Math 120

PSet 7

Oct 24 2024

Contents

Chapter 1 Page 2

1.1 PSet 7 2

Chapter 1

1.1 PSet 7

Question 1

Calculate the given iterated integrals.

1. $\int_0^1 \int_0^1 x\sqrt{1+4y}\,dy\,dx$
2. $\int_0^1 \int_1^2 \frac{xe^x}{y}\,dy\,dx$

Solution:
1)

$$\int_0^1 \int_0^1 x\sqrt{1+4y}\,dy\,dx$$

$$\int_0^1 x\sqrt{1+4y}\,dy$$

$$1+4y=t \quad r=dt$$

$$x\int_0^1 \frac{1}{4}\sqrt{t}dt$$

$$\frac{1}{4}x\int_0^1 \sqrt{t}dt$$

$$\frac{1}{4}x\cdot\frac{2t\sqrt{t}}{3}\bigg|_0^1$$

$$\frac{1}{4}x\cdot\frac{2(1+4y)\sqrt{1+4y}}{3}\Bigg|_0^1$$

$$\frac{x\sqrt{1+4y}(1+4y)}{6}\Bigg|_0^1$$

$$\frac{x\sqrt{1+4}(1+4)}{6}-\frac{x\sqrt{1}1}{6}$$

$$\frac{5x\sqrt{5}}{6}-\frac{x}{6}$$

$$\int_0^1 \frac{5x\sqrt{5}}{6}-\frac{x}{6}dx$$

$$\frac{1}{6}\int_0^1 5\sqrt{5}x-x\,dx$$

$$\frac{1}{6}\left(\int_0^1 5\sqrt{5}x dx - \int_o^1 x\,dx\right)$$

$$\int_0^1 5\sqrt{5}x\,dx \Rightarrow \left.\frac{5\sqrt{5}x^2}{2}\right|_0^1$$

$$\frac{5\sqrt{5}(1)^2}{2} - 0 = \frac{5\sqrt{5}}{2}$$

$$\int_0^1 x\,dx \Rightarrow \left.\frac{x^2}{2}\right|_0^1$$

$$\frac{1}{2} - 0 = \frac{1}{2}$$

$$\frac{1}{6}\left(\frac{5\sqrt{5}}{2} - \frac{1}{2}\right) = \frac{5\sqrt{5}-1}{12}$$

2)

$$\int_0^1 \int_1^2 \frac{xe^x}{y}\,dy\,dx$$

$$xe^x \int_1^2 \frac{1}{y}\,dy$$

$$xe^x \ln(y)\Big|_1^2 \Rightarrow xe^x \ln(2) - xe^x \ln(1) = xe^x \ln(2)$$

$$\ln(2) \int_0^1 xe^x\,dx$$

$$\ln(2)\ (xe^x - e^x)|_0^1$$

$$(\ln(2)e - \ln(2)e) - \left(\ln(2)(0) - \ln(2)e^0\right) = 0 - (-\ln(2)(1)) = \ln(2)$$

Question 2

(a) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^2 e^{-x^2-y^2}\,dy\,dx.$$

(b) Explain why

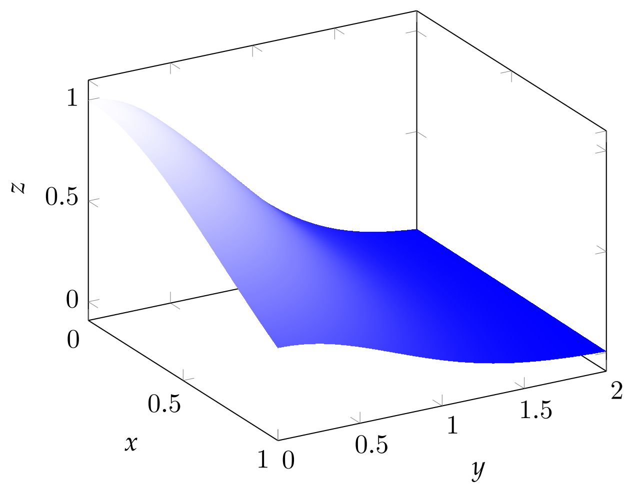
$$\int_0^1 \int_0^2 e^{-x^2-y^2}\,dy\,dx = \int_0^1 e^{-x^2}\,dx \cdot \int_0^2 e^{-y^2}\,dy.$$

(c) Use Desmos to compute

$$\int_0^1 \int_0^2 e^{-x^2-y^2}\,dy\,dx.$$

(Desmos will give a numerical approximation, but this is fine. In fact, there is no way to compute the antiderivatives necessary to get an exact answer.)

Solution:



b)
It is because $e^{-x^2y^2=e^{x^2}\cdot e^{y^2}}$ and the bounds of y are independent of x, so that allows e^{-x^2} to be treated as a constant when integrating with respect to y and vice versa.
c)

$$\int_0^1 \int_0^2 e^{-x^2-y^2}\, dy\, dx \approx 0.6588$$

Question 3

(a) Find the average value of the function $f(x, y) = \sin x \cos y$ on the rectangle $R = [0, \pi] \times [-\pi/2, \pi/2]$.

(b) Use symmetry to find the average value of $f(x, y) = \frac{4 \sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$ on the region $R = [2\pi, 4\pi] \times [2\pi, 6\pi]$. Please explain your answer carefully.

Solution: a)

$$f(x, y) = \sin x \cos y$$

$$R = [0, \pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$f_{avg} = \frac{1}{A(R)} \iint\limits_R f(x, y) dA$$

$$A(R) = (\pi - 0) \times (\frac{\pi}{2} - -\frac{\pi}{2}) = \pi^2$$

$$\frac{1}{\pi^2} \int_0^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos y\, dy\, dx$$

$$\sin x \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos y\, dy$$

$$(\sin x) \sin y \Big|_{\frac{pi}{2}}^{\frac{\pi}{2}}$$

$$(\sin x) \sin\left(\frac{\pi}{2}\right) - (\sin x) \sin\left(\frac{-\pi}{2}\right) = 2 \sin x$$

$$\int_0^{\pi} 2 \sin x dx$$

$$-2 \cos x|_0^{\pi}$$

$$-\cos \pi - (-2) \cos(0) = 4$$

$$\frac{1}{\pi^2}\cdot 4 = \frac{4}{\pi^2}$$

b)

$$f(x,y) = \frac{4\sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3$$

$$R = [2\pi, 4\pi]\times[2\pi, 6\pi]$$

$$f_{avg} = \iint\limits_R f(x,y)dA$$

$$A(R) = [4\pi - 2\pi]\times[6\pi - 2\pi] = 8\pi^2$$

$$\int_{2\pi}^{4\pi}\int_{2\pi}^{6\pi} \frac{4\sin y}{e^{x^2}} - \frac{\cos x}{\ln y} + 3\, dy\, dx$$

$$\iint\limits_R f(x,y)dA - \iint\limits_R \frac{4\sin y}{e^{x^2}}dA - \iint\limits_R \frac{\cos x}{\ln y}dA + \iint\limits_R 3dA$$

$$\int_{2\pi}^{6\pi} 4\sin y\, dy = -4\left[\cos y\right]_{2\pi}^{6\pi} = -4(6\cos\pi - \cos 2\pi) = -4(1-1) = 0$$

$$\iint\limits_R f(x,y)\frac{4\sin y}{e^{x^2}}dA = \int_{2\pi}^{4\pi} \frac{1}{e^{x^2}}dx \times 0 = 0$$

$$\int_{2\pi}^{4\pi} \cos x dx = \sin x|_{2\pi}^{4\pi} = sin 4\pi - \sin 2\pi = 0 - 0 = 0$$

$$\iint\limits_R \frac{\cos x}{\ln y}dA = \int_{2\pi}^{6\pi} \frac{1}{\ln y} \times 0 = 0$$

$$\iint\limits_R 3dA = 3\times A(R) = 3\times 8\pi^2 = 24\pi^2$$

$$\frac{24\pi^2}{8\pi^2} = 3$$

Question 4

In each part, draw the region D, and evaluate the integral.

1. $\iint_D \frac{y}{x^5+1}\, dA$, where D is the region $D = \{(x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x^2\}$.
2. $\iint_D x^3\, dA$, where $D = \{(x,y) \mid 1 \leqslant x \leqslant e,\ 0 \leqslant y \leqslant \ln x\}$.

Solution: 1.

$$\iint_D \frac{y}{x^5+1}\, dA \quad D = \{(x,y) \mid 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant x^2\}$$

$$\int_0^1\int_0^{x^2} \frac{y}{x^5+1}\, dy\, dx$$

$$\frac{1}{x^5+1}\int_0^{x^2} y dy$$

$$\frac{y^2}{2}\Big|_0^{x^2} \Rightarrow \frac{(x^2)^2}{2} - \frac{0}{2} = \frac{x^4}{2}$$

$$\int_0^1 \frac{1}{x^5+1}\times\frac{x^4}{2}dx$$

$$x^5 + 1 = t \quad dt = 5x^4 dx$$

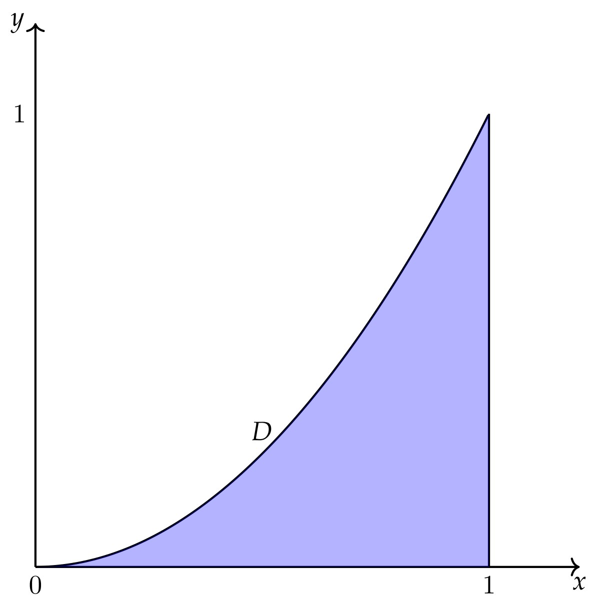
$$\frac{1}{10}\int_0^1 \frac{1}{t} dt$$

$$\frac{1}{10}\ln|t|\Big|_0^1$$

$$\frac{1}{10}|x^5 + 1|\Big|_0^1$$

$$\frac{1}{10}\ln(1^5 + 1) - \frac{1}{10}\ln(1)$$

$$\frac{1}{10}\ln(2) - \frac{1}{10}\ln(1) = \frac{1}{10}\ln(2)$$



2.

$$\iint_D x^3\, dA \quad D = \{(x, y) \mid 1 \leqslant x \leqslant e,\ 0 \leqslant y \leqslant \ln x\}$$

$$\int_1^e \int_0^{\ln x} x^3\, dy\, dx$$

$$x^3 \int_0^{\ln x} 1 dy$$

$$\left(x^3\right) y\Big|_0^{\ln x}$$

$$x^3 \ln x - 0$$

$$uv - \int v\, du$$

$$u = \ln x \quad du = \frac{1}{x}\, dx$$

$$v = \frac{x^4}{4} \quad x^3 dx$$

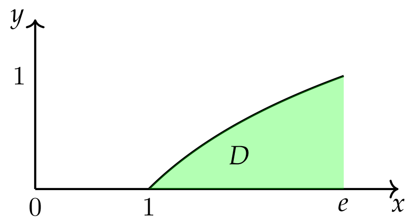
$$\frac{\ln x \cdot x^4}{4} - \int \frac{x^3}{4}\, dx$$

$$\left[\frac{\ln x^4 \cdot x^4}{4} - \frac{x^4}{16}\right]_1^e$$

$$\left(\frac{\ln e \cdot e^4}{4} - \frac{e^4}{16}\right) - \left(\frac{\ln 1 \cdot 1^4}{4} - \frac{1^4}{16}\right)$$

$$\left(\frac{\ln e \cdot e^4}{4}\right) - \left(0 - \frac{1}{16}\right)$$

$$\left(\frac{e^4}{4} - \frac{e^4}{16}\right) + \frac{1}{16}$$

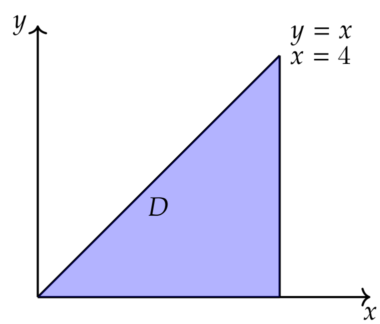


Question 5

Draw the region D. Set up the iterated integrals for both orders of integration. Then evaluate the double integral using the easier order and explain why it's easier.

$$\iint_D x^2 e^{-xy}\, dA \quad \text{where } D \text{ is bounded by } y = x,\ x = 4, \text{ and } y = 0.$$

Solution:



$$\int_0^4 \int_0^x x^2 e^{-yx}\, dy\, dx$$

$$\int_0^4 \int_y^4 x^2 e^{-yx}\, dx\, dy$$

$$x^2 \int_y^4 e^{-yx} dy$$

$$\left(x^2\right) \frac{e^{-yx}}{x}\Big|_0^x \left(x^2\right) \frac{e^{-yx}}{x} - \left(x^2\right) \frac{e^{-0 \cdot x}}{x}$$

$$xe^{-x^2} - x^2 \cdot \frac{1}{x} \Rightarrow -xe^{-x^2} + x$$

$$\int_0^4 -xe^{-x^2} + x\, dx \Rightarrow \int_0^4 -xe^{-x^2}\, dx \int_0^4 x\, dx$$

$$-x^2 = t \quad -2x = dt$$

$$\int_0^4 \frac{1}{2} e^t\, dt \Rightarrow \frac{1}{2} \int_0^4 e^t\, dt$$

$$\frac{1}{2}e^t\Big|_0^4 \Rightarrow \frac{1}{2}^{x^2}\Big|_0^4$$

$$\frac{1}{2}e^{-4^2} - \frac{1}{2}e^{-0^2} \Rightarrow \frac{1}{2}e^{-16} - \frac{1}{2}$$

$$\int_0^4 x\,dx$$

$$\frac{x^2}{2}\Big|_0^4$$

$$\frac{4^2}{2} - \frac{0^2}{2} = \frac{16}{2} = 8$$

$$\int_0^4 -xe^{-x^2} + x\,dx = \frac{1}{2}e^{-16} + \frac{15}{2}$$

Question 6

(a) Find the volume of the solid in the first octant enclosed by the parabolic cylinder $y = 1 - x^2$ and the planes $z = 2 - y$ and $z = y$.

(b) Sketch the solid whose volume is given by the iterated integral

$$\int_0^1 \int_0^{1-x} (2 - y^2)\,dy\,dx.$$

Solution:

a)

$$y = -x^2 \quad z = 2 - y \quad z = y$$

$$x, y, z \geqslant 0$$

$$2 - y = y \Rightarrow 2y = 2 \Rightarrow y = 1 \Rightarrow 0 \leqslant y \leqslant 1 - x^2$$

$$\text{height } = (2 - y) - y \Rightarrow 2 - 2y$$

$$V = \int_0^1 \int_0^{1-x^2} 2 - 2y\,dy\,dx$$

$$\int_0^{1-x^2} 2 - 2y\,dy$$

$$2y - y^2\Big|_0^{1-x^2}$$

$$2\left(1 - x^2\right) - \left(1 - x^2\right) - 2(0) - (0)^2$$

$$2 - 2x^2 - 1 + 2x^2 - x^4$$

$$1 - x^4$$

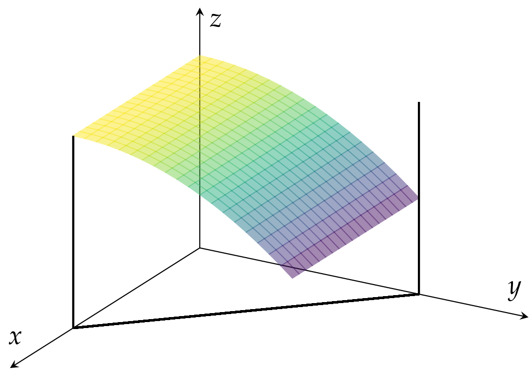
$$\int_0^1 1 - x^4 dx$$

$$x - \frac{x^5}{5}\Big|_0^1$$

$$1 - \frac{1^5}{5} - 0 - \frac{0^5}{5}$$

$$1 - \frac{1}{5} = \frac{4}{5}$$

b)



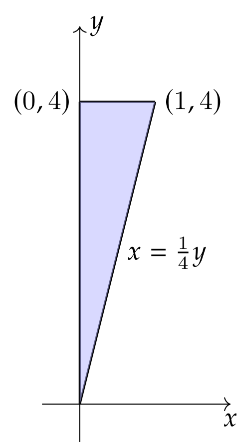
Question 7

Sketch the region of integration and change the order of integration.

1. $\int_0^1 \int_{4x}^4 f(x,y)\,dy\,dx$
2. $\int_0^3 \int_0^{\sqrt{9-y}} f(x,y)\,dx\,dy$
3. $\int_0^4 \int_0^{\ln 2x} f(x,y)\,dy\,dx$

Solution:

a)



$$\int_0^1 \int_{4x}^4 f(x,y)\,dy\,dx$$

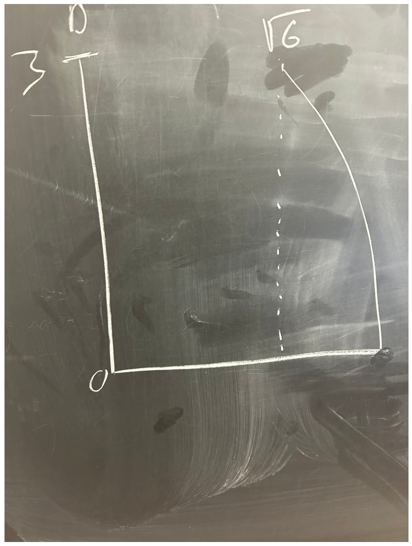
$$\iint\limits_D f(x,y)dA$$

$$D = \{(x,y)|0 \leqslant x \leqslant 1, 4x \leqslant y \leqslant 4\}$$

$$D = \{(x,y)|0 \leqslant y \leqslant 4, 0 \leqslant x \leqslant \frac{1}{4}y\}$$

$$\iint_D f(x,y)dA = \int_0^4 \int_0^{\frac{1}{4}y} f(x,y)\,dx\,dy$$

b)



$$\int_0^3 \int_0^{\sqrt{9-y}} f(x,y)\,dx\,dy$$

$$\iint_D f(x,y)dA$$

$$D = \{(x,y)|0 \leqslant x \leqslant \sqrt{9-y}, 0 \leqslant y \leqslant 3\}$$

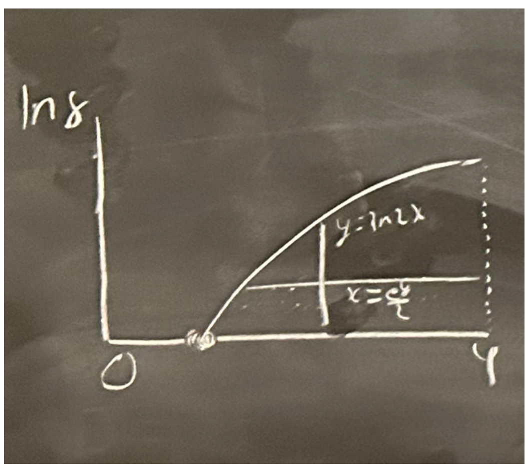
$$x = \sqrt{9-y} \quad x^2 - 9 = -y \quad -x^2 + 9 = y$$

$$-x^2 + 9 = 3 \quad -x^2 = -6 \Rightarrow x^2 = 6 \Rightarrow x = \sqrt{6}$$

$$D = \{(x,y)|0 \leqslant y \leqslant -x^2 + 9, \sqrt{6} \leqslant x \leqslant 3\}$$

$$\int_{\sqrt{6}}^3 \int_0^{-x^2+9} f(x,y)\,dy\,dx$$

c)



$$\int_0^4 \int_0^{ln2x}$$

$$\iint_D f(x,y)dA \quad D = \{0 \leqslant x \leqslant 4, 0 \leqslant y \leqslant lnx\}$$

$$y = ln2x \Rightarrow y = ln2(4) \Rightarrow y = ln8 \Rightarrow 0 = ln2x \Rightarrow ln1 = \ln 2x2x \Rightarrow x = \frac{1}{2}$$

$$\iint_D f(x,y)dA \quad D = \{(x,y)|0 \leqslant y \leqslant \ln 8, \frac{1}{2} \leqslant x \leqslant \frac{e^y}{2}\}$$

$$\int_0^{\ln 8} \int_{\frac{1}{2}}^{\frac{e^y}{2}} f(x,y)\, dx\, dy$$

Question 8

Evaluate the integral

$$\int_0^1 \int_x^1 e^{\frac{x}{y}}\, dy\, dx$$

by reversing the order of integration.

Solution:

$$\int_0^1 \int_0^y e^{\frac{x}{y}} dxdy$$

$$ye^{\frac{x}{y}}\Big|_0^y$$

$$\int_0^1 ye - ydy$$

$$\frac{ey^2}{2} - \frac{y^2}{2}\Big|_0^1$$

$$\frac{e-1}{2}$$

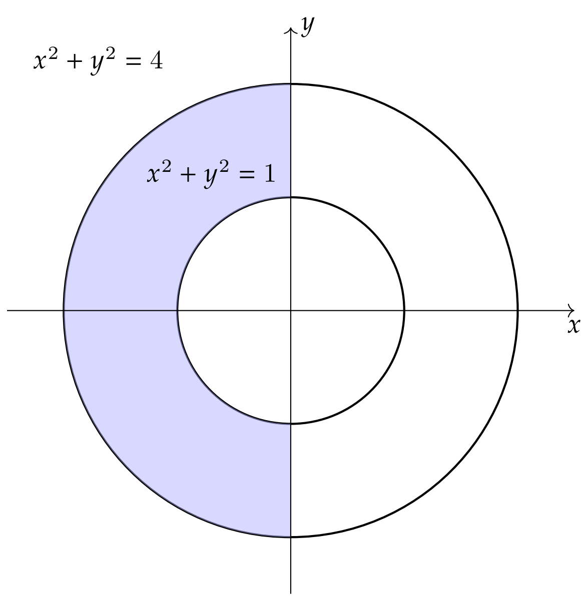
Question 9

Evaluate the given integral by converting to polar coordinates. Be sure to draw the region of integration in each part.

1. $\iint_R (x+y)\, dA$, where R is the region that lies to the left of the y-axis between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

2. $\iint_R ye^x\, dA$, where R is the region in the first quadrant enclosed by the circle $x^2+y^2=25$.

Solution: a)



$$x \leqslant 0 \quad x + y^2 = 1 \quad x^2 + y^2 = 4$$

$$x = r\cos\theta \quad y = r\sin\theta \quad dA = r dr\, d\theta$$

$$R : 1 \leqslant r \leqslant 2$$

$$x + y = r\cos\theta + r\sin\theta \Rightarrow r(\cos\theta \sin\theta)$$

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^2 r(\cos\theta \sin\theta) r\, dr\, d\theta$$

$$\cos\theta + \sin\theta \int_1^2 r^2 dr$$

$$\left.\frac{r^3}{3}\right|_1^2$$

$$\frac{2^3}{3} - \frac{1}{3} = \frac{7}{3}$$

$$\frac{7}{3}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos\theta + \sin\theta\, d\theta$$

$$\frac{7}{3}\sin\theta\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \frac{7}{3}\cos\theta\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

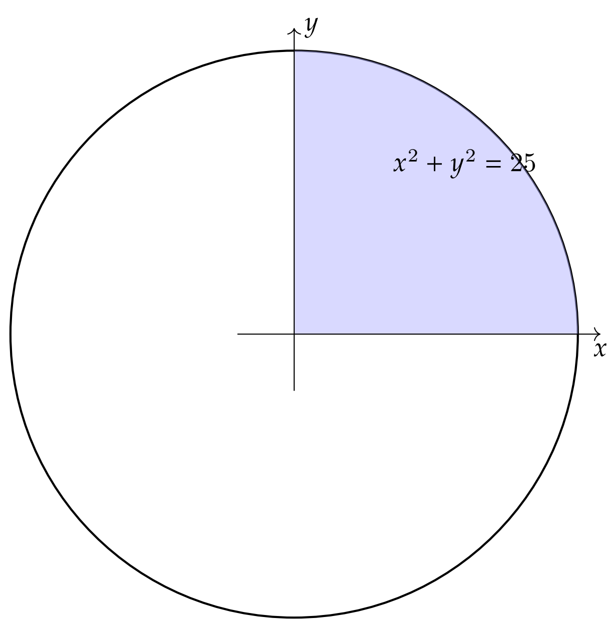
$$\frac{7}{3}(-1) - \frac{7}{3}(1) = -\frac{14}{3}$$

$$-\frac{7}{3}\cos\frac{3\pi}{2} + \frac{7}{3}\cos\frac{\pi}{2}$$

$$-\frac{7}{3}(0) + \frac{7}{3} = 0$$

$$-\frac{14}{3}$$

b)



$$\iint_R ye^x dA$$

$$x = r\cos\theta \quad y = r\sin\theta \quad x^2 + y^2 = 25$$

$$\text{1st quadrant} \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

$$ye^x \Rightarrow r\sin\theta e^{r\cos\theta}$$

$$dA = r\,dr\,d\theta$$

$$\int_0^{\frac{\pi}{2}} \int_0^5 r\sin\theta e^{r\cos\theta}$$

$$I(\theta) = \int_0^5 r^2 \sin\theta\, e^{r\cos\theta}\, dr$$

$$\frac{\partial}{\partial\theta} e^{r\cos\theta} = -r\sin\theta\, e^{r\cos\theta}$$

$$r\sin\theta e^{r\cos\theta} = -\frac{\partial}{\partial\theta} e^{r\cos\theta}$$

$$I = \int_0^{\frac{\pi}{2}} \int_0^5 -r\frac{\partial}{\partial\theta} e^{r\cos\theta}\, dr\, d\theta$$

$$I = -\int_0^5 r\left(\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial\theta} e^{r\cos\theta}\, d\theta\right) dr$$

$$\int_0^{\frac{\pi}{2}} \frac{\partial}{\partial\theta} e^{r\cos}\, d\theta = \left. e^{r\cos\frac{\pi}{2}} \right|_0^{\frac{\pi}{2}}$$

$$e^{r\cos\frac{\pi}{2}} - e^{r\cos\theta} = e^{r\cdot 0} - e^{r\cdot 1} = 1 - e^r$$

$$I = \int_0^5 re^r dr - \int_0^3 r dr$$

$$u = r \quad du = dr \quad v = e^r \quad dv = e^r dr$$

$$\int_0^5 re^r dr = re^r - \int_0^5 e^r dr = re^r - e^r + K$$

$$\int_0^5 re^r dr = [re^r - e^r]_0^5 = \left(5e^5 - e^5\right) - (0 - e^0) = 4e^5 + 1$$

$$\int_0^5 r dr = \left. \frac{1}{2} r^2 \right|_0^5 = \frac{1}{2}(25 - 0) = \frac{25}{2}$$

$$I = (4e^5 + 1) - \frac{25}{2} = 4e^5 - 11.5$$

Question 10

Use polar coordinates to find the volume of the given solid.

(a) Inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.

(b) Bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

Solution: a)

$$x^2 + y^2 + z^2 = 4 \quad x^2 + y^1 = 1$$

$$x = r\cos\theta \quad y = r\sin\theta \quad 0 \leqslant \theta \leqslant 2\pi$$

$$r^2 + z^2 = 4 \quad x^2 + y^2 = 1 \Rightarrow r^2 = 1 \Rightarrow r = 1$$

$$r^2 + z^2 = 4 \Rightarrow r^2 = 4 - z^2 \Rightarrow r = \sqrt{4 - z^2}$$

$$V = \iint_D \left[z_{\text{upper}} - z_{\text{lower}}\right]$$

$$V = \int_0^{2\pi}\int_1^2 r\sqrt{4 - r^2 - \left(-\sqrt{4 - r^2}\right)}r\,dr\,d\theta$$

$$V = 2\int_0^{2\pi}\int_1^2 \sqrt{4 - r^2}\,dr\,d\theta$$

$$4 - r^2 = t \qquad -2r = dt$$

$$-\int_1^2 \frac{1}{2}\sqrt{t}dt \Rightarrow -\frac{1}{2}\int_1^2 \sqrt{t}dt = -\frac{1}{2}\cdot\frac{2t\sqrt{t}}{3}\Bigg|_1^2$$

$$-\frac{1}{2}\cdot\frac{2(4-r^2)\sqrt{4-r^2}}{3}\Bigg|_1^2 \left(-\frac{1}{2}\cdot\frac{2(4-2^2)\sqrt{4-2^2}}{3}\right) - \left(-\frac{1}{2}\cdot\frac{2(4-1^2)\sqrt{4-1^2}}{3}\right)$$

$$V = 2\left(\int_2^{2\pi} d\theta\right)\left(\int_1^2 r\sqrt{4-r^2}dr\right)$$

$$\int_0^{2\pi} d\theta = 2\pi \quad V = 2\cdot 2\pi\cdot\int_1^2 r\sqrt{4-r^2}dr = 4\pi\int_1^2 r\sqrt{4-r^2}dr$$

$$-\left(-\frac{1}{2}\cdot\frac{2(3)\sqrt{3}}{3}\right) \Rightarrow \left(\frac{1}{2}\sqrt{3}\right) \Rightarrow -(-\sqrt{3})$$

$$4\pi\sqrt{3}$$

b)

$$z = 3x^2 3y^2 = 3\left(x^2 + y^2\right) = 3\left(x^2 + y^2\right) = 3r^2$$

$$z = 4 - x^2 - y^2 = 4 - r^2$$

$$3r^2 = 4 - r^2$$

$$4r^2 = 4 \Rightarrow r = 1$$

$$\int_0^{2\pi}\int_0^1 \left(4 - r^2 - 3r^2\right) r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 \left(4 - 4r^2\right) r\,dr\,d\theta$$

$$2r - r^4\Big|_0^1 d\theta \Rightarrow \left(2(1)^2 - 1\right) - 0$$

$$\int_0^{2\pi} 1\,d\theta$$

$$\theta\Big|_0^{2\pi} = 2\pi$$

Question 11

Evaluate the iterated integral

$$\int_0^b \int_{-\sqrt{b^2-y^2}}^0 x^2 y \, dx \, dy$$

by converting to polar coordinates.

Solution:

$$\int_0^b \int_{-\sqrt{b^2-y^2}}^0 x^2 y \, dx \, dy$$

$$y = 0 \quad \text{to} \quad y = b$$

$$x = -\sqrt{b^2 - y^2}\text{to} \quad \text{to } x = 0$$

$$\text{left half of } x^2 + y^2 = b^2$$

$$x = r\cos\theta \quad y = r\sin\theta \quad 0 \leqslant r \leqslant b \quad \frac{\pi}{2} \leqslant \theta\pi$$

$$x^y = (r\cos\theta)^2(r\sin\theta) = r^3\cos\theta\sin\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} \int_0^b \left(r^3\cos^2\theta\sin\theta\right) r \, dr \, d\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} \int_0^b r^4\cos^2\theta\sin\theta \, d\theta \, dr$$

$$\cos^2\theta\sin\theta \int_0^b r^4 \, dr$$

$$\cos^2\theta\sin\theta \frac{r^5}{5}\bigg|_0^b$$

$$\cos^2\theta\sin\theta\frac{b^5}{5} - 0$$

$$\int_{\frac{\pi}{2}}^{\pi} \frac{b^5}{5}\cos^2\sin\theta \, d\theta$$

$$\frac{b^5}{5} int_{\frac{\pi}{2}}^{\pi} \cos^2\sin\theta$$

$$\frac{b^5}{5}\frac{\cos^3\theta}{3}\bigg|_{\frac{\pi}{2}}^{\pi}$$

$$\left(\frac{b^5}{5}\right)\left(\frac{\cos^3\frac{\pi}{2}}{3}\right) - \left(\frac{b^5}{5}\right)\left(\frac{\cos^3\pi}{3}\right)$$

$$0 - \frac{b^5}{5} \cdot \frac{-1}{3}$$

$$\frac{b^5}{15}$$

Question 12

Let D be the disk with center at the origin and radius a.

(a) Use your intuition: what do you expect is the average distance from points on the disk to the origin?

- less than $a/2$
- $a/2$
- between $a/2$ and a
- more than a

Give an intuitive explanation of your answer.

(b) What is the average distance from points in the disk to the origin?

Solution: The area of the disk should be greater on the interval of $\left[\frac{a}{2}, a\right]$ than from $\left[0, \frac{a}{2}\right]$ which means there are more points on the interval of $\left[\frac{a}{2}, a\right]$ meaning hte average distance is on this interval.
b)

$$D = \frac{1}{A} \iint_A d \; da$$

$$D = \frac{1}{A} \int_0^{2\pi} \int_0^a r \cdot r \, dr \, d\theta$$

$$\left.\frac{r^3}{3}\right|_0^a \Rightarrow \frac{a^3}{3} - 0 = \frac{a^3}{3}$$

$$A = a^2\pi$$

$$\frac{1}{a^2\pi} \int_0^{2\pi} \frac{a^3}{3} \, d\theta$$

$$\frac{1}{a^2\pi} \left.\left(\frac{a^3}{3}\right)\right|_0^{2\pi}$$

$$\frac{2a}{3}$$