## Logic

Logic is a language for reasoning. Since logic can help us to reason the mathematical models. It need some rules for mathematical reasoning. Application may be for designing circuits, programming, program verification, et.

## Propositions and Propertional calculus

- Proposition is a fundamental concept of logic.
- Proposition is a declarative sentences that is either true or false, but not both.

  e.g $2+2=5$

  $7-1=6$

  Kathmandu is capital city of Nepal

  are some examples of proposition.

  $x>5$, come here, $3+4$ are some examples which are not propositions.
- Propositions are denoted by using small letters like p, q, r, s ... The truth value of proposition ~~and~~ is denoted by T for True proposition and F for False propositions.
- The logic that deals with proposition is called propositional logic or propositional calculus.

## Logical operators / connectives

- Logical operators are used to connect mathematical statements having one or more propositions by combining the propositions.
- The combinal propositional is called compound propositions.
- The truth table is used to get the relationship between truth values of propositions.

1. Negation (NOT)

Gives a proposition P, negation operator ($\neg$) or ($\sim$) or is used to get negation of P, denoted by $\neg P$ or $\sim P$ called "not P".

eg P: I love birds

$\neg P$: I don't love birds

| P | $\neg P$ |
|---|---|
| T | F |
| F | T |

2. Conjuction (AND)

Gives two propositions p and q, the proposition "p & q" denoted by $p \wedge q$ is the proposition that is true whenever p and q are true, false otherwise.

e.g p = "Ram is intelligent"

q = "Ram is diligent"

$p \wedge q$ = Ram is intelligent and diligent.

| p | q | $p \wedge q$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

3. Disjunction (OR):

Given two proposition p and q, the propositions "p or q" denoted by $p \vee q$ is the proposition that is false whenever both the proposition p and q are false, true otherwise.

e.g - p = "Ram is intelligent"

q = "Ram is diligent"

$p \vee q$ = Ram is intelligent or diligent.

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

construct a Truth Table

1. $\neg p \wedge (p \vee \neg q)$

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \vee \neg q$ | $\neg p \wedge (p \vee \neg q)$ |
|---|---|---|---|---|---|
| T | T | F | F | T | F |
| T | F | F | T | T | F |
| F | T | T | F | F | F |
| F | F | T | T | T | T |

II. $(p \wedge q) \vee (\neg q \wedge r)$

| $p$ | $q$ | $r$ | $\neg q$ | $p \wedge q$ | $\neg q \wedge r$ | $(p \wedge q) \vee (\neg q \wedge$ |
|---|---|---|---|---|---|---|
| T | T | T | F | T | F | T |
| T | T | F | F | T | F | T |
| T | F | T | T | F | T | T |
| T | F | F | T | F | F | F |
| F | T | T | F | F | F | F |
| F | T | F | F | F | F | F |
| F | F | T | T | F | T | T |
| F | F | F | T | F | F | F |

4. Exclusive OR (XOR)

Given two propositions p and q, the proposition exclusive OR of p and q denoted by $p \oplus q$ is the proposition that is true whenever only one of the proposition p and q is true, false otherwise.

e.g: p = "Ram drinks tea in the morning"

q = "Ram drinks coffee in the morning"

$p \oplus q$ - Ram drinks tea or coffee in the morning

| p | q | $p \oplus q$ |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | T |
| F | F | F |

5. Implication

Given two propositional p and q, the proposition implication $p \rightarrow q$, is the proposition that is false when p is true and q is false, true otherwise. Here, p is called "Hypothesis" or "antecedent" or "premise" and q is called "conclusive" or "consequence"

Different terminologies to express $p \rightarrow q$ are like:

1. "if p then q"
2. "q is a consequence of p"
3. "p is sufficient of q"
4. "q if p"
5. "q is necessary for p"
6. "q follows from p"
7. "if p, q"
8. "p implies q"
9. "p only if q"
10. "q whenever p"
11. "q provides p"

| $p$ | $q$ | $p \to q$ |
| --- | --- | --- |
| T | T | T |
| T | F | F |
| F | T | T |
| F | F | T |

## Contrapositive, Inverse and Converse

Some of related implication formed from $p \to q$ are:

Converse: $q \to p$

Inverse: $\neg p \to \neg q$

Contrapositive: $\neg \cancel{p} q \to \neg p$

$p$ = "Today is sunday"
$q$ = "It is hot today"

Implication - "If today is sunday, it is hot"
Converse - "It is hot today only if today is sunday"
Inverse - "If today is not sunday, it is not hot"
Contrapositive - "If it is not hot today, it is not sunday"

# Is Contrapositive same as $p \to q$? verify

| $p$ | $q$ | $p \to q$ | $\neg p$ | $\neg q$ | $\neg q \to \neg p$ |
| --- | --- | --- | --- | --- | --- |
| T | T | T | F | F | T |
| T | F | F | F | T | F |
| F | T | T | T | F | T |
| F | F | T | T | T | T |

$p \to q = \neg q \to \neg p$

Let p, q and r be the positions.

$p$ : "You have the flu"

$q$ : You miss the final examination

$r$ : You pass the course

Express each of the proportions as an English Sentence and construct the truth table.

i) $p \to q$

If you have flu, you miss the final examination

ii) $q \to \neg r$

If you miss the final exam, you will not pass the course

iii) $(p \to \neg r) \vee (q \to \neg r)$

If you have the flu, you will not pass the course or if you miss final exam, you will not pass the course.

| $p$ | $q$ | $r$ | $\neg r$ | $p \to \neg r$ | $q \to \neg r$ | $p \to q$ | $(p \to \neg r) \vee (q \to \neg r)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | F | F | T | |
| T | T | F | T | T | T | T | |
| T | F | T | F | F | T | F | |
| T | F | F | T | T | T | F | |
| F | T | T | F | T | F | T | |
| F | T | F | T | T | T | T | |
| F | F | T | F | T | T | T | |
| F | F | F | T | T | T | T | |

e.g "If you try hard for your, then you(u) will succeec

$p$ = you tried hard for your exam

$q$ = you succeed

case 1 : " You tried hard for your exam" and "you succee

$p$ = True $q$ = True

compound proposition $p \to q$ is True

case 2 : "You tried hard for your exam" but "you failed"

$p$ = True $q$ = False

compound proposition $p \to q$ is False

case 3 : "You haven't tried hard for your exam" and "you succeeded"

$p$ = False $q$ = True

Compound proposition $p \to q$ is True ? why

Because you can make the compound proposition false only when you satisfy the first condition itself i.e., P If that itself not satisfied that we cannot make compound proposition False. "Not False means True"

case 4 : "You haven't tried hard for your exam" and "you failed"

$p$ = false $q$ = false.

6. Biconditional operator

Defn: let p and q be two propositions. The biconditional statement of the form $p \leftrightarrow q$ is the proposition "p ~~and~~ if and only if q"

$p \leftrightarrow q$ is true whenever the truth values of p and q are same.

Some of the technologies used for biconditional are:-

1) "p if and only if q" or "p iff q"
2) "If p then q, and conversely."
3) p is necessary and sufficient for q and vice-versa.

e.g. p = "Today is sunday"
q = "It is hot today"

$p \leftrightarrow q$ = Today is sunday if and only if it is hot day.

| p | q | $p \to q$ | $q \to p$ | $p \leftrightarrow q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

e.g- let p, q and r be the proposition with truth value T, F, T respectively. Evaluate the following

i) $\neg r \vee \neg(p \vee q)$

ii) $\neg(p \vee q) \wedge (\neg r) \vee q$

i) $\neg r \vee \neg(p \vee q)$

| p | q | r | $\neg r$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg r \vee \neg(p \vee q)$ |
|---|---|---|---|---|---|---|
| T | F | T | F | T | F | F |

11) $\neg(p \vee q) \wedge (\neg r) \vee q$

| $p$ | $q$ | $r$ | $\neg r$ | $p \vee q$ | $\neg(p \vee q)$ | $(\neg r) \vee q$ | $\neg(p \vee q) \wedge (\neg r) \vee q$ |
|---|---|---|---|---|---|---|---|
| T | F | T | F | T | F | F | F |

## Precedence of logical operators

Precedence of operators helps us to decide which operator will get evaluated first in a complicated looking compound proposition

e.g $p \rightarrow q \wedge \neg p$

$(p \rightarrow (q \wedge (\neg p)))$

$p \rightarrow q \wedge \neg p$

F

| operators | Names | Preceden |
|---|---|---|
| $\neg$ | Negation | 1 |
| $\wedge$ | conjunction | 2 |
| $\vee$ | Disjunction | 3 |
| $\rightarrow$ | Implication | 4 |
| $\leftrightarrow$ | Biconditional | 5 |

## Tautology and contradiction

**Tautology**: A compound proposition that is always true, no matter what the truth value of the atomic proposition that contain it is called Tautology.

e.g- $p \vee \neg p$ is always true verify

| $p$ | $\neg p$ | $p \vee \neg p$ |
|---|---|---|
| T | F | T |
| F | T | T |

A compound proposition that is always false is called **contradiction**. e.g $p \wedge \neg p$

| $p$ | $\neg p$ | $p \wedge \neg p$ |
|---|---|---|
| T | F | F |
| F | T | F |

A compound proposition that is neither a tautology nor a contradiction is called a contingency.

## Logical Equivalences:

The compound propositions $p$ and $q$ are logically equivalent denoted by $p \Leftrightarrow q$ or $p \equiv q$, if propositions $p \leftrightarrow q$ is a tautology.

Some important logical equivalences:

1. $p \wedge T \Leftrightarrow p$ Identity law

$p \vee F \Leftrightarrow p$

2. $p \wedge F \Leftrightarrow F$ Domination law

$p \vee T \Leftrightarrow T$

3. $p \wedge p \Leftrightarrow p$ Idempotent Law

$p \vee p \Leftrightarrow p$

4. $\neg(\neg p) \Leftrightarrow p$ Double Negation law

5. $p \wedge q \Leftrightarrow q \wedge p$ Commutative law

$p \vee q \Leftrightarrow q \vee p$

6. $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ Associative law

$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

7. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ Distributive law

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

8. $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$ De-Morgan's Law

$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$

9. $p \wedge \neg p \Leftrightarrow F$ Trival Tautology

$p \vee \neg p \Leftrightarrow T$

10. $p \rightarrow q \Leftrightarrow \neg p \vee q$ Implication

11. $p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$ Equivalence

12. $(p \wedge q) \rightarrow r \Leftrightarrow p \rightarrow (q \rightarrow r)$

13. $p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p \Leftrightarrow$ Absurdity

14. $p \wedge (p \vee q) \Leftrightarrow p$ Absorption

15. $p \vee (p \wedge q) \Leftrightarrow p$

1) Truth Table
2) Symbolic Derivation

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

$p \wedge (q \vee r)$

| $p$ | $q$ | $r$ | $q \vee r$ | $p \wedge (q \vee r)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | T | F | T | T |
| T | F | T | T | T |
| T | F | F | F | F |
| F | T | T | T | F |
| F | T | F | T | F |
| F | F | T | T | F |
| F | F | F | F | F |

| $p$ | $q$ | $r$ | $p \wedge q$ | $p \wedge r$ | $(p \wedge q) \vee (p \wedge r)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | T | F | T | F | T |
| T | F | T | F | T | T |
| T | F | F | F | F | F |
| F | T | T | F | F | F |
| F | T | F | F | F | F |
| F | F | T | F | F | F |
| F | F | F | F | F | F |

Hence, $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

Prove that $(P \to q) \wedge (P \to \neg q) \equiv \neg P$ by the help of truth table.

| $P$ | $q$ | $\neg q$ | $P \to q$ | $P \to \neg q$ | $(P \to q) \wedge (P \to \neg q)$ |
|---|---|---|---|---|---|
| T | T | F | T | F | F |
| T | F | T | F | T | F |
| F | T | F | T | T | T |
| F | F | T | T | T | T |

| $\neg P$ |
|---|
| F |
| F |
| T |
| T |

$\therefore$ Hence, $(P \wedge q) \wedge (P \to \neg q) \equiv \neg P$

Show that $\neg(P \to q)$ and $(P \wedge \neg q)$ are logically equivalent by the help of Symbolic derivation.

Solution:

LHS: $\neg(P \to q)$

$\equiv \neg(\neg P \vee q)$ [$\because$ Implication law]

$\equiv \neg(\neg P) \wedge \neg(q)$ [$\because$ De-Morgan's Law]

$\equiv P \wedge \neg q$ [$\because$ Double negation]

$\equiv$ RHS

proved

$(P \to q) \wedge (P \to \neg q) \equiv \neg P$

LHS $= (P \to q) \wedge (P \to \neg q)$

$= \neg P$ (Absurdity law)

Show that $\neg(p \vee (\neg p \wedge q))$ and $(\neg p \wedge \neg q)$ are logically equivalent.

$LHS = \neg(p \vee (\neg p \wedge q))$ De-Morgan's Law

$= \neg p \wedge \neg(\neg p \wedge q)$ Double Negation

$= \neg p \wedge (p \vee \neg q)$ Distributive

$= (\neg p \wedge p) \vee (\neg p \wedge \neg q)$ Trivial Tautology

$= F \vee (\neg p \wedge \neg q)$ (Identity)

$= \neg p \wedge \neg q$

$\equiv (\neg p \wedge \neg q) \vee F \equiv \neg p \wedge \neg q$ Commutative

OR

$LHS = \neg(p \vee (\neg p \wedge q)$

$= \neg[(p \vee \neg p) \wedge (p \vee q)]$ Distributive law

$\equiv \neg(T \wedge (p \vee q))$ Trival Tautology

$\equiv \neg((T \wedge p) \vee (T \wedge q))$ Distributive

$\equiv \neg((p \wedge T) \vee (q \wedge T))$ Commutative

$\equiv \neg(p \vee q)$ Identity

$\equiv \neg p \wedge \neg q$ De-Morgan

Show that $(p \wedge q) \rightarrow (p \vee q)$ is a Tautology.

$(p \wedge q) \rightarrow (p \vee q) \equiv \neg(p \wedge q) \vee (p \vee q)$ Implication Law

$\equiv (\neg p \vee \neg q) \vee (p \vee q)$ De-Morgan's law

$\equiv (\neg p \vee p) \vee (\neg q \vee q)$ Associative

$\equiv (p \vee \neg p) \vee (q \vee \neg q)$ Commutative

$\equiv T \vee T$ Trivial Tautology

$\equiv T$

## Dual of Compound Proposition

The dual of compound proposition that contains only the logical operation $\wedge$, $\vee$ and $\neg$ is the compound proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T. The dual of S is denoted by $S^*$.

Find the dual of each of these compound proposition

a) $p \vee \neg q$

$\Rightarrow S = p \vee \neg q$

$S^* = p \wedge \neg q$

b) $p \wedge (q \vee (r \wedge T))$

$S = p \wedge (q \vee (r \wedge T))$

$S^* = p \vee (q \wedge (r \vee F))$

c) $(p \wedge \neg q) \vee (q \wedge F)$

$S = (p \wedge \neg q) \vee (q \wedge F)$

$S^* = (p \vee \neg q) \wedge (q \vee T)$

d) $(p \vee F) \wedge (q \vee T)$

$S = (p \vee F) \wedge (q \vee T)$

$S^* = (p \wedge T) \vee (q \wedge F)$

eg When does $S^* = S$, where S is a compound proposition?

Solution:

Since S is a compound proposition, let P be proposition then

$S = p \vee p$

The dual of $S \Rightarrow S^* = p \wedge p$

$p \vee p = p \wedge p = p$ (Idempotent law)

$\therefore S = S^*$

(10)

Show that $(s^*)^* = s$ where $s$ is a compound proposition

Solution:

$s$ is a compound proposition Then

let $s = p \wedge \neg q$

Now, $s^* = p \vee \neg q$

$(s^*)^* = p \wedge \neg q$

Predicate

Lets take a statement $x > 3$, there are two parts one is the variable part called "subject" and another is relation part ">3" called predicate. we can denote the statement "$x > 3$" by $P(x)$ where $P$ is a predicate ">3" and $x$ is the variable. once the value is assigned to the propositional function then we can tell whether it is true or false i.e., proposition.

- The logic involving predicate is called predicate or predicate calculus.

# let $P(x)$ denotes the statements "$x > 10$". what are the truth value of $P(12)$ and $P(5)$

$P(x)$ denotes $x > 10$

$\therefore P(12)$ denotes $12 > 10$

which is "true"

$P(5)$ denotes $5 > 10$

which is "false"

# let $Q(x,y)$ denote the statement "$x=y+3$". What are the truth value for the propositions $Q(1,2)$ and $Q(3,0)$?

Solution:

$Q(x,y)$ denotes $x=y+3$

$Q(1,2)$ denotes $1=2+3$
which is "False"

$Q(3,0)$ denotes $3=0+3$
which is "True"

# let $R(x,y,z)$ denote the statement "$x+y=z$". What are the truth value of $R(1,2,3)$ & $R(0,0,1)$?

Solution

$R(x,y,z)$ denotes $x+y=z$

$R(1,2,3)$ denotes $1+2=3$
which is "True"

$R(0,0,1)$ denotes $0+0=1$
which is "False"

## Quantifiers

Quantifiers are the tools to make the propositional function a proposition. Construction of propositions from the predicates using quantifiers is called quantification. The variables that appear in the statement can take different possible values and all the possible values that the variables can take forms a domain called "Universe of Discourse" or "Universal set".

## Types of Quantifiers

1. Universal Quantifiers
2. Existential Quantifier

## Universal Quantifier

It is denoted by ( $\forall$ ) Symbol "for all". The universal quantification of $p(x)$ denoted by for all $x$ $P(x)$

$$\forall x \; P(x)$$

is a proposition. "$P(x)$ is true for all the values of $x$ in the universal Discourse."

We can represent the universal quantification by using the English sentences like

1) for all $x$, $P(x)$ holds
11) for every $x$, $P(x)$ holds
111) for each $x$, $P(x)$ holds

Example: Take the universe a set of all students of CDCSIT

$P(x)$ represents $x$ takes graphics class.

Here universal quantification is $\forall x \; P(x)$ i.e., "all students of CDCSIT take graphics class" is a proposition.

The universal quantification is conjunction of all the propositions that are obtained by assigning the value of the variable in the predicate. Going back to above example if universe of discourse is a set {Ram, Shyam, Hari, Sita} then the truth value of the universal quantification is given by $P(Ram) \wedge P(Shyam) \wedge P(Hari) \wedge P(Sita)$ i.e., it is true only if all the atomic propositions are true.

## Existential Quantifier

Universal quantifier, denoted by $\exists$, is used for existential quantification. The existential quantification of $P(x)$, denoted by $\exists x\, P(x)$, is a proposition "$P(x)$ is true for some values of $x$ in the universe of discourse". The other forms of representation include "there exists $x$ such that $P(x)$ is true" or "$P(x)$ is true for at least one $x$".

Example: For the same problem given in universal quantification $\exists x\, P(x)$ is a proposition is represent like "some students of CDCSIT take graphics class".

The existential quantification is the disjunction of all the propositions that are obtained by assigning the values of the variable from the universe of discourse. So the above example is equivalent to $P(Ram) \vee P(Shyam) \vee P(Hari) \vee P(Sita)$, where all the instances of variables are as in example of universal quantification. Here if at least one of the students takes graphics class then the existential quantification results true.

## Free and Bound variables

when the variable is assigned a value or it is quantified. It is called bound variable. If the variable is not bounded it is called free variables.

Examples: Identify the free and bound variables.

1) $P(x,y)$ both are free variables
2) $P(2,y)$ $y$ is free
3) $P(2,y)$ where $y=4$, both are bounded
4) $\forall x\ P(x)$ $x$ is bounded variable
5) $\forall x\ P(x,y)$, $x$ bounded, $y$ free.

- Expression with no free variable is proposition
- Expression with at least one free variable is predicate?

## order of quantification

Example: Let $L(x,y)$ denotes $x$ loves $y$ where $U_oD$ for $x,y$ is set of all people in the world. Translate the given quantified statement in English

1) $\forall x\ \exists y\ L(x,y)$
   - for all $x$ there exists some $y$ such that $x$ loves $y$.
   Everyone loves someone.

ii) $\exists y\ \forall x\ L(x,y)$
   - There exists some $y$ for all $x$ such that $x$ loves $y$
   i.e., Some ~~is~~ one is loved by every one.

iii) $\forall x\ \forall y\ L(x,y)$
   - for all $x$ and $y$ such that $x$ loves $y$ i.e.,
   everyone loves everybody.

iv) $\exists x\ \exists y\ L(x,y)$
   - There exists some $x$ and some $y$ such that $x$ loves $y$.
   - Someone loves somebody.

## Negation of Quantified Expression

$\forall x\, P(x) \Rightarrow \neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$

$\exists x\, P(x) \Rightarrow \neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$

Example:

Let $P(x)$ denotes $x$ is lovely, UoD for $x$ is girls in Kathmandu

$\forall x\, P(x)$

- Every girl in Kathmandu are lovely

$\exists x\, P(x)$

- Some girls in Kathmandu are lovely.

$\neg \forall x\, P(x)$

- Not all girls in Kathmandu are lovely.

$\neg \exists x\, P(x)$

- All girls in Kathmandu are not lovely.

Translate the sentence into logical expression

1. Not every integer is even.

let $P(x)$ denotes $x$ is even integer. UoD for integer

$\neg \forall x \, P(x)$

Translate "If a person is female and is a parent then this person is Someone is someone's mother" into logical expression whose UoD is set of all peoples.

$\Rightarrow$ let $F(x)$ denote female

$P(x)$ denote parent

UoD = {set of all people}

$M(x,y)$ denotes $x$ is mother of $y$.

$\forall x \, \exists y ((F(x) \wedge P(x) \longrightarrow M(x,y))$

# Mathematical Reasoning

## Rules of Reasoning

To draw conclusion from the given premise we must be apply some well defined steps that helps reaching the conclusion. These steps of reaching the conclusion are provided by rules of inference.

## Rule 1: Modus Ponens (or Law of Detachment)

Whenever two propositions $p$ & $p \to q$ are both true then we confirm $q$ is true i.e.,

$$\begin{array}{l} p \\ \underline{p \to q} \\ \therefore q \end{array}$$

, This rule is value rule of inference because the implication $[p \wedge (p \to q)] \to q$ is a tautology.

| $p$ | $q$ | $p \to q$ | $p \wedge (p \to q)$ | $[p \wedge (p \to q)] \to q$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | F | T |
| F | F | T | F | T |

## Rule 2: Hypothetical Syllogism (Transitive Rule)

Whenever two proposition $p \to q$ and $q \to r$ are both true then we conform that implication $p \to r$ is true.

i.e.,

$$\begin{array}{l} p \to q \\ \underline{q \to r} \\ \therefore p \to r \end{array}$$

, This rule is valid rule of inference because the implication

$[(p \to q) \wedge (q \to r)] \to (p \to r)$ is tautology

similarly,

$$\begin{array}{l} p \to q \\ q \to r \\ r \to s \\ \hline \therefore p \to s \end{array}$$

| $p$ | $q$ | $r$ | $p \to q$ | $p \to r$ | $q \to r$ | $(p \to q) \wedge (q \to r)$ | $[(p \to q) \wedge (q \to r)] \to p \to r$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | T | F | T | T | T | T |
| T | F | F | F | F | T | T | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | T | F | F | T |
| F | F | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T |

## Rule 3: Addition

Due to tautology $p \to (p \vee q)$, rule $\dfrac{p}{\therefore p \vee q}$ is valid rule of inference.

$p \to (p \vee q)$

$\equiv \neg p \vee (p \vee q)$ implication

$\equiv \neg p \vee p \vee q$

$\equiv T \vee q$ Trival tautology

$\equiv T$ Domination

$[(p \to q) \wedge (q \to r)] \to (p \to r)$ 15

$\equiv (p \wedge \neg q) \vee (q \wedge \neg r) \vee (\neg p \vee r)$ Implication.

$\equiv (p \wedge \neg q) \vee \{(q \wedge \neg r) \vee r\} \vee \neg p$ Associative

$\equiv (p \wedge \neg q) \vee \{(q \vee r) \wedge (\neg r \vee r)\} \vee \neg p$ Distributive

$\equiv (p \wedge \neg q) \vee \{(q \vee r) \wedge T\} \vee \neg p$ Trival Tautology

$\equiv \{(p \wedge \neg q) \vee \neg p) \vee (q \vee r) \wedge T$ Associative

$\equiv (p \vee \neg p) \wedge (\neg q \vee \neg p) \vee (q \vee r) \wedge T$ Distributive

$\equiv T \wedge (\neg q \vee \neg p) \vee (q \vee r) \wedge T$ Trival Tautology

$\equiv T \wedge (\neg q \vee q) \vee \neg p \vee r \wedge T$ Associative

$\equiv T \wedge \{T \vee (\neg p \vee r)\} \wedge T$ Distributive

$\equiv T \wedge T \wedge T$ Trivial Tautology

$\equiv T$

**Rule 4: Simplification**

Due to the tautology $(p \wedge q) \to p$, rule $\frac{p \wedge q}{\therefore p}$ is a valid rule of inference.

$(p \wedge q) \to p$

$\equiv \neg(p \wedge q) \vee p$ Implication

$\equiv \neg p \vee \neg q \vee p$ De-Morgan's Law

$\equiv p \vee \neg p \vee \neg q$

$\equiv T \vee \neg q$ Trival Tautology

$\equiv T$

Rule 5: Conjunction

Due to Tautology $[(p) \wedge (q)] \rightarrow (p \wedge q)$, rule

$$\begin{array}{l} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$$

is valid rule of inference.

$[(p) \wedge (q)] \rightarrow (p \wedge q)$

$\equiv (p \wedge q) \rightarrow (p \wedge q)$

$\equiv \neg(p \wedge q) \vee (p \wedge q)$ Implication

$\equiv \neg p \vee \neg q \vee (p \wedge q)$ De-Morgan's Law

Rule 6: Modes Tollens

Due to Tautology $[\neg q \wedge (p \rightarrow q) \rightarrow \neg p)$, rule

$$\begin{array}{l} \neg q \\ \underline{p \rightarrow q} \\ \therefore \neg p \end{array}$$

is valid rule of inference.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \rightarrow q$ | $\neg q \wedge (p \rightarrow q)$ | $\neg q \wedge (p \rightarrow q) \rightarrow \neg p$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T |
| T | F | F | T | T | F | T |
| F | T | T | F | T | F | T |
| F | F | T | T | T | T | T |

$\neg q \wedge (p \rightarrow q) \rightarrow \neg p$

**Rule 7: Disjunctive Syllogism**

Due to Tautology $[(p \vee q) \wedge \neg p)] \rightarrow q$, rule

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$$

| $p$ | $q$ | $\neg p$ | $p \vee q$ | $(p \vee q) \wedge \neg p$ | $[(p \vee q) \wedge \neg p] \rightarrow q$ |
|---|---|---|---|---|---|
| T | T | F | T | F | T |
| T | F | F | T | F | T |
| F | T | T | T | T | T |
| F | F | T | F | F | T |

$[(p \vee q) \wedge \neg p] \rightarrow q$

**Rule 8: Resolution**

Due to Tautology $[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$, rule

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore q \vee r \end{array}$$

is valid rule of inference.

$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$

Example: For the set of premises "If I play football, then I am sore the next day". "I will take rest if I am sore", "I did not take rest". what relevant conclusio can be drawn? Explain the rule of inference to draw the conclusion.

Solution: $p$ = I play football

$q$ = I am sore

$r$ = I will take rest

Hypothesis

i) $p \to q$

ii) $q \to r$

iii) $\neg r$

| Steps | Reason |
| --- | --- |
| i) $p \to q$ | Hypothesis |
| ii) $q \to r$ | Hypothesis |
| iii) $p \to r$ | Transitive from (i) & (ii) |
| iv) $\neg r$ | Hypothesis |
| v) $\neg p$ | Modes Tollen |

Conclusion:

Therefore we can conclude that, "I did not play football".

Example: show that the hypothesis "It is not sunny this afternoon and it is colder than yesterday". "we will go swimming only if it is sunny", "If we do not go summing, then we will take a trip" and "If we take a trip, then we will be home by sunset" Lead to the conclusion "we will be home by sunset".

Solution:

$p$ = It is sunny this afternoon

$q$ = It is cooler

$r$ = we will go swimming

$s$ = we will take a trip

$t$ = we will be home by sunset.

Hypothesis

i) $\neg P \wedge q$

ii) $r \rightarrow p$

iii) $\neg r \rightarrow s$

iv) $s \rightarrow t$

| Steps | Reason |
|---|---|
| i) $\neg p \wedge q$ | Hypothesis |
| ii) $r \rightarrow p$ | Hypothesis |
| iii) $\neg r \rightarrow s$ | Hypothesis |
| iv) $s \rightarrow t$ | Hypothesis |
| v) $\neg r \rightarrow t$ | Transitive from (iii) & (iv) |
| vi) $\neg p$ | Simplification (i) |
| vii) $\neg r$ | Modes Tollens (vi) & (ii) |
| viii) $s$ | Modes Ponen (vii) & (iii) |
| ix) $t$ | Modus Ponen (viii) & (iv) |

**Fallacies:**

The fallacies are argument that are convincing but not correct. So, fallacies produce faulty inference fallancie are contingency rather than tautology.

1) Fallocy of affirming the conclusion

The kind of fallacy has the form

$$\begin{array}{l} q \\ \underline{p \rightarrow q} \\ \therefore p \end{array}$$

i.e., $q \wedge (p \rightarrow q) \rightarrow p$. This is not a tautology, hence it is fallacy.

| $p$ | $q$ | $p \to q$ | $q \wedge (p \to q)$ | $q \wedge (p \to q) \to p$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | T | T | F |
| F | F | T | F | T |

Example:
"If the economy of Nepal is poor, then the education system in Nepal will be poor" " The education system in Nepal is poor. Therefore, "economy of Nepal is poor".

Solution:

$p$ = economy of Nepal is poor
$q$ = Education system in Nepal is poor

$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

(which is fallacy (fallacy of affirming the conclusion) Hence we can conclude economy of Nepal is not poor.

ii) Fallacy of denying the hypothesis

This kind of fallacy has the form

$$\begin{array}{l} \neg p \\ p \to q \\ \hline \therefore \neg q \end{array}$$

i.e., $[\neg p \wedge (p \to q)] \to \neg q$. This is not a tautology hence it is a fallacy.

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \to q$ | $\neg p \wedge (p \to q)$ | $[\neg p \wedge (p \to q)] \to \neg q$ |
|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | F |
| F | F | T | T | T | T | T |

Example: "If today is sunday, then it rains today"
"Today is not sunday". Therefore "it does not rain today"

$p$ = Today is sunday
$q$ = It rains today.

$$\begin{array}{l} p \to q \\ \neg p \\ \hline \therefore \neg q \end{array}$$

III) Begging the question (Circular Reasoning)
If the statement that is used for loop is equivalent to the statement that is being proved then it is called circular reasoning
e.g - Ram is black because he is black.

## Rules of inference for quantified statement

### Universal Instantiation

If the proposition of the form $\forall x\ P(x)$ is supposed to be true then the universal quantifier can be dropped out to get $P(c)$ is true for arbitrary $c$ in the universe of discourse

i.e., $\dfrac{\forall x\ P(x)}{\therefore P(c), \text{ for all } c}$

Here the choosen $c$ must be arbitrary not a specific element from the UoD.

### Existential Instantiation

If the proposition of the form $\exists x\ P(x)$ is supposed to be true then there is an element $c$ in the UoD such that $P(c)$ is true.

i.e., $\dfrac{\exists x\ P(x)}{\therefore P(c) \text{ for same } c}$

Here $c$ is not an arbitrary, it must be scientific such that $P(x)$ is true.

### Extential Generalization

If at least can element $c$ from the UoD make $P(c)$ true then $\exists x\ P(x)$ is true.

i.e., $\dfrac{P(c), \text{ for some } c}{\therefore \exists x\ P(x)}$

# Show that the premises, "Everyone in this discrete mathematics class taken a course in Computer sciene" and Marla is a student in this class" imply the conclusion "Marla has take a course in computess sciene"

Solution:

$D(x)$ denotes $x$ in in discrete mathematics class

$C(x)$ denotes $x$ has taken a course in Cs

Quantified Statement (Hypothesis)

i) $\forall x\, (D(x) \rightarrow C(x))$

ii) $D(Marla)$

iii) $C(Marla)$

| Steps | Result |
|---|---|
| 1) $\forall x\, (D(x) \rightarrow C(x))$ | Hypothesis |
| 2) $D(Marla) \rightarrow C(Marla)$ | Universal Instantiation from (1) |
| 3) $D(Marla)$ | Hypothesis |
| 4) $C(Marla)$ | Modes ponens from (2) & (3) |

# Prove or disprove the validity of the argument "Every living thing is a plant or animal", "Hari's dog is alive and it is not a plant". "All animals have heart. Hence "Hari's dog has a heart".

Solution:

UoD denotes "every living thing is a plant or animal"

$H(x)$ denotes all animals have heart

$P(x) \rightarrow x$ is a plant

$A(x) \rightarrow x$ is animal

$J(x) \rightarrow x$ is alive

1. $\forall x\ (P(x) \vee A(x))$ Hypothesis

2. $J(\text{Hari dog}) \wedge \neg P(\text{Hari dog})$ Hypothesis

3. $J(\text{Hari dog}) \wedge A(\text{Hari dog})$ from (I) & (II)

4. $\forall x\ H(x)$ Hypothesis

5. $H(\text{Haris dog})$ Universal instantiation from (5)

Hypothesis:
I) $\forall x\ (P(x) \vee A(x))$
II) $J(\text{Hari Dog}) \wedge \neg P(\text{Hari Dog})$
III) $\forall x\ H(x)$
IV) $H(\text{Hari Dog})$