Math 120

PSet 9

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Chapter 1

1.1 PSet 9

Question 1

The vector field $\vec{F}$ is shown below in the xy-plane and looks the same in all other horizontal planes. (In other words, $\vec{F}$ is independent of z and its z-component is 0.)

(a) Is $\operatorname{div}\vec{F}$ positive, negative, or zero? Explain.

(b) Determine whether $\operatorname{curl}\vec{F} = \vec{0}$. If not, in which direction does $\operatorname{curl}\vec{F}$ point?

Solution:

(a) The divergence of $\vec{F}$ is negative because the vectors converge toward the origin, indicating a net inward flux. This suggests material is "flowing in" rather than spreading out.

(b) The curl of $\vec{F}$ is zero, as there is no rotational pattern; the vectors are purely radial and do not exhibit any circular motion.

Question 2

Find the curl and divergence of the given vector field.

1. $\vec{F}(x, y, z) = \langle x^2yz, xy^2z, xyz^2 \rangle$
2. $\vec{F}(x, y, z) = e^{xy} \sin z \,\hat{\jmath} + y \arctan(x/z)\,\hat{k}$

Solution:

$$P = x^2yx \quad Q = xy^2z \quad R = xyz^2$$

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\operatorname{div}\vec{F} = 2xyz + 2xyz + 2xyz = 6xyz$$

$$\frac{\partial P}{\partial x} = 2xyz,$$

$$\frac{\partial Q}{\partial y} = 2xyz,$$

$$\frac{\partial R}{\partial z} = 2xyz.$$

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2xyz + 2xyz + 2xyz = 6xyz.$$

$$\left(\operatorname{curl}\vec{F}\right)_x = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = xz^2 - xy^2.$$

$$\left(\operatorname{curl}\vec{F}\right)_y = \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = x^2y - yz^2.$$

$$\left(\operatorname{curl}\vec{F}\right)_z = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2z - x^2z.$$

$$\operatorname{curl}\vec{F} = \left(xz^2 - xy^2\right)\hat{\imath} + \left(x^2y - yz^2\right)\hat{\jmath} + \left(y^2z - x^2z\right)\hat{k}.$$

b)

$$\vec{F}(x,y,z) = e^{xy}\sin z\,\hat{\jmath} + y\arctan\left(\frac{x}{z}\right)\hat{k}$$

$$\begin{aligned} P &= 0, \\ Q &= e^{xy}\sin z, \\ R &= y\arctan\left(\frac{x}{z}\right). \end{aligned}$$

$$\begin{aligned} \frac{\partial P}{\partial x} &= 0, \\ \frac{\partial Q}{\partial y} &= xe^{xy}\sin z, \\ \frac{\partial R}{\partial z} &= -\frac{xy}{x^2+z^2}. \end{aligned}$$

$$\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0 + xe^{xy}\sin z - \frac{xy}{x^2+z^2}.$$

$$\begin{aligned} \left(\operatorname{curl}\vec{F}\right)_x &= \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ &= \arctan\left(\frac{x}{z}\right) - e^{xy}\cos z. \end{aligned}$$

$$\begin{aligned} \left(\operatorname{curl}\vec{F}\right)_y &= \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ &= 0 - \left(y\cdot\frac{z}{x^2+z^2}\right) \\ &= -\frac{yz}{x^2+z^2}. \end{aligned}$$

$$\begin{aligned} \left(\operatorname{curl}\vec{F}\right)_z &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \\ &= ye^{xy}\sin z - 0 \\ &= ye^{xy}\sin z. \end{aligned}$$

$$\operatorname{curl}\vec{F} = \left(\arctan\left(\frac{x}{z}\right) - e^{xy}\cos z\right)\hat{\imath} - \left(\frac{yz}{x^2+z^2}\right)\hat{\jmath} + \left(ye^{xy}\sin z\right)\hat{k}.$$

Question 3

Consider the surface given by $\sin(x-y) - x - y + z = 0$. Find a parametrization of the part of the surface that has $|x| \leqslant 3$ and $|y| \leqslant 3$. Be sure to include the bounds for your parameters.

(a) Look at the parametrization that you found in part (a). It should be of the form $\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$. Explain why it is not possible to find a parametrization of the surface $x^2 - y + z^2 = 1$ that is of the form $\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$.

(b) Find a parametrization of $x^2 - y + z^2 = 1$ that is of the form $\vec{r}(u,v) = \langle u, f(u,v), v\rangle$.

(c) Consider the surface $x^2 - y^2 + \frac{z^2}{4} = 1$. Explain why it is not possible to find a parametrization of this surface that is of the form $\vec{r}(u,v) = \langle u, v, f(u,v)\rangle$, or of the form $\vec{r}(u,v) = \langle f(u,v), u, v\rangle$.

(d) Find a parametrization of the surface $x^2 - y^2 + \frac{z^2}{4} = 1$ of the form

$$\vec{r}(u,v) = \langle f(v)\cos u, v, g(v)\sin u\rangle,$$

where $0 \leqslant u \leqslant 2\pi$ and $-\infty < v < \infty$.

Question 4

Identify and sketch the surface with the given parameterization.

(a) $\vec{r}(u,v) = (2\sin u)\,\hat{\jmath} + (3\cos u)\,\hat{\imath} + v\,\hat{k}, \quad 0 \leqslant u \leqslant \pi, \quad -2 \leqslant v \leqslant 2$

(b) $\vec{r}(u,v) = \langle u\sin v, u^2, u\cos v\rangle, \quad 0 \leqslant u \leqslant 3, \quad 0 \leqslant v \leqslant 2\pi$

Question 5

Consider the surface S described by $x - 4y^2 - z^2 + 3 = 0$.

(a) Find a parametrization of S of the form $\vec{r}_1(u,v) = \langle f(u,v), u, v\rangle$. Give the domain for your parametrization.

(b) Find a parametrization of S of the form $\vec{r}_2(u,v) = \langle v, f(v)\cos u, g(v)\sin u\rangle$. Give the domain for your parametrization.

(c) How must we restrict the parameters (u,v) in part (a) if we only want the part of S that lies in front of the yz-plane, i.e., where $x \geqslant 0$?

(d) How must we restrict the parameters (u,v) in part (b) if we only want the part of S that lies in front of the yz-plane?

Question 6

Find parametric equations for each of the following surfaces.

(a) The part of the plane $z = x + 3$ that lies inside the cylinder $x^2 + y^2 = 1$.

(b) The surface obtained by rotating the curve $x = 4y^2 - y^4$, $-2 \leqslant y \leqslant 2$ about the y-axis.

(c) The ellipsoid $\frac{x^2}{4} + 4y^2 + \frac{z^2}{9} = 1$.

Question 7

Find the tangent plane to the parametric surface $\vec{r}(u,v) = \langle u\sin v, u^2, u\cos v\rangle$ at the point where $u = 1$ and $v = \frac{\pi}{3}$. Write the plane both in the vector form $\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$ and in the form $ax + by + cz = d$.