Signals and Systems

Lecture 12: CT Feedback and Control

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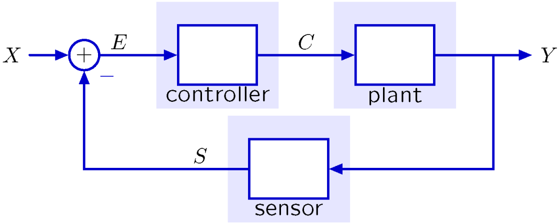
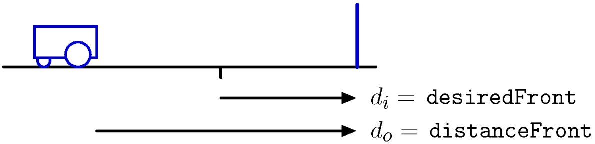
Partly adapted from the materials provided on the MIT OpenCourseWare

Feedback and Control

Feedback: simple, elegant, and robust framework for control.



Last time: robotic driving.



Feedback and Control

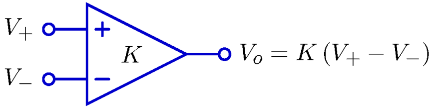
This week: using feedback to enhance performance.

Examples:

- increasing speed and bandwidth
- controlling position instead of speed
- reducing sensitivity to parameter variation
- reducing distortion
- stabilizing unstable systems
  - magnetic levitation
  - inverted pendulum

Op-amps

An "ideal" op-amp has many desireable characteristics.

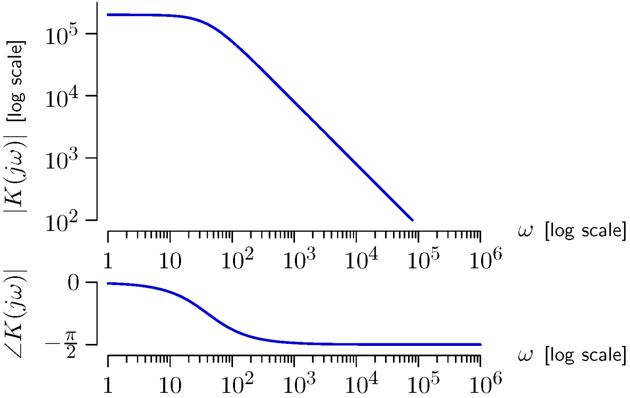


- high speed
- large bandwidth
- high input impedance
- low output impedance
- ...

It is difficult to build a circuit with all of these features.

Op-Amp

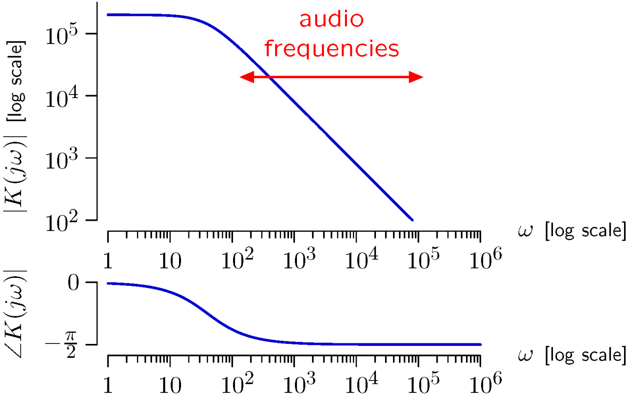
The gain of an op-amp depends on frequency.



Frequency dependence of LM741 op-amp.

Op-Amp

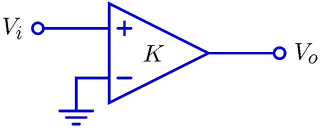
Low-gain at high frequencies limits applications.



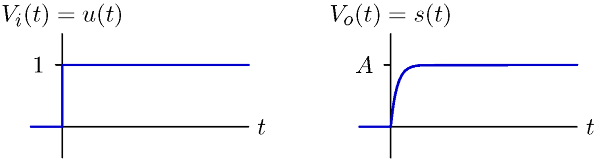
Unacceptable frequency response for an audio amplifier.

Op-Amp

An ideal op-amp has fast time response.

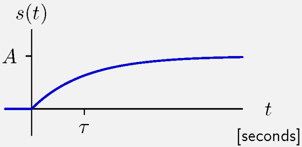


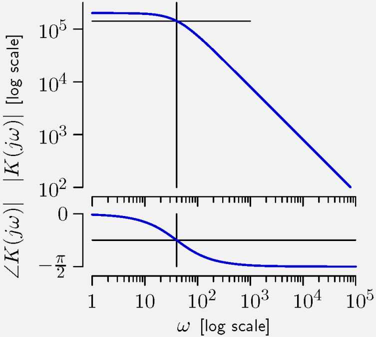
Step response:



Check Yourself

Determine τ for the unit-step response $s(t)$ of an LM741.





1. 40 s
2. $\frac{40}{2\pi}$ s
3. $\frac{1}{40}$ s
4. $\frac{2\pi}{40}$ s
5. $\frac{1}{2\pi\times 40}$ s

0. none of the above

Check Yourself

Determine the step response of an LM741.

System function:

$$K(s) = \frac{\alpha K_0}{s+\alpha}$$

Impulse response:

$$h(t) = \alpha K_0 e^{-\alpha t} u(t)$$

Step response:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = \int_{0}^{t} \alpha K_0 e^{-\alpha\tau} d\tau = \left.\frac{\alpha K_0 e^{-\alpha\tau}}{-\alpha}\right|_0^t = K_0(1-e^{-\alpha t})u(t)$$

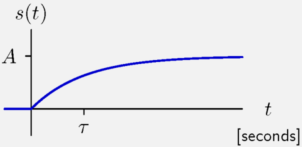
Parameters:

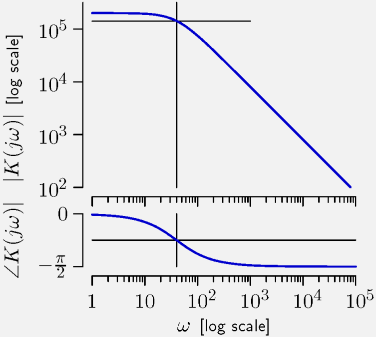
$$A = K_0 = 2\times 10^5$$

$$\tau = \frac{1}{\alpha} = \frac{1}{40}\,\mathrm{s}$$

Check Yourself

Determine τ for the unit-step response $s(t)$ of an LM741.





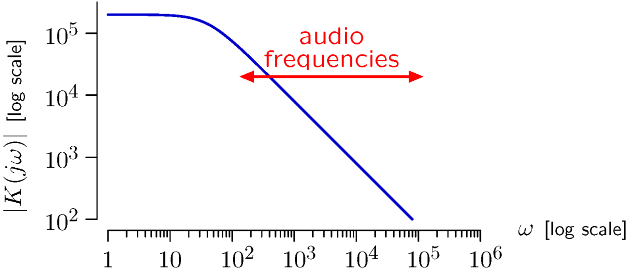
1. 40 s
2. $\frac{40}{2\pi}$ s
3. $\frac{1}{40}$ s
4. $\frac{2\pi}{40}$ s
5. $\frac{1}{2\pi\times 40}$ s

0. none of the above

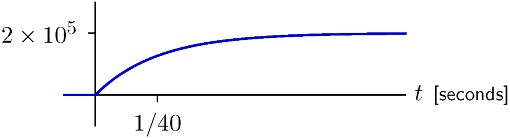
Op-Amp

Performance parameters for real op-amps fall short of the ideal.

Frequency Response: high gain but only at low frequencies.



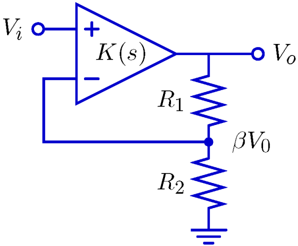
Step Response: slow by electronic standards.



Op-Amp

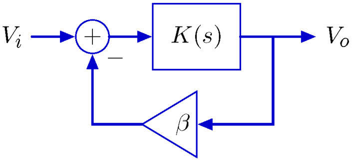
We can use feedback to improve performance of op-amps.

circuit



$$V_- = \beta V_o = \left(\frac{R_2}{R_1 + R_2}\right) V_o$$

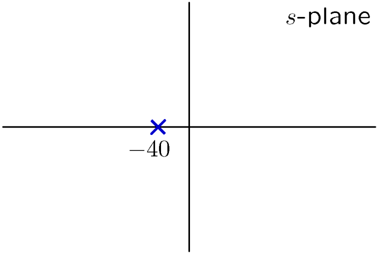
6.003 model



$$\frac{V_o}{V_i} = \frac{K(s)}{1 + \beta K(s)}$$

Dominant Pole

Op-amps are designed to have a dominant pole at low frequencies:
$\rightarrow$ simplifies the application of feedback.

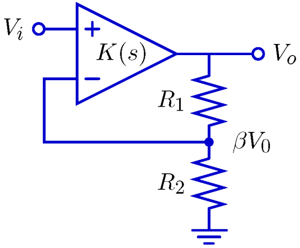


$$\alpha = 40\,\text{rad/s} = \frac{40\,\text{rad/s}}{2\pi\,\text{rad/cycle}} \approx 6.4\,\text{Hz}$$

Improving Performance

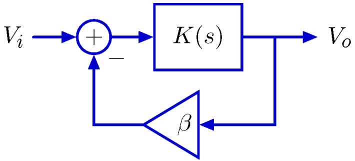
Using feedback to improve performance parameters.

circuit



$$V_- = \beta V_o = \left(\frac{R_2}{R_1+R_2}\right) V_o$$

6.003 model



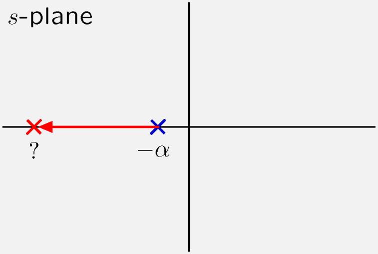
$$\frac{V_o}{V_i} = \frac{K(s)}{1+\beta K(s)}$$

$$= \frac{\frac{\alpha K_0}{s+\alpha}}{1+\beta\frac{\alpha K_0}{s+\alpha}}$$

$$= \frac{\alpha K_0}{s+\alpha+\alpha\beta K_0}$$

Check Yourself

What is the most negative value of the closed-loop pole that can be achieved with feedback?

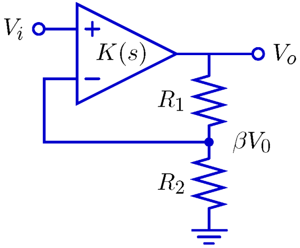


1. $-\alpha(1+\beta)$
2. $-\alpha(1+\beta K_0)$
3. $-\alpha(1+K_0)$
4. $-\infty$
5. none of the above

Improving Performance

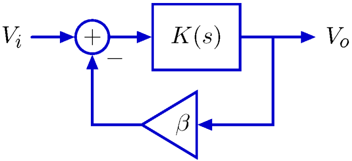
Using feedback to improve performance parameters.

circuit



$$V_- = \beta V_o = \left(\frac{R_2}{R_1 + R_2}\right) V_o$$

6.003 model



$$\frac{V_o}{V_i} = \frac{K(s)}{1 + \beta K(s)}$$

$$= \frac{\frac{\alpha K_0}{s+\alpha}}{1 + \beta \frac{\alpha K_0}{s+\alpha}}$$

$$= \frac{\alpha K_0}{s + \alpha + \alpha\beta K_0}$$

Check Yourself

What is the most negative value of the closed-loop pole that can be achieved with feedback?

Open loop system function: $\frac{\alpha K_0}{s+\alpha}$

$\rightarrow$ pole: $s = -\alpha$.

Closed-loop system function: $\frac{\alpha K_0}{s+\alpha+\alpha\beta K_0}$

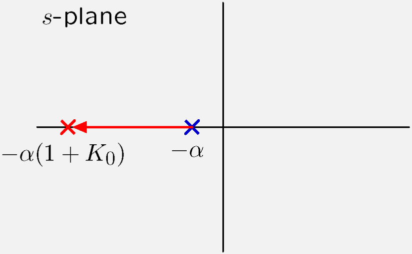
$\rightarrow$ pole: $s = -\alpha(1+\beta K_0)$.

The feedback constant is $0 \leq \beta \leq 1$.

$\rightarrow$ most negative value of the closed-loop pole is $s = -\alpha(1+K_0)$.

Check Yourself

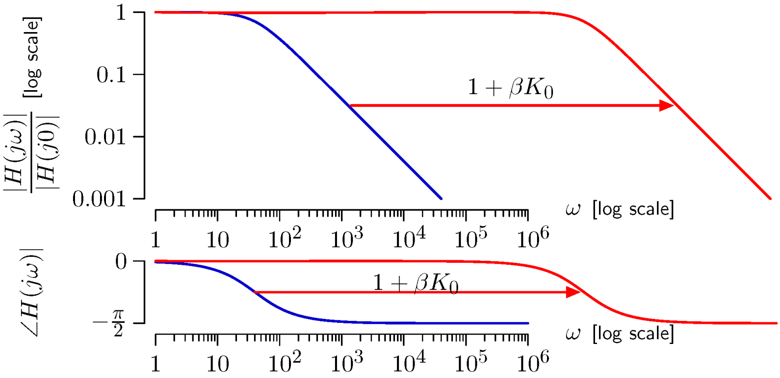
What is the most negative value of the closed-loop pole that can be achieved with feedback? 3



1. $-\alpha(1+\beta)$
2. $-\alpha(1+\beta K_0)$
3. $-\alpha(1+K_0)$
4. $-\infty$
5. none of the above

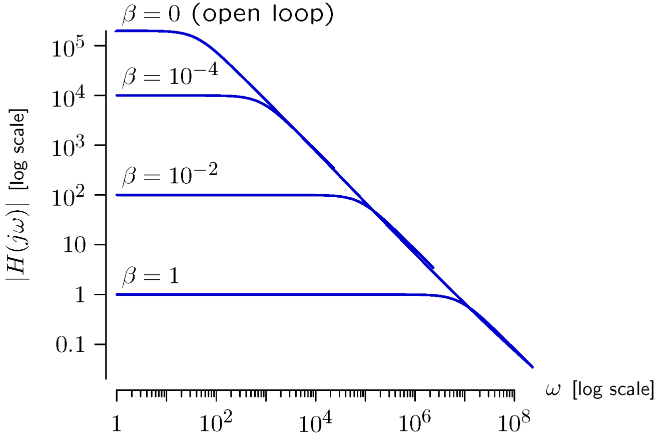
Improving Performance

Feedback extends frequency response by a factor of $1+\beta K_0$ ($K_0 = 2 \times 10^5$).



Improving Performance

Feedback produces higher bandwidths by **reducing** the gain at low frequencies. It trades gain for bandwidth.

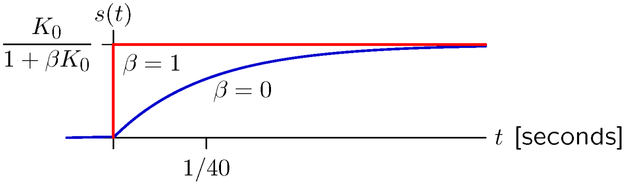


Improving Performance

Feedback makes the time response faster by a factor of $1+\beta K_0$ ($K_0 = 2\times 10^5$).

Step response

$$s(t) = \frac{K_0}{1+\beta K_0}(1-e^{-\alpha(1+\beta K_0)t})u(t)$$

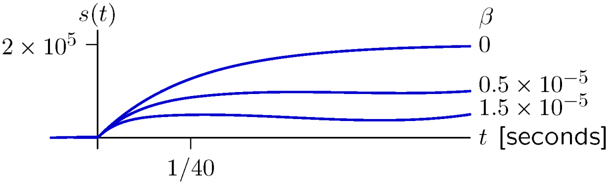


Improving Performance

Feedback produces faster responses by **reducing** the final value of the step response. It trades gain for speed.

Step response

$$s(t) = \frac{K_0}{1+\beta K_0}(1 - e^{-\alpha(1+\beta K_0)t})u(t)$$



The maximum rate of voltage change $\left.\frac{ds(t)}{dt}\right|_{t=0+}$ is not increased.

Improving Performance

Feedback improves performance parameters of op-amp circuits.

- can extend frequency response
- can increase speed

Performance enhancements are achieved through a reduction of gain.

Motor Controller

We wish to build a robot arm (actually its elbow). The input should be voltage $v(t)$, and the output should be the elbow angle $\theta(t)$.

$$v(t) \longrightarrow \boxed{\text{robotic arm}} \longrightarrow \theta(t) \propto v(t)$$

We wish to build the robot arm with a DC motor.

$$v(t) \longrightarrow \boxed{\text{DC motor}} \longrightarrow \theta(t)$$

This problem is similar to the head-turning servo in 6.01 !

Check Yourself

What is the relation between $v(t)$ and $\theta(t)$ for a DC motor?

$$v(t) \longrightarrow \boxed{\text{DC motor}} \longrightarrow \theta(t)$$

1. $\theta(t) \propto v(t)$
2. $\cos\theta(t) \propto v(t)$
3. $\theta(t) \propto \dot{v}(t)$
4. $\cos\theta(t) \propto \dot{v}(t)$
5. none of the above

Check Yourself

What is the relation between $v(t)$ and $\theta(t)$ for a DC motor?

To first order, the rotational speed $\dot{\theta}(t)$ of a DC motor is proportional to the input voltage $v(t)$.

$$v(t) \longrightarrow \boxed{\text{DC motor}} \longrightarrow \theta(t)$$

First-order model: integrator

$$V \longrightarrow \boxed{\gamma\mathcal{A}} \longrightarrow \Theta$$

Check Yourself

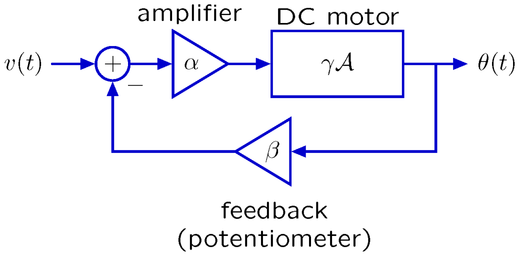
What is the relation between $v(t)$ and $\theta(t)$ for a DC motor?

$$v(t) \longrightarrow \boxed{\text{DC motor}} \longrightarrow \theta(t)$$

1. $\theta(t) \propto v(t)$
2. $\cos\theta(t) \propto v(t)$
3. $\theta(t) \propto \dot{v}(t)$
4. $\cos\theta(t) \propto \dot{v}(t)$
5. none of the above

Motor Controller

Use proportional feedback to control the angle of the motor's shaft.

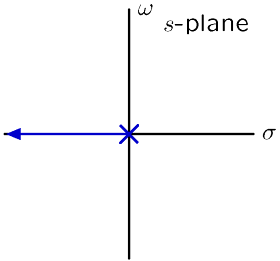


$$\frac{\Theta}{V} = \frac{\alpha\gamma\mathcal{A}}{1+\alpha\beta\gamma\mathcal{A}} = \frac{\alpha\gamma\frac{1}{s}}{1+\alpha\beta\gamma\frac{1}{s}} = \frac{\alpha\gamma}{s+\alpha\beta\gamma}$$

Motor Controller

The closed loop system has a single pole at $s = -\alpha\beta\gamma$.

$$\frac{\Theta}{V} = \frac{\alpha\gamma}{s + \alpha\beta\gamma}$$



As α increases, the closed-loop pole becomes increasingly negative.

Motor Controller

Find the impulse and step response.

The system function is

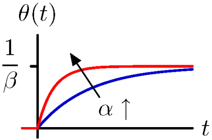
$$\frac{\Theta}{V} = \frac{\alpha\gamma}{s + \alpha\beta\gamma}\,.$$

The impulse response is

$$h(t) = \alpha\gamma e^{-\alpha\beta\gamma t} u(t)$$

and the step response is therefore

$$s(t) = \frac{1}{\beta}\left(1 - e^{-\alpha\beta\gamma t}\right) u(t)\,.$$



The response is faster for larger values of α.

Try it: Demo.

Motor Controller

The speed of a DC motor does not change instantly if the voltage is stepped. There is lag due to rotational inertia.

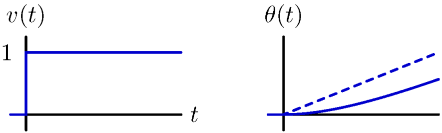
First-order model
integrator

$$V \longrightarrow \boxed{\gamma\mathcal{A}} \longrightarrow \Theta_1$$

Second-order model
integrator with lag

$$V \longrightarrow \boxed{\gamma\mathcal{A}\left(\frac{p\mathcal{A}}{1+p\mathcal{A}}\right)} \longrightarrow \Theta_2$$

Step response of the models:

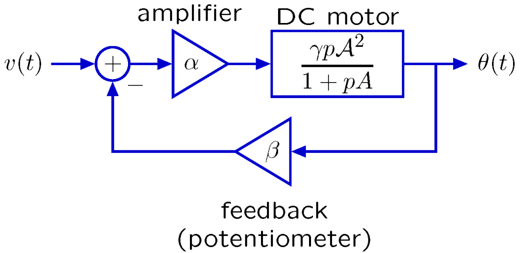


$$\theta_1(t) = \gamma t u(t)$$

$$\theta_2(t) = \big(\gamma t - \tfrac{\gamma}{p}(1 - e^{-pt})\big)u(t)$$

Motor Controller

Analyze second-order model.

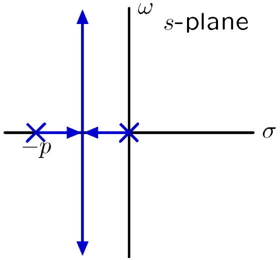


$$\frac{\Theta}{V} = \frac{\frac{\alpha\gamma p\mathcal{A}^2}{1+pA}}{1+\frac{\alpha\beta\gamma p\mathcal{A}^2}{1+pA}} = \frac{\alpha\gamma p\mathcal{A}^2}{1+pA+\alpha\beta\gamma p\mathcal{A}^2} = \frac{\alpha\gamma p}{s^2+ps+\alpha\beta\gamma p}$$

$$s = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - \alpha\beta\gamma p}$$

Motor Controller

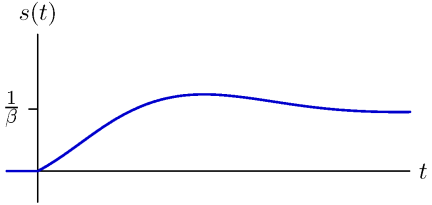
For second-order model, increasing α causes the poles at 0 and $-p$ to approach each other, collide at $s = -p/2$, then split into two poles with imaginary parts.



Increasing the gain α does not increase speed of convergence.

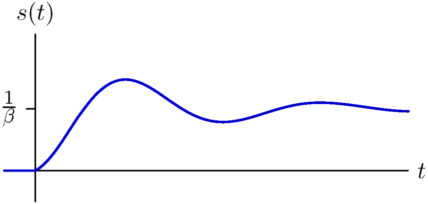
Motor Controller

Step response.



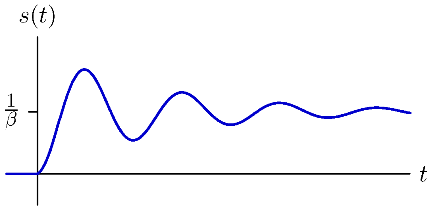
Motor Controller

Step response.



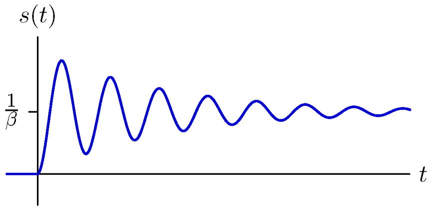
Motor Controller

Step response.



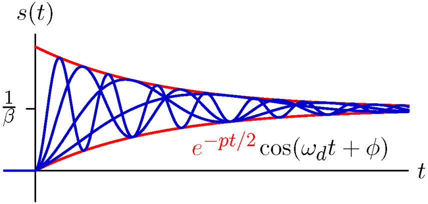
Motor Controller

Step response.



Motor Controller

Step response.



Feedback and Control

Using feedback to enhance performance.

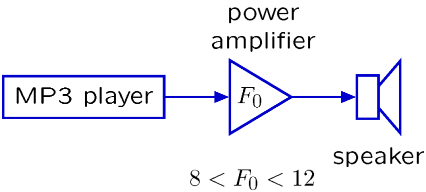
Examples:

- improve performance of an op amp circuit.
- control position of a motor.
- reduce sensitivity to unwanted parameter variation.
- reduce distortions.
- stabilize unstable systems
  - magnetic levitation
  - inverted pendulum

Feedback and Control

Reducing sensitivity to unwanted parameter variation.

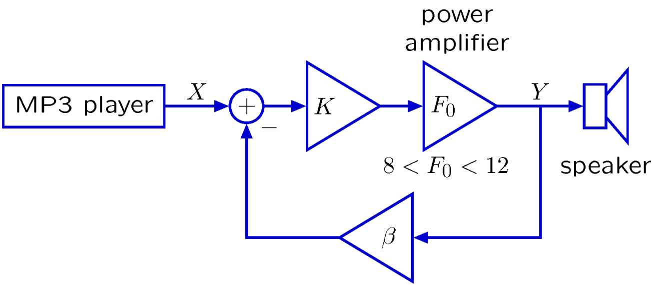
Example: power amplifier



Changes in F_0 (due to changes in temperature, for example) lead to undesired changes in sound level.

Feedback and Control

Feedback can be used to compensate for parameter variation.



$$H(s) = \frac{KF_0}{1 + \beta K F_0}$$

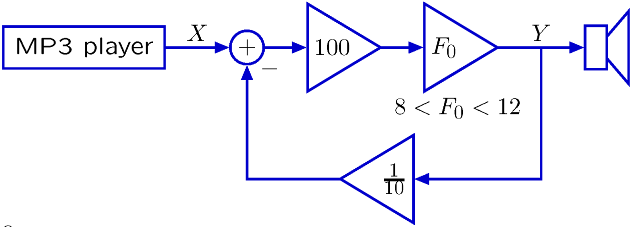
If K is made large, so that $\beta K F_0 \gg 1$, then

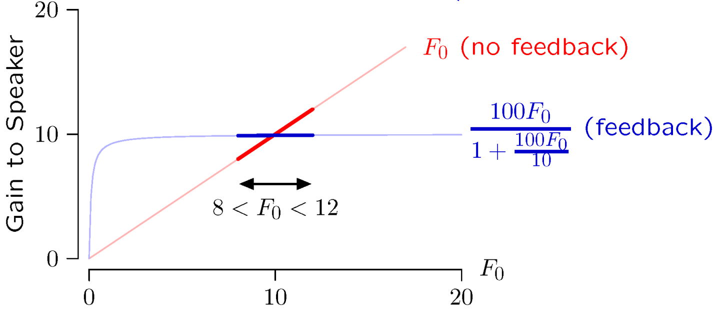
$$H(s) \approx \frac{1}{\beta}$$

independent of K or F_0!

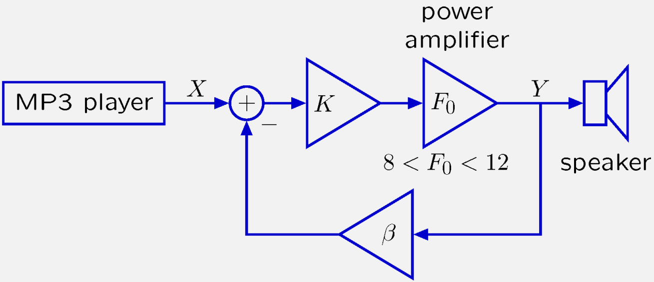
Feedback and Control

Feedback reduces the change in gain due to change in F_0.





Check Yourself



Feedback greatly reduces sensitivity to variations in K or F_0.

$$\lim_{K\to\infty} H(s) = \frac{KF_0}{1+\beta KF_0} \to \frac{1}{\beta}$$

What about variations in β? Aren't those important?

Check Yourself

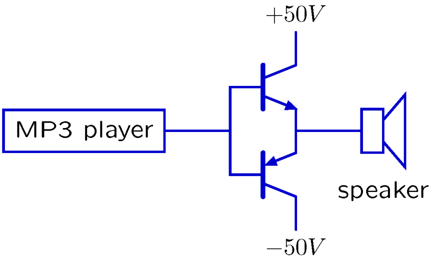
What about variations in β? Aren't those important?

The value of β is typically determined with resistors, whose values are quite stable (compared to semiconductor devices).

Crossover Distortion

Feedback can compensate for parameter variation even when the variation occurs rapidly.

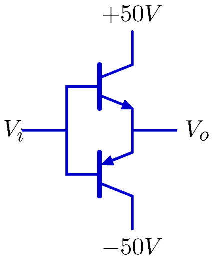
Example: using transistors to amplify power.

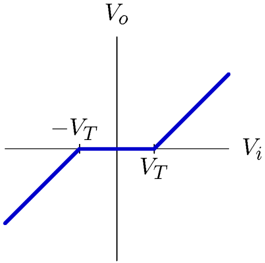


Crossover Distortion

This circuit introduces "crossover distortion."

For the upper transistor to conduct, $V_i - V_o > V_T$.
For the lower transistor to conduct, $V_i - V_o < -V_T$.

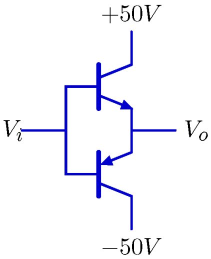


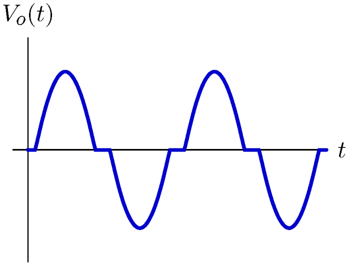


Crossover Distortion

Crossover distortion can have dramatic effects.

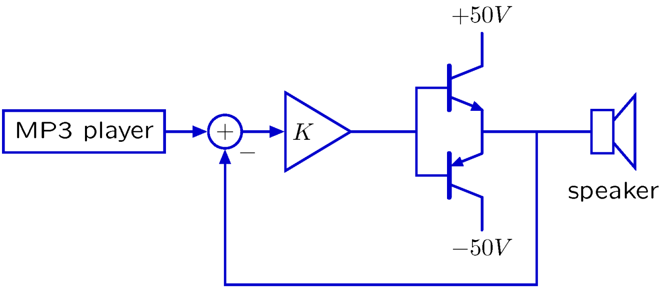
Example: crossover distortion when the input is $V_i(t) = B\sin(\omega_0 t)$.





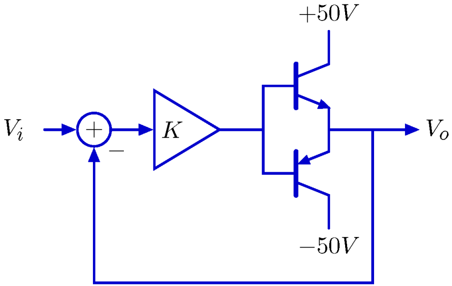
Crossover Distortion

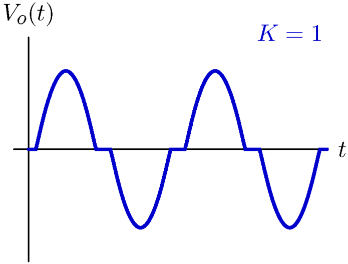
Feedback can reduce the effects of crossover distortion.



Crossover Distortion

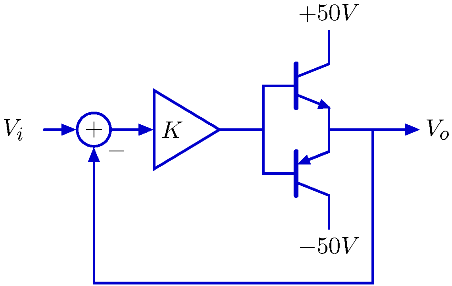
When K is small, feedback has little effect on crossover distortion.

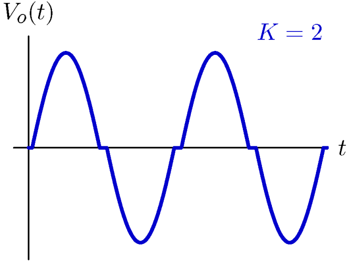




Crossover Distortion

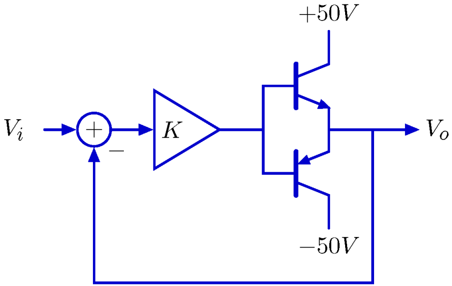
As K increases, feedback reduces crossover distortion.

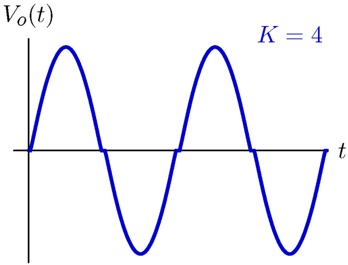




Crossover Distortion

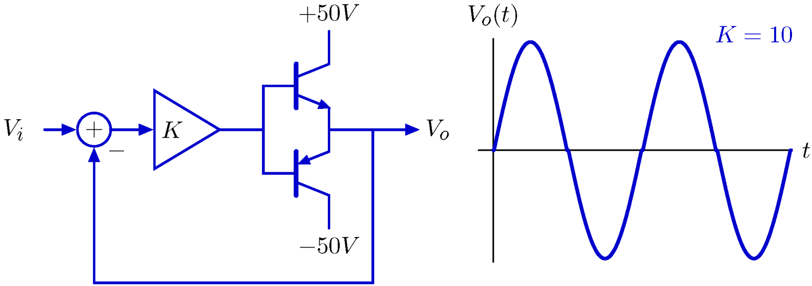
As K increases, feedback reduces crossover distortion.





Crossover Distortion

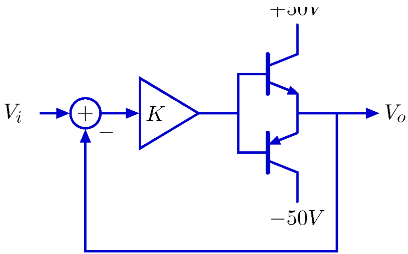
As K increases, feedback reduces crossover distortion.

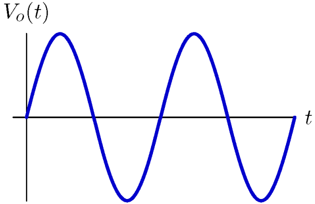


Crossover Distortion

Demo

- original
- no feedback
- $K = 2$
- $K = 4$
- $K = 8$
- $K = 16$
- original





J.S. Bach, Sonata No. 1 in G minor Mvmt. IV. Presto
Nathan Milstein, violin

Feedback and Control

Using feedback to enhance performance.

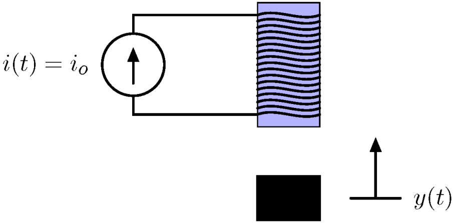
Examples:

- improve performance of an op amp circuit.
- control position of a motor.
- reduce sensitivity to unwanted parameter variation.
- reduce distortions.
- stabilize unstable systems
  - magnetic levitation
  - inverted pendulum

Control of Unstable Systems

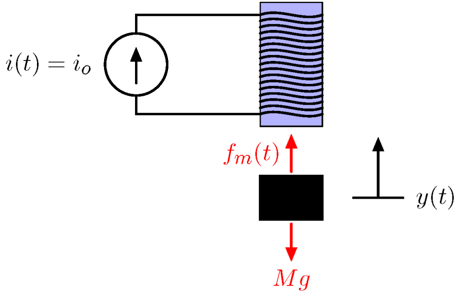
Feedback is useful for controlling **unstable** systems.

Example: Magnetic levitation.



Control of Unstable Systems

Magnetic levitation is unstable.



Equilibrium ($y = 0$): magnetic force $f_m(t)$ is equal to the weight Mg.

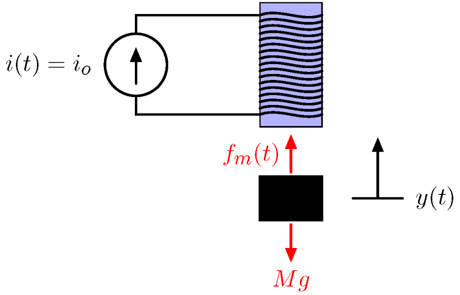
Increase $y \rightarrow$ increased force $\rightarrow$ further increases y.

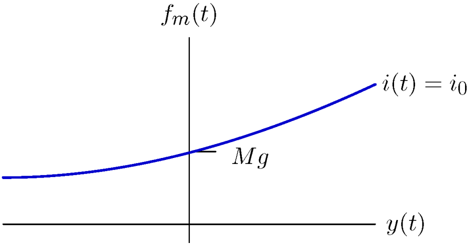
Decrease $y \rightarrow$ decreased force $\rightarrow$ further decreases y.

Positive feedback!

Modeling Magnetic Levitation

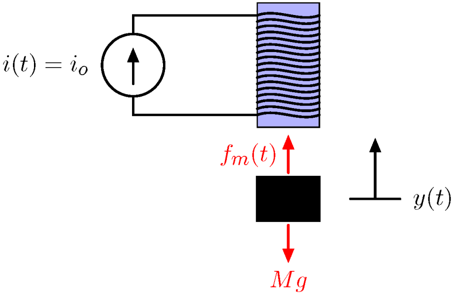
The magnet generates a force that depends on the distance $y(t)$.



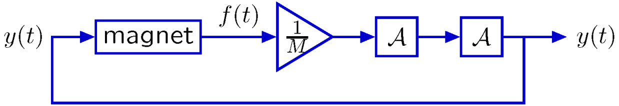


Modeling Magnetic Levitation

The net force accelerates the mass.

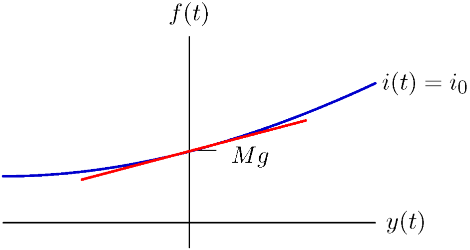


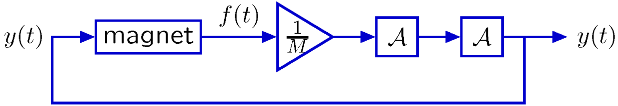
$$f_m(t) - Mg = f(t) = Ma = M\ddot{y}(t)$$



Modeling Magnetic Levitation

Over small distances, magnetic force grows ≈ linearly with distance.

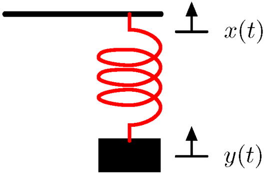


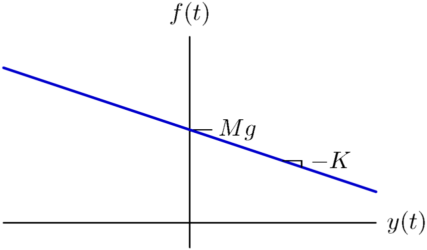


Levitation with a Spring

Relation between force and distance for a spring is opposite in sign.

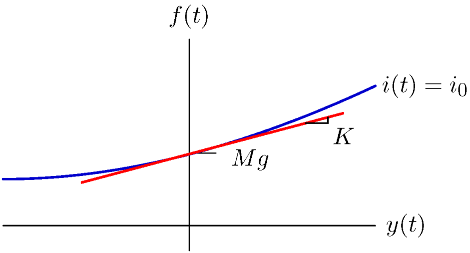
$$F = K\Big(x(t) - y(t)\Big) = M\ddot{y}(t)$$



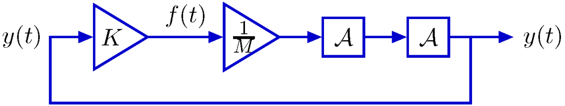


Modeling Magnetic Levitation

Over small distances, magnetic force nearly proportional to distance.



$f(t) \approx Ky(t)$

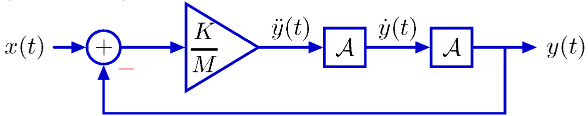


Block Diagrams

Block diagrams for magnetic levitation and spring/mass are similar.

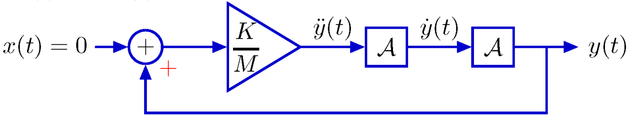
Spring and mass

$$F = K\Big(x(t) - y(t)\Big) = M\ddot{y}(t)$$



Magnetic levitation

$$F = Ky(t) = M\ddot{y}(t)$$

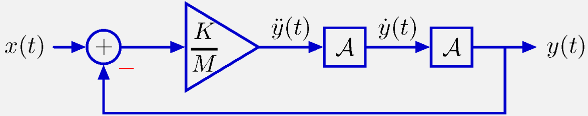


Check Yourself

How do the poles of these two systems differ?

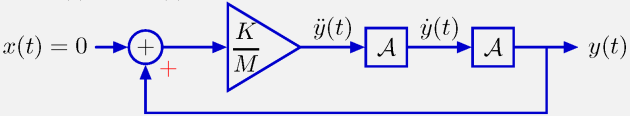
Spring and mass

$$F = K\Big(x(t) - y(t)\Big) = M\ddot{y}(t)$$



Magnetic levitation

$$F = Ky(t) = M\ddot{y}(t)$$



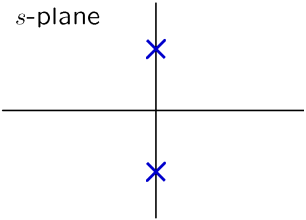
Check Yourself

How do the poles of the two systems differ?

Spring and mass

$$F = K\Big(x(t) - y(t)\Big) = M\ddot{y}(t)$$

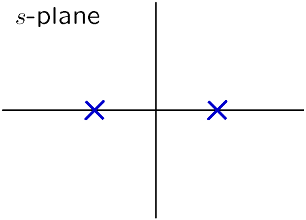
$$\frac{Y}{X} = \frac{\frac{K}{M}}{s^2 + \frac{K}{M}} \;\rightarrow\; s = \pm j\sqrt{\frac{K}{M}}$$



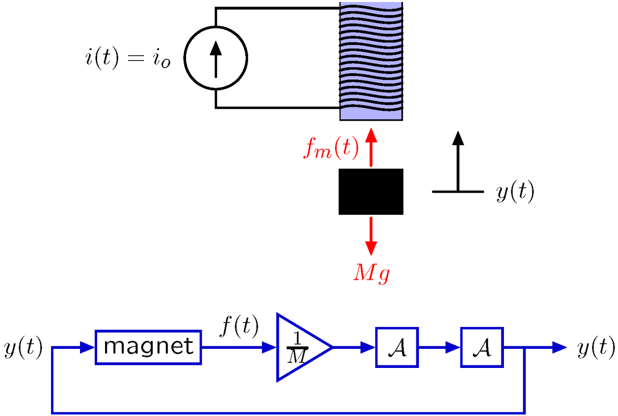
Magnetic levitation

$$F = Ky(t) = M\ddot{y}(t)$$

$$s^2 = \frac{K}{M} \;\rightarrow\; s = \pm\sqrt{\frac{K}{M}}$$

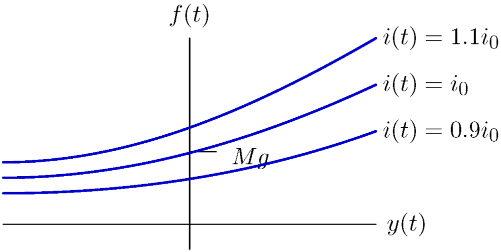


Magnetic Levitation is Unstable



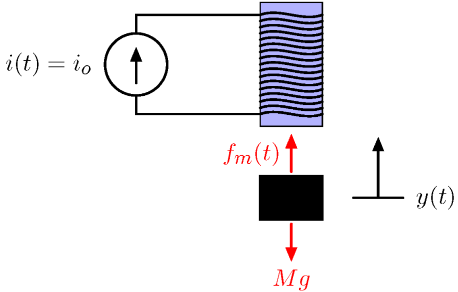
Magnetic Levitation

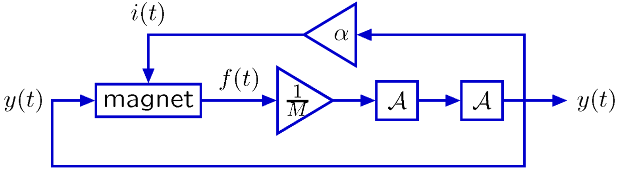
We can stabilize this system by adding an additional feedback loop to control $i(t)$.



Stabilizing Magnetic Levitation

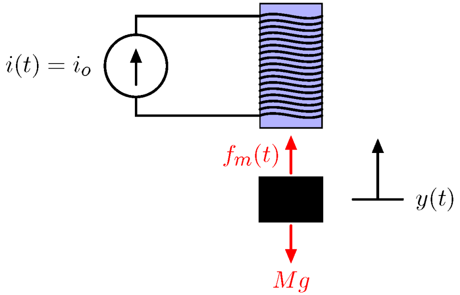
Stabilize magnetic levitation by controlling the magnet current.

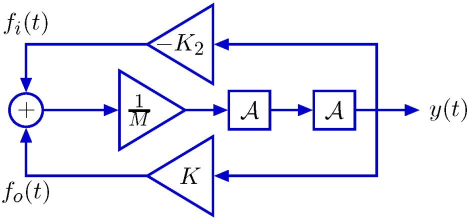




Stabilizing Magnetic Levitation

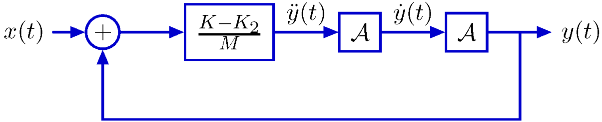
Stabilize magnetic levitation by controlling the magnet current.

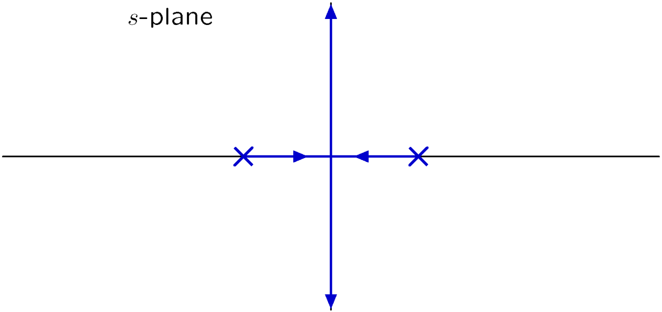




Magnetic Levitation

Increasing K_2 moves poles toward the origin and then onto $j\omega$ axis.

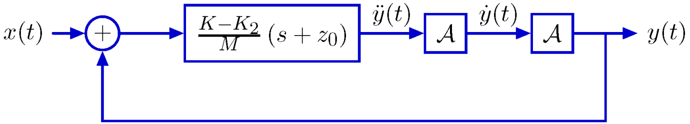


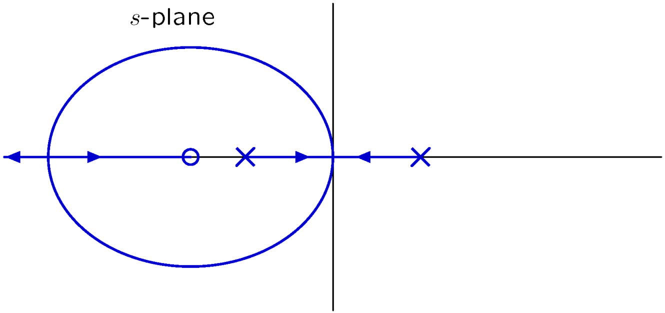


But the poles are still marginally stable.

Magnetic Levitation

Adding a zero makes the poles stable for sufficiently large K_2.

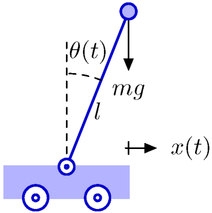




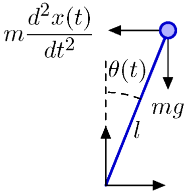
Try it: Demo [designed by Prof. James Roberge].

Inverted Pendulum

As a final example of stabilizing an unstable system, consider an inverted pendulum.



lab frame
(inertial)

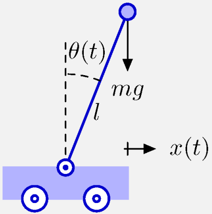


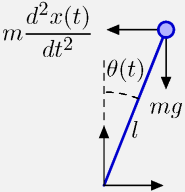
cart frame
(non-inertial)

$$\underbrace{ml^2}_{I}\frac{d^2\theta(t)}{dt^2} = \underbrace{mg}_{\text{force}}\underbrace{l\sin\theta(t)}_{\text{distance}} - \underbrace{m\frac{d^2x(t)}{dt^2}}_{\text{force}}\underbrace{l\cos\theta(t)}_{\text{distance}}$$

Check Yourself: Inverted Pendulum

Where are the poles of this system?

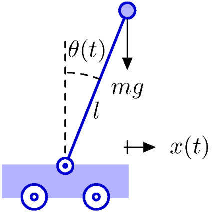


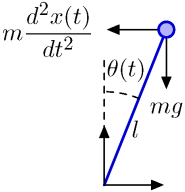


$$ml^2\frac{d^2\theta(t)}{dt^2} = mgl\sin\theta(t) - m\frac{d^2x(t)}{dt^2}l\cos\theta(t)$$

Check Yourself: Inverted Pendulum

Where are the poles of this system?





$$ml^2\frac{d^2\theta(t)}{dt^2} = mgl\sin\theta(t) - m\frac{d^2x(t)}{dt^2}l\cos\theta(t)$$

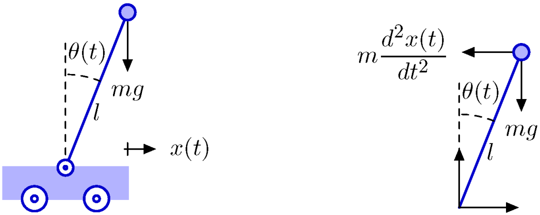
$$ml^2\frac{d^2\theta(t)}{dt^2} - mgl\theta(t) = -ml\frac{d^2x(t)}{dt^2}$$

$$H(s) = \frac{\Theta}{X} = \frac{-mls^2}{ml^2s^2 - mgl} = \frac{-s^2/l}{s^2 - g/l}$$

poles at $s = \pm\sqrt{\frac{g}{l}}$

Inverted Pendulum

This unstable system can be stablized with feedback.



Try it. Demo. [originally designed by Marcel Gaudreau]

Feedback and Control

Using feedback to enhance performance.

Examples:

- improve performance of an op amp circuit.
- control position of a motor.
- reduce sensitivity to unwanted parameter variation.
- reduce distortions.
- stabilize unstable systems
  - magnetic levitation
  - inverted pendulum

Assignments

- Reading Assignment: Chap. 11.0-11.2, Review Chap. 9.7-9.8, 10.7-10.8
- Homework 6