Mathematical Induction

The following steps are used in mathematical induction.

1. Suppose that $P(n)$ be a statement.
2. Show that $P(1)$ and $P(2)$ are true. i.e., $P(n)$ is true for $n=1$ and $n=2$.
3. Assume that $P(K)$ is true i.e., $P(n)$ is true for $n=K$.
4. Show that $P(K+1)$ follows from $P(K)$.

Consider an example $1+2+3+\dots+n = \frac{n(n+1)}{2}$

suppose that $P(n) \equiv 1+2+3+\dots+n = \frac{n(n+1)}{2}$

so, $$P(1) \equiv 1 = \frac{1(1+1)}{2} = 1$$

$$P(2) \equiv 1+2 = 3 = \frac{2(2+1)}{2}$$

So, $P(1)$ and $P(2)$ are true

Assume that $P(K)$ is true. so,

$$1+2+3+\dots+k = \frac{K(k+1)}{2}$$

$$\text{so, } P(k+1) \equiv 1+2+3+\dots+k+(K+1)$$

$$= \frac{k(k+1)}{2} + (K+1) \qquad [\because P(k) \text{ is true}]$$

$$= \frac{(k+1)}{2}(K+2) = \frac{(K+1)(K+2)}{2}$$

which shows that $P(K+1)$ is also true. Hence $P(n)$ is true for all n.

Prove by using method of induction.

a) $1^2+2^2+3^2+\ldots+n^2=\dfrac{n(n+1)(2n+1)}{6}$

b) $1+r+r^2+\ldots+r^{n-1}=\dfrac{1-r^n}{1-r}$; $r\neq 1$

c) $1+r+r^2+\ldots+r^n=\dfrac{1-r^{n+1}}{1-r}$; $r\neq 1$

d) $a+ar+ar^2+\ldots+ar^n=\dfrac{a(1-r^{n+1})}{1-r}$; $r\neq 1$

e) $a+(a+d)+(a+2d)+\ldots+(a+(n-1)d)=\dfrac{n(2a+(n-1)d)}{2}$

f) $3+7+11+\ldots+(4n-1)=n(2n+1)$

g) $2+4+6+\ldots+2n=n(n+1)$

h) $1^2+4^2+7^2+\ldots+(3n-2)^2=\dfrac{n(6n^2-3n-1)}{2}$

i) $3*6+6*9+\ldots+3n(3n+3)=3n(n+1)(n+2)$

j) $1*2+2*3+3*4+\ldots+n(n+1)=\dfrac{n(n+1)(n+2)}{3}$

k) $1*2*3+2*3*4+\ldots+n(n+1)(n+2)=\dfrac{n(n+1)(n+2)(n+3)}{4}$

l) $1+2*3+3*5+\ldots+n(2n-1)=\dfrac{n(n+1)(4n-1)}{6}$

m) $1*3*5+3*5*7+\ldots+(2n-1)(2n+1)(2n+3)=n(n+2)(2n^2+4n-1)$

n) $1^2+(1^2+2^2)+(1^2+2^2+3^2)+\ldots+(1^2+2^2+\ldots+n^2)=\dfrac{n(n+1)^2(n+2)}{12}$

e.g- show by method of induction

$$\frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \dots + \frac{1}{n*(n+1)} = \frac{n}{n+1}$$

solution: suppose

$$P(n) \equiv \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \dots + \frac{1}{n*(n+1)} = \frac{n}{n+1}$$

so, $P(1) \equiv \frac{1}{1*2} = \frac{1}{2} = \frac{1}{1+1}$ and

$$P(2) \equiv \frac{1}{1*2} + \frac{1}{2*3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = \frac{2}{2+1}$$

Assume that $P(k)$ is true. so,

$$P(k) \equiv \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \dots + \frac{1}{k*(k+1)} = \frac{k}{k+1}$$

$$P(K+1) \equiv \frac{1}{1*2} + \frac{1}{2*3} + \frac{1}{3*4} + \dots + \frac{1}{k*(k+1)} + \frac{1}{(k+1)*(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)*(k+2)} \qquad [\because P(k) \text{ is true}]$$

$$= \frac{1}{k+1}\left(k + \frac{1}{k+2}\right)$$

$$= \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right)$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

which shows that $P(k+1)$ is also true.

so, $P(n)$ is true for all n.

o) $1*2^2+2*3^2+\dots+n(n+1)^2=\dfrac{n(n+1)(n+2)(3n+5)}{12}$

p) $3*8+6*11+\dots+3n(3n+5)=3n(n+1)(n+3)$

q) $1+(1+4)+(1+4+7)+\dots+(1+4+7+\dots+(3n-2))=$
$\dfrac{n^2(n+1)}{2}$

r) $2+6+12+20+\dots+\dfrac{n(2n+2)}{2}=\dfrac{n(n+1)(n+2)}{3}$

s) $1+\dfrac{1}{2}+\dfrac{1}{4}+\dots+\dfrac{1}{2^{n-1}}=\dfrac{2^n-1}{2^{n-1}}$

t) $1*4+2*7+3*10+\dots+n(3n+1)=n(n+1)^2$

Q. Prove by method of induction

$$1^3+2^3+3^3+\dots+n^3=\left(\frac{n(n+1)}{2}\right)^2$$

solution: suppose that $P(n) \equiv 1^3+2^3+3^3+\dots+n^3=\left(\frac{n(n+1)}{2}\right)^2$

so, $P(1)=1^3=1=\left(\frac{1(1+1)}{2}\right)^2$

and $P(2)=1^3+2^3=9=\left(\frac{2(2+1)}{2}\right)^2$

Hence $P(1)$ and $P(2)$ are true.

Assume that $P(k)$ is true, so

$$P(k) \equiv 1^3+2^3+3^3+\dots+k^3=\left(\frac{(k+1)k}{2}\right)^2$$

$$P(k+1) \equiv 1^3+2^3+3^3+\dots+k^3+(k+1)^3$$

$$=\left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \qquad [\because P(k) \text{ is true}]$$

$$=(k+1)^2(k^2+4(k+1))/4$$

$$=\left(\frac{(k+1)(k+2)}{2}\right)^2$$

which shows that $P(k+1)$ is also true.

so, $P(n)$ is true for all n.