Math 115 QR

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Contents

Chapter 1

1.1 Problem 1: Stewart 11.2.20

Question 1

Determine whether the series is convergent or divergent by expressing S_n as a telescoping sum. If it is convergent find its sum

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

Solution:

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty} \ln(n) - \ln(n+1)$$

$$S_n = (\ln(1) - \ln(1+1)) + (\ln(2) - \ln(1+2)) + \ldots + (\ln(n) - \ln(n+1))$$

$$S_n = (\ln(1) - \ln(2)) + (\ln(2) - \ln(3)) + \ldots + (\ln(n) - \ln(n+1))$$

$$S_n = (\ln(1) - \ln(n+1))$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (\ln(1) - \ln(n+1)) = -\infty$$

$$\text{diverges}$$

1.2 Problem 2

Question 2

What does the Test for Divergence tell you about convergence or divergence of each of the following series?
(a). $\sum_{n=30}^{\infty} \frac{1}{n^n}$
(b). $\sum_{n=500}^{\infty} (\cos(1)^n)$
(c). $\sum_{n=500}^{\infty} \cos(n)$
(d). $\sum_{k=2}^{\infty} \frac{2^k+1}{3^k+1}$

Solution: (a).

$$\lim_{n\to\infty} \frac{1}{n^n} = 0$$

The Test for Divergence is inconclusive here. This test alone does not tell us whether the series converges or diverges. We would need to use another test to determine convergence or divergence
(b)

$$\lim_{n\to\infty} \cos(1)^n = 0$$

the Test for Divergence is again inconclusive. However, since $\cos(1) < 1$, this series is a geometric series with a ratio less than 1, which converges.
(c)

$$\lim_{n\to\infty} \cos(n) \neq 0$$

The cosine function oscillates between -1 and 1 and does not approach zero as n approaches infinity. Therefore,$\lim_{n\to\infty}$ does not exist. Since the limit of $\cos(n)$ does not approach zero, the Test for Divergence tells us that the series diverges
(d)

$$\lim_{n\to\infty} \frac{2^k + 1}{3^k + 1} = 0$$

Since $\lim_{n\to\infty} \frac{2^k+1}{3^k+1} = 0$, the Test for Divergence is inconclusive here. This test alone does not tell us whether the series converges or diverges

1.3 Problem 3

Question 3

Consider the Alternating Harmonic Series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$, which can be written in summation notation as $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$
(a). Plot the partial sums $s_n = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ versus n for $n = 1, 2, \ldots, 8$
(b). Explain why $S_1 > S_3 > S_5 > \ldots$, ie., why the odd partial sums are decreasing.
(c). Explain why $S_2 < S_4 < S_6 < \ldots$, i.e., why the even partial sums are increasing.
(d). Explain why $\frac{1}{2} \leqslant s_n \leqslant 1$ for all n.
(e). The Monotonic Convergence Theorem states that any increasing sequence that is bounded above must converge, as must any decreasing sequence that is bounded below. Since the odd partial sums of the Alternating Harmonic Series are decreasing and bounded below by $\frac{1}{2}$ the odd partial sums converge. Since the even partial sums are increasing and bounded above by 1, the even partial sums also converge. Explain why the odd partial sums and the even partial sums must converge to the same number. Hint: What is $\lim_{n\to\infty} (s_{2n+1} - s_{2n})$

Solution:

Problem 3 continued

(b)

$$(-1)^{2n+1} = -1$$
$$(-1)^{2n} = 1$$
$$a_{2n} < 0$$
$$a_{2n+1} > 0$$
$$|a_k| > |a_{k+1}|$$
$$S_{2n+3} = S_{2n+1} + a_{2n+2} + a_{2n+3}$$
$$S_{2n+1} > S_{2n+1} + a_{2n+2} + a_{2n+3}$$
$$S_{2n+1} > S_{2n+3}$$

(c)

$$S_{2n} < S_{2n+2}$$
$$a_{2n} < 0$$
$$|a_{2n}| > |a_{2n+2}|$$
$$a_{2n+1} + a_{2n+2} > 0$$
$$S_{2n+2} = S_{2n} + a_{2n+1} + a_{2n+2}$$
$$S_{2n} > S_{2n} + a_{2n+1} + a_{2n+2}$$
$$S_{2n} > S_{2n+2}$$

(d)

$$S_1 = 1$$
$$S_2 = \frac{1}{2}$$
$$S_2 \leqslant S_n \leqslant S_1$$
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots \frac{1}{n}$$
$$|S_1 - S_2| = \frac{1}{2}$$
$$|S_2 - S_3| = \frac{1}{3}$$
$$|S_n - S_{n+1}| = \frac{1}{n+1}$$
$$|S_n - S_{n+1}| > |S_{n+1} - S_{n+2}|$$
$$(-1)^{2n+1} = -1$$
$$(-1)^{2n} = 1$$

The difference between partial sums is decreasing and $S_{2n} > 0 > a_{2n+1}$ for $m \in \mathbb{Z}$. S_n will never pass the lower bounds upper bounds

(e).

$$\lim_{n\to\infty}(s_{2n+1} - s_{2n}) = \lim_{n\to\infty}\frac{1}{n+1} = 0$$

$$\forall n \in \mathbb{Z} s_{2n+1} > S_{2n}$$

$$s_{2n} = \text{ event sum}$$

$$s_{2n+1} = \text{ odd sum}$$

Because $\lim_{n\to\infty}(s_{2n+1} - s_{2n}) = 0$, the difference between S_{2n+1} and S_{2n} is approaching 0, meaning they are the same value. If $n - m = 0$
$n = m$

1.4 Problem 4

Question 4

Use the Comparison Test to decide whether the series

$$\sum_{k=3}^{\infty}\frac{2+(-1)^k}{k}$$

converges or diverges. Explain your reasoning. (In particular, explain why it is valid to use comparison and what you are comparing to).

Solution:

$$\frac{1}{k} \leqslant \frac{2+(-1)^k}{k} \leqslant \frac{3}{k}$$

$$a_k = \frac{1}{k}$$

$$b_k = \frac{2+(-1)^k}{k}$$

$$\sum_{n=3}^{\infty}\frac{1}{k} \text{ diverges}$$

Since the series is always greater that or equal to the harmonic series, which diverges, it can be said that this series also diverges using the comparison test. $a_k \leqslant b_k$ and a_k diverges so b_k diverges as well.

1.5 Problem 5

Question 5

A ball is dropped out of a window from a height of 100 feet. Assume that each time the ball falls from a height h that it rebounds to 30% of height h. Assuming the ball bounces indefinitely, determine the total distance it travels.

Solution:

$$100 + 100 + 2(30) + 2(9) + \ldots$$

$$D = 100 + 0.3(100)2 + (0.3)^2(100)(2) + \ldots (0.3)^n(2)(100)$$

$$100 + \sum_{n=1}^{\infty} 2(100)(0.3^n)$$

$$\sum_{n=1}^{\infty} ar^n = \frac{200}{1-r}$$

$$a = (200)(0.3)^1 = 60$$

$$r = 0.3$$

$$100 + \frac{200}{1-0.3} = 100 + \frac{600}{7} \approx 185.714 \text{ feet}$$