$$f(x_i \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \qquad L = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^N \prod_{i=1}^{N} e^{-\frac{(x_i-\mu)}{2\sigma^2}}$$

$\log(a \cdot b) = \log a + \log b$

$\log a^b = b \log a$

assume epsilon is normally distributed

$$x_i = \mu + \varepsilon_i = \varepsilon_i = x_i - \mu$$

$$l = \log L = N \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$\underset{(0)}{\cancel{N\log 1}} - N\log\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$-N\log\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$= -\frac{N}{2}\log 2\pi - \frac{N}{2}\log \log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$\mu: \frac{dL}{d\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\hat{\mu}) = 0$$

$$\sigma^2\left(\frac{1}{\sigma^2}\sum_{i=1}^{N}(x_i-\hat{\mu})\right) = 0 = \sum_{i=1}^{N}(x_i-\hat{\mu}) \Rightarrow N\hat{\mu} = \sum_{i=1}^{N} x_i \Rightarrow \hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i = \bar{X}$$

$$-N\log\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$$

$$-\frac{N}{2}\log 2\pi - \frac{N}{2}\log \log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2$$

Extra for learning purposes.

Proof: $\hat{\sigma}^2_{MLE}$ is a Biased estimator for σ^2

$$\sigma^2: \frac{dl}{d\sigma^2} = \frac{-N}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{N}(x_i-\mu)^2 = 0$$

$$\hat{\sigma}^2_{MLE} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \underbrace{\hat{\mu}_{ml}}_{\bar{X}})^2$$

note: for the purpose of this proof: $\Sigma = \sum_{i=1}^{N}$

$$E[\hat{\sigma}^2_{MLE}] = E\left[\frac{1}{n}\Sigma(x_i-\bar{x})^2\right] = E\left[\left(\frac{1}{n}\right)\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = \frac{1}{n}E\left[\Sigma(X_i-\bar{X})\right]^2 = \frac{1}{n}E\left[\Sigma((x_i-\bar{x})(x_i-\bar{x}))\right] = \frac{1}{n}E\left[\Sigma(x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right]$$

$$= \frac{1}{n}E\left[\Sigma x_i^2 - \overset{\text{constants}}{\Sigma 2x_i\bar{x}} + \Sigma\bar{x}^2\right] = \frac{1}{n}E\left[\Sigma x_i^2 - 2\bar{x}\Sigma x_i + n\bar{x}^2\right] = \frac{1}{n}E\left[\Sigma X_i^2 - 2\bar{X}n\bar{X} + n\bar{X}^2\right] = \frac{1}{n}E\left[\Sigma X_i^2 - n\bar{X}^2\right]$$

$$-2n\bar{X}^2 + n\bar{X}^2 = \frac{1}{n}\Sigma E(X_i^2) - E(n\bar{X}^2)$$

$$= \frac{1}{n}\Sigma E(X_i^2) - nE(\bar{X}^2) = \frac{1}{n}\Sigma(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)$$

$$\boxed{\begin{aligned}\text{note: } & E(x_i^2) = \sigma^2+\mu^2 \\ & E(\bar{X}^2) = \frac{\sigma^2}{n}+\mu^2\end{aligned}}$$

$$= \frac{1}{n}\, n\sigma^2 + \cancel{n\mu^2} - \sigma^2 - \cancel{n\mu^2} \rightarrow = \sigma^2(n-1)\left(\frac{1}{n}\right.$$

$$\hat{\sigma}^2_{MLE} = \frac{\sigma^2(n-1)}{n}$$

$\therefore$ $\hat{\sigma}^2_{MLE}$ Expected value is not exactly the parameter σ^2. Hence $\hat{\sigma}^2_{MLE}$ is a biased estimation.

$$P\{|\hat{\sigma}^2_{MLE} - \sigma^2| > \varepsilon\} \le \frac{var(\hat{\sigma}^2_{MLE})}{\varepsilon^2} = \frac{var\left(\frac{\sigma^2(n-1)}{n}\right)}{\varepsilon^2} = \frac{var\left(\frac{1}{n}\sigma^2(n-1)\right)}{\varepsilon^2}$$

$=$