Math 120

Final Review

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12.1 Three-Dimensional Coordinate Systems

3D Space

Definition 0.0.1: Three Dimensional Rectangular Coordinate System

The carstesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \left[(x,y,z) \mid x,y,z \in \mathbb{R}\right].$ is the set of all real numbers and is denoted by $\mathbb{R}$

Surfaces and Solids

Note:-
In three-dimensional analytic, an equation in x, y and, z represents a surface in $\mathbb{R}$

Distance and Spheres

Definition 0.0.2: Distance Formula in Three Dimensions

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_1, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

12.2 Vectors

Definition 0.0.3: Vectors

A *vector* indicates a quantity that has both magnitude and direction

Geometric Description of Vectors

Definition 0.0.4: Vector Addition

For vectors **Vector Addition** means the sum of two vectors $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$, with $\mathbf{v}$ placed so its tail starts at the tip of $\mathbf{u}$..

Definition 0.0.5: Triangle Law

For vectors that **Triangle Law** refers to the sum of two vectors representing the third side of a triangle by placing the second vector's tail at the first vector's tip.

Definition 0.0.6: Parallelogram Law

For vectors **Parallelogram Law** refers to the sum of two vectors is representing the diagonal of a parallelogram formed when both vectors originate from the same point.

Definition 0.0.7: Scalar Multiplication

For vectors **Scalar Multiplication** means that if c is a scalar and $\mathbf{v}$ is a vector, the scalar multiple $c\mathbf{v}$ is a vector whose length is $|c|$ times the length of $\mathbf{v}$. Its direction matches $\mathbf{v}$ if $c > 0$, is opposite to $\mathbf{v}$ if $c < 0$, and is $\mathbf{0}$ if $c = 0$ or $\mathbf{v} = \mathbf{0}$.

Components of Vectors

Definition 0.0.8: Vector Representation

Given points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector $\mathbf{a}$ representing the directed segment $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle.$$

Definition 0.0.9: Magnitude of a Vector

The length of a vector $\mathbf{a}$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2} \quad (2D), \quad |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} \quad (3D).$$

Note:-

Vector Addition: Add corresponding components:

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle \quad (2D), \quad \mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle \quad (3D).$$

Vector Subtraction: Subtract corresponding components:

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle \quad (2D), \quad \mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle \quad (3D).$$

Scalar Multiplication: Multiply each component by the scalar:

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle \quad (2D), \quad c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle \quad (3D).$$

Definition 0.0.10: Unit Vector

A vector whose magnitude is 1

12.3 The Dot Product

Dot Product of Two Vectors

Definition 0.0.11: Dot Product

The *dot product* of two vectors $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is a scalar given by:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

Theorem 0.0.1 Angle Between Vectors

The dot product of two vectors $\mathbf{a}$ and $\mathbf{b}$ relates their magnitudes and the cosine of the angle θ between them:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta.$$

This formula shows how the dot product measures both the lengths of the vectors and their alignment in space.

Corollary 0.0.1 Angle Between Vectors

If θ is the angle between the nonzero vectors $\mathbf{a}$ annd $\mathbf{b}$, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Note:-

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$

Direction Angles and Direction Cosines

Definition 0.0.12: Direction Cosines

The direction angles (α, β, γ) of a nonzero vector $\mathbf{a}$ are the angles it makes with the positive x-, y-, and z-axes, respectively. The cosines of these angles $(\cos \alpha, \cos \beta, \cos \gamma)$ are called the direction cosines, and they satisfy:

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Note:-

The vector $\mathbf{a}$ can be expressed in terms of its magnitude and direction cosines as:

$$\mathbf{a} = |\mathbf{a}|\langle \cos \alpha, \cos \beta, \cos \gamma \rangle,$$

and the unit vector in the direction of $\mathbf{a}$ is:

$$\frac{1}{|\mathbf{a}|}\mathbf{a} = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle.$$

Projections

> **Definition 0.0.13: Projections**
>
> The **scalar projection** of **b** onto **a**, denoted $\text{comp}_\mathbf{a}\mathbf{b}$, represents the magnitude of **b** in the direction of **a**. It is the length of the "shadow" of **b** onto **a** and is given by:
>
> $$\text{comp}_\mathbf{a}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}.$$
>
> The **Vector Projection:** The vector projection of **b** onto **a**, denoted $\text{proj}_\mathbf{a}\mathbf{b}$, represents the vector component of **b** in the direction of **a**. It is the scaled vector of **a** that aligns with the projection, given by:
>
> $$\text{proj}_\mathbf{a}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}.$$

12.4 The Cross Product

The Cross Product of Two Vectors

> **Definition 0.0.14: Cross Product**
>
> The **cross product** of two vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ is a vector $\vec{a} \times \vec{b}$ defined as:
>
> $$\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$$
>
> The cross product is a vector:
>
> - **Perpendicular** to both $\vec{a}$ and $\vec{b}$.
>
> - **Magnitude** equal to $\|\vec{a}\|\|\vec{b}\| \sin\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$.
>
> - **Direction** determined by the right-hand rule.

Properties of the Cross Product

Theorem 0.0.2 Resulting Vector
The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both **a** and **b**

Theorem 0.0.3 Geometric Description of the Length of the Cross Product
If θ is the angle between **a** and **b** (s0 $0 \leqslant \theta \leqslant \pi$), then the length of the cross product $\mathbf{a} \times \mathbf{b}$ is given by

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||texbtb| \sin\theta$$

Corollary 0.0.2 Parallellity using Cross Product
Two nonzero vectors **a** and **b** are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = 0$$

> **Note:-**
> The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by **a** and **b**

Triple Products

> **Definition 0.0.15: Triple Product**
>
> The **scalar triple product** of three vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is given by:
> $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}),$$
>
> which can also be expressed as the determinant of the matrix formed by placing the vectors as rows (or columns):
> $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$
>
> - The magnitude of the scalar triple product, $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$, represents the **volume** of the parallelepiped formed by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.
>
> - If $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$, then the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are **coplanar**, meaning they lie in the same plane.

12.5 Equations of Lines and Planes

Lines

> **Definition 0.0.16: Line in 3D Space**
>
> A line in three-dimensional space is defined by:
>
> - A point $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$,
>
> - A direction vector $\mathbf{v} = \langle a, b, c \rangle$.
>
> Its vector equation is:
> $$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v},$$
>
> where t is a scalar parameter.
> In parametric form, the equation becomes:
>
> $$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc.$$

Planes

> **Definition 0.0.17: Plane in 3D Space**
>
> A plane in three-dimensional space is defined by:
>
> - A point $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ on the plane,
>
> - A normal vector $\mathbf{n} = \langle a, b, c \rangle$, orthogonal to the plane.
>
> The vector equation of the plane is:
> $$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0,$$
>
> where $\mathbf{r} = \langle x, y, z \rangle$ is the position vector of any point on the plane.
> The scalar equation of the plane is:
>
> $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$
>
> where (x_0, y_0, z_0) is a point on the plane, and $\mathbf{n} = \langle a, b, c \rangle$ is the normal vector.

> **Note:-**
>
> Two planes are parallel if their normal vectors are parallel
>
> If two planes are not parallel, then they intersect in a straight line and the angle between the two planes is defined as the acute angle between their normal vectors

Distances

> **Definition 0.0.18: Distances**
>
> The distance D from a point $P_1(x_1, y_1, z_1)$ to a plane given by the equation $ax + by + cz + d = 0$ is:
>
> $$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.$$
>
> **Explanation**
>
> - (x_1, y_1, z_1) are the coordinates of the point P_1,
>
> - a, b, c, and d are the constants defining the plane equation,
>
> - $\sqrt{a^2 + b^2 + c^2}$ represents the magnitude of the normal vector $\mathbf{n} = \langle a, b, c \rangle$.

13 Vector Functions

13.1 Vector Functions and Space Curves

Vector-Valued Functions

Definition 0.0.19: Vector-Valued Functions

A **vector-valued function** is a function whose domain is a set of real numbers and whose range is a set of vectors. In three dimensions, a vector-valued function $\mathbf{r}(t)$ can be expressed as:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where $f(t)$, $g(t)$, and $h(t)$ are the component functions of $\mathbf{r}(t)$, and t is the independent variable.

Space Curves

Definition 0.0.20: Space Curves

A **space curve** is the set of all points $C(x, y, z)$ in space traced by a continuous vector-valued function $\mathbf{r}(t)$, where:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle,$$

and t varies over an interval I.
The parametric equations of the space curve are:

$$x = f(t), \quad y = g(t), \quad z = h(t).$$

A space curve represents the path of a moving particle whose position at time t is given by $(f(t), g(t), h(t))$.

13.2 Derivatives and Integrals of Line Functions

Derivatives

Definition 0.0.21: Derivative of a Vector Function

The derivative of a vector function $\mathbf{r}(t)$, denoted as $\mathbf{r}'(t)$, is defined as:

$$\mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t + h) - \mathbf{r}(t)}{h},$$

provided the limit exists.

- $\mathbf{r}'(t)$ represents the **tangent vector** to the curve defined by $\mathbf{r}(t)$ at a point.

- The tangent line to the curve at this point is parallel to $\mathbf{r}'(t)$.

> **Theorem 0.0.4** Derivatives
>
> If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions, then:
>
> $$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

> **Theorem 0.0.5** Orthogonnality from derivative
>
> If $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Integrals

> **Definition 0.0.22: Definite Integral**
>
> The **definite integral** of a continuous vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ over the interval $[a, b]$ is defined as:
>
> $$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt \right)\mathbf{i} + \left(\int_a^b g(t)\,dt \right)\mathbf{j} + \left(\int_a^b h(t)\,dt \right)\mathbf{k}.$$
>
> Conceptually, the integral computes the accumulation of the vector values of $\mathbf{r}(t)$ over the interval, integrating each component function $f(t)$, $g(t)$, and $h(t)$ independently.
>
> The Fundamental Theorem of Calculus for vector functions states:
>
> $$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(b) - \mathbf{R}(a),$$
>
> where $\mathbf{R}(t)$ is an antiderivative of $\mathbf{r}(t)$, satisfying $\mathbf{R}'(t) = \mathbf{r}(t)$.

13.3 Arc Length and Curvature

Arc Length

> **Definition 0.0.23: Arc Length**
>
> The **arc length** L of a curve defined by a vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ over the interval $[a, b]$ is:
>
> $$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt,$$
>
> or equivalently:
>
> $$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt.$$
>
> In compact form, the arc length can be written using the magnitude of the derivative:
>
> $$L = \int_a^b |\mathbf{r}'(t)|\,dt,$$
>
> where $|\mathbf{r}'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$ represents the speed of the particle moving along the curve.

The Arc Length Function

Definition 0.0.24: Arc Length Function

The **arc length function** $s(t)$ measures the length of the curve from the starting point to a point on the curve corresponding to parameter t. For a curve defined by a vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, the arc length function is given by:

$$s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du,$$

where $|\mathbf{r}'(u)|$ represents the magnitude of the derivative of $\mathbf{r}(u)$.
Differentiating $s(t)$ with respect to t yields:

$$\frac{ds}{dt} = |\mathbf{r}'(t)|.$$

This formula connects the arc length function to the speed of motion along the curve.

Curvature

Definition 0.0.25: Curvature

The **curvature** κ of a curve at a given point measures how quickly the direction of the curve changes at that point. It is defined as the magnitude of the rate of change of the unit tangent vector $\mathbf{T}(t)$ with respect to the arc length s:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|,$$

where the unit tangent vector is given by:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

Using the chain rule, curvature can be expressed in terms of the parameter t:

$$\kappa = \frac{\left|\frac{d\mathbf{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|}.$$

Since $\frac{ds}{dt} = |\mathbf{r}'(t)|$, the curvature formula becomes:

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Theorem 0.0.6 Curvature

The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

The Normal and Binormal Vectors

Definition 0.0.26: The Normal Vectors

For a smooth space curve $\mathbf{r}(t)$, the **unit normal vector** $\mathbf{N}(t)$ and the **binormal vector** $\mathbf{B}(t)$ are defined as follows:

The unit normal vector $\mathbf{N}(t)$ is the normalized derivative of the unit tangent vector $\mathbf{T}(t)$:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|},$$

where the unit tangent vector is given by:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}.$$

The vector $\mathbf{N}(t)$ indicates the direction in which the curve is turning.

Definition 0.0.27: Binormal Vectors

For a smooth space curve $\mathbf{r}(t)$, the **unit normal vector** $\mathbf{N}(t)$ and the **binormal vector** $\mathbf{B}(t)$ are defined as follows:

The binormal vector $\mathbf{B}(t)$ is the cross product of the tangent vector $\mathbf{T}(t)$ and the normal vector $\mathbf{N}(t)$:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

The binormal vector $\mathbf{B}(t)$ is orthogonal to both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ and is also a unit vector.

Planes

Definition 0.0.28: Normal Plane

The **normal plane** of a curve C at a point P is the plane determined by the normal vector $\mathbf{N}(t)$ and the binormal vector $\mathbf{B}(t)$. It is orthogonal to the tangent vector $\mathbf{T}(t)$ and consists of all lines passing through P that are orthogonal to $\mathbf{T}(t)$.

Definition 0.0.29: Osculating Planes

The **osculating plane** of a curve C at a point P is the plane determined by the tangent vector $\mathbf{T}(t)$ and the normal vector $\mathbf{N}(t)$. It is the plane that comes closest to containing the part of the curve near P.

Definition 0.0.30: Circle of Curvature

The **circle of curvature**, or **osculating circle**, of C at P is the circle that lies in the osculating plane, passes through P, and has a radius of $\frac{1}{\kappa}$, where κ is the curvature of the curve at P. The center of the circle is located a distance of $\frac{1}{\kappa}$ along the direction of the normal vector $\mathbf{N}(t)$. The circle of curvature best describes how the curve C behaves near P, sharing the same tangent, normal, and curvature at P.

13.4 Motion in Space: Veolcity and Acceleration

Velolcity, Speed and Acceleration

> **Definition 0.0.31: Velocity and Speed**
>
> The **velocity vector** $\mathbf{v}(t)$ of a particle moving through space is defined as the derivative of its position vector $\mathbf{r}(t)$ with respect to time:
>
> $$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t).$$
>
> It represents the direction and rate of change of the particle's position and points along the tangent to the particle's path.
>
> The **speed** of the particle at time t is the magnitude of the velocity vector:
>
> $$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt}.$$
>
> It represents the rate of change of distance with respect to time.

> **Definition 0.0.32: Acceleration Vector**
>
> The **acceleration vector** $\mathbf{a}(t)$ is defined as the derivative of the velocity vector $\mathbf{v}(t)$ with respect to time:
>
> $$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$
>
> It represents the rate of change of the velocity of the particle.

14 Partial Derivatives

14.1 Functions of Several Variables

Functions of Two Variables

> **Definition 0.0.33: Functions of Two Variables**
>
> A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the *domain* of f and its *range* is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

Graphs of Two Variables

> **Definition 0.0.34: Graph of a Function of Two Variables**
>
> Defintion If f is a function of two variables with domain D, then the *graph* of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Level Curves and Contour Maps

> **Definition 0.0.35: Level Curves and Contour Maps**
>
> The *level curves* of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Functions of Three or More Variables

> **Definition 0.0.36: Functions of Three Variables**
>
> A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subseteq \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

14.3 Parital Derivatives

Partial Derivatives of Fuctions of Two Variables

Definition 0.0.37: Partial Derivatives

Defintion Partial Derivative with respect to x

$$f_x(a,b) = g'(a) \quad \text{where} \quad g(x) = f(x,b)$$

Defintion Partial Derivative with respect to y

$$f_y(a,b) = h'(a) \quad \text{where} \quad h(x) = f(a,y)$$

Note:-

If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Note:-

Rule for Finding Partial Derivatives of $z = f(x,y)$

1. To find f_x, regard y as a constant and differentiate $f(x,y)$ with respect to x.

2. To find f_y, regard x as a constant and differentiate $f(x,y)$ with respect to y.

Interpretation of Partial Derivatives

Definition 0.0.38: interpretation of Partial Derivatives

To understand partial derivatives geometrically, think of the equation $z = f(x,y)$ as representing a surface S (the graph of f). If $f(a,b) = c$, then the point $P(a,b,c)$ lies on this surface.

By fixing $y = b$, we focus on the curve C_1 where the vertical plane $y = b$ intersects S. Similarly, fixing $x = a$ gives us the curve C_2, which is where the vertical plane $x = a$ intersects S. Both curves C_1 and C_2 pass through the point P.

The curve C_1 is the graph of the function $g(x) = f(x,b)$, and the slope of its tangent at P is $f_x(a,b)$. The curve C_2 is the graph of $G(y) = f(a,y)$, and the slope of its tangent at P is $f_y(a,b)$.

Thus, the partial derivatives $f_x(a,b)$ and $f_y(a,b)$ represent the slopes of the tangent lines at P along these curves.
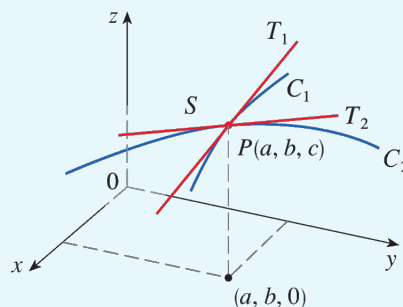
Higher Derivatives

> **Definition 0.0.39: Clairut's Theorem**
>
> Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then
> $$f_{xy}(a, b) = f_{yx}(a, b)$$

14.4 Tangent Planes and Linear Approximations

Tangent Planes

> **Definition 0.0.40: Tangent Plane**
>
> The **tangent plane** to a surface $z = f(x, y)$ at a point $P(x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$, is the plane that best approximates the surface near P. If f has continuous partial derivatives, the equation of the tangent plane is given by:
>
> $$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$
>
> where:
>
> - $f_x(x_0, y_0)$ is the partial derivative of f with respect to x at (x_0, y_0),
>
> - $f_y(x_0, y_0)$ is the partial derivative of f with respect to y at (x_0, y_0).

14.5 The Chain Rule

The Chain Rule

> **Definition 0.0.41: Chain Rule (Case 1)**
>
> Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and
>
> $$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial z}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

> **Definition 0.0.42: Chain Rule (Case 2)**
>
> Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then
>
> $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

> **Definition 0.0.43: Chain Rule (General Case)**
>
> Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and
>
> $$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$
>
> for each $i = 1, 2, \ldots, m$.

Implicit Differentiation

Definition 0.0.44: Implicit Differentiation

Defintion:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial Y}} = -\frac{F_x}{F_y} \qquad \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$$

14.6 Directional Derivatives and their Gradient Vectors

Directional Derivatives

Definition 0.0.45: Directional Derivative

The **directional derivative** of a function $f(x, y)$ at a point (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is the rate of change of f in that direction. It is defined as:

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}.$$

Theorem 0.0.7

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

The Gradient Vector

Definition 0.0.46: The Gradient Vector

If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Note:-

Equation

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

can be rewritten as

$$D_u f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Definition 0.0.47: Gradient of Three Variable Functions

Defintion:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} \tag{1}$$

Maximizing the Directional Derivative

Theorem 0.0.8 Maximizing Directional Derivative

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Tangent Planes to Level Surfaces

Definition 0.0.48: Tanget Planes to Level Surfaces

Consider a surface S defined by $F(x, y, z) = k$, where F is a function of three variables. Let $P(x_0, y_0, z_0)$ be a point on S and C be a curve on S that passes through P. The curve is given by a vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ such that $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S, the equation $F(x(t), y(t), z(t)) = k$ must hold.

By using the Chain Rule to differentiate both sides of this equation, we get:

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

This can be written as a dot product:

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

which means that the gradient ∇F is perpendicular to the tangent vector $\mathbf{r}'(t)$ at P.

At $t = t_0$, the gradient at P, $\nabla F(x_0, y_0, z_0)$, is normal to the tangent plane at P. The equation of the tangent plane is:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0 \tag{2}$$

Note:-

Properties of the Gradient Vecotr

Let f be a differentiable function of two or three variables and suppose that $\nabla f(\mathbf{x}) \neq 0$.

- The directional derivative of f at $\mathbf{x}$ in the direction of a unit vector $\mathbf{u}$ is given by $D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u}$.

- $\nabla f(\mathbf{x})$ points in the direction of maximum rate of increase of f at $\mathbf{x}$, and that maximum rate of change is $|\nabla f(\mathbf{x})|$.

- $\nabla f(\mathbf{x})$ is perpendicular to the level curve or level surface of f through $\mathbf{x}$.

14.7 Maximum and Minimum Values

Local Maximum and Minimum Values

Definition 0.0.49: Local Min and Max

A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Theorem 0.0.9 Critical Point

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Definition 0.0.50: Second Derivatives Test

Suppose the second partial derivatives of f are continuous on a disk with center (a,b), and suppose that $f_x(a,b) = 0$ and $f_y(a,b) = 0$ [so (a,b) is a critical point of f]. Let

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2 \tag{3}$$

(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.

(c) If $D < 0$, then (a,b) is a saddle point of f.

Absolute Maximum and Minimum Values

Definition 0.0.51: Absolute Maxiums and Absolute Minimums

Let (a,b) be a point in the domain D of a function f of two variables. Then $f(a,b)$ is the

- **absolute maximum** value of f on D if $f(a,b) \geqslant f(x,y)$ for all (x,y) in D.

- **absolute minimum** value of f on D if $f(a,b) \leqslant f(x,y)$ for all (x,y) in D.

Definition 0.0.52: Extreme Value Theorem for Functions of Two Variables

If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1,y_1)$ and an absolute minimum value $f(x_2,y_2)$ at some points (x_1,y_1) and (x_2,y_2) in D.

Note:-

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.

2. Find the extreme values of f on the boundary of D.

3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

14.8 Lagrange Multipliers

Definition 0.0.53: Geometric Explanation of Lagrange Multipliers (One Constraint)

To find the maximum or minimum of $f(x,y)$ under a constraint $g(x,y) = k$, the method of Lagrange multipliers ensures that the gradients of f and g are parallel. This happens when $\nabla f = \lambda \nabla g$, where λ is a scalar. For functions of three variables, the same principle applies: the extreme values of $f(x,y,z)$ occur where f's level surface touches the constraint surface defined by $g(x,y,z) = k$.

Definition 0.0.54: Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]:

1. Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Lagrange Multipliers: Two Constraints

Definition 0.0.55: Lagrange Multipliers (Two Constraints)

To find the maximum or minimum of $f(x, y, z)$ under two constraints, $g(x, y, z) = k$ and $h(x, y, z) = c$, we examine the curve where the two constraint surfaces intersect. At an extreme value of f, its gradient ∇f lies in the plane formed by ∇g and ∇h. This gives the relationship:

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

where λ and μ are Lagrange multipliers.

15 Multiple Integrals

15.1 Double Integrals over Rectangles
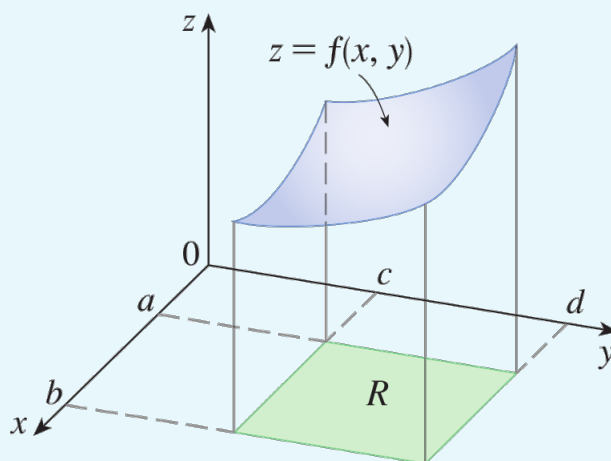
Volumes and Double Integrals

Definition 0.0.56: Volume of a Function

We start with a function f defined over a closed rectangle R in the xy-plane, denoted as:

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b, \, c \leqslant y \leqslant d\}.$$

The goal is to find the volume of the solid S, which lies above the rectangle R and below the graph of the surface $z = f(x, y)$, defined as:

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant z \leqslant f(x, y), \, (x, y) \in R\}.$$
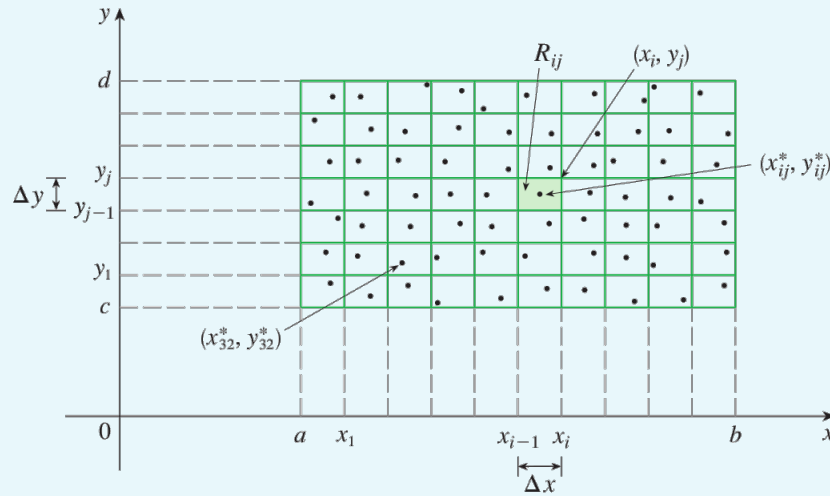
To compute this volume, we first divide the rectangle R into smaller subrectangles. The interval $[a,b]$ is divided into m subintervals of equal width $\Delta x = \frac{b-a}{m}$, and the interval $[c,d]$ is divided into n subintervals of equal width $\Delta y = \frac{d-c}{n}$. The subrectangles are denoted by:
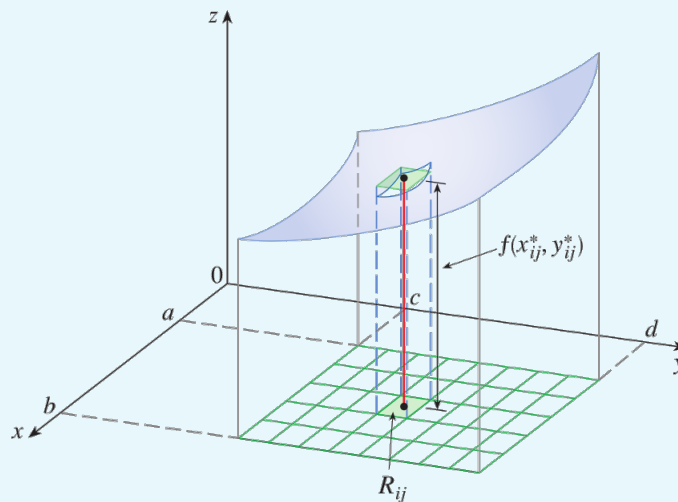
$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] = \{(x,y) \mid x_{i-1} \leqslant x \leqslant x_i,\ y_{j-1} \leqslant y \leqslant y_j\}.$$

Each subrectangle has an area of $\Delta A = \Delta x \Delta y$.

Next, we approximate the volume of the solid S by choosing a sample point (x_{ij}^*, y_{ij}^*) in each subrectangle R_{ij}. Using this point, we approximate the part of S above each R_{ij} by a thin rectangular box (or "column") with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$. The volume of this box is:
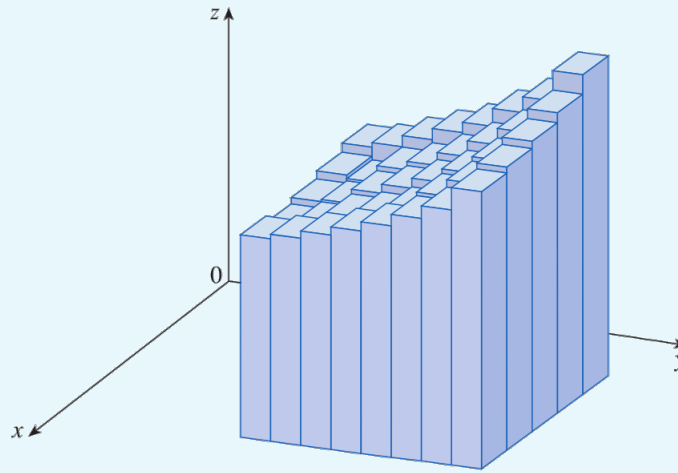
$$f(x_{ij}^*, y_{ij}^*)\Delta A.$$

We then sum the volumes of all these boxes over the entire grid of subrectangles, obtaining an approximation of the total volume V of the solid S:

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

As m and n become larger, our approximation becomes more accurate. The exact volume of S is given by the limit of the double sum as $m, n \to \infty$:

$$\iint_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A \tag{4}$$

This is the formal definition of the volume of the solid S that lies under the graph of the function f and above the rectangle R. By taking the limit, we ensure that the approximation becomes exact.

Defintion: The **double integral** of f over the rectangle R is

$$\iint_R f(x, y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A \tag{5}$$

if this limit exists

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA \tag{6}$$

Iterated Integrals

Definition 0.0.61: Fubini's Theorem

f f is continuous on the rectangle

$$R = \{(x,y) \mid a \leqslant x \leqslant b, \ c \leqslant y \leqslant d\}$$

then

$$\iint\limits_{R} f(x,y) \, dA = \int_a^b \int_c^d f(x,y) \, dy \, dx = \int_c^d \int_a^b f(x,y) \, dx \, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Theorem 0.0.10 Fubinis Theorem

IIf $f(x,y)$ can be expressed as the product of two functions, $f(x,y) = g(x)h(y)$, and the region of integration $R = [a,b] \times [c,d]$ is a rectangle, then the double integral of $f(x,y)$ over R simplifies to the product of two single integrals:

$$\iint_R f(x,y) \, dA = \left(\int_a^b g(x) \, dx \right) \left(\int_c^d h(y) \, dy \right).$$

This result follows from **Fubini's Theorem**, which allows the separation of the double integral into two independent integrals when $f(x,y)$ is separable in x and y.

Average Value

Definition 0.0.62: Average Value

We define the **average value** of a function f of two variables defined on a rectangle R to be

$$f_{\text{avg}} = \frac{1}{A(R)} \iint\limits_{R} f(x,y) \, dA$$

where $A(R)$ is the area of R.
If $f(x,y) \geqslant 0$, the equation

$$A(R) \times f_{\text{avg}} = \iint\limits_{R} f(x,y) \, dA$$
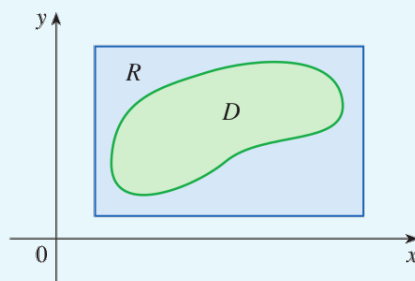
holds true.

15.2 Double Integrals over General Regions

General Regions

Definition 0.0.63: General Regions

Consider a general region D which is bounded, which means that D can be enclosed in a rectangular region R. In order to integrate a function f over D, we define a new function F with domain R by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA$$

where F is given by the above equation.

Definition 0.0.64: Type 1

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\} \tag{7}$$

where g_1 and g_2 are continuous on $[a, b]$.
If f is continuous on a type I region D described by

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then

$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx \tag{8}$$

Definition 0.0.65: Type II

If f is continuous on a type II region D described by

$$D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\} \tag{9}$$

then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy \tag{10}$$

Changing the Order of Integration

Definition 0.0.66: Changing order of Integeration

Changing the Order of Integration allows us to evaluate a double integral by reversing the order of integration. This is particularly useful when one order is difficult or impossible to compute directly. Given a double integral over a region D:

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx,$$

we can rewrite it by reversing the integration order as:

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Here, D must be redefined appropriately to describe the region based on the new order of integration. This transformation simplifies the evaluation when the original order is difficult or impractical.

15.3 Double Integrals in Polar Coordinates

Definition 0.0.67: Rectangle to Polar Coordinates in Double Integerals

If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{11}$$

Definition 0.0.68: Polar Region Double Integration

If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta \tag{12}$$

15.6 Triple Integrals

15.7 Triple Integrals in Cylindrical Coordinates

15.8 Triple Integrals in Spherical Coordinates

16 Vector Calculus

16.1 Vector Fields

Vector Fields in $\mathbb{R}^2$ and $\mathbb{R}^3$

> **Definition 0.0.69: Vector Field**
>
> Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

> **Definition 0.0.70: 3 Dimensional Vector Fields**
>
> Let E be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Gradient Fields

> **Definition 0.0.71: Gradient Fields**
>
> If f is a scalar function of two variables, its gradient ∇f (also called grad f) is defined by
>
> $$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$
>
> Therefore, ∇f is a vector field on $\mathbb{R}^2$ and is called a **gradient vector field**. Similarly, if f is a scalar function of three variables, its gradient is a vector field on $\mathbb{R}^3$, given by
>
> $$\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$$

16.2 Line Integrals

Line Integrals in the Planes

> **Definition 0.0.72: Line Integerals**
>
> If f is defined on a smooth curve C given by
>
> $$x = x(t) \quad y = y(t) \quad a \leqslant t \leqslant b$$
>
> then the **line integral** of f along C is
>
> $$\int_C f(x, y)\, ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \Delta s_i \tag{13}$$
>
> if this limit exists.

Definition 0.0.73: Evaluation of Line Integerals

The length of the curve C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If f is continuous, the following formula can be used to evaluate the line integral of f along C:

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{14}$$

Line Integrals with Respect to x or y

Definition 0.0.74: Line Integerals with Respect to x or y

Two other types of line integrals are obtained by replacing Δs_i by either $\Delta x_i = x_i - x_{i-1}$ or $\Delta y_i = y_i - y_{i-1}$. They are called the **line integrals of f along C with respect to x and y**:

$$\int_C f(x,y)\, dx = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*, y_i^*)\Delta x_i \tag{15}$$

$$\int_C f(x,y)\, dy = \lim_{n \to \infty} \sum_{i=1}^n f(x_i^*, y_i^*)\Delta y_i \tag{16}$$

These line integrals can also be evaluated in terms of a parameter t, where $x = x(t)$, $y = y(t)$, and:

$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))x'(t)\, dt \tag{17}$$

$$\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))y'(t)\, dt \tag{18}$$

Line Integrals in Space

Definition 0.0.75: Line Integerals in Space

Equation:

$$\int_C f(x,y,z)\, ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \tag{19}$$

Line Integrals of Vector Fields; Work

The **work** W done by a continuous vector field $\mathbf{F}$ along a smooth curve C is defined as the line integral of $\mathbf{F}$ along C. If

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k},$$

and C is parameterized by a vector function

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \quad t \in [a, b],$$

then the work is given by:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Equivalently, this can be expressed in terms of the components of $\mathbf{F}$ and the derivatives of $\mathbf{r}(t)$ as:

$$W = \int_a^b \left[P(x(t), y(t), z(t))\frac{dx}{dt} + Q(x(t), y(t), z(t))\frac{dy}{dt} + R(x(t), y(t), z(t))\frac{dz}{dt} \right] dt.$$

Or, in differential form:

$$W = \int_C P\, dx + Q\, dy + R\, dz.$$

16.3 The fundamental Theorem for Line Integrals

The Fundamental Theorem for Line Integrals

> **Definition 0.0.77: Fundamental Theorem for Line Integrals**
>
> If we think of the gradient vector ∇f of a function f of two or three variables as a derivative of f, then the following theorem can be regarded as a version of the Fundamental Theorem for line integrals:
>
> Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then:
>
> $$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$
>
> ### Note 1
>
> Theorem 2 states that we can evaluate the line integral of a conservative vector field (the gradient vector field of the potential function f) simply by knowing the value of f at the endpoints of C. In fact, Theorem 2 says that the line integral of ∇f is the net change in f. For example:
> - If f is a function of two variables and C is a plane curve with initial point $A(x_1, y_1)$ and terminal point $B(x_2, y_2)$, then:
>
> $$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1).$$
>
> - If f is a function of three variables and C is a space curve joining the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$, then:
>
> $$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1).$$
>
> ### Note 2
>
> Under the hypotheses of Theorem 2, if C_1 and C_2 are smooth curves with the same initial points and the same terminal points, then:
>
> $$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}.$$
>
> This highlights that for conservative vector fields, the line integral depends only on the endpoints of the curve, not the path taken between them.

16.4 Green's Theorem

16.5 Curl and Divergence

Curl

> **Definition 0.0.78: Curl**
>
> The **curl** of a vector field $\mathbf{F}$ in $\mathbb{R}^3$, where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, is a vector field that measures the rotational tendency or circulation of $\mathbf{F}$ at a point. Mathematically, it is given by:
>
> $$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}$$
>
> This can be computed using the cross product of the del operator ($\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$) and the vector field $\mathbf{F}$. The result is:
>
> $$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$
>
> Explicitly, the components are:
>
> $$\operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\mathbf{k}$$
>
> This definition captures the curl as the formal representation of the infinitesimal rotation of $\mathbf{F}$ in three-dimensional space.

> **Note:-**
>
> The curl of the gradient of a scalar function f with continuous second-order partial derivatives is always zero:
>
> $$\operatorname{curl}(\nabla f) = 0$$

> **Note:-**
>
> If a vector field $\mathbf{F}$ is defined on all of $\mathbb{R}^3$, has continuous partial derivatives, and satisfies $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

> **Note:-**
>
> The term **curl** is associated with rotations in a vector field. Specifically:
>
> 1. **Curl and Rotation**: The curl vector indicates the axis and direction of rotation of nearby particles, following the right-hand rule. The magnitude of the curl measures the speed of this rotation.
>
> 2. **Irrotational Flow**: If $\operatorname{curl} \mathbf{F} = 0$ at a point P, the vector field is called **irrotational** at P. Here, particles move with the flow but do not rotate about their own axis.
>
> 3. **Rotational Flow**: If $\operatorname{curl} \mathbf{F} \neq 0$, the particles rotate about their axis. The direction of the curl vector points along the axis of rotation.
>
> 4. **Illustration with Fluid Flow**:
>
>   - In a fluid velocity field, a paddle wheel at a point P_1 where $\operatorname{curl} \mathbf{F} \neq 0$ would rotate about its axis, indicating rotational flow.
>   - At a point P_2 where $\operatorname{curl} \mathbf{F} = 0$, the paddle wheel moves with the flow but does not rotate, indicating irrotational flow.

Definition 0.0.79: Divergence

The **divergence** of a vector field $\mathbf{F}$ in $\mathbb{R}^3$, where $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, is a scalar function that measures the rate at which the vector field spreads out (or converges) at a given point. It is defined as:

$$\text{div}\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Alternatively, using the del operator ($\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$), the divergence can be expressed as the dot product of ∇ and $\mathbf{F}$:

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

Key Property

For a vector field $\mathbf{F}$ in $\mathbb{R}^3$ with continuous second-order partial derivatives, the divergence of the curl of $\mathbf{F}$ is always zero:

$$\text{div}(\text{curl}\,\mathbf{F}) = 0$$

This captures the idea that the curl of a vector field does not "spread out" or "converge" in space.

Note:-

1. **Divergence in Fluid Flow**:

   - If $\mathbf{F}(x, y, z)$ represents the velocity of a fluid or gas, $\text{div}\,\mathbf{F}(x, y, z)$ measures the net rate of change of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.
   - It indicates the tendency of the fluid to spread out (diverge) from a point.

2. **Incompressible Flow**:

   - If $\text{div}\,\mathbf{F} = 0$, the fluid is said to be **incompressible**, meaning there is no net outflow or inflow at that point.

3. **Illustration**:

   - **Case 1 ($\text{div}\,\mathbf{F} \neq 0$)**:
     - At a point P_1, if $\text{div}\,\mathbf{F} < 0$, the flow is inward (net inflow).
     - At a point P_2, if $\text{div}\,\mathbf{F} > 0$, the flow is outward (net outflow).
   - **Case 2 ($\text{div}\,\mathbf{F} = 0$)**:
     - The flow is balanced (no net divergence or convergence).

Vector Forms of Green's Theorem

> **Definition 0.0.80: Vector Forms of Green's Theorem**
>
> reen's Theorem relates a line integral around a simple closed curve C to a double integral over the plane region D enclosed by C. It provides two key vector forms:
>
> ### 1. Curl Form (Tangential Component):
>
> The line integral of the tangential component of $\mathbf{F}$ along C is equal to the double integral of the vertical component of curl $\mathbf{F}$ over D:
>
> $$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D (\operatorname{curl} \mathbf{F}) \cdot \mathbf{k}\, dA$$
>
> where
>
> $$(\operatorname{curl} \mathbf{F}) \cdot \mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$
>
> ### 2. Divergence Form (Normal Component):
>
> The line integral of the normal component of $\mathbf{F}$ along C is equal to the double integral of the divergence of $\mathbf{F}$ over D:
>
> $$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_D \operatorname{div} \mathbf{F}\, dA$$
>
> where
>
> $$\operatorname{div} \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$
>
> ### Geometric Interpretation:
>
> - The curl form expresses the relationship between the circulation of $\mathbf{F}$ along the boundary C and the "rotational tendency" of $\mathbf{F}$ over the region D.
>
> - The divergence form relates the flux of $\mathbf{F}$ across C to the net outflow (or inflow) of $\mathbf{F}$ over D.
>
> These two forms of Green's Theorem are fundamental in understanding the relationship between local properties of vector fields and their global integrals.

16.6 Parametric Surfaces and Their Areas

Parametric Surfaces

> **Definition 0.0.81: Parametric Surface**
>
> A *parametric surface* is a surface in three-dimensional space $\mathbb{R}^3$ defined by a vector-valued function $\mathbf{r}(u, v)$, which depends on two parameters u and v. The function is expressed as:
>
> $$\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k},$$
>
> where $x(u, v)$, $y(u, v)$, and $z(u, v)$ are the component functions of $\mathbf{r}$, representing the x-, y-, and z-coordinates of the surface, respectively. These functions are defined over a region D in the uv-plane. The set of all points $(x, y, z) \in \mathbb{R}^3$ that satisfy:
>
> $$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$
>
> as (u, v) varies over D, forms the parametric surface S.

Parametric Equations

> **Definition 0.0.82: Parametric Equations**
>
> For a parametric surface the *parametric equations* are equations that describe the coordinates (x, y, z) of points on the surface as functions of two independent parameters u and v. For a parametric surface S, these equations are given by:
>
> $$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$
>
> where $x(u, v)$, $y(u, v)$, and $z(u, v)$ are the component functions of a vector-valued function $\mathbf{r}(u, v)$. These equations define the spatial coordinates of the surface for every pair of parameters (u, v) in a specified domain D in the uv-plane.

Grid Curves

> **Definition 0.0.83: Grid Curves**
>
> On a parametric surface s *grid curves* are families of curves defined by the vector function $\mathbf{r}(u, v)$. They are obtained by fixing one parameter and varying the other:
> 1. **Curves with $u = u_0$:** When u is held constant, the parametric surface reduces to a curve:
>
> $$\mathbf{r}(u_0, v) = \langle x(u_0, v), y(u_0, v), z(u_0, v) \rangle,$$
>
> which traces a curve C_1 on the surface as v varies.
> 2. **Curves with $v = v_0$:** When v is held constant, the parametric surface reduces to a curve:
>
> $$\mathbf{r}(u, v_0) = \langle x(u, v_0), y(u, v_0), z(u, v_0) \rangle,$$
>
> which traces a curve C_2 on the surface as u varies.
> These two families of curves correspond to horizontal and vertical lines in the uv-plane and form a grid-like structure when plotted on the surface.

Spherical Coordinates

Surfaces of Revolution

> **Definition 0.0.84: Surfaces of Revolution**
>
> A **surface of revolution** is generated by rotating a curve C, defined parametrically or as a function, about a fixed axis in three-dimensional space. The parametric equations of the surface can be expressed as:
>
> $$\begin{aligned} x &= u, \\ y &= r(u) \cos \theta, \\ z &= r(u) \sin \theta, \end{aligned}$$
>
> where:
>
> - u is a parameter describing the curve C,
>
> - $r(u)$ is the radial distance of the curve from the axis of rotation,
>
> - $\theta \in [0, 2\pi]$ is the angle of rotation.
>
> The domain of the parameters u and θ depends on the curve and the extent of rotation.

Tangent Planes

Definition 0.0.85: Tangent Planes

The **tangent plane** to a parametric surface S at a point $P_0(u_0, v_0)$ is the plane that best approximates S near P_0.

If S is defined by a vector-valued function:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

then the tangent plane at P_0 is determined by the two tangent vectors at P_0:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k},$$

$$\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

The tangent plane at P_0 is spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$. A normal vector to the plane is given by:

$$\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v.$$

The equation of the tangent plane can be expressed in the point-normal form:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}(u_0, v_0)) = 0,$$

where $\mathbf{r}(u_0, v_0)$ is the position vector of P_0.

For the tangent plane to exist, the cross product $\mathbf{r}_u \times \mathbf{r}_v$ must be nonzero, ensuring that S is smooth at P_0.

Surface Area for a Parametric Surface

Definition 0.0.86: Surface Area for a Parametric Surface

The **surface area** of a smooth parametric surface S, defined by the vector-valued function:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D,$$

where D is the parameter domain, is given by the integral:

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA,$$

where:

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$$

The cross product $\mathbf{r}_u \times \mathbf{r}_v$ represents a vector orthogonal to the tangent plane at each point on the surface, and its magnitude $|\mathbf{r}_u \times \mathbf{r}_v|$ gives the infinitesimal area of a parallelogram spanned by the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. Integrating this quantity over the parameter domain D yields the total surface area of S.

Surface Area of the Graph of a Function

Definition 0.0.87: Surface Area of the Graph of a Function

The **surface area** of the graph of a function $z = f(x, y)$, where $f(x, y)$ has continuous partial derivatives, over a region D in the xy-plane is given by:

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} \, dA.$$

Explanation

- The parametric representation of the surface is:

$$\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{k}.$$

- The tangent vectors are:

$$\mathbf{r}_x = \mathbf{i} + \frac{\partial f}{\partial x}\mathbf{k}, \quad \mathbf{r}_y = \mathbf{j} + \frac{\partial f}{\partial y}\mathbf{k}.$$

- The magnitude of the cross product of the tangent vectors is:

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$

Integrating this quantity over the region D in the xy-plane gives the total surface area of the graph of $f(x, y)$.

16.7 Surface Integral

Parametric Surfaces

Definition 0.0.88: Surface Integral for Parametric Surfaces

The **surface integral** of a scalar function $f(x, y, z)$ over a parametric surface S, defined by the vector equation:

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}, \quad (u, v) \in D,$$

is given by:

$$\iint_S f(x, y, z)\, dS = \iint_D f(\mathbf{r}(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA.$$

Explanation

- The parameter domain D is divided into subrectangles with dimensions Δu and Δv, and each corresponding surface patch is approximated as a parallelogram in the tangent plane.

- The area of a surface patch is approximated as:

$$\Delta S_{ij} \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u \Delta v.$$

- The surface integral is defined as the limit of a Riemann sum:

$$\iint_S f(x, y, z)\, dS = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}.$$

Key Components

- $\mathbf{r}_u$ and $\mathbf{r}_v$ are the partial derivatives of $\mathbf{r}(u, v)$ with respect to u and v, respectively:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}.$$

- $\mathbf{r}_u \times \mathbf{r}_v$ gives a vector normal to the surface at each point, and $|\mathbf{r}_u \times \mathbf{r}_v|$ represents the infinitesimal surface area element.

This integral evaluates the contribution of $f(x, y, z)$ across the entire surface S.

Graphs of Functions

Definition 0.0.89: Surafe Integrals for Graphs of Functions

The **surface integral** of a scalar function $f(x, y, z)$ over the graph of a function $z = g(x, y)$, where $g(x, y)$ has continuous partial derivatives, is given by:

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\, dA.$$

Explanation

- The graph of the function $z = g(x, y)$ can be regarded as a parametric surface with:

$$x = x, \quad y = y, \quad z = g(x, y).$$

- The tangent vectors to this surface are:

$$\mathbf{r}_x = \mathbf{i} + \frac{\partial g}{\partial x}\mathbf{k}, \quad \mathbf{r}_y = \mathbf{j} + \frac{\partial g}{\partial y}\mathbf{k}.$$

- The cross product of the tangent vectors is:

$$\mathbf{r}_x \times \mathbf{r}_y = -\frac{\partial g}{\partial x}\mathbf{j} - \frac{\partial g}{\partial y}\mathbf{i} + \mathbf{k}.$$

- The magnitude of the cross product is:

$$\left|\mathbf{r}_x \times \mathbf{r}_y\right| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}.$$

By integrating this quantity over the region D in the xy-plane, we account for the contributions of $f(x, y, z)$ over the entire surface S.

Oriented Surfaces

An **oriented surface** is an orientable (two-sided) surface S where it is possible to define a continuous, unit normal vector $\mathbf{n}$ at every point (x, y, z) on the surface, except possibly at boundary points.

Key Properties

- **Two Possible Orientations**: For any orientable surface, there are two choices for the unit normal vector:

    - $\mathbf{n}_1$, the chosen unit normal vector.
    - $\mathbf{n}_2 = -\mathbf{n}_1$, the opposite orientation.

- A surface is called **orientable** if it is possible to assign $\mathbf{n}$ continuously over the entire surface S.

- A classic example of a non-orientable surface is the Möbius strip, which has only one side and no consistent orientation.

Explanation

An oriented surface requires the existence of a consistent way to assign a "positive" or "negative" side across all points on the surface. The orientation is provided by the chosen direction of the normal vector $\mathbf{n}$, which varies smoothly across the surface.

Surface Integrals of Vector Fields; Flux

> **Definition 0.0.91: Flux**
>
> he **surface integral of a vector field** (also called the **flux**) over an oriented surface S with a unit normal vector $\mathbf{n}$ is defined as:
>
> $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS,$$
>
> where:
>
> - $\mathbf{F}$ is a continuous vector field defined on S,
>
> - $\mathbf{n}$ is the unit normal vector to S,
>
> - dS represents the infinitesimal surface area element.
>
> ## Special Case: Surface Defined by $z = g(x, y)$
>
> If the surface S is defined by the graph $z = g(x, y)$, and $\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, the surface integral can be expressed as:
>
> $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA,$$
>
> where D is the projection of the surface onto the xy-plane.
> This formula assumes the upward orientation of S. For a downward orientation, the integral is multiplied by -1.
>
> ## Parametric Form
>
> If the surface S is parameterized by $\mathbf{r}(u, v)$, with tangent vectors:
>
> $$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v},$$
>
> then the flux integral can be written as:
>
> $$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA,$$
>
> where D is the parameter domain.
>
> ## Physical Interpretation
>
> The flux integral measures the total flow of the vector field $\mathbf{F}$ across the surface S, representing quantities like mass flow rate, electric flux, or fluid flow through S.

16.8 Stokes' Theorem

Definition 0.0.92: Stokes' Theorem

Stokes' Theorem relates the surface integral of the curl of a vector field over an oriented surface S to the line integral of the vector field along the boundary curve C of S. Mathematically, it is expressed as:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{r},$$

where:

- S is a piecewise-smooth, oriented surface with unit normal vector $\mathbf{n}$,

- C is the positively oriented, closed boundary curve of S,

- $\mathbf{F}$ is a vector field with continuous partial derivatives,

- $\nabla \times \mathbf{F}$ is the curl of $\mathbf{F}$,

- $d\mathbf{S} = \mathbf{n}\, dS$ is the oriented surface element,

- $d\mathbf{r}$ is the infinitesimal vector along C.

Key Notes

- **Positive Orientation:** The orientation of C is determined by the right-hand rule: when you walk along C with your head pointing in the direction of $\mathbf{n}$, the surface S remains on your left.

- **Special Case:** If S lies flat in the xy-plane and $\mathbf{n} = \mathbf{k}$, Stokes' Theorem reduces to Green's Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$$

Stokes' Theorem provides a fundamental relationship between the circulation of $\mathbf{F}$ along C and the total rotational effects (curl) of $\mathbf{F}$ over the surface S.

Note:-

Stokes' Theorem allows us to compute a surface integral simply by knowing the values of $\mathbf{F}$ on the boundary curve C. This means that if we have another oriented surface with the same boundary curve C, then we get exactly the same value for the surface integral. In general, if S_1 and S_2 are oriented surfaces with the same oriented boundary curve C and both satisfy the hypotheses of Stokes' Theorem, then:

$$\int_{S_1} (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{S_2} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

This fact is useful when it is difficult to integrate over one surface but easy to integrate over the other.

16.9 The Divergence Theorem