

Math 120

PSet 7

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Contents

Chapter 1

Page 2

1.1 PSet 7

2

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Question 1

Calculate the given iterated integrals.

1. $\int_0^1 \int_0^1 x\sqrt{1+4y} \, dy \, dx$

2. $\int_0^1 \int_1^2 \frac{xe^x}{y} \, dy \, dx$

Solution:

1)

$$\int_0^1 \int_0^1 x\sqrt{1+4y} \, dy \, dx$$

$$\int_0^1 x\sqrt{1+4y} \, dy$$

$$1+4y = t \quad r = dt$$

$$x \int_0^1 \frac{1}{4} \sqrt{t} \, dt$$

$$\frac{1}{4}x \int_0^1 \sqrt{t} \, dt$$

$$\frac{1}{4}x \cdot \frac{2t\sqrt{t}}{3} \Big|_0^1$$

$$\frac{1}{4}x \cdot \frac{2(1+4y)\sqrt{1+4y}}{3} \Big|_0^1$$

$$\frac{x\sqrt{1+4y}(1+4y)}{6} \Big|_0^1$$

$$\frac{x\sqrt{1+4}(1+4)}{6} - \frac{x\sqrt{1}1}{6}$$

$$\frac{5x\sqrt{5}}{6} - \frac{x}{6}$$

$$\int_0^1 \frac{5x\sqrt{5}}{6} - \frac{x}{6} \, dx$$

$$\frac{1}{6} \int_0^1 5\sqrt{5}x - x \, dx$$

$$\begin{aligned} & \frac{1}{6} \left(\int_0^1 5\sqrt{5}x dx - \int_0^1 x dx \right) \\ & \int_0^1 5\sqrt{5}x dx \Rightarrow \frac{5\sqrt{5}x^2}{2} \Big|_0^1 \\ & \frac{5\sqrt{5}(1)^2}{2} - 0 = \frac{5\sqrt{5}}{2} \\ & \int_0^1 x dx \Rightarrow \frac{x^2}{2} \Big|_0^1 \\ & \frac{1}{2} - 0 = \frac{1}{2} \\ & \frac{1}{6} \left(\frac{5\sqrt{5}}{2} - \frac{1}{2} \right) = \frac{5\sqrt{5} - 1}{12} \end{aligned}$$

2)

$$\begin{aligned} & \int_0^1 \int_1^2 \frac{xe^x}{y} dy dx \\ & xe^x \int_1^2 \frac{1}{y} dy \\ & xe^x \ln(y) \Big|_1^2 \Rightarrow xe^x \ln(2) - xe^x \ln(1) = xe^x \ln(2) \\ & \ln(2) \int_0^1 xe^x dx \\ & \ln(2) (xe^x - e^x) \Big|_0^1 \\ & (\ln(2)e - \ln(2)e) - (\ln(2)(0) - \ln(2)e^0) = 0 - (-\ln(2)(1)) = \ln(2) \end{aligned}$$

Question 2

(a) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx.$$

(b) Explain why

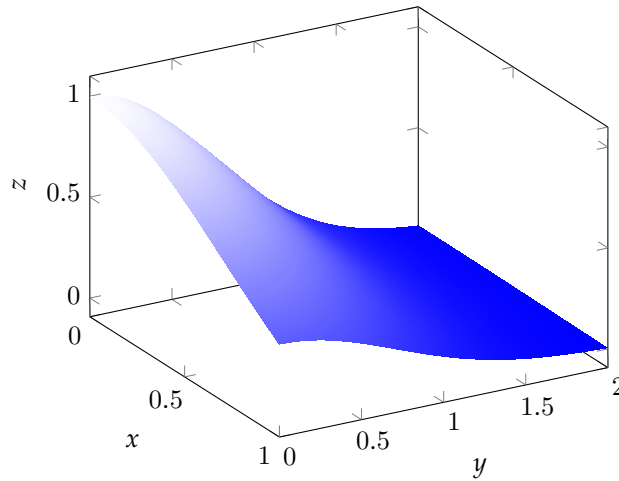
$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx = \int_0^1 e^{-x^2} dx \cdot \int_0^2 e^{-y^2} dy.$$

(c) Use Desmos to compute

$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx.$$

(Desmos will give a numerical approximation, but this is fine. In fact, there is no way to compute the antiderivatives necessary to get an exact answer.)

Solution:



Question 3

- (a) Find the average value of the function $f(x, y) = \sin x \cos y$ on the rectangle $R = [0, \pi] \times [-\pi/2, \pi/2]$.
- (b) Use symmetry to find the average value of $f(x, y) = \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$ on the region $R = [2\pi, 4\pi] \times [2\pi, 6\pi]$. Please explain your answer carefully.

Solution: a)

$$f(x, y) = \sin x \cos y$$

$$R = [0, \pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) dA$$

$$A(R) = (\pi - 0) \times \left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = \pi^2$$

$$\frac{1}{\pi^2} \int_0^\pi \int_{-\pi/2}^{\pi/2} \sin x \cos y dy dx$$

$$\sin x \int_{-\pi/2}^{\pi/2} \cos y dy$$

$$(\sin x) \sin y \Big|_{-\pi/2}^{\pi/2}$$

$$(\sin x) \sin \left(\frac{\pi}{2}\right) - (\sin x) \sin \left(-\frac{\pi}{2}\right) = 2 \sin x$$

$$\int_0^\pi 2 \sin x dx$$

$$-2 \cos x \Big|_0^\pi$$

$$-\cos \pi - (-2) \cos(0) = 4$$

$$\frac{1}{\pi^2} \cdot 4 = \frac{4}{\pi^2}$$

b)

$$f(x, y) = \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$$

$$R = [2\pi, 4\pi] \times [2\pi, 6\pi]$$

$$\begin{aligned}
f_{avg} &= \iint_R f(x, y) dA \\
A(R) &= [4\pi - 2\pi] \times [6\pi - 2\pi] = 8\pi^2 \\
&\int_{2\pi}^{4\pi} \int_{2\pi}^{6\pi} \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3 \, dy \, dx \\
&\iint_R f(x, y) dA - \iint_R \frac{4 \sin y}{e^{x^2}} dA - \iint_R \frac{\cos x}{\ln y} dA + \iint_R 3 dA \\
&\int_{2\pi}^{6\pi} 4 \sin y \, dy = -4 [\cos y]_{2\pi}^{6\pi} = -4(6 \cos \pi - \cos 2\pi) = -4(1 - 1) = 0 \\
&\iint_R f(x, y) \frac{4 \sin y}{e^{x^2}} dA = \int_{2\pi}^{4\pi} \frac{1}{e^{x^2}} dx \times 0 = 0 \\
&\int_{2\pi}^{4\pi} \cos x \, dx = \sin x \Big|_{2\pi}^{4\pi} = \sin 4\pi - \sin 2\pi = 0 - 0 = 0 \\
&\iint_R \frac{\cos x}{\ln y} dA = \int_{2\pi}^{6\pi} \frac{1}{\ln y} \times 0 = 0 \\
&\iint_R 3 dA = 3 \times A(R) = 3 \times 8\pi^2 = 24\pi^2 \\
&\frac{24\pi^2}{8\pi^2} = 3
\end{aligned}$$

Question 4

In each part, draw the region D , and evaluate the integral.

1. $\iint_D \frac{y}{x^5+1} dA$, where D is the region $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\}$.
2. $\iint_D x^3 dA$, where $D = \{(x, y) \mid 1 \leq x \leq e, 0 \leq y \leq \ln x\}$.

Solution: 1.

$$\begin{aligned}
&\iint_D \frac{y}{x^5+1} dA \quad D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\} \\
&\int_0^1 \int_0^{x^2} \frac{y}{x^5+1} dy \, dx \\
&\frac{1}{x^5+1} \int_0^{x^2} y \, dy \\
&\frac{y^2}{2} \Big|_0^{x^2} \Rightarrow \frac{(x^2)^2}{2} - \frac{0}{2} = \frac{x^4}{2} \\
&\int_0^1 \frac{1}{x^5+1} \times \frac{x^4}{2} dx \\
&x^5+1 = t \quad dt = 5x^4 dx \\
&\frac{1}{10} \int_0^1 \frac{1}{t} dt \\
&\frac{1}{10} \ln |t| \Big|_0^1 \\
&5
\end{aligned}$$

$$\begin{aligned} & \frac{1}{10} \left| x^5 + 1 \right|_0^1 \\ & \frac{1}{10} \ln(1^5 + 1) - \frac{1}{10} \ln(1) \\ & \frac{1}{10} \ln(2) - \frac{1}{10} \ln(1) = \frac{1}{10} \ln(2) \end{aligned}$$

2.

$$\iint_D x^3 dA \quad D = \{(x, y) \mid 1 \leq x \leq e, 0 \leq y \leq \ln x\}$$

$$\int_1^e \int_0^{\ln x} x^3 dy dx$$

$$x^3 \int_0^{\ln x} 1 dy$$

$$(x^3) y \Big|_0^{\ln x}$$

$$x^3 \ln x - 0$$

$$uv - \int v du$$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$v = \frac{x^4}{4} \quad x^3 dx$$

$$\frac{\ln x \cdot x^4}{4} - \int \frac{x^3}{4} dx$$

$$\left[\frac{\ln x^4 \cdot x^4}{4} - \frac{x^4}{16} \right]_1^e$$

$$\left(\frac{\ln e \cdot e^4}{4} - \frac{e^4}{16} \right) - \left(\frac{\ln 1 \cdot 1^4}{4} - \frac{1^4}{16} \right)$$

$$\left(\frac{\ln e \cdot e^4}{4} \right) - \left(0 - \frac{1}{16} \right)$$

$$\left(\frac{e^4}{4} - \frac{e^4}{16} \right) + \frac{1}{16}$$

Question 5

Draw the region D . Set up the iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$\iint_D x^2 e^{-xy} dA \quad \text{where } D \text{ is bounded by } y = x, x = 4, \text{ and } y = 0.$$

Solution:

$$\int_0^4 \int_0^x x^2 e^{-yx} dy dx$$

$$\int_0^4 \int_y^4 x^2 e^{-yx} dx dy$$

$$x^2 \int_y^4 e^{-yx} dy$$

$$\begin{aligned}
& (x^2) \frac{e^{-yx}}{x} \Big|_0^x - (x^2) \frac{e^{-yx}}{x} + (x^2) \frac{e^{-0 \cdot x}}{x} \\
& xe^{-x^2} - x^2 \cdot \frac{1}{x} \Rightarrow -xe^{-x^2} + x \\
& \int_0^4 -xe^{-x^2} + x \, dx \Rightarrow \int_0^4 -xe^{-x^2} \, dx + \int_0^4 x \, dx \\
& -x^2 = t \quad -2x = dt \\
& \int_0^4 \frac{1}{2} e^t \, dt \Rightarrow \frac{1}{2} \int_0^4 e^t \, dt \\
& \frac{1}{2} e^t \Big|_0^4 \Rightarrow \frac{1}{2} e^{x^2} \Big|_0^4 \\
& \frac{1}{2} e^{-4^2} - \frac{1}{2} e^{-0^2} \Rightarrow \frac{1}{2} e^{-16} - \frac{1}{2} \\
& \int_0^4 x \, dx \\
& \frac{x^2}{2} \Big|_0^4 \\
& \frac{4^2}{2} - \frac{0^2}{2} = \frac{16}{2} = 8 \\
& \int_0^4 -xe^{-x^2} + x \, dx = \frac{1}{2} e^{-16} + \frac{15}{2}
\end{aligned}$$

Question 6

- (a) Find the volume of the solid in the first octant enclosed by the parabolic cylinder $y = 1 - x^2$ and the planes $z = 2 - y$ and $z = y$.
- (b) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^{1-x} (2 - y^2) \, dy \, dx.$$

Solution:

$$\begin{aligned}
& y = -x^2 \quad z = 2 - y \quad z = y \\
& x, y, z \geq 0 \\
& 2 - y = y \Rightarrow 2y = 2 \Rightarrow y = 1 \Rightarrow 0 \leq y \leq 1 - x^2 \\
& \text{height} = (2 - y) - y \Rightarrow 2 - 2y \\
& V = \int_0^1 \int_0^{1-x^2} 2 - 2y \, dy \, dx \\
& \int_0^{1-x^2} 2 - 2y \, dy \\
& 2y - y^2 \Big|_0^{1-x^2} \\
& 2(1 - x^2) - (1 - x^2)^2 - 2(0) - (0)^2 \\
& 2 - 2x^2 - 1 + 2x^2 - x^4 \\
& 1 - x^4 \\
& 7
\end{aligned}$$

$$\int_0^1 1 - x^4 dx$$

$$x - \frac{x^5}{5} \Big|_0^1$$

$$1 - \frac{1^5}{5} - 0 - \frac{0^5}{5}$$

$$1 - \frac{1}{5} = \frac{4}{5}$$

Question 7

Sketch the region of integration and change the order of integration.

1. $\int_0^1 \int_{4x}^4 f(x, y) dy dx$
2. $\int_0^3 \int_{\sqrt{9-y}}^3 f(x, y) dx dy$
3. $\int_0^4 \int_0^{\ln 2x} f(x, y) dy dx$

Solution:

a)

$$\int_0^1 \int_{4x}^4 f(x, y) dy dx$$

$$\iint_D f(x, y) dA$$

$$D = \{(x, y) | 0 \leq x \leq 1, 4x \leq y \leq 4\}$$

$$D = \{(x, y) | 0 \leq y \leq 4, 0 \leq x \leq \frac{1}{4}y\}$$

$$\iint_D f(x, y) dA = \int_0^4 \int_0^{\frac{1}{4}y} f(x, y) dx dy$$

b)

$$\int_0^3 \int_{\sqrt{9-y}}^3 f(x, y) dx dy$$

$$\iint_D f(x, y) dA \quad D = \{(x, y) | \sqrt{4-y} \leq x \leq 3, 0 \leq y \leq 3\}$$

$$x = \sqrt{9-y} \Rightarrow x^2 = 9-y \Rightarrow x^2 - 9 = -y \Rightarrow -x^2 + 9 = y$$

$$3 = 9 - x^2 \Rightarrow -6 = -x^2 \Rightarrow 6 = x^2 \Rightarrow x = \sqrt{6}$$

$$\iint_D f(x, y) dA \quad D = \{(x, y) | -x^2 + 9 \leq y \leq 3, \sqrt{6} \leq x \leq 3\}$$

$$\int_{\sqrt{6}}^3 \int_{-x^2+9}^3 f(x, y) dy dx$$

c)

$$\int_0^4 \int_0^{\ln 2x} f(x, y) dy dx$$

$$\iint_D f(x, y) dA \quad D = \{0 \leq x \leq 4, 0 \leq y \leq \ln x\}$$

$$y = \ln 2x \Rightarrow y = \ln 2(4) \Rightarrow y = \ln 8 \Rightarrow 0 = \ln 2x \Rightarrow \ln 1 = \ln 2x \Rightarrow x = \frac{1}{2}$$

$$\iint_D f(x, y) dA \quad D = \{(x, y) | 0 \leq y \leq \ln 8, \frac{1}{2} \leq x \leq \frac{e^y}{2}\}$$

$$\int_0^{\ln 8} \int_{\frac{1}{2}}^{\frac{e^y}{2}} f(x, y) dx dy$$

Question 8

Evaluate the integral

$$\int_0^1 \int_x^1 e^{\frac{x}{y}} dy dx$$

by reversing the order of integration.

Solution:

Question 9

Evaluate the given integral by converting to polar coordinates. Be sure to draw the region of integration in each part.

1. $\iint_R (x + y) dA$, where R is the region that lies to the left of the y -axis between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
2. $\iint_R ye^x dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 25$.

Solution: a)

$$x \leq 0 \quad x + y^2 = 1 \quad x^2 + y^2 = 4$$

$$x = r \cos \theta \quad y = r \sin \theta \quad dA = r dr d\theta$$

$$R : 1 \leq r \leq 2$$

$$x + y = r \cos \theta + r \sin \theta \Rightarrow r(\cos \theta + \sin \theta)$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 r(\cos \theta + \sin \theta) r dr d\theta$$

$$(\cos \theta + \sin \theta) \int_1^2 r^2 dr$$

$$\left. \frac{r^3}{3} \right|_1^2$$

$$\frac{2^3}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\frac{7}{3} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\cos \theta + \sin \theta) d\theta$$

$$\frac{7}{3} \sin \theta \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \frac{7}{3} \cos \theta \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$\frac{7}{3}(-1) - \frac{7}{3}(1) = -\frac{14}{3}$$

$$-\frac{7}{3} \cos \frac{3\pi}{2} + \frac{7}{3} \cos \frac{\pi}{2}$$

$$-\frac{7}{3}(0) + \frac{7}{3} = 0$$

$$-\frac{14}{2}$$

b)

Question 10

Use polar coordinates to find the volume of the given solid.

- (a) Inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.
- (b) Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

Question 11

Evaluate the iterated integral

$$\int_0^b \int_{-\sqrt{b^2-y^2}}^0 x^2 y \, dx \, dy$$

by converting to polar coordinates.

Question 12

Let D be the disk with center at the origin and radius a .

- (a) Use your intuition: what do you expect is the average distance from points on the disk to the origin?
- less than $a/2$
 - $a/2$
 - between $a/2$ and a
 - more than a

Give an intuitive explanation of your answer.

- (b) What is the average distance from points in the disk to the origin?