

# Math 120

## PSet 7

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# Chapter 1

## 1.1 PSet 7

### Question 1

Calculate the given iterated integrals.

1.  $\int_0^1 \int_0^1 x\sqrt{1+4y} \, dy \, dx$

2.  $\int_0^1 \int_1^2 \frac{xe^x}{y} \, dy \, dx$

**Solution:**

1)

$$\int_0^1 \int_0^1 x\sqrt{1+4y} \, dy \, dx$$

$$\int_0^1 x\sqrt{1+4y} \, dy$$

$$1+4y = t \quad r = dt$$

$$x \int_0^1 \frac{1}{4} \sqrt{t} \, dt$$

$$\frac{1}{4}x \int_0^1 \sqrt{t} \, dt$$

$$\frac{1}{4}x \cdot \frac{2t\sqrt{t}}{3} \Big|_0^1$$

$$\frac{1}{4}x \cdot \frac{2(1+4y)\sqrt{1+4y}}{3} \Big|_0^1$$

$$\frac{x\sqrt{1+4y}(1+4y)}{6} \Big|_0^1$$

$$\frac{x\sqrt{1+4}(1+4)}{6} - \frac{x\sqrt{1}1}{6}$$

$$\frac{5x\sqrt{5}}{6} - \frac{x}{6}$$

$$\int_0^1 \frac{5x\sqrt{5}}{6} - \frac{x}{6} \, dx$$

$$\frac{1}{6} \int_0^1 5\sqrt{5}x - x \, dx$$

$$\begin{aligned} & \frac{1}{6} \left( \int_0^1 5\sqrt{5}x dx - \int_0^1 x dx \right) \\ & \int_0^1 5\sqrt{5}x dx \Rightarrow \frac{5\sqrt{5}x^2}{2} \Big|_0^1 \\ & \frac{5\sqrt{5}(1)^2}{2} - 0 = \frac{5\sqrt{5}}{2} \\ & \int_0^1 x dx \Rightarrow \frac{x^2}{2} \Big|_0^1 \\ & \frac{1}{2} - 0 = \frac{1}{2} \\ & \frac{1}{6} \left( \frac{5\sqrt{5}}{2} - \frac{1}{2} \right) = \frac{5\sqrt{5} - 1}{12} \end{aligned}$$

2)

$$\begin{aligned} & \int_0^1 \int_1^2 \frac{xe^x}{y} dy dx \\ & xe^x \int_1^2 \frac{1}{y} dy \\ & xe^x \ln(y) \Big|_1^2 \Rightarrow xe^x \ln(2) - xe^x \ln(1) = xe^x \ln(2) \\ & \ln(2) \int_0^1 xe^x dx \\ & \ln(2) (xe^x - e^x) \Big|_0^1 \\ & (\ln(2)e - \ln(2)e) - (\ln(2)(0) - \ln(2)e^0) = 0 - (-\ln(2)(1)) = \ln(2) \end{aligned}$$

### Question 2

(a) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx.$$

(b) Explain why

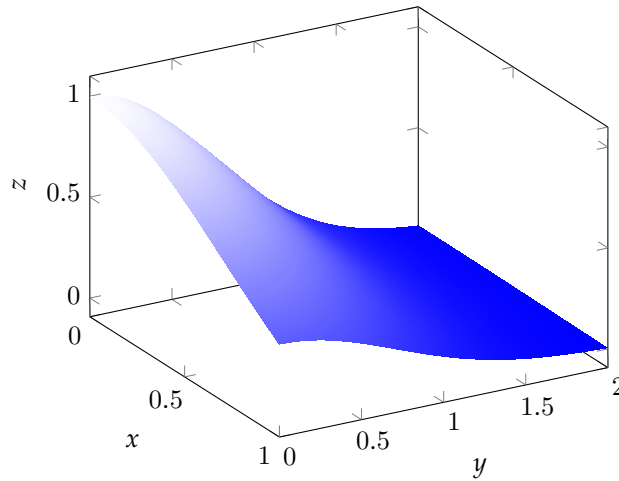
$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx = \int_0^1 e^{-x^2} dx \cdot \int_0^2 e^{-y^2} dy.$$

(c) Use Desmos to compute

$$\int_0^1 \int_0^2 e^{-x^2-y^2} dy dx.$$

(Desmos will give a numerical approximation, but this is fine. In fact, there is no way to compute the antiderivatives necessary to get an exact answer.)

**Solution:**



### Question 3

- (a) Find the average value of the function  $f(x, y) = \sin x \cos y$  on the rectangle  $R = [0, \pi] \times [-\pi/2, \pi/2]$ .
- (b) Use symmetry to find the average value of  $f(x, y) = \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$  on the region  $R = [2\pi, 4\pi] \times [2\pi, 6\pi]$ . Please explain your answer carefully.

**Solution:** a)

$$f(x, y) = \sin x \cos y$$

$$R = [0, \pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) dA$$

$$A(R) = (\pi - 0) \times \left(\frac{\pi}{2} - -\frac{\pi}{2}\right) = \pi^2$$

$$\frac{1}{\pi^2} \int_0^\pi \int_{-\pi/2}^{\pi/2} \sin x \cos y dy dx$$

$$\sin x \int_{-\pi/2}^{\pi/2} \cos y dy$$

$$(\sin x) \sin y \Big|_{-\pi/2}^{\pi/2}$$

$$(\sin x) \sin \left(\frac{\pi}{2}\right) - (\sin x) \sin \left(-\frac{\pi}{2}\right) = 2 \sin x$$

$$\int_0^\pi 2 \sin x dx$$

$$-2 \cos x \Big|_0^\pi$$

$$-\cos \pi - (-2) \cos(0) = 4$$

$$\frac{1}{\pi^2} \cdot 4 = \frac{4}{\pi^2}$$

b)

$$f(x, y) = \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$$

$$R = [2\pi, 4\pi] \times [2\pi, 6\pi]$$

$$\begin{aligned}
f_{avg} &= \iint_R f(x, y) dA \\
A(R) &= [4\pi - 2\pi] \times [6\pi - 2\pi] = 8\pi^2 \\
&\int_{2\pi}^{4\pi} \int_{2\pi}^{6\pi} \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3 \, dy \, dx \\
&\iint_R f(x, y) dA - \iint_R \frac{4 \sin y}{e^{x^2}} dA - \iint_R \frac{\cos x}{\ln y} dA + \iint_R 3 dA \\
&\int_{2\pi}^{6\pi} 4 \sin y \, dy = -4 [\cos y]_{2\pi}^{6\pi} = -4(6 \cos \pi - \cos 2\pi) = -4(1 - 1) = 0 \\
&\iint_R f(x, y) \frac{4 \sin y}{e^{x^2}} dA = \int_{2\pi}^{4\pi} \frac{1}{e^{x^2}} dx \times 0 = 0 \\
&\int_{2\pi}^{4\pi} \cos x \, dx = \sin x \Big|_{2\pi}^{4\pi} = \sin 4\pi - \sin 2\pi = 0 - 0 = 0 \\
&\iint_R \frac{\cos x}{\ln y} dA = \int_{2\pi}^{6\pi} \frac{1}{\ln y} \times 0 = 0 \\
&\iint_R 3 dA = 3 \times A(R) = 3 \times 8\pi^2 = 24\pi^2 \\
&\frac{24\pi^2}{8\pi^2} = 3
\end{aligned}$$

#### Question 4

In each part, draw the region  $D$ , and evaluate the integral.

1.  $\iint_D \frac{y}{x^5+1} dA$ , where  $D$  is the region  $D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\}$ .
2.  $\iint_D x^3 dA$ , where  $D = \{(x, y) \mid 1 \leq x \leq e, 0 \leq y \leq \ln x\}$ .

**Solution:** 1.

$$\begin{aligned}
&\iint_D \frac{y}{x^5+1} dA \quad D = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2\} \\
&\int_0^1 \int_0^{x^2} \frac{y}{x^5+1} dy \, dx \\
&\frac{1}{x^5+1} \int_0^{x^2} y \, dy \\
&\frac{y^2}{2} \Big|_0^{x^2} \Rightarrow \frac{(x^2)^2}{2} - \frac{0}{2} = \frac{x^4}{2} \\
&\int_0^1 \frac{1}{x^5+1} \times \frac{x^4}{2} dx \\
&x^5+1 = t \quad dt = 5x^4 dx \\
&\frac{1}{10} \int_0^1 \frac{1}{t} dt \\
&\frac{1}{10} \ln |t| \Big|_0^1 \\
&5
\end{aligned}$$

$$\begin{aligned} & \frac{1}{10} \left| x^5 + 1 \right|_0^1 \\ & \frac{1}{10} \ln(1^5 + 1) - \frac{1}{10} \ln(1) \\ & \frac{1}{10} \ln(2) - \frac{1}{10} \ln(1) = \frac{1}{10} \ln(2) \end{aligned}$$

2.

$$\iint_D x^3 dA \quad D = \{(x, y) \mid 1 \leq x \leq e, 0 \leq y \leq \ln x\}$$

$$\int_1^e \int_0^{\ln x} x^3 dy dx$$

$$x^3 \int_0^{\ln x} 1 dy$$

$$(x^3) y \Big|_0^{\ln x}$$

$$x^3 \ln x - 0$$

$$uv - \int v du$$

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$v = \frac{x^4}{4} \quad x^3 dx$$

$$\frac{\ln x \cdot x^4}{4} - \int \frac{x^3}{4} dx$$

$$\left[ \frac{\ln x^4 \cdot x^4}{4} - \frac{x^4}{16} \right]_1^e$$

$$\left( \frac{\ln e \cdot e^4}{4} - \frac{e^4}{16} \right) - \left( \frac{\ln 1 \cdot 1^4}{4} - \frac{1^4}{16} \right)$$

$$\left( \frac{\ln e \cdot e^4}{4} \right) - \left( 0 - \frac{1}{16} \right)$$

$$\left( \frac{e^4}{4} - \frac{e^4}{16} \right) + \frac{1}{16}$$

### Question 5

Draw the region  $D$ . Set up the iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$\iint_D x^2 e^{-xy} dA \quad \text{where } D \text{ is bounded by } y = x, x = 4, \text{ and } y = 0.$$

**Solution:**

$$\int_0^4 \int_0^x x^2 e^{-yx} dy dx$$

$$\int_0^4 \int_y^4 x^2 e^{-yx} dx dy$$

$$x^2 \int_y^4 e^{-yx} dy$$

$$\begin{aligned}
& (x^2) \frac{e^{-yx}}{x} \Big|_0^x - (x^2) \frac{e^{-yx}}{x} + (x^2) \frac{e^{-0 \cdot x}}{x} \\
& x e^{-x^2} - x^2 \cdot \frac{1}{x} \Rightarrow -x e^{-x^2} + x \\
& \int_0^4 -x e^{-x^2} + x \, dx \Rightarrow \int_0^4 -x e^{-x^2} \, dx + \int_0^4 x \, dx \\
& -x^2 = t \quad -2x = dt \\
& \int_0^4 \frac{1}{2} e^t \, dt \Rightarrow \frac{1}{2} \int_0^4 e^t \, dt \\
& \frac{1}{2} e^t \Big|_0^4 \Rightarrow \frac{1}{2} e^{x^2} \Big|_0^4 \\
& \frac{1}{2} e^{-4^2} - \frac{1}{2} e^{-0^2} \Rightarrow \frac{1}{2} e^{-16} - \frac{1}{2} \\
& \int_0^4 x \, dx \\
& \frac{x^2}{2} \Big|_0^4 \\
& \frac{4^2}{2} - \frac{0^2}{2} = \frac{16}{2} = 8 \\
& \int_0^4 -x e^{-x^2} + x \, dx = \frac{1}{2} e^{-16} + \frac{15}{2}
\end{aligned}$$

#### Question 6

- (a) Find the volume of the solid in the first octant enclosed by the parabolic cylinder  $y = 1 - x^2$  and the planes  $z = 2 - y$  and  $z = y$ .
- (b) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^{1-x} (2 - y^2) \, dy \, dx.$$

**Solution:**

$$\begin{aligned}
& y = -x^2 \quad z = 2 - y \quad z = y \\
& x, y, z \geq 0 \\
& 2 - y = y \Rightarrow 2y = 2 \Rightarrow y = 1 \Rightarrow 0 \leq y \leq 1 - x^2 \\
& \text{height} = (2 - y) - y \Rightarrow 2 - 2y \\
& V = \int_0^1 \int_0^{1-x^2} 2 - 2y \, dy \, dx \\
& \int_0^{1-x^2} 2 - 2y \, dy \\
& 2y - y^2 \Big|_0^{1-x^2} \\
& 2(1 - x^2) - (1 - x^2)^2 - 2(0) - (0)^2 \\
& 2 - 2x^2 - 1 + 2x^2 - x^4 \\
& 1 - x^4 \\
& 7
\end{aligned}$$



$$\int_0^1 1 - x^4 dx$$

$$x - \frac{x^5}{5} \Big|_0^1$$

$$1 - \frac{1^5}{5} - 0 - \frac{0^5}{5}$$

$$1 - \frac{1}{5} = \frac{4}{5}$$

### Question 7

Sketch the region of integration and change the order of integration.

1.  $\int_0^1 \int_{4x}^4 f(x, y) dy dx$
2.  $\int_0^3 \int_{\sqrt{9-y}}^3 f(x, y) dx dy$
3.  $\int_0^4 \int_0^{\ln 2x} f(x, y) dy dx$

**Solution:**

a)

$$\int_0^1 \int_{4x}^4 f(x, y) dy dx$$

$$\iint_D f(x, y) dA$$

$$D = \{(x, y) | 0 \leq x \leq 1, 4x \leq y \leq 4\}$$

$$D = \{(x, y) | 0 \leq y \leq 4, 0 \leq x \leq \frac{1}{4}y\}$$

$$\iint_D f(x, y) dA = \int_0^4 \int_0^{\frac{1}{4}y} f(x, y) dx dy$$

b)

$$\int_0^3 \int_{\sqrt{9-y}}^3 f(x, y) dx dy$$

$$\iint_D f(x, y) dA \quad D = \{(x, y) | \sqrt{4-y} \leq x \leq 3, 0 \leq y \leq 3\}$$

$$x = \sqrt{9-y} \Rightarrow x^2 = 9-y \Rightarrow x^2 - 9 = -y \Rightarrow -x^2 + 9 = y$$

$$3 = 9 - x^2 \Rightarrow -6 = -x^2 \Rightarrow 6 = x^2 \Rightarrow x = \sqrt{6}$$

$$\iint_D f(x, y) dA \quad D = \{(x, y) | -x^2 + 9 \leq y \leq 3, \sqrt{6} \leq x \leq 3\}$$

$$\int_{\sqrt{6}}^3 \int_{-x^2+9}^3 f(x, y) dy dx$$

c)

$$\int_0^4 \int_0^{\ln 2x} f(x, y) dy dx$$

$$\iint_D f(x, y) dA \quad D = \{0 \leq x \leq 4, 0 \leq y \leq \ln x\}$$

$$y = \ln 2x \Rightarrow y = \ln 2(4) \Rightarrow y = \ln 8 \Rightarrow 0 = \ln 2x \Rightarrow \ln 1 = \ln 2x \Rightarrow x = \frac{1}{2}$$

$$\iint_D f(x, y) dA \quad D = \{(x, y) | 0 \leq y \leq \ln 8, \frac{1}{2} \leq x \leq \frac{e^y}{2}\}$$

$$\int_0^{\ln 8} \int_{\frac{1}{2}}^{\frac{e^y}{2}} f(x, y) dx dy$$

### Question 8

Evaluate the integral

$$\int_0^1 \int_x^1 e^{\frac{x}{y}} dy dx$$

by reversing the order of integration.

**Solution:**

### Question 9

Evaluate the given integral by converting to polar coordinates. Be sure to draw the region of integration in each part.

1.  $\iint_R (x + y) dA$ , where  $R$  is the region that lies to the left of the  $y$ -axis between the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .
2.  $\iint_R ye^x dA$ , where  $R$  is the region in the first quadrant enclosed by the circle  $x^2 + y^2 = 25$ .

**Solution:** a)

$$x \leq 0 \quad x + y^2 = 1 \quad x^2 + y^2 = 4$$

$$x = r \cos \theta \quad y = r \sin \theta \quad dA = r dr d\theta$$

$$R : 1 \leq r \leq 2$$

$$x + y = r \cos \theta + r \sin \theta \Rightarrow r(\cos \theta + \sin \theta)$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_1^2 r(\cos \theta + \sin \theta) r dr d\theta$$

$$(\cos \theta + \sin \theta) \int_1^2 r^2 dr$$

$$\left. \frac{r^3}{3} \right|_1^2$$

$$\frac{2^3}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\frac{7}{3} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (\cos \theta + \sin \theta) d\theta$$

$$\frac{7}{3} \sin \theta \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \frac{7}{3} \cos \theta \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$\frac{7}{3}(-1) - \frac{7}{3}(1) = -\frac{14}{3}$$

$$-\frac{7}{3} \cos \frac{3\pi}{2} + \frac{7}{3} \cos \frac{\pi}{2}$$

$$-\frac{7}{3}(0) + \frac{7}{3} = 0$$

$$-\frac{14}{2}$$

b)

$$\iint_R ye^x dA$$

$$x = r \cos \theta \quad y = r \sin \theta \quad x^2 + y^2 = 25$$

$$\text{1st quadrant} \quad 0 \leq \theta \leq \frac{\pi}{2}$$

$$ye^x \Rightarrow r \sin \theta e^{r \cos \theta}$$

$$dA = r dr d\theta$$

$$\int_0^{\frac{\pi}{2}} \int_0^5 r \sin \theta e^{r \cos \theta}$$

$$I(\theta) = \int_0^5 r^2 \sin \theta e^{r \cos \theta} dr$$

$$\frac{\partial}{\partial \theta} e^{r \cos \theta} = -r \sin \theta e^{r \cos \theta}$$

$$r \sin \theta e^{r \cos \theta} = -\frac{\partial}{\partial \theta} e^{r \cos \theta}$$

$$I = \int_0^{\frac{\pi}{2}} \int_0^5 -r \frac{\partial}{\partial \theta} e^{r \cos \theta} dr d\theta$$

$$I = - \int_0^5 r \left( \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \theta} e^{r \cos \theta} d\theta \right) dr$$

$$\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial \theta} e^{r \cos \theta} d\theta = e^{r \cos \theta} \Big|_0^{\frac{\pi}{2}}$$

$$e^{r \cos \frac{\pi}{2}} - e^{r \cos 0} = e^{r \cdot 0} - e^{r \cdot 1} = 1 - e^r$$

$$I = \int_0^5 r e^r dr - \int_0^3 r dr$$

$$u = r \quad du = dr \quad v = e^r \quad dv = e^r dr$$

$$\int_0^5 r e^r dr = r e^r - \int_0^5 e^r dr = r e^r - e^r + K$$

$$\int_0^5 r e^r dr = [r e^r - e^r]_0^5 = (5e^5 - e^5) - (0 - e^0) = 4e^5 + 1$$

$$\int_0^5 r dr = \frac{1}{2} r^2 \Big|_0^5 = \frac{1}{2} (25 - 0) = \frac{25}{2}$$

$$I = (4e^5 + 1) - \frac{25}{2} = 4e^5 - 11.5$$

### Question 10

Use polar coordinates to find the volume of the given solid.

(a) Inside the sphere  $x^2 + y^2 + z^2 = 4$  and outside the cylinder  $x^2 + y^2 = 1$ .

(b) Bounded by the paraboloids  $z = 3x^2 + 3y^2$  and  $z = 4 - x^2 - y^2$ .

**Solution:** a)

$$\begin{aligned}
 x^2 + y^2 + z^2 &= 4 & x^2 + y^2 &= 1 \\
 x &= r \cos \theta & y &= r \sin \theta & 0 \leq \theta \leq 2\pi \\
 r^2 + z^2 &= 4 & x^2 + y^2 &= 1 \Rightarrow r^2 = 1 \Rightarrow r = 1 \\
 r^2 + z^2 &= 4 \Rightarrow r^2 = 4 - z^2 \Rightarrow r = \sqrt{4 - z^2} \\
 V &= \iint_D [z_{\text{upper}} - z_{\text{lower}}] \\
 V &= \int_0^{2\pi} \int_1^2 r \sqrt{4 - r^2} - (-\sqrt{4 - r^2}) r \, dr \, d\theta \\
 V &= 2 \int_0^{2\pi} \int_1^2 \sqrt{4 - r^2} \, dr \, d\theta \\
 4 - r^2 &= t & -2r &= dt \\
 - \int_1^2 \frac{1}{2} \sqrt{t} \, dt &\Rightarrow -\frac{1}{2} \int_1^2 \sqrt{t} \, dt = -\frac{1}{2} \cdot \frac{2t\sqrt{t}}{3} \Big|_1^2 \\
 -\frac{1}{2} \cdot \frac{2(4 - r^2)\sqrt{4 - r^2}}{3} \Big|_1^2 &\left( -\frac{1}{2} \cdot \frac{2(4 - 2^2)\sqrt{4 - 2^2}}{3} \right) - \left( -\frac{1}{2} \cdot \frac{2(4 - 1^2)\sqrt{4 - 1^2}}{3} \right) \\
 V &= 2 \left( \int_0^{2\pi} d\theta \right) \left( \int_1^2 r \sqrt{4 - r^2} \, dr \right) \\
 \int_0^{2\pi} d\theta &= 2\pi & V &= 2 \cdot 2\pi \cdot \int_1^2 r \sqrt{4 - r^2} \, dr = 4\pi \int_1^2 r \sqrt{4 - r^2} \, dr \\
 - \left( -\frac{1}{2} \cdot \frac{2(3)\sqrt{3}}{3} \right) &\Rightarrow \left( \frac{1}{2} \sqrt{3} \right) \Rightarrow -(-\sqrt{3}) \\
 &4\pi\sqrt{3}
 \end{aligned}$$

### Question 11

Evaluate the iterated integral

$$\int_0^b \int_{-\sqrt{b^2 - y^2}}^0 x^2 y \, dx \, dy$$

by converting to polar coordinates.

### Question 12

Let  $D$  be the disk with center at the origin and radius  $a$ .

(a) Use your intuition: what do you expect is the average distance from points on the disk to the origin?

- less than  $a/2$
- $a/2$
- between  $a/2$  and  $a$
- more than  $a$

Give an intuitive explanation of your answer.

(b) What is the average distance from points in the disk to the origin?