Calculus and Linear Algebra II
SVD

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1 Singular Value Decomposition

Theorem:
Any mxn matrix A has a single value decomposition (SVD)

$$A = U\Sigma V^+$$

where U is the unitary mxm and V unitary nxn.
$\Sigma - mxn$ matrix with only non negative values σ_i on the diagonal and zeros everywhere else. The G_i are called singular values of A, they are uniquely determined by A.

$$A = U\Sigma V^+$$
$$A^+A = (U\Sigma V^+)^+(U\Sigma V^+) = V\Sigma^+U^+U\Sigma V^+ = V\Sigma^+\Sigma V^+$$

$$AA^+ = (U\Sigma V^+)(U\Sigma V^+)^+ = U\Sigma V^+V\Sigma V^+U^+ = U\Sigma^+\Sigma^+U^+$$

$\Sigma^+\Sigma$ and $\Sigma\Sigma^+$ are diagonalizable with $\sigma_1^2 ... \sigma_1^2$ (k = min(m,n)

$$\begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

A^+A and AA^+ are diagonalizable by a unitary matrix

We can conclude:

$\sigma_1^2 \dots \sigma_k^2$ are eigenvalues of both A^+A and AA^+

V has orthonormal eigenvectors of A^+A as columns.

U has orthonormal eigenvectors of AA^+ as columns.

$$Ax = y$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

But what if :

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} A^+ = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

$$A^+A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$$

$$det(\begin{bmatrix} 4-\lambda & 2 \\ 2 & 5-\lambda \end{bmatrix}) =$$

$$(4-\lambda)(5-\lambda) - 4 = 20 + \lambda^2 - 9\lambda - 4 = \lambda^2 - 9\lambda + 16 = 0$$

From this we have the singular values:

$$\sigma_1, \sigma_2 = \sqrt{\frac{9 \pm \sqrt{17}}{2}}$$

2 Linear Differential Equations with constant coefficients

$$a_n y^n + a_{n-1} y^{n-1} + ... + a_1 y' + a_0 y = f(t)$$

Homogeneous Equations :

$$a_n y^n + a_{n-1} y^{n-1} + ... + a_1 y' + a_0 y = 0$$

The General solution is of form :

$$e^{\lambda t} \implies a_n \lambda^n e^{\lambda t} + a_{n-1} \lambda^{n-1} e^{\lambda t} + ... + a_1 \lambda e^{\lambda t} + a_0 e^{\lambda t}$$

$$(a_n \lambda^n + a_{n-1} \lambda^{n-1} + ... + a_1 \lambda + a_0) e^{\lambda t} = 0$$

Example1 :

$$y''' - y' = 0$$

$$\lambda^3 - \lambda = 0 \implies \lambda = 0, +1, -1$$

$$y(t) = C_1e^{0t} + C_2e^t + C_3e^{-t} = C_1 + C_2e^t + C_3e^{-t}$$

Harmonic Oscillator:

$$y'' + y = 0$$

$$\lambda^2 + 1 \implies \lambda = \pm i$$

First case:

$$\lambda - real, distinct \implies C_1e^{\lambda_1 t} + ... + C_ne^{\lambda_n t}$$

Second case:

$$\lambda - a \pm ib \implies y(t) = C_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$$

Third case:

$$y''' + 3y'' + 3y' + y = 0$$

$$\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0 \implies (\lambda + 1)^3 = 0$$

$\lambda = -1$ with algebraic mult. $= 3$

$$y(t) = c_1e^{-t}$$

$$y(t) = (C_1 + C_2t + C_3t^2)e^{-t}$$

Fourth case:

$$\lambda = a \pm ib$$

with multiplity m

$$y(t) = (C_1 + C_2t + ...C_mt^{m-1})e^{at}\cos(bt) + (D_1 + D_2t... + D_mt^{m-1})e^{at}\sin bt$$

Dealing with the right hand side:

$$a_ny^n + a_{n-1}y^{n-1} + ... + a_1y' + a_0y = f(t)$$

f(t) : polynomials, exponentials, sin and cos.

If we have:

$$y''' - y' = t^2 e^{2t} + \sin(t)$$

$$y_{gen}(t) = y_{genhomo}(t) + y_{geninhomo}(t)$$

From above $\lambda = 2$ and for the sin we have: $\lambda = \pm i$
Now the solution will be:

$$e^{2t}(A_1 t^2 A_2 t + A_3) + A_4 \cos(t) + A_5 \sin(t)$$

$$y' = -A_4 \sin(t) + A_5 \cos(t)$$

$$y'' = -A_4 \cos(t) + A_5 \sin(t)$$

$$y''' = A_4 \sin(t) + A_5 \cos(t)$$

$$A_4 \sin(t) - A_5 \cos(t) + A_4 \sin(t) + A_5 \cos(t) = \sin(t)$$

$$-A_5 = 0$$

$$2A_4 = 1$$

$$A_4 = \frac{1}{2}, A_5 = 0$$