Review Exam 1 Concepts:
Foundations of multivariable calculus: vectors, geometry in space, and basic derivatives. Understanding these helps you visualize and tackle more advanced problems later.

1. Vectors
Concept: Vectors represent quantities with both magnitude and direction. They are the building blocks of spatial reasoning in calculus.

- Magnitude: For $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.
- Addition and scalar multiplication: If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and $c \in \mathbb{R}$, then $\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$, $c\mathbf{a} = \langle ca_1, ca_2, ca_3 \rangle$.
- $\overrightarrow{AB}$: If $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$, then $\overrightarrow{AB} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$.

2. Dot Product
Concept: The dot product measures how much two vectors "line up." It relates to projections and angles.

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta.$$

Projection of $\mathbf{a}$ onto $\mathbf{b}$:

$$\text{proj}_{\mathbf{b}}(\mathbf{a}) = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}.$$

If two vectors are perpendicular, the dot product is zero.

3. Cross Product
Concept: The cross product gives a vector perpendicular to both inputs, representing the "area" spanned by two vectors and a direction given by the right-hand rule.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \quad \|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta.$$

Also relates to volumes of parallelepipeds and indicates orientation in space.

4. Planes
Concept: A plane in 3D is defined by a point and a normal vector. The normal captures the plane's orientation.

$$ax + by + cz = d.$$

Normal vector $\langle a, b, c \rangle$ shows how the plane is "tilted."

5. Distances
Concept: Distance formulas measure how far apart objects are in space, crucial for geometry and optimization.
- Distance from point $P_0 = (x_0, y_0, z_0)$ to plane $ax + by + cz = d$:

$$\text{distance} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

- Distance from point $P_0 = (x_0, y_0, z_0)$ to line through $A = (x_1, y_1, z_1)$ with direction vector $\mathbf{v}$:

$$\text{distance} = \frac{\|\overrightarrow{P_0 A} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

6. Derivative of Vector Functions
Concept: A vector function changes with respect to a parameter (often time). Its derivative gives the instantaneous direction and rate of change.

$$\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle.$$

Think of it as velocity if $\mathbf{r}(t)$ is a position.

7. Tangent Line
Concept: The tangent line at a point on a curve shows the curve's immediate direction. It's a linear approximation at a specific point.

$$\text{Tangent line: } \mathbf{r}(t) = \mathbf{r}(t_0) + \mathbf{r}'(t_0)(t - t_0).$$

8. Integrals of Vector Functions
Concept: Integrals summarize accumulation over time or space. For vector functions, it represents accumulated displacement or "area under" a vector curve.

$$\int_a^b \mathbf{r}'(t)\, dt = \mathbf{r}(b) - \mathbf{r}(a).$$

9. Functions of Several Variables
Concept: Surfaces and contours come from functions of two or three variables. If $f(x, y, z)$ is a function of three variables, level surfaces $f(x, y, z) = c$ define 3D shapes.

10. Implicit Differentiation
Concept: When functions aren't given explicitly (like $y = f(x)$), we differentiate implicitly to find relationships between rates of change of variables.
For $F(x, y) = 0$:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

For surfaces defined implicitly by $F(x, y, z) = 0$, partial derivatives follow similarly:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

After these basics, Exam 2 topics incorporate partial derivatives, optimizing functions in multiple variables, and evaluating integrals over paths and areas.

Review Exam 2 Topics:
Building on Exam 1, we now focus on how surfaces change in different directions, find maxima/minima in multiple dimensions, and handle integrals along curves.

1. Directional Derivatives
Concept: The directional derivative tells you how a function changes as you move in a given direction. It's like a "slope" in the direction of a chosen vector.

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}.$$

2. Tangent Plane for $z = f(x, y)$
Concept: The tangent plane is a 2D approximation of a surface near a point. It's the "best linear fit" to a surface at that point.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

3. Critical Points & 2nd-Derivative Test
Concept: Critical points are where a function's slope "flattens out." The 2nd-derivative test classifies these points as peaks, valleys, or saddle points.

$$D = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2.$$

4. Lagrange Multipliers
Concept: A method to find maxima/minima of a function subject to a constraint. Geometrically, it aligns gradients so that at extrema, the surfaces "touch" but do not intersect.

$$\nabla f = \lambda \nabla g.$$

5. Double Integrals
Concept: Double integrals measure volumes under surfaces. Changing to polar coordinates makes circular or radial regions simpler.

$$\iint_R f(x, y)\, dA.$$

In polar: $x = r\cos\theta, y = r\sin\theta$, $dA = r\, dr\, d\theta$.

6. Line Integrals
Concept: Integrating along a path in space. Can represent work done by a force along a path or mass of a wire.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

7. Fundamental Theorem of Line Integrals
Concept: If a vector field is the gradient of some function, then the integral depends only on the endpoints—making calculations much simpler.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\text{end}) - f(\text{start}).$$

8. Vector Fields
Concept: Assigning a vector to every point in space describes flows, fields (like gravity or electricity), and directional tendencies.
If $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$, it is a vector field on $\mathbb{R}^3$.

9. Green's Theorem
Concept: Converts a line integral around a closed curve into a double integral over the region inside. It relates circulation around a boundary to a "curl-like" measure inside the region.

$$\oint_C (P\, dx + Q\, dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

10. Conservative Vector Fields
Concept: If a field is conservative, there's a potential function whose gradient gives the field. This means path independence and simpler integral evaluations.
If $\mathbf{F} = \nabla f$, then $\mathbf{F}$ is conservative.

For the Final, we integrate everything: curl, divergence, and the great theorems linking line, surface, and volume integrals. These theorems unify our understanding of vector fields and surfaces.

Review for the Final Exam:
We now employ curl, divergence, and parameterizations more fully. This big-picture view links line integrals, surface integrals, and volume integrals into powerful theorems.

1. Curl of a Vector Field
Concept: Curl measures the "rotation" of a vector field. A high curl means the field swirls around that point.

$$\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

2. Divergence of a Vector Field
Concept: Divergence measures how much a vector field "spreads out" from a point. Positive divergence means sources (outflow), negative means sinks (inflow).

$$\nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

3. Parametric Plane
Concept: We describe surfaces by parameters. A parametric plane can be given by:

$$\mathbf{r}(u,v) = \mathbf{r}_0 + u\mathbf{r}_u + v\mathbf{r}_v,$$

where $\mathbf{r}_u$ and $\mathbf{r}_v$ are direction vectors.

4. Parametric Surfaces
Concept: More general surfaces (like spheres or saddle shapes) can be described by two parameters (u,v):

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle.$$

This approach makes complex integrals easier.

5. Tangent Planes to Surfaces
Concept: Similar to tangent lines but now for surfaces. If $\mathbf{r}(u,v)$ describes a surface, the tangent vectors are:

$$\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}, \quad \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}.$$

A tangent plane is spanned by $\mathbf{r}_u$ and $\mathbf{r}_v$ at a point.

6. Surface Integral
Concept: Surface integrals extend the idea of double integrals to curved surfaces. For a scalar function f:

$$\iint_S f(x,y,z)\,dS.$$

If S is given by $\mathbf{r}(u,v)$, then $dS = \|\mathbf{r}_u \times \mathbf{r}_v\|\,du\,dv$.

7. Surface Orientation
Concept: Orientation (the direction of the normal vector) matters for flux. For $\mathbf{r}(u,v)$, the normal vector can be $\mathbf{r}_u \times \mathbf{r}_v$ or its negative.

8. Flux Integral
Concept: The flux integral measures how much of a vector field passes through a surface.

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

If S is parameterized by $\mathbf{r}(u,v)$, then

$$\iint_S \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv.$$

9. Stokes' Theorem
Concept: Stokes' Theorem links a line integral around a boundary to a surface integral of the curl.

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS.$$

10. Triple Integrals
Concept: Triple integrals measure volume-based quantities inside 3D regions.

$$\iiint_W f(x,y,z)\,dV.$$

Change of coordinates (cylindrical or spherical) often simplifies these integrals.

11. Cylindrical Coordinates
Concept: Cylindrical coordinates simplify integrals in problems with circular symmetry.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad dV = r\,dr\,d\theta\,dz.$$

12. Spherical Coordinates
Concept: Spherical coordinates simplify integrals over spherical regions.

$$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi, \quad dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

13. Divergence Theorem
Concept: Converts a flux integral over a closed surface into a volume integral of the divergence:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V (\nabla\cdot\mathbf{F})\,dV.$$

14. Remarks
Concept: Choosing the right theorem or coordinate system often simplifies the work. Think strategically!

By now, you've seen how all pieces connect: from simple vector operations to powerful theorems that turn complicated integrals into manageable ones.

Additional Strategies and Tips:
These strategies help you navigate complex problems and choose the best approach.

1. **Use FTLI:** If a line integral is over a gradient field, just find the potential function's values at the endpoints.
2. **Convert to Double/Surface Integrals:** Green's and Stokes' turn line integrals into area/surface integrals when it's simpler.
3. **Use Divergence Theorem:** If dealing with a closed surface, consider switching to a volume integral of divergence.
4. **Linear Approximations:** Tangent planes and vectors help approximate functions locally.
5. **Optimization Tools:** Lagrange multipliers and the 2nd-derivative test help find maxima and minima efficiently.

General Tips:

- **Visualize:** Drawing regions, surfaces, and vector fields aids understanding.
- **Normal Vectors:** Gradients give normals to surfaces defined implicitly.
- **Check Curl/Divergence:** If $\nabla \times \mathbf{F} = 0$, consider potential functions. If $\nabla \cdot \mathbf{F} = 0$, you might simplify flux integrals.
- **Coordinate Changes:** Cylindrical and spherical coordinates handle symmetrical regions more easily.