# Math 120 QR

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# Contents

# Chapter 1

## 1.1 12.1 Notes (Three Dimensional Coodinate Systems)

**Definition 1.1.1: Distance Formula**

Defintion:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

**Definition 1.1.2: Equation of a sphere**

Defintion: An equation of a sphere with center $C(h, k, l)$, and radius $r$ is

$$(x - h)^2 + (y - k)^2 + (z - l)^2$$

In particular, if the center is the origin $O$, than an equation of the sphere is

$$x^2 + y^2 + z^2$$

## 1.2 12.2 Notes (Vectors)

**Definition 1.2.1: Vector Addition**

If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the **sum $\mathbf{u} + \mathbf{v}$** is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

**Definition 1.2.2: Scalar Multiplication**

If $c$ is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple $c\mathbf{v}$** is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c = 0$ or $\mathbf{v} = 0$, then c$\mathbf{v} = 0$

**Example 1.2.1:**

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector **a** with represenation $\overrightarrow{AB}$ is:

$$a = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**Example 1.2.2:**

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$
$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$
$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarily, for three demensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + a_3 + b_3 \rangle$$
$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - a_3 - b_3 \rangle$$
$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

**Note:-**

Properties of vectors: If **a**, **b**, and **c** are vectors in $V_n$ and $c$ and $d$ are scalars than

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $a + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + 0 = \mathbf{a}$
- $\mathbf{a} + \mathbf{a} + \text{-}\mathbf{a} = 0$
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
- $(c + d)a = c\mathbf{a} + d\mathbf{a}$
- $(cd)\mathbf{a} = c(d\mathbf{a})$
- $l\mathbf{a} = \mathbf{a}$

## 1.3 12.3 Notes (Dot Product)

**Definition 1.3.1: Dot Product**

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties of the Dot Product: If $\mathbf{a}, \mathbf{b},$ and $\mathbf{c}$ are vectors in $V_3$ and $c$ is a scalar, then

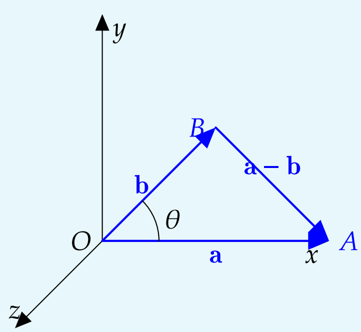
1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

**Definition 1.3.2: Geometric Definition of the Dot Product**

If $\theta$ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

Proof:




$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos\theta$$

Corollary: If $\theta$ is the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

**Note:-**

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if an only if $\mathbf{a} \cdot \mathbf{b} = 0$

**Example 1.3.1** (Direction Angles and Cosines)

The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles $\alpha$, $\beta$, and $\gamma$ (in the interval $[0, \pi]$) that $\mathbf{a}$ makes with the positive $x$-, $y$-, and $z$-axes, respectively .

The cosines of these direction angles, $\cos\alpha$, $\cos\beta$, and $\cos\gamma$, are called the **direction cosines** of the vector $\mathbf{a}$. Using Corollary 6 with $\mathbf{b}$ replaced by $\mathbf{i}$, we obtain:

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|} \tag{1}$$

Similarly, we also have:

$$\cos\beta = \frac{a_2}{|\mathbf{a}|} \quad \text{and} \quad \cos\gamma = \frac{a_3}{|\mathbf{a}|} \tag{2}$$

By squaring the expressions in Equations 8 and 9 and adding, we see that:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \tag{3}$$

We can also use Equations 8 and 9 to write:

$$\mathbf{a} = \langle a_1, a_2, a_3\rangle = \langle |\mathbf{a}|\cos\alpha, |\mathbf{a}|\cos\beta, |\mathbf{a}|\cos\gamma\rangle = |\mathbf{a}|\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$$

Therefore,

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle\cos\alpha, \cos\beta, \cos\gamma\rangle \tag{4}$$

which says that the direction cosines of $\mathbf{a}$ are the components of the unit vector in the direction of $\mathbf{a}$.

**Definition 1.3.3: Projections**

The **scalar projection** of $\mathbf{b}$ onto $\mathbf{a}$ (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. This is denoted by $\text{comp}_{\mathbf{a}}\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leqslant \pi$. The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of $\mathbf{a}$ and $\mathbf{b}$ can be interpreted as the length of $\mathbf{a}$ times the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of $\mathbf{b}$ along $\mathbf{a}$ can be computed by taking the dot product of $\mathbf{b}$ with the unit vector in the direction of $\mathbf{a}$. We summarize these ideas as follows.

**Scalar projection of b onto a:** $\quad \text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

**Vector projection of b onto a:** $\quad \text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

## 1.4 12.4 Notes (Cross Product)

**Definition 1.4.1: Cross Product**

Given two nonzero vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, suppose that a nonzero vector $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Then $\mathbf{a} \cdot \mathbf{c} = 0$ and $\mathbf{b} \cdot \mathbf{c} = 0$, and so:

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \tag{1}$$
$$b_1c_1 + b_2c_2 + b_3c_3 = 0 \tag{2}$$

To eliminate $c_3$, we multiply (1) by $b_3$ and (2) by $a_3$ and subtract:

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0 \tag{3}$$

Equation (3) has the form $pc_1 + qc_2 = 0$, for which an obvious solution is $c_1 = q$ and $c_2 = -p$. So, a solution of (3) is:

$$c_1 = a_2b_3 - a_3b_2$$
$$c_2 = a_3b_1 - a_1b_3$$

Substituting these values into (1) and (2), we then get:

$$c_3 = a_1b_2 - a_2b_1$$

This means that a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ is:

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ and is denoted by $\mathbf{a} \times \mathbf{b}$.

**Definition 1.4.2: Cross Product of two vectors**

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

**Note:-**

Determinant of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

**Note:-**

Determinant of order 3:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

**Definition 1.4.3: Second definition of cross product**

Arithmetic Definition:

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = |a||b|\sin(\theta)$$

$$\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} i - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} j + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 k \end{bmatrix}$$
$$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$

**Example 1.4.1: Proof that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both a**

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\
&= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\
&= a_1a_2b_3 - a_1a_3b_2 - a_2a_1b_3 + a_2a_3b_1 + a_3a_1b_2 - a_3a_2b_1 \\
&= 0
\end{aligned}$$

**Definition 1.4.4: sin definition of cross product**

If $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$$

Proof:

$$\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \quad \text{(by Theorem 12.3.3)} \\
&= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2\theta) \\
&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta
\end{aligned}$$

Taking square roots and observing that $\sqrt{\sin^2\theta} = \sin\theta$ because $\sin\theta \geqslant 0$ when $0 \leqslant \theta \leqslant \pi$, we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

**Note:-**

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = 0$$

**Example 1.4.2: Geometric interpretation of $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$**

If $\mathbf{a}$ and $\mathbf{b}$ are represented by directed line segments with the same inital point, then they determine a parallelogram with base $|\mathbf{a}|$, altitude $\mathbf{b}\sin(\theta)$ and area

$$A = |\mathbf{a}|(|\mathbf{b}| \sin\theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product:
The length of the cross product of $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$

**Note:-**

If we apply the following theorem:
The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$, and

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$$

to the standard basis vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ using $\theta = \frac{\pi}{2}$, we obtain

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

**Note:-**

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and $c$ is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

**Example 1.4.3: Proof of property 5 of cross products**

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, then:

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$

**Definition 1.4.5: Triple Products**

The product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ that occurs in Property 5 is called the *scalar triple product* of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. The area of the base parallelogram is $A = |\mathbf{b}\times\mathbf{c}|$. If $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$, then the height $h$ of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Therefore, the volume of the parallelepiped is

$$V = Ah = |\mathbf{b}\times\mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

Thus, we have proved the following formula: The volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product:

$$V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$$

**Note:-**

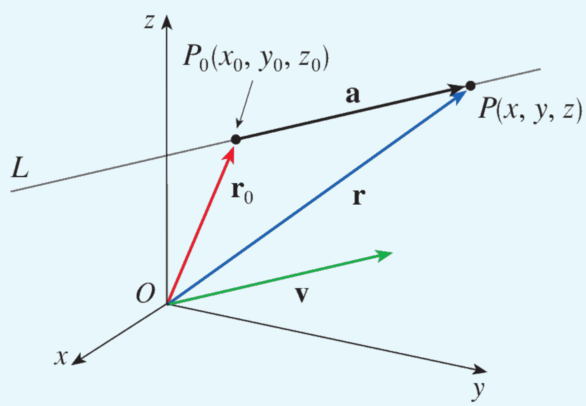
If we use the formula in $V = |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$ and discover that the volume of the parallelepiped determined by a, b, and c is 0, then the vectors must lie in the same plane; that is, they are coplanar

## 1.5 12.5 Notes (Equations of Lines and Planes)

**Definition 1.5.1: Hi**

Likewise, a line $L$ in three-dimensional space is determined when we know a point $P_0(x_0, y_0, z_0)$ on $L$ and a direction for $L$, which is conveniently described by a vector $\mathbf{v}$ parallel to the line. Let $P(x, y, z)$ be an arbitrary point on $L$ and let $\mathbf{r_0}$ and $\mathbf{r}$ be the position vectors of $P_0$ and $P$ (that is, they have representations $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$). If $\mathbf{a}$ is the vector with representation $\overrightarrow{P_0P}$, as in Figure 1, then the Triangle Law for vector addition gives

$$\mathbf{r} = \mathbf{r_0} + \mathbf{a}.$$




Since $\mathbf{a}$ and $\mathbf{v}$ are parallel vectors, there is a scalar $t$ such that $\mathbf{a} = t\mathbf{v}$ Thus

$$r = r_0 + t\mathbf{v}$$

**Note:-**

If the vector $\mathbf{v}$ that gives the direction of the line $L$ is written in component form as

$$\mathbf{v} = \langle a, b, c \rangle,$$

then we have $t\mathbf{v} = \langle ta, tb, tc \rangle$. We can also write $\mathbf{r} = \langle x, y, z \rangle$ and

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle,$$

so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle.$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

**Example 1.5.1: Line example**

Find a vector equation and parametric equations for the line that passes through the point $(5, 1, 3)$ and is parallel to the vector $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$. Here $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

Parametric equations are

$$x = 5 + t \quad y = 1 + 4t \quad z = 3 - 2t$$

**Note:-**

In general, if a vector $\mathbf{v} = \langle a, b, c \rangle$ is used to describe the direction of a line $L$, then the numbers $a$, $b$, and $c$ are called *direction numbers* of $L$. Since any vector parallel to $\mathbf{v}$ could also be used, we see that any three numbers proportional to $a$, $b$, and $c$ could also be used as a set of direction numbers for $L$.

Another way of describing a line $L$ is to eliminate the parameter $t$ from Equations 2. If none of $a$, $b$, or $c$ is 0, we can solve each of these equations for $t$:

$$t = \frac{x - x_0}{a} \quad t = \frac{y - y_0}{b} \quad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called symetric equations of $L$

**Definition 1.5.2: Line segment**

The line segment from $r_0$ to $r_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r_0} + tr_1 \quad 0 \leqslant t \leqslant 1$$

### Definition 1.5.3: Planes

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane. This orthogonal vector $\mathbf{n}$ is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane, and let $\mathbf{r_0}$ and $\mathbf{r}$ be the position vectors of $P_0$ and $P$. Then the vector $\mathbf{r} - \mathbf{r_0}$ is represented by $\overrightarrow{P_0P}$. The normal vector $\mathbf{n}$ is orthogonal to every vector in the given plane. In particular, $\mathbf{n}$ is orthogonal to $\mathbf{r} - \mathbf{r_0}$ and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0 \tag{1.1}$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0} \tag{1.2}$$

These can be reffered to as the **vector equation of the plane**
To obtain a scalar equation for the plane, we write $\mathbf{n} = \langle a, b, c\rangle, \mathbf{r} = \langle x, y, x\rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$. then the vector equation becomes:

$$\langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0$$

Expanding the left side of this equation gives the following:
A **scalar equation of the plane** through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

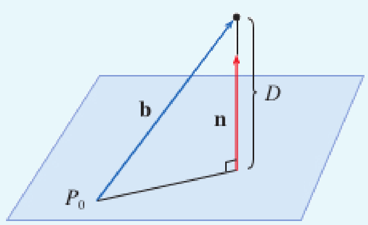
by colecting terms can be rewritten as:

$$ax + by + cz + d = 0$$

### Definition 1.5.4: Distance of a plane

In order to find a formula for the distance $D$ from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$, we let $P_0(x_0, y_0, z_0)$ be any point in the given plane and $\mathbf{b}$ be the vector corresponding to $\overrightarrow{P_0P_1}$. Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0\rangle$$




From Figure, you can see that the distance $D$ from $P_1$ to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c\rangle$. Thus,

$$\begin{aligned} D = |\mathrm{comp}_{\mathbf{n}}\mathbf{b}| &= \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$

## 1.6 12.6 Reading Notes (Cylinders and Quadric Surfaces)

**Definition 1.6.1: Cylinder**

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

**Definition 1.6.2: Quadric Surfaces**

A Quadric Surface is the graph of a second-degree equation in three variables $x$, $y$, and $z$. The most general such equation is

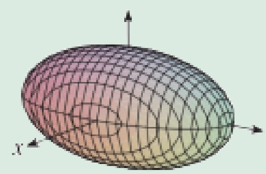
$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0$$

where $A$, $B$, $C$, $\ldots$, $J$ are constants, but by translation and rotation it can be brought into one of the two *standard forms*

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

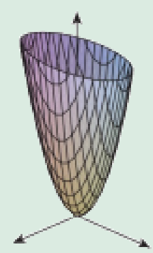
**Example 1.6.1: Graphs of Quadric Surfaces PT 1**

Ellipsoid:



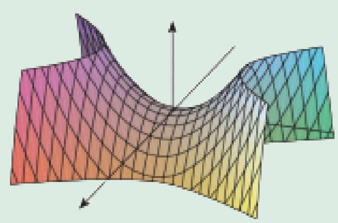

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.



$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

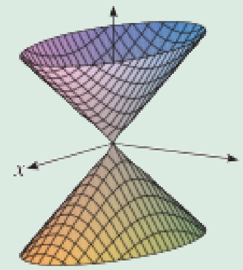
Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.



$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

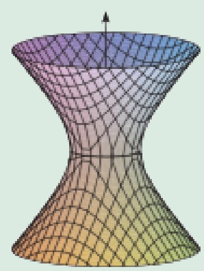
Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.

**Example 1.6.2: Quadric Surfaces Pt 2**



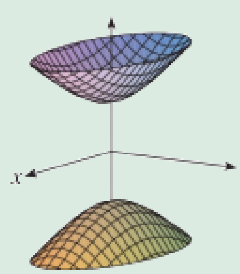

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.




$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

# Chapter 2

## 2.1 13.1 Reading Notes(Vector Functions and Space Curves)

**Definition 2.1.1: Vector Value Functions**

A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions r whose values are three-dimensional vectors. If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\mathbf{r}(t)$, then $f$ , $g$, and $h$ are real-valued functions called the **component functions** of r and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter $t$ to denote the independent variable because it represents time in most applications of vector functions.

**Definition 2.1.2: Limit of Vectors**

The **limit** of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions as follows. If $(\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle)$, then

$$\lim_{t\to a} \mathbf{r}(t) = \left\langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

**Definition 2.1.3: Space Curves**

here is a close connection between continuous vector functions and space curves. Suppose that $f$, $g$, and $h$ are continuous real-valued functions on an interval $I$. Then the set $C$ of all points $(x, y, z)$ in space, where

$$x = f(t) \quad y = g(t) \quad z = h(t)$$

and $(t)$ varies throughout the interval $I$, is called a **space curve**. The equations in are called **parametric equations of** $C$ and $t$ is called a **parameter**.

## 2.2 13.2 Notes (Derivatives and Integrals of Vector Functions)

**Definition 2.2.1: Derivatives**

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

### Definition 2.2.2: Derivatives of vectors pt 2

he following theorem gives us a convenient method for computing the derivative of a vector function $\mathbf{r}$: just differentiate each component of $\mathbf{r}$. **Theorem** If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f$, $g$, and $h$ are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

### Example 2.2.1: Proof of Definition 2.2.2

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\mathbf{r}(t+\Delta t) - \mathbf{r}(t)] \\
&= \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\langle f(t+\Delta t), g(t+\Delta t), h(t+\Delta t)\rangle - \langle f(t), g(t), h(t)\rangle] \\
&= \lim_{\Delta t\to 0} \left\langle \frac{f(t+\Delta t)-f(t)}{\Delta t}, \frac{g(t+\Delta t)-g(t)}{\Delta t}, \frac{h(t+\Delta t)-h(t)}{\Delta t} \right\rangle \\
&= \left\langle \lim_{\Delta t\to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}, \lim_{\Delta t\to 0} \frac{g(t+\Delta t)-g(t)}{\Delta t}, \lim_{\Delta t\to 0} \frac{h(t+\Delta t)-h(t)}{\Delta t} \right\rangle \\
&= \langle f'(t), g'(t), h'(t)\rangle
\end{aligned}$$

A unit vector that has the same direction as the tangent vector is called the **unit tangent vector T** and is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

### Definition 2.2.3: Differentiation Rules

Proof: **Theorem** Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, $c$ is a scalar, and $f$ is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

**Note:-**

We use Formula 4 to prove the following theorem. **Theorem** If $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all $t$.

**Definition 2.2.4: Interation of Vectors**

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of $\mathbf{r}$ in terms of the integrals of its component functions $f$, $g$, and $h$ as follows. (We use the notation of Chapter 5.)

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{n\to\infty} \sum_{j=1}^{n} \mathbf{r}(t_j^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left[ \left( \sum_{i=1}^{n} f(t_i^*)\,\Delta t \right) \mathbf{i} + \left( \sum_{i=1}^{n} g(t_i^*)\,\Delta t \right) \mathbf{j} + \left( \sum_{i=1}^{n} h(t_i^*)\,\Delta t \right) \mathbf{k} \right]$$

and so

$$\int_a^b \mathbf{r}(t)\,dt = \left( \int_a^b f(t)\,dt \right) \mathbf{i} + \left( \int_a^b g(t)\,dt \right) \mathbf{j} + \left( \int_a^b h(t)\,dt \right) \mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function. We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$. We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (antiderivatives).

## 2.3 13.3 Notes (Arc Length and Curvature)

**Definition 2.3.1: Length of a space curve**

Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leqslant t \leqslant b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where $f'$, $g'$, and $h'$ are continuous. If the curve is traversed exactly once as $t$ increases from $a$ to $b$, then it can be shown that its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \tag{2.1}$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{2.2}$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

$$L = \int_a^b |\mathbf{r}'(t)|\, dt \tag{2.3}$$

because, for plane curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

A single curve $C$ can be respresented by more than one vector function. For instance the twisted cube

$$r_1(t) = \langle t, t^2, t^3 \rangle \quad 1 \leqslant t \leqslant 2 \tag{2.4}$$

could also be represented by the function

$$r_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \quad o \leqslant u \leqslant \ln 2 \tag{2.5}$$

where the connection between the parameters $t$ and $u$ is given by $t = e^u$ We say that equations 2.4 and 2.5 are parameterizations of the curve C. If we were to use Equation 2.3 to compute the length of C using Equations 2.4 and 2.5, we would get the same answer. This is because arc length is a geometric property of the curve and hence is independent of the parametrization that is used.

**Definition 2.3.2: Arc Length Function**

Now we supose that the curve $C$ is a cruve given by a vector function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \leqslant t \leqslant b$$

where $\mathbf{r}$' is continuous and $C$ is transvered exactly once as $t$ increases from $a$ to $b$. We define its **arc length functions** by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du \tag{2.6}$$

Thus $s(t)$ is the length of part $C$ between $\mathbf{r}(a)$ and $\mathbf{r}(t)$. If we differentiate both sides of equation 2.6 using part 1 of the Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = |\mathbf{r}(t)| \tag{2.7}$$

It is often useful to **paramterize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system of a particular parametrization.

**Definition 2.3.3: Curvature**

A parametrization $\mathbf{r}(t)$ is called **smooth** on an interval $I$ if $\mathbf{r}$' is continuous and $\mathbf{r}$'$(t) \neq 0$ on $I$. A curve is called smooth if it has a smooth parameterization. A smooth corner has no cusp or sharp corner; when the tangent vector turns it does so continuously.

If $C$ is a smooth curve defined by the vector $\mathbf{r}$, recall that the unit tangent vector $\mathbf{T}$(t) is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The curvature of a curve is

$$k = |\frac{d\mathbf{T}}{ds}| \tag{2.8}$$

where $\mathbf{T}$ is the unit tangent vector

The curvature is easier to compute if it is easier to compute if it is expressed in terms of the paramteter $t$ instead of $s$, so we use the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \Rightarrow k = \left|\frac{d\mathbf{T}}{s}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|$$

but $ds/dt = |\mathbf{r}'(t|)$ from equation 2.7

$$k(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \tag{2.9}$$

The curvature of the curve given by the vector function $\mathbf{r}$ is

$$k(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}^n(t)|}{|\mathbf{r}'(t)|^3} \tag{2.10}$$

**Note:-**

For the special case of a plane curve with equation $y = f(x)$, we choose $x$ as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = 0$, it follows that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f'(x)\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$ and so, by Theorem 10,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \tag{2.11}$$

**Note:-**

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$. We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all $t$, we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$ by Theorem 13.2.4, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$. Note that, typically, $\mathbf{T}'(t)$ is itself not a unit vector. But at any point where $\kappa \neq 0$ we can define the *principal unit normal vector* $\mathbf{N}(t)$ (or simply *unit normal*) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. The vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

**Note:-**

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \quad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

**Definition 2.3.4: Torision**

The **torsion** of a curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Torsion is easier to compute if it is expressed in terms of the parameter $t$ instead of $s$, so we use the Chain Rule to write

$$\frac{d\mathbf{B}}{dt} = \frac{d\mathbf{B}}{ds}\frac{ds}{dt} \quad \text{so} \quad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{\mathbf{B}'(t)}{|\mathbf{r}'(t)|}$$

$$\tau(t) = \frac{-\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|}$$

**Theorem** The torsion of the curve given by the vector function $\mathbf{r}$ is

$$\tau(t) = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

## 2.4 13.4 Notes (Motion in Space: Velocity and Acceleration)

# Chapter 3

## 3.1 14.1 Functions of Several Variables

**Definition 3.1.1: Functions of Two Variables**

Defintion: A function $f$ of two variables is a rule that assigns to each ordered pair of real numbers $(x, y)$ in a set $D$ a unique real number denoted by $f(x, y)$. The set $D$ is the *domain* of $f$ and its *range* is the set of values that $f$ takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

**Definition 3.1.2: Graph of a Function of Two Variables**

Defintion If $f$ is a function of two variables with domain $D$, then the *graph* of $f$ is the set of all points $(x, y, z)$ in $\mathbb{R}^3$ such that $z = f(x, y)$ and $(x, y)$ is in $D$.

**Definition 3.1.3: Level Curves and Contour Maps**

Defintion: The *level curves* of a function $f$ of two variables are the curves with equations $f(x, y) = k$, where $k$ is a constant (in the range of $f$).

**Definition 3.1.4: Functions of Three Variables**

Defintion: **A function of three variables**, $f$, is a rule that assigns to each ordered triple $(x, y, z)$ in a domain $D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$. For instance, the temperature...

## 3.2 14.2 Limits and Continuity

**Definition 3.2.1: Limit of Two Variable Functions**

Defintion: Let $f$ be a function of two variables whose domain $D$ includes points arbitrarily close to $(a, b)$. Then we say that the *limit of* $f(x, y)$ *as* $(x, y)$ *approaches* $(a, b)$ is $L$ and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$, then $|f(x, y) - L| < \varepsilon$.

## 3.3 14.3 Partial Derivatives

**Definition 3.3.1: Partial Derivatives**

Defintion Partial Derivative with respect to x

$$f_x(a,b) = g'(a) \quad \text{where} \quad g(x) = f(x,b)$$

Defintion Partial Derivative with respect to y

$$f_y(a,b) = h'(a) \quad \text{where} \quad h(x) = f(a,y)$$

**Note:-**

If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

**Note:-**

**Rule for Finding Partial Derivatives of $z = f(x,y)$**

1. To find $f_x$, regard $y$ as a constant and differentiate $f(x,y)$ with respect to $x$.
2. To find $f_y$, regard $x$ as a constant and differentiate $f(x,y)$ with respect to $y$.

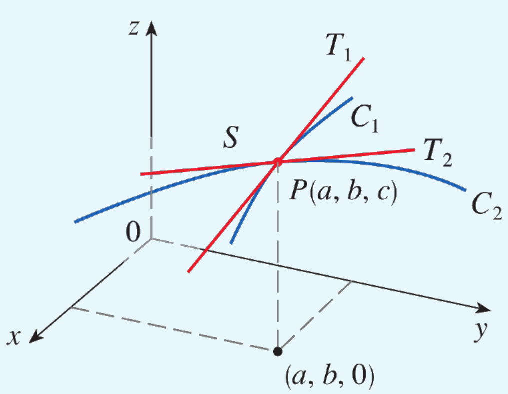
**Definition 3.3.2: interpretation of Partial Derivatives**

To understand partial derivatives geometrically, think of the equation $z = f(x,y)$ as representing a surface $S$ (the graph of $f$). If $f(a,b) = c$, then the point $P(a,b,c)$ lies on this surface.

By fixing $y = b$, we focus on the curve $C_1$ where the vertical plane $y = b$ intersects $S$. Similarly, fixing $x = a$ gives us the curve $C_2$, which is where the vertical plane $x = a$ intersects $S$. Both curves $C_1$ and $C_2$ pass through the point $P$.

The curve $C_1$ is the graph of the function $g(x) = f(x,b)$, and the slope of its tangent at $P$ is $f_x(a,b)$. The curve $C_2$ is the graph of $G(y) = f(a,y)$, and the slope of its tangent at $P$ is $f_y(a,b)$.

Thus, the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ represent the slopes of the tangent lines at $P$ along these curves.



**Definition 3.3.3: Higher Derivatives**

If $f$ is a function of two variables, then its partial derivatives $f_x$ and $f_y$ are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the *second partial derivatives* of $f$. If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation $f_{xy}$ (or $\frac{\partial^2 f}{\partial y \partial x}$) means that we first differentiate with respect to $x$ and then with respect to $y$, whereas in computing $f_{yx}$ the order is reversed.

**Definition 3.3.4: Clairut's Theorem**

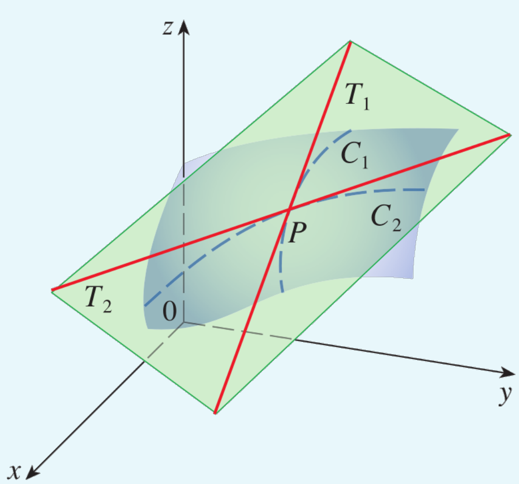
Defintion: Suppose $f$ is defined on a disk $D$ that contains the point $(a, b)$. If the functions $f_{xy}$ and $f_{yx}$ are both continuous on $D$, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

## 3.4 14.4 Tangent Planes and Linear Approximation

**Definition 3.4.1: Tangent Planes**

Let's consider a surface $S$ given by the equation $z = f(x, y)$, where $f$ has continuous first derivatives. Let $P(x_0, y_0, z_0)$ be a point on the surface. Two curves, $C_1$ and $C_2$, are formed by slicing the surface with vertical planes $y = y_0$ and $x = x_0$. These curves pass through the point $P$. The tangent lines to $C_1$ and $C_2$ at $P$ are denoted $T_1$ and $T_2$. The **tangent plane** to the surface at $P$ is the plane that contains both tangent lines $T_1$ and $T_2$.



**Definition 3.4.2: Equation of a tangent plan**

Suppose $f$ has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \tag{3.1}$$

**Definition 3.4.3: Linear Approximations**

If $z = f(x, y)$, then $f$ is **differentiable** at $(a, b)$ if $\Delta z$ can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y \tag{3.2}$$

where $\epsilon_1$ and $\epsilon_2$ are functions of $\Delta x$ and $\Delta y$ such that $\epsilon_1$ and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

## 3.5 14.5 The Chain Rule

**Definition 3.5.1: Chain Rule (Case 1)**

Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of $t$. Then $z$ is a differentiable function of $t$ and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \tag{3.3}$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial z}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \tag{3.4}$$

**Definition 3.5.2: Chain Rule (Case 2)**

Suppose that $z = f(x, y)$ is a differentiable function of $x$ and $y$, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of $s$ and $t$. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \tag{3.5}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \tag{3.6}$$

**Definition 3.5.3: Chain Rule (General Case)**

Suppose that $u$ is a differentiable function of the $n$ variables $x_1, x_2, \ldots, x_n$ and each $x_j$ is a differentiable function of the $m$ variables $t_1, t_2, \ldots, t_m$. Then $u$ is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i} \tag{3.7}$$

for each $i = 1, 2, \ldots, m$.

**Definition 3.5.4: Implicit Differentiation**

Defintion:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial Y}} = -\frac{F_x}{F_y} \tag{3.8}$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \tag{3.9}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z} \tag{3.10}$$

## 3.6 14.6 Directional Derivatives and the Gradient Vector

**Definition 3.6.1: Directional Derivative**

Defintion: The **directional derivative** of $f$ at $(x_0, y_0)$ in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \tag{3.11}$$

if this limit exists.

**Theorem 3.6.1**

If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b \tag{3.12}$$

**Definition 3.6.2: The Gradient Vector**

If $f$ is a function of two variables $x$ and $y$, then the **gradient** of $f$ is the vector function $\nabla f$ defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \tag{3.13}$$

**Note:-**

Equation 3.12 can be rewritten as

$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u} \tag{3.14}$$

**Definition 3.6.3: Gradient of Three Variable Functions**

Defintion:

$$\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{3.15}$$

**Theorem 3.6.2** Maximizing Directional Derivative

Suppose $f$ is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

**Definition 3.6.4: Tanget Planes to Level Surfaces**

Consider a surface $S$ defined by $F(x, y, z) = k$, where $F$ is a function of three variables. Let $P(x_0, y_0, z_0)$ be a point on $S$ and $C$ be a curve on $S$ that passes through $P$. The curve is given by a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ such that $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$. Since $C$ lies on $S$, the equation $F(x(t), y(t), z(t)) = k$ must hold.

By using the Chain Rule to differentiate both sides of this equation, we get:

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

This can be written as a dot product:

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

which means that the gradient $\nabla F$ is perpendicular to the tangent vector $\mathbf{r}'(t)$ at $P$.
At $t = t_0$, the gradient at $P$, $\nabla F(x_0, y_0, z_0)$, is normal to the tangent plane at $P$. The equation of the tangent plane is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{3.16}$$

**Note:-**

**Properties of the Gradient Vecotr**
Let $f$ be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x}) \neq 0$.

- The directional derivative of $f$ at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_{\mathbf{u}} f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.
- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of $f$ at $\mathbf{x}$, and that maximum rate of change is $|\nabla f(\mathbf{x})|$.
- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of $f$ through $\mathbf{x}$.

## 3.7 14.7 Maxium and Minimum Values

**Definition 3.7.1: Local Min and Max**

Defintion: A function of two variables has a **local maximum** at $(a, b)$ if $f(x, y) \leqslant f(a, b)$ when $(x, y)$ is near $(a, b)$. [This means that $f(x, y) \leqslant f(a, b)$ for all points $(x, y)$ in some disk with center $(a, b)$.] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when $(x, y)$ is near $(a, b)$, then $f$ has a **local minimum** at $(a, b)$ and $f(a, b)$ is a **local minimum value**.

**Theorem 3.7.1** Critical Point

If $f$ has a local maximum or minimum at $(a, b)$ and the first-order partial derivatives of $f$ exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

**Definition 3.7.2: Second Derivatives Test**

Suppose the second partial derivatives of $f$ are continuous on a disk with center $(a, b)$, and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [so $(a, b)$ is a critical point of $f$]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \tag{3.17}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $(a, b)$ is a saddle point of $f$.

**Definition 3.7.3: Absolute Maxiums and Absolute Minimums**

Let $(a, b)$ be a point in the domain $D$ of a function $f$ of two variables. Then $f(a, b)$ is the

- **absolute maximum** value of $f$ on $D$ if $f(a, b) \geqslant f(x, y)$ for all $(x, y)$ in $D$.
- **absolute minimum** value of $f$ on $D$ if $f(a, b) \leqslant f(x, y)$ for all $(x, y)$ in $D$.

**Definition 3.7.4: Extreme Value Theorem for Functions of Two Variables**

If $f$ is continuous on a closed, bounded set $D$ in $\mathbb{R}^2$, then $f$ attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points $(x_1, y_1)$ and $(x_2, y_2)$ in $D$.

**Note:-**

To find the absolute maximum and minimum values of a continuous function $f$ on a closed, bounded set $D$:

1. Find the values of $f$ at the critical points of $f$ in $D$.
2. Find the extreme values of $f$ on the boundary of $D$.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

## 3.8 14.8 Lagrange Multipliers

**Definition 3.8.1: Geometric Explanation Lagrange Multipliers (One Constraint)**

The geometric basis of Lagrange's method for two variables involves finding the extreme values of $f(x, y)$ under the constraint $g(x, y) = k$. This means finding the extreme values of $f$ along the level curve defined by $g(x, y) = k$. To maximize $f(x, y)$ subject to this constraint, we look for the largest value of $f$ where its level curve touches the constraint curve at one point. At this point, the gradients of $f$ and $g$ are parallel, which gives the relationship $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar $\lambda$.

This reasoning also applies to functions of three variables. To find extreme values of $f(x, y, z)$ under the constraint $g(x, y, z) = k$, the point $(x, y, z)$ must lie on the level surface defined by $g(x, y, z) = k$. The gradients of $f$ and $g$ are again parallel at the point where the maximum value of $f$ is reached.

To make this precise, consider a curve $C$ on the surface where $g(x, y, z) = k$. The function $f$ has an extreme value at the point $P(x_0, y_0, z_0)$. If we parameterize the curve as $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, then $f$ has an extreme value at $t_0$ when $h'(t_0) = 0$, where $h(t) = f(\mathbf{r}(t))$. Using the Chain Rule, we find that the gradients are again parallel.

**Definition 3.8.2: Method of Lagrange Multipliers**

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]:

1. Find all values of $x$, $y$, $z$, and $\lambda$ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate $f$ at all the points $(x, y, z)$ that result from step 1. The largest of these values is the maximum value of $f$; the smallest is the minimum value of $f$.

**Definition 3.8.3: Lagrange Multipliers Two Constraints**

To find the maximum and minimum values of $f(x, y, z)$ subject to two constraints, $g(x, y, z) = k$ and $h(x, y, z) = c$, we look for extreme values when $(x, y, z)$ lies on the curve formed by the intersection of the level surfaces of $g$ and $h$.

At the point where $f$ has an extreme value, the gradient of $f$, $\nabla f$, is orthogonal to the curve. The gradients of $g$ and $h$ are also orthogonal to this curve, which means $\nabla f$ must lie in the plane formed by $\nabla g$ and $\nabla h$.

Thus, there exist two numbers, $\lambda$ and $\mu$, called Lagrange multipliers, such that $\nabla f(x_0, y_0, z_0)$ is a linear combination of $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$.

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0) \tag{3.18}$$

# Chapter 4

# Multiple Integerals

## 4.1 15.1 Double Integral Over Rectangles

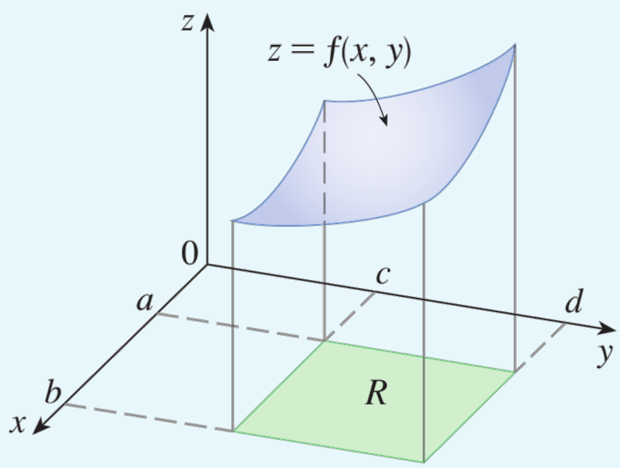
**Definition 4.1.1: Volume of a Function**

We start with a function $f$ defined over a closed rectangle $R$ in the $xy$-plane, denoted as:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}.$$

The goal is to find the volume of the solid $S$, which lies above the rectangle $R$ and below the graph of the surface $z = f(x, y)$, defined as:

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y),\ (x, y) \in R\}.$$

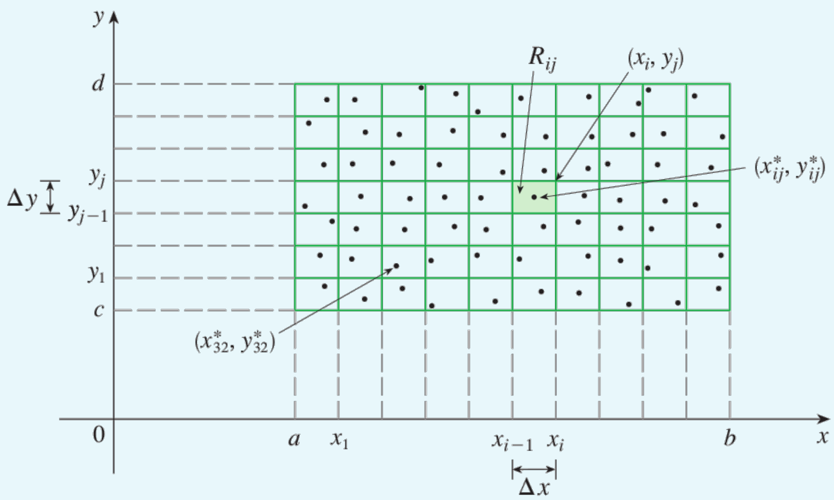


**Definition 4.1.2: Volume of a Function (PART II)**

To compute this volume, we first divide the rectangle $R$ into smaller subrectangles. The interval $[a, b]$ is divided into $m$ subintervals of equal width $\Delta x = \frac{b-a}{m}$, and the interval $[c, d]$ is divided into $n$ subintervals of equal width $\Delta y = \frac{d-c}{n}$. The subrectangles are denoted by:

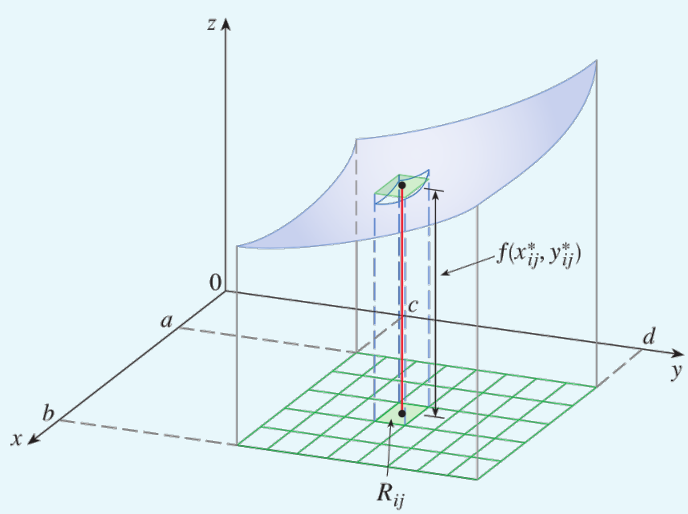
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}.$$

Each subrectangle has an area of $\Delta A = \Delta x \Delta y$.




Next, we approximate the volume of the solid $S$ by choosing a sample point $(x_{ij}^*, y_{ij}^*)$ in each subrectangle $R_{ij}$. Using this point, we approximate the part of $S$ above each $R_{ij}$ by a thin rectangular box (or "column") with base $R_{ij}$ and height $f(x_{ij}^*, y_{ij}^*)$. The volume of this box is:

$$f(x_{ij}^*, y_{ij}^*)\Delta A.$$



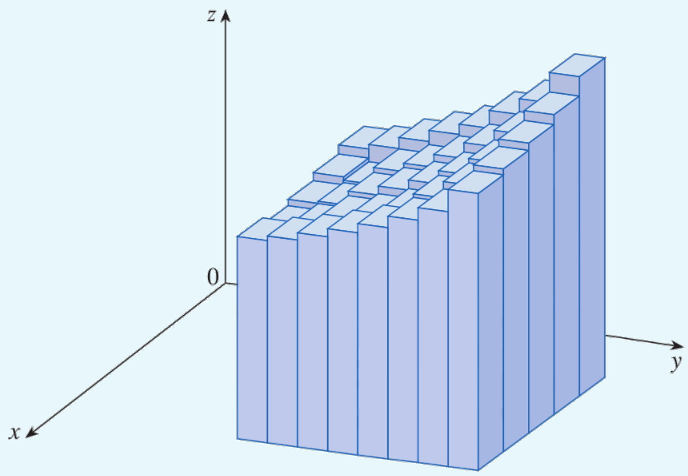
### Definition 4.1.3: Volume of a Function (PART III)

We then sum the volumes of all these boxes over the entire grid of subrectangles, obtaining an approximation of the total volume $V$ of the solid $S$:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

As $m$ and $n$ become larger, our approximation becomes more accurate. The exact volume of $S$ is given by the limit of the double sum as $m, n \to \infty$:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \tag{4.1}$$




This is the formal definition of the volume of the solid $S$ that lies under the graph of the function $f$ and above the rectangle $R$. By taking the limit, we ensure that the approximation becomes exact.

### Definition 4.1.4: Double Integeral

Defintion: The **double integral** of $f$ over the rectangle $R$ is

$$\iint\limits_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \tag{4.2}$$

if this limit exists

### Definition 4.1.5: Equation for Volume

If $f(x,y) \geqslant 0$, then the volume $V$ of the solid that lies above the rectangle $R$ and below the surface $z = f(x,y)$ is

$$V = \iint\limits_R f(x,y)\,dA \tag{4.3}$$

### Definition 4.1.6: Midpoint Rule for Double Integerals

Equation:

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\,\Delta A \tag{4.4}$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\bar{y}_j$ is the midpoint of $[y_{j-1}, y_j]$.

### Definition 4.1.7: Fubini's Theorem

If $f$ is continuous on the rectangle

$$R = \{(x,y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$$

then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy dx = \int_c^d \int_a^b f(x,y)\,dx dy$$

More generally, this is true if we assume that $f$ is bounded on $R$, $f$ is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

### Definition 4.1.8: Average Value

We define the **average value** of a function $f$ of two variables defined on a rectangle $R$ to be

$$f_{\text{avg}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

where $A(R)$ is the area of $R$.
If $f(x,y) \geqslant 0$, the equation

$$A(R) \times f_{\text{avg}} = \iint_R f(x,y)\,dA$$

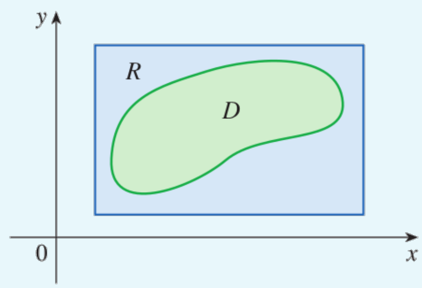
holds true.

## 4.2 15.2 Double Integrals over General Regions

**Definition 4.2.1: General Regions**

Consider a general region $D$ which is bounded, which means that $D$ can be enclosed in a rectangular region $R$. In order to integrate a function $f$ over $D$, we define a new function $F$ with domain $R$ by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$




$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA$$

where $F$ is given by the above equation.

**Definition 4.2.2: Type 1**

A plane region $D$ is said to be of **type I** if it lies between the graphs of two continuous functions of $x$, that is,

$$D = \{(x,y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\} \tag{4.5}$$

where $g_1$ and $g_2$ are continuous on $[a,b]$.
If $f$ is continuous on a type I region $D$ described by

$$D = \{(x,y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy dx \tag{4.6}$$

**Definition 4.2.3: Type II**

If $f$ is continuous on a type II region $D$ described by

$$D = \{(x,y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\} \tag{4.7}$$

then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx dy \tag{4.8}$$

**Note:-**

**Properties of Integerals**

- $$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA \tag{4.9}$$

- $$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA \quad \text{where } c \text{ is a constant.} \tag{4.10}$$

- If $f(x,y) \geqslant g(x,y)$ for all $(x,y) \in D$, then
$$\iint_D f(x,y)\,dA \geqslant \iint_D g(x,y)\,dA \tag{4.11}$$

- If $D = D_1 \cup D_2$ where $D_1$ and $D_2$ don't overlap except perhaps on their boundaries, then
$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA \tag{4.12}$$

- If we integrate the constant function $f(x,y) = 1$ over a region $D$, we get the area of $D$
$$\iint_D 1\,dA = A(D) \tag{4.13}$$

- If $m \leqslant f(x,y) \leqslant M$ for all $(x,y) \in D$, then
$$m \cdot A(D) \leqslant \iint_D f(x,y)\,dA \leqslant M \cdot A(D) \tag{4.14}$$

## 4.3 Double Integrals in Polar Coodinates

**Definition 4.3.1: Rectangle to Polar Coordinates in Double Integerals**

If $f$ is continuous on a polar rectangle $R$ given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{4.15}$$

**Definition 4.3.2: Polar Region Double Integration**

If $f$ is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{4.16}$$

# Chapter 5

## 5.1 16.1 Vector Fields

**Definition 5.1.1: Vector Field**

Defintion: Let $D$ be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point $(x, y)$ in $D$ a two-dimensional vector $\mathbf{F}(x, y)$.

**Definition 5.1.2: 3 Dimensional Vector Fields**

Defintion: Let $E$ be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point $(x, y, z)$ in $E$ a three-dimensional vector $\mathbf{F}(x, y, z)$.

**Definition 5.1.3: Gradient Fields**

Defintion: If $f$ is a scalar function of two variables, its gradient $\nabla f$ (also called grad $f$) is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

Therefore, $\nabla f$ is a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Similarly, if $f$ is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$, given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

## 5.2 16. 2 Line Integerals

**Definition 5.2.1: Line Integerals**

If $f$ is defined on a smooth curve $C$ given by

$$x = x(t) \quad y = y(t) \quad a \leqslant t \leqslant b$$

then the **line integral** of $f$ along $C$ is

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i \tag{5.1}$$

if this limit exists.

**Definition 5.2.2: Evaluation of Line Integerals**

The length of the curve $C$ is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If $f$ is continuous, the following formula can be used to evaluate the line integral of $f$ along $C$:

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{5.2}$$

**Definition 5.2.3: Line Integrals with Respect to x or y**

Two other types of line integrals are obtained by replacing $\Delta s_i$ by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$. They are called the **line integrals of $f$ along $C$ with respect to $x$ and $y$**:

$$\int_C f(x, y)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i \tag{5.3}$$

$$\int_C f(x, y)\, dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i \tag{5.4}$$

These line integrals can also be evaluated in terms of a parameter $t$, where $x = x(t)$, $y = y(t)$, and:

$$\int_C f(x, y)\, dx = \int_a^b f(x(t), y(t))x'(t)\, dt \tag{5.5}$$

$$\int_C f(x, y)\, dy = \int_a^b f(x(t), y(t))y'(t)\, dt \tag{5.6}$$

**Definition 5.2.4: Line Integrals in Space**

Equation:

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{5.7}$$

# Chapter 6

# CPSC 201 Midterm 2 Review

## 6.1 Booleans

**Definition 6.1.1: Truth table**

Text:

1. Table of all the possible trith values returned by a boolean expressoin from all the possible inputs

**Definition 6.1.2: Sum of Products**

Text:

1. Isolate rows where teh output is 1/true and ignore rows where output is 0/false
2. For each true row, write a sub-expression that takes the AND of the all the variables together while taking the NOT for any variable whose input value is 0