Math 115 QR
PSet 2

Alex Hernandez Juarez

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Contents

Chapter 1

1.1 Problem 3: Stewart 11.11.6

Question 1

(a) approximate f by a Taylor polynomial with degree n and number a.
(b) Use Taylor's inequality to estimate the accuracy of the approximation $f(x) = T_n(x)$ when x lies in the given interval

Solution:
(a)

$$f(x) = \sin(x)$$
$$a = \frac{\pi}{6}$$
$$n = 4$$
$$0 \leqslant x \leqslant \frac{\pi}{3}$$
$$f'(x) = \cos(x)$$
$$f''(x) = -\sin(x)$$
$$f^{(3)}(x) = -\cos(x)$$
$$f^{(4)}(x) = \sin(x)$$
$$P_4(x) = \sin\left(\frac{\pi}{6}\right) + \cos\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + \frac{-\sin\left(\frac{\pi}{6}\right)}{2}\left(x - \frac{\pi}{6}\right)^2 + \frac{-\cos\left(\frac{\pi}{6}\right)}{6}\left(x - \frac{\pi}{6}\right)^3 + \frac{\sin\left(\frac{\pi}{6}\right)}{24}\left(x - \frac{\pi}{6}\right)^4$$
$$T_4(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{6}\right)^4$$

(b)

$$\left|f(x) = P_n(x)\right| \leqslant \frac{M(x-a)^{n+1}}{(n+1)!}$$
$$\left|f(x) = P_n(x)\right| \leqslant \frac{1(0 - \frac{\pi}{6})^5}{(5)!} \approx 0.000327953194429$$
$$\left|f(x) = P_n(x)\right| \leqslant \frac{1(\frac{\pi}{3} - \frac{\pi}{6})^5}{(5)!} \approx 0.000327953194429$$

1.2 Problem 4: Stewart 11.20

Question 2

(a) approximate f by a Taylor polynomial with degree n and number a.
(b) Use Taylor's inequality to estimate the accuracy of the approximation $f(x) = T_n(x)$ when x lies in the given interval

Solution:
(a)

$$f(x) = x\ln(x)$$

$$a = 1$$

$$n = 3$$

$$0.5 \leqslant x \leqslant 1.5$$

$$f'(x) = 1 \cdot \ln(x) + x\frac{1}{x} = \ln(x) + 1$$

$$f''(x) = \frac{1}{x} \cdot 1 + 0\ln(x) = \frac{1}{x} = x^{-1}$$

$$f^{(3)}(x) = -x^{-2}$$

$$P_3(x) = \ln(1) + (\ln(1) + 1) + (x-1) + \frac{(1)^{-1}}{2}(x-1)^2 + \frac{-(1)^{-2}}{6}(x-1)^3$$

$$T_3(x) = 0 + (x-1) + \frac{1}{2}(x-1)^2 + -\frac{1}{6}(x-1)^3$$

(b)

$$f^{(4)} = 2(x)^{-3}$$

$$\left|f(x) = P_n(x)\right| \leqslant \frac{M(x-a)^{n+1}}{(n+1)!}$$

$$M = 2(0.5)^{-3} = 16$$

$$|f(x) = P_n(x)| \leqslant \frac{16(0.5-1)^4}{4!} \approx 0.0416666666667$$

$$|f(x) = P_n(x)| \leqslant \frac{16(1.5-1)^4}{4!} \approx 0.0416666666667$$

1.3 Problem 5

Question 3

(a) Use a second degree Taylor polynomial to approximate $\sqrt[5]{34}$. (It's up to you to pick a suitable function to approximate and a suitable base for the approximation.) You should check that your approximation is reasonable
(b) Use the Taylor Error Bound formula to find an upper bound on the error for the approximation you made in part (a), and verify (using a calculator or computer) that your answer to part (a) is indeed within this error bound
(c) Suppose you use the same Taylor polynomial as in part (a) to approximate $\sqrt[5]{30}$. Will the error bound be the same? Why or why not? If not, find an upper bound on the error in using the Taylor polynomial to approximate $\sqrt[5]{34}$.

Solution: (a)

$$a = 32$$
$$f(x) = (x)^{\frac{1}{5}}$$
$$f'(x) = \frac{1}{5}(x)^{\frac{-4}{5}}$$
$$f''(x) = \frac{-4}{25}x^{\frac{-9}{5}}$$
$$P_2(x) = 32^{\frac{1}{5}} + \frac{1}{5}(32)^{-\frac{4}{5}}(x-32) - \frac{4}{50}(32)^{-\frac{9}{5}}(x-32)^2$$
$$P_2(34) = 32^{\frac{1}{5}} + \frac{1}{5}(32)^{-\frac{4}{5}}(34-32) - \frac{4}{50}(32)^{-\frac{9}{5}}(34-32)^2 \approx 2.024375$$

(b)

$$[32, 34]$$
$$f^{(3)}(x) = \frac{36}{125}x^{-\frac{14}{5}}$$
$$M = \frac{36}{125}\,(32)^{-\frac{14}{5}}$$
$$\left|f(x) = P_n(x)\right| \leqslant \frac{M(x-a)^{n+1}}{(n+1)!}$$
$$\left|f(x) = P_n(x)\right| \leqslant \frac{\frac{36}{125}\,(32)^{-\frac{14}{5}}\,(34-32)^{2+1}}{(2+1)!} \approx 0.0000234375$$
$$|\sqrt[5]{34} - 2.024375| = 0.0000224584998851$$
$$0.0000224584998851 < 0.0000234375$$

(c)

$$[30, 32]$$
$$f^{(3)}(x) = \frac{36}{125}x^{-\frac{14}{5}}$$
$$M = \frac{36}{125}\,(1)^{-\frac{14}{5}}$$
$$\left|f(x) = P_n(x)\right| \leqslant \frac{M(x-a)^{n+1}}{(n+1)!}$$
$$\left|f(x) = P_n(x)\right| \leqslant \frac{\frac{36}{125}\,(1)^{-\frac{14}{5}}\,|30-32|^{2+1}}{(2+1)!} \approx 0.384$$

It will be different because the M value is different

1.4 Problem: 6 Stewart 11.11.24

Question 4

Use the information from Exercise 16 to estimate sin(38) correct to five decimal places

Solution:

$$38^\circ = \frac{\pi}{180} \cdot 38 = \frac{19\pi}{90} = \frac{2\pi}{45} + \frac{\pi}{6}$$

$\frac{19\pi}{90}$ is in the range of $\left(0, \frac{\pi}{3}\right)$ so the approximation from question 3 should apply

$$0.00001 \geqslant \frac{1\left(\frac{19\pi}{90} - \frac{\pi}{6}\right)^{n+1}}{(n+1)!}$$

$$0.00001 \geqslant \frac{1\left(\frac{19\pi}{90} - \frac{\pi}{6}\right)^{4+1}}{(4+1)!} \approx 4.4223745988 \cdot 10^7$$

$$T_4(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\left(\frac{2\pi}{45} + \frac{\pi}{6}\right) - \frac{\pi}{6}\right) - \frac{1}{4}\left(\left(\frac{2\pi}{45} + \frac{\pi}{6}\right) - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(\left(\frac{2\pi}{45} + \frac{\pi}{6}\right) - \frac{\pi}{6}\right)^3 + \frac{1}{48}\left(\left(\frac{2\pi}{45} + \frac{\pi}{6}\right) - \frac{\pi}{6}\right)^4$$

$$T_4(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{2\pi}{45}\right) - \frac{1}{4}\left(\frac{2\pi}{45}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{2\pi}{45}\right)^3 + \frac{1}{48}\left(\frac{2\pi}{45}\right)^4 \approx 0.61566$$

1.5 Problem 7

Question 5

Let $P_n(x)$ be the n^{th} degree taylor polynomial for $f(x) = e^x$ based at $x = 0$. Use the error bound Taylor's inequality to show that $\lim_{n\to\infty} |f(x) - P_n(x)| = 0$.

Solution:

$$P_n(x) = e^0 + e^0(x) + \frac{e^0}{2}(x)^2 + \frac{e^0}{6}(x)^3 + \frac{e^0}{24}(x)^4 + \frac{e^0}{120}(x)^5 + \ldots \frac{e^0}{n!}(x)^n$$

$$P_n(x) = 1 + (x) + \frac{1}{2}(x)^2 + \frac{1}{6}(x)^3 + \frac{1}{24}(x)^4 + \frac{1}{120}(x)^5 + \ldots \frac{1}{n!}x^n$$

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}(x)^k$$

$$|f(x) - P_n(x)| \leqslant \frac{M(x-a)^{n+1}}{(n+1)!}$$

$$|f(x) - P_n(x)| \leqslant \frac{e^x(x)^{n+1}}{(n+1)!}, \text{where } |e^z| \leqslant M \text{ for all } z \text{ between } 0 \text{ and } x$$

$$\lim_{n\to\infty} |f(x) - P_n(x) = ?$$

$$\lim_{n\to\infty} \frac{e^x \cdot x^{n+1}}{(n+1)!}$$

$$e^x \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

$$|f(x) - P_n(x)| \leqslant \frac{e^x(x)^{n+1}}{(n+1)!}, \text{where } |e^0| = M \text{ between } -\infty \text{ and } 0 \text{ for x}$$

$$\lim_{n\to\infty} |f(x) - P_n(x) = ?$$

$$\lim_{n\to\infty} \frac{e^0 \cdot x^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

1.6 Problem 8

Question 6

Let $P_n(x)$ be the n^{th} degree taylor polynomial for $f(x) = \frac{1}{1-x}$ based at $x = 0$. It turns out that $\lim_{n\to\infty} P_n(x) = f(x)$ for all x in the interval $(-1, 1)$ Can you show this using the method of the previous problem? If not, what goes wrong?

Solution:

$$f(x) = \frac{1}{1-x} = (1-x)^{-1}$$

$$f'(x) = (1-x)^{-2}$$

$$f''(x) = 2(1-x)^{-3}$$

$$f^n(x) = (n!)(1-x)^{-(n+1)}$$

$$P_n(x) = (1-x)^{-1} + (1-x)^{-2}(x) + \frac{2(1-x)^{-3}}{2}(x)^2 + \ldots + \frac{(n!)(1-x)^{-(n+1)}}{n!}(x)^n$$

$$P_n(x) = \sum_{k=0}^{n} (1)^{-(n+1)}(x)^k$$

$$|f(x) - P_n(x)| \leqslant \frac{M(x-a)^{n+1}}{(n+1)!}$$

$$M = (n+1)!(1-x)^{-(n+2)}$$

$$|f(x) - P_n(x)| \leqslant \frac{\left((n+1)!(1-x)^{-(n+2)}\right)(x-0)^{n+1}}{(n+1)!}$$

$$|f(x) - P_n(x)| \leqslant \frac{\left((n+1)!(1-x)^{-(n+2)}\right)(x)^{n+1}}{(n+1)!}$$

$$|f(x) - P_n(x)| \leqslant \left((1-x)^{-(n+2)}\right)(x)^{n+1}$$

$$\lim_{n\to\infty} \left((1-x)^{-(n+2)}\right)(x)^{n+1}$$

$$\lim_{n\to\infty} \left((1-x)^{-(n+2)}\right)(x)^{n+1}$$

$$\left((1-x)^{-(n+2)}\right)(x)^{n+1} = \frac{x^{n+1}}{(1-x)^{n+2}} = \frac{1}{1-x}\left(\frac{x}{1-x}\right)^{n+1}$$

$$\lim_{n\to\infty} \left(\frac{x}{1-x}\right)^{n+1}$$

$$t = \frac{x}{1-x} < 1$$

$$\frac{x}{1-x} > 1 \text{ on interval } \left[\frac{1}{2}, 1\right]$$
$$\frac{x}{1-x} > 1 \text{ on interval } \left(-1, \frac{1}{2}\right)$$
$$\lim_{n\to\infty} t^{n+1}$$
$$\lim_{n\to\infty} t^{n+1} = 0 \text{ on interval } -1 < x < \frac{1}{2}$$
$$\lim_{n\to\infty} t^{n+1} = \infty \text{ on interval } \frac{1}{2} \leqslant x \leqslant 1$$
$$\text{on the interval of } -1 < x < \frac{1}{2} \lim_{n\to\infty} \left(\frac{x}{1-x}\right)^{n+1} \text{ limit approaches } 0$$
$$\text{on the interval of } \frac{1}{2} < x < 1 \lim_{n\to\infty} \left(\frac{x}{1-x}\right)^{n+1} \text{ limit approaches } \infty$$

On the interval of $-1 < x < \frac{1}{2}$, $\lim_{n\to\infty} \frac{1}{1-x} \left(\frac{x}{1-x}\right)^{n+1} = 0$, however this is not true on the interval of $\frac{1}{2} \leqslant x \leqslant 1$, as on this interval the limit approaches ∞. This means that $P_n(x) = f(x)$ is only true on the interval of $-1 < x < \frac{1}{2}$.