

Math 120

PSet 7

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Contents

Chapter 1

Page 2

1.1 PSet 7

2

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Question 1

Evaluate the scalar line integral

$$\int_C (3x + y) ds,$$

where C is the line segment from $(-1, 3)$ to $(4, 2)$.

Question 2

In this problem we will sketch part of the argument that a scalar line integral $\int_C f ds$ is independent of the parameterization of C that we choose to compute the integral. Suppose $\vec{r}_1(t)$, $a \leq t \leq b$, and $\vec{r}_2(t)$, $c \leq t \leq d$, are two smooth parameterizations of the same smooth curve C . Assuming that both parameterizations are in the same direction it can be shown that $\vec{r}_2(t) = \vec{r}_1(w(t))$, for some increasing function $w(t)$ satisfying $w(c) = a$ and $w(d) = b$. If this is the case, show that

$$\int_a^b f(\vec{r}_1(t)) |\vec{r}'_1(t)| dt = \int_c^d f(\vec{r}_2(t)) |\vec{r}'_2(t)| dt$$

for any continuous function f .

Question 3

Sketch the vector field $\vec{F}(x, y) = xy\hat{i} + \frac{1}{2}\hat{j}$.

Question 4

Given the contour diagram for a function f shown below, in which dark colors correspond to low values of f and light colors correspond to high values of f , sketch the gradient vector field $\vec{F} = \nabla f$.

Question 5

A thin wire has the shape of the curve C parameterized by $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 4\pi$, where x , y , and z are measured in centimeters. The linear density of the wire is given by $\rho(x, y, z) = x^2z$ grams per centimeter. Find the mass of the wire.

Question 6

Let \vec{F} be the vector field shown below, and let C be the unit circle, oriented clockwise. Is the vector line integral

$$\int_C \vec{F} \cdot d\vec{r}$$

positive, negative, or zero? Explain your reasoning.

Question 7

Evaluate the line integral

$$\int_C \sin x \, dx + \cos y \, dy$$

where C consists of the top half of the circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(-1, 0)$ and the line segment from $(-1, 0)$ to $(-2, 3)$. (Remember that when you see an integral that looks like

$$\int_C P(x, y) \, dx + \int_C Q(x, y) \, dy$$

it is a shorthand notation for

$$\int_C \vec{F}(\vec{r}(t)) \cdot d\vec{r}$$

where $\vec{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$. The analogous thing is true in three dimensions.)

Question 8

Compute the line integral of the vector field

$$\vec{F}(x, y) = \frac{x}{\sqrt{x^2 + y^2}} \hat{i} + \frac{y}{\sqrt{x^2 + y^2}} \hat{j}$$

along the parabola $x = 1 + y^2$ from $(2, -1)$ to $(2, 1)$.

Question 9

Evaluate the line integral of the vector field

$$\vec{F}(x, y, z) = (x + y)\hat{i} + (y - z)\hat{j} + z^2\hat{k}$$

along the path parameterized by

$$\vec{r}(t) = t^2\hat{i} + t^3\hat{j} + t^2\hat{k}, \quad 0 \leq t \leq 1.$$

Question 10

For each of the following vector fields \vec{F} and curves C , find a function f such that $\vec{F} = \nabla f$ and use this function to evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

along the given directed curve C .

1. $\vec{F}(x, y) = \langle x^2, y^2 \rangle$, C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.
2. $\vec{F}(x, y, z) = \langle e^y, xe^y, (z + 1)e^z \rangle$, $C : \vec{r}(t) = \langle t, t^2, t^3 \rangle$, $0 \leq t \leq 1$.

Question 11

Clairaut's Theorem implies that if the vector field $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is conservative and P , Q , and R have continuous first-order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

1. Use the statement above to show that the vector line integral

$$\int_C x \, dx + 2x \, dy + xz \, dz$$

is not independent of path.

2. Find two directed curves C_1 and C_2 that start at the same point and end at the same point, such that

$$\int_{C_1} x \, dx + 2x \, dy + xz \, dz \neq \int_{C_2} x \, dx + 2x \, dy + xz \, dz.$$

Question 12

The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\vec{r} = \langle x, y, z \rangle$ is

$$\vec{F}(\vec{r}) = K \frac{\vec{r}}{|\vec{r}|^3},$$

where K is a constant. Find the work done on the particle as it moves along the straight line from $(0, 3, 0)$ to $(1, 3, 2)$ in two ways:

1. Parameterize the line segment, and compute

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

directly.

2. Although \vec{F} is not defined at the origin, it turns out that \vec{F} is conservative on its domain. Find a potential function f , and use the Fundamental Theorem of Line Integrals to compute the work done on the particle.