Calculus and Linear Algebra II
Matrix Decomposition

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May 3, 2023

LU Decomposition

$$A\bar{x} = \bar{b}$$

A - n x n matrix and b $\in \mathbb{R}^n$.
A is invertible (det $\neq 0$) $\implies \exists$ unique solution

$$\bar{x} = A^{-1}\bar{b}$$

$$T_1 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} T_2 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \lambda & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} T_3 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \lambda & & & \\ & & & 1 & & \\ & & \lambda & & 1 & \\ & & & & & 1 \end{bmatrix}$$

Gauss Elimination
We bring the system $A\bar{x} = \bar{b}$ to the upper triangular form (forward elimination):

$$A = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

To bring to this form, we have:

$$M_N, M_{N-1} \dots M_1$$

$$A\bar{x} = M_N, M_{N-1} \dots M_1\bar{b}$$

$M_i = \{T1, T2, T3\}$ Upper triangle : M A = U, $A = M^{-1}U$

Assume that all T1 and T2 operations are done:

$$T_3 = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \lambda & & & \\ & & & 1 & & \\ & & \lambda & & 1 & \\ & & & & & 1 \end{bmatrix} T_3^{-1} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \lambda & & & \\ & & & 1 & & \\ & & -\lambda & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$\lambda rowi + rowj$ $\qquad\qquad\qquad\qquad$ $-\lambda rowi + rowj$

Theorem:

Let A be an invertible n x n matrix. Then we can decompose PA = LU, where L is the lower triangle, U- upper triangular and P is a matrix that permutes A.

Example 1

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Gaussian elemination

$$T = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$TA = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$$

$$A = T^{-1} \cdot \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$$

Now we have:

$$L = \begin{bmatrix} L11 & 0 \\ L21 & L22 \end{bmatrix} U = \begin{bmatrix} U11 & U12 \\ 0 & U22 \end{bmatrix}$$

$$LU = \begin{bmatrix} L11U11 & L11U12 \\ L21U11 & L21U12 + L22U22 \end{bmatrix} = A$$

L11 U11 = 1

L11 U12 = 3

L21 U11 = 2

L21 U12+L22U22 = 4

From which we get:

L11 = L22 = 1, U11 = 1, U12 = 3, L21 = 2 , $2 \cdot 3 + U22 = 4$, U22 =-2

$$L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} U = \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$$

Applications

Solving linear systems of equations:

$$A\bar{x} = \bar{b}$$

A = L U

$$L \cdot U\bar{x} = \bar{b} \implies U\bar{x} = \bar{y}$$

Solving we have:

$$L\bar{y} = \bar{b} \implies \bar{y}$$

Then solve

$$U\bar{x} = \bar{y}$$

Gram-Schmidt orthonormalization

$\mathbb{R}^n$, basis {U1,U2,...U3}
How to construct an orthonormal basis:
{V1,V2,...V3} ?

Step 1:

$\bar{U1} = \bar{V1}$

$$\bar{V1} = \frac{\bar{U1}}{||\bar{U1}||}$$

Step 2:

$$\bar{V2} = \alpha\bar{U2} + \beta\bar{V1}$$
$$< \bar{V1}, \bar{V2} >= 0$$
$$< \bar{V1}, \alpha\bar{U2} + \beta\bar{V1} >= \alpha < \bar{V1}, \bar{U2} > +\beta < \bar{V1}, \bar{V1} >= 0$$

$$\beta = -\alpha < \bar{V1}, \bar{U2} >$$
$$\alpha = 1 \implies \beta = - < \bar{V1}, \bar{U2} >$$

$$\bar{V2} = \bar{U2} - < \bar{V1}, \bar{U2} > \bar{V1}$$

repeat:
Step j:

$$\bar{V_j} = \bar{U_j} - \sum_{k=1}^{j-1} < \bar{U_j}, \bar{V_k} > \bar{V_k}$$

span $\{\bar{U_1}...\bar{U_n}\} = span\{\bar{V}1, ...\bar{V_n}\}$

QR Decomposition

A - n x n matrix. A = $(U_1, U_2, ...U_n)$ where U_i - linearly independent - basis. Gram - Schmidt - Q = $(V_1, ...V_2)$, where Q is orthogonal matirx. $(Q^{-1} = Q^T)$ and $V_i - orthonormal$

A = QR

$$\bar{U_1} =< \bar{V}1, \bar{U}1 > \bar{V}1$$
$$\bar{U_2} =< \bar{V}1, \bar{U}2 > \bar{V}1 + < \bar{V}2, \bar{U}2 > \bar{V}2$$
$$...$$
$$\bar{U_n} =< \bar{V}1, \bar{U}n > \bar{V}1 < \bar{V}2, \bar{U}n > \bar{V}2 + \cdots + < \bar{V}n, \bar{U}n > \bar{V}n$$

Theorem:

Any real n x n Matrix can be written as A = QR, where Q is orthogonal and R is upper triangular.

If we want to solve a system of equations:

$$A\bar{x} = \bar{b} \implies QR\bar{x} = \bar{b}$$
$$Q^T QR\bar{x} = Q^T \bar{b}$$
$$R\bar{x} = Q^T \bar{b}$$

Remember that Q is orthongonal such that $Q^T = Q^{-1}$