

Math 120

PSet 2

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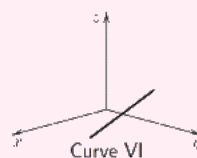
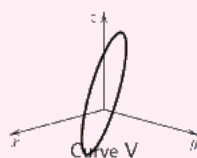
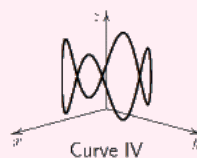
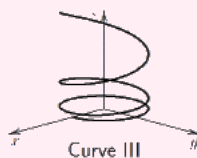
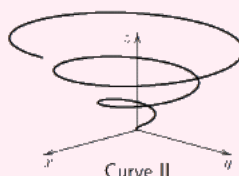
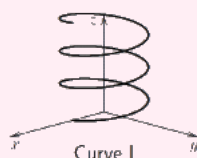
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Chapter 1

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Question 1



- (a) $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$
- (b) $\vec{r}(t) = t \langle \cos t, \sin t, t \rangle$
- (c) $\vec{r}(t) = \langle \cos t, \sin t, t^3 \rangle$
- (d) $\vec{r}(t) = \langle \cos(t^3), \sin(t^3), t^3 \rangle$
- (e) $\vec{r}(u) = \langle \cos u, \sin u, 1 + \sin(4u) \rangle$
- (f) $\vec{r}(u) = \langle \cos u, \sin u, 1 + 4 \sin(u) \rangle$
- (g) $\vec{r}(t) = \langle 2 \cos t, 1 + 4 \cos t, 3 \cos t \rangle$

Solution:

Question 2

2. Find a vector function that represents the curve of intersection of the plane $z = -2$ and the sphere $x^2 + (y - 1)^2 + (z + 1)^2 = 9$.

Solution:

$$\begin{aligned}x^2 + (y - 1)^2 + ((-2) + 1)^2 &= 9 \\x^2 + (y - 1)^2 &= 8 \\r &= 2\sqrt{2} \\x(t) &= 2\sqrt{2} \cos(t) \\y - 1 &= 2\sqrt{2} \sin(t) \Rightarrow y = 2\sqrt{2} \sin(t) + 1 \\\vec{r}(t) &= \langle 2\sqrt{2} \cos(t), 2\sqrt{2} \sin(t) + 1, -2 \rangle\end{aligned}$$

Question 3

Consider the vector-valued function $\vec{r}_1(t) = \langle 2 \sin t, -3 \cos t, 0 \rangle$, $0 \leq t \leq 2\pi$.

- (a) Sketch the plane curve given by $\vec{r}_1(t)$.
- (b) Compute and draw on your sketch from part (a) the position vector $\vec{r}_1\left(\frac{2\pi}{3}\right)$ and the tangent vector $\vec{r}'_1\left(\frac{2\pi}{3}\right)$.
- (c) The vector-valued function $\vec{r}_2(t) = \langle 2 \cos(3t), -3 \sin(3t) \rangle$ parameterizes the same curve. Find the smallest $t^* > 0$ such that $\vec{r}_2(t^*) = \vec{r}_1\left(\frac{2\pi}{3}\right)$, and compute $\vec{r}'_2(t^*)$. Explain how and why $\vec{r}'_2(t^*)$ differs from the tangent vector $\vec{r}'_1\left(\frac{2\pi}{3}\right)$ you computed in part (b).

Solution:

Question 4

Find parametric equations for the tangent line to the curve parameterized by

$$x = 2t + 1, \quad y = e^{t^2-4}, \quad z = \ln(1+t^2)$$

at the point $(5, 1, \ln 5)$.

Solution:

$$\begin{aligned}x(t) &= 2t + 1 & y(t) &= e^{t^2-4} & z(t) &= \ln(1+t^2) \\x'(t) &= 2 & y'(t) &= 2te^{t^2} & z'(t) &= \frac{2t}{1+t^2} \\5 &= 2t + 1 \Rightarrow 4 = 2t \Rightarrow t = 2 \\x'(2) &= 2 & y'(2) &= 4e^{4t} & z'(2) &= \frac{4}{5}\end{aligned}$$

Question 5

- (a) Evaluate the integral $\int (\tan t \hat{i} + \sin^2 t \hat{j} + \sec^2 t \tan t \hat{k}) dt$.
- (b) Suppose a particle is at the point $(-2, 1, 4)$ at time $t = 0$, and moves according to the velocity function $\vec{v}(t) = \tan t \hat{i} + \sin^2 t \hat{j} + \sec^2 t \tan t \hat{k}$. Find the particle's position at time $t = \frac{\pi}{4}$.

Solution:

Question 6

Consider the curve parameterized by $\vec{r}(t) = \langle e^{2t}, e^{-2t}, \sqrt{8t} \rangle$, $0 \leq t \leq 1$.

- (a) Sketch the projections of $\vec{r}(t)$ in the xy -, zx -, and yz -planes.
- (b) Find the length of the curve. *Hint:* To integrate, you will need to write $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$ as a perfect square.

Solution:

a)

$$x(t) = e^{2t} \quad y(t) = e^{-2t}$$

$$x \cdot y = e^{2t} \cdot e^{-2t} = 1$$

$$x(t) = e^{2t} \quad z(t) = \sqrt{8t}$$

b)

$$L = \int_0^1 \|\vec{r}'(t)\| dt$$

$$\vec{r}(t) = \langle e^{2t}, e^{-2t}, \sqrt{8t} \rangle$$

$$\vec{r}'(t) = \left\langle \frac{d}{dt}(e^{2t}), \frac{d}{dt}(e^{-2t}), \frac{d}{dt}(\sqrt{8t}) \right\rangle$$

$$\vec{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, \sqrt{8} \rangle$$

$$L = \|\vec{r}'(t)\| = \int_0^1 \sqrt{(2e^{2t})^2 + (-2e^{-2t})^2 + (\sqrt{8})^2} dt$$

$$(2e^{2t})^2 + (-2e^{-2t})^2 + (\sqrt{8})^2 = (2e^{2t})^2 + (2e^{-2t})^2 + 8$$

$$(2e^{2t})^2 + (2e^{-2t})^2 + 8 = (2e^{2t} + 2e^{-2t})^2$$

$$L = \int_0^1 \sqrt{(2e^{2t} + 2e^{-2t})^2} dt \Rightarrow \int_0^1 (2e^{2t} + 2e^{-2t}) dt$$

$$2e^{2t} + 2e^{-2t} = 4 \cosh(2t)$$

$$L = \int_0^1 4 \cosh(2t) dt \Rightarrow 2 \sinh(2t) \Big|_0^1 \rightarrow 2 \sinh(2(1)) - 2 \sinh(2(0))$$

$$L = 2 \sinh(2) - 2 \sinh(0)$$

Question 7

Let C be the curve of intersection of the cylinder $x^2 + y^2 = 4$ and the plane $2x + y + z = 4$.

- (a) Find a parameterization of C .
- (b) Write down an integral for the length of C .

- (c) Find the length accurate to five decimal places by using Desmos: <https://www.desmos.com/calculator>. (Click on the keyboard icon, then “functions”, then “Misc”, to find the integral symbol.)

Solution:

a)

$$x^2 + y^2 = 4 \quad r = 2$$

$$x(t) = 2 \cos(t) \quad y(t) = 2 \sin(t)$$

$$2(2 \cos(t)) + 2 \sin(t) + z = 4 \Rightarrow z = 4 - 4 \cos(t) - 2 \sin(t)$$

$$x^2 + y^2 = 4 \quad r = 2 \quad z(t) = 4 - 4 \cos(t) - 2 \sin(t)$$

$$L = \int_a^b \sqrt{\left(\frac{d}{dt}x(t)\right)^2 + \left(\frac{d}{dt}y(t)\right)^2 + \left(\frac{d}{dt}z(t)\right)^2}$$

$$\vec{r}'(t) = \langle \rangle$$

Question 8

Find the velocity and position vectors of a particle that has acceleration given by

$$\vec{a}(t) = 2\hat{i} + 6t\hat{j} + 12t^2\hat{k},$$

and initial velocity and position given by

$$\vec{v}(0) = \hat{i} \quad \text{and} \quad \vec{r}(0) = \hat{j} - \hat{k}.$$

Solution:

Question 9

Consider the function $f(x, y) = \frac{\sqrt{y}-3x}{\ln(4-x^2-y^2)}$.

- Find and sketch the domain of f .
- On your sketch from part (a), mark where $f(x, y) = 0$, and indicate the region(s) where $f(x, y)$ is positive and negative.

Solution:

Question 10

Here are several surfaces.

Match each function with its graph. Justify your answers.

(a) $f(x, y) = x^2$

(b) $f(x, y) = \sqrt{x^2 + y^2}$

(c) $f(x, y) = e^{x^2+y^2} - 1$

(d) $f(x, y) = y \sin x$

(e) $f(x, y) = \sin(x + y)$

(f) $f(x, y) = \sin\left(\sqrt{x^2 + y^2}\right)$

Solution:

Question 11

Draw a contour map of the function $f(x, y) = x^2 e^{-y}$ showing several level curves.

Question 12

Match the function with its graph (labeled A-F below) and with its contour map (labeled I-VI). Give reasons for your choices.

(a) $z = e^x \cos y$

(b) $z = \sin x - \sin y$

(c) $z = \frac{x-y}{1+x^2+y^2}$

