Math 120

PSet 10

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Contents

Chapter 1

1.1 PSet 10

Question 1

Find the surface area of the part of the paraboloid $x = 3y^2 + 3z^2$ satisfying $x \leqslant 3$.

Solution:

$$y = r\cos\theta, \quad z = r\sin\theta, \quad x = 3r^2$$

$$A = \iint_{\text{Domain}} \left\| \frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} \right\| dr d\theta$$

$$\mathbf{S}(r,\theta) = (3r^2, r\cos\theta, r\sin\theta)$$

$$\frac{\partial \mathbf{S}}{\partial r} = (6r, \cos\theta, \sin\theta), \quad \frac{\partial \mathbf{S}}{\partial \theta} = (0, -r\sin\theta, r\cos\theta)$$

$$\frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} = (r, -6r^2\cos\theta, -6r^2\sin\theta)$$

$$\left\| \frac{\partial \mathbf{S}}{\partial r} \times \frac{\partial \mathbf{S}}{\partial \theta} \right\| = r\sqrt{1+36r^2}$$

$$A = \int_0^{2\pi} \int_0^1 r\sqrt{1+36r^2} dr d\theta$$

$$u = 1 + 36r^2, \quad du = 72r dr, \quad r dr = \frac{du}{72}$$

$$\int_{r=0}^{r=1} r\sqrt{1+36r^2} dr = \frac{1}{72}\int_{u=1}^{u=37} \sqrt{u} du = \frac{1}{72}\left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=37}$$

$$\frac{1}{72}\left[\frac{2}{3}u^{3/2}\right]_{u=1}^{u=37} = \frac{1}{108}\left(37^{3/2} - 1\right)$$

$$A = \frac{\pi}{54}\left(37^{3/2} - 1\right)$$

$$37^{3/2} = \sqrt{37^3} = 37\sqrt{37}$$

$$A = \frac{\pi}{54}\left(37\sqrt{37} - 1\right)$$

Question 2

In this problem, we'll find the surface area of the part of the cylinder $x^2 + y^2 = 9$ that lies between the planes $x + y + z = -6$ and $x + y + z = 5$.

(a) Find a parametrization $\vec{r}_1(u)$ of the curve C_1 of intersection of the plane $x + y + z = 5$ with the cylinder $x^2 + y^2 = 9$. Also find a parametrization $\vec{r}_2(u)$ of the curve C_2 of intersection of the plane $x + y + z = -6$ with the cylinder.

(b) Write down a parametrization $\vec{s}(u, v)$ of the part of the cylinder that lies between the two planes. The curves C_1 and C_2 should be two grid curves of the parametrization, and the bounds on the parameters should be of the form $a \leqslant v \leqslant b$ and $c \leqslant u \leqslant d$ for constants a, b, c, and d.

(c) Use the parametrization you found to calculate the surface area of the part of the cylinder that lies between the two planes.

(d) Does your answer from (c) make sense? Give an intuitive geometric reason that the surface area you found is the same as the surface area of a cylinder of radius 3 and height 11.

Solution:

(a)

$$x = 3\cos u, \quad y = 3\sin u$$
$$z = 5 - x - y = 5 - (3\cos u + 3\sin u)$$
$$\vec{r}_1(u) = (3\cos u,\ 3\sin u,\ 5 - 3\cos u - 3\sin u)$$
$$z = -6 - x - y = -6 - (3\cos u + 3\sin u)$$
$$\vec{r}_2(u) = (3\cos u,\ 3\sin u,\ -6 - 3\cos u - 3\sin u)$$

(b)

$$\vec{s}(u, v) = (3\cos u,\ 3\sin u,\ v - 3\cos u - 3\sin u)$$
$$0 \leqslant u \leqslant 2\pi, \quad -6 \leqslant v \leqslant 5$$

(c)

$$\frac{\partial \vec{s}}{\partial u} = (-3\sin u,\ 3\cos u,\ 3\sin u - 3\cos u)$$
$$\frac{\partial \vec{s}}{\partial v} = (0,\ 0,\ 1)$$
$$\frac{\partial \vec{s}}{\partial u} \times \frac{\partial \vec{s}}{\partial v} = (3\cos u,\ 3\sin u,\ 0)$$
$$\left\| \frac{\partial \vec{s}}{\partial u} \times \frac{\partial \vec{s}}{\partial v} \right\| = \sqrt{(3\cos u)^2 + (3\sin u)^2} = 3$$
$$A = \int_{v=-6}^{5} \int_{u=0}^{2\pi} 3\, du\, dv = 3 \cdot \left(\int_{u=0}^{2\pi} du \right) \cdot \left(\int_{v=-6}^{5} dv \right) = 3 \cdot 2\pi \cdot 11 = 66\pi$$

(d)

$$A = 2\pi r h = 2\pi \times 3 \times 11 = 66\pi$$

Question 3

Evaluate the surface integral

$$\iint_S (x^2 + y^2 + z^2)\, dS$$

where S is the surface of the solid cylinder defined by the inequalities $x^2 + z^2 \leqslant 1$ and $0 \leqslant y \leqslant 5$. Note that S consists of a hollow cylinder and two disks.

Solution:

$$I = \iint_S (x^2 + y^2 + z^2)\, dS$$

$$x^2 + z^2 \leqslant 1, \quad 0 \leqslant y \leqslant 5$$

$$S = S_{\text{cyl}} \cup S_{\text{top}} \cup S_{\text{bot}}$$

$$\text{For } S_{\text{cyl}}, \quad x = \cos\theta, \quad z = \sin\theta, \quad y = y, \quad \theta \in [0, 2\pi), \quad y \in [0, 5]$$

$$\vec{r}(\theta, y) = (\cos\theta, y, \sin\theta)$$

$$\vec{r}_\theta = (-\sin\theta, 0, \cos\theta), \quad \vec{r}_y = (0, 1, 0)$$

$$\vec{r}_\theta \times \vec{r}_y = (-\cos\theta, 0, -\sin\theta), \quad |\vec{r}_\theta \times \vec{r}_y| = 1$$

$$dS_{\text{cyl}} = d\theta\, dy$$

$$x^2 + y^2 + z^2 = \cos^2\theta + y^2 + \sin^2\theta = 1 + y^2$$

$$I_{\text{cyl}} = \int_0^{2\pi} \int_0^5 (1 + y^2)\, dy\, d\theta$$

$$\int_0^5 (1 + y^2)\, dy = \int_0^5 1\, dy + \int_0^5 y^2\, dy = [y]_0^5 + \left[\frac{y^3}{3}\right]_0^5 = 5 + \frac{125}{3} = \frac{140}{3}$$

$$I_{\text{cyl}} = \int_0^{2\pi} \frac{140}{3}\, d\theta = \frac{140}{3} \cdot 2\pi = \frac{280\pi}{3}$$

$$\text{For } S_{\text{top}}, \quad x = r\cos\theta, \quad z = r\sin\theta, \quad y = 5, \quad r \in [0, 1], \quad \theta \in [0, 2\pi)$$

$$dS_{\text{top}} = r\, dr\, d\theta$$

$$x^2 + y^2 + z^2 = r^2 + 25$$

$$I_{\text{top}} = \int_0^{2\pi} \int_0^1 (r^2 + 25) r\, dr\, d\theta$$

$$\int_0^1 (r^2 + 25) r\, dr = \int_0^1 (r^3 + 25r)\, dr = \left[\frac{r^4}{4} + \frac{25r^2}{2}\right]_0^1 = \frac{1}{4} + \frac{25}{2} = \frac{51}{4}$$

$$I_{\text{top}} = \int_0^{2\pi} \frac{51}{4}\, d\theta = \frac{51}{4} \cdot 2\pi = \frac{51\pi}{2}$$

$$\text{For } S_{\text{bot}}, \quad x = r\cos\theta, \quad z = r\sin\theta, \quad y = 0$$

$$x^2 + y^2 + z^2 = r^2$$

$$dS_{\text{bot}} = r\,dr\,d\theta$$

$$I_{\text{bot}} = \int_0^{2\pi}\int_0^1 r^3\,dr\,d\theta$$

$$\int_0^1 r^3\,dr = \left[\frac{r^4}{4}\right]_0^1 = \frac{1}{4}$$

$$I_{\text{bot}} = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{1}{4}\cdot 2\pi = \frac{\pi}{2}$$

$$I = I_{\text{cyl}} + I_{\text{top}} + I_{\text{bot}} = \frac{280\pi}{3} + \frac{51\pi}{2} + \frac{\pi}{2}$$

$$I = \frac{560\pi}{6} + \frac{153\pi}{6} + \frac{3\pi}{6} = \frac{716\pi}{6} = \frac{358\pi}{3}$$

$$\boxed{\frac{358\pi}{3}}$$

Question 4

Find the mass of a thin funnel in the shape of a cone $z = \sqrt{x^2+y^2}$, $1 \leqslant z \leqslant 4$, if its density function is $\rho(x, y, z) = z + 2$.

Solution:

$$z = \sqrt{x^2+y^2}, \quad 1 \leqslant z \leqslant 4$$

$$\rho(x, y, z) = z + 2$$

$$\begin{cases} x = z\cos\theta \\ y = z\sin\theta \\ z = z \end{cases}$$

$$\mathbf{r}(\theta, z) = (z\cos\theta, z\sin\theta, z)$$

$$\mathbf{r}_\theta = (-z\sin\theta, z\cos\theta, 0)$$

$$\mathbf{r}_z = (\cos\theta, \sin\theta, 1)$$

$$\mathbf{N} = \mathbf{r}_\theta \times \mathbf{r}_z = (z\cos\theta, z\sin\theta, -z)$$

$$|\mathbf{N}| = \sqrt{(z\cos\theta)^2 + (z\sin\theta)^2 + (-z)^2} = z\sqrt{2}$$

$$dS = z\sqrt{2}\,d\theta\,dz$$

$$M = \int_{\theta=0}^{2\pi} \int_{z=1}^{4} \rho(z)\, dS$$

$$M = \int_{z=1}^{4} \int_{\theta=0}^{2\pi} (z+2)\, z\sqrt{2}\, d\theta\, dz$$

$$\int_{\theta=0}^{2\pi} d\theta = 2\pi$$

$$M = 2\pi\sqrt{2} \int_{z=1}^{4} (z+2)z\, dz$$

$$M = 2\pi\sqrt{2} \int_{z=1}^{4} (z^2+2z)\, dz$$

$$\int (z^2+2z)\, dz = \frac{1}{3}z^3 + z^2$$

$$\left[\frac{1}{3}(4)^3 + (4)^2\right] - \left[\frac{1}{3}(1)^3 + (1)^2\right]$$

$$\left(\frac{64}{3} + 16\right) - \left(\frac{1}{3} + 1\right)$$

$$36$$

$$M = 2\pi\sqrt{2} \times 36$$

$$M = 72\pi\sqrt{2}$$

Question 5

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}, \quad \text{where } \vec{F} = \langle x, y, 2z \rangle$$

and S is the part of the paraboloid $z = 4 - x^2 - y^2$, oriented downwards, that lies above the unit square $[0,1] \times [0,1]$.

Solution:

$$\iint_S \vec{F} \cdot d\vec{S}, \quad \vec{F} = \langle x, y, 2z \rangle$$

$$z = 4 - x^2 - y^2$$

$$\vec{n} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right)$$

$$\frac{\partial f}{\partial x} = -2x, \quad \frac{\partial f}{\partial y} = -2y$$

$$\vec{n} = (-2x, -2y, -1)$$

$$d\vec{S} = \vec{n}\, dx\, dy = (-2x, -2y, -1)\, dx\, dy$$

$$\vec{F} = (x, y, 2z) = \big(x, y, 2(4 - x^2 - y^2)\big)$$

$$\vec{F} \cdot d\vec{S} = (x, y, 2z) \cdot (-2x, -2y, -1)\, dx\, dy$$

$$\vec{F} \cdot d\vec{S} = [-2x^2 - 2y^2 - 2(4 - x^2 - y^2)]\, dx\, dy$$

$$\vec{F} \cdot d\vec{S} = [-2x^2 - 2y^2 - 8 + 2x^2 + 2y^2]\, dx\, dy = -8\, dx\, dy$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D (-8)\, dx\, dy = -8 \iint_D dx\, dy$$

$$\iint_S \vec{F} \cdot d\vec{S} = -8 \times 1 = -8$$

$$\iint_S \vec{F} \cdot d\vec{S} = -8$$

Question 6

Evaluate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}, \quad \text{where } \vec{F} = \langle -z, x, y \rangle$$

and S is the part of the unit sphere $x^2 + y^2 + z^2 = 1$ in the first octant, oriented upwards.

Solution:

$$\vec{F} = \langle -z, x, y \rangle$$

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi$$

$$0 \leqslant \phi \leqslant \frac{\pi}{2}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

$$\vec{r}_\phi = \frac{\partial \vec{r}}{\partial \phi} = \begin{pmatrix} \cos\phi\cos\theta \\ \cos\phi\sin\theta \\ -\sin\phi \end{pmatrix}, \quad \vec{r}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \begin{pmatrix} -\sin\phi\sin\theta \\ \sin\phi\cos\theta \\ 0 \end{pmatrix}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{pmatrix} \sin^2\phi\cos\theta \\ \sin^2\phi\sin\theta \\ \cos\phi\sin\phi \end{pmatrix}$$

$$d\vec{S} = \begin{pmatrix} \sin^2\phi\cos\theta \\ \sin^2\phi\sin\theta \\ \cos\phi\sin\phi \end{pmatrix} d\phi d\theta$$

$$\vec{F} = \langle -z, x, y \rangle = \langle -\cos\phi, \sin\phi\cos\theta, \sin\phi\sin\theta \rangle$$

$$\vec{F} \cdot d\vec{S} = (-\cos\phi)\left(\sin^2\phi\cos\theta\right) + \left(\sin\phi\cos\theta\right)\left(\sin^2\phi\sin\theta\right) + \left(\sin\phi\sin\theta\right)\left(\cos\phi\sin\phi\right)$$

$$\vec{F} \cdot d\vec{S} = \cos\phi\sin^2\phi(-\cos\theta + \sin\theta) + \sin^3\phi\cos\theta\sin\theta$$

$$\int_0^{\frac{\pi}{2}} (-\cos\theta + \sin\theta)\, d\theta = 0, \quad \int_0^{\frac{\pi}{2}} \sin 2\theta\, d\theta = 1$$

$$\int_0^{\frac{\pi}{2}} \sin^3\phi\, d\phi = \frac{4}{3}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \frac{1}{2}\cdot\frac{4}{3} = \frac{2}{3}$$

$$\boxed{\frac{2}{3}}$$

Question 7

Find the flux of $\vec{F}(x,y,z) = z\hat{i} + y\hat{j} + x\hat{k}$ across the helicoid

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle, \quad 0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 2\pi,$$

oriented upward.

Solution:

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$$

$$\vec{r}_u = \langle \cos v, \sin v, 0\rangle, \quad \vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$$

$$\vec{n} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \langle \sin v, -\cos v, u\rangle$$

$$\vec{F}(x,y,z) = z\,\hat{i} + y\,\hat{j} + x\,\hat{k}, \quad \vec{F}(\vec{r}(u,v)) = \langle v, u\sin v, u\cos v\rangle$$

$$\vec{F}\cdot\vec{n} = v\sin v - u\sin v\cos v + u^2\cos v$$

$$\Phi = \iint_D (\vec{F}\cdot\vec{n})\, du\, dv = \int_0^{2\pi}\int_0^1 \left(v\sin v - u\sin v\cos v + u^2\cos v\right) du\, dv$$

$$\int_0^1 v\sin v\, du = v\sin v, \quad \int_0^1 u\sin v\cos v\, du = \frac{1}{2}\sin v\cos v, \quad \int_0^1 u^2\cos v\, du = \frac{1}{3}\cos v$$

$$\Phi = \int_0^{2\pi} \left(v\sin v - \frac{1}{2}\sin v\cos v + \frac{1}{3}\cos v\right) dv$$

$$\sin v\cos v = \frac{1}{2}\sin 2v$$

$$\Phi = \int_0^{2\pi} \left(v\sin v - \frac{1}{4}\sin 2v + \frac{1}{3}\cos v\right) dv$$

$$\int_0^{2\pi} v\sin v\, dv = -2\pi, \quad \int_0^{2\pi} \sin 2v\, dv = 0, \quad \int_0^{2\pi} \cos v\, dv = 0$$

$$\Phi = -2\pi - \frac{1}{4}(0) + \frac{1}{3}(0)$$

$$\Phi = -2\pi$$

Question 8

Let S be the part of the elliptical cylinder $y^2 + 4z^2 = 4$ that lies above the xy-plane and between the planes $x = -2$ and $x = 2$. Let S have the upward orientation; that is, let S be oriented so that the normal vectors have positive z-component.

(a) Find a parameterization of S.

(b) Does your parameterization match the given orientation of S? Explain.

(c) Let $\vec{F}$ be the vector field

$$\vec{F}(x,y,z) = e^{x^2y^2z^2}\hat{\imath} + x^2y\hat{\jmath} + z^2e^{x/5}\hat{k}.$$

Find the flux of $\vec{F}$ across the oriented surface S.

Solution:

$$\text{(a)}\quad \vec{r}(x,\theta) = \langle x, 2\sin\theta, \cos\theta\rangle\,, \quad -2 \leqslant x \leqslant 2, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

$$\text{(b)}\quad \vec{r}_x = \langle 1,0,0\rangle\,, \quad \vec{r}_\theta = \langle 0, 2\cos\theta, -\sin\theta\rangle\,, \quad \vec{N} = \vec{r}_x \times \vec{r}_\theta = \langle 0, \sin\theta, 2\cos\theta\rangle$$

$$\text{(c)}\quad F_x = e^{4x^2\sin^2\theta\cos^2\theta}, \quad F_y = 2x^2\sin\theta, \quad F_z = \cos^2\theta e^{x/5}$$

$$\vec{F}\cdot\vec{N} = 2x^2\sin^2\theta + 2\cos^3\theta e^{x/5}$$

$$\Phi = \int_{x=-2}^{2}\int_{\theta=0}^{\frac{\pi}{2}} \left(2x^2\sin^2\theta + 2\cos^3\theta e^{x/5}\right) d\theta\, dx$$

$$\int_{\theta=0}^{\frac{\pi}{2}} \sin^2\theta\, d\theta = \frac{\pi}{4}, \quad \int_{\theta=0}^{\frac{\pi}{2}} \cos^3\theta\, d\theta = \frac{2}{3}$$

$$\Phi = \int_{x=-2}^{2} \left(2x^2\cdot\frac{\pi}{4} + 2\cdot\frac{2}{3}e^{x/5}\right) dx$$

$$\int_{x=-2}^{2} x^2 dx = \frac{16}{3}, \quad \int_{x=-2}^{2} e^{x/5} dx = 5\left(e^{2/5} - e^{-2/5}\right)$$

$$\Phi = \frac{\pi}{2}\cdot\frac{16}{3} + \frac{4}{3}\cdot 5\left(e^{2/5} - e^{-2/5}\right)$$

$$\Phi = \frac{8\pi}{3} + \frac{20}{3}\left(e^{2/5} - e^{-2/5}\right)$$

Question 9

For each of the following parameterizations $\vec{r}(u,v)$, and vector fields $\vec{F}(x,y,z)$:

i. Describe the surface S that is parameterized by $\vec{r}(u,v)$.

ii. Describe in words the positive orientation of S given by the family of unit normal vectors $\vec{n} = (\vec{r}_u \times \vec{r}_v)/|\vec{r}_u \times \vec{r}_v|$. That is, give a description of which way the normal vectors point for this orientation.