|^2 = |v|^2 + |w|^2$$

Solution: when $v \perp w$

Definition 1.2.4: Cross Product

The cross product of two vectors v and w, $v * w$ is a vector u defined by $u \perp v$ and $u \perp w$.
Direction of u is given by the right hand rule
Magnitude: $|u|$ = Area of the parallelogram spanned by v and w.

1.3 Day 2 Reading notes

Definition 1.3.1: Vector Addition

If **u** and **v** are vectors positioned so the initial point of **v** is at the terminal point of **u**, then the **sum u + v** is the vector from the initial point of **u** to the terminal point of **v**.

Definition 1.3.2: Scalar Multiplication

If c is a scalar and **v** is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of **v** and whose direction is the same as **v** if $c > 0$ and is opposite to **v** if $c = 0$ or $\mathbf{v} = 0$, then $c\mathbf{v} = 0$

Example 1.3.1:

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with represenation $\overrightarrow{AB}$ is:

$$a = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 1.3.2:

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarily, for three demensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Note:-

Properties of vectors: If **a**, **b**, and **c** are vectors in V_n and c and d are scalars than

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $a + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + 0 = \mathbf{a}$
- $\mathbf{a} + \mathbf{a} + \text{-}\mathbf{a} = 0$

- $c(\mathbf{a}+\mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
- $(c+d)a = c\mathbf{a} + d\mathbf{a}$
- $(cd)\mathbf{a} = c(d\mathbf{a})$
- $l\mathbf{a} = \mathbf{a}$

1.4 Day 3 Reading notes

Example 1.4.1:

$\in$

1.5 Day 3 Class notes

Example 1.5.1:

Use the geometric def of cross product to calculate:
$i \times (i+j)$
$(i+j) \times (i-j)$

Definition 1.5.1: Second definition of dot product

Arithmetic Definition:

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = |a||b|\sin(\theta)$$

$$\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} i - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} j - \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 k \end{bmatrix}$$

$$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

$$|a \times b|^2 = (a_2b_3 - a_3b_2) - (a_1b_3 - a_3b_1) + (a_1b_2 - a_2b_1)$$
$$|a \times b|^2 = (a_2b_3 - a_3b_2) - (a_1b_3 - a_3b_1) + (a_1b_2 - a_2b_1)$$

Question 8

Calculate the corss product of $v = \langle -1, 3, 4 \rangle$ and $w = \langle 2, 1, -2 \rangle$

Solution:

$$v \times w = \begin{bmatrix} i & j & k \\ -1 & 4 & 4 \\ 2 & 1 & -2 \end{bmatrix} = 10i + 6j - 7k$$

Question 9

Calculate the area of the triangle with vertices P(1,0,1), Q (-3, 4, 0) and R(2,1,0)

Solution:

$$\overrightarrow{RP} = \langle -1, -1, 1 \rangle$$

$$\overrightarrow{RQ} = \langle 5, 0, 0 \rangle$$

$$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{bmatrix} i & j & k \\ -1 & -1 & 1 \\ -5 & 0 & 0 \end{bmatrix} = -5j - 5k$$

$$\text{area of } \delta PQR = \frac{1}{2}|\overrightarrow{RP \times \overrightarrow{RQ}}| = \frac{1}{2}\sqrt{5^2 + 5^2} = \frac{5\sqrt{2}}{2}$$

Question 10

What is special about: P(1,4,2), 1(3,3,5), R(9,1,11), and S(5, 11, 9)

Solution: If $(\overrightarrow{AC} \times \overrightarrow{AB} \parallel (\overrightarrow{AD} \times \overrightarrow{AC}))$ then A, B, C, D are on the same plane

Question 11

1. If $u \times w = u \times v$ then $w = v$
2. If $u \cdot v = 0$ and $u \times v = \vec{0}$ then either $u = \vec{0}$ or $v = \vec{0}$

Solution:
1. False

$$u \times (u + w) = u \times w$$

2. True
The reason is because the dot product is $|u||v|\cos(\theta)$ and the cross product is $|u||v|\sin(\theta)$

Question 12

Find the distance of a point p to the line passing through A and B

Solution:

$$= \frac{1}{2}\text{dist}(P, AB) \cdot |AB|$$

$$\text{dist}(P, AB) = \frac{|\overrightarrow{AB} \times \overrightarrow{AP}|}{|\overrightarrow{AB}}$$

Note:-

Line in 2D equation: $ax + by = c$
Line in 3D: Equation of a line passing through (x_0, y_0, z_0) and parallel to $\vec{v} = \langle a, b, c \rangle$ is

- $x = x_0 + at$
- $y = y_0 + bt$
- $z = z_0 + zt$

Note:-

Equation of a plane passing through (x_0, y_0, z_0) and orthogonal to $\vec{n} = \langle a, b, c \rangle$ is $(r - r_0) \cdot n = 0$
$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$
$ax + by + cz + d = 0$

Question 13

What is the normal of the plane $3x + z + 2 = 0$?

Solution:

$$\vec{n} = \langle 3, 0, 1 \rangle$$

Question 14

Is the line $x = 2t$, $y = 1 + 3t$, $z = 2 + 4t$ parallel to the plane $x - 2y + z = 7$

1.6 Integration Review

Example 1.6.1:

$$\int \frac{x}{\sqrt{1 - x^2}} dx$$

$$v = 1 - x^2$$

$$dv = -2x dx$$

$$\int \frac{-\frac{1}{2}}{\sqrt{v}} dv$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{v}}$$

Example 1.6.2:

$$(3x^2 5) e^{x^3 + 5x} dx$$

$$u = x^3 + 5x$$

$$du = 3x^2 5 dx$$

$$= \int e^u du$$

$$= e^{x^3 + 5x} c$$

Example 1.6.3:

$$\cos^5(2t)dt$$

$$u = 2t$$

$$du = 2dt$$

$$= \frac{1}{2}\cos^3(u)du = \frac{1}{2}\cos^4(u) + \cos(u)du$$

$$= \frac{1}{2}\int (1 - \sin^2(u))^2 \cdot \cos(u)du$$

$$v = \sin(u)$$

$$dv = \cos(u)du$$

Example 1.6.4:

$$xe^{2x}dx$$

$$x' = 1$$

$$\int e^{2x}dx = \frac{1}{2}e^{2x} + c$$

$$\int vdv = uv + \int vdu$$

$$v = x \longrightarrow dv = 1dx$$

$$dv = e^{2x}dx \longrightarrow v = \frac{1}{2}e^{2x}$$

$$\int = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}dx$$

Example 1.6.5:

$$\int t\sin(2t)dt$$

$$t' = 1$$

$$\int \sin(2t)dt = -\frac{1}{2}\cos(2t)$$

$$\int vdv = uv + \int vdv$$

$$v = t \longrightarrow dv = 1dt$$

$$dv = \sin(2t)dx \longrightarrow v = -\frac{1}{2}\cos(2t)$$

$$\int = -\frac{1}{2}\cos(2t) - \int \frac{1}{2}e^{2x}dx$$

1.7 Day 4 Class Notes

Example 1.7.1: Warm up 1

True or False
If Q_1R are not on P and $\vec{n} \cdot \overrightarrow{QR} = 0$ than Q_1R are on the same side of (P). True: Since the dot product of the normal and $\overrightarrow{QR}$ are 0 than it is parallel to the plane

Example 1.7.2: Warm up 2

What is the shape (in 3D) of

$$\textbf{a)}\ x^2 + y^2 = 1 \quad \textbf{b)}\ x^2 + y^2 + z^2 = 4$$

a) Cylinder
b) Sphere

1.8 Day 5 Class Notes

Definition 1.8.1: Tangent Line

iven a cruve $r(t)$, and a point $r(t_0)$ Equation of the tangent line passing through $r(t_0)$ is

$$l(t) = r(t_0) + t \cdot r'(t_0)$$

$$r(t) = \langle \cos(t), \sin(t), 0 \rangle \quad \text{for } t \in [0, 2\pi]$$

Note:-

Differential Rules:

1. $(u \pm v)' = u' \pm v'$
2. $cu' = (cu)'$
3. $(u \cdot v)' = u' \cdot v + u \cdot v'$
4. $(u \times v)' = u' \times v + u \times v'$
5. $(u_0 f)' = u'(f)f'$

Definition 1.8.2

If a curve is parameterized by $r(t)$ for $\leqslant t \leqslant b$ then its (arc)length is

$$L = \int_a^b |r'(t)| dt$$

$$\langle -2\cos(t), 0, -2\sin(t) \rangle$$

$$\int_{-\pi}^{\frac{\pi}{2}} |\langle -2\sin(t), -4, 2\cos(t) \rangle$$

1.9 Day 7 Notes

Example 1.9.1 (Warm Up)

Calculate first and second derivatives of $f(x,y) = 2x^2 - 3xy^2$ and $f(x,y) = \sin(x-y) + \sin(x+y)$
Chain rule compute $\frac{d}{dt}$ in 2 ways (chain rule, and subtitution)

$$
\begin{aligned}
f(x,y) &= 2x^2 - 3xy^2 \\
f'(x) &= 3x^2 - 3y^2 \\
f'(y) &= -6xy \\
f''(x) &= 6x \\
f''(y) &= -6y
\end{aligned}
$$

1.10 Day 8 Notes

Definition 1.10.1: Gradient

Gradient of $f(x,y)$

$$\triangledown f = \langle f_x, f_y \rangle$$

RMK:
Given a direction (unit) $\vec{u}$, we have

$$
\begin{aligned}
D_{\vec{u}} f(x_0, y_0) &= \vec{u} \cdot \langle f_x\left(x_0, y_0\right), f_y\left(x_0, y_0\right)\rangle \\
&= \vec{u} \cdot \vec{\triangledown} f\left(x_0, y_0\right)
\end{aligned}
$$

Example 1.10.1:

Given $\left(x_0, y_0\right)$ what is the direction that f increases/decreases teh most?
f increases the most in the direction $\frac{\triangledown f\left(x_0, y_0\right)}{|\triangledown f|}$
f decreases the most in the direction $\frac{-\triangledown f}{|\triangledown f|}\left(x_0, y_0\right)$