- PROBABILITY AND STOCHASTIC PROCESSES CAT I Takeaway

Random Walk and Gambler's Ruin Problem

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Question 1: Peter is in jail and has 20 dollars; he can get out on bail if he has 64 dollars. A guard agrees to make a series of bets with him. If Peter bets A dollar, he wins A dollar with probability 0.55 and loses A dollar with probability 0.45.

Find the probability that he wins 8 dollars before losing all of his money if: a. he bets 5 dollars each time (timid strategy).

q(z)=(1-(q/p)$^{i)/(1-(q/p)}$N)

```
i <- 5
N <- 8
p <- .55
q <- .45

result_timid <- (1-(q/p)^i) / (1-(q/p)^N)

result_timid

## [1] 0.7924987
```

Therefore, the probability of Peter winning 8 dollars with a timid strategy is 0.7924987 or 79.2%

b. he bets, each time, as much as possible but not more than necessary to bring his fortune up to 64 dollars (bold strategy).

q0=0 q1=0.55(q2)+0.45(q0) q2=0.55(q4)+0.45(q0) q4=0.55(q8)+0.45(q0) q8=1

```
start_dollars <- 20
out_on_bail <- c()

for (j in 20:100000) {
  samples <- c(start_dollars)
  for (x in 20: 100) {
    samples <- append(samples, sample(c(sum(samples),-sum(samples)), size=20,
replace=T, prob=c(.55,.45)))
    if (sum(samples) >= 64){
      out_on_bail <- append(out_on_bail, 1)
      break
    } else if (sum(samples) <= 0) {
```

```
      break
    }
  }
}

sum(out_on_bail) / 100000

## [1] 0.64234
```

The probability of Peter winning with a bold strategy is 0.64394 or 64.39%

```
q_0 <- 0
q_64 <- 20

q_4 <- 0.55*(q_64)+0.45*(q_0)
q_2 <- 0.55*(q_4)+0.45*(q_0)
result_bold <- 0.55*(q_2)+0.45*(q_0)

result_bold

## [1] 3.3275
```

c. Which strategy gives Smith the better chance of getting out of jail?

The timid strategy is better than the bold strategy, therefore Timid is the best strategy to use.

Question 2: Derive the formulas for the Gambler's Ruin Problem for situation when $p = q$ and $p \neq q$.

Answer: Consider Peter who starts with an intial luck of

$$ $1$$

and then on each successive gamble he will either win

$$ $1 $$

or loose

$$ $1 $$

Therefore p stands for the probability of wining while q is the probability of loosing which is represented as

$$q = 1 - p$$

In general the idea is

$$0 < i < N$$

While the game proceeds

$$Rn: n >= 0$$

forms a random walk.

$$Rn = \Delta 1 + \cdots + \Delta n, R0 = i,$$

Where

$$\Delta 1$$

forms an i.i.d. sequence of r.v.s distributed as

$$P(\Delta = i) = p, P(\Delta = -1) = q = 1 - P$$

Let

$$ Pi = min{n≥0: Rn Σ{0,N}|Ro = i$$

when either

$$Rn = 0(whenthegamestops)$$

gambler is ruined or

$$Rn = N(whenPeterwins)$$

$$Pi = PPi + 1 + \Sigma Pi - 1$$

$$1 \leq i \leq N - 1$$

$$\Delta 1 = 1 or \Delta 1 = -1$$

For

$$P + q = 1$$

$$PPi + \Sigma pi = PPi + 1 + qPi + 1$$

$$= Pi + 1 - Pi = q/p(Pi - Pi - 1)$$

therefore,

$$P2 - Pi = (q/p)(P1 - Po) = (q/p)P1 for Po = 0$$

so:

$$P3 - P2 = (q/p)(P2 - P1) = (q/p)^2$$

Generally,

$$Pi + 1 - Pi = (q/p)^i P1$$

$$0 < i < N$$

Thus

$$Pi + 1 - P1 = \Sigma^i k = 1(Pk + 1 - Pk)$$

$$\Sigma^i k = 1(q/p)^k P1$$

Which gies

$$Pi + 1 = P1 + P1\Sigma^i k = 1(q/p)^k = Pi\Sigma^i k = 0(q/p)^k$$

$$P1 \frac{1 - (q/p)^i + 1}{1 - q/p} ifP \neq qP1(i + 1)ifp = q = 0.5$$

Using geometric series equation

$$\Sigma^i n = o = \frac{1 - a^{(i+1)}}{1 - a}$$

for any a and integer i≥1

Choosing

$$i = N - 1andPN = 1$$

$$1 = PN = P1 \frac{1 - (q/p)^N}{1 - q/p} ifP \neq q1, Nifp = q = 0.5$$

we therefore conclude

$$P1 = \frac{1 - (q/p)}{1 - q/p}, ifP \neq q\frac{i}{N}, ifp = q = 0.5$$

Please Note

$$1 - Pi$$

is the probability of ruin

Question 2b) Also describe how to becoming infinitely rich or getting ruined in the same Gambler's Ruin setting

Answer

$$ifP > 0.5$$

then

$$\frac{P}{q} < 1$$

thus

$$P1 = \frac{1 - (q/p)}{1 - q/p}, ifP \neq q\frac{i}{N}, ifp = q = 0.5$$

No winning value in this case

$$= limPi = 1 - \left(\frac{q}{p}\right)^i > 0, P > 0.5$$

No winning value in this case.Therefore he is in favor of winning If

$$P \leq 0.5, then \frac{q}{p} \geq 1$$

thus

$$limPi = O, P \leq 0.5$$

Peter is not in favor of being ruined

$$N \to \infty$$

Therefore it means that P>0.5 is on Peter's favor

Question 3:

Tom starts with $5, and p = 0.63: What is the probability that Tom obtains a fortune of N = 12 without going broke?

$$i = 5, N = 12 \text{ and } q = 1 - p = 0.37$$

hence

$$\frac{q}{p} = \frac{37}{63}$$

$$P_2 = \frac{1 - (37/63)^5}{1 - (37/63)^{12}} = \frac{0.93012}{0.99832} = 0.9317$$

What is the probability that Tom will become infinitely rich? If Tom instead started with i = $2, what is the probability that he would go broke?

$$i = 2$$

$$\left(\frac{q}{p}\right)^i = \left(\frac{37}{63}\right)^2 = 0.3449$$

Question 5

Collins bought a share of stock for

$$$10$$

, and it is believed that the stock price moves (day by day) as a simple random walk with p = 0.6. What is the probability that Collins' stock reaches the high value of

$$$25$$

before the low value of

$$$4$$

?

Answer a = 25 b = 4 p = 0.6 q = 0.4

$$ p(a) = (\frac{1-(\frac{q}{p})^b{1-(\frac{q}{p})^(a+b)}})$$

```
a <- 25
b <- 4
p <- 0.6
q <- 0.4

P <- (1-(q/p)^b)/(1 -(q/p)^(a+b))

P

## [1] 0.8024754
```