

Math 120

PSet 8

Oct 31 2024

Contents

Chapter 1

Page 2

1.1 PSet 8

2

Chapter 1

1.1 PSet 8

Question 1

Let $\vec{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$, and let C be the triangle from $(0, 0)$ to $(2, 6)$ to $(2, 0)$ to $(0, 0)$. Use Green's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$. (Check the orientation of the curve before applying the theorem.)

Solution:

$$\begin{aligned}\vec{F}(x, y) &= \langle y^2 \cos x, x^2 + 2y \sin x \rangle \\ \int_C \vec{F} \cdot d\vec{r} &= - \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx dy \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x} (x^2 + 2y \sin x) = 2x + 2y \cos x \\ \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y} (y^2 \cos x) = 2y \cos x \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= [2x + 2y \cos x] - [2y \cos x] = 2x \\ \int_C \vec{F} \cdot d\vec{r} &= - \iint_D 2x dx dy \\ \int_{y=0}^{y=3x} 2x dy &= 2x(3x - 0) = 6x^2 \\ - \int_{x=0}^2 6x^2 dx &= -6 \int_0^2 x^2 dx = -6 \left[\frac{x^3}{3} \right]_0^2 = -6 \left(\frac{8}{3} \right) = -16 \\ \int_C \vec{F} \cdot d\vec{r} &= -16\end{aligned}$$

Question 2

Let $P(x, y) = x - x^2 y^3$ and $Q(x, y) = xy^2$, and let C be the circle $x^2 + y^2 = 4$, oriented counterclockwise.

- (a) Compute $\int_C \vec{F} \cdot d\vec{r}$ directly, by parameterizing C and finding the line integral.
- (b) Compute $\int_C \vec{F} \cdot d\vec{r}$ using Green's Theorem.

Solution:

a)

$$x(t) = 2 \cos t, \quad y(t) = 2 \sin t, \quad t \in [0, 2\pi]$$

$$dx = -2 \sin t \, dt, \quad dy = 2 \cos t \, dt$$

$$\begin{aligned} P &= x - x^2 y^3 = 2 \cos t - (2 \cos t)^2 (2 \sin t)^3 \\ &= 2 \cos t - 4 \cos^2 t \cdot 8 \sin^3 t = 2 \cos t - 32 \cos^2 t \sin^3 t \end{aligned}$$

$$Q = xy^2 = (2 \cos t)(2 \sin t)^2 = 2 \cos t \cdot 4 \sin^2 t = 8 \cos t \sin^2 t$$

$$\begin{aligned} P \, dx &= (2 \cos t - 32 \cos^2 t \sin^3 t) (-2 \sin t \, dt) \\ &= -4 \cos t \sin t \, dt + 64 \cos^2 t \sin^4 t \, dt \end{aligned}$$

$$Q \, dy = (8 \cos t \sin^2 t) (2 \cos t \, dt) = 16 \cos^2 t \sin^2 t \, dt$$

$$P \, dx + Q \, dy = [-4 \cos t \sin t + 64 \cos^2 t \sin^4 t + 16 \cos^2 t \sin^2 t] \, dt$$

$$P \, dx + Q \, dy = [-4 \cos t \sin t + 16 \cos^2 t \sin^2 t (1 + 4 \sin^2 t)] \, dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} [-4 \cos t \sin t + 16 \cos^2 t \sin^2 t (1 + 4 \sin^2 t)] \, dt$$

$$I = I_1 + I_2 + I_3$$

$$I_1 = \int_0^{2\pi} -4 \cos t \sin t \, dt = -2 \int_0^{2\pi} \sin 2t \, dt = 0$$

$$I_2 = 16 \int_0^{2\pi} \cos^2 t \sin^2 t \, dt$$

$$I_3 = 64 \int_0^{2\pi} \cos^2 t \sin^4 t \, dt$$

$$\cos^2 t \sin^2 t = \frac{1}{4} \sin^2 2t = \frac{1}{8} (1 - \cos 4t)$$

$$I_2 = 16 \cdot \frac{1}{8} \int_0^{2\pi} (1 - \cos 4t) \, dt = 2 \left[t - \frac{\sin 4t}{4} \right]_0^{2\pi} = 4\pi$$

$$\sin^4 t = (\sin^2 t)^2 = \left(\frac{1 - \cos 2t}{2} \right)^2 = \frac{1 - 2 \cos 2t + \cos^2 2t}{4}$$

$$\cos^2 t = \frac{1 + \cos 2t}{2}$$

$$\cos^2 t \sin^4 t = \frac{1 + \cos 2t}{2} \cdot \frac{1 - 2 \cos 2t + \cos^2 2t}{4} = \frac{(1 + \cos 2t)(1 - 2 \cos 2t + \cos^2 2t)}{8}$$

$$(1 + \cos 2t)(1 - 2 \cos 2t + \cos^2 2t) = (1)(1 - 2 \cos 2t + \cos^2 2t) + \cos 2t(1 - 2 \cos 2t + \cos^2 2t)$$

$$= 1 - 2 \cos 2t + \cos^2 2t + \cos 2t - 2 \cos^2 2t + \cos^3 2t$$

$$= 1 - \cos 2t - \cos^2 2t + \cos^3 2t$$

$$\cos^2 t \sin^4 t = \frac{1 - \cos 2t - \cos^2 2t + \cos^3 2t}{8}$$

$$I_3 = 64 \int_0^{2\pi} \cos^2 t \sin^4 t \, dt = 8 \int_0^{2\pi} (1 - \cos 2t - \cos^2 2t + \cos^3 2t) \, dt$$

$$\int_0^{2\pi} 1 \, dt = 2\pi$$

$$\int_0^{2\pi} \cos 2t \, dt = 0$$

$$\int_0^{2\pi} \cos^2 2t \, dt = \int_0^{2\pi} \frac{1 + \cos 4t}{2} \, dt = \pi$$

$$\cos^3 2t = \frac{3 \cos 2t + \cos 6t}{4} \implies \int_0^{2\pi} \cos^3 2t \, dt = 0$$

$$I_3 = 8(2\pi - 0 - \pi + 0) = 8 \cdot \pi = 8\pi$$

$$\int_C \vec{F} \cdot d\vec{r} = I_1 + I_2 + I_3 = 0 + 4\pi + 8\pi = \boxed{12\pi}$$

Part (b)

$$\int_C P \, dx + Q \, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \, dy$$

$$\frac{\partial Q}{\partial x} = y^2$$

$$\frac{\partial P}{\partial y} = -3x^2 y^2$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 + 3x^2 y^2 = y^2(1 + 3x^2)$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (r \sin \theta)^2 (1 + 3r^2 \cos^2 \theta) = r^2 \sin^2 \theta (1 + 3r^2 \cos^2 \theta)$$

$$\int_0^{2\pi} \int_0^2 r^2 \sin^2 \theta (1 + 3r^2 \cos^2 \theta) r \, dr \, d\theta = \int_0^{2\pi} \sin^2 \theta \left(\int_0^2 r^3 (1 + 3r^2 \cos^2 \theta) \, dr \right) d\theta$$

$$I = \int_0^{2\pi} \sin^2 \theta \left(\int_0^2 r^3 \, dr + 3 \cos^2 \theta \int_0^2 r^5 \, dr \right) d\theta$$

$$\int_0^2 r^3 \, dr = \left[\frac{r^4}{4} \right]_0^2 = \frac{16}{4} = 4$$

$$\int_0^2 r^5 \, dr = \left[\frac{r^6}{6} \right]_0^2 = \frac{64}{6} = \frac{32}{3}$$

$$I = \int_0^{2\pi} \sin^2 \theta \left(4 + 3 \cos^2 \theta \cdot \frac{32}{3} \right) d\theta = \int_0^{2\pi} \sin^2 \theta (4 + 32 \cos^2 \theta) d\theta$$

$$I = \int_0^{2\pi} (4 \sin^2 \theta + 32 \sin^2 \theta \cos^2 \theta) d\theta$$

$$I_1 = 4 \int_0^{2\pi} \sin^2 \theta \, d\theta = 4 \cdot \pi$$

$$I_2 = 32 \int_0^{2\pi} \sin^2 \theta \cos^2 \theta \, d\theta = 32 \cdot \frac{1}{8} \int_0^{2\pi} (1 - \cos 4\theta) \, d\theta = 4(2\pi - 0) = 8\pi$$

$$I = I_1 + I_2 = 4\pi + 8\pi = \boxed{12\pi}$$

Question 3

Use Green's Theorem to find the area enclosed by the parametric curve $\vec{r}(t) = \langle \sin t, \sin 2t \rangle$, $0 \leq t \leq \pi$.

Solution:

$$\begin{aligned}
 A &= \frac{1}{2} \int_C x \, dy - y \, dx \Rightarrow \frac{1}{2} \int_C \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right) dt \\
 x &= \sin t \quad y = \sin 2t \quad \frac{dx}{dt} = \cos t \quad \frac{dy}{dt} = 2 \cos t \\
 \sin 2t &= 2 \sin t \cos t \quad \cos 2t = \cos^2 t - \sin^2 t \\
 x \frac{dy}{dt} - y \frac{dx}{dt} &= 2 \sin t (\cos^2 t - \sin^2 t) - 2 \sin t \cos t (\cos t) \\
 &= 2 \sin t (\cos^2 t - \sin^2 t - \cos^2 t) = -2 \sin^3 t \\
 A &= \frac{1}{2} \int_0^\pi -2 \sin^3 t \, dt = - \int_0^\pi \sin^3 t \, dt \\
 \sin^3 t &= \sin t (1 - \cos^2 t) = \sin t - \sin t \cos^2 t \\
 \int_0^\pi \sin t \, dt - \int_0^\pi \sin t \cos^2 t \, dt \\
 \int_0^\pi \sin t \, dt &\Rightarrow -\cos t \Big|_0^\pi = 2 \\
 \int_0^\pi \sin t \cos^2 t \, dt \\
 u &= \cos t \quad du = -\sin t \, dt \\
 - \int_0^\pi u^2 \, du &\Rightarrow -\frac{u^3}{3} \Big|_0^\pi \\
 -\frac{\cos^3 t}{3} \Big|_0^\pi &= \frac{2}{3} \\
 A &= 2 - \frac{2}{3} = \frac{4}{3}
 \end{aligned}$$

Question 4

Consider the vector field $\vec{F} = -\frac{y}{x^2+y^2} \hat{i} + \frac{x}{x^2+y^2} \hat{j}$.

- Show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ at every point in the domain of \vec{F} .
- Let C be the short arc of the circle $x^2 + y^2 = 2$ from $(1, 1)$ to $(-1, 1)$. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ directly, by parameterizing the curve and computing $\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$.
- Integrate $P(x, y) = -\frac{y}{x^2+y^2}$ with respect to x , and check that the partial derivative of the result with respect to y is $Q(x, y) = \frac{x}{x^2+y^2}$. You have now found a function f such that $\nabla f = \vec{F}$.

What is the domain of this function f ? Is it the same as the domain of \vec{F} ?

- Use your answer to part (c) and the Fundamental Theorem of Line Integrals to check your answer to part (b).
- Now let C be the circle of radius R centered at the origin, oriented counterclockwise. Compute $\oint_C \vec{F} \cdot d\vec{r}$. Explain why your answer doesn't contradict the statement that the integral of a conservative vector field around any closed curve must be zero. Hint: Look carefully at the domain of the potential function f you found in part (b).

Question 5

Again consider the vector field $\vec{F} = -\frac{y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$. Let C_1 be any closed curve going counterclockwise around the origin, such as the orange curve below. Let C_2 be a circle, centered around the origin, with radius less than the shortest distance between C_1 and the origin. (This condition guarantees that the two curves don't intersect.) Let D be the region between the two curves.

- Explain why Green's Theorem applies on the region D .
- The boundary of D is the union of the two curves C_1 and $-C_2$, where by $-C_2$ we mean the inside circle oriented clockwise. Since $\int_{-C_2} \vec{F} \cdot d\vec{r} = -\int_{C_2} \vec{F} \cdot d\vec{r}$, Green's Theorem implies that

$$\int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} = \iint_D (Q_x - P_y) dA.$$

Use the results of Problem # 4 above to determine the value of $\int_{C_1} \vec{F} \cdot d\vec{r}$.

Question 6

Let $\vec{F} = \langle 2y - x^2, 4x + ye^{\cos y} \rangle$, and let C be the curve $y = x^2 - 9$, $-3 \leq x \leq 3$, oriented from left to right.

- Parameterize the curve C , and write the vector line integral $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$. Do not try to compute this integral directly!
- Let C^* be the line segment along the x -axis from $(3, 0)$ to $(-3, 0)$. Compute $\int_{C^*} \vec{F} \cdot d\vec{r}$.
- Let D be the region bounded by the parabola $y = x^2 - 9$ and the x -axis. Compute $\iint_D (Q_x - P_y) dA$.
- Use your answers to (b) and (c) to compute $\int_C \vec{F} \cdot d\vec{r}$.