

# Math 120 QR

Alex Hernandez Juarez

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# Chapter 1

## 1.1 12.1 Notes (Three Dimensional Coordinate Systems)

### Definition 1.1.1: Distance Formula

Defintion:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$



### Definition 1.1.2: Equation of a sphere

Defintion: An equation of a sphere with center  $C(h, k, l)$ , and radius  $r$  is

$$(x - h)^2 + (y - k)^2 + (z - l)^2$$

In particular, if the center is the origin  $O$ , than an equation of the sphere is

$$x^2 + y^2 + z^2$$



## 1.2 12.2 Notes (Vectors)

### Definition 1.2.1: Vector Addition

If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors positioned so the initial point of  $\mathbf{v}$  is at the terminal point of  $\mathbf{u}$ , then the **sum**  $\mathbf{u} + \mathbf{v}$  is the vector from the initial point of  $\mathbf{u}$  to the terminal point of  $\mathbf{v}$ .



### Definition 1.2.2: Scalar Multiplication

If  $c$  is a scalar and  $\mathbf{v}$  is a vector, then the **scalar multiple**  $c\mathbf{v}$  is the vector whose length is  $|c|$  times the length of  $\mathbf{v}$  and whose direction is the same as  $\mathbf{v}$  if  $c > 0$  and is opposite to  $\mathbf{v}$  if  $c < 0$ . If  $c = 0$  or  $\mathbf{v} = \mathbf{0}$ , then  $c\mathbf{v} = \mathbf{0}$ .



**Example 1.2.1:**

Given the points  $A(x_1, y_1, z_1)$  and  $B(x_2, y_2, z_2)$ , the vector  $\mathbf{a}$  with representation  $\overrightarrow{AB}$  is:

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

**Example 1.2.2:**

If  $\mathbf{a} = \langle a_1, a_2 \rangle$  and  $\mathbf{b} = \langle b_1, b_2 \rangle$ , then:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

**Note:-**

Properties of vectors: If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_n$  and  $c$  and  $d$  are scalars then

- $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
- $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
- $\mathbf{a} + \mathbf{0} = \mathbf{a}$
- $\mathbf{a} + \mathbf{a} + -\mathbf{a} = \mathbf{0}$
- $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
- $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
- $(cd)\mathbf{a} = c(d\mathbf{a})$
- $l\mathbf{a} = \mathbf{a}$



## 1.3 12.3 Notes (Dot Product)

### Definition 1.3.1: Dot Product

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , then the **dot product** of  $\mathbf{a}$  and  $\mathbf{b}$  is the number  $\mathbf{a} \cdot \mathbf{b}$  given by

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Properties of the Dot Product: If  $\mathbf{a}, \mathbf{b}$ , and  $\mathbf{c}$  are vectors in  $V_3$  and  $c$  is a scalar, then

1.  $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2.  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3.  $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4.  $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5.  $\mathbf{0} \cdot \mathbf{a} = 0$

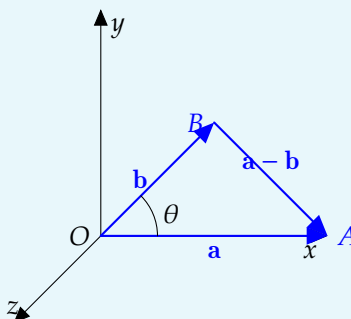


### Definition 1.3.2: Geometric Definition of the Dot Product

If  $\theta$  is the angle between vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

Proof:



$$|AB|^2 = |OA|^2 + |OB|^2 - 2|OA||OB| \cos \theta$$

Corollary: If  $\theta$  is the angle between nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$ , then

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$



### Note:-

Two vectors  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal if and only if  $\mathbf{a} \cdot \mathbf{b} = 0$



### Example 1.3.1 (Direction Angles and Cosines)

The **direction angles** of a nonzero vector  $\mathbf{a}$  are the angles  $\alpha$ ,  $\beta$ , and  $\gamma$  (in the interval  $[0, \pi]$ ) that  $\mathbf{a}$  makes with the positive  $x$ -,  $y$ -, and  $z$ -axes, respectively.

The cosines of these direction angles,  $\cos \alpha$ ,  $\cos \beta$ , and  $\cos \gamma$ , are called the **direction cosines** of the vector **a**. Using Corollary 6 with **b** replaced by **i**, we obtain:

$$\cos \alpha = \frac{\mathbf{a} \cdot \mathbf{i}}{|\mathbf{a}||\mathbf{i}|} = \frac{a_1}{|\mathbf{a}|} \quad (1)$$

Similarly, we also have:

$$\cos \beta = \frac{a_2}{|\mathbf{a}|} \quad \text{and} \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|} \quad (2)$$

By squaring the expressions in Equations 8 and 9 and adding, we see that:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad (3)$$

We can also use Equations 8 and 9 to write:

$$\mathbf{a} = \langle a_1, a_2, a_3 \rangle = \langle |\mathbf{a}| \cos \alpha, |\mathbf{a}| \cos \beta, |\mathbf{a}| \cos \gamma \rangle = |\mathbf{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

Therefore,

$$\frac{1}{|\mathbf{a}|} \mathbf{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \quad (4)$$

which says that the direction cosines of **a** are the components of the unit vector in the direction of **a**.

### Definition 1.3.3: Projections

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number  $|\mathbf{b}| \cos \theta$ , where  $\theta$  is the angle between **a** and **b**. This is denoted by  $\text{comp}_{\mathbf{a}} \mathbf{b}$ . Observe that it is negative if  $\pi/2 < \theta \leq \pi$ . The equation

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = |\mathbf{a}|(|\mathbf{b}| \cos \theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}| \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|} \cdot \mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**. We summarize these ideas as follows.

**Scalar projection of b onto a:**  $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$

**Vector projection of b onto a:**  $\text{proj}_{\mathbf{a}} \mathbf{b} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a}$



## 1.4 12.4 Notes (Cross Product)

### Definition 1.4.1: Cross Product

Given two nonzero vectors  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , suppose that a nonzero vector  $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$  is perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ . Then  $\mathbf{a} \cdot \mathbf{c} = 0$  and  $\mathbf{b} \cdot \mathbf{c} = 0$ , and so:

$$a_1c_1 + a_2c_2 + a_3c_3 = 0 \quad (1)$$

$$b_1c_1 + b_2c_2 + b_3c_3 = 0 \quad (2)$$

To eliminate  $c_3$ , we multiply (1) by  $b_3$  and (2) by  $a_3$  and subtract:

$$(a_1b_3 - a_3b_1)c_1 + (a_2b_3 - a_3b_2)c_2 = 0 \quad (3)$$

Equation (3) has the form  $pc_1 + qc_2 = 0$ , for which an obvious solution is  $c_1 = q$  and  $c_2 = -p$ . So, a solution of (3) is:

$$c_1 = a_2b_3 - a_3b_2$$

$$c_2 = a_3b_1 - a_1b_3$$

Substituting these values into (1) and (2), we then get:

$$c_3 = a_1b_2 - a_2b_1$$


This means that a vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$  is:

$$\langle c_1, c_2, c_3 \rangle = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

The resulting vector is called the **cross product** of  $\mathbf{a}$  and  $\mathbf{b}$  and is denoted by  $\mathbf{a} \times \mathbf{b}$ . 


### Definition 1.4.2: Cross Product of two vectors

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$  and  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$  then the **cross product** of  $\mathbf{a}$  and  $\mathbf{b}$  is:

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$



#### Note:-

Determinant of order 2:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$


#### Note:-

Determinant of order 3:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$


### Definition 1.4.3: Second definition of cross product

Arithmetic Definition:

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{bmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = |\mathbf{a}||\mathbf{b}| \sin(\theta) \\ &= \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} i - \begin{bmatrix} a_1 & a_3 \\ b_1 & b_3 \end{bmatrix} j + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} k \\ &= (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k \end{aligned}$$

The vector  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$



### Example 1.4.1: Proof that $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$

$$\begin{aligned} (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} &= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} a_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} a_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} a_3 \\ &= a_1(a_2b_3 - a_3b_2) - a_2(a_1b_3 - a_3b_1) + a_3(a_1b_2 - a_2b_1) \\ &= a_1a_2b_3 - a_1a_3b_2 - a_2a_1b_3 + a_2a_3b_1 + a_3a_1b_2 - a_3a_2b_1 \\ &= 0 \end{aligned}$$



### Definition 1.4.4: sin definition of cross product

If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b}$  (so  $0 \leq \theta \leq \pi$ ), then the length of the cross product  $\mathbf{a} \times \mathbf{b}$  is given by:

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin(\theta)$$

Proof:

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\ &= a_2^2b_3^2 - 2a_2a_3b_2b_3 + a_3^2b_2^2 + a_3^2b_1^2 - 2a_1a_3b_1b_3 + a_1^2b_3^2 + a_1^2b_2^2 - 2a_1a_2b_1b_2 + a_2^2b_1^2 \\ &= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \cos^2 \theta \quad (\text{by Theorem 12.3.3}) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1 - \cos^2 \theta) \\ &= |\mathbf{a}|^2|\mathbf{b}|^2 \sin^2 \theta \end{aligned}$$

Taking square roots and observing that  $\sqrt{\sin^2 \theta} = \sin \theta$  because  $\sin \theta \geq 0$  when  $0 \leq \theta \leq \pi$ , we have

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$





**Note:-**

Two nonzero vectors  $\mathbf{a}$  and  $\mathbf{b}$  are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = 0$$

**Example 1.4.2: Geometric interpretation of  $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$** 

If  $\mathbf{a}$  and  $\mathbf{b}$  are represented by directed line segments with the same initial point, then they determine a parallelogram with base  $|\mathbf{a}|$ , altitude  $|\mathbf{b}| \sin(\theta)$  and area

$$A = |\mathbf{a}|(|\mathbf{b}| \sin \theta) = |\mathbf{a} \times \mathbf{b}|$$

Thus we have the following way of interpreting the magnitude of a cross product:

The length of the cross product of  $\mathbf{a} \times \mathbf{b}$  is equal to the area of the parallelogram determined by  $\mathbf{a}$  and  $\mathbf{b}$

**Note:-**

If we apply the following theorem:

The vector  $\mathbf{a} \times \mathbf{b}$  is orthogonal to both  $\mathbf{a}$  and  $\mathbf{b}$ , and

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

to the standard basis vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  using  $\theta = \frac{\pi}{2}$ , we obtain

$$\begin{array}{lll} \mathbf{i} \times \mathbf{j} = \mathbf{k} & \mathbf{j} \times \mathbf{k} = \mathbf{i} & \mathbf{k} \times \mathbf{i} = \mathbf{j} \\ \mathbf{j} \times \mathbf{i} = -\mathbf{k} & \mathbf{k} \times \mathbf{j} = -\mathbf{i} & \mathbf{i} \times \mathbf{k} = -\mathbf{j} \end{array}$$

**Note:-**

If  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  are vectors and  $c$  is a scalar, then

1.  $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2.  $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3.  $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4.  $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5.  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6.  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

**Example 1.4.3: Proof of property 5 of cross products**

If  $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ ,  $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ , and  $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$ , then:

$$\begin{aligned} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\ &= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 \\ &= (a_2b_3 - a_3b_2)c_1 + (a_3b_1 - a_1b_3)c_2 + (a_1b_2 - a_2b_1)c_3 \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \end{aligned}$$



### Definition 1.4.5: Triple Products

The product  $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$  that occurs in Property 5 is called the *scalar triple product* of the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ .

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The geometric significance of the scalar triple product can be seen by considering the parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ . The area of the base parallelogram is  $A = |\mathbf{b} \times \mathbf{c}|$ . If  $\theta$  is the angle between  $\mathbf{a}$  and  $\mathbf{b} \times \mathbf{c}$ , then the height  $h$  of the parallelepiped is  $h = |\mathbf{a}| \cos \theta$ . (We must use  $|\cos \theta|$  instead of  $\cos \theta$  in case  $\theta > \pi/2$ .) Therefore, the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}| |\mathbf{a}| \cos \theta = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Thus, we have proved the following formula: The volume of the parallelepiped determined by the vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is the magnitude of their scalar triple product:

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$



### Note:-

If we use the formula in  $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$  and discover that the volume of the parallelepiped determined by  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  is 0, then the vectors must lie in the same plane; that is, they are coplanar.

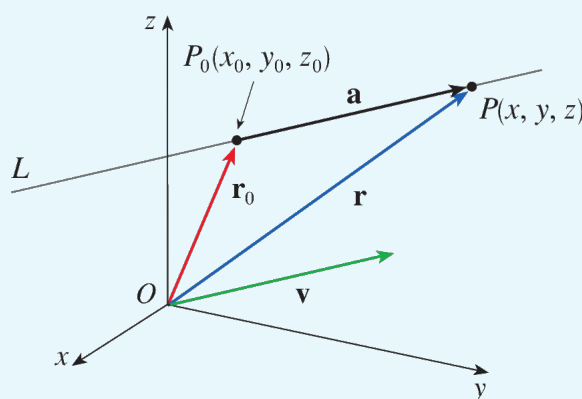


## 12.5 Notes (Equations of Lines and Planes)

### Definition 1.5.1: Hi

Likewise, a line  $L$  in three-dimensional space is determined when we know a point  $P_0(x_0, y_0, z_0)$  on  $L$  and a direction for  $L$ , which is conveniently described by a vector  $\mathbf{v}$  parallel to the line. Let  $P(x, y, z)$  be an arbitrary point on  $L$  and let  $\mathbf{r}_0$  and  $\mathbf{r}$  be the position vectors of  $P_0$  and  $P$  (that is, they have representations  $\overrightarrow{OP_0}$  and  $\overrightarrow{OP}$ ). If  $\mathbf{a}$  is the vector with representation  $\overrightarrow{P_0P}$ , as in Figure 1, then the Triangle Law for vector addition gives

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{a}.$$



Since  $\mathbf{a}$  and  $\mathbf{v}$  are parallel vectors, there is a scalar  $t$  such that  $\mathbf{a} = t\mathbf{v}$ . Thus

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$



**Note:-**

If the vector  $\mathbf{v}$  that gives the direction of the line  $L$  is written in component form as

$$\mathbf{v} = \langle a, b, c \rangle,$$

then we have  $t\mathbf{v} = \langle ta, tb, tc \rangle$ . We can also write  $\mathbf{r} = \langle x, y, z \rangle$  and

$$\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle,$$

so the vector equation (1) becomes

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle.$$

Two vectors are equal if and only if corresponding components are equal. Therefore we have the three scalar equations:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

**Example 1.5.1: Line example**

Find a vector equation and parametric equations for the line that passes through the point  $(5, 1, 3)$  and is parallel to the vector  $\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ . Here  $\mathbf{r}_0 = \langle 5, 1, 3 \rangle = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k}$  and  $\mathbf{v} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ , so the vector equation (1) becomes

$$\mathbf{r} = (5\mathbf{i} + \mathbf{j} + 3\mathbf{k}) + t(\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$$

or

$$\mathbf{r} = (5 + t)\mathbf{i} + (1 + 4t)\mathbf{j} + (3 - 2t)\mathbf{k}$$

Parametric equations are

$$x = 5 + t \quad y = 1 + 4t \quad z = 3 - 2t$$

**Note:-**

In general, if a vector  $\mathbf{v} = \langle a, b, c \rangle$  is used to describe the direction of a line  $L$ , then the numbers  $a$ ,  $b$ , and  $c$  are called *direction numbers* of  $L$ . Since any vector parallel to  $\mathbf{v}$  could also be used, we see that any three numbers proportional to  $a$ ,  $b$ , and  $c$  could also be used as a set of direction numbers for  $L$ .

Another way of describing a line  $L$  is to eliminate the parameter  $t$  from Equations 2. If none of  $a$ ,  $b$ , or  $c$  is 0, we can solve each of these equations for  $t$ :

$$t = \frac{x - x_0}{a} \quad t = \frac{y - y_0}{b} \quad t = \frac{z - z_0}{c}$$

Equating the results, we obtain

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called symmetric equations of  $L$

**Definition 1.5.2: Line segment**

The line segment from  $\mathbf{r}_0$  to  $\mathbf{r}_1$  is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$



### Definition 1.5.3: Planes

A plane in space is determined by a point  $P_0(x_0, y_0, z_0)$  in the plane and a vector  $\mathbf{n}$  that is orthogonal to the plane. This orthogonal vector  $\mathbf{n}$  is called a **normal vector**. Let  $P(x, y, z)$  be an arbitrary point in the plane, and let  $\mathbf{r}_0$  and  $\mathbf{r}$  be the position vectors of  $P_0$  and  $P$ . Then the vector  $\mathbf{r} - \mathbf{r}_0$  is represented by  $\overrightarrow{P_0P}$ . The normal vector  $\mathbf{n}$  is orthogonal to every vector in the given plane. In particular,  $\mathbf{n}$  is orthogonal to  $\mathbf{r} - \mathbf{r}_0$  and so we have

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0 \quad (1.1)$$

which can be rewritten as

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0 \quad (1.2)$$

These can be referred to as the **vector equation of the plane**

To obtain a scalar equation for the plane, we write  $\mathbf{n} = \langle a, b, c \rangle$ ,  $\mathbf{r} = \langle x, y, z \rangle$ , and  $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ . then the vector equation becomes:

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Expanding the left side of this equation gives the following:

A **scalar equation of the plane** through the point  $P_0(x_0, y_0, z_0)$  with normal vector  $\mathbf{n} = \langle a, b, c \rangle$  is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

by collecting terms can be rewritten as:

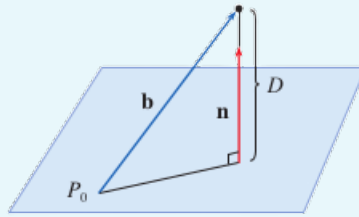
$$ax + by + cz + d = 0$$



### Definition 1.5.4: Distance of a plane

In order to find a formula for the distance  $D$  from a point  $P_1(x_1, y_1, z_1)$  to the plane  $ax + by + cz + d = 0$ , we let  $P_0(x_0, y_0, z_0)$  be any point in the given plane and  $\mathbf{b}$  be the vector corresponding to  $\overrightarrow{P_0P_1}$ . Then

$$\mathbf{b} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$$



From Figure, you can see that the distance  $D$  from  $P_1$  to the plane is equal to the absolute value of the scalar projection of  $\mathbf{b}$  onto the normal vector  $\mathbf{n} = \langle a, b, c \rangle$ . Thus,

$$\begin{aligned} D &= |\text{comp}_{\mathbf{n}} \mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \\ &= \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{|(ax_1 + by_1 + cz_1) - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}} \end{aligned}$$



## 1.6 12.6 Reading Notes (Cylinders and Quadric Surfaces)

### Definition 1.6.1: Cylinder

A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.



### Definition 1.6.2: Quadric Surfaces

A Quadric Surface is the graph of a second-degree equation in three variables  $x$ ,  $y$ , and  $z$ . The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0$$

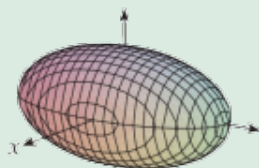
where  $A, B, C, \dots, J$  are constants, but by translation and rotation it can be brought into one of the two *standard forms*

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$



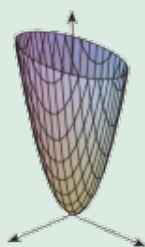
### Example 1.6.1: Graphs of Quadric Surfaces PT 1

Ellipsoid:



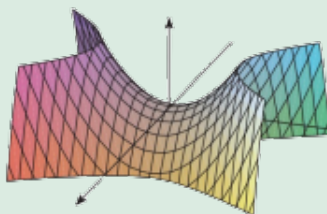
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All traces are ellipses. If  $a = b = c$ , the ellipsoid is a sphere.



$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.

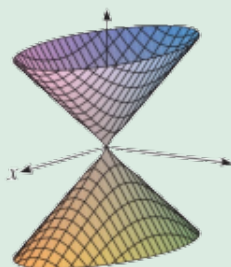


$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where  $c < 0$  is illustrated.

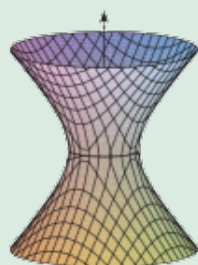


### Example 1.6.2: Quadric Surfaces Pt 2



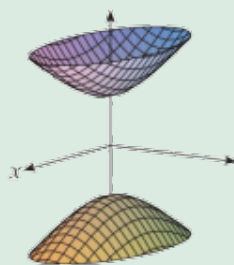
$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses. Vertical traces in the planes  $x = k$  and  $y = k$  are hyperbolas if  $k \neq 0$  but are pairs of lines if  $k = 0$ .



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.



$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Horizontal traces in  $z = k$  are ellipses if  $k > c$  or  $k < -c$ . Vertical traces are hyperbolas. The two minus signs indicate two sheets.



# Chapter 2

## 2.1 13.1 Reading Notes (Vector Functions and Space Curves)

### Definition 2.1.1: Vector Value Functions

A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors. We are most interested in vector functions  $\mathbf{r}$  whose values are three-dimensional vectors. If  $f(t)$ ,  $g(t)$ , and  $h(t)$  are the components of the vector  $\mathbf{r}(t)$ , then  $f$ ,  $g$ , and  $h$  are real-valued functions called the **component functions** of  $\mathbf{r}$  and we can write

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter  $t$  to denote the independent variable because it represents time in most applications of vector functions.

### Definition 2.1.2: Limit of Vectors

The **limit** of a vector function  $\mathbf{r}$  is defined by taking the limits of its component functions as follows. If  $(\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle)$ , then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \left\langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \right\rangle$$

provided the limits of the component functions exist.

### Definition 2.1.3: Space Curves

here is a close connection between continuous vector functions and space curves. Suppose that  $f$ ,  $g$ , and  $h$  are continuous real-valued functions on an interval  $I$ . Then the set  $C$  of all points  $(x, y, z)$  in space, where

$$x = f(t) \quad y = g(t) \quad z = h(t)$$

and  $(t)$  varies throughout the interval  $I$ , is called a **space curve**. The equations in are called **parametric equations of  $C$**  and  $t$  is called a **parameter**.

## 2.2 13.2 Notes (Derivatives and Integrals of Vector Functions)

### Definition 2.2.1: Derivatives

The derivative  $\mathbf{r}'$  of a vector function  $\mathbf{r}$  is defined in much the same way as for real-valued functions:

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$



### Definition 2.2.2: Derivatives of vectors pt 2

The following theorem gives us a convenient method for computing the derivative of a vector function  $\mathbf{r}$ : just differentiate each component of  $\mathbf{r}$ . **Theorem** If  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ , where  $f$ ,  $g$ , and  $h$  are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$



### Example 2.2.1: Proof of Definition 2.2.2

$$\begin{aligned} \mathbf{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\mathbf{r}(t + \Delta t) - \mathbf{r}(t)] \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\langle f(t + \Delta t), g(t + \Delta t), h(t + \Delta t) \rangle - \langle f(t), g(t), h(t) \rangle] \\ &= \lim_{\Delta t \rightarrow 0} \left\langle \frac{f(t + \Delta t) - f(t)}{\Delta t}, \frac{g(t + \Delta t) - g(t)}{\Delta t}, \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\ &= \left\langle \lim_{\Delta t \rightarrow 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{g(t + \Delta t) - g(t)}{\Delta t}, \lim_{\Delta t \rightarrow 0} \frac{h(t + \Delta t) - h(t)}{\Delta t} \right\rangle \\ &= \langle f'(t), g'(t), h'(t) \rangle \end{aligned}$$

A unit vector that has the same direction as the tangent vector is called the **unit tangent vector**  $\mathbf{T}$  and is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$



### Definition 2.2.3: Differentiation Rules

Proof: **Theorem** Suppose  $\mathbf{u}$  and  $\mathbf{v}$  are differentiable vector functions,  $c$  is a scalar, and  $f$  is a real-valued function. Then

1.  $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2.  $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3.  $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4.  $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
5.  $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$
6.  $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

(Chain Rule)

### Note:-

We use Formula 4 to prove the following theorem. **Theorem** If  $|\mathbf{r}(t)| = c$  (a constant), then  $\mathbf{r}'(t)$  is orthogonal to  $\mathbf{r}(t)$  for all  $t$ .



#### Definition 2.2.4: Iteration of Vectors

The **definite integral** of a continuous vector function  $\mathbf{r}(t)$  can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of  $\mathbf{r}$  in terms of the integrals of its component functions  $f$ ,  $g$ , and  $h$  as follows. (We use the notation of Chapter 5.)

$$\begin{aligned}\int_a^b \mathbf{r}(t) dt &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \mathbf{r}(t_j^*) \Delta t \\ &= \lim_{n \rightarrow \infty} \left[ \left( \sum_{i=1}^n f(t_i^*) \Delta t \right) \mathbf{i} + \left( \sum_{i=1}^n g(t_i^*) \Delta t \right) \mathbf{j} + \left( \sum_{i=1}^n h(t_i^*) \Delta t \right) \mathbf{k} \right]\end{aligned}$$

and so

$$\int_a^b \mathbf{r}(t) dt = \left( \int_a^b f(t) dt \right) \mathbf{i} + \left( \int_a^b g(t) dt \right) \mathbf{j} + \left( \int_a^b h(t) dt \right) \mathbf{k}$$

This means that we can evaluate an integral of a vector function by integrating each component function. We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t) dt = \mathbf{R}(t) \Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where  $\mathbf{R}$  is an antiderivative of  $\mathbf{r}$ , that is,  $\mathbf{R}'(t) = \mathbf{r}(t)$ . We use the notation  $\int \mathbf{r}(t) dt$  for indefinite integrals (antiderivatives).



## 2.3 13.3 Notes (Arc Length and Curvature)

### Definition 2.3.1: Length of a space curve

Suppose that the curve has the vector equation  $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ ,  $a \leq t \leq b$ , or, equivalently, the parametric equations  $x = f(t)$ ,  $y = g(t)$ ,  $z = h(t)$ , where  $f'$ ,  $g'$ , and  $h'$  are continuous. If the curve is traversed exactly once as  $t$  increases from  $a$  to  $b$ , then it can be shown that its length is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt \quad (2.1)$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \quad (2.2)$$

Notice that both of the arc length formulas (1) and (2) can be put into the more compact form

$$L = \int_a^b |\mathbf{r}'(t)| dt \quad (2.3)$$

because, for plane curves  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ ,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves  $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ ,

$$|\mathbf{r}'(t)| = |f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

A single curve  $C$  can be represented by more than one vector function. For instance the twisted cube

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle \quad 1 \leq t \leq 2 \quad (2.4)$$

could also be represented by the function

$$\mathbf{r}_2(u) = \langle e^u, e^{2u}, e^{3u} \rangle \quad 0 \leq u \leq \ln 2 \quad (2.5)$$

where the connection between the parameters  $t$  and  $u$  is given by  $t = e^u$ . We say that equations 2.4 and 2.5 are parameterizations of the curve  $C$ . If we were to use Equation 2.3 to compute the length of  $C$  using Equations 2.4 and 2.5, we would get the same answer. This is because arc length is a geometric property of the curve and hence is independent of the parametrization that is used.



### Definition 2.3.2: Arc Length Function

Now we suppose that the curve  $C$  is a curve given by a vector function


$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \quad a \leq t \leq b$$

where  $\mathbf{r}'$  is continuous and  $C$  is traversed exactly once as  $t$  increases from  $a$  to  $b$ . We define its **arc length functions** by

$$s(t) = \int_a^t |\mathbf{r}'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du \quad (2.6)$$

Thus  $s(t)$  is the length of part  $C$  between  $\mathbf{r}(a)$  and  $\mathbf{r}(t)$ . If we differentiate both sides of equation 2.6 using part 1 of the Fundamental Theorem of Calculus, we obtain

$$\frac{ds}{dt} = |\mathbf{r}'(t)| \quad (2.7)$$

It is often useful to **parameterize a curve with respect to arc length** because arc length arises naturally from the shape of the curve and does not depend on a particular coordinate system of a particular parametrization. 

### Definition 2.3.3: Curvature

A parametrization  $\mathbf{r}(t)$  is called **smooth** on an interval  $I$  if  $\mathbf{r}'$  is continuous and  $\mathbf{r}'(t) \neq 0$  on  $I$ . A curve is called smooth if it has a smooth parameterization. A smooth corner has no cusp or sharp corner; when the tangent vector turns it does so continuously.

If  $C$  is a smooth curve defined by the vector  $\mathbf{r}$ , recall that the unit tangent vector  $\mathbf{T}(t)$  is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

The curvature of a curve is

$$k = \left| \frac{d\mathbf{T}}{ds} \right| \quad (2.8)$$

where  $\mathbf{T}$  is the unit tangent vector


The curvature is easier to compute if it is expressed in terms of the parameter  $t$  instead of  $s$ , so we use the chain rule

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt} \Rightarrow k = \left| \frac{d\mathbf{T}}{ds} \right| = \left| \frac{d\mathbf{T}/dt}{ds/dt} \right|$$

but  $ds/dt = |\mathbf{r}'(t)|$  from equation 2.7

$$k(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} \quad (2.9)$$

The curvature of the curve given by the vector function  $\mathbf{r}$  is

$$k(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \quad (2.10) $$

#### Note:-

For the special case of a plane curve with equation  $y = f(x)$ , we choose  $x$  as the parameter and write  $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$ . Then  $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$  and  $\mathbf{r}''(x) = f''(x)\mathbf{j}$ . Since  $\mathbf{i} \times \mathbf{j} = \mathbf{k}$  and  $\mathbf{j} \times \mathbf{j} = 0$ , it follows that  $\mathbf{r}'(x) \times \mathbf{r}''(x) = f'(x)\mathbf{k}$ . We also have  $|\mathbf{r}'(x)| = \sqrt{1 + [f'(x)]^2}$  and so, by Theorem 10,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} \quad (2.11)$$

**Note:-**

At a given point on a smooth space curve  $\mathbf{r}(t)$ , there are many vectors that are orthogonal to the unit tangent vector  $\mathbf{T}(t)$ . We single out one by observing that, because  $|\mathbf{T}(t)| = 1$  for all  $t$ , we have  $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$  by Theorem 13.2.4, so  $\mathbf{T}'(t)$  is orthogonal to  $\mathbf{T}(t)$ . Note that, typically,  $\mathbf{T}'(t)$  is itself not a unit vector. But at any point where  $\kappa \neq 0$  we can define the *principal unit normal vector*  $\mathbf{N}(t)$  (or simply *unit normal*) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

We can think of the unit normal vector as indicating the direction in which the curve is turning at each point. The vector

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

**Note:-**

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\begin{aligned} \mathbf{T}(t) &= \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} & \mathbf{N}(t) &= \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} & \mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\ \kappa &= \left| \frac{d\mathbf{T}}{ds} \right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \end{aligned}$$

**Definition 2.3.4: Torision**

The **torsion** of a curve is

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}$$

Torsion is easier to compute if it is expressed in terms of the parameter  $t$  instead of  $s$ , so we use the Chain Rule to write

$$\begin{aligned} \frac{d\mathbf{B}}{dt} &= \frac{d\mathbf{B}}{ds} \frac{ds}{dt} \quad \text{so} \quad \frac{d\mathbf{B}}{ds} = \frac{d\mathbf{B}/dt}{ds/dt} = \frac{\mathbf{B}'(t)}{|\mathbf{r}'(t)|} \\ \tau(t) &= \frac{-\mathbf{B}'(t) \cdot \mathbf{N}(t)}{|\mathbf{r}'(t)|} \end{aligned}$$

**Theorem** The torsion of the curve given by the vector function  $\mathbf{r}$  is

$$\tau(t) = \frac{[\mathbf{r}'(t) \times \mathbf{r}''(t)] \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2}$$

## 2.4 13.4 Notes (Motion in Space: Velocity and Acceleration)

# Chapter 3

## 3.1 14.1 Functions of Several Variables

### Definition 3.1.1: Functions of Two Variables

Definition: A function  $f$  of two variables is a rule that assigns to each ordered pair of real numbers  $(x, y)$  in a set  $D$  a unique real number denoted by  $f(x, y)$ . The set  $D$  is the *domain* of  $f$  and its *range* is the set of values that  $f$  takes on, that is,  $\{f(x, y) \mid (x, y) \in D\}$ .



### Definition 3.1.2: Graph of a Function of Two Variables

Definition If  $f$  is a function of two variables with domain  $D$ , then the *graph* of  $f$  is the set of all points  $(x, y, z)$  in  $\mathbb{R}^3$  such that  $z = f(x, y)$  and  $(x, y)$  is in  $D$ .



### Definition 3.1.3: Level Curves and Contour Maps

Definition: The *level curves* of a function  $f$  of two variables are the curves with equations  $f(x, y) = k$ , where  $k$  is a constant (in the range of  $f$ ).



### Definition 3.1.4: Functions of Three Variables

Definition: A **function of three variables**,  $f$ , is a rule that assigns to each ordered triple  $(x, y, z)$  in a domain  $D \subseteq \mathbb{R}^3$  a unique real number denoted by  $f(x, y, z)$ . For instance, the temperature...



## 3.2 14.2 Limits and Continuity

### Definition 3.2.1: Limit of Two Variable Functions

Definition: Let  $f$  be a function of two variables whose domain  $D$  includes points arbitrarily close to  $(a, b)$ . Then we say that the *limit of  $f(x, y)$  as  $(x, y)$  approaches  $(a, b)$*  is  $L$  and we write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if for every number  $\varepsilon > 0$  there is a corresponding number  $\delta > 0$  such that if  $(x, y) \in D$  and  $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ , then  $|f(x, y) - L| < \varepsilon$ .



### 3.3 14.3 Partial Derivatives

#### Definition 3.3.1: Partial Derivatives

Definition Partial Derivative with respect to  $x$

$$f_x(a, b) = g'(a) \quad \text{where} \quad g(x) = f(x, b)$$

Definition Partial Derivative with respect to  $y$

$$f_y(a, b) = h'(a) \quad \text{where} \quad h(x) = f(a, y)$$



#### Note:-

If  $z = f(x, y)$ , we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$



#### Note:-

**Rule for Finding Partial Derivatives of  $z = f(x, y)$**

1. To find  $f_x$ , regard  $y$  as a constant and differentiate  $f(x, y)$  with respect to  $x$ .
2. To find  $f_y$ , regard  $x$  as a constant and differentiate  $f(x, y)$  with respect to  $y$ .



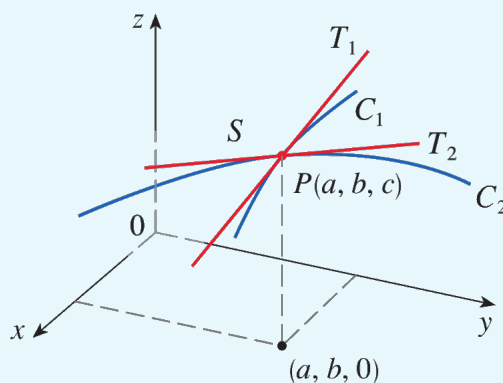
#### Definition 3.3.2: interpretation of Partial Derivatives

To understand partial derivatives geometrically, think of the equation  $z = f(x, y)$  as representing a surface  $S$  (the graph of  $f$ ). If  $f(a, b) = c$ , then the point  $P(a, b, c)$  lies on this surface.

By fixing  $y = b$ , we focus on the curve  $C_1$  where the vertical plane  $y = b$  intersects  $S$ . Similarly, fixing  $x = a$  gives us the curve  $C_2$ , which is where the vertical plane  $x = a$  intersects  $S$ . Both curves  $C_1$  and  $C_2$  pass through the point  $P$ .

The curve  $C_1$  is the graph of the function  $g(x) = f(x, b)$ , and the slope of its tangent at  $P$  is  $f_x(a, b)$ . The curve  $C_2$  is the graph of  $G(y) = f(a, y)$ , and the slope of its tangent at  $P$  is  $f_y(a, b)$ .

Thus, the partial derivatives  $f_x(a, b)$  and  $f_y(a, b)$  represent the slopes of the tangent lines at  $P$  along these curves.



### Definition 3.3.3: Higher Derivatives

If  $f$  is a function of two variables, then its partial derivatives  $f_x$  and  $f_y$  are also functions of two variables, so we can consider their partial derivatives  $(f_x)_x$ ,  $(f_x)_y$ ,  $(f_y)_x$ , and  $(f_y)_y$ , which are called the *second partial derivatives* of  $f$ . If  $z = f(x, y)$ , we use the following notation:

$$\begin{aligned}(f_x)_x &= f_{xx} = f_{11} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} \\(f_x)_y &= f_{xy} = f_{12} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} \\(f_y)_x &= f_{yx} = f_{21} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y} \\(f_y)_y &= f_{yy} = f_{22} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Thus the notation  $f_{xy}$  (or  $\frac{\partial^2 f}{\partial y \partial x}$ ) means that we first differentiate with respect to  $x$  and then with respect to  $y$ , whereas in computing  $f_{yx}$  the order is reversed.



### Definition 3.3.4: Clairut's Theorem

Defintion: Suppose  $f$  is defined on a disk  $D$  that contains the point  $(a, b)$ . If the functions  $f_{xy}$  and  $f_{yx}$  are both continuous on  $D$ , then

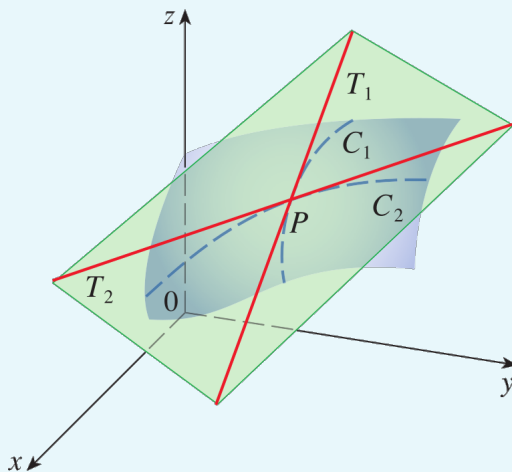
$$f_{xy}(a, b) = f_{yx}(a, b)$$



## 3.4 14.4 Tangent Planes and Linear Approximation

### Definition 3.4.1: Tangent Planes

Let's consider a surface  $S$  given by the equation  $z = f(x, y)$ , where  $f$  has continuous first derivatives. Let  $P(x_0, y_0, z_0)$  be a point on the surface. Two curves,  $C_1$  and  $C_2$ , are formed by slicing the surface with vertical planes  $y = y_0$  and  $x = x_0$ . These curves pass through the point  $P$ . The tangent lines to  $C_1$  and  $C_2$  at  $P$  are denoted  $T_1$  and  $T_2$ . The **tangent plane** to the surface at  $P$  is the plane that contains both tangent lines  $T_1$  and  $T_2$ .





### Definition 3.4.2: Equation of a tangent plan

Suppose  $f$  has continuous partial derivatives. An equation of the tangent plane to the surface  $z = f(x, y)$  at the point  $P(x_0, y_0, z_0)$  is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \quad (3.1)$$

### Definition 3.4.3: Linear Approximations

If  $z = f(x, y)$ , then  $f$  is **differentiable** at  $(a, b)$  if  $\Delta z$  can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y \quad (3.2)$$

where  $\epsilon_1$  and  $\epsilon_2$  are functions of  $\Delta x$  and  $\Delta y$  such that  $\epsilon_1$  and  $\epsilon_2 \rightarrow 0$  as  $(\Delta x, \Delta y) \rightarrow (0, 0)$ .

## 3.5 14.5 The Chain Rule

### Definition 3.5.1: Chain Rule (Case 1)

Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(t)$  and  $y = h(t)$  are both differentiable functions of  $t$ . Then  $z$  is a differentiable function of  $t$  and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \quad (3.3)$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \quad (3.4)$$

### Definition 3.5.2: Chain Rule (Case 2)

Suppose that  $z = f(x, y)$  is a differentiable function of  $x$  and  $y$ , where  $x = g(s, t)$  and  $y = h(s, t)$  are differentiable functions of  $s$  and  $t$ . Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} \quad (3.5)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} \quad (3.6)$$

### Definition 3.5.3: Chain Rule (General Case)

Suppose that  $u$  is a differentiable function of the  $n$  variables  $x_1, x_2, \dots, x_n$  and each  $x_j$  is a differentiable function of the  $m$  variables  $t_1, t_2, \dots, t_m$ . Then  $u$  is a function of  $t_1, t_2, \dots, t_m$  and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i} \quad (3.7)$$

for each  $i = 1, 2, \dots, m$ .

### Definition 3.5.4: Implicit Differentiation

Defintion:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \quad (3.8)$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \quad (3.9)$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z} \quad (3.10)$$



## 3.6 14.6 Directional Derivatives and the Gradient Vector

### Definition 3.6.1: Directional Derivative

Defintion: The **directional derivative** of  $f$  at  $(x_0, y_0)$  in the direction of a unit vector  $\mathbf{u} = \langle a, b \rangle$  is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} \quad (3.11)$$

if this limit exists.



### Theorem 3.6.1

If  $f$  is a differentiable function of  $x$  and  $y$ , then  $f$  has a directional derivative in the direction of any unit vector  $\mathbf{u} = \langle a, b \rangle$  and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b \quad (3.12)$$

### Definition 3.6.2: The Gradient Vector

If  $f$  is a function of two variables  $x$  and  $y$ , then the **gradient** of  $f$  is the vector function  $\nabla f$  defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} \quad (3.13)$$



### Note:-

Equation 3.12 can be rewritten as

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u} \quad (3.14)$$



### Definition 3.6.3: Gradient of Three Variable Functions

Defintion:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k} \quad (3.15)$$



### Theorem 3.6.2 Maximizing Directional Derivative

Suppose  $f$  is a differentiable function of two or three variables. The maximum value of the directional derivative  $D_{\mathbf{u}}f(\mathbf{x})$  is  $|\nabla f(\mathbf{x})|$  and it occurs when  $\mathbf{u}$  has the same direction as the gradient vector  $\nabla f(\mathbf{x})$ .

### Definition 3.6.4: Tangent Planes to Level Surfaces

Consider a surface  $S$  defined by  $F(x, y, z) = k$ , where  $F$  is a function of three variables. Let  $P(x_0, y_0, z_0)$  be a point on  $S$  and  $C$  be a curve on  $S$  that passes through  $P$ . The curve is given by a vector function  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$  such that  $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$ . Since  $C$  lies on  $S$ , the equation  $F(x(t), y(t), z(t)) = k$  must hold.

By using the Chain Rule to differentiate both sides of this equation, we get:

$$\frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial z} \frac{dz}{dt} = 0$$

This can be written as a dot product:

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

which means that the gradient  $\nabla F$  is perpendicular to the tangent vector  $\mathbf{r}'(t)$  at  $P$ .

At  $t = t_0$ , the gradient at  $P$ ,  $\nabla F(x_0, y_0, z_0)$ , is normal to the tangent plane at  $P$ . The equation of the tangent plane is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \quad (3.16)$$

### Note:-

#### Properties of the Gradient Vector

Let  $f$  be a differentiable function of two or three variables and suppose that  $\nabla f(\mathbf{x}) \neq 0$ .

- The directional derivative of  $f$  at  $\mathbf{x}$  in the direction of a unit vector  $\mathbf{u}$  is given by  $D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$ .
- $\nabla f(\mathbf{x})$  points in the direction of maximum rate of increase of  $f$  at  $\mathbf{x}$ , and that maximum rate of change is  $|\nabla f(\mathbf{x})|$ .
- $\nabla f(\mathbf{x})$  is perpendicular to the level curve or level surface of  $f$  through  $\mathbf{x}$ .



## 3.7 14.7 Maximum and Minimum Values

### Definition 3.7.1: Local Min and Max

Definition: A function of two variables has a **local maximum** at  $(a, b)$  if  $f(x, y) \leq f(a, b)$  when  $(x, y)$  is near  $(a, b)$ . [This means that  $f(x, y) \leq f(a, b)$  for all points  $(x, y)$  in some disk with center  $(a, b)$ .] The number  $f(a, b)$  is called a **local maximum value**. If  $f(x, y) \geq f(a, b)$  when  $(x, y)$  is near  $(a, b)$ , then  $f$  has a **local minimum** at  $(a, b)$  and  $f(a, b)$  is a **local minimum value**.



### Theorem 3.7.1 Critical Point

If  $f$  has a local maximum or minimum at  $(a, b)$  and the first-order partial derivatives of  $f$  exist there, then  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$ .

### Definition 3.7.2: Second Derivatives Test

Suppose the second partial derivatives of  $f$  are continuous on a disk with center  $(a, b)$ , and suppose that  $f_x(a, b) = 0$  and  $f_y(a, b) = 0$  [so  $(a, b)$  is a critical point of  $f$ ]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 \quad (3.17)$$

- (a) If  $D > 0$  and  $f_{xx}(a, b) > 0$ , then  $f(a, b)$  is a local minimum.
- (b) If  $D > 0$  and  $f_{xx}(a, b) < 0$ , then  $f(a, b)$  is a local maximum.
- (c) If  $D < 0$ , then  $(a, b)$  is a saddle point of  $f$ .



### Definition 3.7.3: Absolute Maxiums and Absolute Minimums

Let  $(a, b)$  be a point in the domain  $D$  of a function  $f$  of two variables. Then  $f(a, b)$  is the

- **absolute maximum** value of  $f$  on  $D$  if  $f(a, b) \geq f(x, y)$  for all  $(x, y)$  in  $D$ .
- **absolute minimum** value of  $f$  on  $D$  if  $f(a, b) \leq f(x, y)$  for all  $(x, y)$  in  $D$ .



### Definition 3.7.4: Extreme Value Theorem for Functions of Two Variables

If  $f$  is continuous on a closed, bounded set  $D$  in  $\mathbb{R}^2$ , then  $f$  attains an absolute maximum value  $f(x_1, y_1)$  and an absolute minimum value  $f(x_2, y_2)$  at some points  $(x_1, y_1)$  and  $(x_2, y_2)$  in  $D$ .



#### Note:-

To find the absolute maximum and minimum values of a continuous function  $f$  on a closed, bounded set  $D$ :

1. Find the values of  $f$  at the critical points of  $f$  in  $D$ .
2. Find the extreme values of  $f$  on the boundary of  $D$ .
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.



## 3.8 14.8 Lagrange Multipliers

### Definition 3.8.1: Geometric Explanation Lagrange Multipliers (One Constraint)

The geometric basis of Lagrange's method for two variables involves finding the extreme values of  $f(x, y)$  under the constraint  $g(x, y) = k$ . This means finding the extreme values of  $f$  along the level curve defined by  $g(x, y) = k$ . To maximize  $f(x, y)$  subject to this constraint, we look for the largest value of  $f$  where its level curve touches the constraint curve at one point. At this point, the gradients of  $f$  and  $g$  are parallel, which gives the relationship  $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$  for some scalar  $\lambda$ .

This reasoning also applies to functions of three variables. To find extreme values of  $f(x, y, z)$  under the constraint  $g(x, y, z) = k$ , the point  $(x, y, z)$  must lie on the level surface defined by  $g(x, y, z) = k$ . The gradients of  $f$  and  $g$  are again parallel at the point where the maximum value of  $f$  is reached.

To make this precise, consider a curve  $C$  on the surface where  $g(x, y, z) = k$ . The function  $f$  has an extreme value at the point  $P(x_0, y_0, z_0)$ . If we parameterize the curve as  $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ , then  $f$  has an extreme value at  $t_0$  when  $h'(t_0) = 0$ , where  $h(t) = f(\mathbf{r}(t))$ . Using the Chain Rule, we find that the gradients are again parallel.



### Definition 3.8.2: Method of Lagrange Multipliers

To find the maximum and minimum values of  $f(x, y, z)$  subject to the constraint  $g(x, y, z) = k$  [assuming that these extreme values exist and  $\nabla g \neq 0$  on the surface  $g(x, y, z) = k$ ]:

1. Find all values of  $x$ ,  $y$ ,  $z$ , and  $\lambda$  such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate  $f$  at all the points  $(x, y, z)$  that result from step 1. The largest of these values is the maximum value of  $f$ ; the smallest is the minimum value of  $f$ .



### Definition 3.8.3: Lagrange Multipliers Two Constraints

To find the maximum and minimum values of  $f(x, y, z)$  subject to two constraints,  $g(x, y, z) = k$  and  $h(x, y, z) = c$ , we look for extreme values when  $(x, y, z)$  lies on the curve formed by the intersection of the level surfaces of  $g$  and  $h$ .

At the point where  $f$  has an extreme value, the gradient of  $f$ ,  $\nabla f$ , is orthogonal to the curve. The gradients of  $g$  and  $h$  are also orthogonal to this curve, which means  $\nabla f$  must lie in the plane formed by  $\nabla g$  and  $\nabla h$ .

Thus, there exist two numbers,  $\lambda$  and  $\mu$ , called Lagrange multipliers, such that  $\nabla f(x_0, y_0, z_0)$  is a linear combination of  $\nabla g(x_0, y_0, z_0)$  and  $\nabla h(x_0, y_0, z_0)$ .

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

(3.18)



## Chapter 4

# Multiple Integrals

### 4.1 15.1 Double Integral Over Rectangles

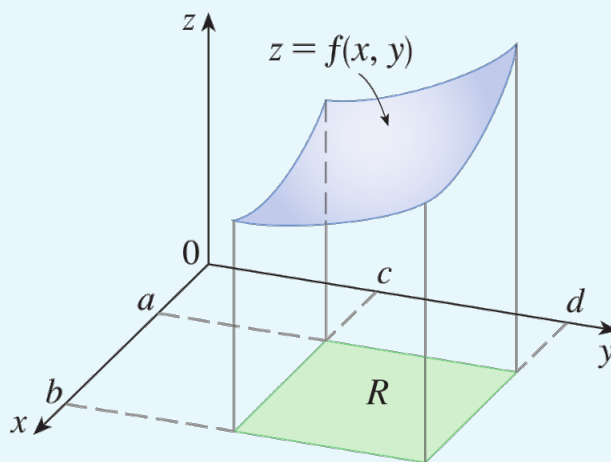
#### Definition 4.1.1: Volume of a Function

We start with a function  $f$  defined over a closed rectangle  $R$  in the  $xy$ -plane, denoted as:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}.$$

The goal is to find the volume of the solid  $S$ , which lies above the rectangle  $R$  and below the graph of the surface  $z = f(x, y)$ , defined as:

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y), (x, y) \in R\}.$$

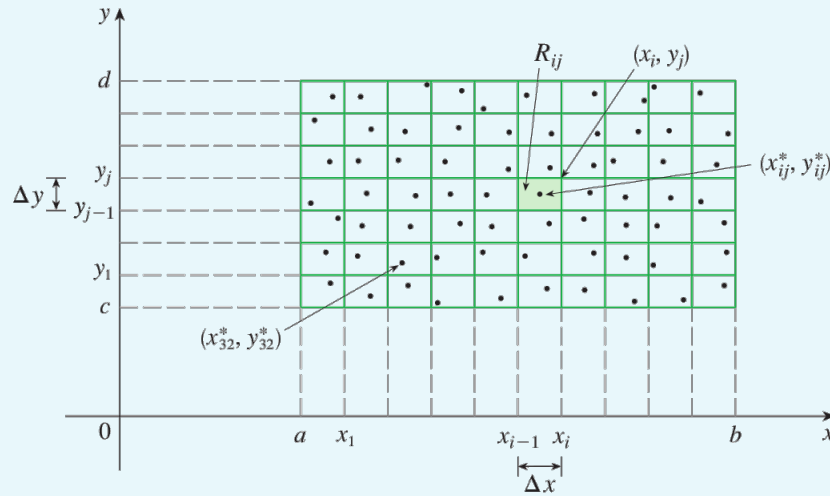


### Definition 4.1.2: Volume of a Function (PART II)

To compute this volume, we first divide the rectangle  $R$  into smaller subrectangles. The interval  $[a, b]$  is divided into  $m$  subintervals of equal width  $\Delta x = \frac{b-a}{m}$ , and the interval  $[c, d]$  is divided into  $n$  subintervals of equal width  $\Delta y = \frac{d-c}{n}$ . The subrectangles are denoted by:

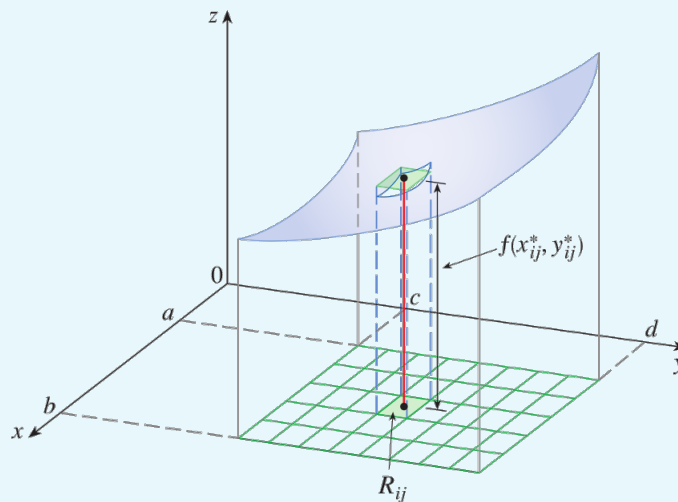
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x, y) \mid x_{i-1} \leq x \leq x_i, y_{j-1} \leq y \leq y_j\}.$$

Each subrectangle has an area of  $\Delta A = \Delta x \Delta y$ .



Next, we approximate the volume of the solid  $S$  by choosing a sample point  $(x_{ij}^*, y_{ij}^*)$  in each subrectangle  $R_{ij}$ . Using this point, we approximate the part of  $S$  above each  $R_{ij}$  by a thin rectangular box (or “column”) with base  $R_{ij}$  and height  $f(x_{ij}^*, y_{ij}^*)$ . The volume of this box is:

$$f(x_{ij}^*, y_{ij}^*) \Delta A.$$



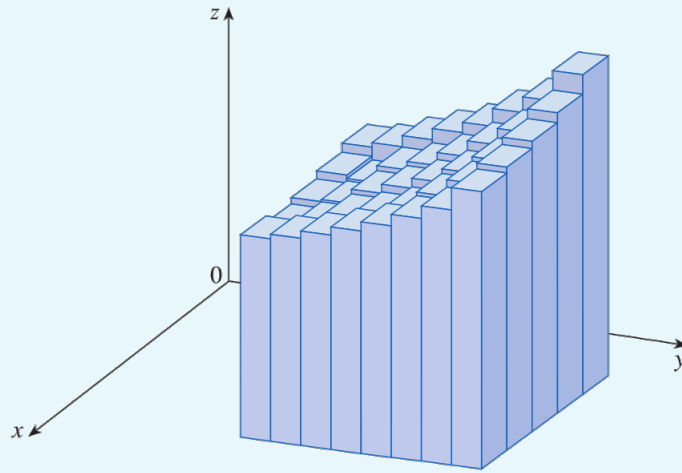
### Definition 4.1.3: Volume of a Function (PART III)


We then sum the volumes of all these boxes over the entire grid of subrectangles, obtaining an approximation of the total volume  $V$  of the solid  $S$ :

$$V \approx \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A.$$

As  $m$  and  $n$  become larger, our approximation becomes more accurate. The exact volume of  $S$  is given by the limit of the double sum as  $m, n \rightarrow \infty$ :

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A \quad (4.1)$$




This is the formal definition of the volume of the solid  $S$  that lies under the graph of the function  $f$  and above the rectangle  $R$ . By taking the limit, we ensure that the approximation becomes exact. 

### Definition 4.1.4: Double Integral

Definition: The **double integral** of  $f$  over the rectangle  $R$  is

$$\iint_R f(x, y) dA = \lim_{m, n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta A \quad (4.2)$$

if this limit exists 

### Definition 4.1.5: Equation for Volume

If  $f(x, y) \geq 0$ , then the volume  $V$  of the solid that lies above the rectangle  $R$  and below the surface  $z = f(x, y)$  is

$$V = \iint_R f(x, y) dA \quad (4.3)$$



#### Definition 4.1.6: Midpoint Rule for Double Integrals

Equation:

$$\iint_R f(x, y) dA \approx \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \Delta A \quad (4.4)$$

where  $\bar{x}_i$  is the midpoint of  $[x_{i-1}, x_i]$  and  $\bar{y}_j$  is the midpoint of  $[y_{j-1}, y_j]$ .



#### Definition 4.1.7: Fubini's Theorem

If  $f$  is continuous on the rectangle

$$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$$

then

$$\iint_R f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

More generally, this is true if we assume that  $f$  is bounded on  $R$ ,  $f$  is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.



#### Definition 4.1.8: Average Value

We define the **average value** of a function  $f$  of two variables defined on a rectangle  $R$  to be

$$f_{\text{avg}} = \frac{1}{A(R)} \iint_R f(x, y) dA$$

where  $A(R)$  is the area of  $R$ .

If  $f(x, y) \geq 0$ , the equation

$$A(R) \times f_{\text{avg}} = \iint_R f(x, y) dA$$

holds true.

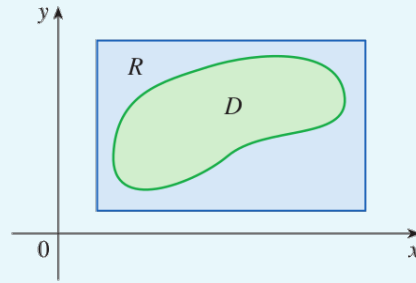


## 4.2 15.2 Double Integrals over General Regions

### Definition 4.2.1: General Regions

Consider a general region  $D$  which is bounded, which means that  $D$  can be enclosed in a rectangular region  $R$ . In order to integrate a function  $f$  over  $D$ , we define a new function  $F$  with domain  $R$  by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$



$$\iint_D f(x, y) dA = \iint_R F(x, y) dA$$

where  $F$  is given by the above equation.



### Definition 4.2.2: Type I

A plane region  $D$  is said to be of **type I** if it lies between the graphs of two continuous functions of  $x$ , that is,

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\} \quad (4.5)$$

where  $g_1$  and  $g_2$  are continuous on  $[a, b]$ .

If  $f$  is continuous on a type I region  $D$  described by

$$D = \{(x, y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

$$\iint_D f(x, y) dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) dy dx \quad (4.6)$$



### Definition 4.2.3: Type II

If  $f$  is continuous on a type II region  $D$  described by

$$D = \{(x, y) \mid c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\} \quad (4.7)$$

then

$$\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) dx dy \quad (4.8)$$



**Note:-**

### Properties of Integrals

•

$$\iint_D [f(x, y) + g(x, y)] dA = \iint_D f(x, y) dA + \iint_D g(x, y) dA \quad (4.9)$$

•

$$\iint_D c f(x, y) dA = c \iint_D f(x, y) dA \quad \text{where } c \text{ is a constant.} \quad (4.10)$$

• If  $f(x, y) \geq g(x, y)$  for all  $(x, y) \in D$ , then

$$\iint_D f(x, y) dA \geq \iint_D g(x, y) dA \quad (4.11)$$

• If  $D = D_1 \cup D_2$  where  $D_1$  and  $D_2$  don't overlap except perhaps on their boundaries, then

$$\iint_D f(x, y) dA = \iint_{D_1} f(x, y) dA + \iint_{D_2} f(x, y) dA \quad (4.12)$$

• If we integrate the constant function  $f(x, y) = 1$  over a region  $D$ , we get the area of  $D$

$$\iint_D 1 dA = A(D) \quad (4.13)$$

• If  $m \leq f(x, y) \leq M$  for all  $(x, y) \in D$ , then

$$m \cdot A(D) \leq \iint_D f(x, y) dA \leq M \cdot A(D) \quad (4.14)$$



## 4.3 Double Integrals in Polar Coordinates

### Definition 4.3.1: Rectangle to Polar Coordinates in Double Integrals

If  $f$  is continuous on a polar rectangle  $R$  given by  $0 \leq a \leq r \leq b$ ,  $\alpha \leq \theta \leq \beta$ , where  $0 \leq \beta - \alpha \leq 2\pi$ , then

$$\iint_R f(x, y) dA = \int_{\alpha}^{\beta} \int_a^b f(r \cos \theta, r \sin \theta) r dr d\theta \quad (4.15)$$



### Definition 4.3.2: Polar Region Double Integration

If  $f$  is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r \cos \theta, r \sin \theta) r dr d\theta \quad (4.16)$$



# Chapter 5

## 5.1 16.1 Vector Fields

### Definition 5.1.1: Vector Field

Defintion: Let  $D$  be a set in  $\mathbb{R}^2$  (a plane region). A **vector field on  $\mathbb{R}^2$**  is a function  $\mathbf{F}$  that assigns to each point  $(x, y)$  in  $D$  a two-dimensional vector  $\mathbf{F}(x, y)$ .



### Definition 5.1.2: 3 Dimensional Vector Fields

Defintion: Let  $E$  be a subset of  $\mathbb{R}^3$ . A **vector field on  $\mathbb{R}^3$**  is a function  $\mathbf{F}$  that assigns to each point  $(x, y, z)$  in  $E$  a three-dimensional vector  $\mathbf{F}(x, y, z)$ .



### Definition 5.1.3: Gradient Fields

Defintion: If  $f$  is a scalar function of two variables, its gradient  $\nabla f$  (also called  $\text{grad } f$ ) is defined by

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

Therefore,  $\nabla f$  is a vector field on  $\mathbb{R}^2$  and is called a **gradient vector field**. Similarly, if  $f$  is a scalar function of three variables, its gradient is a vector field on  $\mathbb{R}^3$ , given by

$$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$



## 5.2 16. 2 Line Integrals

### Definition 5.2.1: Line Integrals

If  $f$  is defined on a smooth curve  $C$  given by

$$x = x(t) \quad y = y(t) \quad a \leq t \leq b$$

then the **line integral** of  $f$  along  $C$  is

$$\int_C f(x, y) ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta s_i \quad (5.1)$$

if this limit exists.



### Definition 5.2.2: Evaluation of Line Integrals

The length of the curve  $C$  is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

If  $f$  is continuous, the following formula can be used to evaluate the line integral of  $f$  along  $C$ :

$$\int_C f(x, y) ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \quad (5.2)$$

### Definition 5.2.3: Line Integrals with Respect to $x$ or $y$

Two other types of line integrals are obtained by replacing  $\Delta s_i$  by either  $\Delta x_i = x_i - x_{i-1}$  or  $\Delta y_i = y_i - y_{i-1}$ . They are called the **line integrals of  $f$  along  $C$  with respect to  $x$  and  $y$** :

$$\int_C f(x, y) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta x_i \quad (5.3)$$

$$\int_C f(x, y) dy = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*) \Delta y_i \quad (5.4)$$

These line integrals can also be evaluated in terms of a parameter  $t$ , where  $x = x(t)$ ,  $y = y(t)$ , and:

$$\int_C f(x, y) dx = \int_a^b f(x(t), y(t)) x'(t) dt \quad (5.5)$$

$$\int_C f(x, y) dy = \int_a^b f(x(t), y(t)) y'(t) dt \quad (5.6)$$

### Definition 5.2.4: Line Integrals in Space

Equation:

$$\int_C f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt \quad (5.7)$$