Math 120

PSet 8

Oct 31 2024

Contents

Chapter 1

1.1 PSet 8

Question 1

Let $\vec{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$, and let C be the triangle from $(0, 0)$ to $(2, 6)$ to $(2, 0)$ to $(0, 0)$. Use Green's Theorem to evaluate $\int_C \vec{F} \cdot d\vec{r}$. (Check the orientation of the curve before applying the theorem.)

Solution:

$$\vec{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy$$

$$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}(x^2 + 2y \sin x) = 2x + 2y \cos x$$

$$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}(y^2 \cos x) = 2y \cos x$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = [2x + 2y \cos x] - [2y \cos x] = 2x$$

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_D 2x\, dx\, dy$$

$$\int_{y=0}^{y=3x} 2x\, dy = 2x(3x - 0) = 6x^2$$

$$-\int_{x=0}^{2} 6x^2\, dx = -6 \int_0^2 x^2\, dx = -6 \left[\frac{x^3}{3} \right]_0^2 = -6 \left(\frac{8}{3} \right) = -16$$

$$\int_C \vec{F} \cdot d\vec{r} = -16$$

Question 2

Let $P(x, y) = x - x^2 y^3$ and $Q(x, y) = xy^2$, and let C be the circle $x^2 + y^2 = 4$, oriented counterclockwise.

(a) Compute $\int_C \vec{F} \cdot d\vec{r}$ directly, by parameterizing C and finding the line integral.

(b) Compute $\int_C \vec{F} \cdot d\vec{r}$ using Green's Theorem.

Question 3

Use Green's Theorem to find the area enclosed by the parametric curve $\vec{r}(t) = \langle \sin t, \sin 2t \rangle$, $0 \leqslant t \leqslant \pi$.

Solution:

$$A = \frac{1}{2}_C x\,dy - y\,dx \Rightarrow \frac{1}{2}\int_C \left(x\frac{dy}{dy} - y\frac{dx}{dt}\right) dt$$

$$x = \sin t \quad y = \sin 2t \quad \frac{dx}{dt} = \cos t \quad \frac{dy}{dt} = -2\cos t$$

$$\sin 2t = 2\sin t\cos t \quad \cos 2t = \cos^2 t - \sin^2 t$$

$$x\frac{dy}{dt} - y\frac{dx}{dt} = 2\sin t\left(\cos^2 t - \sin^2 t\right) - 2\sin t\cos t(\cos t)$$

$$= 2\sin t\left(\cos^2 t - \sin^2 t - \cos^2 t\right) = -2\sin^3 t$$

$$A = \frac{1}{2} - 2\sin^3 t\,dt = -\int \sin^3 t\,dt$$

$$\sin^3 t = \sin t\left(1 - \cos^2 t\right) = \sin t - \sin t\cos^2 t$$

$$\int_0^\pi \sin t\,dt - \int_0^\pi \sin t\cos^2 t$$

$$\int_0^\pi \sin t\,dt \Rightarrow -\cos t|_0^\pi = 2$$

Question 4

Consider the vector field $\vec{F} = -\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$.

(a) Show that $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ at every point in the domain of $\vec{F}$.

(b) Let C be the short arc of the circle $x^2 + y^2 = 2$ from $(1,1)$ to $(-1,1)$. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ directly, by parameterizing the curve and computing $\int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$.

(c) Integrate $P(x,y) = -\frac{y}{x^2+y^2}$ with respect to x, and check that the partial derivative of the result with respect to y is $Q(x,y) = \frac{x}{x^2+y^2}$. You have now found a function f such that $\nabla f = \vec{F}$.

What is the domain of this function f? Is it the same as the domain of $\vec{F}$?

(d) Use your answer to part (c) and the Fundamental Theorem of Line Integrals to check your answer to part (b).

(e) Now let C be the circle of radius R centered at the origin, oriented counterclockwise. Compute $\oint_C \vec{F}\cdot d\vec{r}$. Explain why your answer doesn't contradict the statement that the integral of a conservative vector field around any closed curve must be zero. Hint: Look carefully at the domain of the potential function f you found in part (b).

Question 5

Again consider the vector field $\vec{F} = -\frac{y}{x^2+y^2}\hat{\imath} + \frac{x}{x^2+y^2}\hat{\jmath}$. Let C_1 be any closed curve going counterclockwise around the origin, such as the orange curve below. Let C_2 be a circle, centered around the origin, with radius less than the shortest distance between C_1 and the origin. (This condition guarantees that the two curves don't intersect.) Let D be the region between the two curves.

(a) Explain why Green's Theorem applies on the region D.

(b) The boundary of D is the union of the two curves C_1 and $-C_2$, where by $-C_2$ we mean the inside circle oriented clockwise. Since $\int_{-C_2}\vec{F}\cdot d\vec{r} = -\int_{C_2}\vec{F}\cdot d\vec{r}$, Green's Theorem implies that

$$\int_{C_1}\vec{F}\cdot d\vec{r} - \int_{C_2}\vec{F}\cdot d\vec{r} = \iint_D (Q_x - P_y)\,dA.$$

Use the results of Problem # 4 above to determine the value of $\int_{C_1} \vec{F} \cdot d\vec{r}$.

Question 6

Let $\vec{F} = \langle 2y - x^2, 4x + ye^{\cos y} \rangle$, and let C be the curve $y = x^2 - 9$, $-3 \leqslant x \leqslant 3$, oriented from left to right.

(a) Parameterize the curve C, and write the vector line integral $\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$. Do not try to compute this integral directly!

(b) Let C^* be the line segment along the x-axis from $(3, 0)$ to $(-3, 0)$. Compute $\int_{C^*} \vec{F} \cdot d\vec{r}$.

(c) Let D be the region bounded by the parabola $y = x^2 - 9$ and the x-axis. Compute $\iint_D (Q_x - P_y)\, dA$.

(d) Use your answers to (b) and (c) to compute $\int_C \vec{F} \cdot d\vec{r}$.