Math 120 Cheat Sheet

3D Coordinate Systems

Distance: $|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$
Sphere: $(x-h)^2+(y-k)^2+(z-l)^2 = r^2$

Vectors

Properties: $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$
$c(\mathbf{a}+\mathbf{b}) = c\mathbf{a}+c\mathbf{b}$
Normalize vector : $\frac{\vec{n}}{|n|}$

Dot Product

Formula: $\mathbf{a}\cdot\mathbf{b} = a_1b_1+a_2b_2+a_3b_3$
Angle: $\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$
Orthogonality: $\mathbf{a}\cdot\mathbf{b} = 0$
Projections:

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Positive Dot Product: The angle θ between the vectors is acute. The vectors point in the same general direction.
Zero Dot Product: The angle θ between the vectors is right. The vectors are perpendicular (orthogonal) to each other.
Negative Dot Product: The angle θ between the vectors is obtuse. The vectors point in opposite general directions.

Cross Product

Formula:

$$\mathbf{a}\times\mathbf{b} = \langle a_2b_3-a_3b_2, a_3b_1-a_1b_3, a_1b_2-a_2b_1\rangle$$

Magnitude: $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$
Triple Product: $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$
If **Triple Product** $= 0$ than they are coplanar.

Lines and Planes

Line: $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$
Plane: $\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{r_0}$
$ax+by+cz+d = 0$
Distance: '

$$D = \frac{|a(x_1-x_0)+b(y_1-y_0)+c(z_1-z_0)|}{\sqrt{a^2+b^2+c^2}}$$

Curve Length: $s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2}\,du$

Equation for Planes containing three points

$$\vec{PQ}\times\vec{PR}$$

Normal Vector of a Plane

$$ax+by+cz+d = 0 \quad \text{with normal vector} \quad \vec{n} = \langle a,b,c\rangle$$

Plane containing L_1 and parallel to L_2

$$\vec{n} = v_1\times v_2 \Rightarrow \vec{n}\cdot(\langle x,y,z\rangle - v_1(t))$$

Vector Value Functions

Form:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i}+g(t)\mathbf{j}+h(t)\mathbf{k}$$

Limit:

$$\lim_{t\to a}\mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\right\rangle$$

Derivative: $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$
Unit Tangent Vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
Orthogonality: If $|\mathbf{r}(t)| = c$, then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.
Definite Integral: $\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b)-\mathbf{R}(a)$
Length: $L = \int_a^b |\mathbf{r}'(t)|\,dt$

Curvature

Form: $s(t) = \int_a^t |\mathbf{r}'(u)|du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2}du$

Motion in space

Form: $\mathbf{v}(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h} = \mathbf{r}'(t)$

Partial Derivatives

Form: $f_x(a,b) = g'(a) \quad \text{where} \quad g(x) = f(x,b)$
Defintion:

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Notation:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1f = D_xf$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2f = D_yf$$

Rules
To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.
To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.
Higher Derivatives:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 z}{\partial y\partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 z}{\partial x\partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Clairaut's Theorem $f_{xy}(a,b) = f_{yx}(a,b)$

Chain Rule

Example:

$$\frac{d}{dt}\left(x^2(t)y(t)\right) = \langle 2x(t)y(t), x^2(t)\rangle \cdot \langle x'(t), y'(t)\rangle$$

Case 1:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Case 2:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

General Form:

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

Implicit Differentiation

Form:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Three variables:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

Directional Derivative and Gradient Vector

Directional Derivative:

$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b$$

Gradient For 2 variables

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Gradient for 3 Variables

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Properties of Gradient Vecotr

$\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.

Tangent Planes and Surfaces

Form:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

Derivative Rules

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

$$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)g(x) + f(x)g'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Intersection of Two Planes

Plane Equations:

$a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$

1. Direction Vector

$$\vec{d} = \langle b_1c_2 - b_2c_1,\ c_1a_2 - c_2a_1,\ a_1b_2 - a_2b_1\rangle$$

2. Point on Line

Set one variable to 0, solve for the other two.

3. Parametric Line Equation

$$\vec{r}(t) = \langle x_0, y_0, z_0\rangle + t\langle d_x, d_y, d_z\rangle$$

Component Form:

$$x = x_0 + t \cdot d_x$$
$$y = y_0 + t \cdot d_y$$
$$z = z_0 + t \cdot d_z$$

Plane Containing Line L and Point A

1. Line L in Parametric Form:

$$\vec{r}(t) = \langle x_1, y_1, z_1\rangle + t\langle d_x, d_y, d_z\rangle$$

2. Point A

$$A = (x_0, y_0, z_0)$$

3. Normal Vector of the Plane:

$$\vec{n} = \vec{d} \times \overrightarrow{PA}$$

where $\overrightarrow{PA} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle$.

4. Equation of the Plane:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where $\vec{n} = \langle a, b, c\rangle$.

Distance from a Point to a Line

Line L in Parametric Form: $\vec{r}(t) = \langle x_1, y_1, z_1\rangle + t\langle d_x, d_y, d_z\rangle$

Point P $P = (x_0, y_0, z_0)$

1. Vector from Line Point to P: $\overrightarrow{QP} = \langle x_0 - x_1, y_0 - y_1, z_0 - z_1\rangle$

2. Distance Formula: $d = \frac{|\overrightarrow{QP} \times \vec{d}|}{|\vec{d}|}$

where $\overrightarrow{QP} \times \vec{d}$ is the cross product of $\overrightarrow{QP}$ and the direction vector $\vec{d} = \langle d_x, d_y, d_z\rangle$.