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DISCRETE STRUCTURE

Introduction

Discrete Mathematics deals with discrete objects.
Discrete object are those objects that can be counted.
For example:- Trees, integers, houses, etc.

Logic

Logic is a language for reasoning. Since logic can help us to reason the mathematical models. It need some rules associated with logic. So that we can apply those rules for mathematical reasoning. Application maybe for designing circuit, programming, program verification, etc.

Propositions and Propositional Calculus

- Proposition is a fundamental concept of logic.
- Proposition is a declarative sentence that is either true or false, but not both.
e.g. $2+2=5$ (false); $7-1=6$ (true); Kathmandu is capital city of Nepal (true) are some of the examples of proposition.

$x > 5$; come here; $3+4$ are some examples which are not propositions.

- Propositions are denoted by using small letters like p, q, r, s, \dots . The truth value of proposition is denoted by T for true proposition and F for false proposition.

→ The logic that deals with proposition is called propositional logic or propositional calculus.

Logical Operators / Connectives

→ Logical operators are used to connect mathematical statements having one or more propositions by combining the propositions.

→ The combinational propositional is called compound proposition.

→ The truth table is used to get the relationship between truth values of propositions.

1. Negation (NOT)

Gives a proposition p , negation operator (\neg) is used to get negation of p , denoted by $\neg p$ called "not p ".

Eg:- $p = \text{I love birds}$

$\neg p = \text{I don't love birds}$.

p	$\neg p$
T	F
F	T

2. Conjunction (AND)

Gives two propositions p and q , the proposition " $p \wedge q$ " denoted by $p \wedge q$ is the proposition that is true whenever p and q are true, false otherwise.

Eg:- $p = \text{"Ram is intelligent"}$

$q = \text{"Ram is diligent"}$

$p \wedge q = \text{Ram is intelligent and diligent.}$

P	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

3. Disjunction (OR):

Given two propositions p and q , the proposition " p OR q " denoted by $p \vee q$ is the proposition that is false whenever both the proposition p and q are false, true otherwise.

Eg:- p = "Ram is intelligent"

q = "Ram is deligent"

$p \vee q$ = "Ram is Intelligent or deligent".

P or q	1 such p v q			
T	T	T	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	F

Construct a truth table

1. $\neg p \wedge (p \vee \neg q)$

P	q	$\neg p$	$\neg q$	$(p \vee \neg q)$	$\neg p \wedge (p \vee \neg q)$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	F
F	F	T	T	T	T

$$11. (p \wedge q) \vee (\neg q \wedge r)$$

P	q	r	$\neg q$	$p \wedge q$	$\neg q \wedge r$	$(p \wedge q) \vee (\neg q \wedge r)$
T	T	T	F	T	F	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	F	T	F	F	F
F	T	T	F	F	F	F
F	T	F	F	F	F	F

4. Exclusive OR (XOR)

Given two propositions p and q, the proposition exclusive or of p and q denoted by $p \oplus q$ is the propositions that is true whenever only one of the propositions p and q is true, false otherwise.

Eg:- p = "Ram drinks coffee in the morning".

q = "Ram drinks tea in the morning".

$p \oplus q$ = Ram drinks tea or coffee in the morning

P	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

5. Implication

Given two propositional p and q , the proposition implication $p \rightarrow q$, is the proposition that is false when p is true and q is false, true otherwise. Here p is called "Hypothesis" or "antecedent" or "premise" and q is called "conclusive" or "consequence".

Different terminologies to express $p \rightarrow q$ are like:-

1. "if p then q "
2. " q is consequence of p "
3. " p is sufficient of q "
4. " q if p "
5. " q is necessary for p "
6. " q follows from p "
7. "if p , q "
8. " p implies q "
9. " p only if q "
10. " q whenever p "
11. " q provides p "

P	q	$p \rightarrow q$	Truth value of $p \rightarrow q$
T	T	T	
T	F	F	" p is true but q is false" = F
F	T	T	" p is false and q is true" = T
F	F	T	" p is false and q is false" = T

Converse, Inverse and Contrapositive

Some of related implication formed from $p \rightarrow q$ are:-

Converse: $q \rightarrow p$

Inverse: $\neg p \rightarrow \neg q$

Contrapositive: $\neg q \rightarrow \neg p$

$P = \text{"today is Sunday"}$

$q = \text{"it is not today"}$

Implication = If today is Sunday, it is hot.

Converse :- It is hot today only if today is Sunday.

Inverse :- If today is not Sunday, it is not hot.

Contrapositive :- If it is not hot today, it is not Sunday.

Is contrapositive same as $p \rightarrow q$? Verify.

P	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

$$p \rightarrow q = \neg q \rightarrow \neg p$$

Q. Let p, q and r be the positions.

$P = \text{"You have the flu"}$

$q = \text{You } \cancel{\text{have}} \text{ miss the final examination"}$

$r = \text{"You pass the course".}$

Express each of the proportion as an English sentence and construct the truth table.

① $p \rightarrow q$

If You have flu, you miss final examination.

(ii) $q \rightarrow \neg r$

If You miss the final exam, you will not pass the course.

(iii) $(p \rightarrow \neg r) \vee (\neg q \rightarrow \neg r)$

If you have the flu, you will not pass the course or if you miss final exam, you will not pass the course.

 $\neg (p \wedge q) \Leftrightarrow$

P	q	r	$\neg r$	$p \rightarrow \neg r$	$\neg q \rightarrow \neg r$	$p \rightarrow q$	$(p \rightarrow \neg r) \vee (\neg q \rightarrow \neg r)$
T	T	T	F	F	F	T	F
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	T	F	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	T	T	T	T	T

Note:-

To translate english sentence to the proposition symbolic form, follows these steps:-

1. Restart the given sentence into building block sentence.
2. Give the symbol to each sentence and substitute the symbol using connectives:-

Q. If it is snowing then I will go to the beach.

$p \rightarrow$ "It is snowing" $q =$ "I will go to beach"

$\Rightarrow p \rightarrow q$

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P if q

- Q. You can access the internet from campus, if you are a computer science major or you are not the freshman.

\Rightarrow

$P =$ You can access the internet from campus.

$q =$ You are a computer science major

$r =$ You are the freshman

$$\Rightarrow (q \vee \neg r) \rightarrow P$$

Construct the truth table:

$$(P \leftrightarrow q) \oplus (\neg p \rightarrow q) \vee (q \rightarrow \neg r)$$

P	q	r	$\neg p$	$\neg r$	$P \leftrightarrow q$	$\neg p \rightarrow q$	$q \rightarrow \neg r$	\oplus
T	T	T	F	F	T	T	F	T
T	T	F	F	T	T	T	F	T
T	F	T	F	F	F	T	T	T
T	F	F	F	T	F	T	T	T
F	T	T	T	F	F	T	T	T
F	T	F	F	T	T	F	T	T
F	F	T	T	F	T	T	F	T
F	F	F	T	T	F	T	T	T

Function F is True if and only if P is True and Q is False or P is False and Q is True.

Now we have to find the value of $P \rightarrow q$.

"Value of $P \rightarrow q$ = P if Q is True else T"

U

$P \rightarrow q$

6. Biconditional

Given propositions p and q , the biconditional $p \leftrightarrow q$ is a proposition that is true when $p \wedge q$ have same truth values. Alternatively $p \leftrightarrow q$ is true whenever both $p \rightarrow q$ and $q \rightarrow p$ are true. Some of the technologies used for biconditional are:-

- (1) "p if and only if q"
- (2) "if p then q and conversely"
- (3) "p is necessary and sufficient for q".

Eg:- $p =$ "today is sunday"
 $q =$ "it is hot today"

$p \leftrightarrow q$: today is sunday if and only if it is hot day.

	P	q	$P \rightarrow q$	$q \rightarrow P$	$P \leftrightarrow q$
	T	T	T	T	T
	T	F	F	T	F
	F	T	T	F	F

let p , q and r be the proposition with truth value T, F, T respectively. Evaluate the following:

$$\text{i) } \neg r \vee \neg(p \vee q)$$

$$\text{ii) } \neg(p \vee q) \wedge (\neg r) \vee q$$

$$\text{D) } \neg r \vee \neg(p \vee q)$$

$$\begin{array}{ccccccc} p & q & r & \neg r & p \vee q & \neg(p \vee q) & \neg r \vee \neg(p \vee q) \\ \text{T} & \text{F} & \text{T} & \text{F} & \text{T} & \text{F} & \text{T} \end{array}$$

$$\text{(ii) } \neg(p \vee q) \wedge (\neg r) \vee q$$

$$\begin{array}{ccccccc} p & q & r & \neg r & p \vee q & \neg(p \vee q) & (\neg r) \vee q \quad \neg(p \vee q) \wedge (\neg r) \vee q \\ \text{T} & \text{F} & \text{T} & \text{F} & \text{F} & \text{T} & \text{F} \end{array}$$

Tautology and Contradiction

A compound proposition that is always true, no matter what the truth values of the atomic propositions that contain it is called tautology.

For example:-

$p \vee \neg p$ is always true verify

p	$\neg p$	$p \vee \neg p$
T	F	T
T	F	T
F	T	T
F	T	T

A compound proposition that is always false is called contradiction. For eg:- $p \wedge \neg p$.

P
T
F

$\neg p$
F
T

$p \wedge \neg p$
F
F

A compound proposition that is neither a tautology nor a contradiction is called Contingency.

Logical Equivalences:

The compound propositions p and q are logically equivalent denoted by $p \Leftrightarrow q$ or $p \equiv q$, if proposition $p \Leftrightarrow q$ is tautology.

Some important logical equivalences:-

1. $p \wedge T \Leftrightarrow p$ Identity Law
2. $p \vee F \Leftrightarrow p$ "
3. $p \wedge F \Leftrightarrow F$ Domination Law
4. $p \vee T \Leftrightarrow T$ "
5. $p \wedge p \Leftrightarrow p$ Idempotent Law
6. $p \vee p \Leftrightarrow p$ "
7. $\neg(\neg p) \Leftrightarrow p$ Double negation law
8. $p \wedge q \Leftrightarrow q \wedge p$ Commutative law
9. $p \vee q \Leftrightarrow q \vee p$ "
10. $(p \wedge q) \vee r \Leftrightarrow p \wedge (q \vee r)$ Associative law
11. $(p \vee q) \wedge r \Leftrightarrow p \vee (q \wedge r)$ "
12. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ Distributive law
13. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ "
14. $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ De-Morgan's Law
15. $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ "

1. $p \wedge \neg p \Leftrightarrow F$ Trivial Tautology

$$17. p \vee \neg p \Leftrightarrow T$$

$$18. p \rightarrow q \Leftrightarrow \neg p \vee q$$

Implication

$$19. p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$$

Equivalence

$$20. (p \wedge q) \rightarrow r \Leftrightarrow p \rightarrow (q \rightarrow r)$$

Exportation

$$21. (p \rightarrow q) \wedge (p \rightarrow \neg q) \Leftrightarrow \neg p$$

Absurdity

$$22. p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$$

Contraposition

$$23. p \wedge (p \vee q) \Leftrightarrow p$$

Absorption

$$24. p \vee (p \wedge q) \Leftrightarrow p$$

"

① Truth Table

② Symbolic Derivation

$$1. p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

$$p \wedge (q \vee r)$$

$$\begin{array}{cccc} p & q & r & q \vee r \\ \hline T & T & T & T \\ T & T & F & F \\ T & F & T & T \\ T & F & F & F \\ F & T & T & T \\ F & T & F & F \\ F & F & T & T \\ F & F & F & F \end{array}$$

$$p \wedge (q \vee r) \vdash q \quad 1$$

$$T \Leftrightarrow T \vee q \quad 2$$

$$T \Leftrightarrow q \vee q \quad 3$$

$$T \Leftrightarrow q \vee q \quad 4$$

$$F \Leftrightarrow (q \vee q) \vdash F$$

$$F \Leftrightarrow p \wedge q \quad 5$$

$$F \Leftrightarrow p \wedge q \quad 6$$

$$F \Leftrightarrow \neg (p \wedge q) \quad 7$$

$$F \Leftrightarrow \neg (p \wedge q) \quad 8$$

$$(p \wedge q) \Leftrightarrow (\neg p \vee q) \quad 9$$

$$(\neg p \vee q) \Leftrightarrow (\neg p) \vee q \quad 10$$

$$(\neg p) \vee q \Leftrightarrow (p \wedge q) \vdash F \quad 11$$

$$(\neg p) \vee q \Leftrightarrow (p \wedge q) \vdash F \quad 12$$

$$(\neg p) \vee q \Leftrightarrow (p \wedge q) \vdash F \quad 13$$

$$(p \wedge q) \vee (p \wedge r)$$

Truth table for $(p \wedge q) \vee (p \wedge r)$

T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	(F) T	T	T
T	F	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Hence, $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$$q \vee r \equiv (p \wedge q) \rightarrow (p \wedge r)$$

Prove that: - $(p \rightarrow q) \wedge (p \rightarrow \neg q) \equiv \neg p$ by the help of truth table.

P	q	$\neg q$	$(p \rightarrow q)$	$(p \rightarrow \neg q)$	$(p \rightarrow q) \wedge (p \rightarrow \neg q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	F	T	T	T	T

$$\neg p$$

From Q

From P

$$T$$

$$T$$

Q is true -> P is false

Product with 1st

(NobP)

$$(p \rightarrow q) \vee q \equiv 1$$

$$(p \rightarrow q) \vee \neg q \equiv$$

$$\text{Hence, } (p \rightarrow q) \wedge (p \rightarrow \neg q) \equiv \neg p$$

$$(p \rightarrow q) \vee (q \rightarrow q) \equiv$$

$$(p \rightarrow q) \vee F \equiv$$

$$p \rightarrow q \equiv 1 \vee (p \rightarrow q) \equiv$$

$$(p \rightarrow q) \equiv$$

Show that $\neg(p \rightarrow q)$ and $(p \wedge \neg q)$ are logically equivalent by the help of symbolic derivation.

Solution:-

$$\begin{aligned}
 \text{LHS} &: \neg(p \rightarrow q) \\
 &\equiv \neg(\neg p \vee q) \quad [\because \text{Implication law}] \\
 &\equiv \neg(\neg p) \wedge \neg(q) \quad [\because \text{De Morgan's law}] \\
 &\equiv p \wedge \neg q \quad [\because \text{double negation}] \\
 &\equiv \text{RHS}
 \end{aligned}$$

proved!

$$\star (p \rightarrow q) \wedge (p \rightarrow \neg q) \equiv \neg p$$

$$\begin{aligned}
 \text{LHS} &\equiv (p \rightarrow q) \wedge (p \rightarrow \neg q) \\
 &\times \equiv (\neg p \vee q) \wedge (\neg p \vee \neg q) \quad [\text{Implication law}] \\
 &\equiv [\neg p \vee q] \wedge [\neg p \vee \neg q] \quad [\text{double negation}] \\
 &\equiv \neg p \quad [\text{Absurdity law}]
 \end{aligned}$$

Show that $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ are logically equivalent.

$$\begin{aligned}
 \text{LHS} &\equiv \neg(p \vee (\neg p \wedge q)) \\
 &\equiv \neg p \wedge \neg(\neg p \wedge q) \quad \text{De Morgan's Law} \\
 &\equiv \neg p \wedge (p \vee \neg q) \quad \text{double negation} \\
 &\equiv \neg p \wedge p \vee \neg q \\
 &\equiv \underline{\underline{F \vee \neg q}} \\
 &\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) \quad \text{Distribution} \\
 &\equiv F \vee (\neg p \wedge \neg q) \quad (\text{Double Negation}) \\
 &\equiv \neg p \wedge \neg q \quad (\text{Top + Tautology}) \\
 &\equiv \neg p \wedge \neg q \quad (\text{Identity}) \\
 &\equiv \neg p \wedge \neg q \vee F \quad (\text{Commutative})
 \end{aligned}$$

$$\text{LHS: } \neg(p \vee (\neg p \wedge q))$$

$$\equiv \neg[(p \vee \neg p) \wedge (p \vee q)]$$

(distributive law)

$$\equiv \neg(T \wedge (p \vee q))$$

Tr. tautology.

$$\equiv \neg((T \wedge p) \vee (\neg T \wedge q))$$

Distributive

$$\equiv \neg((p \wedge T) \vee (q \wedge \neg T))$$

Commutative

$$\equiv \neg(p \wedge q)$$

Identity

$$\equiv \neg p \wedge \neg q$$

De Morgan's

$$\text{proved!}$$

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

$$\Leftrightarrow \neg(p \wedge q) \rightarrow (p \vee q)$$

~~$\neg(\neg p \vee q) \rightarrow$~~

$$\equiv \neg(\neg(p \wedge q)) \vee (p \vee q)$$

[Implication Law]

$$\equiv (\neg \neg p \vee \neg q) \vee (p \vee q)$$

[De Morgan's Law]

$$\equiv (\neg \neg p \vee \neg q)$$

$$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$$

[Implication Law]

$$\equiv (\neg p \vee \neg q) \vee (p \vee q)$$

[De Morgan's Law]

$$\equiv (\neg p \vee p) \vee (\neg q \vee q)$$

[Associative]

$$\equiv (p \vee \neg p) \vee (q \vee \neg q)$$

[Commutative]

$$\equiv T \vee T$$

[Trival. Tautology]

$$\equiv T$$

$$(T \vee p) \wedge (T \vee q)$$

$$(T \vee p) \wedge (T \vee q)$$

$$(T \wedge p) \vee (T \wedge q)$$

Dual of compound proposition

The dual of compound proposition that contains only the logical operators \wedge , \vee and \neg is the compound proposition obtained by replacing each \vee by \wedge , each \wedge by \vee , each T by F and each F by T. The dual of S is denoted by S^* .

Find the dual of each of these compound proposition.

$$\textcircled{a} \quad p \vee \neg q$$

$$\Rightarrow S = p \vee \neg q$$

$$S^* = p \wedge \neg q$$

$$\textcircled{b} \quad p \wedge (q \vee (r \wedge T))$$

$$S = p \wedge (q \vee (r \wedge T))$$

$$S^* = p \vee (q \wedge (r \vee F))$$

$$\textcircled{c} \quad (p \wedge \neg q) \vee (q \wedge F)$$

$$S = (p \wedge \neg q) \vee (q \wedge F)$$

$$S^* = (p \vee \neg q) \wedge (q \vee F)$$

$$\textcircled{d} \quad (p \vee F) \wedge (q \vee T)$$

$$S = (p \vee F) \wedge (q \vee T)$$

$$S^* = (p \wedge T) \vee (q \wedge F)$$

When does $s^* = s$, where s is a compound proposition?

Solution:- Since s is a compound proposition, let p be proposition then

$$s = p \vee p \quad \text{The dual of } s \Rightarrow s^* = p \wedge p \quad \text{if}$$

$$p \vee p = p \wedge p = p \quad (\text{Idempotent law})$$

$$\therefore s = s^*$$

Show that $(s^*)^* = s$ where s is a compound proposition.

Solution:-

If s is a compound proposition then,

$$\text{Let } s = p \wedge q$$

Now,

$$s^* = p \vee q$$

$$(s^*)^* = p \wedge q$$

$$\therefore (s^*)^* = s$$

proved!

Predicate:

lets take a statement $x > 3$, there are two parts one is the variable part called "subject" and another is relation part " > 3 " called predicate. We can denote the statement " $x > 3$ " by $P(x)$ where P is a predicate " > 3 " and x is the variable. Once the value is assigned to the propositional function then we can tell whether it is true or false i.e. proposition.

→ The logic involving predicates is called predicate logic or predicate calculus.

Let $P(x)$ denotes the statement " $x > 10$ ". What are the truth value of $P(12)$ and $P(5)$?

Solution:-

$$P(x) = x > 10$$

$$P(12) = 12 > 10 \quad (\text{Truth})$$

$$P(5) = 5 > 10 \quad (\text{False})$$

$P(x)$ denotes $x > 10$

$P(12)$ denotes $12 > 10$

which is "true"

$P(5)$ denotes $5 > 10$

which is "false"

Let $\mathcal{Q}(x, y)$ denote the statement " $x = y + 3$ ". What are the truth value for the propositions $\mathcal{Q}(1, 2)$ and $\mathcal{Q}(3, 0)$?

Solution:-

$\mathcal{Q}(x, y)$ denotes $x = y + 3$

$\mathcal{Q}(1, 2)$ denotes $1 = 2 + 3$

which is "False"

$\mathcal{Q}(3, 0)$ denotes $3 = 0 + 3$

which is "True"

Let $R(x, y, z)$ denote the statement " $x + y = z$ ". What are the truth value of $R(1, 2, 3)$ & $R(0, 0, 1)$?

Solution:-

$R(x, y, z)$ denotes $x + y = z$

$R(1, 2, 3)$ denotes $1 + 2 = 3$

which is "True"

$R(0, 0, 1)$ denotes $0 + 0 = 1$

which is "False"

Quantifiers

Quantifiers are the tools to make the propositional function a proposition. Construction of propositions from the predicates using quantifiers is called quantification. The variables that appear in the statement can take different possible values and all the possible values that the variables can take forms a domain called "Universe of Discourse" or "Universal set".

Types of Quantifiers:

1. Universal Quantifier
2. Existential Quantifier

Universal Quantifier

It is denoted by (\forall) symbol "for all".
The universal Quantification of $p(x)$ denoted by
for all x $p(x)$ is

$$\forall x \in P(x)$$

is a proposition. " $p(x)$ is true for all the values of x in the Universe of Discourse".

We can represent the universal quantification by using the English sentences like ① "for all x $p(x)$ holds".

① "for every x ~~is~~ $p(x)$ holds."

② "for each x $p(x)$ holds".

Example:- Express the statement "all students of CSIT takes discrete mathematics class", where $V_o D$ is set of all CSIT students.

$\Rightarrow D(x)$ denotes x takes discrete mathematics class.

What is the truth value of $\exists x P(x)$ where $P(x)$ is the statement " $x^2 > 10$ " and the UoD consists of the integer not exceeding 4?

$\Rightarrow P(x)$ denotes " $x^2 > 10$ "

$$\text{UoD} = \{0, 1, 2, 3, 4\}$$

$\exists x P(x)$ (True)

Because: $P(2)$.

$$4^2 > 10$$

$$16 > 10$$

(True)

The true value of $\exists x P(x)$ is true.

Free and Bound Variables

When the variable is assigned a value or it is quantified, it is called Bound variable. If the variable is not bounded then it is called free variables.

Examples:-

Identify the free and bound variables:

1. $P(x, y)$, both are free variables.

2. $P(2, y)$, y is free.

3. $P(2, y)$ where $y = 4 \Rightarrow$ both bounded.

4. $\forall x P(x)$, x is bounded variable.

5. $\forall x P(x, y)$, x bounded, y free

Expression with no free variable is proposition?

Expression with at least one free variables is predicate?

Orders of Quantification:-

Example:-

Let $L(x, y)$ denotes x loves y where $V \cup D$ for x, y is set of all people in the world.

Translate the given quantified statement in English.

$$\text{i) } \forall x \exists y L(x, y)$$

\rightarrow for all x there exist some y , such that x loves y . i.e. everyone loves someone.

$$\text{ii) } \exists y \forall x L(x, y)$$

\rightarrow There exists some y for all x such that x loves y . i.e. someone is loved by everyone.

$$\text{iii) } \forall x \forall y L(x, y)$$

\Rightarrow for all x and y such that x loves y .
i.e. everyone loves everybody.

$$\text{iv) } \exists x \exists y L(x, y)$$

\Rightarrow there exist some x and some y such that x loves y .
someone loves somebody

Negation of Quantified Expression.

$$\forall x P(x) \Rightarrow \neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\exists x P(x) \Rightarrow \neg \exists x P(x) \equiv \forall x \neg P(x),$$

Example:-

Let $P(x)$ denotes x is lovely, $\forall x$ for x is girls in Kathmandu.

$$\neg \forall x P(x)$$

\Rightarrow Every girl in Kathmandu are lovely.

$$\exists x P(x)$$

\Rightarrow Some girls in Kathmandu are lovely.

$$\neg \forall x P(x)$$

\Rightarrow Not all girls in Kathmandu are lovely.

$$\neg \exists x P(x)$$

\Rightarrow All girls in Kathmandu are not lovely.

Translate the sentence into logical expression.

Ex. "not every integer is even."

Let $P(x)$ denotes x is even, $\cup D$ for integers

$$\neg \forall x P(x)$$

Translate "If a person is female and is a parent then this person is someone's mother". into logical expression where $\cup D$ is set of all peoples. Let

\Rightarrow let $F(x)$ denotes female

$P(x)$ denotes parent

$\cup D = \{ \text{set of all people} \}$
 $M(x, y)$ denotes x is mother of y .

$$\forall x \exists y ((F(x) \wedge P(x)) \rightarrow M(x, y))$$

Mathematical Reasoning.

Rules of Reasoning

To draw conclusion from the given premise we must be able to apply some well defined steps that helps in reaching the conclusion. These steps of reaching the conclusion are provided by rule of inference.

Rule 1: Modus Ponens (or Law of Detachment)

Whenever two propositions $p \wedge p \rightarrow q$ are both true then we confirm q is true i.e.

P

$P \rightarrow q$, this rule is valid rule of inference because the implication $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.

P	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Rule 2: Hypothetical Syllogism (Transitive Rule)

Whenever two proposition $p \rightarrow q$ and $q \rightarrow r$ are both true then we confirm that implication $p \rightarrow r$ is true.

i.e. $p \rightarrow q$

$\frac{q \rightarrow r}{\therefore p \rightarrow r}$, this rule is valid rule of inference because the implication

$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ is tautology
 Similarly, $p \rightarrow q$

$\neg p \rightarrow q$

~~Carol~~ ~~substitution~~ $q \rightarrow r$ $\neg (p \wedge q) \vee (\neg p \vee r)$

Example: $r \rightarrow s \quad (\neg r \vee q) \vee [(\neg q \wedge p) \wedge (\neg s \wedge q)] \vdash r \wedge s$

Conjugation, $P \rightarrow S$ (vap) \downarrow (top) \downarrow vane \downarrow P

Each student I graphed a curve for each function.

$$P \rightarrow q \quad r \quad p \rightarrow q \quad p \rightarrow r \quad q \rightarrow r \quad [(p \rightarrow q) \wedge (q \rightarrow r)] \quad [(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

$\vdash \neg \neg p \rightarrow \neg \neg q$ [$\vdash \neg \neg p$] $\vdash \neg \neg q$

T T F T F F F T

T F T F T T T T T T

T F F F T T T T

Er wird darüber nicht viel zu sagen haben, was er sich nicht selbst erzählt.

$F \left(\frac{1}{2} E_8 \oplus \text{TA} \right) T \oplus F^2 T \oplus V \oplus F^2 p \right) \oplus T \left(\frac{1}{2} E_8 \oplus \text{TA} \right) T$

وَلِمَنْجَانَةِ تَرْكِيَّةِ كَوْكَبِيَّةِ مُهَاجِرَةِ مُهَاجِرَةِ مُهَاجِرَةِ

Exhibit A) in (top) v (bottom) is

Rule 3 :- Addition

$$\text{rotation}(\text{rep}) = \text{PR}(\text{rep}) \times (\text{Car} \nabla \text{op} \text{P})^{\frac{1}{2}}$$

Due to tautology $P \rightarrow (P \vee q)$ rule P T in 1

rule of inference

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$$P \rightarrow (P \vee q)$$

$$\equiv \exists P \vee (\neg P \wedge Q) \quad (\text{implication})$$

$$= \overline{f} \circ (\overline{f})^{-1} = \text{id}_{\mathcal{P}(X)}$$

\equiv T_{Va} $\{T_{\text{real}} \text{ to } \text{abs}\}$

$\equiv T$ T Dynamics

Dominican

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$$\cancel{[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)}$$

$$\equiv \top [(p \rightarrow q) \wedge (q \rightarrow r)] \vee (p \rightarrow r) \quad [\because \text{Implication law}]$$

$$\equiv \top [(p \rightarrow q) \wedge (q \rightarrow r)] \vee (\neg p \vee r) \quad [\because \text{Implication law}]$$

$$\equiv \top [(p \rightarrow q) \vee \neg (q \rightarrow r) \vee (\neg p \vee r)] \quad [\because \text{De Morgan's law}]$$

$$\equiv \top [(\neg p \vee q) \vee \neg (\neg q \vee r) \vee (\neg p \vee r)] \quad [\text{Implication law}]$$

$$\equiv [(\neg p \wedge \neg q) \vee (\neg (\neg q) \wedge \neg r) \vee (\neg p \vee r)] \quad [\text{De Morgan's law}]$$

$$\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r) \quad [\text{Double Negation}]$$

$$\equiv \top$$

$$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$$

$$\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r) \quad [\text{Implication}]$$

$$\equiv (p \wedge \neg q) \vee [(q \wedge \neg r) \vee r] \vee \neg p \quad [\text{Associative}]$$

$$\equiv (p \wedge \neg q) \vee [(q \vee r) \wedge (\neg r \vee r)] \vee \neg p \quad [\text{Distributive}]$$

$$\equiv (p \wedge \neg q) \vee [(q \vee r) \wedge T] \vee \neg p \quad [\text{Trivial Tautology}]$$

$$\equiv [(p \wedge \neg q) \vee \neg p] \vee (q \vee r) \wedge T \quad [\text{Associative}]$$

$$\equiv (p \vee \neg p) \wedge (\neg q \vee \neg p) \vee (q \vee r) \wedge T \quad [\text{Distributive}]$$

$$\equiv T \wedge (\neg q \wedge \neg p) \vee (q \vee r) \wedge T \quad (\text{Trivial tautology})$$

$$\equiv T \wedge (\neg q \wedge \neg p) \vee \neg p \vee r \wedge T \quad (\text{Associative})$$

$$\equiv T \wedge T \vee (\neg p \vee r) \wedge T \quad (\text{Dominative})$$

$$\equiv T \wedge T \wedge T \quad (\text{Trivial tautology})$$

$$\equiv T$$

Rule 4:- Simplification:-

Due to the tautology $(p \wedge q) \rightarrow p$, rule
 $\frac{p \wedge q}{\therefore p}$ is a valid rule of inference.

$$\begin{aligned} & (p \wedge q) \rightarrow p \\ & \equiv \neg(p \wedge q) \vee p \\ & \equiv \neg p \vee \neg q \vee p \\ & \equiv \neg p \vee \neg p \vee \neg q \\ & \equiv \neg \top \vee \neg q \\ & \equiv \neg \top \\ & \equiv \bot \end{aligned}$$

[Implication]
 [De-Morgan's Law]
 (Trivial tautology)
 (Trivial tautology)

Rule 5: Conjunction

Due to the tautology $[(p) \wedge (q)] \rightarrow (p \wedge q)$; rule
 $\frac{p \quad q}{\therefore p \wedge q}$ is valid rule of inference.

$$\begin{aligned} & [(p) \wedge (q)] \rightarrow (p \wedge q) \\ & \equiv (p \wedge q) \rightarrow (p \wedge q) \\ & \equiv \neg(p \wedge q) \vee (p \wedge q) \\ & \equiv \neg p \vee \neg q \vee (p \wedge q) \end{aligned}$$

[Implication]

[De-Morgan's law]

Rule 6: Modus Tollens

Due to tautology $\neg q \wedge (p \rightarrow q) \rightarrow \neg p$; rule $\frac{\neg q}{\neg p}$

is valid rule of inference.

P	q	$\neg p$	$\neg q$	$(p \rightarrow q)$	$\neg q \wedge (p \rightarrow q)$	$\neg q \wedge (p \rightarrow q) \rightarrow \neg p$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	(T or F or T or F)	T	F	F	F	T
F	(F or T or T or F)	T	T	F	T	T

$$\neg q \wedge (p \rightarrow q) \rightarrow \neg p$$

$$\neg q \wedge (p \rightarrow q) \vdash [(\neg q) \wedge (p \rightarrow q)]$$

which is true below if $p \rightarrow q$.

$$(p \rightarrow q) \leftrightarrow [(\neg p) \vee (q)]$$

$$[(\neg p) \vee (q)] \vdash [(\neg p) \wedge (\neg q)]$$

$$[(\neg p) \wedge (\neg q)] \vdash [(\neg q) \wedge (p \rightarrow q)]$$

Rule 7: Disjunction by logic

Due to tautology $[(p \vee q) \wedge \neg p] \rightarrow q$, rule $\frac{p \vee q}{\neg p \quad q}$

p	q	$\neg p$	$p \vee q$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \rightarrow q$
T	T	F	T	F	T
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	F	F	T

$$[(p \vee q) \wedge \neg p] \rightarrow q$$

Rule 8: Resolution

Due to tautology $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$,
rule $p \vee q$ is valid rule of inference

$$\frac{\neg p \vee r}{\therefore q \vee r}$$

$$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

$$p \rightarrow [q \wedge (\neg p \vee r)]$$

Example:-

For the set of premises "I play football, then I am sore the next day", "I will take rest if I am sore", "I did not take rest". What relevant conclusion can be drawn? Explain the rule of inference to draw the conclusion.

Solution:-

$$\begin{aligned} p &= \text{I play football} \\ q &= \text{I am sore} \\ r &= \text{I } \cancel{\text{will}} \text{ take rest} \end{aligned}$$

Hypothesis:-

- (i) $p \rightarrow q$
- (ii) ~~$\neg q \rightarrow r$~~
- (iii) $\neg r$

- Steps
- (i) $p \rightarrow q$
 - (ii) $q \rightarrow r$
 - (iii) $p \rightarrow \neg r$
 - (iv) $\neg r$
 - (v) $\neg p$

Reason:-

Hypothesis.

Hypothesis

Transitive from (i) & (ii)

Hypothesis

Modes Tollen

Conclusion:-

Therefore we can conclude that, "I did not play football."

Example:-

Show that the hypothesis "It is not sunny this afternoon and it is cooler than yesterday". "We will go swimming only if it is sunny", "If we do not go swimming, then we will take a trip" and "If we take a trip, then we will be home by sunset" lead to the conclusion "We will be home by sunset".

Solution:-

$p = \text{It is sunny this afternoon.}$

$q = \text{It is cooler.}$

$r = \text{We will go swimming.}$

$s = \text{We will take a trip.}$

$t = \text{We will be home by sunset.}$

Hypothesis

$$\textcircled{I} \quad \neg p \wedge q$$

$$\textcircled{II} \quad r \rightarrow p$$

$$\textcircled{III} \quad \neg r \rightarrow s$$

$$\textcircled{IV} \quad s \rightarrow t$$

Steps

$$\textcircled{I} \quad \neg p \wedge q$$

$$\textcircled{II} \quad r \rightarrow p$$

$$\textcircled{III} \quad \neg r \rightarrow s$$

$$\textcircled{IV} \quad s \rightarrow t$$

$$\textcircled{V} \quad \neg r \rightarrow t$$

$$\textcircled{VI} \quad \neg p$$

$$\textcircled{VII} \quad \neg r$$

$$\textcircled{VIII} \quad s$$

$$\textcircled{IX} \quad t$$

Reason.

Hypothesis

Hypothesis

Hypothesis

Hypothesis

Transitive from \textcircled{III} & \textcircled{IV}

Simplification \textcircled{I} .

(Modus Tollens)

$\textcircled{VI} \wedge \textcircled{VII}$

Modus Ponens $\textcircled{VIII} \wedge \textcircled{IX}$

Modus Ponens $\textcircled{VII} \wedge \textcircled{IV}$

Fallacies:-

The fallacies are argument that are convincing but not correct. So fallacies produce faulty inference. Fallacies are contingency rather than tautology.

① Fallacy of affirming the conclusion:

The kind of fallacy has the form $\frac{p \rightarrow q}{p}$

i.e. $q \wedge (p \rightarrow q) \rightarrow p$. This is not a tautology, hence it is fallacy.

P	q^*	$p \rightarrow q$	$q \wedge (p \rightarrow q)$	$q \wedge (p \rightarrow q) \rightarrow p$
T	T	T	T	T
T	F	F	F	F
F	T	T	F	F
F	F	T	F	T

Example:-

"If the economy of Nepal is poor, then the education system in Nepal will be poor."

"The education system in Nepal is poor. Therefore, Economy of Nepal is poor."

Solution:-

p = economy of Nepal is poor

q = education system in Nepal is poor.

$$p \rightarrow q$$

$$\underline{q}$$

$\therefore p$ (which is fallacy (fallacy of affirming the conclusion)). Hence we can conclude economy of Nepal is not poor.

⑥ Fallacy of denying the hypothesis

This kind of fallacy has the form $\frac{\neg p}{\neg q}$

i.e. $[(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q]$. This is not a tautology hence it is a fallacy.

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg p \wedge (p \rightarrow q)$	$[(\neg p \wedge (p \rightarrow q)) \rightarrow \neg q]$
T	F	F	T	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	T	F
F	F	T	T	T	T	T

Example:-

"If today is Sunday, then it rains today". "Today is not Sunday". Therefore, "it does not rain today".

\Rightarrow p : today is sunday
 q : it rains today.

$$P \rightarrow q$$

$$\frac{\neg p}{\neg q}$$

$$\therefore \frac{\neg p}{\neg q}$$

(iii) Begging the question (Circular Reasoning)

If the statement that is used for loop is equivalent to the statement that is being proved then it is called circular reasoning.

Example:-

Ram is black because he is black.

Rules of inference for Quantified Statement

Universal Instantiation:-

If the proposition of the form $\forall x P(x)$ is supposed to be true then the universal quantifier can be dropped out to get $P(c)$ is true for arbitrary c in the universe of discourse.

i.e. $\forall x P(x)$

$\therefore P(c)$, for all c

Universal Generalization:-

If all the instances of x makes $P(c)$ true, then $\forall x P(x)$ is true. This can be written as,

$P(c)$ for all c ,

$\therefore \forall x P(x)$

Here the chosen c must be arbitrary not a specific element from the UoD.

Existential Instantiation:

If the proposition of the form $\exists x P(x)$ is supposed to be true then there is an element c in the UoD such that $P(c)$ is true.

$$\text{i.e. } \exists x P(x) \\ \therefore P(c) \text{ for some } c$$

Here c is not an arbitrary, it must be specific such that $P(c)$ is true.

Existential Generalization

If at least one element c from the UoD makes $P(c)$ true then $\exists x P(x)$ is true.

$$\text{i.e. } P(c), \text{ for some } c \\ \therefore \exists x P(x)$$

Show that the premises, "Everyone in this discrete mathematics class taken a course in computer Science" and "Marla is a student in this class" imply the conclusion "Marla has taken a course in Computer Science".

Solution:-

$\exists x$ $D(x)$ denotes x is in discrete mathematics class

$C(x)$ denotes x has taken a course in cs.

Quantified Statement (Hypothesis)

- ① $\forall x (D(x) \rightarrow C(x))$
- ② $D(\text{Marla})$

(3) C (Marla)

Steps

- (1) $\forall x(P(x) \rightarrow ((x)))$
 (2) $D(\text{Marla}) \rightarrow (C(\text{Marla}))$

- (3) $D(\text{Marla})$
 (4) $C(\text{Marla})$

Result:

Hypothesis.

Universal instantiation.

from (1).

Hypothesis

Modus ponens from (2)
 + (3).

Prove or disprove the validity of the argument,
 "every living thing is a plant or animal.", "Mar's dog is alive and it is not a plant.", "All animals have heart"; hence "Mar's dog has a heart".

Solution:- UD denotes "every living thing is a plant or animal".

$H(x)$ denotes All animals have heart.

$P(x) \rightarrow x$ is plant, $A(x) \rightarrow x$ is animal, $J(x) \rightarrow x$ is alive.

1. $\forall x(P(x) \vee A(x))$

Hypothesis.

2. $J(\text{Mar's dog}) \wedge \neg p(\text{Mar's dog})$

Hypothesis.

3. $J(\text{Mar's dog}) \wedge A(\text{Mar's dog})$

from (1) & (2).

4. $\forall x h(x)$

Hypothesis

5. $H(\text{Mar's Dog})$

Universal instantiation from (5)

(1) $\forall x(P(x) \vee A(x))$

Hypothesis: (i) $J(\text{Mar's Dog}) \wedge \neg p(\text{Mar's Dog})$

(ii) $\forall x h(x)$

(iv) $h(\text{Mar's dog})$.

Proving Theorem

1. Direct Proof:-

We prove the implication $p \rightarrow q$, where we start assuming that the hypothesis i.e. p is true and using rule of inference, theorems etc. If q becomes true, then the argument becomes valid. This is known as direct proof.

Example:-

If a and b are odd integers, then $a+b$ is even integer.

Solution:

By the defⁿ of odd integers
 $\Rightarrow n = 2k + 1$

By the defⁿ of even integers.

and Here,

$$a = 2k + 1 \quad (\because \text{By defn}) \checkmark$$

$$z = 2l + 1 \quad (\because \dots) \quad (2) \times A \checkmark \text{ (N) } 9 \text{ (N) }$$

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$$a+b = 2k+1 + 2l+1$$

$$= 2k' + 2l + 2$$

$$= 2(L+L+1)$$

$= Qm$ ($\because (k+l+2) = m$ is any integers).

$\therefore a+b = 2m$ which is given by the defⁿ of even integers.

Example :-

If m, n are divisible by 3, mn is divisible by 9.

Solution :-

By the definition of divisible, by 3
 $\rightarrow m, n / 3 = \{1, 2, 3, \dots\}$

By the definition of divisible by 9
 $\rightarrow m, n / 9 = \{1, 2, 3, \dots\}$

Here,

$$m = 3k$$

$$n = 3l$$

So,

$$\therefore mn = 3k \cdot 3l$$

$$= 9(kl)$$

$= 9n$ ($\because kl = n$ is any integer).

Show that the square of an even number is an even number using direct proof.

Solution :-

By definition of ~~the~~ even number.

$$n = 2k$$

By definition of square of even number.
 $n^2 = (2k)^2$

Here,

$$n = 2k \text{ from definition.}$$

So,

$$n = 2k$$

Squaring we get.

$$n^2 = 4k^2 \quad (\text{where } k \text{ is an even integer}).$$

$$= 4m$$

$\therefore n^2 = 4m$ ($\because k^2 = m$, which is given by defⁿ of square of even num.)

Use direct proof to show that every odd integer is the difference of two squares.

~~Ans:-~~

By definition of odd integers

$$n = 2k + 1$$

By definition of difference square,

$$n = \cancel{(2k+1)}^{\cancel{2}} - (2l+1)^2$$

∴ Now

$$\begin{aligned} n &= (2k+1)^2 - (2l+1)^2 \\ &= 4k^2 + 4k + 1 - 4l^2 - 4l - 1 \\ &= 4k^2 + 4k - 4l^2 - 4l \end{aligned}$$

Indirect Proof (Proof by Contradiction)

If we prove the implication $p \rightarrow q$ by assuming the conclusion is false i.e. $\cancel{p \rightarrow q} \equiv \cancel{q \rightarrow p}$ is known as proof by contradiction or indirect proof.

Example:-

If the product of two integers $a \times b$ is even, then either a is even or b is even.

Solution:-

$\cancel{q} = a$ and b are both odd.

$\cancel{p} =$ product of two integers $a \times b$ is odd.

We know,

$$a = 2k + 1$$

$$(2k+1) \times (2l+1) = 2kl + 2k + 2l + 1$$

Because $2k + 2l$ is always even, so $2kl + 2k + 2l + 1$ is odd.

Now,

$$\begin{aligned}
 ab &= (2k+1)(2l+1) \\
 &= 2k+2l+2 \quad 4kl+2k+2l+1 \\
 &= 2(k+l+1) + 2(2kl+k+l)+1 \\
 &= 2m+1 \quad (m = 2kl+k+l+1; m \text{ is any integer})
 \end{aligned}$$

we know that.

$$7q \rightarrow 7p \equiv p \rightarrow q$$

\therefore We can conclude that product of two integers $a \in b$ is even, then either a is even or b is even.

Prove that if n is an integer and $3n+2$ is odd, then n is odd.

Solution:-

By definition of integer.

 $7p = 0$ n is not odd. evenP.y. $7q = (3n+2)$ is even.

We know,

$$n = \cancel{2k+1} + 2k$$

Now,

$$\begin{aligned}
 3n+2 &= 3(\cancel{2k+1}) + 2 \quad 3n+2 - 3(k) + 2 \\
 &= 6k+3+2 \quad = 3k+2 \\
 &= 6k+5 \quad = \cancel{2k+1} + 2k + k + 1 + 1 \\
 &= (2k+1) + (k+2)
 \end{aligned}$$

$$3n+2 = 3(2k+1) + 2$$

$$\begin{aligned}
 &- \cancel{2k+1} + 2 \\
 &= 3k+5 - 2(k+2) = 2m
 \end{aligned}$$

Proofs by Contradiction

The steps in proof of implication $p \rightarrow q$ by contradiction are:-

i) Assume $p \wedge \neg q$ are true.

ii) Try to show the above assumption is false.

Example:-

If a^2 is an even number, then a is an even number.

Solution:-

Assume a is an odd number and a^2 is an even number.

(By defⁿ of odd integers)

Then,

$$a^2 = 4k^2 + 4k + 1$$

$$= 2(2k^2 + 2k) + 1$$

$$= 2m + 1$$

(where $2k^2 + 2k = m$; is any integer)

So, by contradiction, we can prove that

If a is an even number then a^2 is an even number.

Give a proof by contradiction of the theorem "If $3n+2$ is odd, then n is odd".

Sol:-

Assume that n is an even number and $3n+2$ is odd.

By defⁿ of even integers
 $\bullet n = 2k$

Then,

$$\begin{aligned}
 3n+2 &= 3(2k)+2 \\
 &= 6k+2 \\
 &= 2(3k+1) \\
 &= 2m \quad (3k+1 = m ; m \text{ is any integer})
 \end{aligned}$$

So, by contradiction, we can prove that, if $3n+2$ is odd, then n is odd.

Show that if n is an integer and n^3+5 is odd, then n is even using

- i) a proof by contraposition
- ii) a proof by contradiction

Sol:

i) by contraposition

$\neg p \Rightarrow n \rightarrow \text{integer}$

$\neg p \Rightarrow n^3+5 \text{ is even}$

$\neg q \Rightarrow n \text{ is odd}$.

Then,

$$n = 2k+1$$

(By defⁿ of even integer)

$$\neg q = 2k+1$$

$$\neg p = n^3+5$$

$$= (2k+1)^3+5$$

$$= 8k^3 + 6k^2 + 3k + 1 + 5$$

$$= 8k^3 + 12k^2 + 6k + 6$$

$$= 2(4k^3 + 6k^2 + 3k + 3)$$

$$= 2m \quad (4k^3 + 6k^2 + 3k + 3 = m ; m \text{ is an integer})$$

\therefore we know that

$$\neg q \rightarrow \neg p \equiv p \rightarrow q$$

\therefore we can conclude that if n is an integer and $n^3 + 5$ is odd, then n is even.

(ii) by contradiction.

Assume n is odd and $n^3 + 5$ is odd.

Then,

$$n = 2k + 1$$

(By defⁿ of odd integers)

$$\begin{aligned} n^3 + 5 &= (2k+1)^3 + 5 \\ &= 8k^3 + 12k^2 + 6k + 1 + 5 \\ &= 8k^3 + 12k^2 + 6k + 6 \\ &= 2(4k^3 + 6k^2 + 3k + 3) \\ &= 2m \quad (m = 4k^3 + 6k^2 + 3k + 3 \text{ is any integer}) \end{aligned}$$

So by contradiction, we can prove that, if $n^3 + 5$ is odd then n is even.

Prove that if n is an even perfect square, then $n+2$ is not a perfect square using.

i) Direct proof method

ii) Indirect proof method

iii) Proof by contradiction

Solution:-

i) Direct proof method.

By defⁿ of perfect square.

$$n = a^2$$

Then,

$$n+2 = (a+1)^2 + 2$$

$$\text{Since } n \text{ is a perfect square} \rightarrow n = k^2$$

$$\therefore n+2 = (k^2) + 2 \\ = k^2 + 2$$

which is not a perfect square.

$\therefore n+2 = k^2 + 2$ is not a perfect square by the definition of perfect square.

ii) Indirect proof method (Contraposition).

$\neg p = n$ is not a perfect square.
 $\neg q =$ perfect square.

Then,

$$n+2 = (k+2)^2 \quad (\text{By def. of perfect square})$$

$$\begin{aligned} \neg \neg q &= (k+2)^2 \\ &= k^2 + 4 \\ &= m^2 \quad ((k+2)^2 = m^2; m \text{ is any integer}) \end{aligned}$$

\therefore we can conclude that if n is a perfect square, then $n+2$ is not a perfect square.

iii) Proof by contradiction

Assume m is a perfect square and $n+2$ is not a perfect square.

Proof by Cases :-

The implications of the form $(p_1 \vee p_2 \vee \dots \vee p_n) \rightarrow q$ can be proved by using the tautology $(p_1 \vee p_2 \vee \dots \vee p_n) \rightarrow q \Leftarrow [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \dots \wedge (p_n \rightarrow q)]$, i.e. we can show every implication $(p_i \rightarrow q)$ true for $i=1, 2, \dots, n$.

Example:-

if $x^2 > 3$, then $x^2 > 9$, where x is a real number.

Solution :-

Consider two cases.

(i) $x > 3$

(ii) $x > 3$

Since $|x|$ is an absolute value of x (i), the value of x is x when $x \geq 0$. (ii) The value of x is $-x$ when $x < 0$.

- i) if $x > 3$, $x^2 > 9$ which is true.
- ii) if $x < 3$, $x^2 > 9$ which is true.

Hence, if $x > 3$ then $x^2 > 9$.

Proof of Equivalence

We can prove the equivalence i.e. $p \Leftrightarrow q$ by showing $p \rightarrow q$ and $q \rightarrow p$ both.

Example:-

Show that if n is a positive integer, then n is even iff $7n+4$ is even.

Solution:-

Consider $p = n$ is even

$q = 7n+4$ is even.

We have,

$$n = 2k \quad (\text{By def. of even integer})$$

$$\text{Then, } q = 7(2k)+4$$

$$= 14k + 4$$

$$= 2(7k+2)$$

$$= 2m \quad (7k+2 = m; m \text{ is any integer})$$

$\therefore p \rightarrow q$ is true.

Again,

$7p = n$ is odd.

$q = 7n + 4$ is even.

We know,

$$n = QK + 1$$

$$\therefore q = 7(QK+1) + 4 \\ = 14QK + 7 + 4$$

$q = 7n+4$ is even

$p = n$ is even

We have,

$$n = QK$$

$$\begin{aligned} q &= 7(QK) + 4 \\ &= 14K + 4 \\ &= 2(7K + 2) \\ &= 2m \quad (7K + 2 = m; m \text{ is any integer}) \end{aligned}$$

$\therefore q \rightarrow p$ is true.

We have,

$p \rightarrow q$ is true & $q \rightarrow p$ is also true

$\therefore p \leftrightarrow q$ is true

Existence Proof :-

A proof of a proposition of the form $\exists x P(x)$ is called an existence proof.

constructive existence proof.

\Rightarrow Here some element 'a' is found to show $P(a)$ is true.

non-constructive existence proof:-

\Rightarrow Here, we do not provide 'a' such that $P(a)$ is true but prove that $\exists x P(x)$ is true in different way.

Example:-

Prove by using constructive existence proof that there are 100 consecutive positive integers that are not perfect squares.

Solution :

$$50^2 = 2500$$

$$51^2 = 2601$$

\therefore The 100 consecutive positive integers that are not perfect squares are:-
 \rightarrow 2501 to 2600.

Proof using non-constructive existence proof that there exists rational numbers x & y such that x^y is rational.

Ans :-

$$\text{Let } x = \sqrt{2}, y = \sqrt{2}$$

Then

$$x^y = (\sqrt{2})^{\sqrt{2}}$$

(i) If $(\sqrt{2})^{\sqrt{2}}$ is rational, we are done.

(ii) If $(\sqrt{2})^{\sqrt{2}}$ is irrational.

Then

$$x = (\sqrt{2})^{\sqrt{2}}, y = \sqrt{2}$$

$$x^y = ((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} = 2 \text{ which is rational.}$$

Vacuous Proof:-

A proof of $p \rightarrow q$ that uses the fact that p is false.

Example:

Show that proposition $p(0)$ is true, where $P(n)$ is " $\forall n \geq 1 \text{ then } n^2 > n$ ".

Solution.

$$\text{Given, } p = n \geq 1 \\ q = n^2 > n$$

when $n=0$,

$$P(0) = 0 > 1 \text{ (false)}$$

$$\cancel{q(0) = 0^2 > 0 \text{ (false)}}$$

$$\therefore P = n < 1.$$

and,

$$\therefore q = n^2 > n.$$

\therefore We can conclude that if $n > 1$ then $n^2 > n$.

Trivial Proof:-

A proof of $p \rightarrow q$ that uses the fact that q is true.

Example:-

let $P(n)$ be "If a and b are the integers with $a \geq b$ then $a^n \geq b^n$ ", where, the domain consists of all integers. Show that $P(0)$ is true.

Solution:-

$$p: a \geq b \quad \text{where } a, b \text{ are the integers}$$

$$q: a^n \geq b^n$$

then,

$$\text{When } P(0) = \cancel{a \geq b}$$

$$\text{when } n=0$$

$$P(0) = a^0 \geq b^0 = 1 \geq 1 \text{ (True),}$$

\therefore We can conclude that if a & b are the integers with $a \geq b$ then $a^n \geq b^n$.

Mathematical Induction

Principle of mathematical Induction

Steps:-

- (i) Basis Step:- Show $P(n_0)$ is true.
- (ii) Inductive Hypothesis: Assume $P(k)$ is true for $k=n$.
- (iii) Inductive Step:- Show that $P(k+1)$ is true on the basis of Inductive Hypothesis.

Example:-

Use mathematical induction to prove that

$$1+2+\dots+n = \frac{n(n+1)}{2}$$

Solution

1. Basis steps: $n=0$

$$\frac{0(0+1)}{2} = 0$$

$$n=1$$

$$\frac{1(1+1)}{2} = 1 \quad \text{which is true for } n=1$$

2. Inductive Hypothesis:

Assume $P(k)$ is true

$$\text{i.e. } P(k) : \frac{k(k+1)}{2} = \frac{k^2+k}{2}$$

$$= \frac{k(k+1)}{2} = 1+2+\dots+k.$$

3. Inductive step:

$$P(k+1) = 1+2+3+\dots+k+(k+1)$$

$$= \frac{k(k+1)+(k+1)}{2} = \frac{(k+1)(k+2)}{2} \quad (\text{By induction method.})$$

which is true, hence proved.

Example:-

Use mathematical induction to prove that.

$$1+2+2^2+\dots+2^n = 2^{n+1}-1$$

Solution:

1. Basis step: $n=1$

$$1 = 2^{1+1}-1 = 2^2-1 = 3$$

$$2^{0+1} = 2-1 = 1$$

which is true
for $n=1$.

2. Inductive hypothesis:-

Assume $P(k)$ is true.

$$\text{i.e. } P(k) = 2^{k+1}-1 ; k=1+2+\dots+2^k$$

3. Inductive step.

$$P(k+1) = 1+2+\dots+2^k+2^{k+1}$$

$$= 2^{k+1}-1+2$$

$$= \cancel{2^{k+1}(1-1)} \cdot 2^k \cdot 2^1 - 1 + 2^{k+1}$$

$$= 2^{k+1} - 1 + 2^k$$

Prove that $1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n! = (n+1)! - 1$, whenever n is an integer.

Solution

1. Basic step:-

$$n=0$$

$$(0+1)! - 1 = 0$$

$$n=1$$

$$(1+1)! - 1 = 1$$

which is true for $n=1$.

2. Inductive hypothesis

Assume $P(k)$ is true

$$\text{i.e. } P(k) = 1 \cdot 1! + 2 \cdot 2! + \dots + k \cdot k! \\ = (k+1)! - 1$$

3. Inductive steps.

$$P(k+1) = 1 \cdot 1! + 2 \cdot 2! + \dots + (k+1) \cdot (k+1)! \\ = (k+1)! - 1 + (k+1) \cdot (k+1)!$$

$$= (k+1)! \cdot (1 + (k+1)) - 1$$

$$= (k+1)! \cdot (k+2) - 1$$

$$= (k+2)! - 1 \quad \text{By induction method}$$

Strong Induction (Second principle of Mathematical Induction)

Steps:-

1. Basis steps: Show $P(n_0)$ is true.
2. Inductive hypothesis (Strong): Assume $P(k)$ is true for all $n_0 \leq k \leq n$.
3. Inductive step: Show based on assumption that $P(k+1)$ is true.

Eg:- Prove that 2 divides n^2+n whenever n is a positive integer.

Solution:

1. Basis step:-

$$n=0$$

$$P(n_0) = 0$$

$$P(n_0) = \frac{1^2+1}{2} = \frac{2}{2} = 1$$

which is true for $n=1$.

2. Inductive hypothesis (Strong).

Assume $P(k)$ is true for all $n_0 \leq k \leq n$.

$$\text{i.e. } P(k) = \frac{k^2+k}{2}$$

3. Inductive step:-

$$P(k+1) = \frac{(k+1)^2+(k+1)}{2}$$

$$= \frac{k^2+2k+1+k+1}{2} = \frac{k^2+3k+2}{2}$$

\therefore from inductive hypothesis: k^2+k is divisible by 2

$$= \frac{k^2+k}{2} + \frac{2(k+1)}{2} = \frac{k^2+k+2(k+1)}{2}$$

$$= \frac{2k^2+2k+2m}{2} \quad (m=k^2+k)$$

$$= 2(m+k+1) = 2l$$

which is divisible by 2.

Hence, it is true, proved!

Recursive Definition:-

Steps:-

1. Basis Step:-

Specify the value of the function at base (generally 0 or 1)

2. Recursive Step:-

Specify the rule for finding the value of function using the value of a function already found.

Example:-

Give a recursive definition of a sequence $\{a_n\}_{n=1,2,\dots,n}$ if $a_1 = 10$.

Basis Step:-

$n=1$

$$a_n = 10^n \quad \therefore a_1 = 10$$

$$(1+x)^n + (1+x)^{n-1} = (1+x)^n$$

Recursive Step:-

$$a_n = 10 a_{n-1}$$

∴ The recursive definition of sequence is $a_n = 10 a_{n-1}$

$$\begin{aligned} (1+x)^m &= m! \\ 2^2 &= (1+x+1)^2 \end{aligned}$$

Give a recursive defⁿ of the sequence $\{a_n\}_{n=1,2,3\dots}$
if

(a) $a_n = 6n$

(b) $a_n = 2n + 1$

(c) $a_n = 5$

(d) $a_n = n^2$

Sol:-

(a) $a_n = 6n$

Basis step:-

$n=1$.

$a_n = 6n ; a_1 = 6$

Recursive step steps:-

$a_n = 6 + (a_{n-1})$

\therefore The recursive definition of sequence is $a_n = 6 + (a_{n-1})$

(b) $a_n = 2n + 1$

Basis step:-

$n=1$

$a_n = 2n + 1 \Rightarrow a_1 = 3$

Recursive steps:-

$a_n = 2 + (a_{n-1})$

\therefore The recursive definition of sequence is
 $a_n = 2 + (a_{n-1})$,

$$\textcircled{C} \quad a_n = 5$$

Basis steps:-

$$\textcircled{a} \quad n=1$$

$$a_n = 5 ; a_1 = 5.$$

Recursive step:-

$$a_n = 5$$

\therefore The ~~given~~ recursive definition of sequence
 $a_n = 5$

$$\textcircled{D} \quad a_n = n^2$$

Basis steps:-

$$n=1$$

$$a_n = n^2 ; a_1 = 1 ; a_2 = 4 ; a_3 = 9 \dots$$

Recursive step:-

$$a_n = (2n+1) + (a_{n-1})$$

\therefore The recursive definition of sequence is

$$a_n = (2n+1) + (a_{n-1})$$

Recurrence Relation

→ A recurrence relation for a sequence of any is an equation that express a_n in terms of one or more of the previous terms of the sequence namely a_0, a_1, \dots, a_{n-1} for all integers n with $n \geq n_0$, where n_0 is a non-negative integer.

→ A sequence is called a solution of recurrence relation if its terms satisfies the recurrence relation.

Example:-

For $a_n = 3a_{n-1}$ and $a_0 = 1$ find a_1, a_2, a_3, a_4 & so on.

Solution:-

$$a_0 = 1$$

$$a_n = 3a_{n-1}$$

Then

$$a_1 = 3 \cdot a_0 = 3(1) = 3$$

$$a_2 = 3 \cdot a_1 = 9$$

$$a_3 = 3 \cdot a_2 = 27$$

$$a_4 = 3 \cdot a_3 = 81$$

$$a_5 = 3 \cdot a_4 = 243$$

Find the recurrence relation to find the total amount after 30 yrs if a person deposits Rs. 10,000 in a saving account at a bank yielding 11% per year with interest compounded annually.

Sol:-

Basis step:

$$n=1.$$

$$\begin{aligned} a_1 &= 10,000 \times 11\% + 10,000 \\ &= 10,000 + 1100 \\ &= 11,100 \\ &= a_0 + 0.11 a_0 \end{aligned}$$

$$\text{let } a_0 = 10,000$$

Recursive step:

$$a_n = \cancel{a_{n-1}} + (a_{n-1}) - \cancel{a_0} + 0.11 a_0 \cdot n$$

$$a_n = a_{n-1} + 0.11 \times a_{n-1}$$

$$\text{or, } a_n = (1.11)^n P_0$$

$$a_1 = 11,100$$

$$\begin{aligned} a_2 &= \cancel{11\%} \text{ of } a_1 + a_1 \\ &= (1.11)^2 a_0 \end{aligned}$$

$$a_3 = (1.11)^3 a_0$$

Solving Recurrence Relations.

1. Linear Homogeneous Recurrence Relation of Degree k with constant Coefficients.

linear Homogeneous Recurrence Relation of Degree k with constant coefficient with recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \text{ where, } c_1, c_2, \dots, c_k \text{ are real numbers and } c_k \neq 0.$$

The above relation is linear since, right hand side is the sum of the multiples of previous terms.

of the sequence. It is homogeneous because no terms occurs without being multiple of some a_j .

All the coefficients of the terms are constant because they do not depend on n . And the degree of the relation is k because a_n is expressed in terms of previous k terms of the sequence.

$$(I) P_n = (1-1) P_{n-1}$$

$$(II) a_n = a_{n-5}$$

$$(III) a_n = a_{n+1} + a_{n-2}$$

$$(IV) b_n = 2b_{n-1} + 1 \quad (\times) \text{ (because it's not homogeneous)}$$

$$(V) b_n = n b_{n-1} \quad (\times) \text{ (because there is no constant)}$$

Solving linear Homogeneous Recurrence Relation of Degree K with constant coefficients.

In solving the recurrence relation of this type, we approach to look for the solution of the form $a_n = r^n$, where r is a constant. $a_n = r^n$ is a solution of a recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$ if and only if $r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$. When we divide both sides by r^{n-k} and transpose the right hand side we have,

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0.$$

Here, we can say $a_n = r^n$ is a solution iff r is the solution of eq $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$. which is called characteristic eq of the recurrence relation. This eq is called characteristic root.

Theorem 1: Let c_1 and c_2 be real numbers. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2 . Thus the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \dots$ where α_1 and α_2 are constants.

Example:-

Solve the recurrence relation $a_n = a_{n+1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$.

Solution :-

Characteristics eq :-

$$\begin{aligned} r^2 - c_1 r - c_2 &= 0 \\ \Rightarrow r^2 - r - 6 &= 0. \end{aligned}$$

$$\text{or, } r_1^2 = 3; r_2 = -2$$

The solution of this recurrence relation is

$$a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot (-2)^n$$

$$\therefore a_n = \alpha_1 \cdot 3^n + \alpha_2 \cdot (-2)^n$$

We know that,

$$a_0 = 3^0 + \alpha_2 \cdot (-2)^0 = 1 + \alpha_2$$

$$\alpha_1 \cdot (3)^0 + \alpha_2 \cdot (-2)^0 = 3 \quad \text{--- (1)}$$

$$\alpha_1 = 6$$

$$\alpha_1 \cdot (3)^1 + \alpha_2 \cdot (-2)^1 = 6 \quad \text{--- (2)}$$

From (1) & (2)

$$\alpha_1 + \alpha_2 = 3$$

$$3\alpha_1 - 2\alpha_2 = 6$$

$$\therefore \alpha_1 = 12/5$$

$$\alpha_2 = 8/5$$

$$\{a_n\} \text{ is } a_n = \frac{12}{5} (3)^n + \frac{8}{5} (-2)^n$$

Exercise

What is the solution of the recurrence relation
 $a_n = a_{n+1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution:-

Characteristic eq :-

$$r^2 - r - 2 = 0$$

$$r^2 - r - 2 = 0$$

$$\text{or, } r^2 - r - 2 = 0$$

$$\text{or, } r(r-1) + (r-2) = 0$$

$$\text{or, } (r-2)(r+1) = 0$$

$$\therefore r = 1 \text{ or } r = -1$$

$$r_2 = -1$$

The solution of the recurrence eqⁿ is.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$\therefore a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n \quad \text{--- (1)}$$

We know that,

$$a_0 = 2$$

$$\alpha_1 (2)^0 + \alpha_2 (-1)^0 = 2 \quad \text{--- (2)}$$

$$a_1 = 7$$

$$\alpha_1 (2)^1 + \alpha_2 (-1)^1 = 7 \quad \text{--- (3)}$$

From (1) & (2),

$$\alpha_1 + \alpha_2 = 2 \quad \text{or} \quad \alpha_1 = 2 - \alpha_2$$

$$2\alpha_1 - \alpha_2 = 7$$

$$\text{or}, \quad 2(2 - \alpha_2) - \alpha_2 = 7$$

$$\text{or}, \quad 4 - 2\alpha_2 - \alpha_2 = 7$$

$$\text{or}, \quad 4 - 3\alpha_2 = 7$$

$$\text{or}, \quad 3\alpha_2 = -3$$

$$\text{or}, \quad \alpha_2 = -1.$$

Putting value of α_1 and α_2 , we get.

$$a_n = 2(2)^n + (-1)(-1)^n$$

When $n=0$

$$a_0 = 2(2)^0 + (-1)(-1)^0 = 3 - 1 = 2$$

$$a_1 = 2(2)^1 + (-1)(-1)^1 = 6 + 1 = 7.$$

\therefore The solution of recurrence relation f_n is $a_n = 3(2)^n + (-1)(-1)^n$,

Exercise:-

Find the explicit formula for the fibon acci numbers. [Use $f_n = f_{n-1} + f_{n-2}$ as recursive def' and $f_0 = 0$ and $f_1 = 1$ as initial condition].

Solution:-

Recursive Relation:-

$$f_n = f_{n-1} + f_{n-2}$$

Characteristic eq:-

$$r^2 - r - 1 = 0$$

$$\therefore r = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$\therefore r_1 = \frac{-1 + \sqrt{5}}{2} \quad r_2 = \frac{-1 - \sqrt{5}}{2}$$

The sol' of recursive relation is.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$a_n = \alpha_1 \left(\frac{-1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left(\frac{-1 - \sqrt{5}}{2} \right)^n$$

We Know,

$$\alpha_0 = 0.$$

~~Ans~~ 1.

$$\therefore \alpha_1 = -\alpha_2 \quad \textcircled{1}$$

$$\alpha_1 = 1$$

$$\alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{-1-\sqrt{5}}{2} \right) = 1$$

$$\alpha_1 \cdot \frac{-\alpha_1 + \alpha_1 \sqrt{5}}{2} + \frac{-\alpha_2}{2} - \frac{\alpha_2 \sqrt{5}}{2} = 1$$

$$\alpha_1 \cdot \frac{\alpha_2}{2} - \frac{\alpha_2 \sqrt{5}}{2} + \frac{-\alpha_2}{2} - \frac{\alpha_2 \sqrt{5}}{2} = 1$$

$$\alpha_1 - \alpha_2 = \frac{2}{\sqrt{5} \cdot 2}$$

$$\alpha_1, \alpha_2 = -\frac{1}{\sqrt{5}}$$

$$\therefore \alpha_1 = \frac{1}{\sqrt{5}}$$

Putting value of α_1 & α_2 , we get.

$$\{a_n\} \text{ is } a_n = \frac{1}{\sqrt{5}} \left(\frac{-1+\sqrt{5}}{2} \right)^n + \left(\frac{-1}{\sqrt{5}} \right) \left(\frac{-1-\sqrt{5}}{2} \right)^n$$

Theorem 2:

Let c_1 and c_2 be real numbers with $c_2 \neq 0$.

Suppose that $r^2 - c_1r - c_2 = 0$ has only one root r_0 . Then the sequence $\{a_n\}$ is a solⁿ of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$ for $n = 0, 1, 2, \dots$ where α_1 and α_2 are constant.

Example:-

Solve the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$

Solution:-

Characteristic eqⁿ:

$$r^2 - 2r + 1 = 0$$

$$\text{or, } r^2 - r - r + 1 = 0$$

$$\text{or, } r(r-1) - 1(r-1) = 0$$

$$\text{or, } (r-1)(r-1) = 0$$

$$\therefore r_1 = 1; r_2 = 1$$

The solⁿ of recurrence relation is

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

$$\text{we have, } a_0 = 3$$

$$\therefore \alpha_1 \cdot 1^0 + \alpha_2 \cdot 1^0 = 3$$

$$\text{or, } \alpha_1 + \alpha_2 = 3 \quad \text{--- (1)}$$

$$a_1 = 6$$

$$\alpha_1 \cdot (1)^1 + \alpha_2 \cdot (1)^1 = 6$$

$$\alpha_1 + \alpha_2 = 6$$

$$\alpha_2 = 3$$

$$\therefore \{a_n\} \text{ is } a_n = 3(1)^n + 3 \cdot n(1)^n$$

What is the solution of the recurrence relation

$a_n = 6a_{n-1} - 9a_{n-2}$ with initial condition $a_0 = 1$ &

$$a_1 = 6?$$

$$\Delta a_n^2 =$$

Characteristic eq is

$$r^2 - C_1 r + C_2 = 0$$

$$\text{or, } r^2 - 6r + 9 = 0$$

$$\text{or, } 2r^2 - 3r + 3r + 9 = 0$$

$$\text{or, } r(r-3) - 3(r-3) = 0$$

$$\therefore (r-3)(r-3) = 0$$

$$\therefore r_1 = 3$$

$$r_2 = 3$$

The solⁿ of recurrence relation is

$$a_n = \alpha_1 (r_1)^n + \alpha_2 n (r_2)^{n-1} \geq 0$$

When,

$$a_0 = 1$$

$$\alpha_1 (3)^0 + \alpha_2 0 (3)^0 = 1.$$

$$\alpha_1 = 1$$

$$\alpha_1 = 6$$

$$\alpha_1 (3)^1 + \alpha_2 1 (3)^1 = 6 = 0, \text{ so } \alpha_2 = 0$$

$$3\alpha_1 + 3\alpha_2 = 6$$

$$\alpha_2 = 1$$

$\therefore \{a_n\}$ is $a_n = 1(3)^n + 1^n (3)^n$,

Theorem 3 :-

Let c_1, c_2, \dots, c_k be real numbers suppose $r^k - c_1 r^{k-1} - \dots - c_{k-1} r - c_k = 0$ has k distinct roots r_1, r_2, \dots, r_k .

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $n=0, 1, 2, \dots$ take

$$c_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \text{ if}$$

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n \text{ for } n=0, 1, 2, \dots$$

where $\alpha_1, \dots, \alpha_k$ are constant.

Example:-

Solve the following recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 8a_{n-3} \text{ for } n \geq 3, a_0 = 3, a_1 = 6 \text{ and } a_2 = 9$$

(characteristic Eq :-)

$$r^3 - 2r^2 - r + 2 = 0$$

$$\text{or, } r^2(r-2) - 1(r-2) = 0$$

$$\text{or, } (r^2 - 1)(r-2) = 0$$

$$\text{or, } (r+1)(r-1)(r-2) = 0$$

$$\therefore r_1 = -1$$

$$r_2 = 1$$

$$r_3 = 2$$

The solⁿ of recurrence relation is :-

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$$

$$\text{or, } a_n = \alpha_1(-1)^n + \alpha_2 1^n + \alpha_3 2^n$$

where,

$$a_0 = 3.$$

$$\text{as } \alpha_1(-1)^0 + \alpha_2(1)^0 + \alpha_3(2)^0 = 3$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 3 \quad \textcircled{1}$$

$$\text{when } a_1 = 6$$

$$\alpha_1(-1)^1 + \alpha_2(1)^1 + \alpha_3(2)^1 = 6$$

$$-\alpha_1 + \alpha_2 + 2\alpha_3 = 6 \quad \textcircled{2}$$

when $\alpha_2 = 9$

$$\begin{aligned} \textcircled{1} \quad \alpha_1(-1)^2 + \alpha_2(1)^2 + \alpha_3(2)^2 &= 5 \\ \alpha_1 + \alpha_2 + 4\alpha_3 &= 9 \quad \textcircled{1u} \end{aligned}$$

From $\textcircled{1} + \textcircled{1u}$

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 3 \quad \textcircled{1}, \quad \alpha_1 + \alpha_2 = 4 - 3\alpha_3 - \textcircled{2} \\ -\alpha_1 + \alpha_2 + 2\alpha_3 &= 6 \\ \alpha_1 + \alpha_2 + 4\alpha_3 &= 9 \end{aligned}$$

From $\textcircled{2}$

$$(3 - \alpha_3) + 4(\alpha_3) = 9$$

$$3 - \alpha_3 + 4\alpha_3 = 9$$

$$3 + 3\alpha_3 = 9$$

$$-3 + 9 = 3\alpha_3$$

$$\alpha_3 = 2$$

\therefore from $\textcircled{1} + \textcircled{1u}$

$$\alpha_1 + \alpha_2 = 1$$

$$\underline{-\alpha_1 + \alpha_2 = 2}$$

$$2\alpha_2 = 3$$

$$\therefore \alpha_2 = \frac{3}{2}$$

$$\therefore \alpha_1 = 3 - 2 - \frac{3}{2}$$

$$= 1 - \frac{3}{2} = -\frac{1}{2}$$

$$\therefore \text{sum } a_n = -\frac{1}{2}(-1)^n + \frac{3}{2}(1)^n + 2(2)^n$$

Theorem 4:-

Let c_1, c_2, \dots, c_k be real numbers. Suppose that $r^k - c_1 r^{k-1} - \dots - c_{k-1} r - c_k = 0$ has k distinct roots

r_1, r_2, \dots, r_t with multiplicity m_1, m_2, \dots, m_k respectively, so that $m_i \geq 1$ for $i=1, 2, \dots, t$ and $m_1 + m_2 + \dots + m_k = k$.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$

if

$$a_n = (\alpha_{1,0} + \alpha_{1,1} r_1^n + \dots + \alpha_{1,m_1-1} r_1^{m_1-1}) r_1^n$$

$$+ (\alpha_{2,0} + \alpha_{2,1} r_2^n + \dots + \alpha_{2,m_2-1} r_2^{m_2-1}) r_2^n.$$

$$+ \dots + (\alpha_{t,0} + \alpha_{t,1} r_t^n + \dots + \alpha_{t,m_t-1} r_t^{m_t-1}) r_t^n.$$

for $n=0, 1, 2, \dots$ where α_{ij} are constant for $1 \leq i \leq t$ and $0 \leq j \leq m_i-1$.

Example:

Solve the recurrence relation $a_n = 5a_{n-1} - 7a_{n-2} - \frac{3}{2}a_{n-3}$ for $n \geq 3$, $a_0 = 1$, $a_1 = 7$ and $a_2 = 15$.

Sol:-

Characteristic eqn:-

$$r^3 - 5r^2 + 7r - 3 = 0$$

$$\text{or, } r^3 - r^2 - 4r^2 + 4r + 3r - 3 = 0$$

$$\text{or, } r^2(r-1) - 4r(r-1) + 3(r-1) = 0$$

$$\text{or, } (r-1)(r^2 - 4r + 3) = 0$$

$$\text{or, } (r-1)(r^2 - r - 3r + 3) = 0$$

$$\text{or, } (r-1)r(r-1) - 3(r-1)^2 = 0$$

$$\text{or, } (r-1)(r-1)(r-3) = 0$$

$$\therefore r_1 = 1, r_2 = 1, r_3 = 3$$

$$\therefore r_1 = 1 \text{ and } m_1 = 2$$

$$r_2 = 3 \text{ and } m_2 = 1$$

Since $r_1 = 1$ and $m_1 = 2$

$$\therefore a_n = (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{2,0} + \alpha_{2,1}n + \dots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n$$

$$a_n = [(\alpha_{1,0}) + (\alpha_{1,1})n] 2^n + (\alpha_{2,0}) 3^n$$

We have

$$\alpha_{1,0} = 1 \quad (1)$$

$$\alpha_{1,1} = 9 \quad (2)$$

$$\alpha_{2,0} = 15 \quad (3)$$

~~$$\alpha_{1,0} = [\alpha_{1,0} + \alpha_{1,1} \cdot 0] 1^0 + \alpha_{2,0} 3^0 = 1 + 15 = 16$$~~

~~$$[\alpha_{1,0} + \alpha_{1,1} \cdot 1] 1^1 = 9$$~~

~~$$[\alpha_{1,0} + \alpha_{1,1} \cdot 2] 1^2 + \alpha_{2,0} 3^2 = 15$$~~

~~$$(\alpha_{1,0} + \alpha_{1,1} \cdot 0) 1^0 + \alpha_{2,0} 3^0 = 1 + 15 = 16$$~~

~~$$\alpha_{1,0} + \alpha_{2,0} = 1 \quad (4)$$~~

~~$$(\alpha_{1,0} + \alpha_{1,1} \cdot 1) 1^1 + \alpha_{2,0} 3^1 = 9 + 15 = 24$$~~

~~$$\alpha_{1,0} + \alpha_{1,1} + 3\alpha_{2,0} = 24 \quad (5)$$~~

~~$$(\alpha_{1,0} + \alpha_{1,1} \cdot 2) 1^2 + \alpha_{2,0} 3^2 = 15$$~~

~~$$\alpha_{1,0} + 2\alpha_{1,1} + 9\alpha_{2,0} = 15 \quad (6)$$~~

From (i) & (ii)

$$2\alpha_{1,0} + 2\alpha_{1,1} + 6\alpha_{2,0} = 18$$

$$\alpha_{1,0} + 2\alpha_{1,1} + 3\alpha_{2,0} = 15$$

$$\alpha_{1,0} - 3\alpha_{2,0} = 3 \quad \text{(iv)}$$

From (i) & (v)

$$\alpha_{1,0} - 3\alpha_{2,0} = 3$$

$$\alpha_{1,0} + \alpha_{2,0} = 2$$

$$8 = 2\alpha_{2,0} : 2$$

$$\alpha_{2,0} = \frac{-1}{2}$$

$$\therefore \alpha_{1,0} = 1 + \frac{1}{2} = \frac{3}{2}$$

$$\alpha_{1,1} = 9 - 3\alpha_{2,0} - \alpha_{1,0}$$

$$= 9 + \frac{3}{2} - \frac{3}{2}$$

$$= 9$$

The solution of sequence is :-

$$\{a_n\} \text{ is } a_n = \left(\frac{3}{2} + 3 \cdot n \right) 1^n + \left(\frac{-1}{2} \right) 3^n$$

For $n=0$,

$$a_0 = \frac{3}{2} - \frac{1}{2} = 1 \quad (1+0 \cdot 1, 1+0, 1)$$

$$S_1 = (C_1 - 1)(1 - 3, 1 + 1, 3 + 0, 1)$$

$$(1) \rightarrow S_1 = (S_1, 1 + 1, 3 + 0, 1) -$$

$$(2) \rightarrow S_2 = S_1 + 1, 3 + 0, 1$$

$$(3) \rightarrow S_3 = S_2 + 1, 3 + 0, 1$$

Exercise:-

Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} \text{ with initial conditions}$$

$$a_0 = 1, a_1 = -2 \text{ and } a_2 = -1.$$

solution:-

Characteristic eq is

$$\begin{aligned} r^3 + 3r^2 + 3r + 1 &= 0 \\ \cancel{r^2(r+1)^2} + \cancel{1} &= 0 \\ r^2(r+1)(r+1) &= 0 \end{aligned}$$

$$\therefore r = -1 \text{ and } m = 3$$

$$\therefore a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)r^n$$

$$\therefore a_n = (\alpha_{1,0} + \alpha_{1,1}n + \dots + \alpha_{1,m-1}n^{m-1})r^n$$

$$\therefore a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)(-1)^n$$

We have,

$$a_0 = 1$$

$$a_1 = -2$$

$$a_2 = -1$$

$$(\alpha_{1,0} + \alpha_{1,1} \cdot 0 + \alpha_{1,2} \cdot 0^2)(-1)^0 = 1$$

$$\alpha_{1,0} = 1 \quad \text{--- (1)}$$

$$(\alpha_{1,0} + \alpha_{1,1} \cdot 1 + \alpha_{1,2} \cdot 1^2)(-1)^1 = -2$$

$$-(\alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2}) = -2 \quad \text{--- (2)}$$

$$\cancel{\alpha_{1,0} + \alpha_{1,1} + \alpha_{1,2}} = 2 \quad \text{--- (3)}$$

$$\alpha_{1,1} + \alpha_{1,2} = -2 \quad \text{--- (4)}$$

$$(\alpha_{1,0} + \alpha_{1,1} 2 + \alpha_{1,2} 2^2) (-1)^2 = -1$$

$$\alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2} = -1. \quad \text{--- (III)}$$

from (I) & (III)

$$\cancel{-\alpha_{1,0} - \alpha_{1,1} - 4\alpha_{1,2}} = -2$$

$$\cancel{\alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}} = -1$$

$$\alpha_{1,1} + 3\alpha_{1,2} = -3$$

from (I) & (II)

$$-1 - \alpha_{1,1} - \alpha_{1,2} = -2$$

$$-\alpha_{1,1} - \alpha_{1,2} = -1$$

$$\alpha_{1,1} + \alpha_{1,2} = 1$$

$$\text{or, } \alpha_{1,1} = 1 - \alpha_{1,2}.$$

Now,

in (III)

$$1 + 2 - 2\alpha_{1,2} + 4\alpha_{1,2} = -1$$

$$1 + 2 + 2\alpha_{1,2} = -1$$

$$3 + 2\alpha_{1,2} = -1$$

$$2\alpha_{1,2} = -4$$

$$\alpha_{1,2} = -2$$

$$\text{or, } \alpha_{1,1} = 1 - (-2) = 3.$$

i. The sol. of sequence $\{a_n\}_{n \geq 1}$

$$a_n = [1 + 3n + (-2)n^2] (-1)^n,$$

2. Solving linear Non-homogeneous Recurrence Relation of degree K with constant coefficients.

The recurrence relation of the form

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$ where c_1, c_2, \dots, c_n are real numbers and $f(n)$ is a function depending upon n . The recurrence relation preceding $f(n)$ is called associated homogeneous recurrence relation.

Theorem 5:-

If $\{a_n^{(P)}\}$ is a particular solution of the non homogeneous linear recurrence relation with constant coefficient $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f_n$ then every solution of the form $\{a_n^{(P)} + a_n^{(H)}\}$ where $a_n^{(H)}$ is a solution of a associated homogeneous recurrence relation.

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}.$$

Example:-

Find all the solutions of the recurrence relation $a_n = 4a_{n-1} + n^2$. Also find the solution of relation with initial condition $a_1 = 1$.

Solution,

here the associated recurrence relation is

$$a_n = 4a_{n-1}.$$

Characteristic eq:

~~$r^2 - 4r + 1 = 0$~~

~~$r^2 - 2r - 2r + 1 = 0$~~

$$\therefore a_n = \alpha(4)^n$$

$$\{a_n\} \propto 4^n.$$

Now,

$f(n) = n^2$ is a polynomial of degree 2, a trial solution is a quadratic function in n , say $p_n = a_n^2 + b_n + c$ where a, b, c are constants.

To determine whether there are any solutions of the form, suppose that $p_n = a_n^2 + b_n + c$ is such a solution. Then the eq² is $a_n^2 - 4a_n + 2 + n^2$ becomes.

$$\begin{aligned} a_n^2 + b_n + c &= 4(a_{n-1}^2 + b_{n-1} + c) + n^2 \\ &= [4a_{n-1}^2 + 4b_{n-1} + 4c] + n^2 \\ &= [4a_{n-1}^2 - 4a_{n-1} + 4b_{n-1} + 4c] + n^2 \\ &= (4a_n^2 - 4a + 4b_n - 4b + 4c) + n^2 \\ &= (4a+1)n^2 + (4b-1)n + \\ &\quad (4a+1)n^2 + (-2+4b) + (4a-4b+4c) \end{aligned}$$

$a_n^2 + b_n + c$ is a sol².

~~$a = 4a+1 \Rightarrow 6a = -1/4$~~

~~$b = -8a+4b \Rightarrow 2+4b \Rightarrow b = 1/2$~~

~~$c = 4a - 4b + 4c \Rightarrow -1 - 2 + 4c \Rightarrow c = 3/4$~~

~~$a = 4a+1 \therefore a = -1/3$~~

~~$b = -8a+4b \therefore b = -8/3$~~

~~$c = 4a - 4b + 4c \therefore c = -20/27$~~

$$\therefore \{a_n\} = a_n^2 + b_n + c = -\frac{1}{3}n^2 - \frac{8}{3}n - \frac{20}{27}$$

Now,

the solution of $\{a_n\}$ is

$$\{a_n^{(P)} + a_n^{(I)}\} =$$

$$\therefore a_n = \alpha 4^n + \frac{-1}{3}n^2 - \frac{8}{3}n - \frac{20}{27}$$

Now,
 $a_1 = 2.$

$$a_n = \alpha(4)^n + \left(-\frac{1}{3}n^2 - \frac{8}{5}n - \frac{20}{27} \right)$$

$$\alpha(4)^1 + \left(-\frac{1}{3} \cdot 1^2 - \frac{8}{5} \cdot 1 - \frac{20}{27} \right) = 2$$

$$4\alpha + \left(-\frac{53}{27} \right) = 2$$

$$\alpha = \frac{53}{27}$$

$$\alpha = \frac{20}{27}$$

\therefore The final solution of $\{a_n\}$

$$a_n = \frac{20}{27}(4)^n + (an^2 + bn + c)$$

Exercise :-

Find all the solutions of the recurrence relation
 $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$.

Solution:

here the associated recurrence relation is

$$a_n = 3a_{n-1}$$

Characteristic eqⁿ:

$$r - 3 = 0$$

$$r = 3$$

$$\therefore a_n = \alpha(3)^n$$

$$\{a_n\} \subset \alpha 3^n.$$

Now, $f(n) = 2n$ is a polynomial of degree 1,
 a trial solution of a linear function is n ,
 say $p_n = a_n + c$ where a & c are constants.

To determine whether there are solution of
 the form, suppose $p_n = a_n + c$ is such soln.
 Then eqⁿ is $a_n = 3a_{n-1} + 2n$ becomes.

$$\begin{aligned} a_n + c &= 3[a(n-1) + c] + 2n \\ &= 3[a_n - a + c] + 2n \\ &= 3a_n - 3a + 3c + 2n - 3a \\ &= 3a_n + 3n + 3c \quad n(3a + 2) + 3c(3a + 3) \\ &= (3a + 2)n + 3c(3a + 3) \end{aligned}$$

$\therefore a_n + c$ is soln if

$$3a + 2 = 0 \quad \text{or} \quad 2a = -2 \Rightarrow a = -\frac{2}{2} = -1$$

$$3c + 3 = 0 \quad \text{or} \quad c = -\frac{3}{3} = -1$$

$$3c - c = -3 \quad \text{or} \quad c = -3/2$$

$$\therefore \{a_n(p_n)\} = a_n + c = -n - \frac{3}{2}$$

Now,

The solution of $\{a_n\}$ is

$$\{a_n(p) + a_n(n)\}$$

$$\therefore a_n = \alpha 3^n - n - \frac{3}{2}$$

Now,

$$a_1 = 3$$

$$a_1 = \alpha 3^1 - 1 - \frac{3}{2}$$

$$\text{or, } 3 = \alpha 3^1 - 1 - \frac{3}{2}$$

$$\text{or, } 3 = \alpha 3 - \frac{5}{2}$$

$$\text{or, } 3 = 3\alpha - \frac{5}{2} \quad |+5/2 \quad | \times 2$$

$$\text{or, } 3\alpha = \frac{11}{2}$$

$$\therefore \alpha = \frac{11}{6}$$

∴ Final solution of $\{a_n\}$ is

$$a_n = \frac{11}{6} (3)^n - n - \frac{3}{2}$$

Theorem 6:-

Suppose that if a_n satisfies the linear non-homogeneous recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + f(n)$, where, c_1, c_2, \dots, c_k are real numbers and $f(n) = (b_t n^t + b_{t-1} n^{t-1} + \dots + b_0 n + b_0) s^n$, where b_0, b_1, \dots, b_t are real numbers.

When s is not a root of the characteristic eqⁿ of the associated linear homogeneous recurrence relation, there is a particular solution of the form $(P_t n^t + P_{t-1} n^{t-1} + \dots + P_0) s^n$.

When s is a root of the characteristic eqⁿ and its multiplicity is m , there is a particular solution of the form $n^m (P_t n^t + P_{t-1} n^{t-1} + \dots + P_0) s^n$.

Example:-

Find a solution of the recurrence relation $a_n = 2a_{n-1} + n 2^n$.

Sol :-

here the associated recurrence relation is $a_n = 2a_{n-1}$

Characteristic eqⁿ is

$$r - 2 = 0$$

$$r = 2$$

\therefore Solⁿ of a_n is $\alpha 2^n$

$$f(n) = n \cdot 2^n$$

\therefore The function is $p = (b_1 n + b_0) s^n$

$$\begin{aligned} & \text{Solving } -n^m (P_t n^t + P_{t-1} n^{t-1} + \dots + P_0) s^n \\ &= n^m (P_t n^t + P_{t-1} n^{t-1} + \dots + P_0) s^n \\ &= n^m (P_t n^t + P_{t-1} n^{t-1} + \dots + P_0) 2^n \\ &= n^m (P_t n^t + P_{t-1} n^{t-1} + \dots + P_0) 2^n \end{aligned}$$

- 2 marks

- now if we multiply both sides by 2 we get
 $2(a+b+c+d) = 2n$ which is our required equation

$$2(a+b+c+d) = 2n \quad \text{or} \quad n = \frac{2(a+b+c+d)}{2}$$

and now here we have to find the value of each term
 which is the sum of all the numbers in a row.

$$2(a+b+c+d+e+f+g+h+i+j+k+l+m)$$

now if we multiply each term by 2 we get
 which is the sum of all the numbers in a row.

$$2(a+b+c+d+e+f+g+h+i+j+k+l+m) = 2n$$

which is the sum of all the numbers in a row.

$$2(a+b+c+d+e+f+g+h+i+j+k+l+m) = 2n$$

now if we multiply each term by 2 we get
 which is the sum of all the numbers in a row.

$$2(a+b+c+d+e+f+g+h+i+j+k+l+m) = 2n$$

$$0 = 0 - 0$$

$$0 = 0 - 0$$

so it is true

$$0 = 0 - 0$$

$2(a+b+c+d+e+f+g+h+i+j+k+l+m) = 2n$

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Find the solⁿ of the recurrence relation ~~$a_n = 6a_{n-1} - 9a_{n-2} + n3^n$~~
 $a_n = 6a_{n-1} - 9a_{n-2} + n3^n$.

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Counting

Introduction

- **Combination** Combinatorics is the study of arrangement of objects.
- **Enumeration** the counting of objects with certain properties is an important part of combinatorics.
- we must count objects to determine the complexity of the algorithm, to determine there are enough telephone numbers to meet demand etc.

Basis of Counting

Sum Rule:

If a task can be done in n_1 ways and a second task in n_2 ways and if these tasks cannot be done at same time, then there are $n_1 + n_2$ ways to do one of these tasks.

$$\text{i.e. } |A_1 \cup A_2 \cup A_3 \cup \dots \cup A_m| = |A_1| + |A_2| + \dots + |A_m|.$$

Example:-

In how many ways we can draw a heart or a diamond from an ordinary deck of playing cards?

The number of ways to draw diamond (n_1) = 13

The number of ways to draw heart (n_2) = 13.

∴ The total number of ways ($n_1 + n_2$) = 26.

Example:-

In how many ways we can get a sum of 4 or of 8 when two distinguishable dice are rolled?

Solution:- By sum rule.

No. of ways we can get a sum of 4 is

$$(n_1) = 3$$

No. of ways we can get a sum of 8 is

$$(n_2) = 5$$

∴ The total number of ways $(n_1 + n_2) = 8$.

Product Rule

If a task can be done in n_1 ways and a second task in n_2 ways, after the first task has been done, then there are $n_1 \cdot n_2$ ways to do the work that consists both the task.

$$\text{i.e. } |A_1 \times A_2 \times \dots \times A_k| = |A_1| \cdot |A_2| \cdot \dots \cdot |A_k|$$

Example:-

An office building contains 27 floors and has 32 offices on each floor. How many offices are there in the building?

Solution:-

By Product rule;

$$\text{no. of floors } (n_1) = 27$$

$$\text{no. of offices } (n_2) = 32$$

$$\text{Total no. of offices} = n_1 \cdot n_2 = 27 \times 32$$

How many different three-letter initials with none of the letters can be repeated can people have?

Solution:-

The different ways in which three-letter initials with none of the letters can be repeated can people have = $26 \times 25 \times 24$.

The inclusion-exclusion Principle:-

When two tasks can be done at the same time, we cannot use the sum rule to count the number of ways to do one of the two tasks. Adding the number of ways to do each task leads to an over count, since the ways to do both tasks are counted twice. To correctly count the number of ways to do one of just two tasks, we add the number of ways to do both tasks. This technique is called principle of inclusion-exclusion.

Let A_1 and A_2 be sets and let T_1 be the task of choosing elements from A_1 , and T_2 be the task of choosing an element from A_2 . The number of ways to do either T_1 or T_2 is.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

Exercise:-

How many bit strings of length ~~eight~~⁸ either start with a 1 bit or end with two bits 00?

Solution:-

A_1 = # bit strings starting with 1

A_2 = bit strings ending with 00

$$A_1 = 2^7 \text{ ways.} = 1$$

$$A_2 = 2^6 \text{ ways.} = 0$$

$$|A_1 \cup A_2| = 1 + 0 = 2^5 \text{ ways.}$$

By principle of inclusion-exclusion principle.

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$$

$$= 2^7 + 2^6 - 2^5$$

$$= 160 \text{ ways.}$$

Pigeonhole Principle:-

The pigeonhole Principle (Theorem):-

If $k+1$ or more objects are placed into k boxes then there is at least one box containing two or more of the objects.

Proof:-

We use proof by contradiction. Suppose that $k+1$ or more boxes are placed into k boxes and no boxes such that there are no two objects in a box. This contradicts our assumption. So, there is at least one box containing two or

more of the objects.

Example:- Show that if there are 30 students in a class, then at least two have last name that begins with same letter.

$$\text{total number of students} = 30$$

$$\text{number of alphabets} = 26$$

By pigeon hole principle, ~~at least~~
there are ~~one~~ two students having last name
that begins with same letter.

Example:-

How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

$$\text{Total scores} = \text{Minimum score} + (\text{Maximum score} - \text{Minimum score})$$

Solution:-

$$n = \text{no. of students in class.}$$

$$\text{Score in exam} = 0 \text{ to } 100$$

By pigeon hole principle,

~~At least~~ There must be at least 102 students.

The generalised pigeonhole principle

If N objects are placed into K boxes there
more is at least $\lceil N/K \rceil$ objects.

Proof:-

Suppose one of the boxes contains $\lceil N/K \rceil$ or
more objects. Then every box contain at most
 $\lceil N/K \rceil - 1$ objects. So the total number of objects
is at most $K(\lceil N/K \rceil - 1)$.

$$\text{But } \lceil N/K \rceil - 1 < N/K$$

Thus, the total number of object is less than
 $K(N/K)$ is less than N . This is a contradiction.
Hence the proof:

Examples:-

Find the minimum number of people among
100 people who were born in the same month

Solution:-

$$\lceil N \rceil \text{ total number of people} = 100$$

$$\text{no. of month } [x] = 12$$

By generalized pigeonhole principle,

$$\begin{aligned}\lceil N/x \rceil &= \lceil 8.33 \rceil \\ &= 9,\end{aligned}$$

If a class has 24 students, what is the maximum number of possible grading that must be done to ensure that there are at least two students with the same grade.

Solution :-

$$N = 24$$

$$N - K = ?$$

$$N/K \geq 2$$

$$10 - 0$$

We know that, By generalised pigeonhole principle,

$$N = K(r-1) + L$$

$$\text{or, } 24 = K(2-1) + 1$$

$$\text{or, } 24 = K + 1$$

$$\therefore K = 23$$

What is the minimum number of students required in a class to be sure that at least one will receive the same grade; if there are five possible grades A, B, C, D, E?

Solution :-

$$N = ?$$

$$K = 5$$

$$N/K = ?$$

Acc. to G.P.P.

$$N = K[4K-1] + 1$$

$$24 = ?$$

$$= 5[5-1] + 1$$

$$24 = ?$$

$$= 25$$

$$24 = 5[5-1] + 1$$

Permutations and Combinations:-

Permutations.

A permutation of a set of distinct objects is an ordered arrangement of these objects. An ordered arrangement of n elements of a set is called r -permutation.

$$\therefore P(n, r) = \frac{n!}{(n-r)!} \quad \text{and } P(n, n) = n!$$

Example:-

How many ways are there to select a first-prize winner, a second-prize winner and a third-prize winner from 100 different people who have entered a contest?

Solution:-

$$P(n, r) = \frac{n!}{(n-r)!} = \frac{100!}{(100-3)!} = \frac{100!}{97!}$$

- # How many permutations of the letters ABCDEFGH contains the string ABC.

Solution:-

$$n = 8$$

$$r = 6$$

$$P(n, n) = 6! = 720$$

Suppose that a ~~salesman~~ saleswoman has to visit eight different ^{cities} ~~towns~~. She must begin \rightarrow has her in a specified city, but she can visit other seven cities in any order she wishes. How many possible orders can be the saleswoman use when visiting these cities?

Solution:-

$$n = 8$$

$$r = 7$$

$$P(n, n) = 7!$$

2. Show $\#$ by mathematical induction that

$$\sum_{j=0}^n ar^j = a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r - 1} \text{ where } r \neq 1.$$

$r \neq 1$, and n is a non-negative integer.

Combinations:-

An r -combination of elements of a set is an unordered selection of r elements from the set.

$$C(n, r) = \frac{n!}{r!(n-r)!}$$

Example:-

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another college?

Solution

$$\text{total no. of member}(n) = 10$$

$$r = 5$$

$$\therefore C(10, 5) = \frac{10!}{5!(10-5)!}$$

$\therefore 252$ ways are there to select five players from a 10-member tennis team.

Binomial Coefficients

Theorem:-

Let x and y be variables, and let n be non-negative integer. Then,

$$0020032 = \binom{2y}{x} = \binom{x}{n}$$

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y^1 + \dots + \binom{n}{n-1} x^1 y^{n-1} + \binom{n}{n} y^n$$

1. What is the expansion of $(x+y)^4$?

$$(x+y)^4 = \sum_{j=0}^4 \binom{4}{j} x^{4-j} y^j$$

$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y^1 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$

$$= x^4 + 4x^3 y + 6x^2 y^2 + 4x y^3 + y^4$$

2. What is the coefficient of $x^{12} y^{13}$ in the expansion of $(x+y)^{25}$?

Solution:-

The coefficient of $x^{12} y^{13}$ is :

$$j = 13$$

$$n-j = 12$$

$$\text{and } n = 12 + 13 = 25.$$

$$\therefore \binom{n}{j} = \binom{25}{13} = 5200300$$

3. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x - 3y)^{25}$?

Solution:-

$$(2x - 3y)^{25}$$

The coefficient of $x^{12}y^{13}$ is:-

$$j = 13$$

$$\therefore \binom{n}{j} = \binom{25}{13} = -3.39 \times 10^{16},$$

$$= \binom{25}{13} (2)^{12} x^{12} (-3)^{13} y^{13} = -3.39 \times 10^{16},$$

Corollary :-

let n be a non-negative integer, Then,

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Proof:-

Using the binomial theorem with $x=1$ and $y=1$.

$$2^n = (x+y)^n = (1+1)^n$$

Then the binomial expansion will be.

$$(1+1)^n = \left(\sum_{k=0}^n \binom{n}{k} 1^{n-k} i^k \right) = \sum_{k=0}^n \binom{n}{k}$$

proved!

STANDARDS

STANDARDS

(Corollary 2:-)

Let n be a non-negative integer. Then.

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$$

Proof:-

$$\text{Let } x=1, y=(-1).$$

$$0^n = (1+(-1))^n =$$

∴ Using binomial theorem,

$$\sum_{k=0}^n \binom{n}{k} (1)^{n-k} (-1)^k = \sum_{k=0}^n \binom{n}{k} \times 0 \\ = 0$$

proved!

(Corollary 3:-)

Let n be a non-negative integer. Then,

$$\sum_{k=0}^n (2)^k \binom{n}{k} = 3^n.$$

Proof:-

Using binomial theorem with $x=1$ & $y=2$.

$$\sum_{k=0}^n \binom{n}{k} (1)^{n-k} (2)^k = (1+2)^n = 3^n$$

$$3^n = (1+2)^n$$

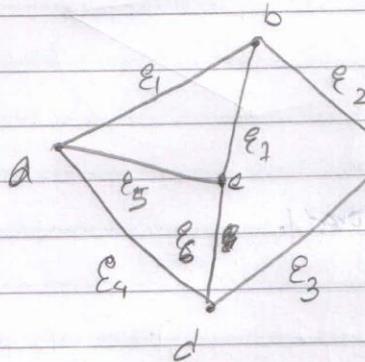
$$(1+2)^n = \sum_{k=0}^n \binom{n}{k} (1)^{n-k} (2)^k$$

$$= 2^k \sum_{k=0}^n \binom{n}{k}$$

prob 1.

Unit - 4: Graphs

$$n(SH) = 2$$

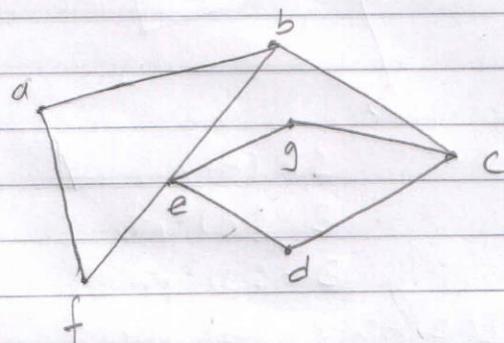


$G(V, E)$

$E = \{(a, b), (b, c), (c, d), (d, a), (a, e), (b, e), (d, e)\}$
 $V = \{a, b, c, d, e\}$

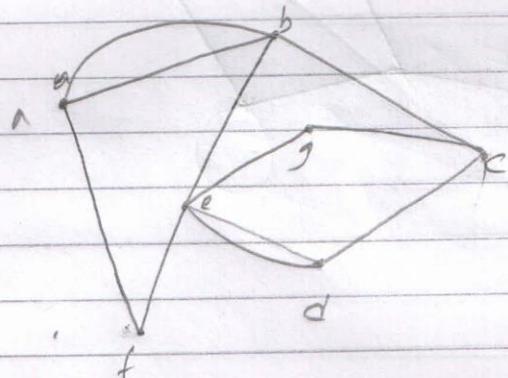
Types of Graph

1. Simple Graph.

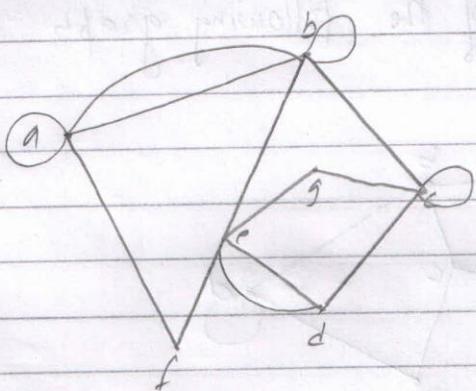


G_1

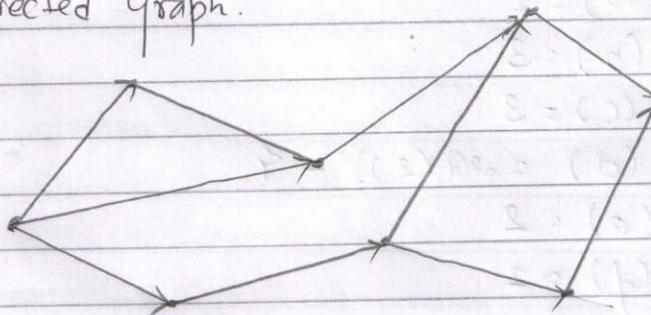
2. Multigraph



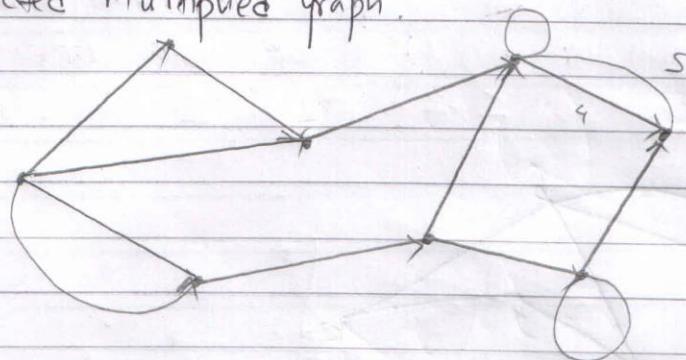
3. Pseudograph



3. Directed Graph



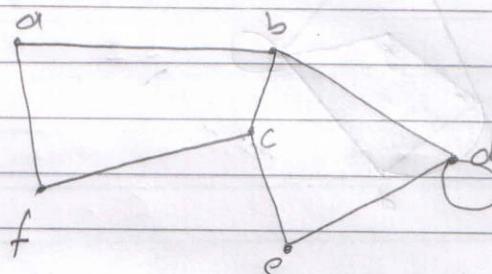
4. Directed Multiplied Graph.



Graph Terminology.

Degree of vertices:-

Find the degree of the following graph.



$$\deg(a) = 2$$

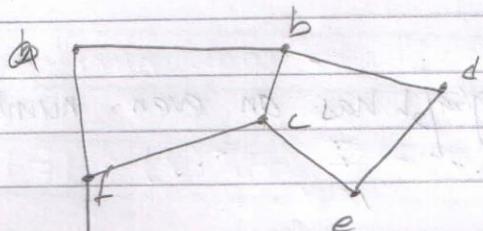
$$\deg(b) = 3$$

$$\deg(c) = 3$$

$$\deg(d) = 2 + (2) = 4$$

$$\deg(e) = 2$$

$$\deg(f) = 2$$

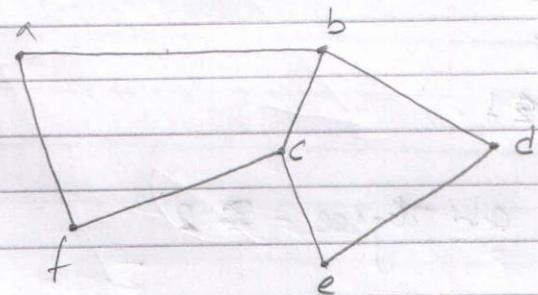


g) pendent

Theorem: The Handshaking Theorem

Let $G = (V, E)$ be an undirected graph with e edges.
Then,

$$2e = \sum_{v \in V} \deg(v)$$



$$\deg(a) = 2$$

$$\deg(b) = 3$$

$$\deg(c) = 3$$

$$\deg(d) = 2$$

$$\deg(e) = 2$$

$$\deg(f) = 2$$

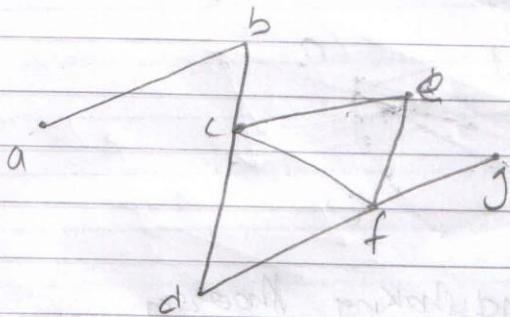
$$\text{Total deg} = 14$$

$$e = ?$$

$$\therefore \deg : \propto e$$

Theorem :

An undirected graph has an even number of vertices of odd degree.



$$\deg(a) = 2$$

$$\deg(b) = 3$$

$$\deg(c) = 3$$

$$\deg(d) = 2$$

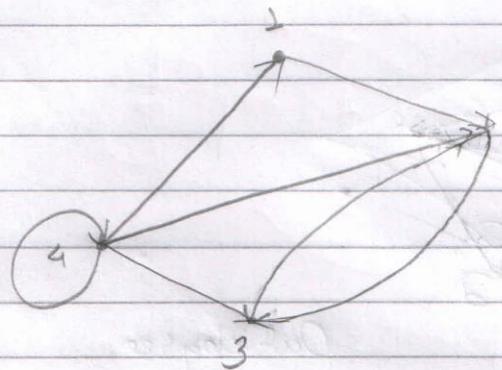
$$\deg(e) = 3$$

$$\deg(f) = 3$$

$$\deg(g) = 2$$

Vertices with odd degree = 2

Find the in-degree and out-degree in the following graph.



In-degree

$$\deg^-(a)$$

Out-degree

$$\deg^+(a)$$

} Representation

In-degree

$$\begin{aligned}\deg^-(1) &= 1 \\ \deg^-(2) &= 3 \\ \deg^-(3) &= 2 \\ \deg^-(4) &= 1\end{aligned}$$

Out-degree

$$\begin{aligned}\deg^+(1) &= 1 \\ \deg^+(2) &= 1 \\ \deg^+(3) &= 1 \\ \deg^+(4) &= 3\end{aligned}$$

classmate

Date _____

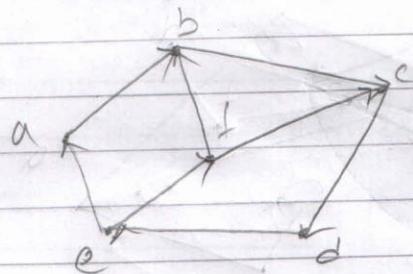
Page _____

Theorem:-

$G(V, E)$ be a graph with directed edges.

Then,

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$$



In degree

$$\deg^-(a) = 1$$

$$\deg^-(a) = 1$$

$$\deg^-(b) = 2$$

$$\deg^-(c) = 2$$

$$\deg^-(d) = 2$$

$$\deg^-(e) = 2$$

$$\deg^-(f) = 1$$

Out degree

$$\deg^+(a) = 1$$

$$\deg^+(b) = 2$$

$$\deg^+(c) = 1$$

$$\deg^+(d) = 2$$

$$\deg^+(e) = 1$$

$$\deg^+(f) = 2$$

$$\therefore \sum \deg^-(v) = 8$$

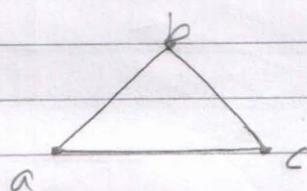
$$\sum \deg^+(v)$$

Complete graph

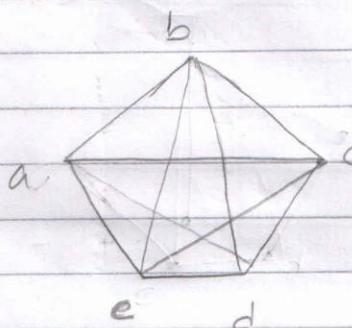
K_n

K_1

K_2



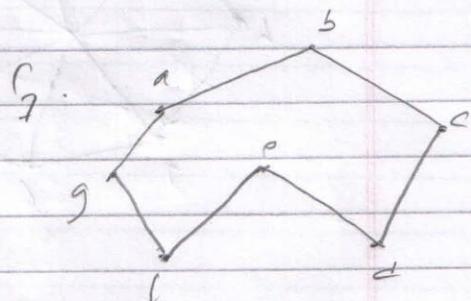
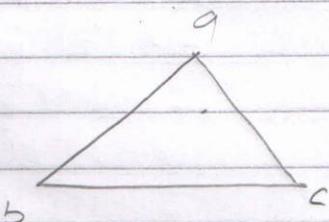
K_3



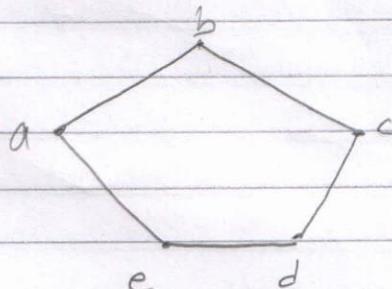
Cycles:-

C_n

C_3

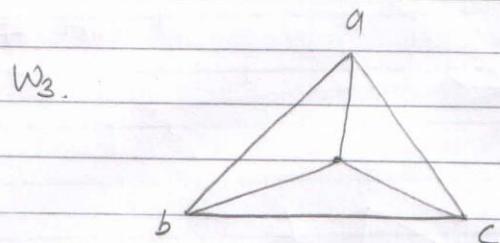


C_5

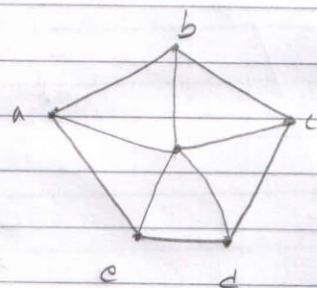


wheels :-

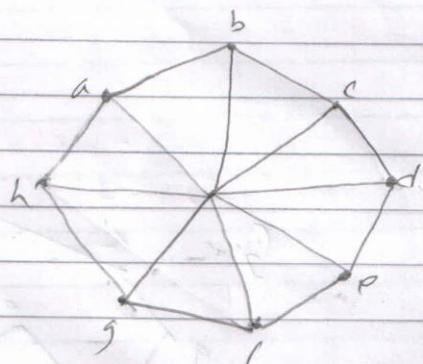
W_n



W₅



W₇

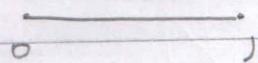


n - Cubes :-

$$Q_n \rightarrow 2^n$$

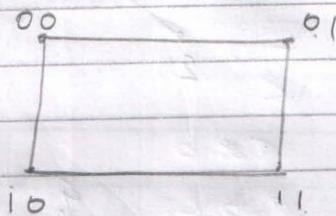
Q₁

$$\text{no. of vertices} = 2^1 = 2$$



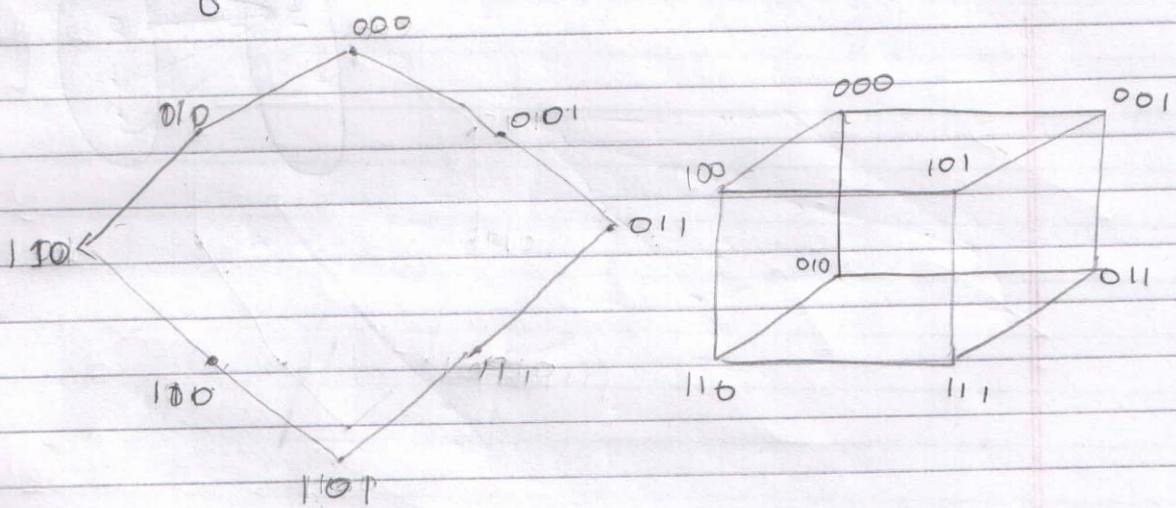
Q₂

$$\text{no. of vertices} = 2^2 = 4$$



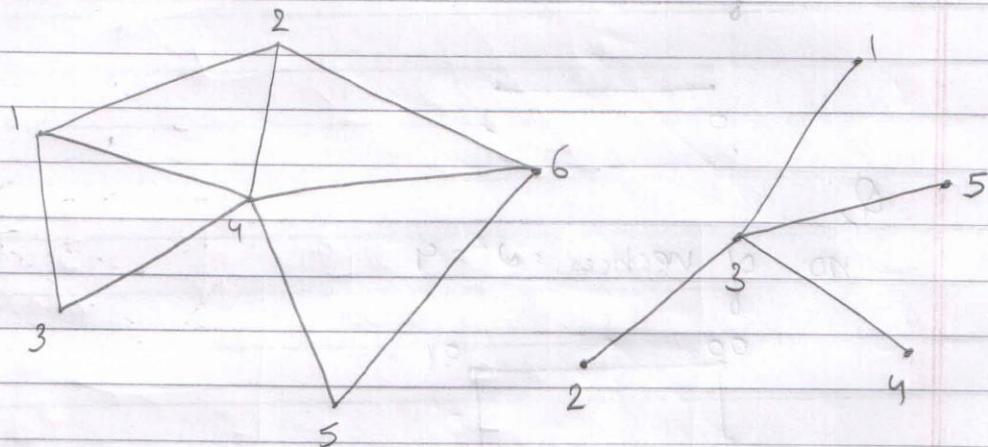
Q₃

$$\text{no. of vertices} = 2^3 = 8$$



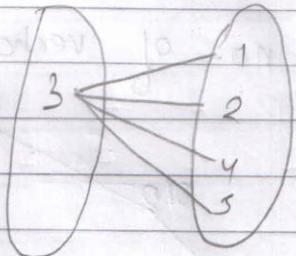
Bipartite Graph.

Are the graphs below bipartite?



9,

g.

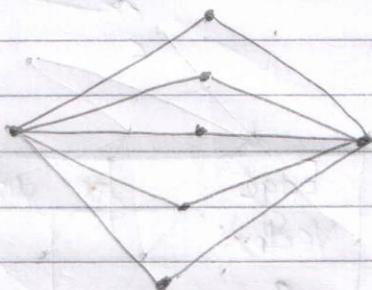
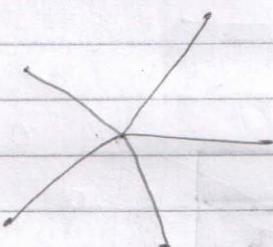


Complete Bipartite Graph.

$K_{m,n}$.

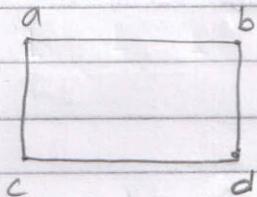
$K_{1,5}$.

$K_{2,5}$.



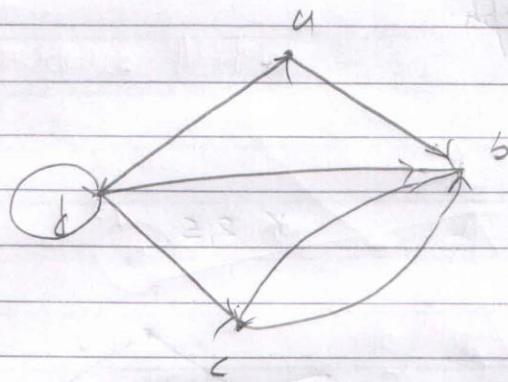
Graph Representation :-

1. Adjacency List :-



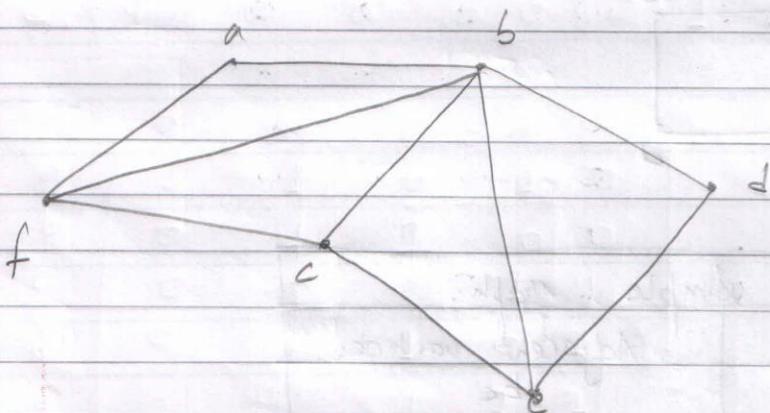
Edge for simple graph.

Vertex	Adjacent vertices
a	bc
b	- ad
c	ad
d	bc

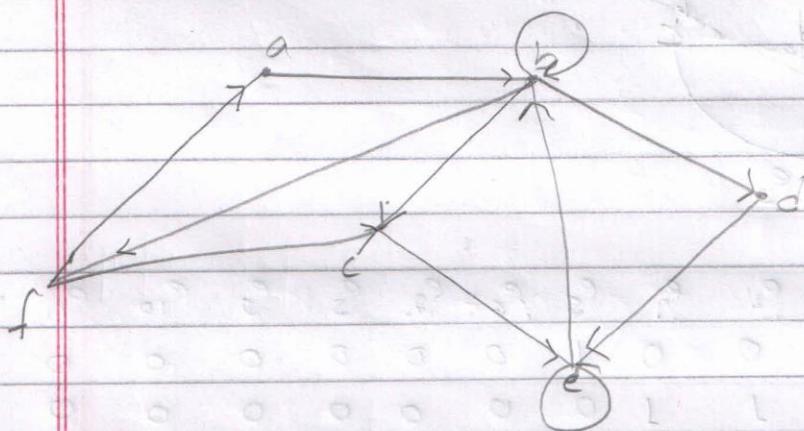


Edge	for simple graph
Vertex	Adjacent graph
a	b
b	c
c	b , d , b
d	a, b, c

2. Adjacency Matrix :-

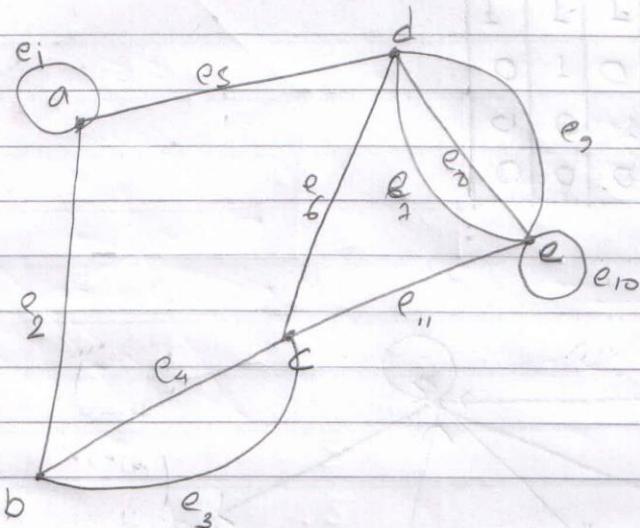


	a	b	c	d	e	f
a	0	1	0	0	0	1
b	1	0	1	1	1	1
c	0	1	0	1	1	1
d	0	1	0	0	1	0
e	0	1	1	1	0	0
f	1	1	1	0	0	0



	a	b	c	d	e	f
a	0	1	0	0	0	0
b	0	1	1	1	0	1
c	0	0	0	0	1	0
d	0	0	0	0	1	0
e	0	1	0	0	1	0
f	1	0	1	0	0	0

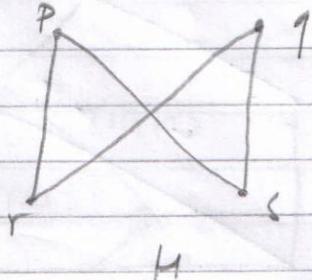
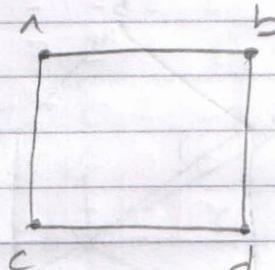
3. Incidence Matrix.



	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}
a	1	1	0	0	1	0	0	0	0	0	0
b	0	1	1	1	0	0	0	0	0	0	0
c	0	0	1	1	0	1	0	0	0	0	1
d	0	0	0	0	1	1	1	1	1	0	0
e	0	0	0	0	0	0	1	1	1	1	1

	a	b	c	d	e	
a	1	0	0	1	0	
b	0	1	1	0	0	
c	0	0	1	0	1	
d	0	0	1	0	3	
e	0	0	1	3	0	

Isomorphic Graph :-



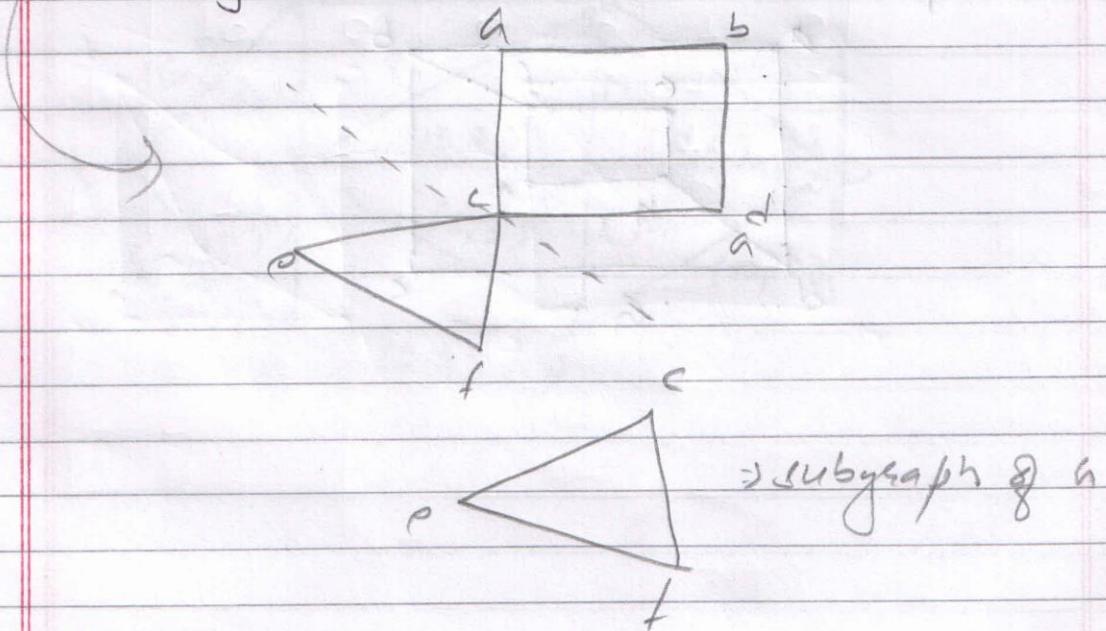
$$G \quad = \quad H$$

Diagram showing a mapping between the vertices of G and H:

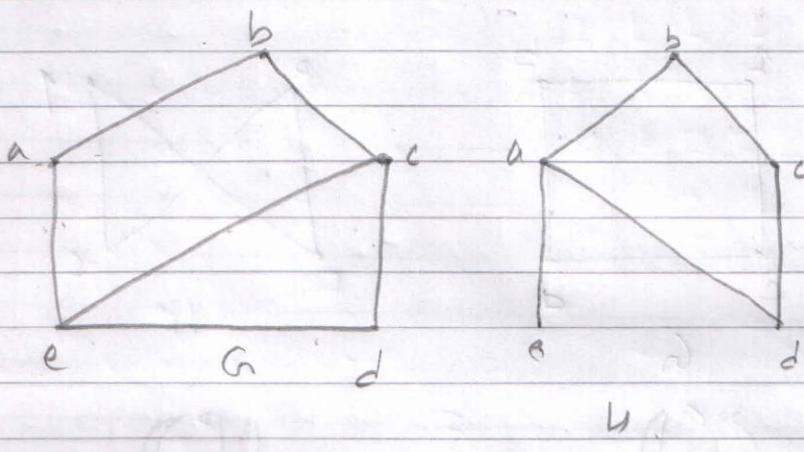
- a → p
- b → q
- c → r
- d → s
- g → t

Invariant Rule

1. no. of vertices must be equal.
2. no. of edges must be equal.
3. no. of degree of vertices must be equal.
4. Subgraph consisting of same degree must be same.
5. Adjacent matrix must be same.



Show that the graphs displayed below are not isomorphic.

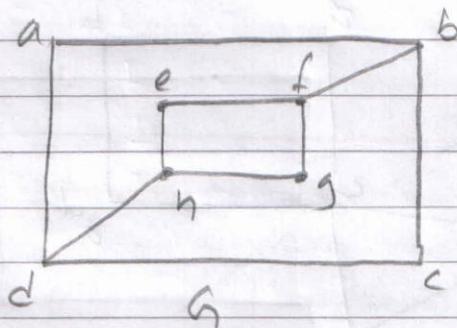


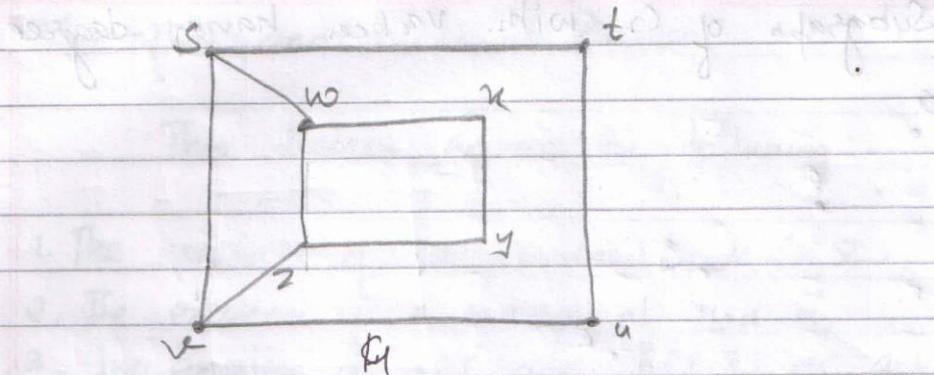
no. of vertices of $G = 6$
no. of vertices of $H = 6$

no. of edges of $G \neq$ no. of edges of H
i.e., $6 \neq 5$

Hence graphs are not isomorphic.

Determine whether the graphs shown below are isomorphic or not.





1. No. of vertices of G = No. of vertices of H = 8.

2. No. of edges of G = No. of edges of H = 10.

3. Degree of vertices of G

$$\deg(a) = 2$$

$$\deg(b) = 3$$

$$\deg(c) = 2$$

$$\deg(d) = 3$$

$$\deg(e) = 2$$

$$\deg(f) = 3$$

$$\deg(g) = 2$$

$$\deg(h) = 3$$

Degree of vertices of H .

$$\deg(s) = 3$$

$$\deg(t) = 2$$

$$\deg(u) = 2$$

$$\deg(v) = 3$$

$$\deg(w) = 3$$

$$\deg(x) = 2$$

$$\deg(y) = 2$$

$$\deg(z) = 3$$

degree of vertices of G = degree of vertices of H
as there are equal no. of vertices having degree
2 & 3.

Subgraph of G with vertices having degree 2.

a.

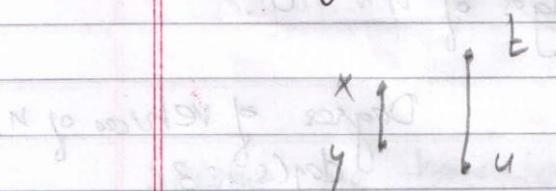
e

g

c

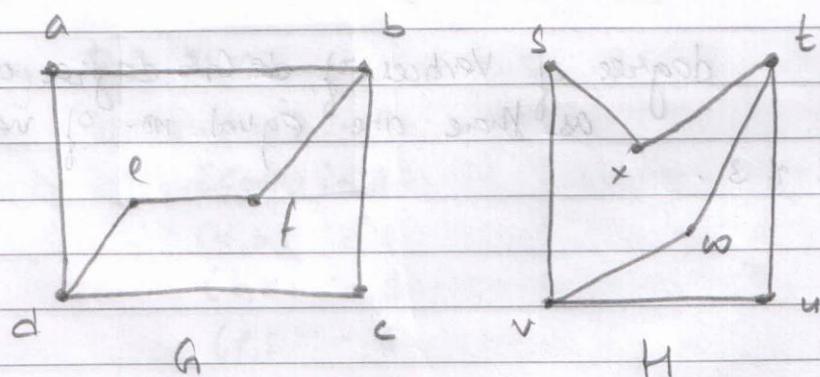
~~8. N is path to odd \Rightarrow 3 (6 vertices to odd)~~

Subgraph of H with vertices having degree 2 is,



Hence subgraph are not same they are not isomorphic.

Determine whether the graph shown below are isomorphic or not.



1 no. of vertices of G = no. of vertices of H . = 7

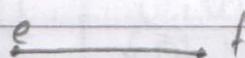
2 no. of edges of G = no. of edges of H = 6

3	degree of vertices of G .	degree of vertices of H
	$\deg(a) = 2$	$\deg(s) = 2$
	$\deg(b) = 3$	$\deg(t) = 3$
	$\deg(c) = 2$	$\deg(u) = 2$
	$\deg(d) = 3$	$\deg(v) = 3$
	$\deg(e) = 2$	$\deg(w) = 2$
	$\deg(f) = 2$	$\deg(x) = 2$

degree of vertices of G = degree of vertices of H
as there are equal no. of vertices having
degree 2 & 3.

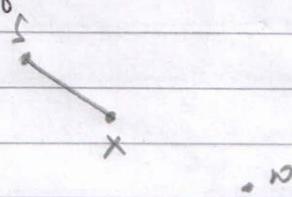
4. subgraph of G with vertices of degree 2.

a.



c

subgraph of H with vertices of degree 2.



• " and 2 vertices

Since these are one straight line, common they are equal.

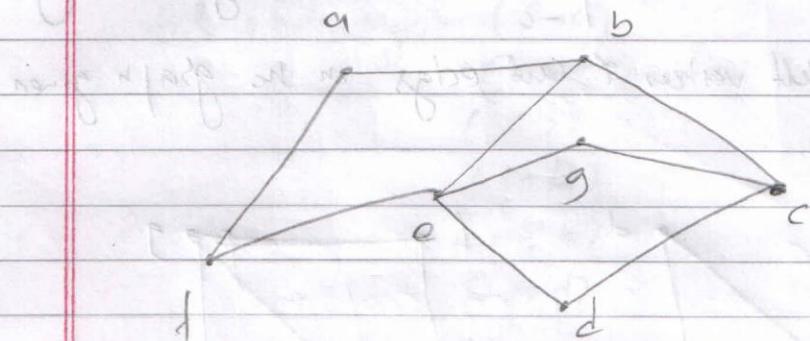
5. Adjacency matrix of G .

	a	b	c	d	e	f
a	0	1	0	1	0	0
b	1	0	1	0	0	1
c	0	1	0	1	0	0
d	0	1	0	1	0	0
e	0	0	0	1	0	1
f	0	1	0	0	1	0

Adjacency matrix of H

Graph Connectivity

(optional) upto two bad (shortest path algorithm) notes



From the graph given above show the paths and circuit of length 4.

\Rightarrow Path:

a-b-c-g-e

a-f-e-g-c

a-b-e-g-c

b-c-d-e-f

a-f-e-d-c

b-c-g-e-f

b-e-d-c-g

f-e-g-c-b

and so on.

Circuit:

a-b-e-f-a

b-e-g-c-b

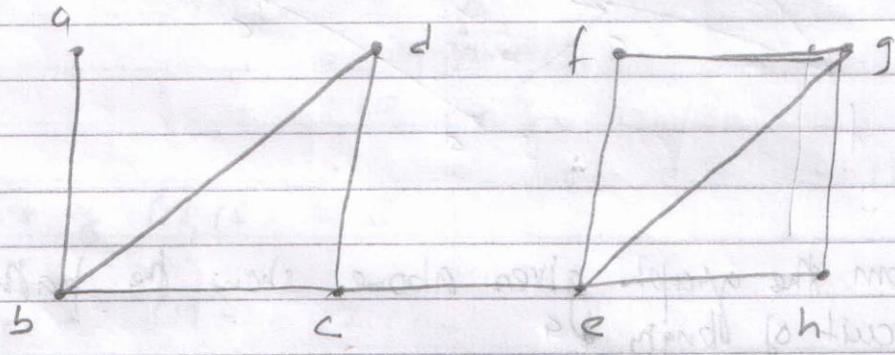
e-g-c-d-e

e-b-c-d-e

and so on. and so on.

Cut vertices (articulation points) and cut edge (bridge)

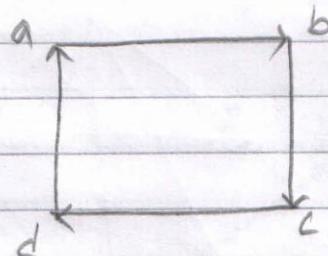
Find out the cut vertices & cut edges in the graph given below:



cut vertices = {a, e, g, b}

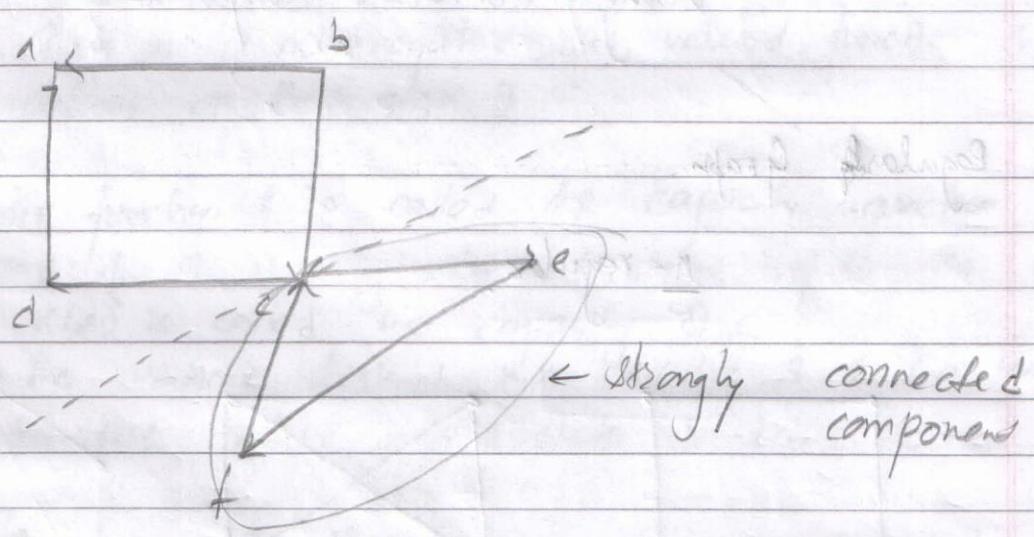
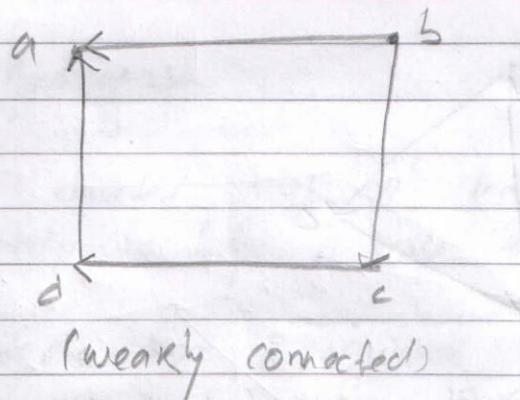
cut edges = {(c,e), (a,o)}

Connectedness in directed graphs.

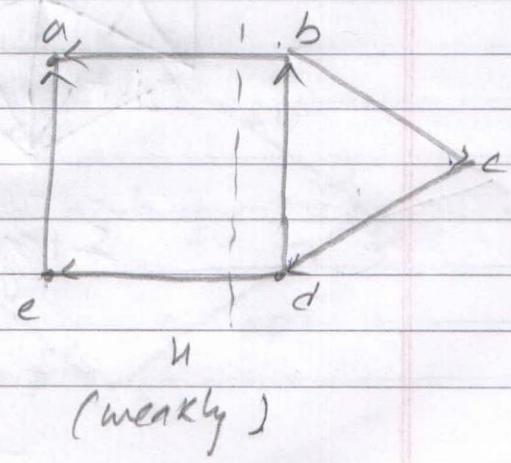
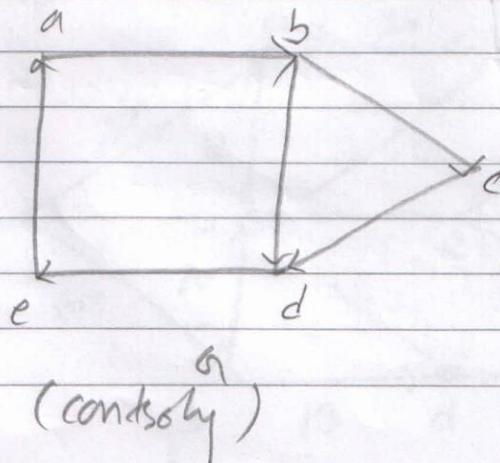


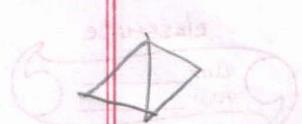
~~every~~ vertex having path
to go to every vertex

strongly connected.



Are the directed graphs shown below strongly connected?
Are they weakly connected? Determine the strongly connected components if any.

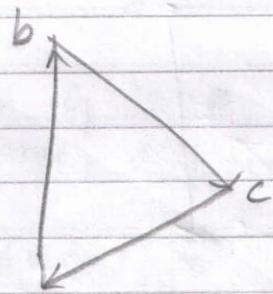




classmate

Date _____

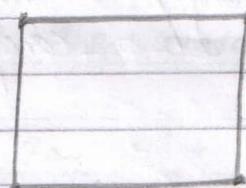
Page _____



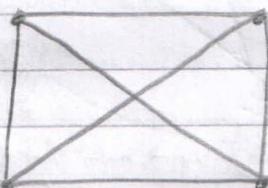
strongly connected component of
 n

Regular Graph.

n-regular.

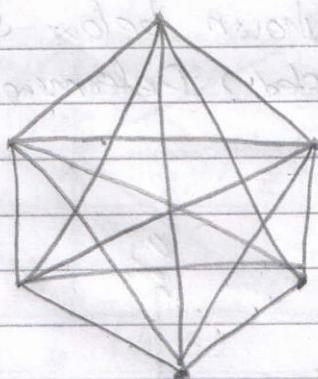


4-regular



3-regular

5-regular



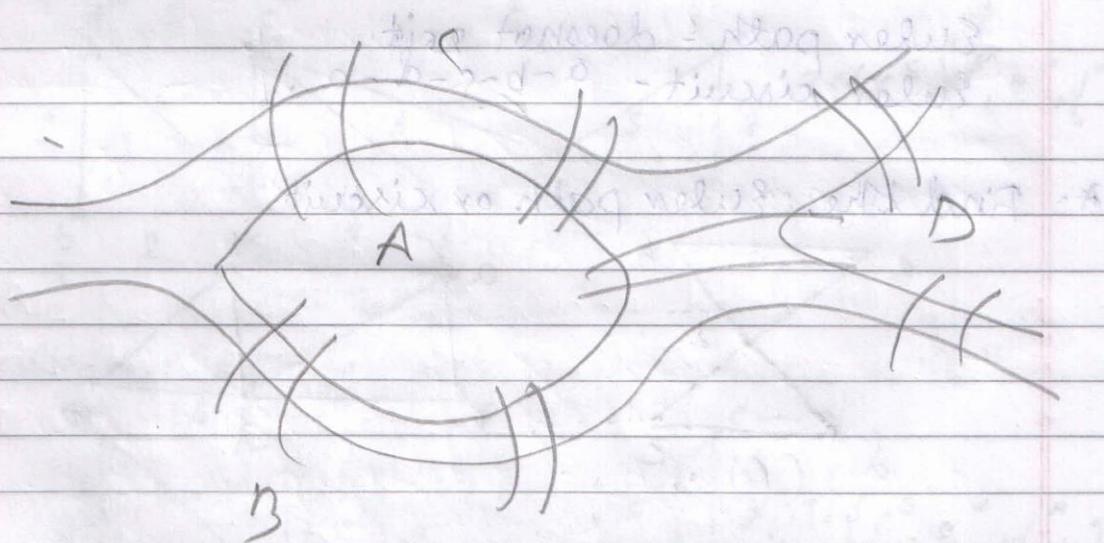
5-regular

Now many vertices does a regular graph of degree 4 with 10 edges have now.

minimum algorithm
principle

time and edge value

Euler paths & circuits.



b-a-d-c-b-c-a

fixation

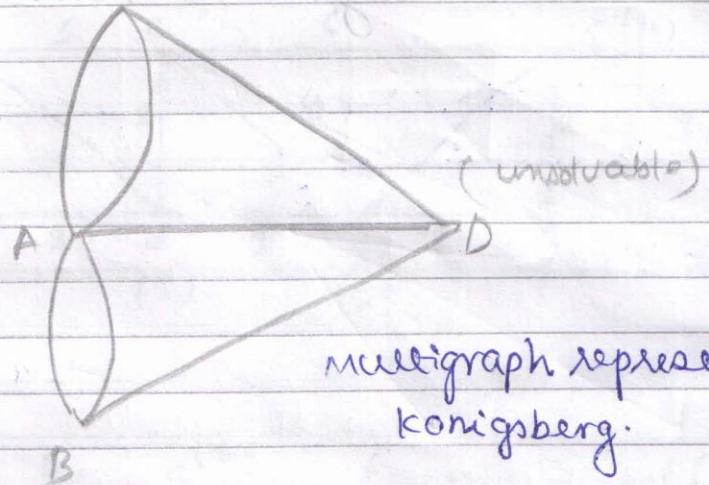
d-e-c-b-g-d-e

value

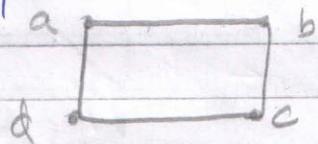
fixate

value

value



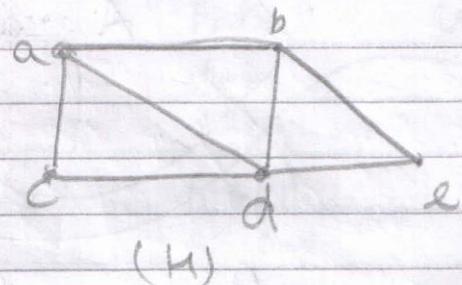
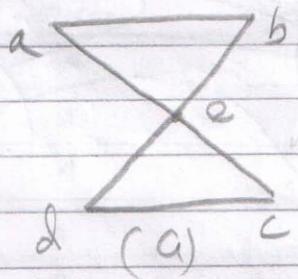
Euler path and circuit



Euler path - does not exist.

Euler circuit - a-b-c-d-a

* Find the Euler path or circuit



Euler path:

not exist

a-c-d-a-b-e-d

Euler circuit:

a-b-e-d-c-a

no exist.

Necessary and sufficient condition for Euler circuits and paths.

Theorem': A connected multigraph has an Euler circuit if and only if each of its vertices have even degree.

necessary condition :- A connected multigraph has an Euler circuit if degree of every vertex is even.

Theorem:

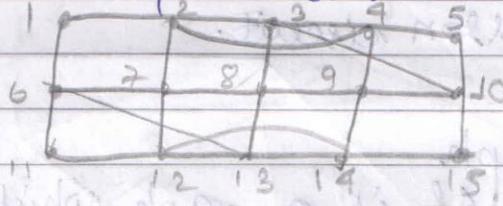
A connected multigraph has an Euler path but not circuit if and only if it has exactly two vertices of odd degree.

necessary

sufficient:

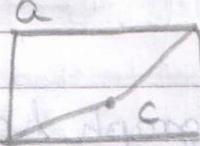


e.g. find Euler path or circuit.

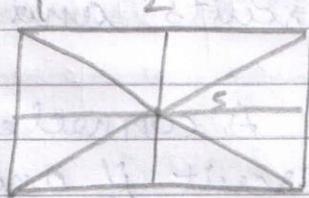


1 - 6 - 11 - 12 - 7 - 6 - 13 - 12 - 14 - 13 - 8 - 9 - 10 -
15 - 14 - 9 - 4 - 5 - 10 - 3 - 4 - 2 - 3 - 8 - 7 - 2 - 1

Hamilton paths.



a - b - c - d - e



1 - 2 - 3 - 5 - 9 - 6 - 5 - 8 - 7 - 5 - 4 - 1 - 3

Theorem: If G is a finite graph with $n \geq 3$ vertices such that the degree of every vertex in G is at least $\frac{n}{2}$, then G has a Hamilton circuit.

Dirac's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $\frac{n}{2}$, then G has a Hamilton circuit.

Ore's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices u and v in G , then G has a Hamilton circuit.

Planar Graph

It is a graph which can be drawn without crossing any edges.

Euler's formula:

Let G be a connected planar simple graph with ' e ' edges and ' v ' vertices. Let r be the number of regions in a planar representation of G , then

$$r = v - e + 2.$$

Q(1) Suppose that a connected planar graph has 30 edges. If a planar representation of these graphs divides the plane into 20 regions. How many vertices does this graph have?

Here, $e = 30$
 $r = 20$
 $v = ?$

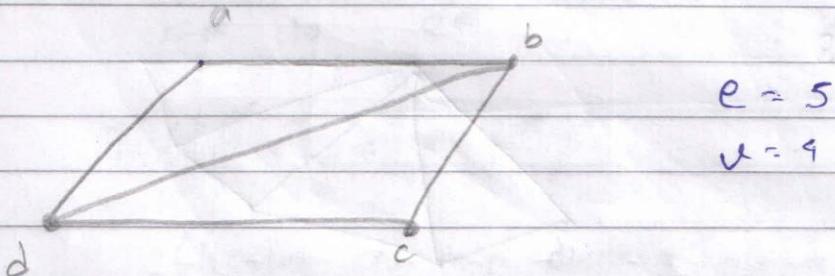
By Euler's theorem,

$$\begin{aligned} r &= e - v + 2 \\ \text{or } v &= e - r + 2 \\ &= 30 - 20 + 2 \\ &= 12 \end{aligned}$$

Hence, there are 12 vertices.

Corollary 1:-

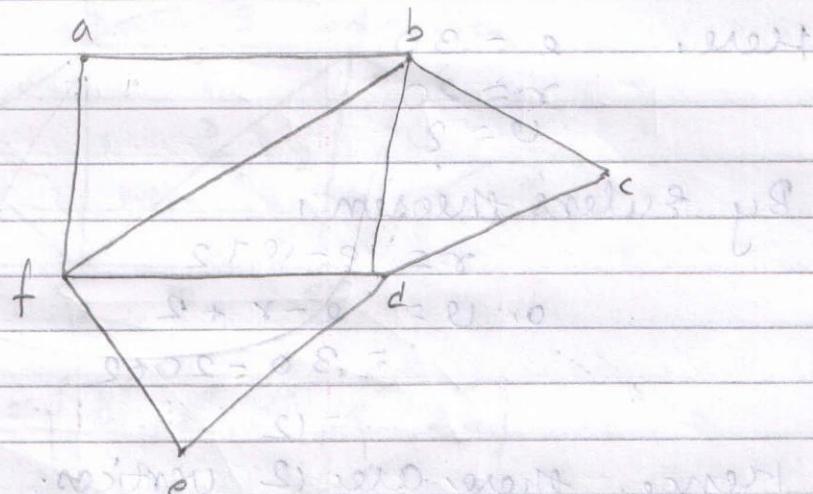
If G is a connected planar graph with e edges and v vertices where $v \geq 3$, then $e \leq 3v - 6$



$$\begin{aligned} \therefore e &\leq 3v - 6 \\ 5 &\leq 12 - 6 \\ 5 &\leq 6 \quad " \end{aligned}$$

Corollary

If G is a connected planar graph, then G has a vertex of degree not exceeding 5.



$$\deg(a) = 2$$

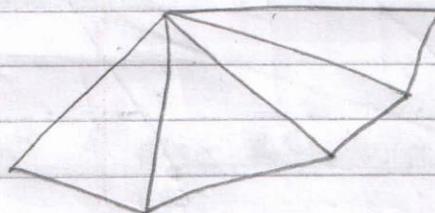
$$\deg(b) = 4$$

$$\deg(c) = 2$$

$$\deg(d) = 4$$

$$\deg(e) = 2$$

$$\deg(f) = 3$$



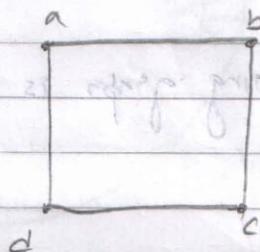
$$2 - 4 \times 2 = 0$$

$$2 - 5 \times 2 = 2$$

$$2 \geq 2$$

Corollary 3. If a connected planar simple graph has

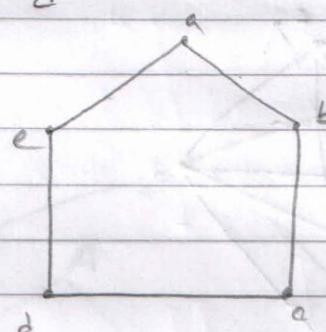
e edges and v vertices with $v \geq 3$ and no circuits of length 3. Then, $e \leq 2v - 4$.



$$4 \leq 2 \times 4 - 4$$

$$4 \leq 8 - 4$$

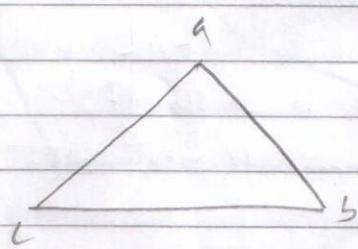
4 ≤ 4 True



$$5 \leq 2 \times 5 - 4$$

$$5 \leq 10 - 4$$

5 ≤ 6. True.

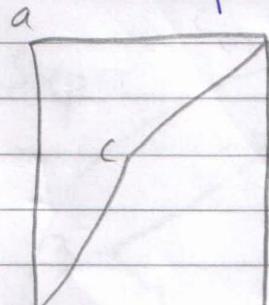


$$3 \leq 2 \times 3 - 4$$

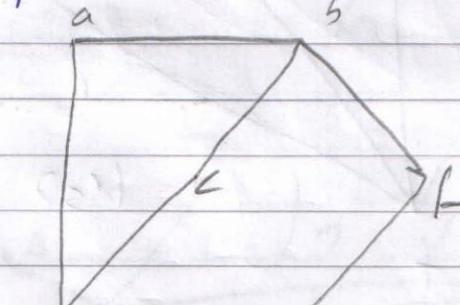
$$3 \leq 6 - 4$$

3 ≤ 2 which is false

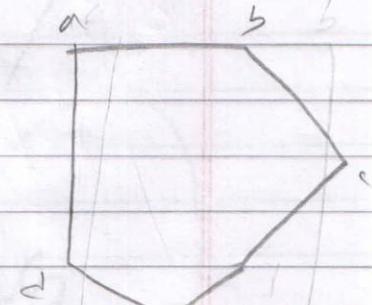
Homeomorphic graphs:-



G_1



G_2



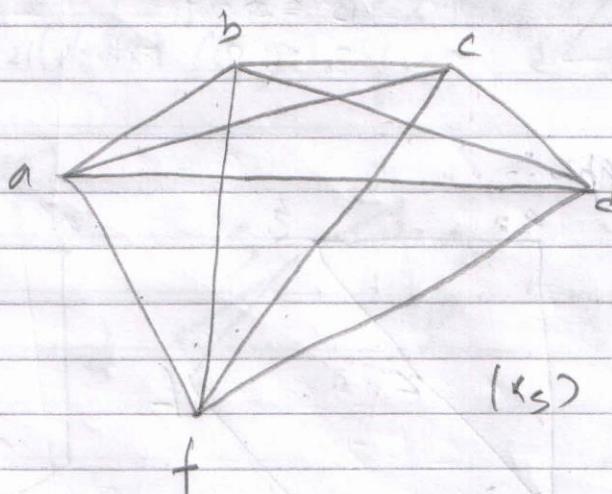
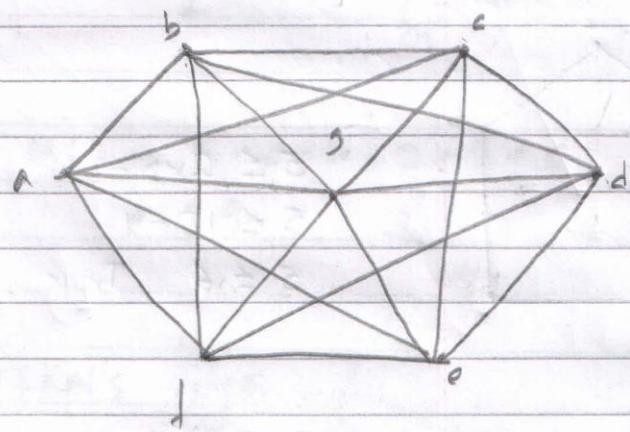
G_3

G_1 and G_2 are homeomorphic to G_3

Kuratowski's theorem:- (Planarity testing algorithm)

A graph is non-planar if and only if it contains subgraph homeomorphic to $K_{3,3}$ or K_5 .

Determine whether the following graph is planar or not?



(K_5)

The graph is non-planar.

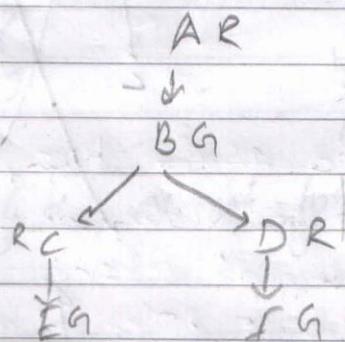
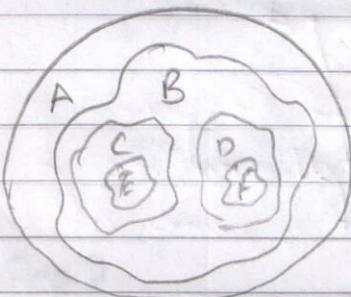
Graph coloring

Chromatic number

The four color Theorem

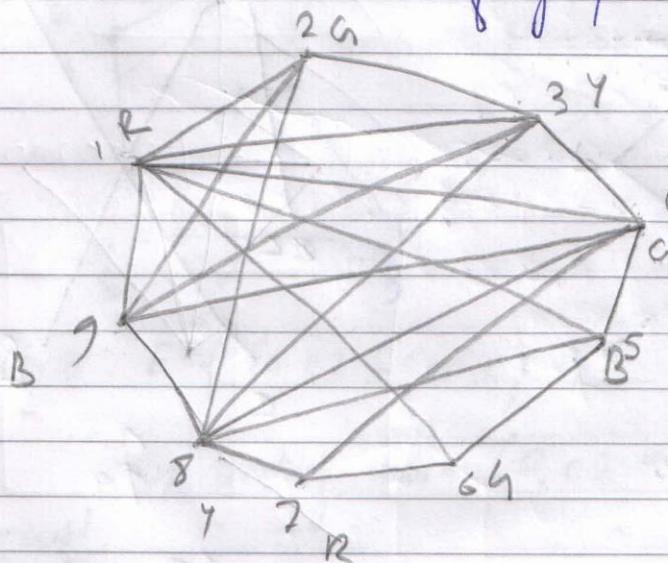
The chromatic number of a planar graph is no greater than four.

F.g:-



Chromatic number = 2

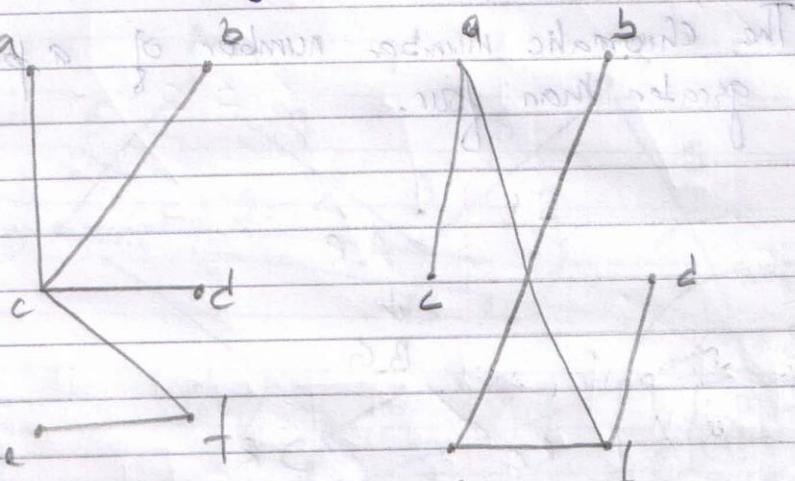
Find the chromatic no. of graph below:-



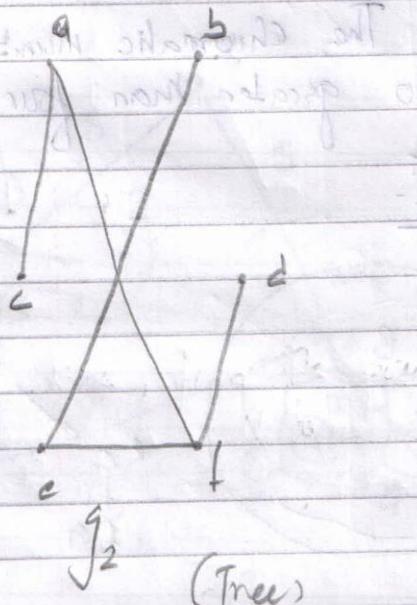
1, 2, 3, 4, 5, 6, 7, 8, 9 (L)
(L)
8, 9, 10, 11, 12, 13, 14, 15, 16, 17 (L)

Trees

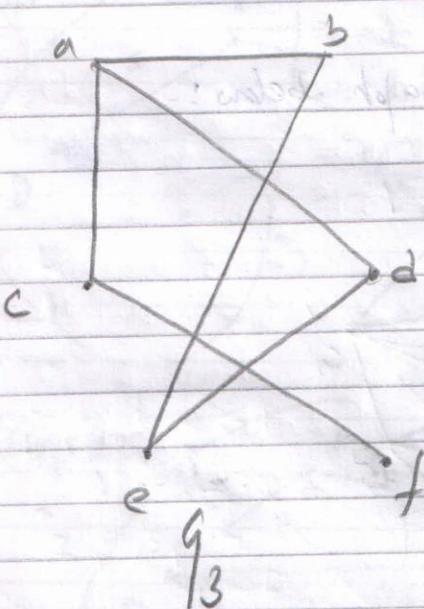
which of the graph show below are trees?



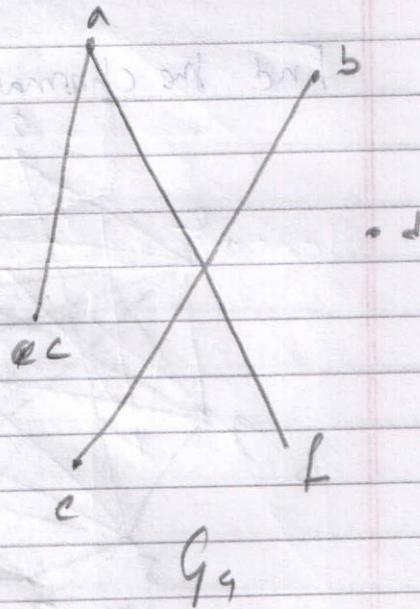
G_1 (Tree)



G_2 (Tree)



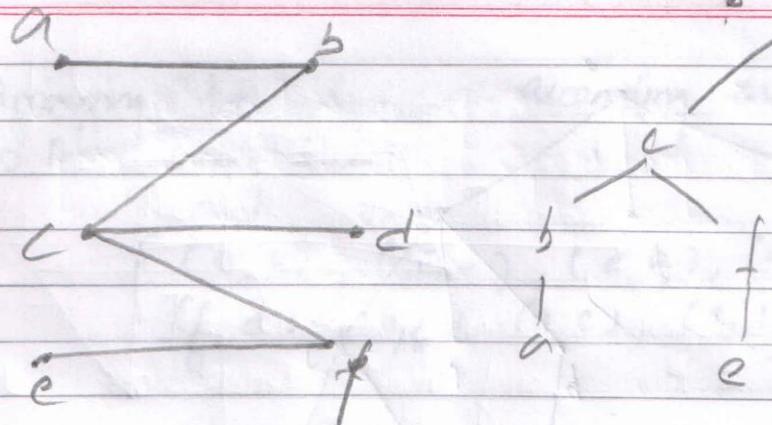
G_3 (Graph)



G_4 (Forest)

classmate
Date _____
Page _____

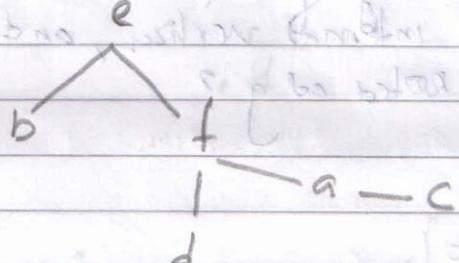
Unrooted



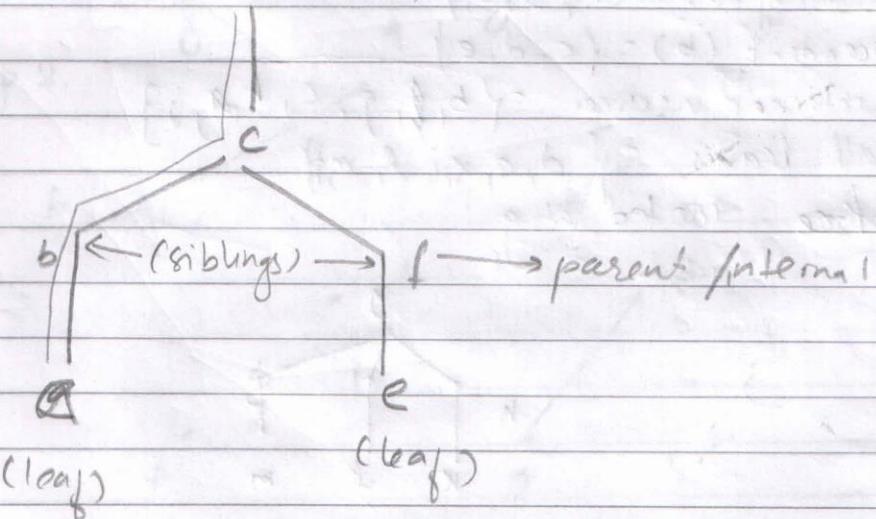
Rooted

~~Rooted graph.~~

Rooted graph G_2 of edges below will be

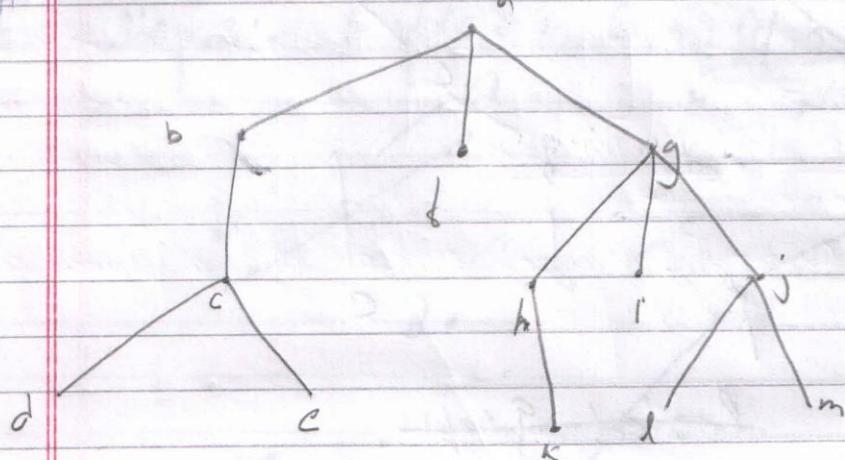


d (root)



ancestor of (a) = {b, c, d}
descendants of (d) = {c, b, a, f, e}

#



In this rooted graph tree, find the parent of c , children of g , the sibling of h , all ancestor of e , all descendants of b , all internal vertices, and all leaves. What is the subtree rooted at g ?

parent of $(c) \rightarrow \{b\}$

children of $(g) \rightarrow \{h, i, j\}$

sibling of $(h) \rightarrow \{i, j\}$

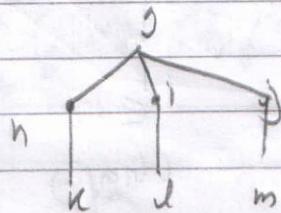
ancestor of $(e) = \{c, b, a\}$

descendant $(b) = \{c, d, e\}$

all internal vertices = $\{b, f, g, c, h, i, j\}$

all leaves = $\{d, e, k, l, m\}$

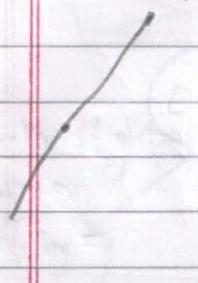
subtree rooted at g :



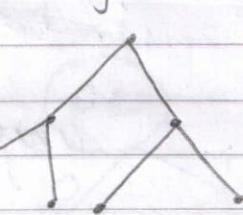
m-ary tree

If node has m children it is called m-ary tree
if each node has 2 children it is called binary tree

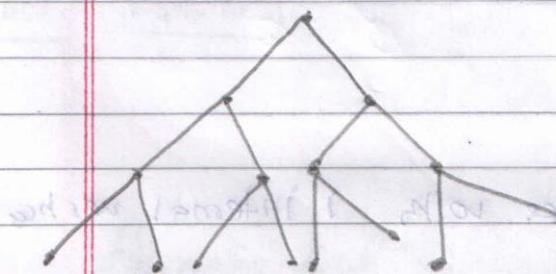
1-ary



2-ary / binary



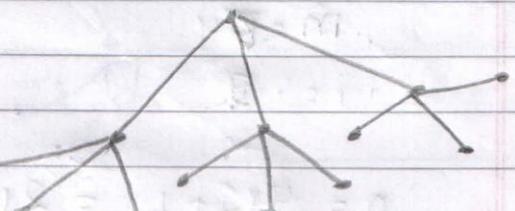
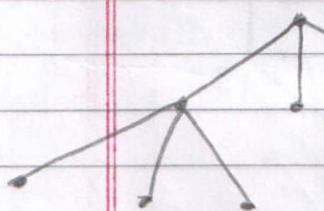
full m-ary tree (2-ary)



full binary tree

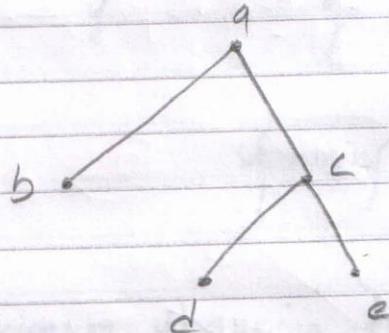
3-ary tree

full 3-ary tree



3.09.2020
Spiral Notebooks
Theorem:-

An undirected graph is a tree if and only if there is a unique simple path between any two graph vertices.

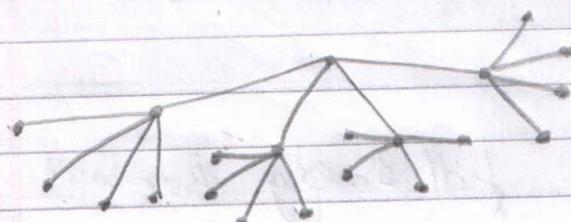


unique path (simple):
 $d - c - e$
 $- e - c - a - b$
 $= d - c - a - b$

Properties of Trees :-

Theorem:-

A full m-ary tree with i internal vertices contains $n = m^i + 1$ vertices.

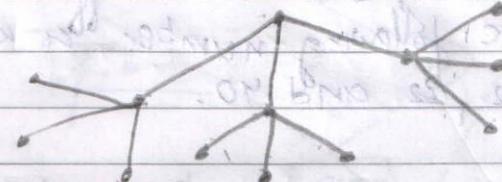


$$m = 4$$

$$i = 3$$

$$\therefore n = 4 \times 5 + 1 = 21$$

Theorem :- A tree with n vertices has $n-1$ edges.



$$\text{vertices } (n) = 13$$

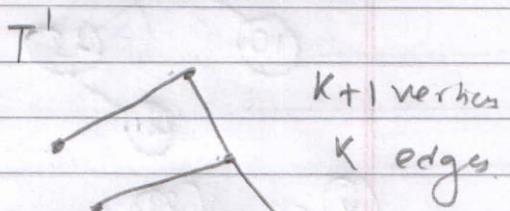
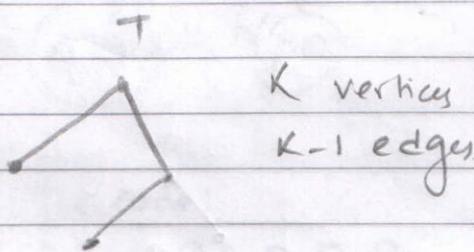
$$\text{edges} = 12 \Rightarrow (n-1),$$

$$\text{Base Step: } n = 1$$

$$\text{edge} = 0$$

Inductive Hypothesis : K vertices $= K-1$ edges

Inductive Step : $K+1$ vertices $= K$ vertex edges

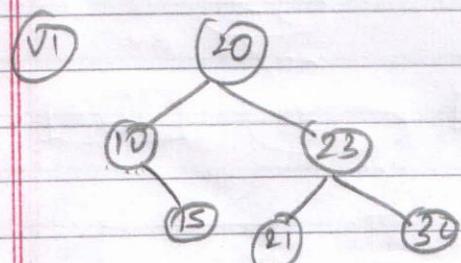
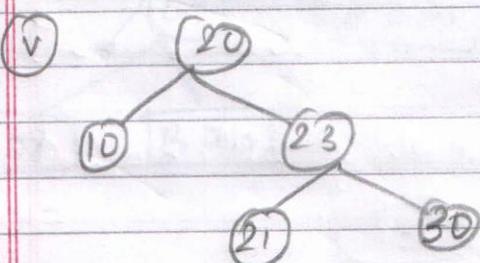
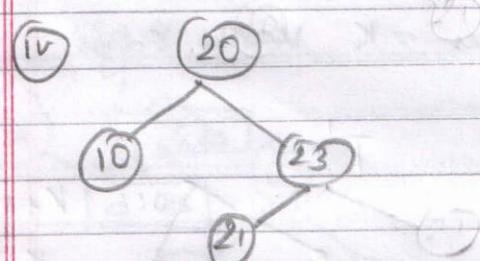
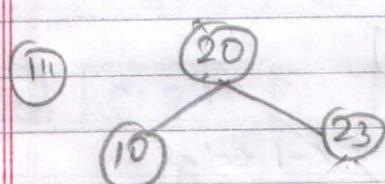
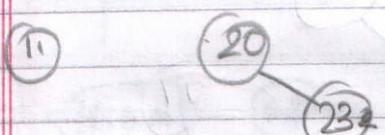
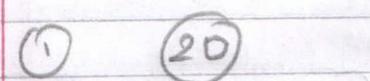


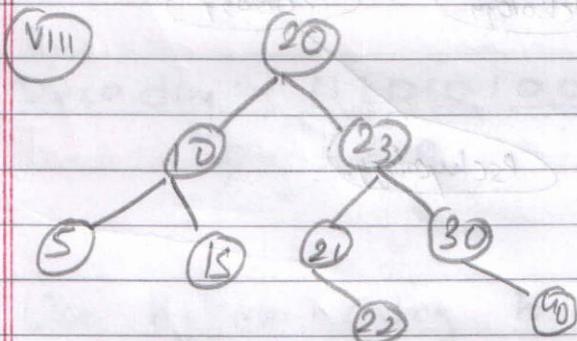
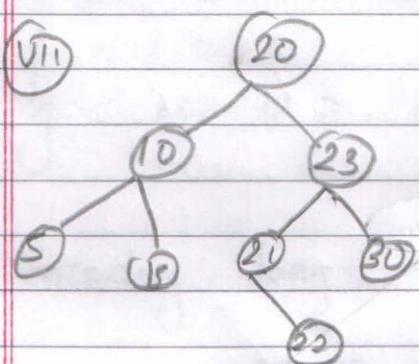
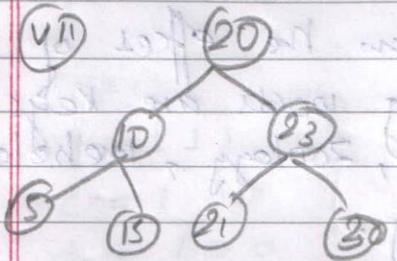
Applications of Trees:- Binary Search Tree (BST)

- # Starting with empty BST, show the effect of successively adding the following number of keys 20, 23, 10, 21, 30, 15, 2, 22 and 40.

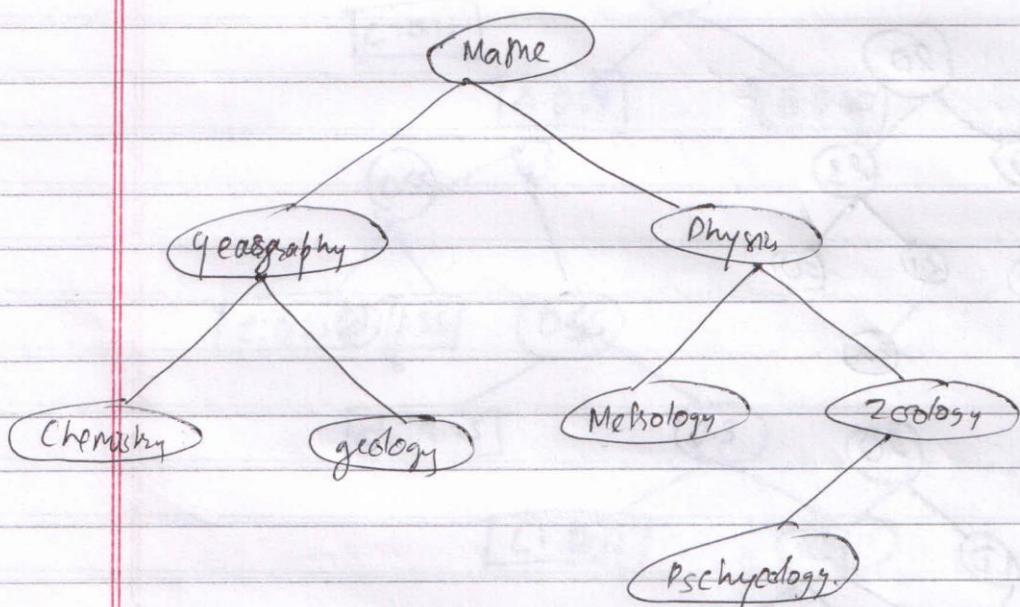
less than \Rightarrow left child

more than \Rightarrow right child



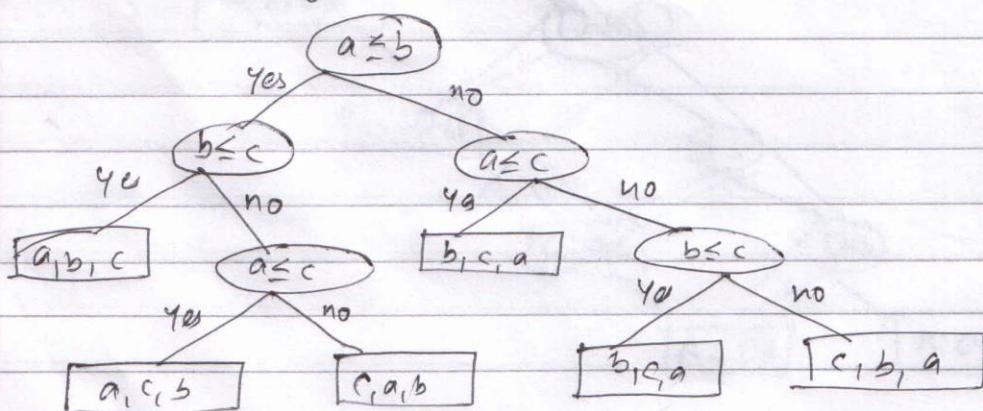


Starting with empty BST, show the effect of successively adding the following words as keys : mathematics, physics, geography, zoology, methology, geology, psychology & chemistry.



Decision Tree.

Decision tree of sorting 3 nos.



Prefix codes.

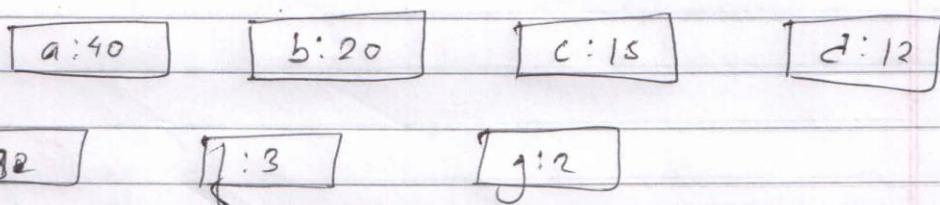
Huffman code (tree) (Greedy paradigm / algorithm)

Eg: $C = \{a, b, c, d, e, f, g\}$

$f(c) = \{40, 20, 15, 12, 8, 3, 2\}$

Construct the Huffman tree and list out prefix codes for characters.

Step 1:



Step 2:

Arranging the characters with the frequencies in ascending order.

* Prefix codes :-

$a = 0$

$b = 111$

$c = 110$

$d = 100$

$e = 1011$

$f = 10101$

$g = 10100$

Encoding : bag

= 1110 10100

Decoding :- 111010100
 $\Rightarrow b \ a \ g$

Use Huffman coding to encode the following symbols with the frequency frequencies listed.

A: 0.08 , B: 0.10 , C: 0.12 , D: 0.15 , E: 0.20 ,
F: 0.35.

What is the average number of bits used to encode a character.

Step 1 .

A: 0.08

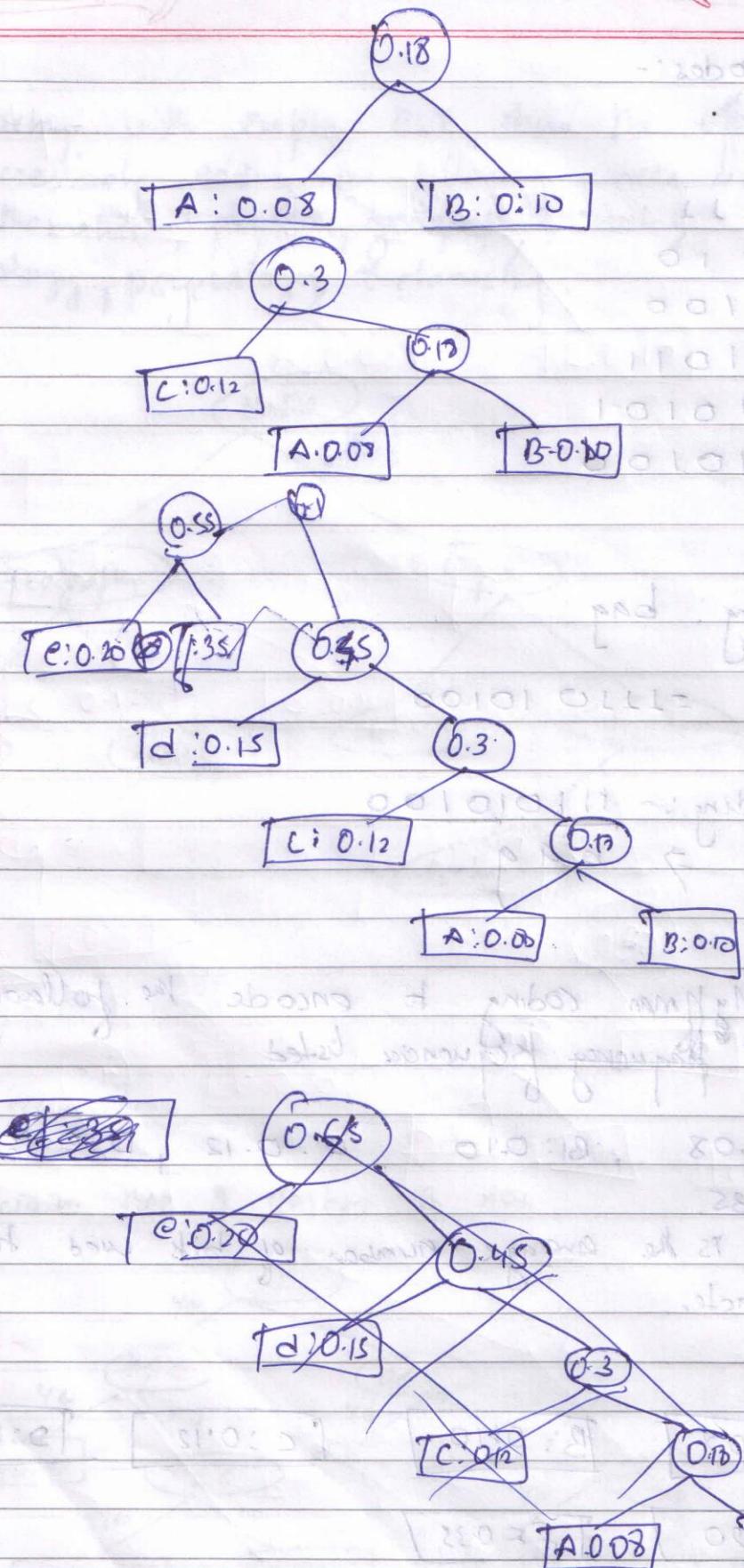
B: 0.10

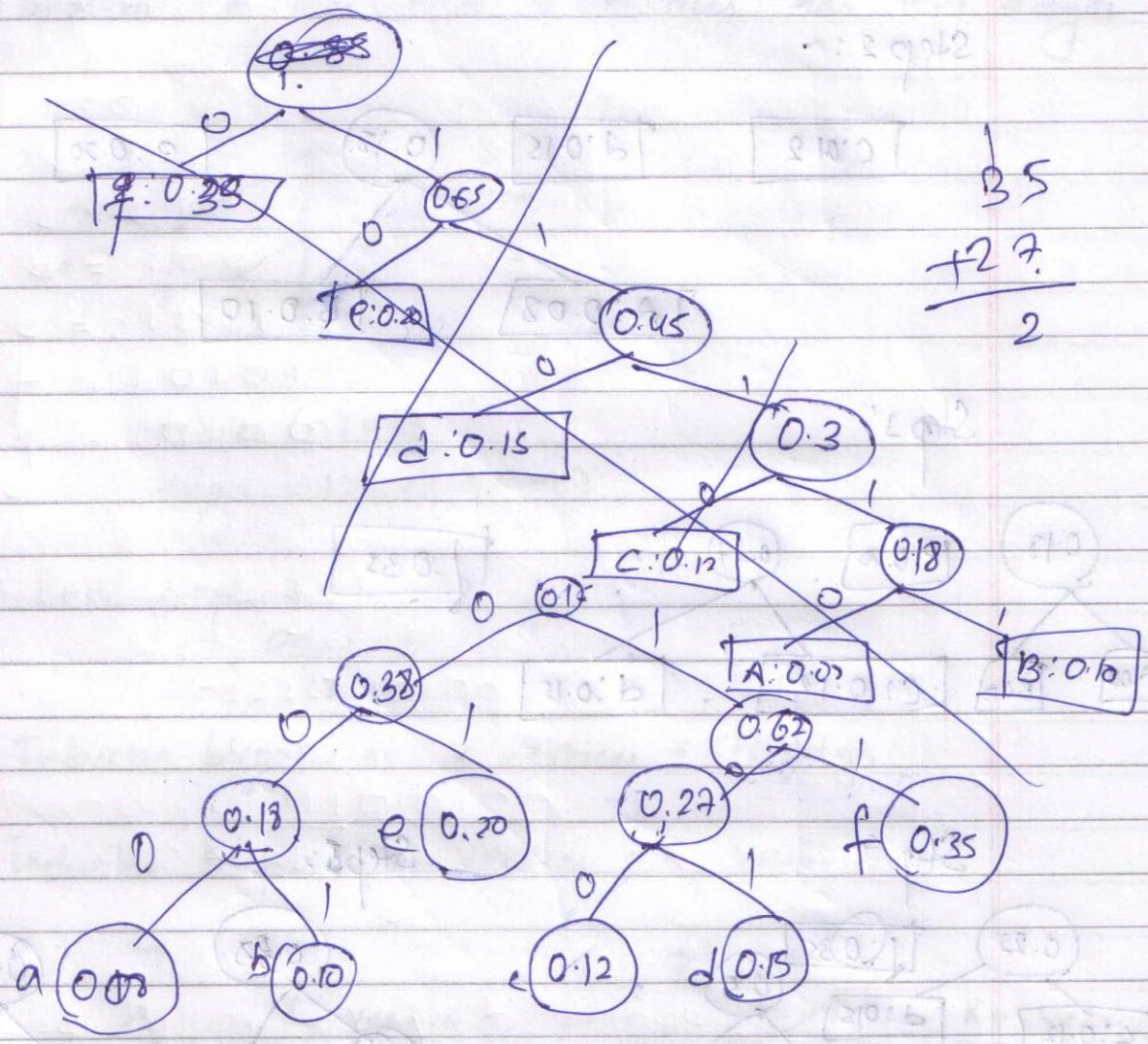
C: 0.12

D: 0.15

E: 0.20

F: 0.35





Prefix code!

$$a = 000$$

$$b = 001$$

$$C = 100$$

$$d = 101$$

$$e = \text{bool}$$

f 2 | 1

\therefore The no. of bits = $3+3+3+3+2+2$

$$\text{Average} = \frac{3+3+3+3+2+2}{6} = \frac{16}{6} = \frac{8}{3},$$



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Date _____

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Step 2:

C: 0.12

D: 0.15

0.18

e: 0.20

f: 0.35

A: 0.08

B: 0.10

Step 3:

0.18

e: 0.20

0.27

f: 0.33

A: 0.0

B: 0.0

C: 0.12

d: 0.15

Step 4:

0.22

f: 0.35

0.3

c: 0.12

d: 0.15

A: 0.08

B: 0.10

Step 5:

0.33

0.62

A: 0.06

B: 0.0

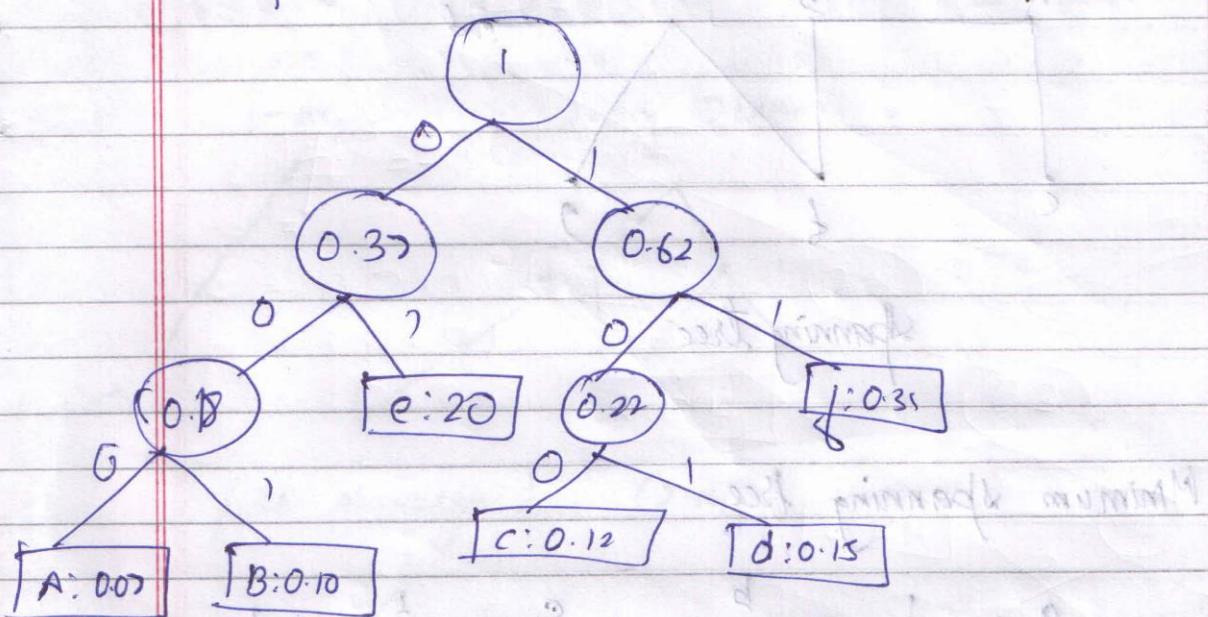
c: 0.12

d: 0.15

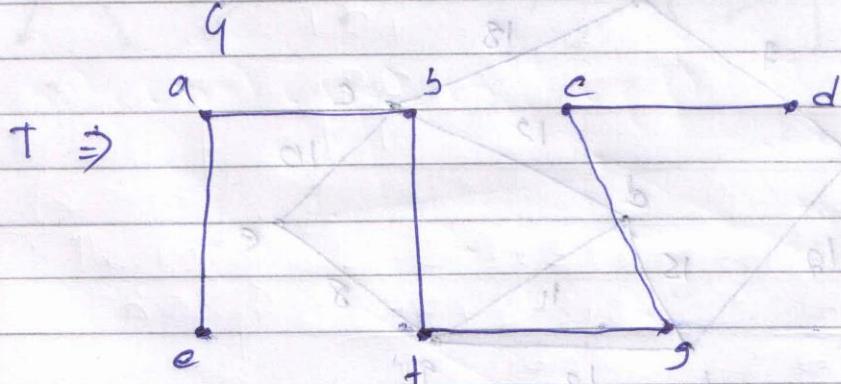
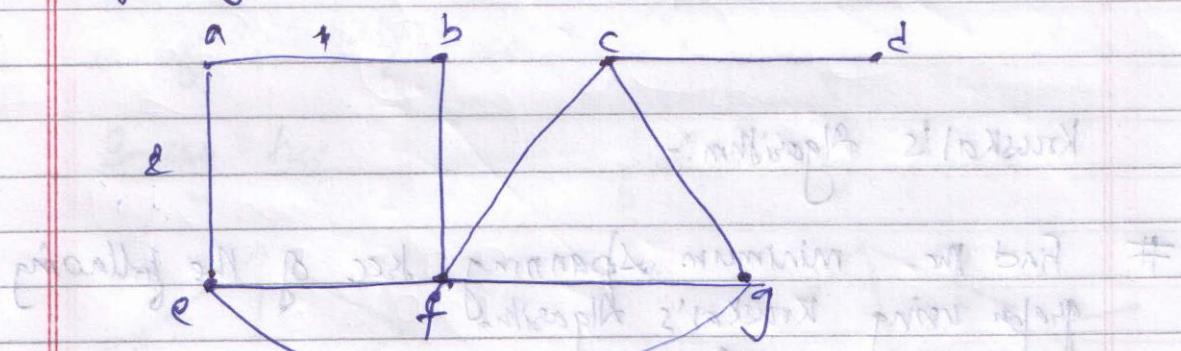
$$0.08 + 0.10 + 0.12 + 0.15 = 0.45$$

$$\frac{0.45}{5} = 0.09$$

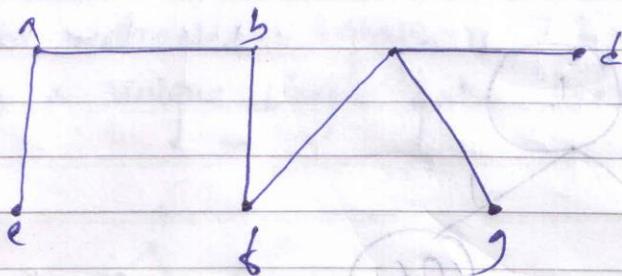
Step 6:



Spanning tree :-

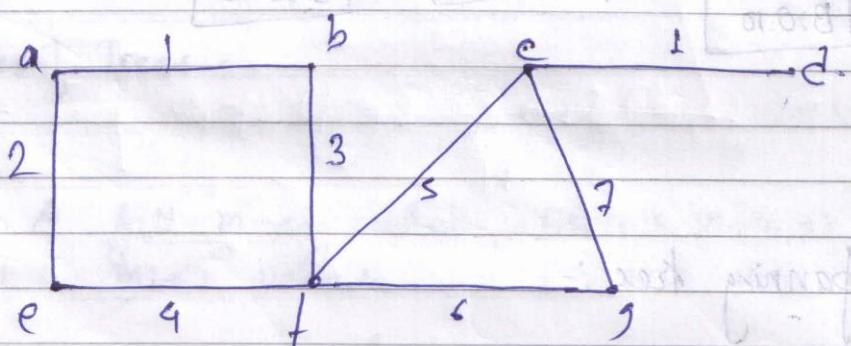


Spanning tree.



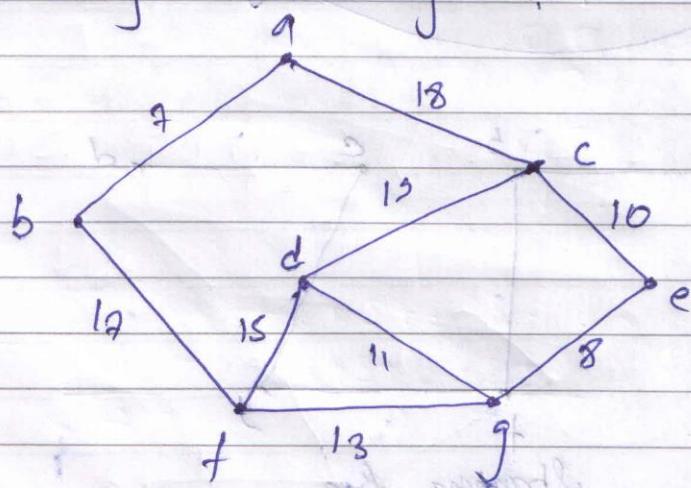
Spanning tree

Minimum Spanning Tree:



Kruskal's Algorithm:-

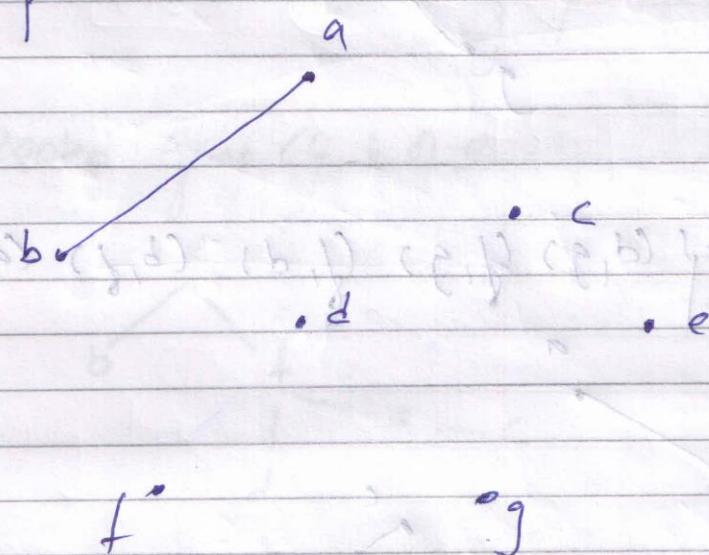
find me minimum Spanning Tree of the following graph using Kruskal's Algorithm



Arranging the edges in ascending order with respect to their weight.

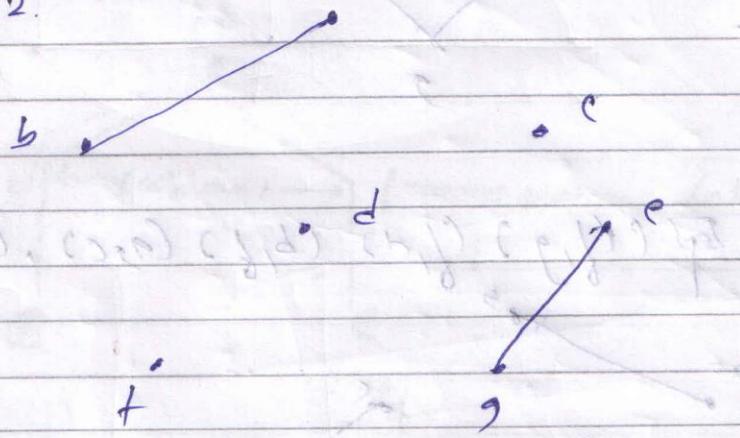
$$\{ (a,b), (e,g), (c,e), (d,g) \} \{ f,g \}, \\ \{ f,d \}, \{ b,f \}, \{ a,c \}, \{ c,d \}, \{$$

Step 1:



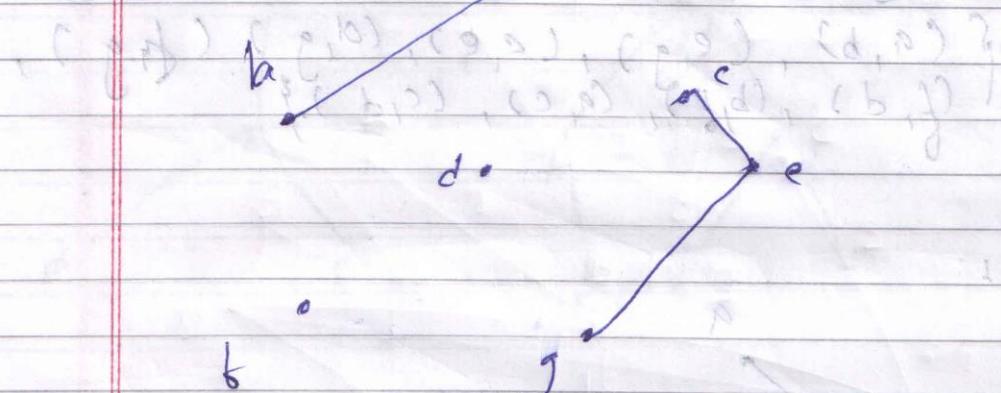
Selecting minimum edge from the set, E.

Step 2.

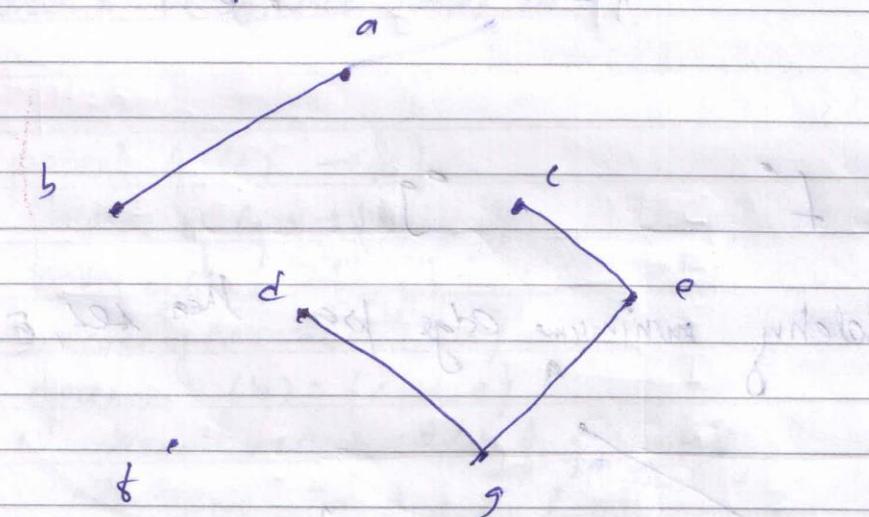


$$\{ e,g \}, \{ c,e \}, \{ d,g \}, \{ f,g \} \{ f,d \}, \{ b,f \}, \{ a,c \}, \{ g,d \}$$

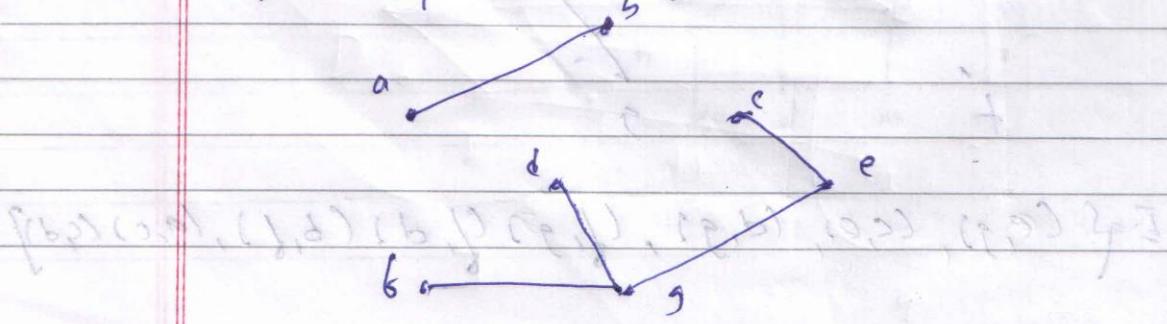
Step 3: $E = \{ (c, e), (d, g), (f, g), (f, d), (b, f), (a, c), (c, d) \}$



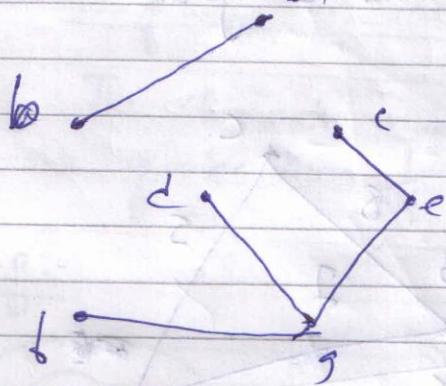
Step 4: $E = \{ (d, g), (f, g), (f, d), (b, f), (a, c), (c, d) \}$



Step 5: $E = \{ (f, g), (f, d), (b, f), (a, c), (c, d) \}$



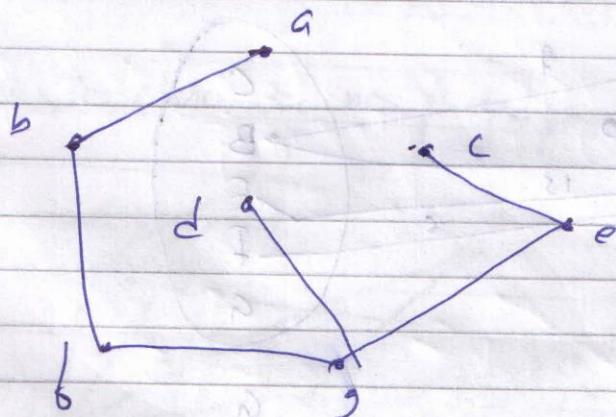
Step 6: $E = \{(f,d), (b,f), (g,c), (c,d)\}$



Discarding (f,d) as it forms a circuit.

Step 7:

$$E = \{(b,f), (g,c), (c,d)\}$$



Step 8: $E = \{(g,c), (c,d)\}$

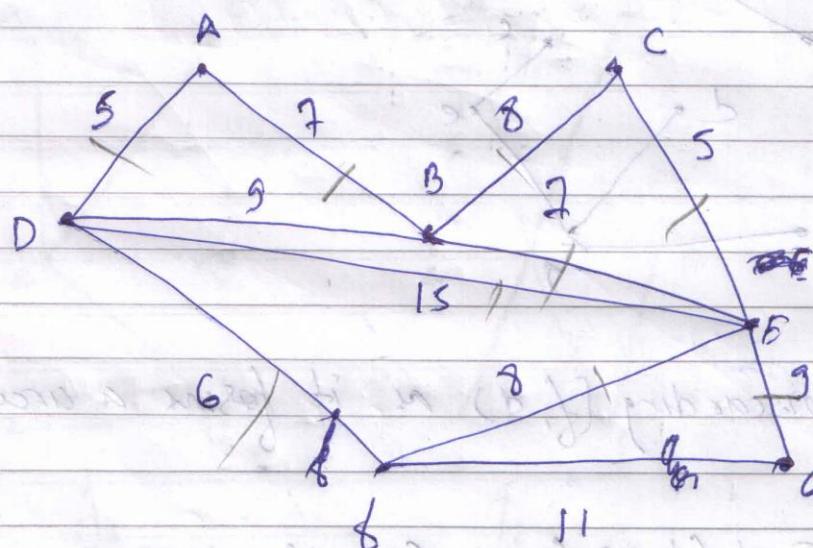
Discarding both (g,c) & (c,d) as it forms circuit.
 \therefore

$$\begin{aligned} MST &= 7 + 17 + 13 + 8 + 10 + 11 \\ &= 66 \end{aligned}$$

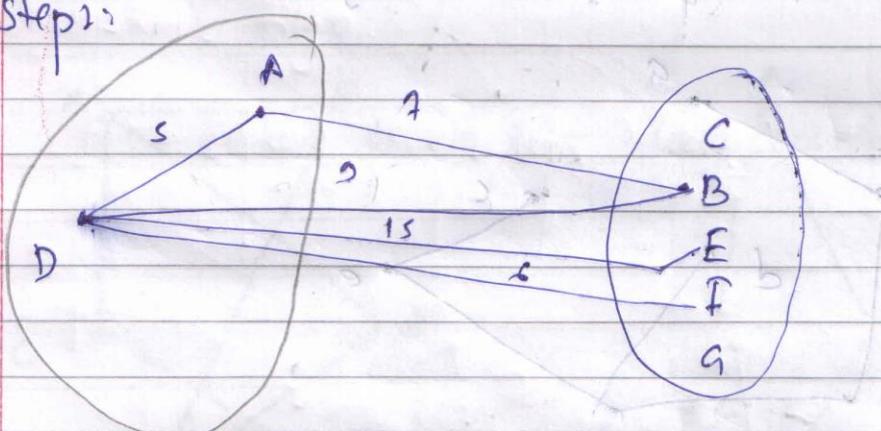
Graph Cutting Algorithm

Priam's Algorithm (Cutting graph)

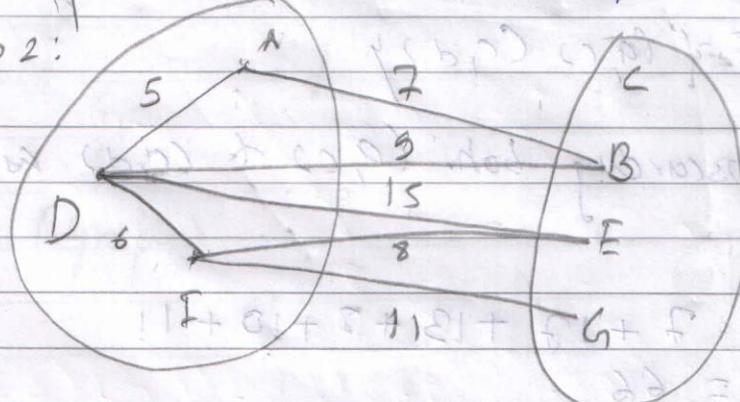
Find the MST using Priam's Algorithm



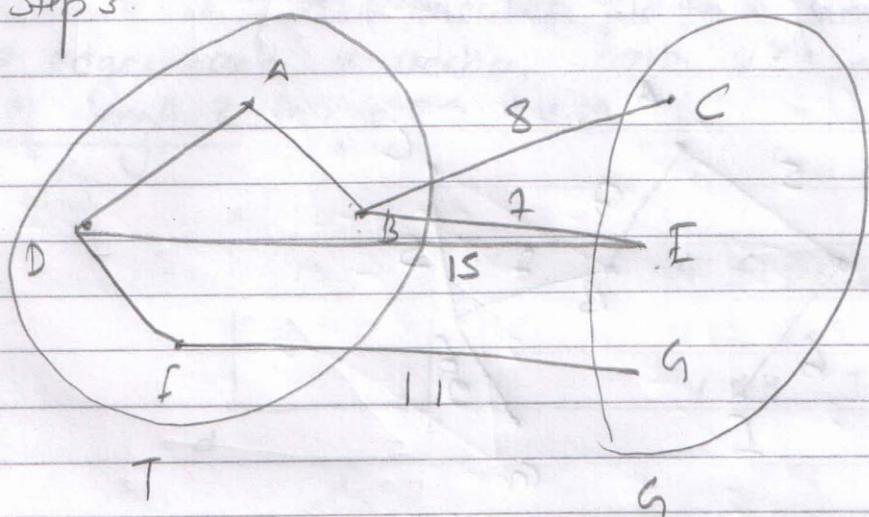
Step 1:



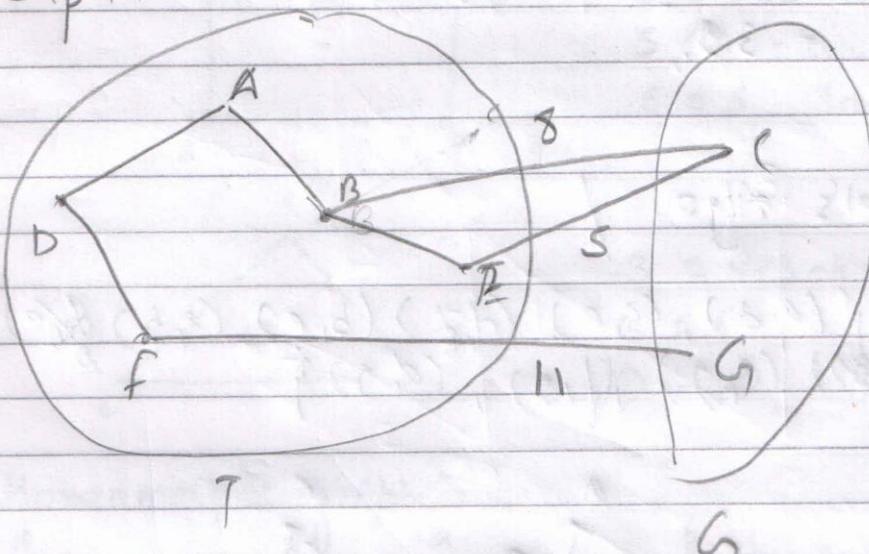
Step 2:



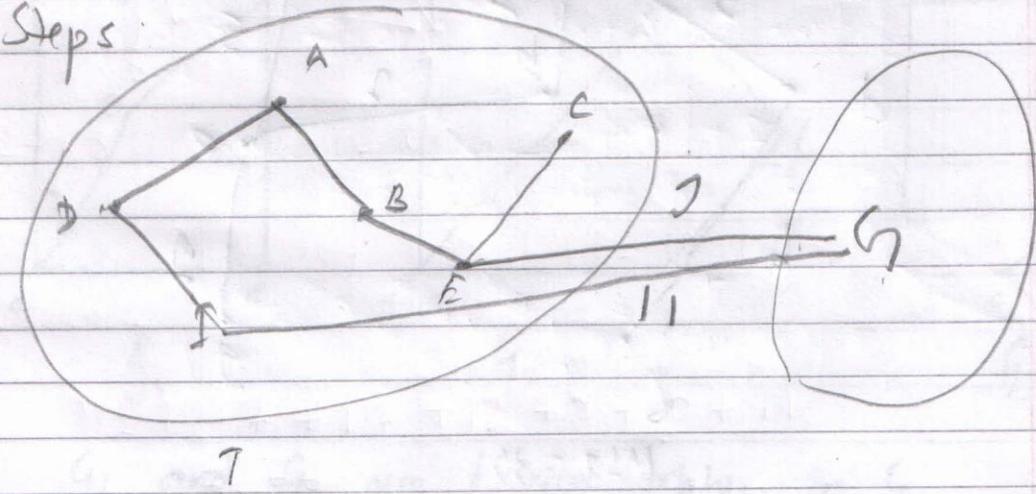
Step 3.



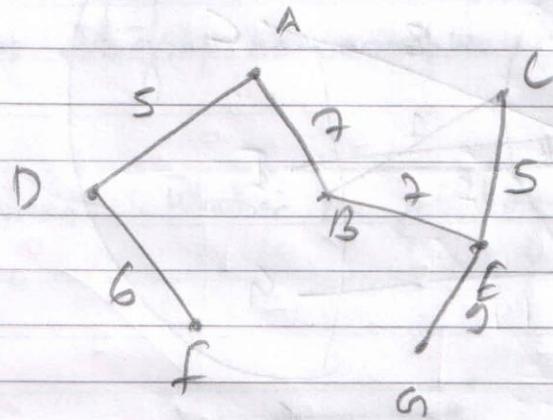
Step 4:



Step 5



Step 6:
The

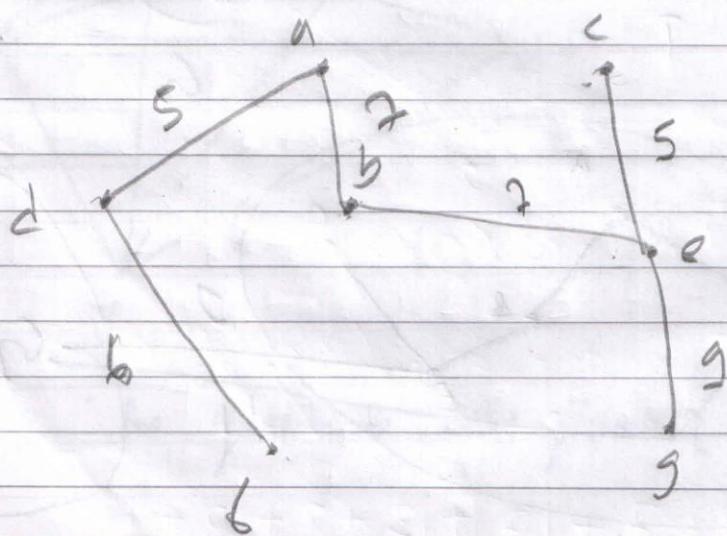


$$\text{MST} = 5 + 6 + 7 + 7 + 2 + 5 \\ = 35,$$

Kruskal's Algo.

$$E = \{(a,c), (c,e), (d,f), (b,e), (a,b), (e,f), (b,d), (e,g), (f,g), (d,g)\}$$

Step 7.

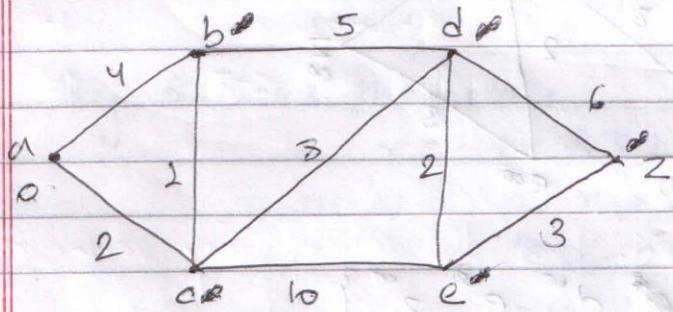


$$\text{MST} = 35$$

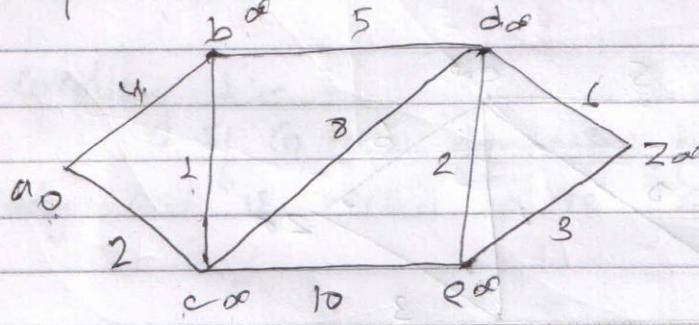
Step 1:

Shortest path Algorithm:-

Dijkstra's Algorithm,



Step 1: For source vertex 'a':-



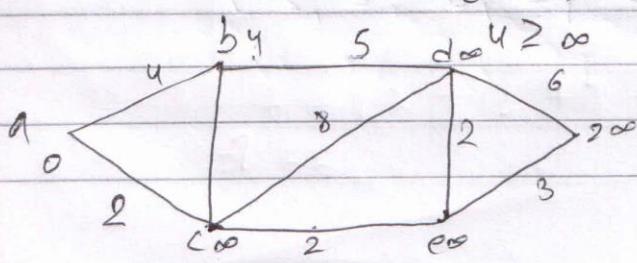
$$T = \{a^{\infty}, b^{\infty}, c^{\infty}, d^{\infty}, e^{\infty}, f^{\infty}\}$$

Choosing minimum distance value.

Step 2:-

Value of 'a' + distance of edges $\geq \infty$

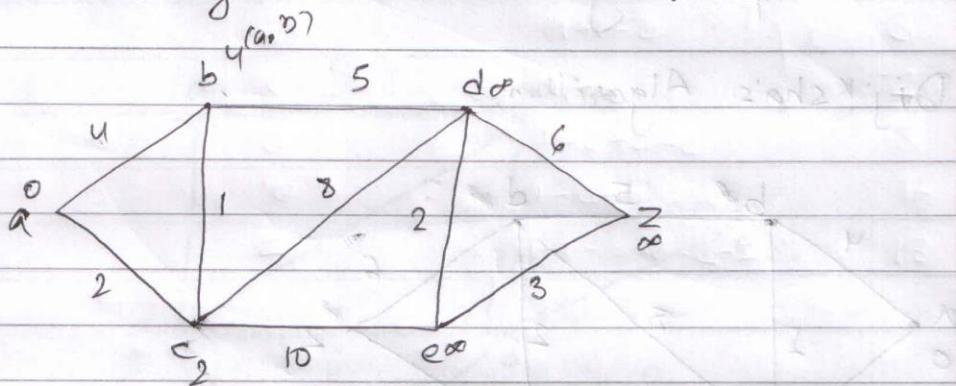
$$0 + 4 \geq \infty$$



$$T = \{b^4, c^{\infty}, d^{\infty}, e^{\infty}, f^{\infty}\}$$

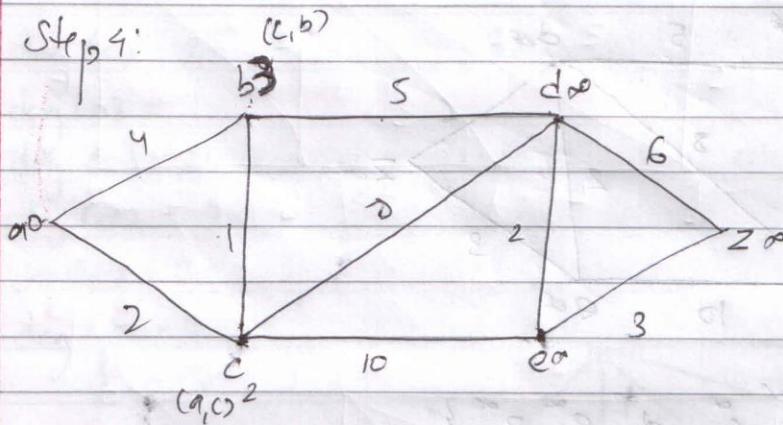
$$T = \{ b^3, c^\infty, d^\infty, e^\infty, 2^\infty \}$$

Step 3: choosing min distance value.



$$T = \{ b^3, \cancel{c}, d^\infty, \cancel{e}^\infty, 2^\infty \}$$

Step 4:

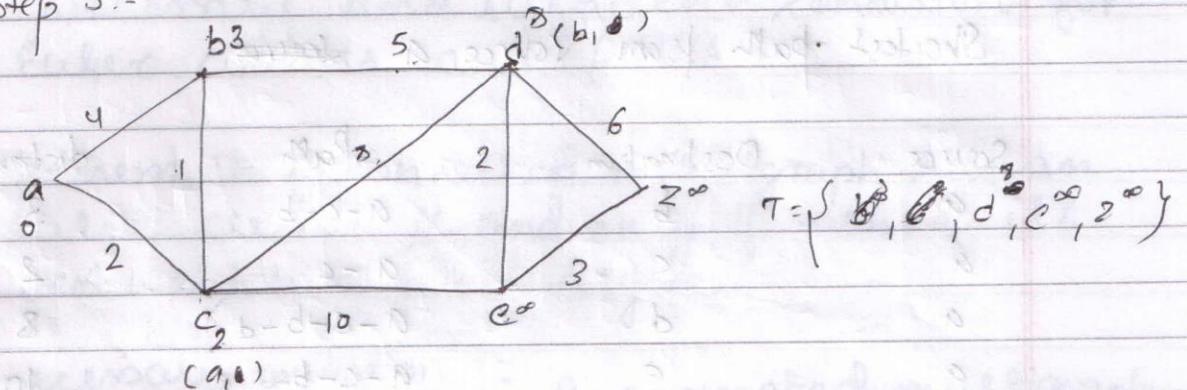
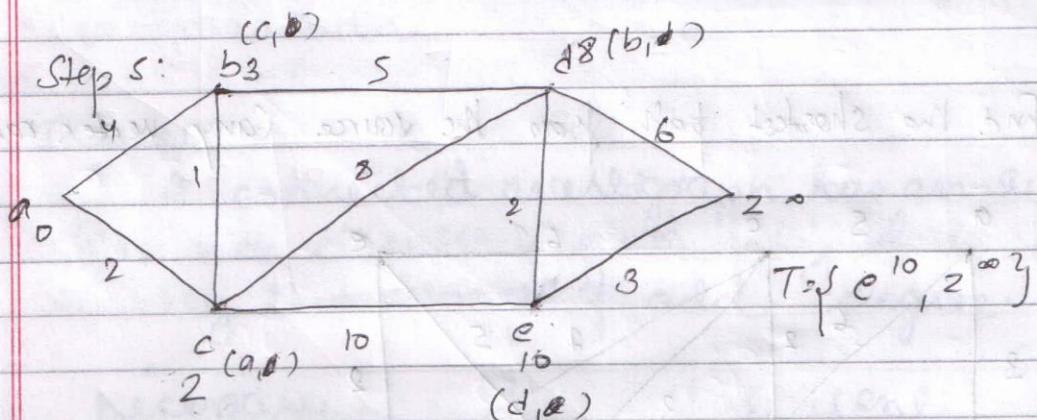
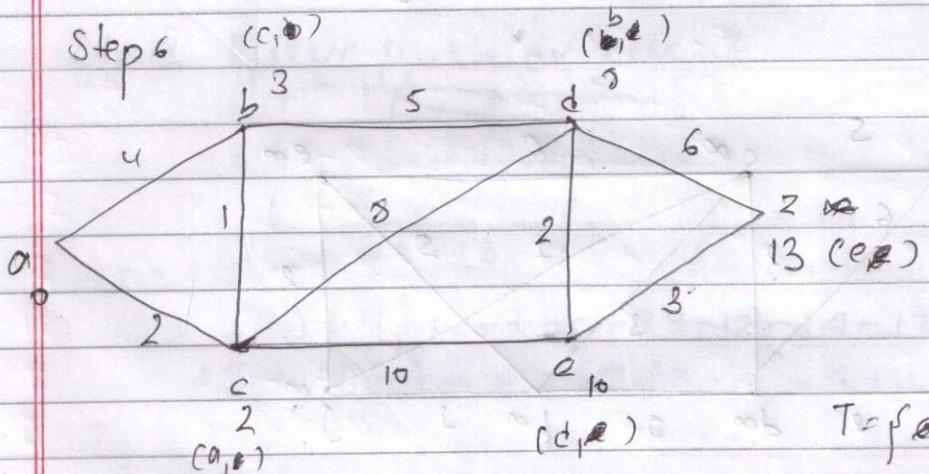


distance of c + wt. of edge \leq distance of b.

$$2 + 1 \leq 4$$

\therefore distance of b is 3

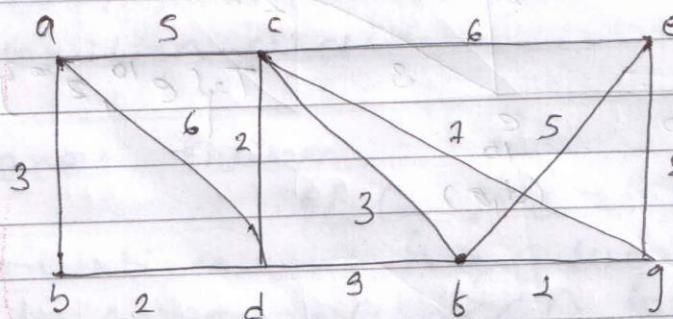
$$T = \{ b^3, \cancel{c}, d^\infty, e^\infty, 2^\infty \}$$

Step 5: - $(C_1 \oplus)$ Step 5: - $(C_1 \oplus)$ Step 6: - $(C_1 \oplus)$ 

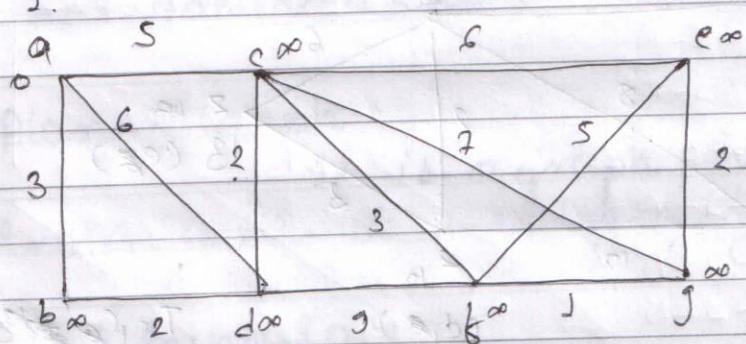
Shortest path from source a ~~to e~~.

Source	Destination	Path	distance
a	b	a-c-b	3
a	c	a-c	2
a	d	a-c-b-d	8
a	e	a-c-b-d-e	10
a	z	a-c-b-d-e-z	13

find the shortest path from the source. Any vertex randomly?

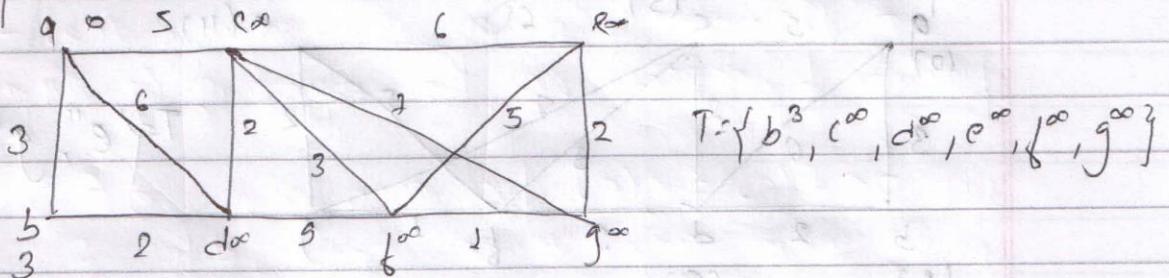


Step 1.

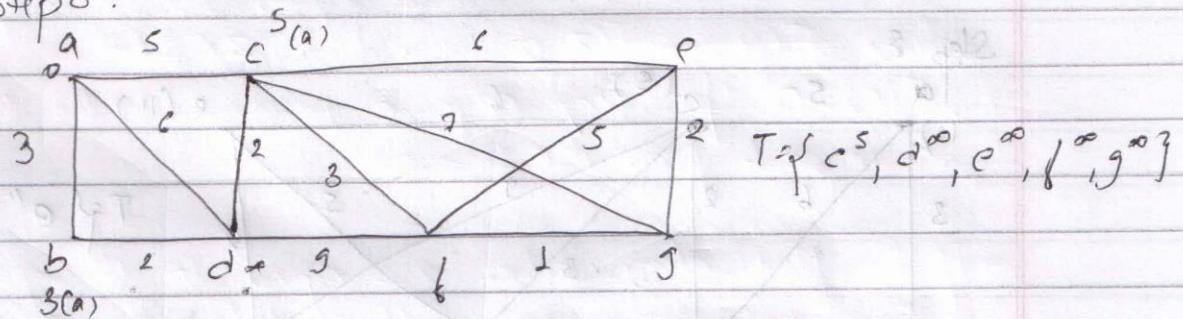


$$T = \{a^\infty, b^\infty, c^\infty, d^\infty, e^\infty, f^\infty, g^\infty\}$$

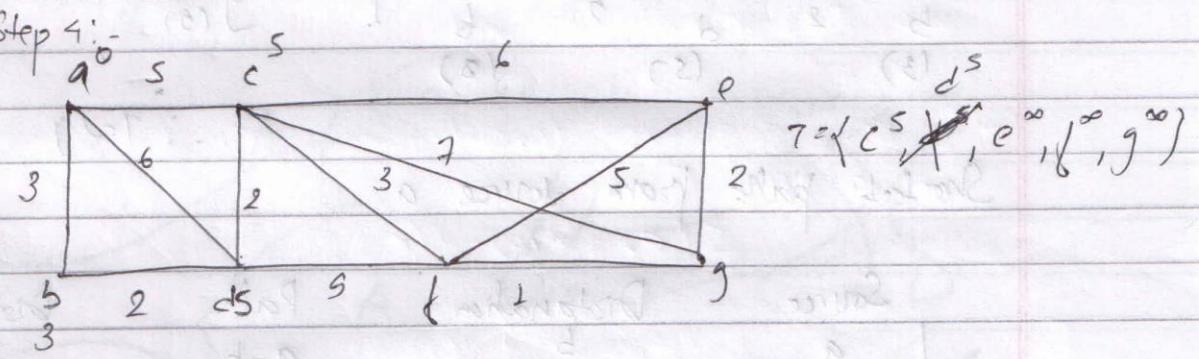
Step 2:



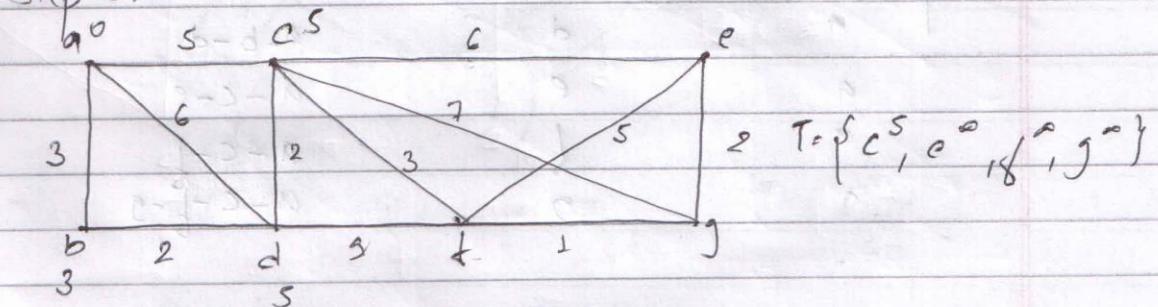
Step 3:



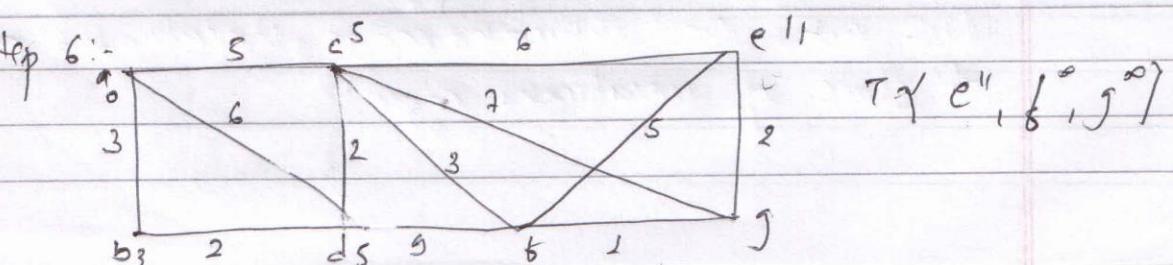
Step 4:-



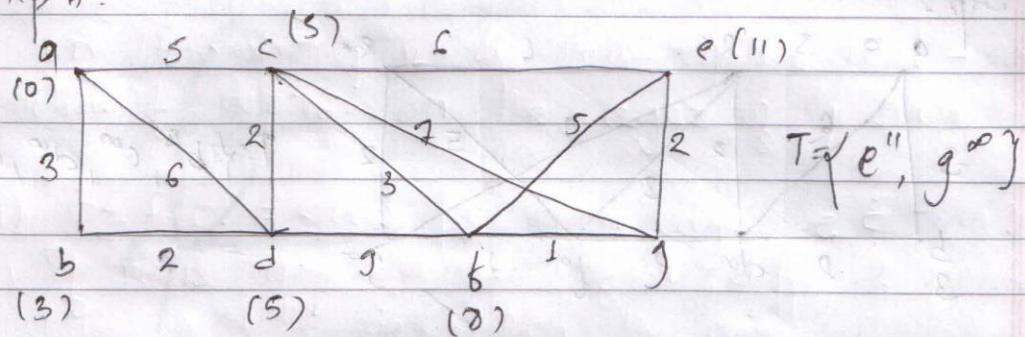
Step 5:-



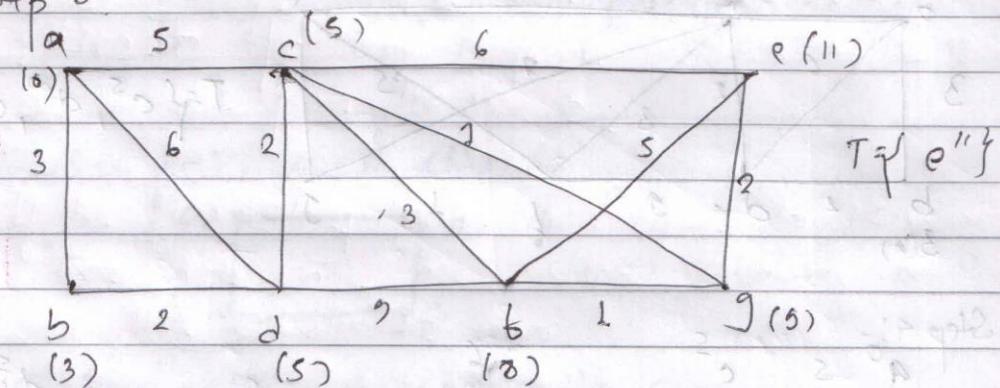
Step 6:-



Step 7:



Step 8:



Shortest path from source a

Source	Destination	Path	Distance
a	b	a-b	3
a	c	a-c	5
a	d	a-b-d	5
a	e	a-c-e	11
a	f	a-c-f	8
a	g	a-c-f-g	9

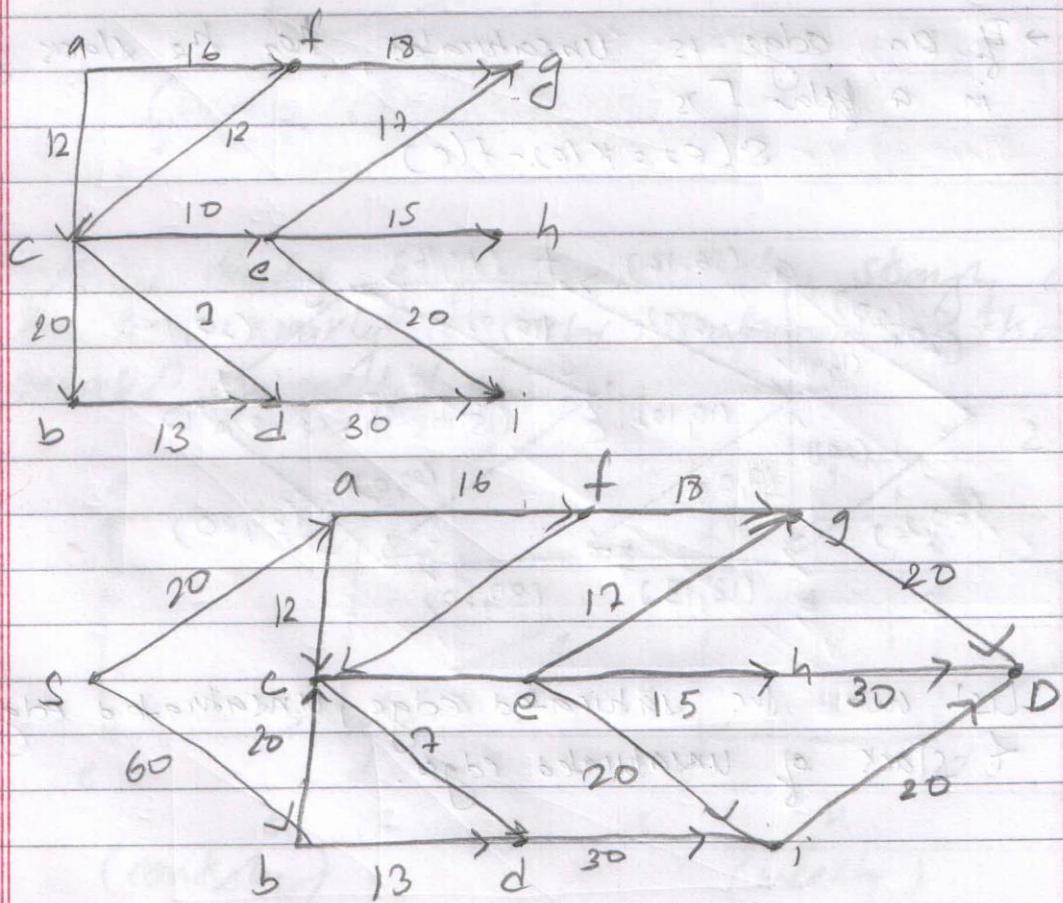
Network flow graph

A directed graph 'G' that is weakly connected and contains no loops. is called network ~~graph~~.

1. There are two distinguished vertices $S \leftarrow D$ of G , called source and sink of G respectively.
2. There is a non-negative real-valued function k defined on the edge of G .

→ the function k is called the capacity function of G and if e is any element edge of G , the value $k(e)$ is called the capacity of e .

→ the vertices distinct from source S & the sink D are called intermediate vertices.



Flow :-

A flow in G is a non-negative real-valued function f defined on the edges of G such that

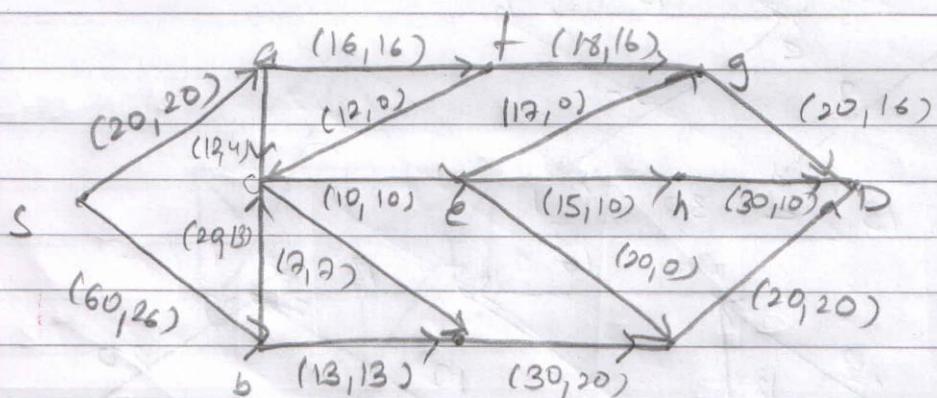
- (i) $0 \leq f(e) \leq k(e)$ for each edge $e \in E(G)$.
- (ii) If x is any vertex of G , different from the source or the sink, then the sum all values $f(x, y)$ such that $y \in A(x)$ must equal the sum of all values $f(x, z) + f(z, x)$ such that $z \in B(A)$.
- (iii) $f(e) = 0$ for any edge e incident to the source or incident from the sink.

→ If the flow along edges are equal to its capacity, then such edges are called saturated edges & if no! then unsaturated edges.

→ If an edge is unsaturated, then the slack of e in a flow f is

$$S(e) = k(e) - f(e)$$

Eg:-



List out the saturated edges, unsaturated edges & slack of unsaturated edges.

Solution :-

saturated edges :- $(S \rightarrow A)$

$(A \rightarrow F)$

$(C \rightarrow E)$

$(I \rightarrow D)$

$(B \rightarrow D)$

$(C \rightarrow D)$

unsaturated edges :- $(S \rightarrow B)$

$(C \rightarrow J)$

$(A \rightarrow C)$

$(E \rightarrow H)$

$(C \rightarrow B)$

$(H \rightarrow D)$

$(F \rightarrow C)$

$(G \rightarrow D)$

$(D \rightarrow I)$

$(E \rightarrow I)$

$(F \rightarrow J)$

$$\text{slack of } (60, 26) = k(e) - f(e)$$

$$= 60 - 26$$

$$= 34$$

$$\text{slack of } (12, 4) = k(e) - f(e)$$

$$= 12 - 4$$

$$= 8$$

$$\text{slack of } (20, 13) = k(e) - f(e)$$

$$= 20 - 13$$

$$= 7$$

$$\text{slack of } (12, 0) = k(e) - f(e)$$

$$= 12 - 0 \quad (\text{P} - 2)$$

$$= 12 \quad (+ \text{RN})$$

(3 - 2)

$$\text{slack of } (30, 20) = k(e) - f(e)$$

$$= 30 - 20$$

$$= 10 \quad (\text{b} - 2)$$

$$\text{slack of } (18, 16) = 2$$

$$\text{slack of } (17, 0) = 17$$

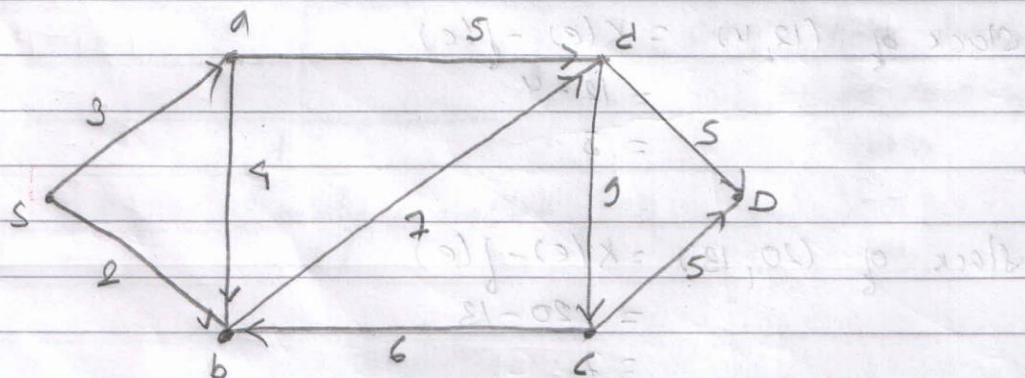
$$\text{slack of } (15, 10) = 5$$

$$\text{slack of } (20, 0) = 20$$

$$\text{slack of } (20, 16) = 4$$

$$\text{slack of } (30, 10) = 20$$

S-D cuts



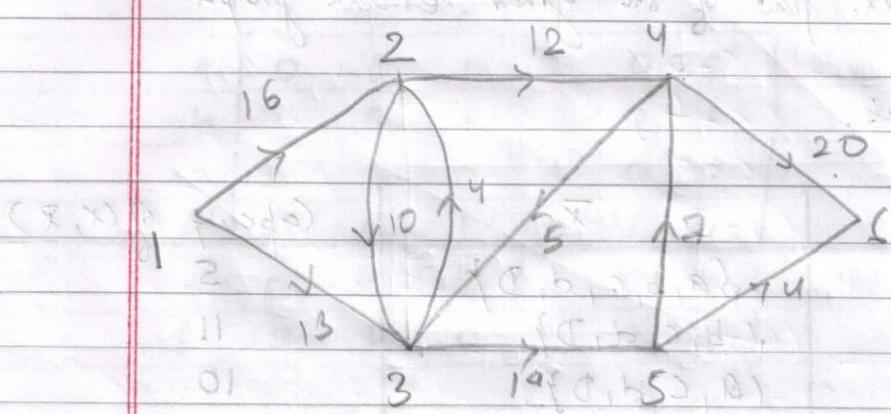
Determine the max. flow of the given network graph.

Possible S-D cuts.

X	\bar{X}	Capacity of (x, \bar{x})
$\{S\}$	$\{a, b, c, d, D\}$	5
$\{S, a\}$	$\{b, c, d, D\}$	11
$\{S, b\}$	$\{a, c, d, D\}$	10
$\{S, c\}$	$\{a, b, d, D\}$	16
$\{S, d\}$	$\{a, b, c, D\}$	19
$\{S, a, b\}$	$\{c, d, D\}$	12
$\{S, a, c\}$	$\{b, d, D\}$	22
$\{S, b, d\}$	$\{a, c, D\}$	17
$\{S, b, c\}$	$\{a, d, D\}$	15
$\{S, a, d\}$	$\{b, c, D\}$	20
$\{S, a, b, c\}$	$\{d, D\}$	17
$\{S, c, d\}$	$\{S, a, b, D\}$	21
$\{S, b, c, d\}$	$\{a, D\}$	13
$\{S, a, c, d\}$	$\{b, D\}$	22
$\{S, a, b, d\}$	$\{c, D\}$	14
$\{S, a, b, c, d\}$	$\{D\}$	10

min capacity is the maximal flow. = 5

$$P_f = Q_f D$$



X	X'	Capacity (x, x')
$\{1\}$	$\{2, 3, 4, 5, 6\}$	29
$\{1, 2\}$	$\{3, 4, 5, 6\}$	35
$\{1, 3\}$	$\{2, 4, 5, 6\}$	34
$\{1, 4\}$	$\{2, 3, 5, 6\}$	54
$\{1, 5\}$	$\{2, 3, 4, 6\}$	40
$\{1, 2, 3\}$	$\{4, 5, 6\}$	26
$\{1, 2, 4\}$	$\{3, 5, 6\}$	48
$\{1, 2, 5\}$	$\{3, 4, 6\}$	46
$\{1, 3, 4\}$	$\{2, 5, 6\}$	54
$\{1, 3, 5\}$	$\{2, 4, 6\}$	31
$\{1, 4, 5\}$	$\{2, 3, 6\}$	58
$\{1, 2, 3, 4\}$	$\{5, 6\}$	34
$\{1, 2, 4, 5\}$	$\{3, 6\}$	52
$\{1, 3, 4, 5\}$	$\{2, 6\}$	44
$\{1, 2, 3, 5\}$	$\{4, 6\}$	23
$\{1, 2, 3, 4, 5\}$	$\{6\}$	24

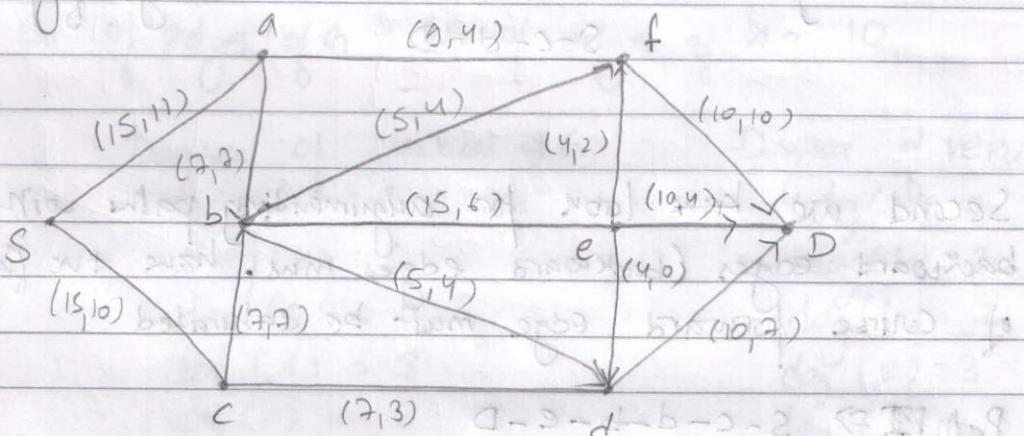
$\min_{\text{max}} \text{Cap} = 24$

Max-flow and min-cut theorem

This theorem asserts the following :-

1. The existence of a minimal cut (X, \bar{X}) .
2. The existence of a maximal flow F .
3. The equality of $|F|$ and $k(X, \bar{X})$ for any maximal flow f and any minimal cut (X, \bar{X})

Eg:-



→ First of all look for an augmenting path where if possible all edges are forward edges & they have to be unsaturated.

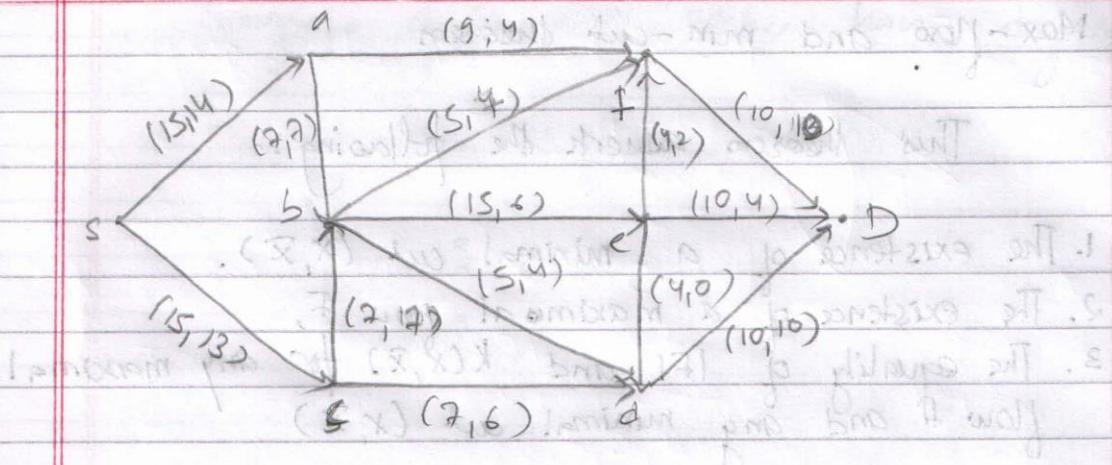
Path P_1 :- $S - c - d - D$.

Now, calculating the slack of each edge of the path P_1 .

slack of $(S-c)$:- $5 - 10$)

" " $(c-d)$:- $4 - 3$)

" " $(d-D)$:- 3)



Adding min. slack i.e. 3 at the edges of path
 $S-C-D-D$

Second we have look for augmenting paths with some backward edges (backward edges must have the flag & of course forward edge must be saturated)

Path P1 $\Rightarrow S-C-D-B-E-D$

Path P2 $\Rightarrow S-A-1-E-D$

Path P3 $\Rightarrow S-A-f-B-E-D$

Now choosing Path P1 $\Rightarrow S-C-D-B-E-D$

slack ($S-C$) :- 2

$(C,D) :- 1$

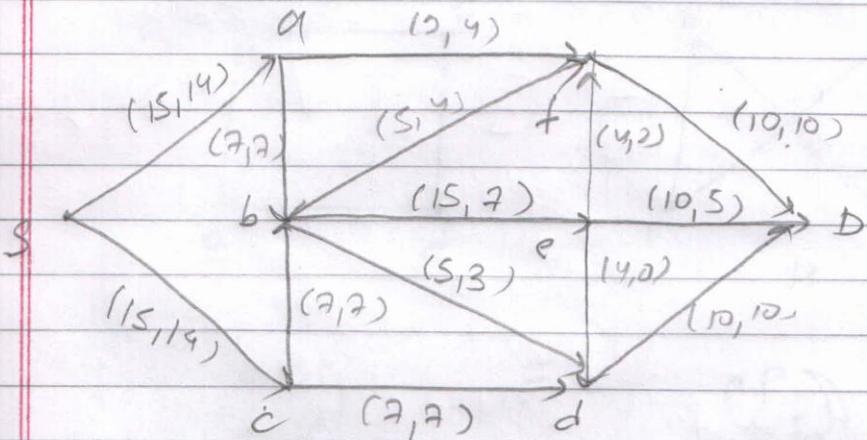
$(D,B) :- 1$

$(B,E) :- 3 \because (3-2) = 1/2$

$(E,D) :- 6 - (5-2) = 3$

$2 - (1-0) = 1$

Now increasing the flow of edges by minimum slack
and decreasing the flow of backward edges by minimum slack



Again taking Path P II: $S \rightarrow a \rightarrow f \rightarrow e \rightarrow D$.

$$\text{Slack of } (S-a) = 1$$

$$\text{", } (a-f) = 5$$

$$(f-e) = 2$$

$$(e-D) = 5$$

Again increasing the flow of edges by min. slack &
decreasing the flow of backward edges by min. slack.

