Math 120

PSet 2

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Contents

Chapter 1

1.1 PSet 2

Question 1

Consider the line L_1 given by $x + 2y = 7$ and the line L_2 given by $5x - y = 2$.

1. There are two unit vectors that are parallel to L_1. What are they?
2. There are two unit vectors that are perpendicular to L_1. What are they?
3. Find the acute angle between the lines L_1 and L_2. First find an exact expression and then approximate to the nearest degree.

Solution:

a)

$$L_1 = x + 2y = 7$$

$$y = -\frac{1}{2}x + \frac{7}{2}$$

$$v_1 = (1, m_1) = (1, -\frac{1}{2})$$

$$|v_1| = \sqrt{1^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$u_1 = \frac{1}{\frac{\sqrt{5}}{2}}\left(1, -\frac{1}{2}\right) = \frac{2}{\sqrt{5}}\left(1, -\frac{1}{2}\right) = \left(\frac{\sqrt{2}}{5}, -\frac{1}{\sqrt{5}}\right)$$

$$-u_1 = \left(-\frac{\sqrt{2}}{5}, \frac{1}{\sqrt{5}}\right)$$

b)

c)

$$\vec{v_1} = \langle 1, -2 \rangle$$

$$\vec{v_2} = \langle 5, 1 \rangle$$

$$\cos(\theta) = \frac{v_1 \cdot v_2}{|v_1||v_2|}$$

$$\cos(\theta) = \frac{-2 + 5}{\sqrt{526}} = \frac{3}{\sqrt{130}}$$

$$\theta = \arccos\left(\frac{3}{\sqrt{130}}\right)$$

Question 2

Find all values of x such that the angle between the vectors $\langle 1, -1, 0\rangle$ and $\langle 2, x, 1\rangle$ is $\frac{\pi}{3}$.

Solution:

$$v_1 = \langle 1, -1, 0\rangle$$

$$v_2 = \langle 2, x, 1, \rangle$$

$$\cos(\frac{\pi}{2}) = \frac{1}{2}$$

$$\frac{1}{2} = \frac{v_1 \cdot v_2}{|v_1||v_2} = \frac{2 - x}{(\sqrt{2})\sqrt{5 + x^2}}$$

$$4 - 2x = \sqrt{10 + 2x^2}$$

$$10 + 2x^2 = 16 - 16x + 4x^2$$

$$-2x^2 + 16x - 6 = 0$$

$$x^2 - 8x + 3 = 0$$

$$x = \frac{8 \pm \sqrt{(-8)^2 - 4(1)(3)}}{2 \cdot 1}$$

$$x = \frac{8 \pm \sqrt{52}}{2}$$

$$x = 4 - \sqrt{13}$$

Question 3

Find the scalar and vector projections of $\vec{b} = \hat{\imath} + \hat{\jmath}$ onto $\vec{a} = -\hat{\imath} + 3\hat{\jmath}$, and illustrate your answers with a sketch.

Solution:
Scalar Projection:

$$\vec{b} = \hat{\imath} + \hat{\jmath}$$

$$\vec{a} = \hat{\imath} + 3\hat{\jmath}$$

$$\text{comp}_a\mathbf{b} = \frac{a \cdot b}{|a|}$$

$$\frac{a \cdot b}{|a|} = \frac{2}{\sqrt{10}} = \frac{2\sqrt{10}}{10} = \frac{\sqrt{10}}{5}$$

Vector Projection:

$$\text{proj}_a\mathbf{b} = \left(\frac{a \cdot b}{|a|^2}\right) a$$

$$\left(\frac{a \cdot b}{|a|^2}\right) a = \frac{2}{\sqrt{10}^2} a = \frac{2}{10} a = \frac{1}{5} a$$

$$\frac{1}{5} a = \frac{1}{5}(-\hat{\imath}, 3\hat{\jmath}) = \langle -\frac{1}{5}\hat{\imath}, \frac{3}{5}\hat{\jmath}\rangle$$

Question 4

Find two vectors of length 2 that are orthogonal to both $\vec{v} = \langle 2, 4, 4\rangle$ and $\vec{w} = \langle 1, -1, -3\rangle$.

Solution:

$$v \times w = \langle 4(-3) - 4(-1), 4(1) - 2(-3), 2(-1) - 4(1)\rangle = \langle -8, 10, -6\rangle$$

$$|u| = \sqrt{(-8)^2 + 10^2 + (-6)^2} = \sqrt{200}$$
$$2 = x \cdot \sqrt{200}$$
$$x = \frac{2}{\sqrt{200}} = \frac{1}{5\sqrt{2}}$$
$$u_1 = \frac{1}{5\sqrt{2}}\langle -8, 10, -6\rangle$$
$$u_2 = -\frac{1}{5\sqrt{2}}\langle -8, 10, -6\rangle$$

Question 5

Let $\vec{a} = \langle 3, 1, 0\rangle$. Find all vectors $\vec{b} = \langle b_1, b_2, b_3\rangle$ such that $\vec{a} \times \vec{b}$ is parallel to the z-axis and pointing in the positive z direction. Illustrate with a sketch, in which all vectors are drawn as position vectors, i.e., with the tail at the origin.

Solution:

$$a \times b = \langle 0, 0, c\rangle$$
$$\langle 3, 1, 0\rangle \times \langle b_1, b_2, b_3\rangle = \langle 0, 0, c\rangle$$
$$\langle 3, 1, 0\rangle \times \langle b_1, b_2, b_3\rangle = \langle 1(b_3) - 0(b_2), 0(b_1) - 3(b_3), 3(b_2) - 1(b_1)\rangle$$
$$\langle 3, 1, 0\rangle \times \langle b_1, b_2, b_3\rangle = \langle b_3, -3b_3, 3b_2 - b_1\rangle$$
$$3b_2 - 3b_1 > 0$$
$$b_3 = 0$$
$$-3b_3 = 0$$

Question 6

Consider the four points in $\mathbb{R}^3$, $K(1, 2, 3)$, $L(1, 3, 6)$, $M(3, 8, 6)$, and $N(3, 7, 3)$.

1. Show that the vectors $\overrightarrow{KL}$, $\overrightarrow{KM}$, and $\overrightarrow{KN}$ are coplanar. Explain why this means that K, L, M, and N all lie in the same plane.
2. From part (a), we know that K, L, M, and N are the vertices of a quadrilateral. Explain how you can tell that this quadrilateral is actually a parallelogram.
3. Find the area of the parallelogram with vertices K, L, M, and N.

Solution: a)

$$\overrightarrow{KL} = \langle 1 - 1, 3 - 2, 6 - 3\rangle = \langle 0, 1, 3\rangle$$
$$\overrightarrow{KM} = \langle 3 - 1, 8 - 2, 6 - 3\rangle = \langle 2, 6, 3\rangle$$
$$\overrightarrow{KN} = \langle 3 - 1, 7 - 2, 3 - 3\rangle = \langle 2, 5, 0\rangle$$
$$\overrightarrow{KL} \cdot (\overrightarrow{KM} \times \overrightarrow{KN}) = \overrightarrow{KL} \times \langle 6(0) - 3(5), 3(2) - 2(0), 2(5) - 6(2)\rangle$$
$$\overrightarrow{KL} \cdot (\overrightarrow{KM} \times \overrightarrow{KN}) = \overrightarrow{KL} \times \langle -15, 6, -2\rangle$$
$$\overrightarrow{KL} \cdot (\overrightarrow{KM} \times \overrightarrow{KN}) = \overrightarrow{KL} \times \langle -15, 6, -2\rangle = 0(15) + 6(1) + 3(-2) = 0$$
$$|\overrightarrow{KL} \cdot (\overrightarrow{KM} \times \overrightarrow{KN})| = 0$$

They are coplanar because the volume determined by the vectors is 0, therefore they must lie on the same plane.
b).

Question 7

Find the vector equation and parametric equations for the line through the point $(1, 2, -2)$ parallel to the line $x = t - 2$, $y = -2t + 1$, $z = 3$.

Solution:

$$x = t - 2 \quad y = -2t + 1 \quad z = 3$$

$$\vec{d} = \langle 1, -2, 0 \rangle$$

$$\vec{r}(t) = \vec{r}_0 + t\vec{d}$$

$$\vec{r}(t) = \langle 1, 2, -2 \rangle + t\langle 1, -2, 0 \rangle$$

$$x(t) = 1 + t \quad y(t) = 2 - 2t \quad z(t) = -2$$

Vector Equation: $\vec{r}(t) = \langle 1, 2, -2 \rangle + t\langle 1, -2, 0 \rangle$
Parametric Equation: $x(t) = 1 + t$, $y(t) = 2 - 2t$, $z(t) = -2$

Question 8

Consider the lines $L_1 : x = t + 3$, $y = 2t - 1$, $z = -t$, and $L_2 : x = t - 1$, $y = t - 4$, $z = -t + 4$. Determine whether the L_1 and L_2 are parallel, skew, or intersecting. If they intersect, find the point of intersection.

Solution:

$$d_1 = \langle 1, 2, -1 \rangle \quad \langle d_2 = 1, 1, -1 \rangle$$

$$\frac{1}{1} \neq \frac{2}{1} \neq \frac{-1}{-1}$$

$$t_1 + 3 = t_2 - 1 \quad 2t_1 - 1 = t_2 - 4 \quad -t_1 = -t_2 + 4$$

$$t_1 + 3 = t_2 - 1 \Rightarrow t_1 - t_2 = -4$$

$$-t_1 = -t_2 + 4 \Rightarrow t_1 = t_2 - 4$$

$$(t_2 - 4) - t_2 = -4 \Rightarrow -4 = -4$$

$$2t_1 - 1 = t_2 - 4$$

$$2(t_2 - 4) - 1 = t_2 - 4$$

$$2t_2 - 9 = t_2 - 4$$

$$t_2 = 5$$

$$t_1 = t_2 - 4 = 5 - 4 = 1$$

$$x_1 = 1 + 3 = 4 \quad y_1 = 2(1) - 1 = 1 \quad z_1 = -1$$

Point of Intersection: (4, 1, - 1)

Question 9

Consider the planes $x + y + 2z = 4$ and $2x - y - 2z = 1$.

1. Find a vector equation for the line of intersection of the planes.
2. Find the angle between the planes. First find an exact expression and then approximate to the nearest degree.

Solution:
a)

$$n_1 = \langle 1, 1, 2 \rangle \quad n_2 = \langle 2, -1, -2 \rangle$$

$$d = n_1 \times n_2 = \langle 1(-2) - 2(-1), 2(2) - 1(-2), 1(-1) - 1(2) \rangle = \langle -4, 6, -3 \rangle$$

$$P_1 : x + y + 2z = 4 \quad P_2 : 2x - y - 2z = 1$$

$$x + y + 2(0) = 4 \Rightarrow x + y = 4 \Rightarrow x = -y + 4$$

$$2x - y - 2(0) = 1 \Rightarrow 2x - y = 1$$

$$2(-y - 4) - y = 1 \Rightarrow -2y - 8 - y = 1 \Rightarrow -3y = 9 \Rightarrow y = -3$$

$$x = -(-3) + 4 = 7$$

$$(7, 9, 0)$$

$$r = (7, 9, 0) + t\langle -4, 6, -3 \rangle$$

b)

$$n_1 = \langle 1, 1, 2 \rangle \quad n_2 = \langle 2, -1, -2 \rangle$$

$$\cos(\theta) = \frac{n_1 \cdot n_2}{|n_1||n_2|} = \frac{1(2) + 1(-1) + 2(-2)}{\sqrt{1^2 + 1^2 + 2^2}\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{-1}{\sqrt{6}\sqrt{9}} = -\frac{1}{3\sqrt{6}}$$

$$\theta = \arccos(-\frac{1}{3\sqrt{6}})$$

Question 10

Let P be the plane $x + y + 2z = 1$ and let A be the point $(1, 1, 1)$.

(a) Find an equation of the plane through point A parallel to plane P.

(b) Find a vector equation for the line through the point A which is perpendicular to the plane P. Call this line L.

(c) Find the point of intersection of the line L (from part (b)) and the plane P.

(d) Find the point on the plane P closest to the point A, and then find the shortest distance from the point A to the plane P.