Robert Morris

1. Mathematically formulate and solve for expressions for the MME estimators for parameters of the Normal Distribution.

Normal Distribution

$$X \sim N(\mu, \sigma^2) \rightarrow f(x) = \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right) e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$$
$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (y+\mu) e^{\frac{-y^2}{2\sigma^2}}\,dy$$
$$= \mu + \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} y e^{\frac{-(y)^2}{2\sigma^2}}\,dy$$
$$= \mu + \frac{1}{\sqrt{2\pi}\,\sigma} \text{ goes to } \rightarrow 0$$
$$= \boxed{\mu}$$

Note: $y = x-\mu$, $dy = dx$, $x = y+\mu$

First Moment $= E(X_i) = \mu$

$$E(X) = \mu = \frac{1}{n}\sum_{i=1}^{n} X_i$$
$$\boxed{\hat{\mu}_{mm} = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}}$$

Second Moment $= Var(X) = E(X^2) = \sigma^2 + \mu^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

$$\hat{\sigma}^2_{mm} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \mu^2$$
$$\hat{\sigma}^2_{mm} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$
$$\boxed{\hat{\sigma}^2_{mm} = \frac{1}{n}\sum_{i=1}^{n} (X_i - \bar{X})^2}$$

$$Var(X) = E((x-\mu)^2)$$
$$E((x-\mu)^2) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma^2} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}\,dx$$
$$= \frac{1}{\sqrt{2\pi}\,\sigma^2} \int_{-\infty}^{\infty} y^2 e^{-y^2/2\sigma^2}\,dy$$

Note: $y = x-\mu$, $dy = dx$, $x = y+\mu$

$u = y$
$dv = y e^{-y^2/2\sigma^2} dy$
$du = dy$
$v = -\sigma^2 e^{-y^2/2\sigma^2}$

$$\int u\,dv = uv - \int v\,du \Rightarrow$$
$$\frac{1}{\sqrt{2\pi}\,\sigma^2}\left[-\sigma^2 y e^{\frac{-y^2}{2\sigma^2}}\Big|_{-\infty}^{\infty} + \sigma^2 \int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,dy\right]$$
$$= 0 + \sigma^2 \cdot 1 = \boxed{\sigma^2}$$

Questions:

1. How does your MME estimator impact your sampled data if there are a small number of sensor measurements?

I believe that with a smaller data set, the mme distribution deviated further from the mean.

2. How does the quality of your sampled data change as the number of sensor measurements increases?

as the sampled data grows, the quality of the distribution of measurements resembles more closely that of the original mean and variance's gaussian distribution.