Calculus and Linear Algebra II
Real and Unitary Matrices

Getuar Rexhepi

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1 Real Matrices

Real Symmetric matrices have only real eigenvalues

Ex. For

$$f : \mathbb{R} \to R$$

$\nabla f(a) = 0$ stationary points
The Hessian

$$Hf = \left[\frac{\partial f(a)}{\partial^2 x_i \partial x_j} \right]$$

$F \in C^2 \implies$ Hessian is a real symmetric matrix
$V_1...V_2$ the orthonormal basis of eigenvectors $h = \sum_{i=1} C_i V_1$

$$< H, h_F(a) \sum c_i V_i >=< H, \sum c_i h_F(a) V_i >=$$

$$< \sum_{j=1} C_j V_j, \sum \lambda_i c_i V_i >= \sum_{j=1} \sum_{i=1} C_j C_i \lambda_i < V_j, V_i >$$

Kronecker Delta:

$$< V_j, V_i >= \delta_{ji} = 0, j \neq i, 1, j = i$$

If all $\lambda_i > 0$, then:

$$< h, Hf(a)h >> 0 \forall h$$

If all $\lambda_i < 0$, then:

$$< h, Hf(a)h >< 0 \forall h$$

If some $\lambda_i > 0$: There exist h for which $< h, Hf(a)h >> 0$
If some $\lambda_i = 0$: There exist h for which $< h, Hf(a)h >= 0$
If some $\lambda_i < 0$: There exist h for which $< h, Hf(a)h >< 0$
For the above we have a saddle point.
If some $\lambda_i = 0$ the test is inconclusive.

2 Unitary Matrices

Recall : Normal matrices A can be diagonalized with orthonormal eigenvectors :

$$\bar{v_1}, ...\bar{v_n}$$

with $<\bar{v_i}, \bar{v_j}>= \delta_{ij}$

Diagonalization $\Lambda = V^{-1}AV$

$$V = \begin{bmatrix} \bar{v_1} \\ \bar{v_2} \\ \dots \\ \bar{v_n} \end{bmatrix}$$

$$V^+V = \begin{bmatrix} \bar{v_1} \\ \bar{v_2} \\ \dots \\ \bar{v_n} \end{bmatrix} \begin{bmatrix} \bar{v_1} | \bar{v_2} | \dots | \bar{v_n} \end{bmatrix} = \begin{bmatrix} <\bar{v_1}, \bar{v_1}><\bar{v_1}, \bar{v_2}...><\bar{v_1}, \bar{v_n}> \\ <\bar{v_2}, \bar{v_1} <\bar{v_2}, \bar{v_2}...><\bar{v_2}, \bar{v_n}> \\ <\bar{v_n}, \bar{v_1}> ..., ..., ...><\bar{v_m}, \bar{v_n}> \end{bmatrix}$$

Definition: An nxn matrix U is called unitary if $U^{-1} = U^+$
Properties :

1. Unitary matrices are those that diagonalize normal matrices

2. They preserve lengths

   $$|Ux|^2 =< Ux, Ux >=< x, U^+Ux >= id =< x, x >= |x|^2$$

3. The preserve angles

   $$< Ux, Uy >=< x, U^+Uy >=< x, y >$$

Such transformations are called isometries (preserve geometry)

4. U is normal

   $$U^+U = I = UU^+ \implies$$

diagonalizable with an orthonormal basis of eigenvectors

5.

   $$1 = detI = detU^+detU = detU^*detU = |detU|^2 \implies |detU| = 1$$

where we have : $detU = e^{i\phi}$, $\phi \in [0, 2\pi]$

6. Eigenvalues ? $\bar{x}$ - eigenvector, λ -eigenvalue

   $$|\lambda|^2|x|^2 = |\lambda x|^2 =< \lambda x, \lambda x >=< Ux, Ux >=< x, x >= |x|^2 \neq 0$$

By this :

$$|\lambda|^2 \implies |\lambda| = 1$$

A matrix U is unitary $\iff$ it can be written as e^{iH}, with Hermittian H.

$$H = \begin{bmatrix} \lambda_1, ..., ..., 0 \\ 0, \lambda_2, ..., 0 \\ 0, 0, ..., \lambda_n \end{bmatrix}$$

$$e^{iH} = \begin{bmatrix} e^{i\phi_1}, .., ..., 0 \\ 0, ..., e^{i\phi_2}, ...0 \\ 0, ..., ..., e^{i\phi_n} \end{bmatrix} = \begin{bmatrix} \phi_1, .., ..., 0 \\ 0, ..., \phi_2, ...0 \\ 0, ..., ..., \phi_n \end{bmatrix}$$

Orthonormal matrices:

Def: A real nxn matrix Q is called Orthogonal if $Q^{-1} = Q^{+}$. Orthogonal matrices diagonalize real symmetric matrices.

$$detQ = \pm 1$$

all eigen values are ± 1

Orthonormal matrices represent rotations and reflections:

$$detQ = 1$$

Q - orientation preserving

$$detQ = -1$$

Q - orientation reversing.

Orientation reversing = analogy to clockwise and counterclockwise

Non-Diagonalizable matrices

$\lambda_i : geom.multiplicity < alegbraic.multiplicity$

Take eigenvectors x:

$$Ax = \lambda x$$

generalized Eigenvectors

$$Ay_1 = \lambda y_1 + x$$

$$Ay_2 = \lambda y_2 + y_1$$

$$...$$

Jordan blocks of order n = number of eigenvalues in a diagonal

$$\begin{bmatrix} \lambda_1, .., ..., 0 \\ 0, ..., \lambda_2, ...0 \\ 0, ..., ..., \lambda_3 \\ 0, ..., ..., ..., \lambda_4 \\ .... \end{bmatrix}$$