Unit-4
Solutions of Linear Algebraic Equations:

System of linear Equations:

A general set of m linear equations and n unknowns is:

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

This can be written in matrix form as;

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

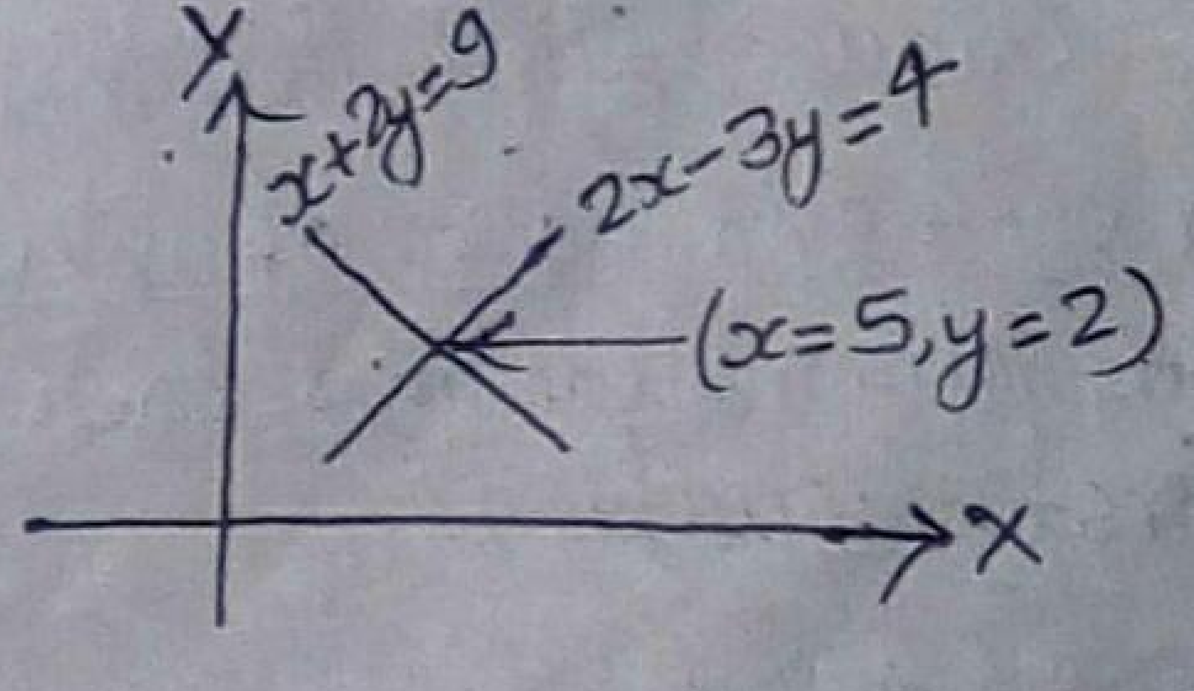
Which is of the form $AX = B$
where A is coefficient matrix, B is called right hand side matrix & x is called solution vector.

Existence of solution of system of linear Equations:-

There are four conditions in solution of system of linear equations:-
i) Unique solution ii) No solution iii) ILL-condition
iv) No-unique solution.

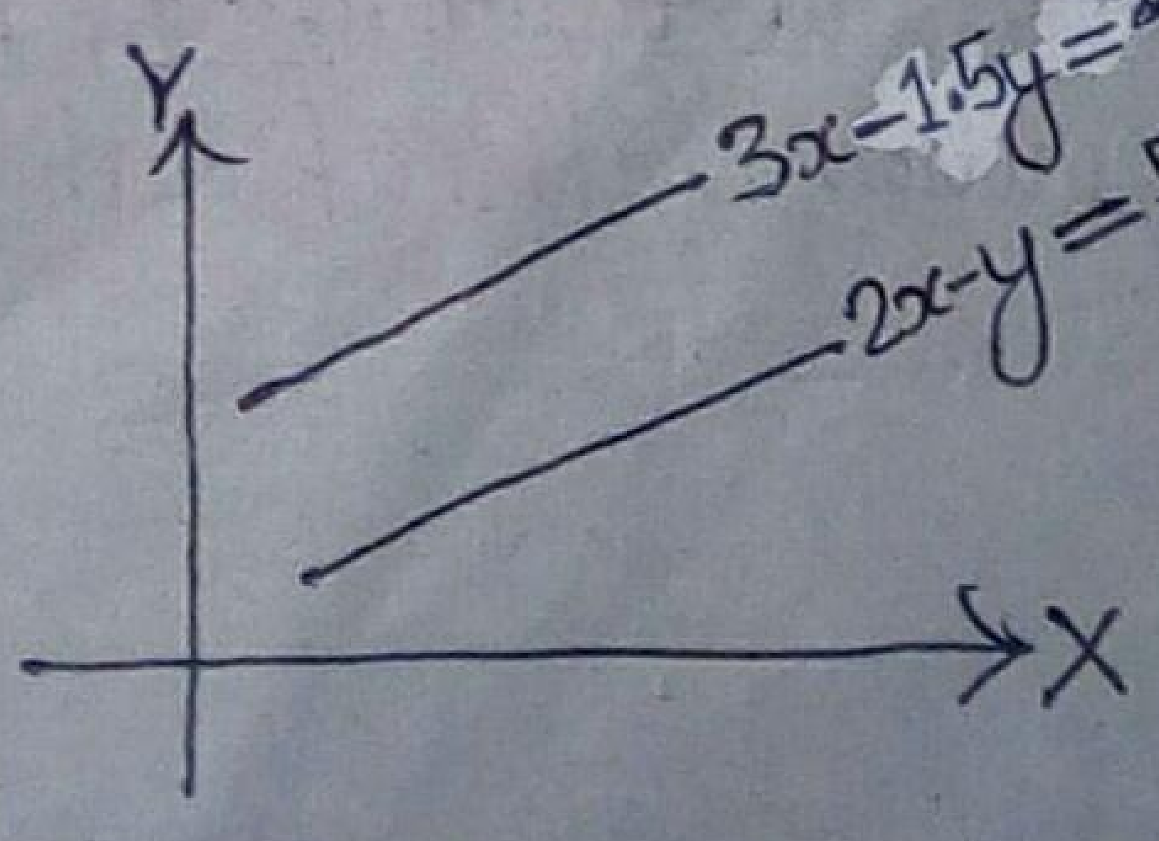
i) Unique solution:

Since the equation of lines $x+2y=9$ and $2x-3y=4$ have no other solution than $(x=5, y=2)$. So the solution is said to be unique.



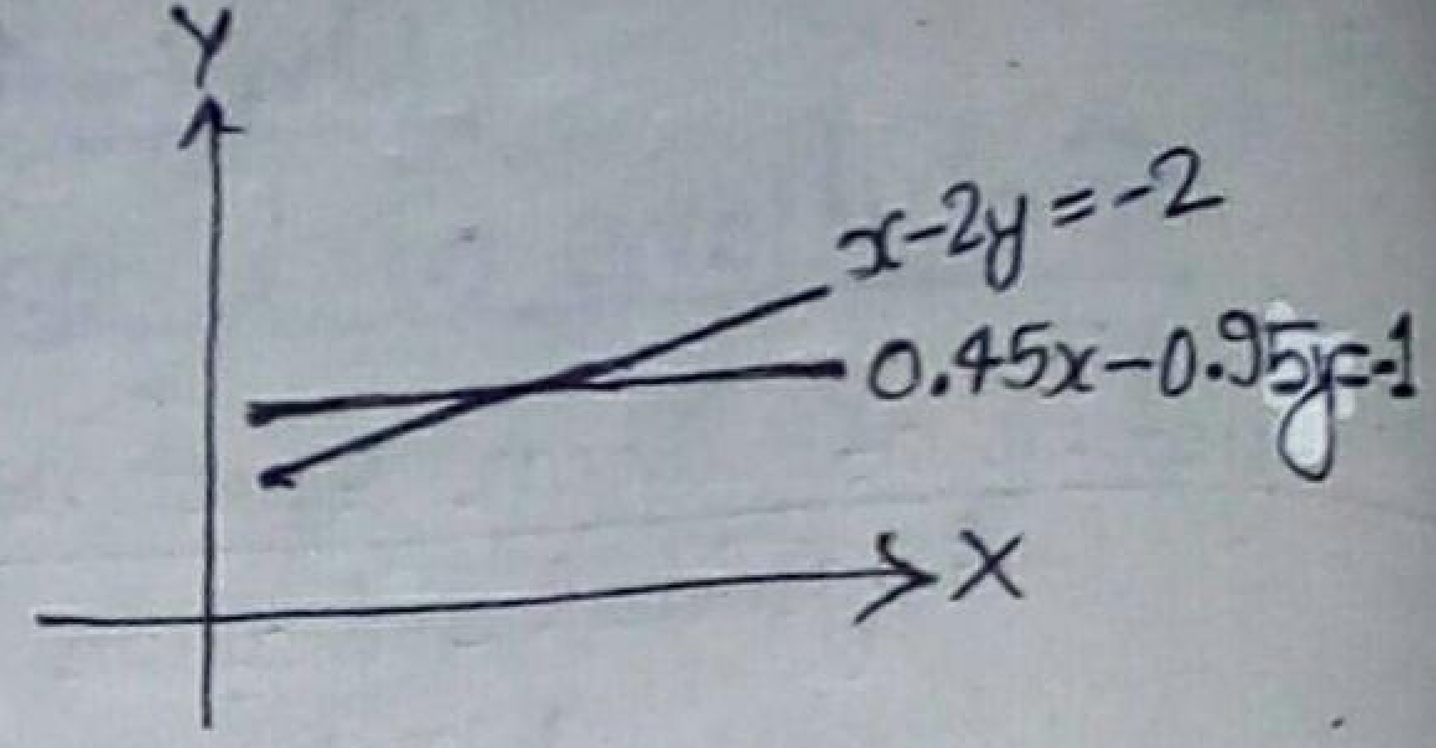
ii) No-solution:

The two lines $3x-1.5y=4$ and $2x-y=5$ are parallel to eachother so they never intersect. Thus there is no-solution.



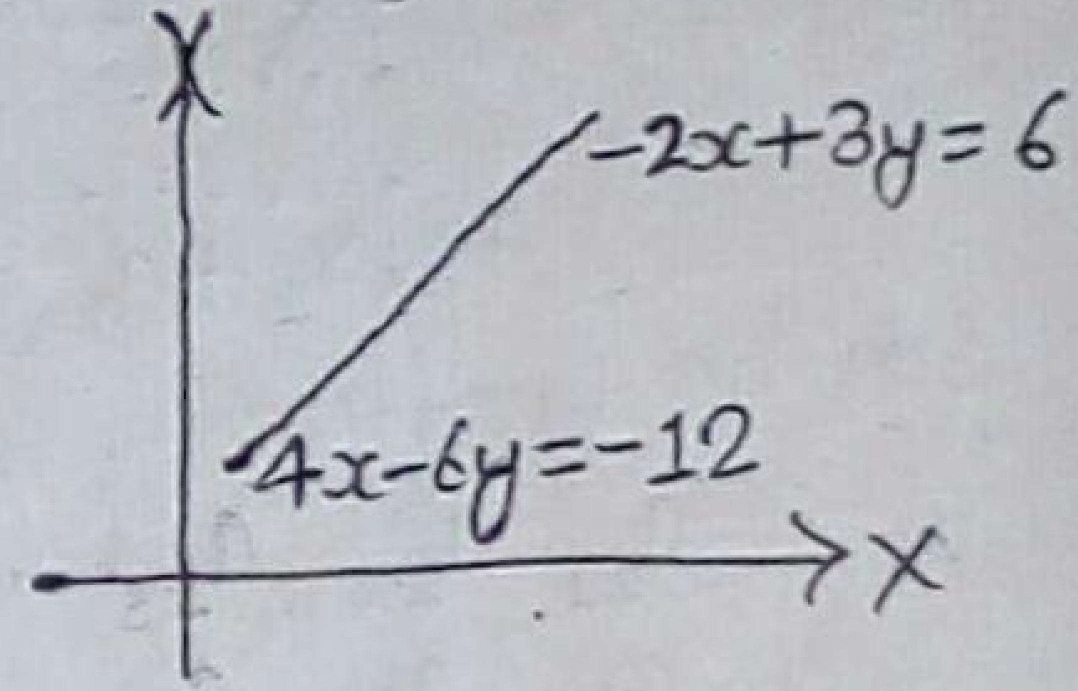
iii) Ill-condition:

It has a solution but it is very difficult to identify the exact point of line intersect. This is called Ill-condition and such systems are ill-conditioned systems.



iv) No-unique solution:

If one system is scalar multiple of other then both system form a single line. In this case one line overlaps another forming many intersection points. Hence in this case there are many solutions.



@ Solving system of linear equations:-

Solution techniques of system of linear equations can be fundamentally divided into two groups;

A) Elimination Approach (Direct method)

B) Iterative Approach.

A) Direct Methods For Solving System of Linear Equations:-

@. Gauss Elimination Method:-

1) Naive (or Basic) Gauss Elimination Method:-

Gauss Elimination method consists of following two steps:

i) Forward elimination of unknowns:- In this step, the unknowns are eliminated from each equation starting with the first equation.

ii) Back Substitution:- In this step, starting from the last equation, value of each of the unknowns is found.

Example: Use Naive Gauss elimination to solve

$$x_1 - 3x_2 + x_3 = 4$$
$$2x_1 - 8x_2 + 8x_3 = -2$$
$$-6x_1 + 3x_2 - 15x_3 = 9$$

Solution

Given equations are;

$$x_1 - 3x_2 + x_3 = 4$$
$$2x_1 - 8x_2 + 8x_3 = -2$$
$$-6x_1 + 3x_2 - 15x_3 = 9$$

Representing the above equations in matrix form, we get

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}$$

i) Forward Elimination of Unknowns

i.e, making matrix as upper triangular matrix

i.e, all elements below diagonal elements are zero.

Now; Performing $R_2 = R_2 - 2R_1$, the resulting matrix is:

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix}$$

Performing $R_3 = R_3 + 6R_1$, the resulting matrix is;

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & -15 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 33 \end{bmatrix}$$

Performing $R_3 = R_3 - 15/2\, R_2$, the resulting matrix is.

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & 0 & 54 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ -108 \end{bmatrix}$$

ii) Back Substitution:-

Now we can solve the above equations by back substitution starting from the third row,

$$54x_3 = -108$$
$$\Rightarrow x_3 = -2$$

Substituting value of x_3 in the second row,

$$-2x_2 + 6x_3 = -10$$
$$\text{or, } -2x_2 - 12 = -10$$
$$\Rightarrow x_2 = -1$$

Substituting the value of x_2 and x_3 in first row,

$$x_1 - 3x_2 + x_3 = 4$$
$$\Rightarrow x_1 - 3\times(-1) + (-2) = 4$$
$$\Rightarrow x_1 = 4 - 1$$
$$\Rightarrow x_1 = 3$$

Hence the solution is $[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

Algorithm:

1. Start
2. Read Dimension of System of equations, Say n
3. Read coefficients of matrix row-wise.
4. Read RHS vector.
5. Perform forward elimination as below;

For gauss elimination with partial pivoting which comes after this method has also same algorithm. We just add following three lines and go to else part excluding if part.
→ Find Largest of a[p][k] for p=k, k+1, ...n.
→ Swap row k and row p in coefficient matrix
→ Swap row k and row p in RHS vector.

```
For k=1 to n-1
    pivot = a[k][k]
    if (pivot <0.000001)
        Display the message "Method failed".
    else
        for i=k+1 to n
            term = a[i][k]/pivot.
Multiply row k of coefficient matrix by "term" and subtract it from row i.
Multiply row k of B matrix by "term" and subtract it from row i.
        End for
    End for
```

6. Perform back substitution as below;

```
x[n] = b[n] / a[n][n]
for i=n-1 to 1
    sum = 0
    for j= i+1 to n
        sum = sum + a[i][j] * x[j]
    End for
    x[i] = (b[i] - sum) / a[i][i]
End for
```

7. Display solution vector and terminate.

Drawbacks of Naive Gauss Elimination Method:

i) <u>Division by zero</u> → This method may suffer from division by zero problems at the beginning of the each step of forward elimination. This occurs when pivot element is zero.

ii) <u>Round-off Error</u> → Round-off Error can be a serious problem when there are large number of equations as errors propagate. Also, if there is subtraction of floating point numbers from each other, it may create large errors.

2) Gauss Elimination with Partial Pivoting:

This method avoids division by zero problem and round-off errors which was problem for naive gauss elimination method.

→ In this method, if there are n equations, then there are $n-1$ forward elimination steps, similar to Naive Gauss elimination.

→ At the beginning of the k^{th} step of forward elimination, one finds the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots, |a_{nk}|.$$

i.e, maximum of a_{11}, a_{21} & a_{31} for 1st column and so on.

→ Then if the maximum of these values is $|a_{pk}|$ in p^{th} row, $k \le p \le n$, then switch rows p and k. i.e interchange and replace 1st row by maximum having row

→ The other steps of forward elimination are same as the Naive Gauss elimination method.

→ The back substitution steps stay exactly the same as the Naive Gauss elimination method.

Example: Solve following system of linear equations by using gauss elimination with partial pivoting.

$$2x_1 + x_2 + x_3 = 5$$
$$4x_1 - 6x_2 = -2$$
$$-2x_1 + 7x_2 + 2x_3 = 9$$

Solution:-

Given equations are;

$$2x_1 + x_2 + x_3 = 5$$
$$4x_1 - 6x_2 = -2$$
$$-2x_1 + 7x_2 + 2x_3 = 9.$$

Representing the above eqations in matrix form, we get

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

i) Forward Elimination of Unknowns:

Since, largest absolute value among a_{11}, a_{12} and a_{13} is 4.
So, switch row 1 and row 2. i.e, Performing $R1 \leftrightarrow R2$, the resulting matrix is;

$$\begin{bmatrix} 4 & -6 & 0 \\ 2 & 1 & 1 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 9 \end{bmatrix}$$

Perform $R_2 = R_2 - \frac{1}{2} R_1$, the resulting matrix is;

$$\begin{bmatrix} 4 & -6 & 0 \\ 0 & 4 & 1 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 9 \end{bmatrix}.$$

Perform $R_3 = R_3 + \frac{1}{2} R_1$, the resulting matrix is;

$$\begin{bmatrix} 4 & -6 & 0 \\ 0 & 4 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 8 \end{bmatrix}$$

elimination गर्नेबेला अर्को column मा पुग्दा maximum वाला apply गर्ने

Since, largest absolute value among a_{22}, a_{23} is 4, we do not need to switch rows.
So, directly perform $R_3 = R_3 - R_2$, the resulting matrix is;

$$\begin{bmatrix} 4 & -6 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 2 \end{bmatrix}$$

ii) Back Substitution:

We can now solve the above equations by back substitution starting from the third row;

$$x_3 = 2$$

Substituting the value of x_3 in second row:

$$4x_2 + x_3 = 6$$
$$4x_2 = 4$$
$$\Rightarrow x_2 = 1.$$

Substituting value of x_2 and x_3 in first equation

$$4x_1 - 6x_2 = -2$$
$$\Rightarrow 4x_1 = 4$$
$$\Rightarrow x_1 = 1$$

Hence the solution is; $[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

Note:- Algorithm is same as in Naive Gauss Elimination method but just a small change which is discussed already in same algorithm.

(b) Gauss-Jordan Method: [V.Imp]

This method is the variation of Gauss elimination method. The differences between Gauss Elimination method and Gauss Jordan method are described below:

→ All rows are normalized by dividing them by their pivot element and when an unknown is eliminated from an equation, it is also eliminated from all other equations. Hence the elimination step results in an identity matrix rather than a triangular matrix.

→ Back substitution is not required. We can obtain solution directly from identity matrix obtained from elimination step.

General matrix form of Gauss-Jordan Method is as follows;

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

where, b_1, b_2, b_3 are constants and remaning are coefficients of given equations.

Example: Solve the following system of linear equations by Gauss-Jordan Method.

$$2x_1 - x_2 + 4x_3 = 15$$
$$2x_1 + 3x_2 - 2x_3 = 1$$
$$3x_1 + 2x_2 - 4x_3 = -4$$

Solution:
The augmented matrix is:

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 15 \\ 2 & 3 & -2 & 1 \\ 3 & 2 & -4 & -4 \end{array}\right]$$

Performing $R_1 = R_1/2$

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$$\left[\begin{array}{ccc|c} 1 & -1/2 & 2 & 15/2 \\ 2 & 3 & -2 & 1 \\ 3 & 2 & -4 & -4 \end{array}\right]$$

normalize = 1 बनाउने

pivot element लाई पहिला normalize गर्ने र pivot element बाहेक बचेका pivot column element लाई pivot row को basis मा subtract गरि 0 पार्ने ।

Performing $R_2 = R_2 - 2R_1$ and $R_3 = R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 2 & 15/2 \\ 0 & 4 & -6 & -14 \\ 0 & 7/2 & -10 & -53/2 \end{array}\right]$$

i.e, pivot element लाई सधैँ पहिला 1 बनाउने अनि तल माथि 0 बनाई identity matrix निकाल्ने

Performing $R_2 = R_2/4$.

$$\left[\begin{array}{ccc|c} 1 & -1/2 & 2 & 15/2 \\ 0 & 1 & -3/2 & -7/2 \\ 0 & 7/2 & -10 & -53/2 \end{array}\right]$$

Performing $R_1 = R_1 + \frac{1}{2}R_2$ and $R_3 = R_3 - \frac{7}{2}R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5/4 & 23/4 \\ 0 & 1 & -3/2 & -7/2 \\ 0 & 0 & -19/4 & -57/4 \end{array}\right]$$

Performing $R_3 = -\frac{4}{19}R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5/4 & 23/4 \\ 0 & 1 & -3/2 & -7/2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Performing $R_1 = R_1 - \frac{5}{4}R_3$ and $R_2 = R_2 + \frac{3}{2}R_3$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Hence, the solution of above system of linear equation is:

$x_1 = 2$, $x_2 = 1$ & $x_3 = 3$.

Algorithm:

1. Start
2. Read Dimension of System of equations, say n
3. Read coefficients of augmented matrix row-wise
4. Perform row operations to convert LHS of augmented matrix into identity matrix as:

```
For k=1 to n
    pivot = a[k][k]
  for p=1 to n+1   //Normalize row k.
      a[k][p] = a[k][p] / pivot.
  for i=1 to n.
      term = a[i][k]
    if (i≠k)
      Multiply row k by "term" and subtract it
                                 from row i.
  End for
End for
```

Just replace n+1 by 2n for algorithm of matrix inversion other code is same as it is. So we will not write it again in matrix inversion topic

5. Display solution vector and terminate.

Note:- Steps 1, 2, 3 and 5 are same as we used before differ is step 4.

© Matrix Inversion:

Inverse matrix can be computed by using Gauss-Jordan method in following two steps:

Step1: Augment the coefficient matrix with identity matrix as below:

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{31} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Step2: Apply Gauss-Jordan method to the augment matrix to reduce coefficient matrix to identify matrix as below;

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a'_{11} & a'_{12} & a'_{13} \\ 0 & 1 & 0 & a'_{21} & a'_{22} & a'_{23} \\ 0 & 0 & 1 & a'_{31} & a'_{32} & a'_{33} \end{array}\right]$$

Now, the right hand side of above augmented matrix is the inverse of original coefficient matrix.

Example: Find the inverse of the matrix $\begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$

Solution:

Augmented matrix is: $\left[\begin{array}{cc|cc} 7 & 3 & 1 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right]$.

Perform $R_1 = \frac{1}{7} R_1$

$$\left[\begin{array}{cc|cc} 1 & 3/7 & 1/7 & 0 \\ 5 & 2 & 0 & 1 \end{array}\right].$$

Perform $R_2 = R_2 - 5R_1$

$$\left[\begin{array}{cc|cc} 1 & 3/7 & 1/7 & 0 \\ 0 & -1/7 & -5/7 & 1 \end{array}\right]$$

Perform $R_2 = -7R_2$

$$\left[\begin{array}{cc|cc} 1 & 3/7 & 1/7 & 0 \\ 0 & 1 & 5 & -7 \end{array}\right]$$

Perform $R_1 = R_1 - \frac{3}{7} R_2$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 3 \\ 0 & 1 & 5 & -7 \end{array}\right].$$

Therefore the inverse of matrix A is $\begin{bmatrix} -2 & 3 \\ 5 & -7 \end{bmatrix}$.

d) Matrix Factorization:

i) Do little LU Decomposition: [Imp]

Coefficient matrix A of a system of linear equations can be decomposed into two triangular matrices L and U such that A = LU.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & 0 \\ l_{21} & l_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

→ The Dolittle algorithm assumes that all the diagonal elements of L are unity (i.e, 1).

i.e, $l_{11}=1, l_{22}=1, \ldots, l_{nn}=1$.

So we get

$$u_{11}=a_{11},\ u_{12}=a_{12}, \ldots, u_{1n}=a_{1n}. \quad \text{— (P)}$$

→ We can determine the elements of L and U as follows:-

row column

(ii)

(a). If $i \le j$, then:

$$u_{ij}=a_{ij}-\sum_{k=1}^{i-1} l_{ik}u_{kj} \quad \text{where, } j=1,2,3,\ldots,n.$$

& we know $u_{11}=a_{11}, u_{12}=a_{12}, \ldots, u_{1n}=a_{1n}$

(b) & If $i>j$, then:

$$l_{ij}=\frac{1}{u_{jj}}\left(a_{ij}-\sum_{k=1}^{j-1} l_{ik}u_{kj}\right)$$

where, $j=1,2,\ldots,i-1$.

& $l_{11}=1, l_{22}=1, \ldots, l_{nn}=1$

& $l_{i1}=\dfrac{a_{i1}}{u_{11}}$

short 5 marks type question

Example1: Factorize the following matrix using Dolittle LU decomposition.

$$[A]=\begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

Solution

$$[A]=[L][U]=\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

where,

$$u_{11}=a_{11}=25,\ u_{12}=a_{12}=5,\ u_{13}=a_{13}=1$$

$$l_{21}=\frac{a_{21}}{u_{11}}=\frac{64}{25}=2.56$$

$$u_{22}=a_{22}-l_{21}u_{12}=8-2.56\times5=-4.8$$

$$u_{23}=a_{23}-l_{21}u_{13}=1-2.56\times1=-1.56$$

$$l_{31}=\frac{a_{31}}{u_{11}}=\frac{144}{25}=5.76$$

$$l_{32} = \frac{1}{u_{22}}(a_{32} - l_{31}u_{12}) = \frac{12 - 5.76 \times 5}{4.8} = 3.5$$

$$u_{33} = (a_{33} - l_{31}u_{13} - l_{32}u_{23}) = 1 - 5.76 \times 1 - 3.5 \times (-1.56)$$
$$= 0.7$$

Now,

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix} \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

Long 10 marks type Questions

Example 2: Solve the following system of equations by using Dolittle LU decomposition method.

$$3x_1 + 2x_2 + x_3 = 10$$
$$2x_1 + 3x_2 + 2x_3 = 14$$
$$x_1 + 2x_2 + 3x_3 = 14$$

Factorization सम्म same अघि के जस्तो गर्ने

Solution:

Coefficient Matrix $[A] = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Decompose the coefficient matrix using Dolittle decomposition as below:

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

where,

$$u_{11} = a_{11} = 3, \quad u_{12} = a_{12} = 2, \quad u_{13} = a_{13} = 1$$

$$l_{21} = \frac{a_{21}}{u_{11}} = \frac{2}{3}$$

$$u_{22} = a_{22} - l_{21}u_{12} = 3 - \frac{2}{3} \times 2 = \frac{5}{3}$$

$$u_{23} = 2 - \frac{2}{3} = \frac{4}{3}$$

$$l_{31} = \frac{a_{31}}{u_{11}} = \frac{1}{3}, \quad l_{32} = \frac{1}{u_{22}}(a_{32} - l_{31}u_{12}) = \frac{2 - \frac{2}{3}}{\frac{5}{3}} = \frac{4}{5}$$

$$u_{33} = (a_{33} - l_{31}u_{13} - l_{32}u_{23}) = 3 - \frac{1}{3} - \frac{16}{15} = \frac{24}{15} = \frac{8}{5}$$

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Thus, $[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix}$

write l's in column wise

write value of u's in row wise

Now, Solve $[L][Z] = [C]$ by using forward substitution.

i.e, $$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 4 \end{bmatrix}$$

We get,

$$z_1 = 10$$

$$z_2 = 14 - \frac{2}{3} \times z_1 = \frac{22}{3}$$

$$z_3 = 4 - \frac{1}{3} \times z_1 - \frac{4}{5} \times z_2 = 14 - \frac{10}{3} - \frac{88}{15} = \frac{24}{5}$$

Again solve $[U][X] = [Z]$ by using backward substitution

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 24/5 \end{bmatrix}$$

We get,

$$x_3 = \frac{24/5}{8/5} = 3$$

$$x_2 = \frac{22/3 - 4/3 \times 3}{5/3} = 2$$

$\because \frac{5}{3} x_2 + \frac{4}{3} x_3 = \frac{22}{3}$

or, $\frac{5}{3} x_2 = \frac{22}{3} - \frac{4}{3} x_3$

or, $x_2 = \frac{22/3 - 4/3 \times 3}{5/3}$

$$x_1 = \frac{10 - 2 \times 2 - 1 \times 3}{3} = 1$$

ii) Cholesky Method:

In case A is symmetric, the LU Decomposition can be modified so that the upper factor is the transpose of lower one (or vice versa) then, we can factorize matrix A as:

$$[A] = [L][L]^T \quad \text{or} \quad [A] = [U^T][U].$$

Thus,

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} u_{11} & 0 & \cdots & 0 \\ u_{12} & u_{22} & \cdots & 0 \\ \vdots & & & \\ u_{1n} & u_{2n} & \cdots & u_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Like DoLittle LU decomposition, multiplying the matrices of right hand side and comparing with coefficient matrix of left hand side, we get,

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2} \qquad \text{where, } i = 1, 2, \ldots, n.$$

If $j > i$, $\quad u_{ij} = \dfrac{1}{u_{ii}}\left(a_{ij} - \displaystyle\sum_{k=1}^{i-1} u_{ki} u_{kj}\right).$

This decomposition is called the Cholesky's factorization or method of square roots.

Example1: Factorize the given matrix by Cholesky decomposition.

$$[A] = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 80 & 44 \\ 7 & 44 & 89 \end{bmatrix}$$

Solution:-

Let,

$$[A] = [L][U] = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$u_{11} = \sqrt{a_{11}} = \sqrt{1} = 1$$

$$u_{12} = \frac{a_{12}}{u_{11}} = \frac{4}{1} = 4$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{7}{1} = 7$$

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{80-16} = 8$$

$$u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} = \frac{44 - 4\times 7}{8} = 2$$

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{89 - 49 - 4} = \sqrt{36} = 6.$$

Thus, two factors of coefficient matrix are:

$$[A] = [L][U] = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 8 & 0 \\ 7 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

write calculated data's column wise serially

write row-wise

Example 2: Solve the following system of equations by using Cholesky decomposition technique.

$$4x_1 + 10x_2 + 8x_3 = 44$$

$$10x_1 + 26x_2 + 26x_3 = 128$$

$$8x_1 + 26x_2 + 61x_3 = 214$$

Solution:

Let, $[A] = [L][U] = \begin{bmatrix} u_{11} & 0 & 0 \\ u_{12} & u_{22} & 0 \\ u_{13} & u_{23} & u_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

where,

$$u_{11} = \sqrt{a_{11}} = \sqrt{4} = 2.$$

$$u_{12} = \frac{a_{12}}{u_{11}} = \frac{10}{2} = 5$$

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{8}{2} = 4$$

$$u_{22} = \sqrt{a_{22} - u_{12}^2} = \sqrt{26 - 5^2} = 1$$

$$u_{23} = \frac{a_{23} - u_{12}u_{13}}{u_{22}} = \frac{26 - 5\times 4}{1} = 6.$$

$$u_{33} = \sqrt{a_{33} - u_{13}^2 - u_{23}^2} = \sqrt{61 - 4^2 - 6^2} = 3.$$

Consider if any calculation mistakes or errors, understand method of solving

Thus, two factors of coefficient matrix are:

$$[A] = [L][U] = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 5 & 6 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

Now, solve the equation $[L][Z]=[C]$.

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 5 & 1 & 0 \\ 4 & 6 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 44 \\ 128 \\ 214 \end{bmatrix}$$

Thus, we get,

$$z_1 = \frac{44}{2} = 22$$

$$z_2 = 128 - 5z_1 = 128 - 5 \times 22 = 18$$

$$z_3 = \frac{214 - 4z_1 - 6z_2}{3} = 6$$

Now, solve the Equation $[U][X]=[Z]$

$$\Rightarrow \begin{bmatrix} 2 & 5 & 4 \\ 0 & 1 & 6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 22 \\ 18 \\ 6 \end{bmatrix}$$

Thus we get

$x_1 = -8$

$x_2 = 6$

& $x_3 = 2$.

B> Iterative methods for solving systems of Linear Equations:

Iterative methods are eaiser to implement on high-performance computers than that of direct methods. Due to round-off errors, direct methods become less efficient than iterative methods, when they are applied to large systems. The amount of storage space required for iterative solutions on computer is far less than the one required for direct methods.

For the iterative solution of a system, one starts with an arbitrary starting vector x_0 and computes a sequence of iterates x_m for $m = 1, 2, 3, \ldots$

(a) Jacobi Iteration Method:

The basic idea behind this method is essentially same as that for fixed point method. This method is used to solve system of equations in which diagonal elements of coefficient matrix are non-zero. Each diagonal element is solved to determine an approximate value. The process is then iterated until it converges.

Let A be an $n \times n$ nonsingular matrix and $Ax = b$ is the system to be solved. That is, we have to solve system of equations,

$$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$$

This can be written as;

$$x_1 = \frac{b_1 - (a_{12}x_2 + \ldots + a_{1n}x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21}x_1 + \ldots + a_{2n}x_n)}{a_{22}}$$

$$\vdots$$

$$x_n = \frac{b_n - (a_{n1}x_1 + a_{n2}x_2 + \ldots)}{a_{nn}}$$

Now, we can compute values of $x_1, x_2, x_3, \ldots, x_n$ by using initial guess.

Then Jacobi's iteration method is used to find new values of unknown using previously computed values as below:

$$x_i^{k+1} = \frac{\left(b_i - \sum_{j \neq i} a_{ij} x_j^k\right)}{a_{ii}} \quad \text{where, } k = 0, 1, \ldots$$

At the end of each iteration, the <u>absolute relative approximate error</u> for each x_i is calculated as below;

Simply Error

$$|\epsilon_a|_i = \left|\frac{x_i^{new} - x_i^{old}}{x_i^{new}}\right| \times 100$$

where, x_i^{new} is the recently obtained value of x_i and x_i^{old} is previous value of x_i. When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, the iterations are stopped.

<u>Example:</u>- Use the Jacobi iteration method to obtain the solution of the following equations.

$$6x_1 - 2x_2 + x_3 = 11$$
$$x_1 + 2x_2 - 5x_3 = -1$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$

<u>Solution:</u>

Rewriting the given system of equations for unknowns with largest coefficient on left hand side as: (This condition is not necessary). We can also solve directly

Third eqn taken for x_2. Since we are taking larger coeff on left first. Since coeff of x_2 are -2, 2 & 7, so 7 is largest & 3rd eqn is selected.

$$x_1 = \frac{2x_2 - x_3 + 11}{6}$$

$$x_2 = \frac{2x_1 - 2x_3 + 5}{7}$$

$$x_3 = \frac{x_1 + 2x_2 + 1}{5}$$

This is not power it is just a notation for initial guess

Now, the initial guesses $(x_1)^0 = (x_1)^0 = (x_3)^0 = 0$, then we calculate $(x_1)^1$, $(x_2)^1$ and $(x_3)^1$:

Since initial guess are 0

$$(x_1)^1 = \frac{2(x_2)^0 - (x_3)^0 + 11}{6} = \frac{2\times0 - 0 + 11}{6} = 1.833$$

$$(x_2)^1 = \frac{2(x_1)^0 - 2(x_3)^0 + 5}{7} = \frac{2\times0 - 2\times0 + 5}{7} = 0.714$$

$$(x_3)^1 = \frac{(x_1)^0 + 2(x_2)^0 + 1}{5} = \frac{0 + 2\times0 + 1}{5} = 0.200$$

Now we use these values in first iteration, to calculate the values for the 2nd iteration:

$$(x_1)^2 = \frac{2(x_2)^1 - (x_3)^1 + 11}{6} = \frac{2\times0.714 - 0.2 + 11}{6} = 2.038$$

$$(x_2)^2 = \frac{2(x_1)^1 - 2(x_3)^1 + 5}{7} = \frac{2\times1.833 - 2\times0.2 + 5}{7} = 1.181$$

$$(x_3)^2 = \frac{(x_1)^1 + 2(x_2)^1 + 1}{5} = \frac{1.833 + 2\times0.714 + 1}{5} = 0.852$$

Now we continue these iterations in table until we reach desired accuracy.

$$\text{Error} = \frac{\text{new value} - \text{previous value}}{\text{new value}}$$
initially 1

Iteration \ Values	x_1	x_2	x_3	E_1	E_2	E_3
1.	1.833	0.714	0.200	1	1	1
2.	2.038	1.181	0.852	0.1004	0.3951	0.7653
3.	2.004	1.001	1.038	0.0224	0.1214	0.2107
4.	1.994	0.990	1.001	0.0401	0.0515	0.0402
5.	1.997	0.998	0.999	0.0051	0.0111	0.0367
6.	1.999	0.999	0.999	0.0012	0.0025	0.0035
7.	1.9998	1.000	0.9994	0.0018	0.0025	0.0035
8.	2.000	1.000	0.9999	0.0001	0.0000	0.0017
9.	2.000	1.000	1.000	0.0000	0.0000	0.0005

Thus, solution is: $x_1 = 2$, $x_2 = 1$ and $x_3 = 1$.

Note: Finally summerizing this method we can use short and eaiser way to solve this type of problem as in second example below:

Second Example:

Use the Jacobi iteration method to obtain the solution of the following system of equations:

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3.$$

Solution,

From given equations, we can obtain.

$$x_1 = -\frac{1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3$$

$$x_2 = \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3$$

$$x_3 = \frac{3}{7} - \frac{2}{7}x_1 + \frac{1}{7}x_2$$

Let, initial guesses are $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$.

Iterations \ Values	x_1	x_2	x_3	E_1	E_2	E_3
1.	−0.200	0.222	−0.429	1	1	1
2.	0.146	0.203	−0.517	2.3736	0.0940	0.1713
3.	0.192	0.328	−0.416	0.2380	0.3810	0.2436
4.	0.180	0.332	−0.420	0.0596	0.0011	0.0113
5.	0.185	0.329	−0.428	0.0239	0.0092	0.0085
6.	0.186	0.330	−0.422	0.0051	0.0056	0.0040
7.	0.185	0.330	−0.422	0.0014	0.0003	0.0000
8.	0.185	0.330	−0.422	0.0002	0.0002	0.0000

Thus, solution is: $x_1 = 0.185$, $x_2 = 0.330$ and $x_3 = -0.422$.

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(b). Gauss Seidel Method:

It is in version of Gauss Jacobi Iteration method. The difference is that, the values of $x_i^{(k)}$ obtained in the k^{th} iteration remains unchanged until the entire $(k+1)^{th}$ iteration has been calculated. But with the Gauss-Seidel method, we use the new values as soon as they are known.

Example: Use the Gauss Seidel method to obtain the solution of the following equations:

$$6x_1-2x_2+x_3=11$$
$$x_1+2x_2-5x_3=-1$$
$$-2x_1+7x_2+2x_3=5.$$

Solution:

We rewrite the equations such that each equation has the unknown with largest coefficient on the left hand side.

$$x_1=\frac{2x_2-x_3+11}{6}, \quad x_2=\frac{2x_1-2x_3+5}{7}, \quad x_3=\frac{x_1+2x_2+1}{5}$$

(This is difference we do not assume finding element)

Assume the initial guesses $(x_2)^0=(x_3)^0=0$, then calculate $(x_1)^1$:

$$(x_1)^1=\frac{2(x_2)^0-(x_3)^0+11}{6}=\frac{2(0)-(0)+11}{6}=1.833$$

Now in Jacobi method we directly find $(x_2)^1$ and $(x_3)^1$ using initial guess but in method we use $(x_1)^1=1.833$ (new values as soon as known is used) and $(x_3)^0=0$ to calculate $(x_2)^1$.

$$(x_2)^1=\frac{2(x_1)^1-2(x_3)^0+5}{7}=\frac{2(1.833)-2(0)+5}{7}=1.238$$

Similarly we use $(x_1)^1=1.833$ and $(x_2^1)=1.238$ to calculate $(x_3)^1$.

$$(x_3)^1=\frac{(x_1)^1+2(x_2)^1+1}{5}=\frac{(1.833)+2(1.238)+1}{5}=1.062$$

Now we repeat the same procedure for the 2nd iteration as:

$$(x_1)^2=\frac{2(x_2)^1-(x_3)^1+11}{6}=\frac{2(1.238)-(1.062)+11}{6}=2.069$$

$$(x_2)^2 = \frac{2(x_1)^2 - 2(x_3)^1 + 5}{7} = \frac{2(2.069) - 2(1.062) + 5}{7} = 1.002$$

$$(x_3)^2 = \frac{(x_1)^2 + 2(x_2)^2 + 1}{5} = \frac{(2.069) + 2(1.002) + 1}{5} = 1.015$$

Continuing the above iterative procedure in table as:

Iteration \ Values	x_1	x_2	x_3	E_1	E_2	E_3
1.	1.833	1.238	1.062	1	1	1
2.	2.069	1.002	1.015	0.1141	0.2355	0.0463
3.	1.998	0.995	0.998	0.0355	0.0251	0.0170
4.	1.999	1.000	1.000	0.0005	0.005	0.002
5.	2.000	1.000	1.000	0.0010	0	0

Thus, solution is: $x_1 = 2$, $x_2 = 1$ and $x_3 = 1$.

Note:- As we did in Jacobi method we can neglect excessive theory and calculation part after understanding solving method and can make shorter in just by solving in table.

Eigen Vectors and Eigen Values of a Matrix:-

$$AX = \lambda X$$

X — Eigen vector
λ — Eigen value
A — Matrix.

$$Ax - \lambda x = 0$$
$$(A - \lambda I)x = 0$$

Power Method.

$$Y = AX$$

$$X = \frac{1}{k} Y, \quad \text{where } k \text{ is the element of } Y \text{ with largest magnitude.}$$

Inverse Power Method:

$$A^{-1}X = \frac{1}{\lambda} X$$

We will understand better with following examples:-

Example 1: Find the dominant eigenvalue and corresponding eigenvectors of the matrix given below by using power method.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:

Assume that initial guess for eigenvector as

$$X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Iteration-1

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 2$, New value of X can be calculated as;

$$X = \frac{1}{k} Y = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$E_1 = \left| \frac{1-0}{1} \right| = 1 \quad E_2 = \left| \frac{0.5-1}{0.5} \right| = 1 \quad E_3 = \left| \frac{0-0}{0} \right| = 0.$$

Iteration-2

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 2.5$, New value of X can be calculated as:

$$X = \frac{1}{k} Y = \frac{1}{2.5} \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix}$$

$$E_1 = \left| \frac{0.8-1}{0.8} \right| = 0.25 \quad E_2 = \left| \frac{1-0.5}{1} \right| = 0.5 \quad E_3 = \left| \frac{0-0}{0} \right| = 0$$

Iteration-3

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.8 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 2.6 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 2.8$, New value of X can be calculated as:

$$X = \frac{1}{k} Y = \frac{1}{2.8} \begin{bmatrix} 2.8 \\ 2.6 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix}$$

$$E_1 = \left| \frac{1-0.8}{1} \right| = 0.2 \quad E_2 = \left| \frac{0.93-1}{0.93} \right| = 0.0752 \quad E_3 = \left| \frac{0-0}{0} \right| = 0$$

Iteration-4

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.93 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 2.93$, New value of X can be calculated as:

$$X = \frac{1}{k} Y = \frac{1}{2.93} \begin{bmatrix} 2.86 \\ 2.93 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix}$$

$$E_1 = \left| \frac{1-1}{1} \right| = 0 \quad E_2 = \left| \frac{0.99-0.93}{0.99} \right| = 0.0606 \quad E_3 = \left| \frac{0-0}{0} \right| = 0$$

Iteration-5

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.98 \\ 2.99 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 2.99$, New value of X can be calculated as

$$X = \frac{1}{k} Y = \frac{1}{2.99} \begin{bmatrix} 2.98 \\ 2.99 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$|E_1| = \left| \frac{1-1}{1} \right| = 0 \quad E_2 = \left| \frac{1-0.99}{1} \right| = 0.01 \quad E_3 = \left| \frac{0-0}{0} \right| = 0$$

Iteration-6

$$Y = AX = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 3$, New value of X can be calculated as;

$$X = \frac{1}{k} Y = \frac{1}{3} \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$E_1 = \left| \frac{1-1}{1} \right| = 0 \quad E_2 = \left| \frac{1-1}{1} \right| = 0 \quad E_3 = \left| \frac{0-0}{0} \right| = 0$$

Thus largest eigenvalue is 3, and corresponding eigenvector is

$$X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Example 2:- Find the smallest eigen value and corresponding eigenvectors of the matrix given below by using power method.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 6 \end{bmatrix}$$

⇒ Note that finding smallest eigenvalue of A is equivalent to finding dominant eigenvalue of A^{-1}.

Solution:

Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 6 \end{bmatrix}$

We can obtain easily inverse of A by using Gauss-Jordan method.

$$B = A^{-1} = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix}$$

Now, use power method to find dominant eigen value of A^{-1}.

Assume initial is: $X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$

Iteration-1

$$Y = BX = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

⇒ k=1, New value of X can be calculated as,

$$X = \frac{1}{k} Y = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix}$$

Iteration-2

$$Y = BX = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} -0.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 21 \\ -17.5 \\ 4.5 \end{bmatrix}$$

⇒ k = 21, New value of X can be calculated as:

$$X = \frac{1}{k} Y = \frac{1}{21} \begin{bmatrix} 21 \\ -17.5 \\ 4.5 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.833 \\ 0.214 \end{bmatrix}$$

Iteration-3

$$Y = BX = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.833 \\ 0.214 \end{bmatrix} = \begin{bmatrix} -37.924 \\ 31.639 \\ -8.118 \end{bmatrix}$$

$\Rightarrow k = -37.924$, New value of X can be calculated as;

$$X = \frac{1}{k} Y = \frac{1}{-37.924} \begin{bmatrix} -37.924 \\ 31.639 \\ -8.118 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.834 \\ 0.214 \end{bmatrix}$$

Iteration-4

$$Y = BX = \begin{bmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -0.834 \\ 0.214 \end{bmatrix} = \begin{bmatrix} -37.942 \\ 31.654 \\ -8.122 \end{bmatrix}$$

$\Rightarrow k = -37.924$, New value can be calculated as;

$$X = \frac{1}{k} Y = \frac{1}{-37.942} \begin{bmatrix} -37.942 \\ 31.654 \\ -8.122 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.834 \\ 0.214 \end{bmatrix}$$

Thus, dominant eigenvalue of A^{-1} is -37.942 and corresponding eigenvector is:

$$\begin{bmatrix} 1 \\ -0.834 \\ 0.214 \end{bmatrix}.$$

& Smallest eigen value of A is $\frac{1}{-37.942} = -0.0264$ and

corresponding eigen-vector is: $\begin{bmatrix} 1 \\ -0.834 \\ 0.214 \end{bmatrix}$.