

Math 120

PSet 3

Sep 19 2024

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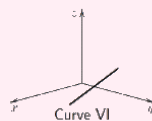
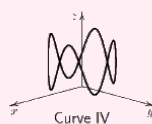
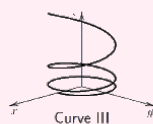
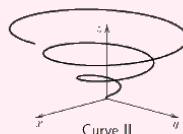
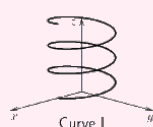
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Chapter 1

1.1 PSet 3

Question 1



Find the curve parameterized by each vector-valued function. Justify your answers

- (a) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$
- (b) $\vec{r}(t) = t \langle \cos t, \sin t, t \rangle$
- (c) $\vec{r}(t) = \langle \cos t, \sin t, t^3 \rangle$
- (d) $\vec{r}(t) = \langle \cos(t^3), \sin(t^3), t^3 \rangle$
- (e) $\vec{r}(u) = \langle \cos u, \sin u, 1 + \sin(4u) \rangle$
- (f) $\vec{r}(u) = \langle \cos u, \sin u, 1 + 4 \sin(u) \rangle$
- (g) $\vec{r}(t) = \langle 2 \cos t, 1 + 4 \cos t, 3 \cos t \rangle$

Solution:

Equation a should make a helix with fixed radi lengths so it goes with curve I.

Equation b is similar to equation a but would have increasing ring sizes as t increases so it goes with curve II.

Equation c should be III because the radi is constant but it grows much faster.

Equation d is similar to equation a except the helix would get rings faster. It would still go with curve I.

Equation e would go with curve IV because the x and y portions should form circles but due to $1 + \sin(4u)$ there should also be oscillation in the z axis.

Equation f would go with curve V because it has oscillating height with periods that match the y -axis.

Equation g would go with curve VI because all components are proportional to $\cos t$ which suggests a straight line.

Question 2

Find a vector function that represents the curve of intersection of the plane $z = -2$ and the sphere $x^2 + (y - 1)^2 + (z + 1)^2 = 9$.

Solution:

$$\begin{aligned}x^2 + (y - 1)^2 + ((-2) + 1)^2 &= 9 \\x^2 + (y - 1)^2 &= 8 \\r &= 2\sqrt{2} \\x(t) &= 2\sqrt{2} \cos(t) \\y - 1 &= 2\sqrt{2} \sin(t) \Rightarrow y = 2\sqrt{2} \sin(t) + 1 \\\vec{r}(t) &= \langle 2\sqrt{2} \cos(t), 2\sqrt{2} \sin(t) + 1, -2 \rangle\end{aligned}$$

Question 3

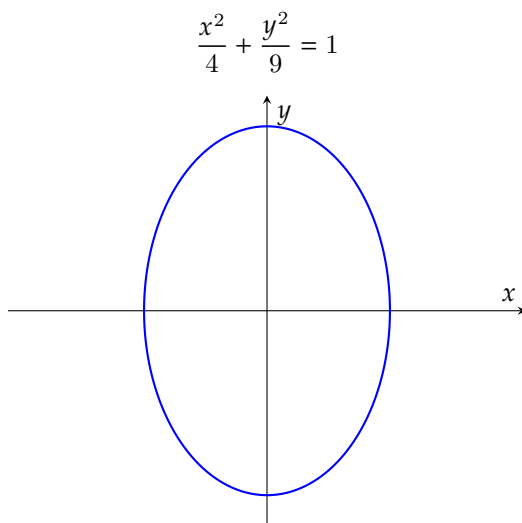
Consider the vector-valued function $\vec{r}_1(t) = \langle 2 \sin t, -3 \cos t, 0 \rangle$, $0 \leq t \leq 2\pi$.

- Sketch the plane curve given by $\vec{r}_1(t)$.
- Compute and draw on your sketch from part (a) the position vector $\vec{r}_1\left(\frac{2\pi}{3}\right)$ and the tangent vector $\vec{r}'_1\left(\frac{2\pi}{3}\right)$.
- The vector-valued function $\vec{r}_2(t) = \langle 2 \cos(3t), -3 \sin(3t) \rangle$ parameterizes the same curve. Find the smallest $t^* > 0$ such that $\vec{r}_2(t^*) = \vec{r}_1\left(\frac{2\pi}{3}\right)$, and compute $\vec{r}'_2(t^*)$. Explain how and why $\vec{r}'_2(t^*)$ differs from the tangent vector $\vec{r}'_1\left(\frac{2\pi}{3}\right)$ you computed in part (b).

Solution:

a)

$$\begin{aligned}x &= 2 \sin t \quad y = -3 \cos t \\ \left(\frac{x}{2}\right)^2 + \left(\frac{y}{-3}\right)^2 &= \sin^2 t + \cos^2 t = 1\end{aligned}$$



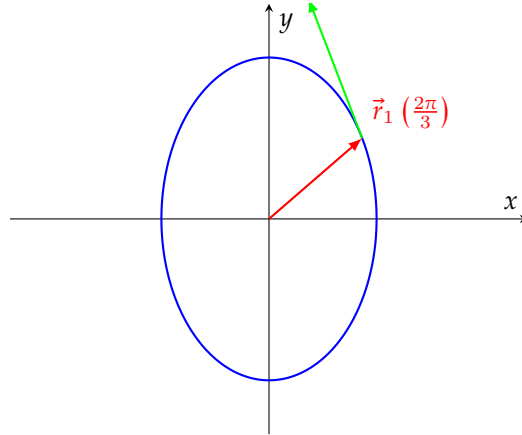
b)

$$\begin{aligned}\vec{r}_1\left(\frac{2\pi}{3}\right) &= \left\langle 2 \sin\left(\frac{2\pi}{3}\right), -3 \cos\left(\frac{2\pi}{3}\right) \right\rangle = \left\langle \sqrt{3}, \frac{3}{2} \right\rangle \\\vec{r}'_1(t) &= \langle 2 \cos(t), 3 \sin(t) \rangle\end{aligned}$$

$$\vec{r}'_1\left(\frac{2\pi}{3}\right) = \langle 2 \cos\left(\frac{2\pi}{3}\right), 3 \sin\left(\frac{2\pi}{3}\right) \rangle$$

$$\vec{r}'_1\left(\frac{2\pi}{3}\right) = \langle 2\left(-\frac{1}{2}\right), 3\left(\frac{\sqrt{3}}{2}\right) \rangle$$

$$\vec{r}'_1\left(\frac{2\pi}{3}\right) = \langle -1, \frac{3\sqrt{3}}{2} \rangle$$



c)

$$r_2(t^*) = \langle 2 \cos(3t^*), -2 \sin(3t^*) \rangle = \langle \sqrt{3}, \frac{3}{2} \rangle$$

$$2 \cos(3t^*) = \sqrt{3} \quad -3 \sin(3t^*) = \frac{3}{2}$$

$$\cos(3t^*) = \frac{\sqrt{3}}{2} \Rightarrow 3t^* = \frac{\pi}{6}, \frac{11\pi}{6}$$

$$t^* = \frac{\pi}{18}, \frac{11\pi}{18}$$

$$-3 \sin(3t^*) = \frac{3}{2} \Rightarrow \sin(3t^*) = -\frac{1}{2}$$

$$3t^* = \frac{-7\pi}{6}, \frac{11\pi}{6}$$

$$t^* = \frac{7\pi}{18}, \frac{11\pi}{18}$$

$$t^* = \frac{11\pi}{18}$$

$$r'_2(t) = \langle -6 \sin(3t), -9 \cos(3t) \rangle$$

$$r'_2\left(\frac{11\pi}{18}\right) = \langle 3, \frac{-9\sqrt{3}}{2} \rangle$$

Tangent vectors differ because $r_2(t)$ traces the same curve more rapidly which affects the magnitude and direction of the tangent vector.

Question 4

Find parametric equations for the tangent line to the curve parameterized by

$$x = 2t + 1, \quad y = e^{t^2-4}, \quad z = \ln(1+t^2)$$

at the point $(5, 1, \ln 5)$.

Solution:

$$x(t) = 2t + 1 \quad y(t) = e^{t^2-4} \quad z(t) = \ln(1+t^2)$$

$$x'(t) = 2 \quad y'(t) = 2te^{t^2-4} \quad z'(t) = \frac{2t}{1+t^2}$$

$$5 = 2t + 1 \Rightarrow 4 = 2t \Rightarrow t = 2$$

$$x'(2) = 2 \quad y'(2) = 4e^{2^2-4} = 4 \quad z'(2) = \frac{4}{5}$$

$$x : 5 + 2t \quad y : 1 + 4t \quad z : \ln(5) + \frac{4}{5}t$$

Question 5

(a) Evaluate the integral $\int (\tan t \hat{i} + \sin^2 t \hat{j} + \sec^2 t \tan t \hat{k}) dt$.

(b) Suppose a particle is at the point $(-2, 1, 4)$ at time $t = 0$, and moves according to the velocity function $\vec{v}(t) = \tan t \hat{i} + \sin^2 t \hat{j} + \sec^2 t \tan t \hat{k}$. Find the particle's position at time $t = \frac{\pi}{4}$.

Solution:

a)

$$\int (\tan t \hat{i} + \sin^2 t \hat{j} + \sec^2 t \tan t \hat{k}) dt = \int \tan t \hat{i} dt + \int \sin^2 t \hat{j} dt + \int \sec^2 t \tan t \hat{k} dt$$

$$\int \tan t \hat{i} dt = \hat{i} \int \frac{\sin t}{\cos t} dt$$

$$x = \cos(t)$$

$$\hat{i} \int \tan t dt = \hat{i} \int -\frac{1}{x} dx = -\ln|x| + k$$

$$(-\ln|x| + k)\hat{i} = (-\ln|\cos(t)| + a)\hat{i}$$

$$\hat{j} \int \sin^2 t dt = \hat{j} \int \frac{1 - 2\cos 2t}{2} dt$$

$$\hat{j} \int \frac{1 - 2\cos 2t}{2} dt \Rightarrow \hat{j} \frac{1}{2} \int 1 - 2\cos 2t dt$$

$$\hat{j} \frac{1}{2} \int 1 - 2\cos 2t dt = \hat{j} \frac{1}{2} t - \hat{j} \frac{1}{2} \int \cos 2t dt$$

$$\hat{j} \frac{1}{2} t - \hat{j} \int \cos 2t dt = \left(\frac{1}{2} t - \frac{\sin(2t)}{4} + b \right) \hat{j}$$

$$\hat{k} \int \sec^2 t \tan t dt$$

$$\tan t = u \quad \sec^2 t dt = du$$

$$\int u du = \frac{u^2}{2} + c \Rightarrow \left(\frac{\tan^2 t}{2} + c \right) \hat{k}$$

$$\int \left(\tan t \hat{i} + \sin^2 t \hat{j} + \sec^2 t \tan t \hat{k} \right) dt = (-\ln |\cos(x)| + a) \hat{i} + \left(\frac{1}{2}t - \frac{\sin(2t)}{4} + b \right) \hat{j} + \left(\frac{\tan^2 t}{2} + c \right) \hat{k}$$

b)

$$(-\ln |\cos(0)| + a), \left(\frac{1}{2}t - \frac{\sin(2(0))}{4} + b \right), \left(\frac{\tan^2(0)}{2} + c \right) = (-2, 1, 4)$$

$$-\ln(1) + a, 0 - \frac{0}{4} + b, \frac{0}{2} + c = (-2, 1, 4)$$

$$a = -2 \quad b = 1 \quad c = 4$$

$$\left(-\ln \left| \cos \left(\frac{\pi}{4} \right) \right| - 2 \right), \left(\frac{1}{2}t - \frac{\sin(2(\frac{\pi}{4}))}{4} + 1 \right), \left(\frac{\tan^2(\frac{\pi}{4})}{2} + 4 \right) = \left(-\ln \left(\frac{\sqrt{2}}{2} \right) - 2, \frac{\pi}{8} + \frac{3}{4}, \frac{9}{2} \right)$$

Question 6

Consider the curve parameterized by $\vec{r}(t) = \langle e^{2t}, e^{-2t}, \sqrt{8t} \rangle$, $0 \leq t \leq 1$.

(a) Sketch the projections of $\vec{r}(t)$ in the xy -, zx -, and yz -planes.

(b) Find the length of the curve. *Hint:* To integrate, you will need to write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$ as a perfect square.

Solution:

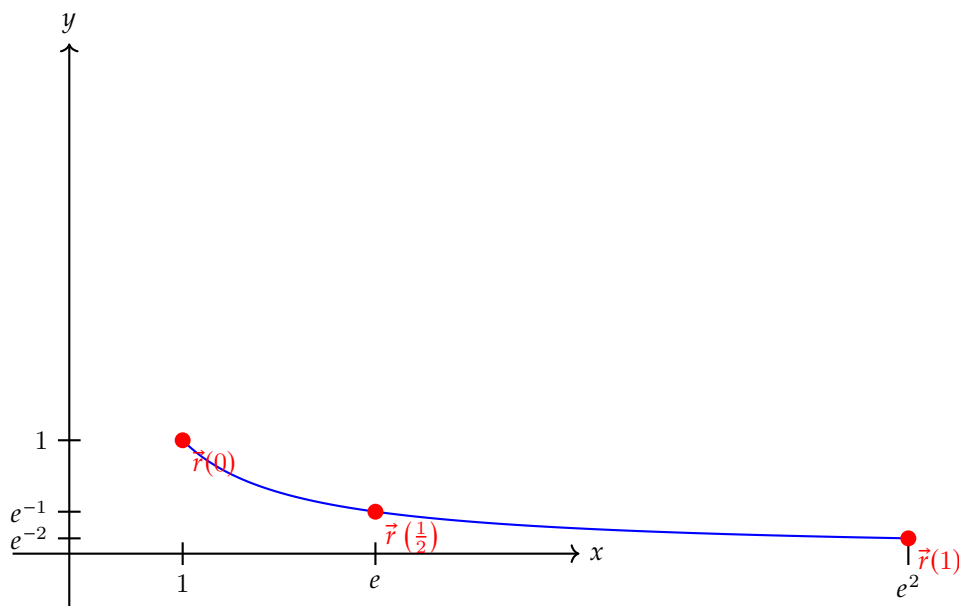
a)

$$t = 0 \quad \vec{r}(0) = \langle 1, 1, 0 \rangle$$

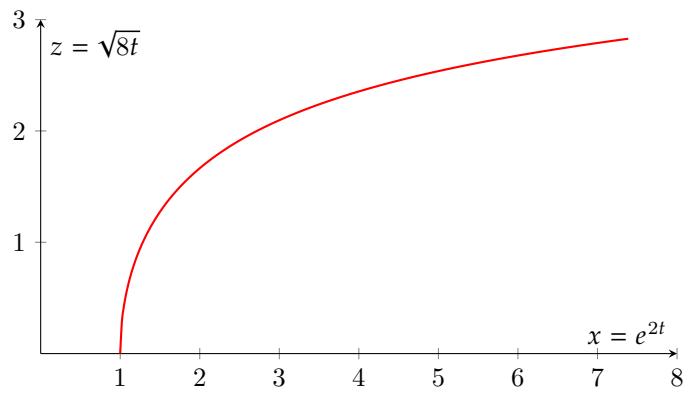
$$t = \frac{1}{2} \quad \vec{r}\left(\frac{1}{2}\right) = \langle e, e^{-1}, \sqrt{2} \rangle$$

$$t = 1, \vec{r}(1) = \langle e^2, e^{-2}, 2\sqrt{2} \rangle$$

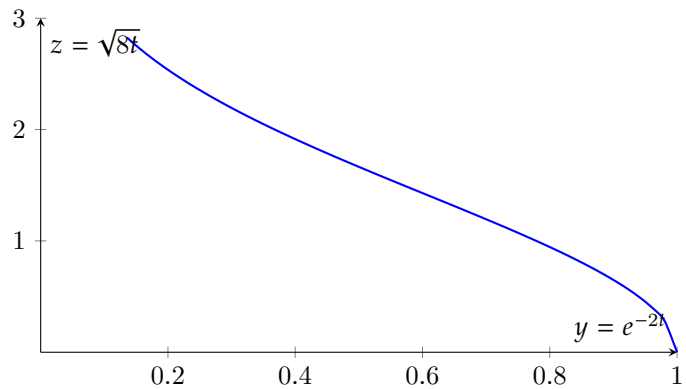
xy



xz



yz



b)

$$L = \int_0^1 \|\vec{r}'(t)\| dt$$

$$\vec{r}(t) = \langle e^{2t}, e^{-2t}, \sqrt{8t} \rangle$$

$$\vec{r}'(t) = \left\langle \frac{d}{dt}(e^{2t}), \frac{d}{dt}(e^{-2t}), \frac{d}{dt}(\sqrt{8t}) \right\rangle$$

$$\vec{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, \sqrt{8} \rangle$$

$$L = |\vec{r}'(t)| = \int_0^1 \sqrt{(2e^{2t})^2 + (-2e^{-2t})^2 + (\sqrt{8})^2} dt$$

$$(2e^{2t})^2 + (-2e^{-2t})^2 + (\sqrt{8})^2 = (2e^{2t})^2 + (-2e^{-2t})^2 + 8$$

$$(2e^{2t})^2 + (-2e^{-2t})^2 + 8 = (2e^{2t} + 2e^{-2t})^2$$

$$L = \int_0^1 \sqrt{(2e^{2t} + 2e^{-2t})^2} dt \Rightarrow \int_0^1 (2e^{2t} + 2e^{-2t}) dt$$

$$\cosh = \frac{e^t + e^{-t}}{2}$$

$$2e^{2t} + 2e^{-2t} = 4 \cosh(2t)$$

$$L = \int_0^1 4 \cosh(2t) dt \Rightarrow 2 \sinh(2t) \Big|_0^1 \rightarrow 2 \sinh(2(1)) - 2 \sinh(2(0))$$

$$L = 2 \sinh(2) - 2 \sinh(0)$$

Question 7

Let C be the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $2x + y + z = 4$.

- Find a parameterization of C .
- Write down an integral for the length of C .
- Find the length accurate to five decimal places by using Desmos: <https://www.desmos.com/calculator>. (Click on the keyboard icon, then “functions”, then “Misc”, to find the integral symbol.)

Solution:

a)

$$x^2 + y^2 = 4 \quad r = 2$$

$$x(t) = 2 \cos(t) \quad y(t) = 2 \sin(t)$$

$$2(2 \cos(t)) + 2 \sin(t) + z = 4 \Rightarrow z = 4 - 4 \cos(t) - 2 \sin(t)$$

$$x^2 + y^2 = 4 \quad r = 2 \quad z(t) = 4 - 4 \cos(t) - 2 \sin(t)$$

b)

$$L = \int_a^b \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2 + \left(\frac{d}{dt}z(t)\right)^2}$$

$$\vec{r}'(t) = \langle -2 \sin t, 2 \cos t, 4 \sin t - 2 \cos t \rangle$$

$$L = \int_a^b \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + (4 \sin t - 2 \cos t)^2}$$

$$L = \int_a^b \sqrt{4 \sin^2 t + 4 \cos^2 t + 16 \sin^2 t - 16 \sin t \cos t + 4 \cos^2 t}$$

$$L = \int_a^b \sqrt{20 \sin^2 t + 8 \cos^2 t - 16 \sin t \cos t}$$

c)

$$\approx 22.64159$$

Question 8

Find the velocity and position vectors of a particle that has acceleration given by

$$\vec{a}(t) = 2\hat{i} + 6t\hat{j} + 12t^2\hat{k},$$

and initial velocity and position given by

$$\vec{v}(0) = \hat{i} \quad \text{and} \quad \vec{r}(0) = \hat{j} - \hat{k}.$$

Solution:

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$$

$$\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$$

$$\vec{a}(t) = 2\hat{i} + 6t\hat{j} + 12t^2\hat{k}$$

$$\vec{v}(t) = \int \vec{a}(t)dt = \int 2\hat{i} + 6t\hat{j} + 12t^2\hat{k}dt$$

$$\int 2\hat{i} + 6t\hat{j} + 12t^2\hat{k} = (2t + a)\hat{i} + (3t^2 + b)\hat{j} + (4t^3 + c)\hat{k}$$

$$\vec{v}(0) = (2(0) + a)\hat{i}, (3(0)^2 + b)\hat{j}, (4(0)^{3+c})\hat{k} = \langle i, 0, 0 \rangle$$

$$\vec{v}(t) = (2t + 1)\hat{i} + (3t^2)\hat{j} + (4t^3)\hat{k}$$

$$\vec{r}(t) = \int \vec{v}(t)dt = \int (2t + 1)\hat{i} + (3t^2)\hat{j} + (4t^3)\hat{k} dt$$

$$\int (2t + 1)\hat{i} + (3t^2)\hat{j} + (4t^3)\hat{k} dt = (t^2 + t + a_2)\hat{i} + (t^3 + b_2)\hat{j} + (t^4 + c_2)\hat{k}$$

$$\vec{r}(0) = ((0)^2 + (0) + a_2)\hat{i} + ((0)^3 + b_2)\hat{j} + ((0)^4 + c_2)\hat{k} = (a_2)\hat{i} + (b_1)\hat{j} + (c_2)\hat{k}$$

$$(a_2)\hat{i} + (b_1)\hat{j} + (c_2)\hat{k} = \langle 0, \hat{j}, -\hat{k} \rangle$$

$$a_2 = 0 \quad b_2 = 1 \quad c_2 = -1$$

$$\vec{r}(t) = (t^2 + t)\hat{i} + (t^3 + 1)\hat{j} + (t^4 - 1)\hat{k}$$

Question 9

Consider the function $f(x, y) = \frac{\sqrt{y-3x}}{\ln(4-x^2-y^2)}$.

- Find and sketch the domain of f .
- On your sketch from part (a), mark where $f(x, y) = 0$, and indicate the region(s) where $f(x, y)$ is positive and negative.

Solution:

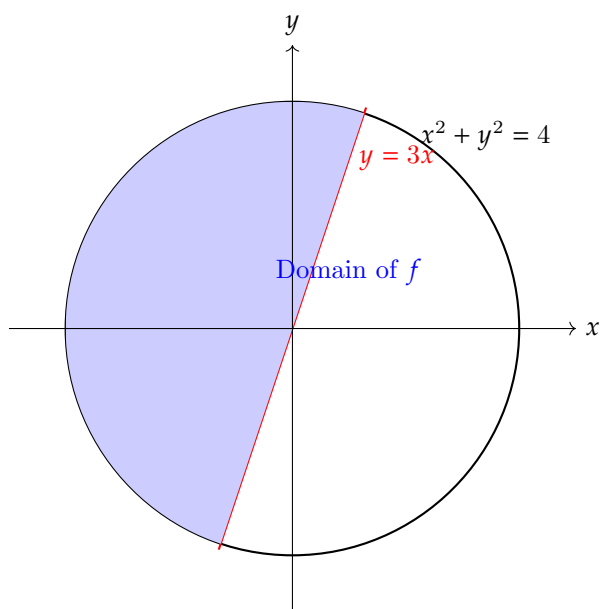
a)

$$(x, y) = \frac{\sqrt{y-3x}}{\ln(4-x^2-y^2)}$$

$$y \geq 3x$$

$$4 - x^2 - y^2 > 0$$

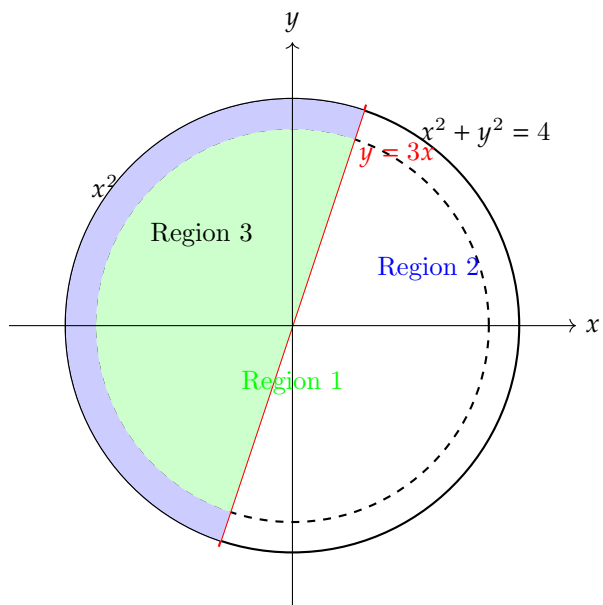
$$\text{domain: } x + y^2 < 4 \quad \text{and} \quad y \geq 3x$$



b)

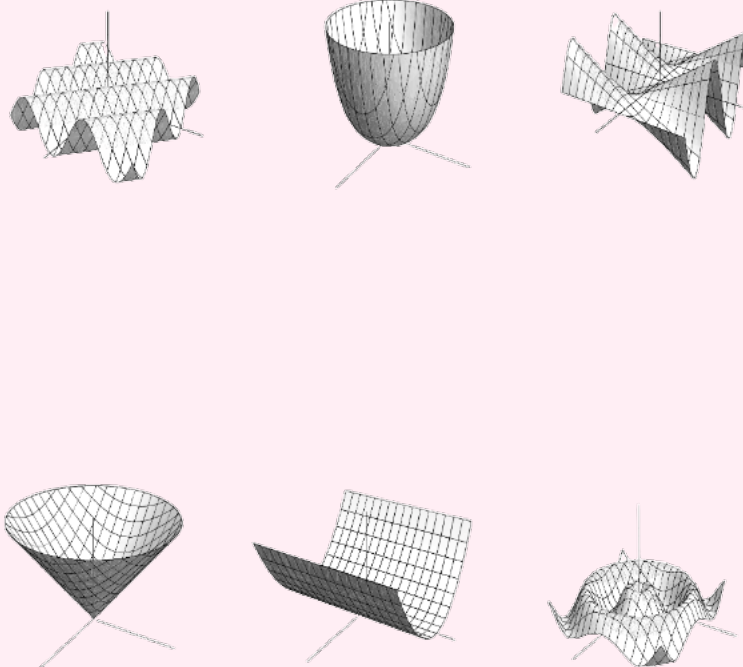
Positive: $y > 3x$ and $x^2 + y^2 < 4$

Negative: $y > 3x$ and $0 < 4 - x^2 - y^2 < 1$



Question 10

Here are several surfaces.



Match each function with its graph. Justify your answers.

- (a) $f(x, y) = x^2$
- (b) $f(x, y) = \sqrt{x^2 + y^2}$
- (c) $f(x, y) = e^{x^2+y^2} - 1$
- (d) $f(x, y) = y \sin x$
- (e) $f(x, y) = \sin(x + y)$
- (f) $f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$

Solution:

Equation a goes with Surface V because it should be parabolic along the x-axis and independent of y.

Equation b goes with surface IV because the value of z should increase linearly with radical distance which should make a cone like shape

Equation c goes with surface II z increases exponentially as x and y increase which should make for something cone like but that grows faster which should make a steep smooth rise

Equation d goes with surface III because the function should oscillate in the x-direction while increasing due to y values so

Equation e goes with surface I because it depends on the sum of x and y and would have a constant phase along lines $x + y$ equals a constant. So there should be a diagonal wave pattern.

Equation f goes with surface VI because it should still have waves that depend on distance from origin, meaning there should be ripples.

Question 11

Draw a contour map of the function $f(x, y) = x^2 e^{-y}$ showing several level curves.

Solution:

$$f(x, y) = x^2 e^{-y}$$

$$x^2 e^{-y} = k$$

$$e^{-y} = \frac{k}{x^2}$$

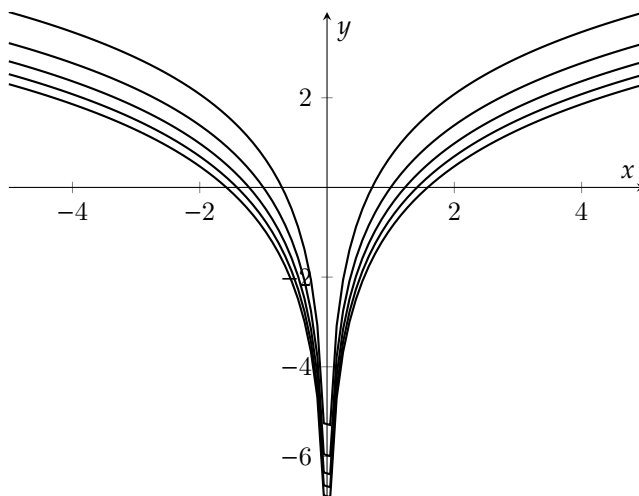
$$\ln(e^{-y}) = \ln\left(\frac{k}{x^2}\right)$$

$$-y = \ln(k) - \ln(x^2) \Rightarrow y = -\ln(k) + 2\ln(x)$$

$x^2 > 0$ for all x in the original function definition so the equation is actually

$$-y = \ln(k) - \ln(x^2) \Rightarrow y = -\ln(k) + 2\ln(|x|)$$

$$y = -\ln(k) + 2\ln(|x|)$$



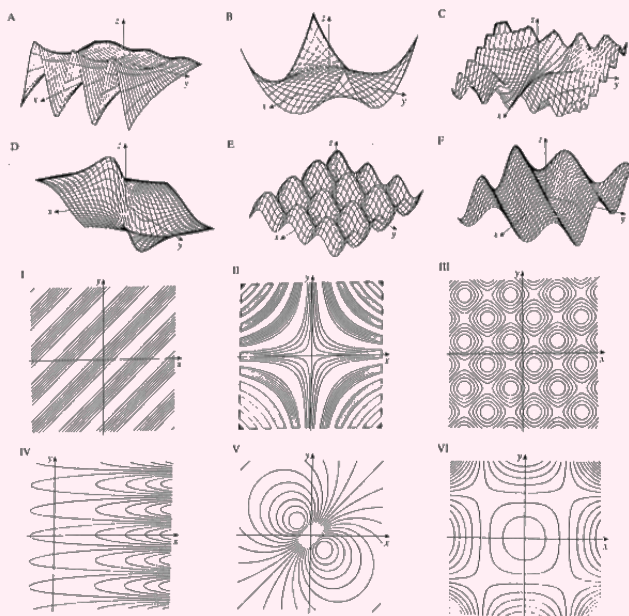
Question 12

Match the function with its graph (labeled A-F below) and with its contour map (labeled I-VI). Give reasons for your choices.

(a) $z = e^x \cos y$

(b) $z = \sin x - \sin y$

(c) $z = \frac{x-y}{1+x^2+y^2}$



Solution:

a)

e^x is an exponential function in the x -direction, meaning that as x increases the value of z grow rapidly. $\cos(y)$ means that there are oscillations in the y -direction causing wave-like behavior along the y -axis.

Graph A shows an exponential rise in the x -direction with some oscillations in the y -direction. Contour IV because of the oscillations and because it is what graph A would look like from the top.

b) $\sin x - \sin y$ would have oscillations along the x -direction and y -direction. These oscillations would be of the same size as there are fixed values that this equation can result in.

The graph is E for this reason. Contour is III because it is a top view of the graph E and the circles are the same size which is a trait you would expect.

c)

numerator of $x - y$ suggests a linear slope or difference between x and y , so one side will be positive and the other negative.

The denominator makes the effect of the numerator decrease as x and y increase since it outgrows them. So the graph of this function will have a positive peak and a negative peak near the origin and then it should level out on the sides.

This is why the graph is D. It is contour v because of the increasing size of the ring-like shapes as you move away from the origin which is a trait of graph d.