A New Alpha Power Type-1 Family of Distributions and Modelling the Overdispersed Count Outcome: Supplementary Material

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1 Special cases of NAPF1 of distributions

The analytical and graphical presentations of the the special cases of the NAPF1 of distributions are given in this section. The $F\left(z; \alpha, \boldsymbol{\xi}\right)$ and $f\left(z; \alpha, \boldsymbol{\xi}\right)$ of the NAPF1 are given by:

$$F\left(z; \alpha, \boldsymbol{\xi}\right) = 1 - \alpha^{-\left(\frac{G(z;\boldsymbol{\xi})}{1-G(z;\boldsymbol{\xi})}\right)}, \qquad \alpha > 1, z \in \mathbb{R}, \tag{1}$$

and

$$f\left(z; \alpha, \boldsymbol{\xi}\right) = \frac{\log(\alpha) g(z;\boldsymbol{\xi})}{\left[1 - G(z;\boldsymbol{\xi})\right]^2} \alpha^{-\left(\frac{G(z;\boldsymbol{\xi})}{1-G(z;\boldsymbol{\xi})}\right)}, \qquad \alpha > 1, z \in \mathbb{R}, \tag{2}$$

respectively.

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1.1 A new alpha power type-1-Lomax distribution

Let Z be a Lomax random variable with the CDF:

$$G\left(z;\boldsymbol{\xi}\right)=1-\left(1+\lambda z\right)^{-\beta}, \qquad \lambda,\beta>0, z\geq 0, \tag{3}$$

where λ and β are the scale and shape parameters, respectively.
Corresponding to Eq. (3), the PDF $g\left(z;\boldsymbol{\xi}\right)$, survival function $S\left(z;\boldsymbol{\xi}\right)$, hazard function $h\left(z;\boldsymbol{\xi}\right)$, and cumulative hazard function $H\left(z;\boldsymbol{\xi}\right)$ are given by

$$g\left(z;\boldsymbol{\xi}\right)=\frac{\lambda\beta}{\left(1+\lambda z\right)^{(1+\beta)}},$$

$$S\left(z;\boldsymbol{\xi}\right)=\left(1+\lambda z\right)^{-\beta},$$

$$h\left(z;\boldsymbol{\xi}\right)=\frac{\lambda\beta}{\left(1+\lambda z\right)},$$

and

$$H\left(z;\boldsymbol{\xi}\right)=\beta\log\left(1+\lambda z\right),$$

respectively, and $\boldsymbol{\xi}=(\lambda,\beta)^T$.

By inserting Eq. (3) into Eq. (1), we obtain the CDF of NAPF1-Lomax distribution as follows:

$$F\left(z;\alpha,\boldsymbol{\xi}\right)=1-\alpha^{-\left((1+\lambda z)^{\beta}-1\right)}, \qquad \alpha>1,\beta,\lambda>0, z\geq 0. \tag{4}$$

In addition, the $f\left(z;\alpha,\boldsymbol{\xi}\right)$, $S\left(z;\alpha,\boldsymbol{\xi}\right)$, and $h\left(z;\alpha,\boldsymbol{\xi}\right)$ can be obtained by using the general forms.

$$f\left(z;\alpha,\boldsymbol{\xi}\right)=\frac{\log(\alpha)\left(\lambda\beta/\left(1+\lambda z\right)^{(1+\beta)}\right)}{\left(1+\lambda z\right)^{-2\beta}}\alpha^{-\left((1+\lambda z)^{\beta}-1\right)}, \qquad z>0, \tag{5}$$

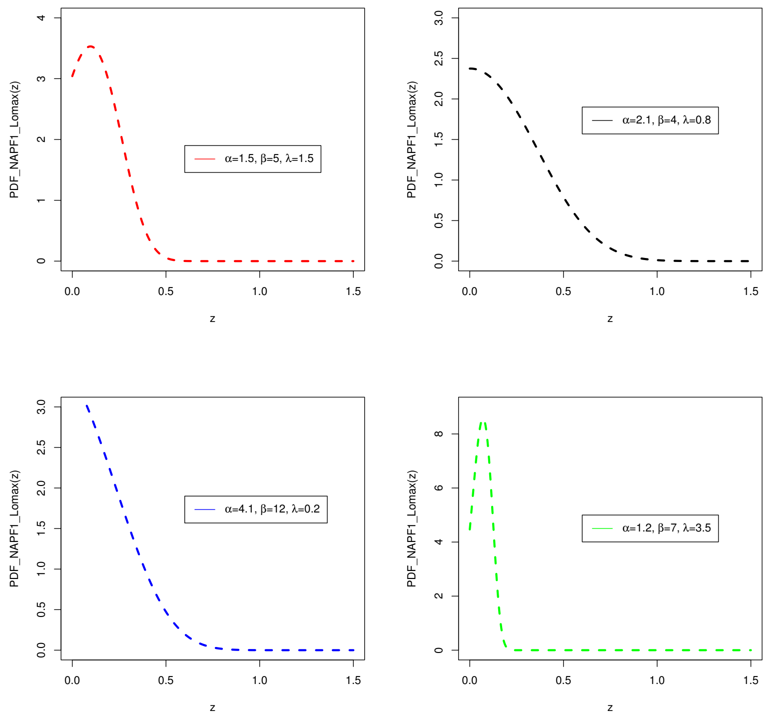
$$S\left(z;\alpha,\boldsymbol{\xi}\right)=\alpha^{-\left(1+\lambda z)^{\beta}-1\right)},$$

and

$$h\left(z;\alpha,\boldsymbol{\xi}\right)=\frac{\log(\alpha)\left(\lambda\beta/\left(1+\lambda z\right)^{(1+\beta)}5\right)}{\left(1+\lambda z\right)^{-2\beta}},$$

respectively.

(a) Visual illustration of the $f\left(z;\alpha,\boldsymbol{\xi}\right)$ of the NAPF1-Lomax.



(b) Visual illustration of the $h\left(z;\alpha,\boldsymbol{\xi}\right)$ of the NAPF1-Lomax.

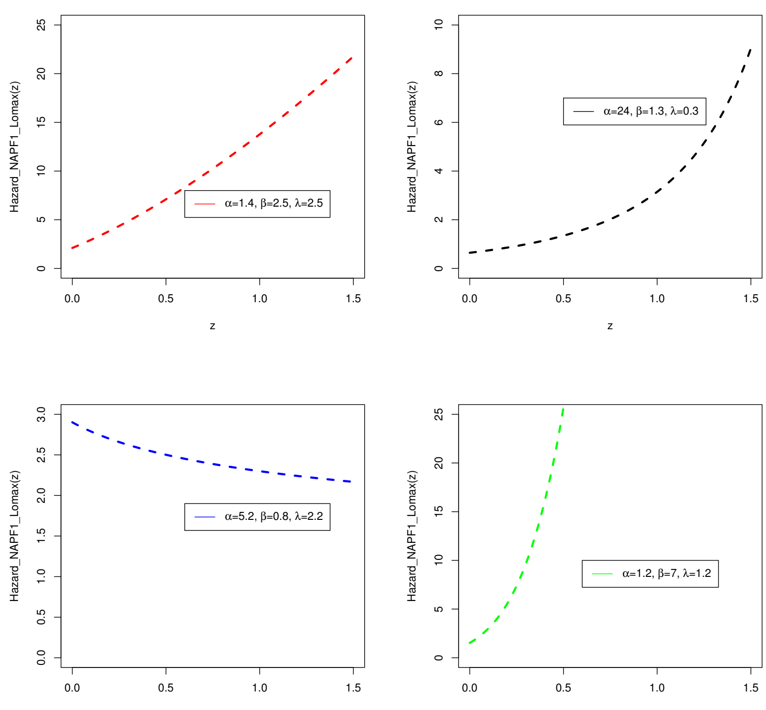


Figure 1: Plots of the $f\left(z;\alpha,\boldsymbol{\xi}\right)$ and $h\left(z;\alpha,\boldsymbol{\xi}\right)$ for the NAPF1-Lomax distribution for different scenarios.

Figure 1 displays the plots of $f(z;\alpha,\boldsymbol{\xi})$ (a) and $h(z;\alpha,\boldsymbol{\xi})$ (b) for the NAPF1-Lomax for different scenarios: (i) $\alpha = 1.5, \beta = 5, \lambda = 1.5$, (ii) $\alpha = 2.1, \beta = 4, \lambda = 0.8$, (iii) $\alpha = 4.1, \beta = 12, \lambda = 0.2$, (iv) $\alpha = 1.2, \beta = 7, \lambda = 3.5$ and (i) $\alpha = 1.4, \beta = 2.5, \lambda = 2.5$, (ii) $\alpha = 24, \beta = 1.3, \lambda = 0.3$, (iii) $\alpha = 5.2, \beta = 0.8, \lambda = 2.2$, (iv) $\alpha = 1.2, \beta = 7, \lambda = 1.2$, respectively. The graphs of the PDF and the hazard rate for the NAPF1-Lomax model seem to have decreasing, increasing-decreasing-constant, and increasing patterns.

1.2 A new alpha power type-1-Gompertz distribution

Let Z be a Gompertz random variable with the CDF:

$$G(z;\boldsymbol{\xi}) = 1 - e^{-\lambda/\beta\left(e^{\beta z}-1\right)}, \qquad \lambda, \beta > 0, z \geq 0, \tag{6}$$

where λ and β are the scale and shape parameters, respectively.
Based on Eq. (6), the $g(z;\boldsymbol{\xi})$, $S(z;\boldsymbol{\xi})$, $h(z;\boldsymbol{\xi})$, and $H(z;\boldsymbol{\xi})$ are given by:

$$g(z;\boldsymbol{\xi}) = \lambda e^{\beta z} e^{-\lambda/\beta\left(e^{\beta z}-1\right)},$$

$$S(z;\boldsymbol{\xi}) = e^{-\lambda/\beta\left(e^{\beta z}-1\right)},$$

$$h(z;\boldsymbol{\xi}) = \lambda e^{\beta z},$$

and

$$H(z;\boldsymbol{\xi}) = \lambda/\beta\left(e^{\beta z}-1\right),$$

respectively.
By inserting Eq. (6) into Eq. (1), we obtain the CDF of the NAPF1-Gompertz (NAPF1-Gomp, for short) distribution as follows:

$$F(z;\alpha,\boldsymbol{\xi}) = 1 - \alpha^{-\left(e^{\lambda\beta^{-1}\left(e^{\beta z}-1\right)}-1\right)}, \qquad \alpha > 1, \lambda, \beta > 0, z \geq 0. \tag{7}$$

In addition, the $f(z;\alpha,\boldsymbol{\xi})$, $S(z;\alpha,\boldsymbol{\xi})$, $h(z;\alpha,\boldsymbol{\xi})$, and $H(z;\alpha,\boldsymbol{\xi})$ can be obtained by using the general forms.

$$f(z;\alpha,\boldsymbol{\xi}) = \lambda\log(\alpha) e^{\beta z} e^{\lambda/\beta\left(e^{\beta z}-1\right)} \alpha^{-\left(e^{\lambda\beta^{-1}\left(e^{\beta z}-1\right)}-1\right)}, \qquad z > 0, \tag{8}$$

$$S(z;\alpha,\boldsymbol{\xi}) = \alpha^{-\left(e^{\lambda\beta^{-1}\left(e^{\beta z}-1\right)}-1\right)},$$

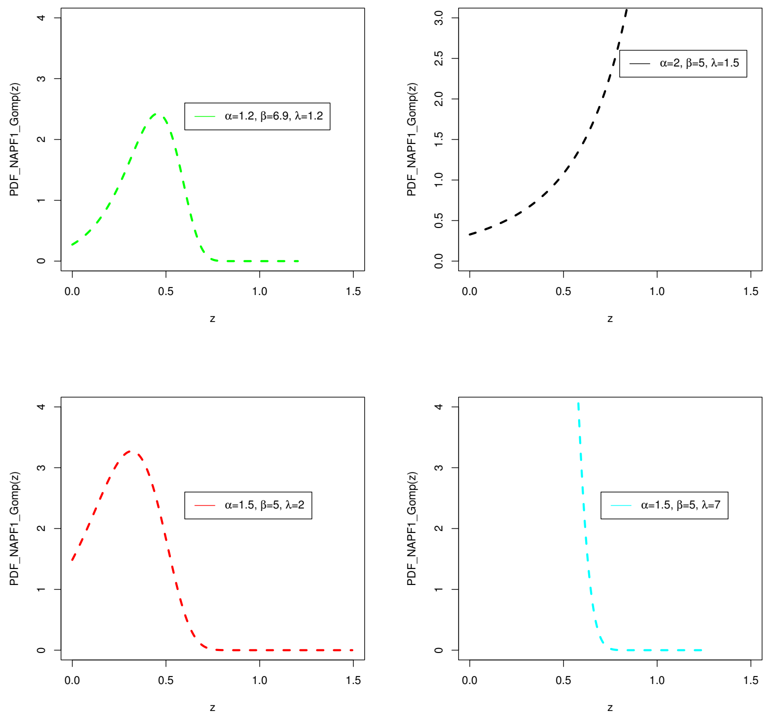
$$h(z;\alpha,\boldsymbol{\xi}) = \lambda\log(\alpha) e^{\beta z} e^{\lambda/\beta\left(e^{\beta z}-1\right)},$$

and

$$H(z;\alpha,\boldsymbol{\xi}) = -\log\left(\alpha^{-\left(e^{\lambda\beta^{-1}\left(e^{\beta z}-1\right)}-1\right)}\right),$$

respectively.

(a) Visual illustration of the $f(z; \alpha, \boldsymbol{\xi})$ of the NAPF1-Gomp.



(b) Visual illustration of the $h(z; \alpha, \boldsymbol{\xi})$ of the NAPF1-Gomp.

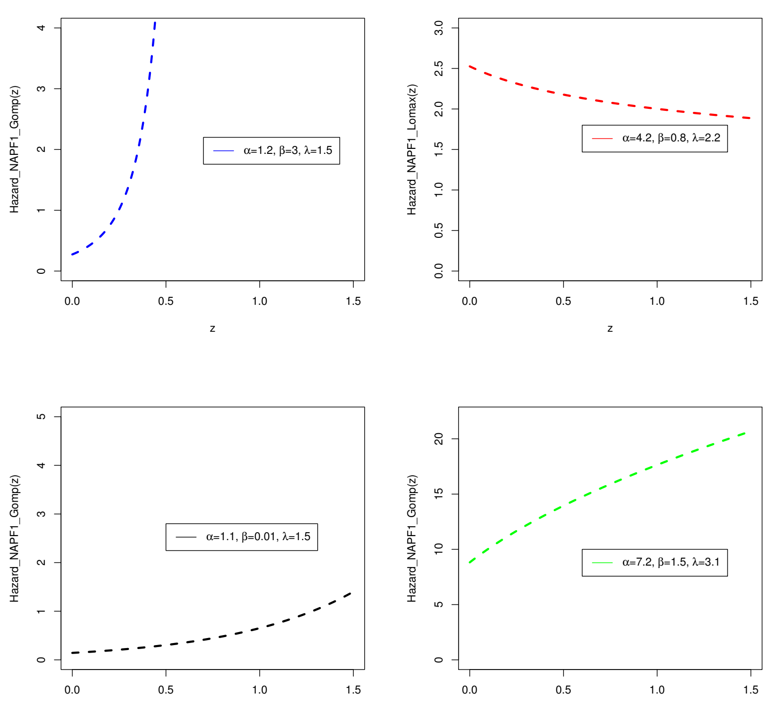


Figure 2: Plots of the $f(z; \alpha, \boldsymbol{\xi})$ and $h(z; \alpha, \boldsymbol{\xi})$ for the NAPF1-Gomp distribution for different scenarios.

Figure 2 displays the plots of $f(z;\alpha,\boldsymbol{\xi})$ (a) and $h(z;\alpha,\boldsymbol{\xi})$ (b) for the NAPF1-Gomp for different scenarios: (i) $\alpha=1.2, \beta=6.9, \lambda=1.2$, (ii) $\alpha=2, \beta=5, \lambda=1.5$, (iii) $\alpha=1.5, \beta=5, \lambda=2$, (iv) $\alpha=1.5, \beta=5, \lambda=7$ and (i) $\alpha=1.2, \beta=3, \lambda=1.5$, (ii) $\alpha=4.2, \beta=0.8, \lambda=2.2$, (iii) $\alpha=1.1, \beta=0.01, \lambda=1.5$, (iv) $\alpha=7.2, \beta=1.5, \lambda=3.1$, respectively. The figure shows right-skewed, increasing, increasing-decreasing, and bathtub patterns.

The mathematical expressions of the $F\left(z;\hat{\alpha},\hat{\boldsymbol{\xi}}\right)$ and $S\left(z;\hat{\alpha},\hat{\boldsymbol{\xi}}\right)$ of NAPF1-Weib are given by

$$F\left(z;\hat{\alpha},\hat{\boldsymbol{\xi}}\right)=1-32.000^{-\left(e^{0.012z^{3.566}}-1\right)}, \qquad z\geqslant 0,$$

and

$$S\left(z;\hat{\alpha},\hat{\boldsymbol{\xi}}\right)=32.000^{-\left(e^{0.012z^{3.566}}-1\right)}, \qquad z>0,$$

respectively.

The quantile-quantile (Q-Q) plots of the estimated CDF and $S\left(z;\hat{\alpha},\hat{\boldsymbol{\xi}}\right)$ including the Total Time on Test (TTT), the plot for estimated CDF are given in Figure 3 below.

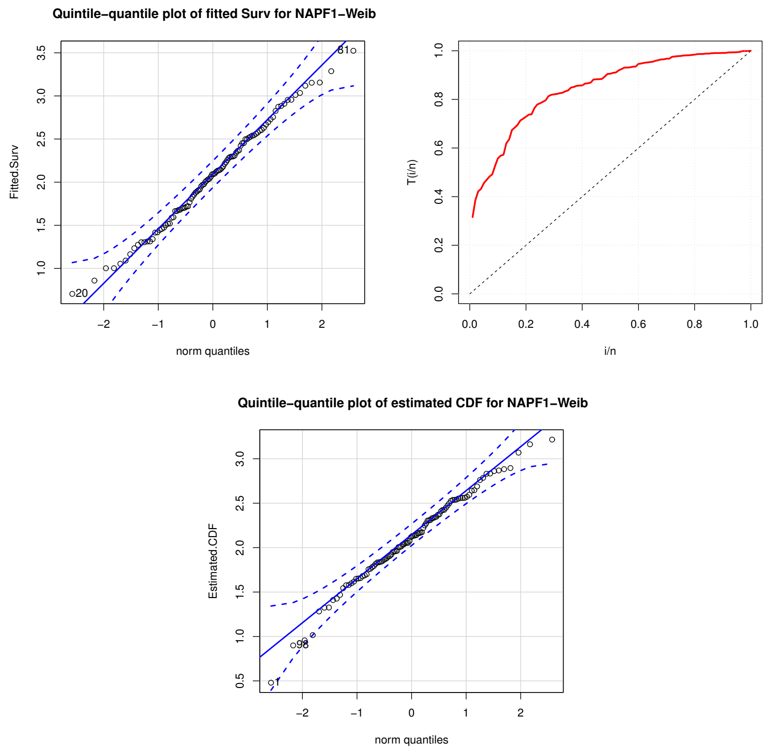


Figure 3: Plot of Q-Q for estimated CDF and $S\left(z;\hat{\alpha},\hat{\boldsymbol{\xi}}\right)$ and TTT for estimated CDF for the NAPF1-Weib distribution.

From Figure 3, it is clearly observed that there is no potential influential observation and the data is linear and normal enough. In addition, the TTT plot shows the model has an increasing shape in this data set.

1.3 Numerical and graphical description of the CHF data

The numerical visualization in Table 1 shows 121 (36.7%) of the patients among the total number of patients who were not first visited by doctors in their follow-up. This indicates that there are excess zeros in the data. And it is clearly seen that the observed variance (2.0) is greater than the observed mean (1.2) which shows that there is an overdispersion in the data. The graphical elicitation via histogram in Fig. (4) also shows that most of the observations are skewed (concentrated) to zero.

Table 1: Numerical visualization of Zero-inflation and overdispersion.

Zero-inflation		Overdispersion		
# visits	Frequency	Percent	Mean	Variance
0	121	36.7		
1	98	29.7		
2	67	20.3		
3	19	5.8		
4	11	3.3		
5	9	2.7		
6	3	0.9		
7	2	0.6		
Total	330	100.0	1.2	2.0

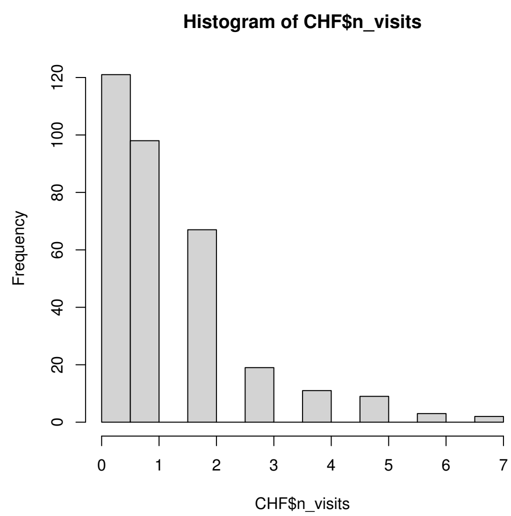


Figure 4: The Histogram of the number of visits for CHF patients.

Table 2: Frequentist model Comparison of P, NB, ZIP, ZINB, and COMP models. The table summarizes the parameter estimates and standard errors (SE.) together with model comparison criteria results.

		P	NB	ZIP
Effect	Parameter	Estimate (SE.)	Estimate (SE.)	Estimate (SE.)
Intercept	ξ_0	-0.155(0.395)	-0.694(0.274)$*$	-0.594(0.310)
sex-male	ξ_1	-0.309(0.141)$*$	-0.203(0.101)$*$	0.033(0.117)
time	ξ_2	0.570(0.056)$*$	0.315(0.031)$*$	0.368(0.029)$*$
Place-Urban	ξ_3	-0.015(0.141)	-0.019(0.097)	-0.019(0.110)
wieghts	ξ_4	0.006(0.007)	0.005(0.004)	0.004(0.005)
heart-disease-yes	ξ_5	0.240(0.142)	0.155(0.095)	0.027(0.110)
-2logL		530.6	959.4	448.8
AIC		1075.2	973.4	923.5
BIC		1101.8	1000.0	972.9
		ZINB	COMP	
Effect	Parameter	Estimate (SE.)	Estimate (SE.)	
Intercept	ξ_0	-2.911(1.224)$*$	-0.785(0.375)$*$	
sex-male	ξ_1	1.067(0.467)$*$	-0.167(0.142)$*$	
time	ξ_2	0.711(0.139)$*$	0.302(0.034)$*$	
Place-Urban	ξ_3	-0.040(0.428)	0.035(0.127)	
wieghts	ξ_4	-0.010(0.020)	0.004(0.006)	
heart-disease-yes	ξ_5	-0.742(0.475)	0.059(0.128)	
-2logL		448.8	476.3	
AIC		921.5	964.7	
BIC		967.1	987.5	

2 Results for simulated data

Additionally, as displayed in the supplementary material, the graphical presentation of the Posterior distribution of the Parameters, the chain convergence for simulated data, marginal densities of the parameters, and the 95% highest posterior densities (HPD) for both link scales (log and logit) are given in Fig.s (5) and (6), respectively. In the two figures, the 95% HPD shows the parameters interval where they are significant. Thus, the red lined cross like interval shows the significant parameters.

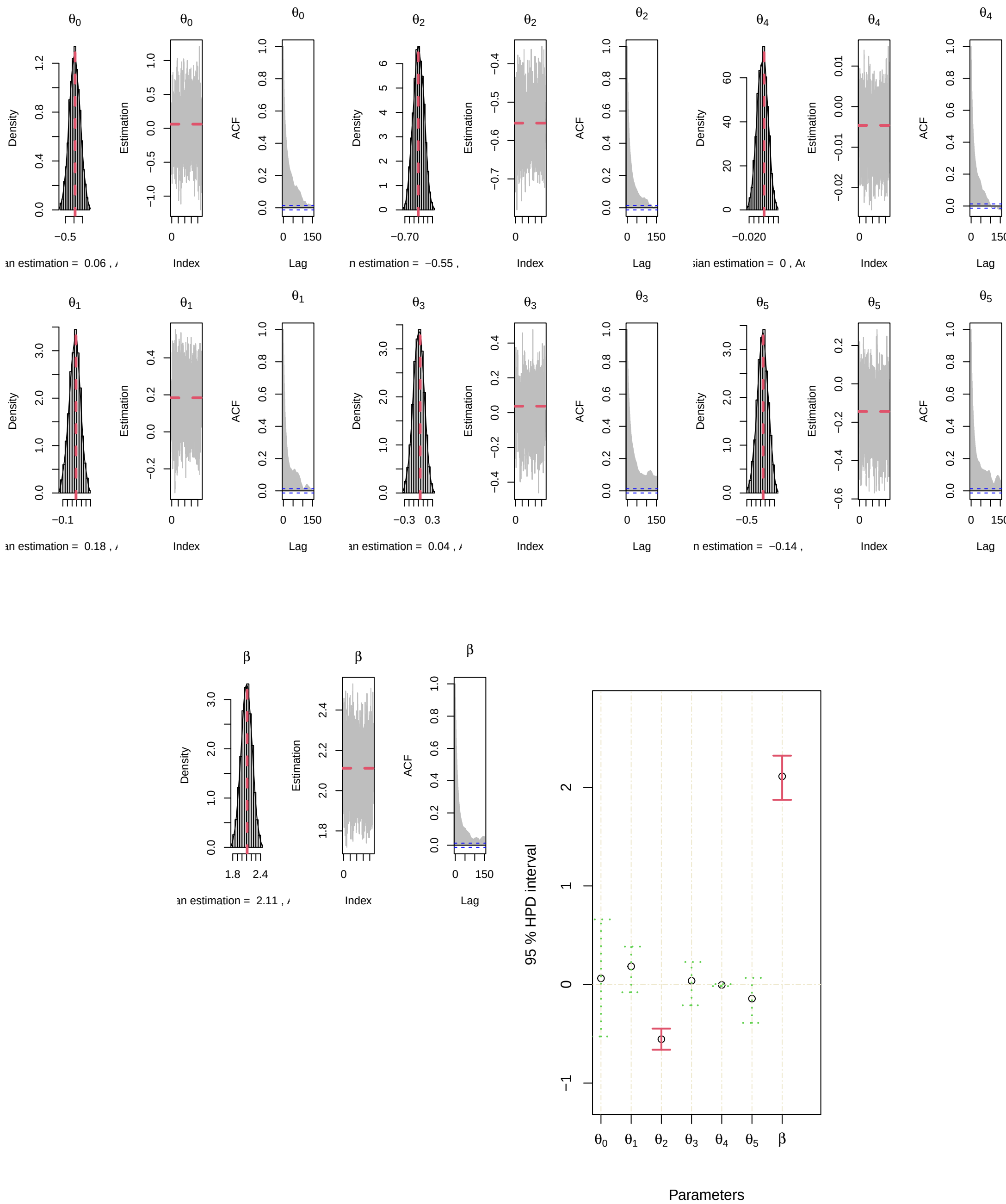


Figure 5: Plot of the Posterior distribution of the Parameters, the chain convergence for simulated data and marginal densities of the parameters, and the 95% HPD for the log scale.

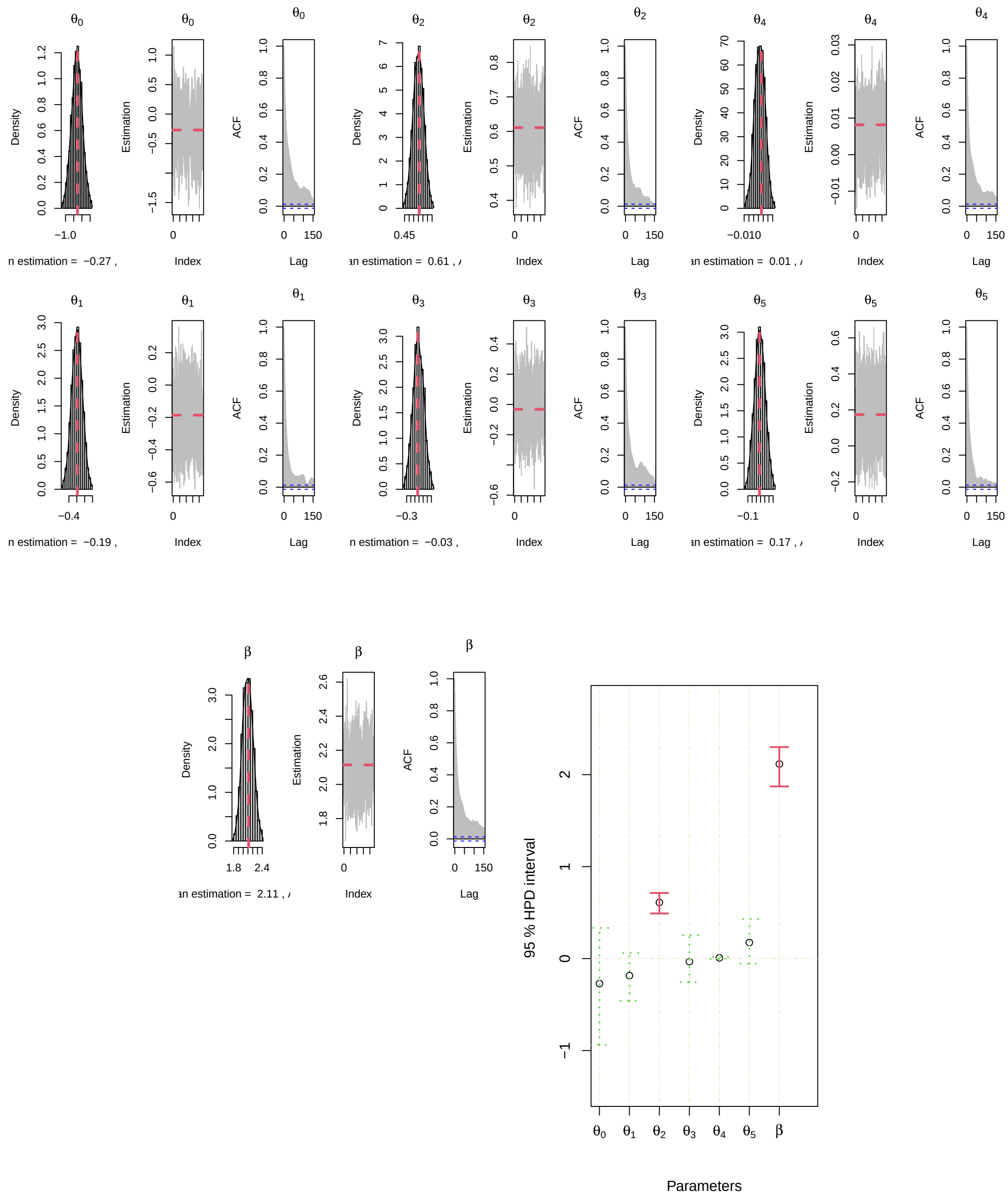


Figure 6: Plot of the Posterior distribution of the Parameters, the chain convergence for simulated data and marginal densities of the parameters, and the 95% HPD for the logit link scale.