Math 120 QR

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Contents

Chapter 1

1.1   12.1 notes

> **Definition 1.1.1: Distance Formula**
>
> Defintion:
>
> $$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

> **Definition 1.1.2: Equation of a sphere**
>
> Defintion: An equation of a sphere with center $C(h, k, l)$, and radius r is
>
> $$(x - h)^2 + (y - k)^2 + (z - l)^2$$
>
> In particular, if the center is the origin O, than an equation of the sphere is
>
> $$x^2 + y^2 + z^2$$

1.2   12.2 Notes

> **Definition 1.2.1: Vector Addition**
>
> If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the **sum $\mathbf{u} + \mathbf{v}$** is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

> **Definition 1.2.2: Scalar Multiplication**
>
> If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple $c\mathbf{v}$** is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c = 0$ or $\mathbf{v} = 0$, then $c\mathbf{v} = 0$.

> **Example 1.2.1:**
>
> Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ with represenation $\overrightarrow{AB}$ is:
>
> $$a = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarily, for three demensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Note:-

Properties of vectors: If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalars than

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $a + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + 0 = \mathbf{a}$
- $\mathbf{a} + \mathbf{a} + \text{-}\mathbf{a} = 0$
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
- $(c + d)a = c\mathbf{a} + d\mathbf{a}$
- $(cd)\mathbf{a} = c(d\mathbf{a})$
- $l\mathbf{a} = \mathbf{a}$

1.3   12.3 Notes

Definition 1.3.1: Dot Product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties of the Dot Product: If $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
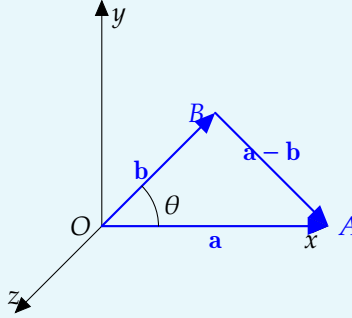5. $\mathbf{0} \cdot \mathbf{a} = 0$

Definition 1.3.2: Geometric Definition of the Dot Product

If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

Proof:

$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta$$

Corollary: If θ is the angle between nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Note:-

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if an only if $\mathbf{a} \cdot \mathbf{b} = 0$

Example 1.3.1 (Direction Angles and Cosines)

The **direction angles** of a nonzero vector $\mathbf{a}$ are the angles α, β, and γ (in the interval $[0, \pi]$) that $\mathbf{a}$ makes with the positive x-, y-, and z-axes, respectively .

The cosines of these direction angles, $\cos \alpha$, $\cos \beta$, and $\cos \gamma$, are called the **direction cosines** of the vector $\mathbf{a}$. Using Corollary 6 with $\mathbf{b}$ replaced by $\mathbf{i}$, we obtain:

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|} \tag{1}$$

Similarly, we also have:

$$\cos \beta = \frac{a_2}{|\mathbf{a}|} \quad \text{and} \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|} \tag{2}$$

By squaring the expressions in Equations 8 and 9 and adding, we see that:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \tag{3}$$

We can also use Equations 8 and 9 to write:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle = |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

Therefore,

$$\frac{1}{|\mathbf{a}|} \mathbf{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \tag{4}$$

which says that the direction cosines of $\mathbf{a}$ are the components of the unit vector in the direction of $\mathbf{a}$.

Definition 1.3.3: Projections

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}|\cos\theta$, where θ is the angle between **a** and **b**. This is denoted by $\text{comp}_\mathbf{a}\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leqslant \pi$. The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**. We summarize these ideas as follows.

Scalar projection of b onto a: $\quad\text{comp}_\mathbf{a}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of b onto a: $\quad\text{proj}_\mathbf{a}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\right)\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

1.4 Day 3 Class notes

Example 1.4.1:

Use the geometric def of cross product to calculate:
$i \times (i + j)$
$(i + j) \times (i - j)$

Definition 1.4.1: Second definition of dot product

Arithmetic Definition:

$$a \times b = \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = |a||b|\sin(\theta)$$

$$\begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} i - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} j - \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2k \end{bmatrix}$$

$$= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

$$|a \times b|^2 = (a_2b_3 - a_3b_2) - (a_1b_3 - a_3b_1) + (a_1b_2 - a_2b_1)$$

$$|a \times b|^2 = (a_2b_3 - a_3b_2) - (a_1b_3 - a_3b_1) + (a_1b_2 - a_2b_1)$$