Signals and Systems

Lecture 6: Z Transform

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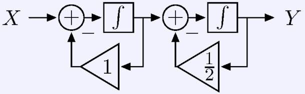
Partly adapted from the materials provided on the MIT OpenCourseWare

Review: Laplace Transform

Relations among representations.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\,u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

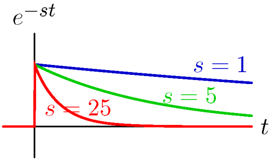
Review: Initial Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$ then

$$x(0^+) = \lim_{s\to\infty} sX(s)\,.$$

Consider $\lim_{s\to\infty} sX(s) = \lim_{s\to\infty} s\int_{-\infty}^{\infty} x(t)e^{-st}dt = \lim_{s\to\infty}\int_0^{\infty} x(t)\,se^{-st}dt.$

As $s \to \infty$ the function e^{-st} shrinks towards 0.



Area under e^{-st} is $\frac{1}{s}$ $\rightarrow$ area under se^{-st} is 1 $\rightarrow$ $\lim_{s\to\infty} se^{-st} = \delta(t)$!

$$\lim_{s\to\infty} sX(s) = \lim_{s\to\infty}\int_0^{\infty} x(t)se^{-st}dt \to \int_0^{\infty} x(t)\delta(t)dt = x(0^+)$$

(the 0^+ arises because the limit is from the right side.)

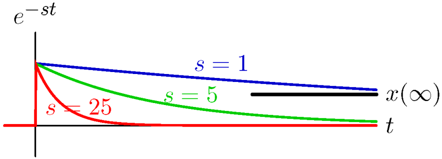
Review: Final Value Theorem

If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \to \infty$

$$x(\infty) = \lim_{s \to 0} sX(s)\,.$$

Consider $\lim_{s\to 0} sX(s) = \lim_{s\to 0} s \int_{-\infty}^{\infty} x(t)e^{-st}dt = \lim_{s\to 0} \int_{0}^{\infty} x(t)\, se^{-st}dt.$

As $s \to 0$ the function e^{-st} flattens out. But again, the area under se^{-st} is always 1.



As $s \to 0$, area under se^{-st} monotonically shifts to higher values of t (e.g., the average value of se^{-st} is $\frac{1}{s}$ which grows as $s \to 0$).

In the limit, $\lim_{s\to 0} sX(s) \to x(\infty)\,.$

Z Transform

Z transform is discrete-time analog of Laplace transform.

Furthermore, you already know about Z transforms
(we just haven't called them Z transforms) !

Example: Fibonacci system

difference equation	$y[n] = x[n] + y[n-1] + y[n-2]$
operator expression	$Y = X + \mathcal{R}Y + \mathcal{R}^2Y$
system functional	$\frac{Y}{X} = \frac{1}{1-\mathcal{R}-\mathcal{R}^2}$
unit-sample response	$h[n]:\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Check Yourself

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What is the relation between system functional and $h[n]$?

Check Yourself

system functional $\qquad \dfrac{Y}{X} = \dfrac{1}{1 - \mathcal{R} - \mathcal{R}^2}$

unit-sample response $\qquad h[n]:\;\; 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

Expand functional in a series:

$$\begin{array}{r|llllll}
 & 1 & +\mathcal{R} & +2\mathcal{R}^2 & +3\mathcal{R}^3 & +5\mathcal{R}^4 & +8\mathcal{R}^5 \quad + \cdots \\ \cline{2-7}
1 - \mathcal{R} - \mathcal{R}^2 & 1 & & & & & \\
 & 1 & -\mathcal{R} & -\mathcal{R}^2 & & & \\ \cline{2-4}
 & & \mathcal{R} & +\mathcal{R}^2 & & & \\
 & & \mathcal{R} & -\mathcal{R}^2 & -\mathcal{R}^3 & & \\ \cline{3-5}
 & & & 2\mathcal{R}^2 & +\mathcal{R}^3 & & \\
 & & & 2\mathcal{R}^2 & -2\mathcal{R}^3 & -2\mathcal{R}^4 & \\ \cline{4-6}
 & & & & 3\mathcal{R}^3 & +2\mathcal{R}^4 & \\
 & & & & 3\mathcal{R}^3 & -3\mathcal{R}^4 & -3\mathcal{R}^5 \\ \cline{5-7}
 & & & & & \cdots &
\end{array}$$

$$\begin{aligned}
\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2} &= 1 + \mathcal{R} + 2\mathcal{R}^2 + 3\mathcal{R}^3 + 5\mathcal{R}^4 + 8\mathcal{R}^5 + 13\mathcal{R}^6 + \cdots \\
&= h[0] + h[1]\mathcal{R} + h[2]\mathcal{R}^2 + h[3]\mathcal{R}^3 + h[4]\mathcal{R}^4 + \cdots \\
&= \sum_n h[n]\mathcal{R}^n
\end{aligned}$$

Check Yourself

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$$\frac{Y}{X} = \sum_n h[n]\mathcal{R}^n$$

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$$\frac{Y}{X} = \sum_n h[n]\mathcal{R}^n$$

What's the relation between $H(z)$ and $h[n]$?

Check Yourself

Series expansion of system functional:

$$\frac{Y}{X} = \sum_n h[n]\mathcal{R}^n$$

Substitute $\mathcal{R} \rightarrow \frac{1}{z}$:

$$H(z) = \sum_n h[n]z^{-n}$$

Check Yourself

Example: Fibonacci system

difference equation $y[n] = x[n] + y[n-1] + y[n-2]$

operator expression $Y = X + \mathcal{R}Y + \mathcal{R}^2 Y$

system functional $\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$

unit-sample response $h[n]:\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

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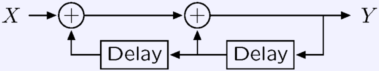
What's the relation between $H(z)$ and $h[n]$?

$$H(z) = \sum_n h[n] z^{-n}$$

Concept Map: Discrete-Time Systems

Multiple representations of DT systems.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Unit-Sample Response

$$h[n]: \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

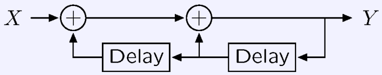
System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1 - z - z^2}$$

Concept Map: Discrete-Time Systems

Relation between Unit-Sample Response and System Functional.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1-\mathcal{R}-\mathcal{R}^2}$$

series

Unit-Sample Response

$h[n]: \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

Difference Equation

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System Function

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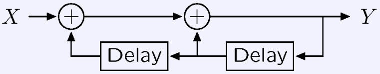
Concept Map: Discrete-Time Systems

Relation between System Functional and System Function.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Unit-Sample Response

$$h[n]: \;\; 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

$$\mathcal{R} \rightarrow \frac{1}{z}$$

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1 - z - z^2}$$

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Relation between Unit-Sample Response and System Function.

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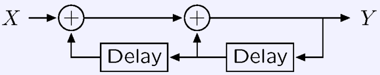

System Functional

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series

Unit-Sample Response

$h[n]:\ \ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$

$\mathcal{R} \rightarrow \frac{1}{z}$

Z transform

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

System Function

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Check Yourself

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unit-sample response $h[n]:\ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

$$\frac{Y}{X} = \sum_n h[n]\mathcal{R}^n$$

$$H(z) = \sum_n h[n]z^{-n} \qquad \leftarrow \text{ Z transform!}$$

Z Transform

Z transform is discrete-time analog of Laplace transform.

Z transform maps a function of discrete time n to a function of z.

$$X(z) = \sum_{n} x[n] z^{-n}$$

There are two important variants:

Unilateral

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

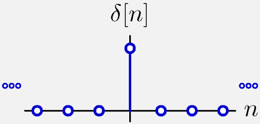
Bilateral

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Differences are analogous to those for the Laplace transform.

Check Yourself

Find the Z transform of the unit-sample signal.



Check Yourself

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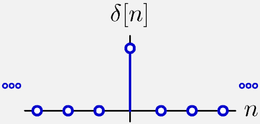


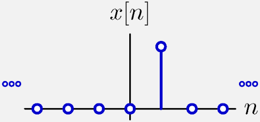
$x[n] = \delta[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[0]z^{0} = 1$$

$\mathcal{Z}\{\delta[n]\} = 1$, analogous to $\mathcal{L}\{\delta(t)\} = 1$.

Check Yourself

Find the Z transform of a delayed unit-sample signal.



Check Yourself

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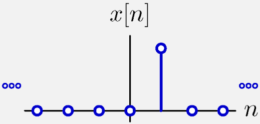


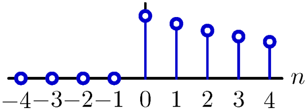
$$x[n] = \delta[n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = x[1]z^{-1} = z^{-1}$$

Z Transforms

Example: Find the Z transform of the following signal.

$$x[n] = \left(\tfrac{7}{8}\right)^n u[n]$$



$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1 - \frac{7}{8} z^{-1}} = \frac{z}{z - \frac{7}{8}}$$

provided $\left|\frac{7}{8} z^{-1}\right| < 1$, i.e., $|z| > \frac{7}{8}$.

Z Transforms

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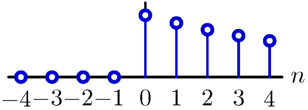

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} u[n] = \sum_{n=0}^{\infty} \left(\frac{7}{8}\right)^n z^{-n} = \frac{1}{1 - \frac{7}{8}z^{-1}} = \frac{z}{z - \frac{7}{8}}$$

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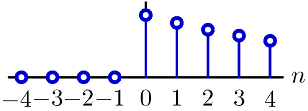
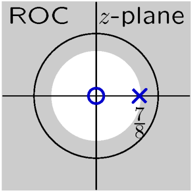
$$x[n] = \left(\tfrac{7}{8}\right)^n u[n]$$



$$\frac{z}{z - \frac{7}{8}}$$



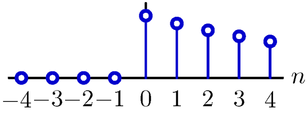
Shape of ROC

Regions of converge for Z transform are delimited by circles.

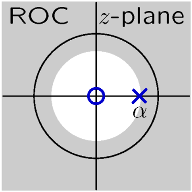
Example: $x[n] = \alpha^n u[n]$

$$
\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} \alpha^n u[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \frac{1}{1-\alpha z^{-1}}\,; \qquad \left|\alpha z^{-1}\right| < 1 \\
&= \frac{z}{z-\alpha}\,; \qquad |z| > |\alpha|
\end{aligned}
$$

$x[n] = \alpha^n u[n]$



$$\frac{z}{z-\alpha}$$



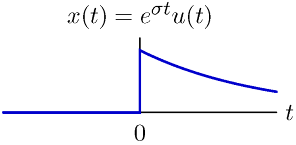
Shape of ROC

Regions of converge for Laplace transform delimited by vertical lines.

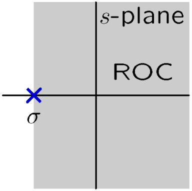
Example: $x(t) = e^{\sigma t}u(t)$

$$X(s) = \int_{-\infty}^{\infty} e^{\sigma t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{\sigma t}e^{-st}dt$$

$$= \frac{1}{s-\sigma}\,; \qquad \mathrm{Re}(s) > \mathrm{Re}(\sigma)$$



$$\frac{1}{s-\sigma}$$



Distinguishing Features of Transforms

Most-important feature of Laplace transforms is the derivative rule:

$$x(t) \leftrightarrow X(s)$$

$$\dot{x}(t) \leftrightarrow sX(s)$$

$\rightarrow$ allows us to use Laplace transforms to solve differential equations.

Similarly, most-important feature of Z transforms is the delay rule:

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1}X(z)$$

$\rightarrow$ allows us to use Z transforms to solve difference equations.

Distinguishing Features of Transforms

Delay property

$$x[n] \leftrightarrow X(z)$$
$$x[n-1] \leftrightarrow z^{-1}X(z)$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

Let $y[n] = x[n-1]$ then

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} x[n-1]z^{-n}$$

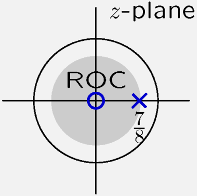
Substitute $m = n - 1$

$$Y(z) = \sum_{m=-\infty}^{\infty} x[m]z^{-m-1} = z^{-1}X(z)$$

Check Yourself

What DT signal has the following Z transform?

$$\frac{z}{z - \frac{7}{8}}\,; \quad |z| < \frac{7}{8}$$



Check Yourself

If

$$Y(z) = \frac{z}{z - \frac{7}{8}}\,; \quad |z| < \frac{7}{8}$$

then $y[n]$ corresponds to the unit-sample response of

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z - \frac{7}{8}}\,.$$

The difference equation for this system is

$$y[n+1] - \frac{7}{8}y[n] = x[n+1]\,.$$

Convergence **inside** $|z| = \frac{7}{8}$ corresponds to a left-sided (non-causal) response. Solve by iterating backwards in time:

$$y[n] = \frac{8}{7}\left(y[n+1] - x[n+1]\right)$$

Check Yourself

Solve by iterating backwards in time:

$$y[n] = \frac{8}{7}\,(y[n+1] - x[n+1])$$

Start "at rest":

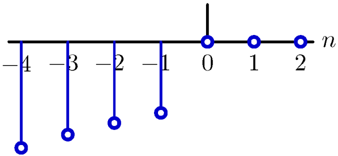
n	$x[n]$	$y[n]$
>0	0	0
0	1	0
-1	0	$-\left(\frac{8}{7}\right)$
-2	0	$-\left(\frac{8}{7}\right)^2$
-3	0	$-\left(\frac{8}{7}\right)^3$
$\cdots$		$\cdots$
n		$-\left(\frac{8}{7}\right)^{-n}$

$$y[n] = -\left(\frac{8}{7}\right)^{-n}\;;\quad n < 0 = -\left(\frac{7}{8}\right)^{n} u[-1-n]$$

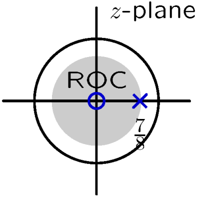
Check Yourself

Plot

$$y[n] = -\left(\frac{7}{8}\right)^n u[-1-n]$$



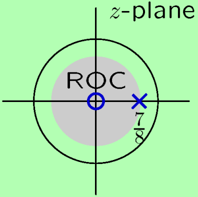
$$\frac{z}{z-\frac{7}{8}}$$



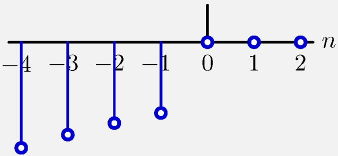
Check Yourself

What DT signal has the following Z transform?

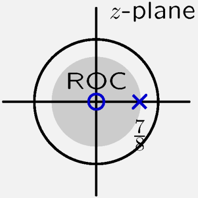
$$\frac{z}{z-\frac{7}{8}}\,; \quad |z| < \frac{7}{8}$$



$$y[n] = -\left(\tfrac{7}{8}\right)^n u[-1-n]$$



$$\frac{z}{z-\frac{7}{8}}$$



Check Yourself

Find the inverse transform of

$$X(z) = \frac{-3z}{2z^2 - 5z + 2}$$

given that the ROC includes the unit circle.

Check Yourself

Find the inverse transform of

$$X(z) = \frac{-3z}{2z^2 - 5z + 2}$$

given that the ROC includes the unit circle.

Expand with partial fractions:

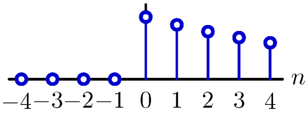
$$X(z) = \frac{-3z}{2z^2 - 5z + 2} = \frac{1}{2z - 1} - \frac{2}{z - 2}$$

Not at standard form!

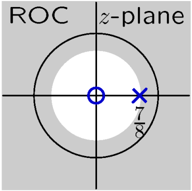
Check Yourself

Standard forms:

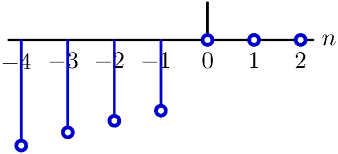
$$x[n] = \left(\tfrac{7}{8}\right)^n u[n]$$



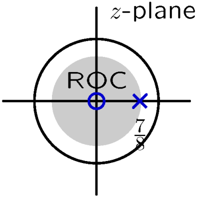
$$\frac{z}{z-\frac{7}{8}}$$



$$y[n] = -\left(\tfrac{7}{8}\right)^n u[-1-n]$$



$$\frac{z}{z-\frac{7}{8}}$$



Check Yourself

Find the inverse transform of

$$X(z) = \frac{-3z}{2z^2 - 5z + 2}$$

given that the ROC includes the unit circle.

Expand with partial fractions:

$$X(z) = \frac{-3z}{2z^2 - 5z + 2} = \frac{1}{2z-1} - \frac{2}{z-2}$$

Not at standard form!

Expand it differently: as a standard form:

$$X(z) = \frac{-3z}{2z^2 - 5z + 2} = \frac{2z}{2z-1} - \frac{z}{z-2} = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2}$$

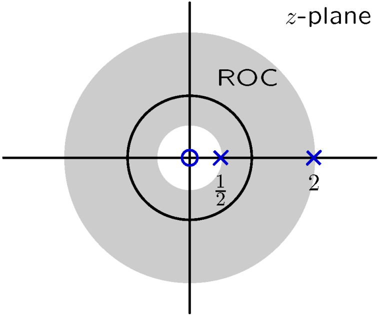
Standard form: a pole at $\frac{1}{2}$ and a pole at 2.

Check Yourself

Ratio of polynomials in z:

$$X(z) = \frac{-3z}{2z^2 - 5z + 2} = \frac{z}{z - \frac{1}{2}} - \frac{z}{z - 2}$$

– a pole at $\frac{1}{2}$ and a pole at 2.



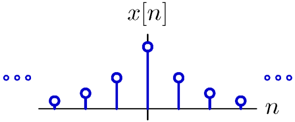
Region of convergence is "outside" pole at $\frac{1}{2}$ but "inside" pole at 2.

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-1-n]$$

Check Yourself

Plot.

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-1-n]$$



Check Yourself

Alternatively, stick with non-standard form:

$$X(z) = \frac{-3z}{2z^2 - 5z + 2} = \frac{1}{2z - 1} - \frac{2}{z - 2}$$

Make it look more standard:

$$X(z) = \frac{1}{2} z^{-1} \frac{z}{z - \frac{1}{2}} - 2z^{-1} \frac{z}{z - 2}$$

Check Yourself

Alternatively, stick with non-standard form:

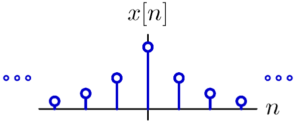
$$X(z) = \frac{-3z}{2z^2 - 5z + 2} = \frac{1}{2z-1} - \frac{2}{z-2}$$

Make it look more standard:

$$X(z) = \frac{1}{2} z^{-1} \frac{z}{z - \frac{1}{2}} - 2 z^{-1} \frac{z}{z-2}$$

Now

$$\begin{aligned} x[n] &= \frac{1}{2}\mathcal{R}\left\{ \left(\frac{1}{2}\right)^n u[n] \right\} + 2\mathcal{R}\left\{ +2^n u[-1-n] \right\} \\ &= \frac{1}{2}\left\{ \left(\frac{1}{2}\right)^{n-1} u[n-1] \right\} + 2\left\{ +2^{n-1} u[-n] \right\} \\ &= \left\{ \left(\frac{1}{2}\right)^n u[n-1] \right\} + \left\{ +2^n u[-n] \right\} \end{aligned}$$



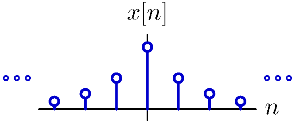
Check Yourself

Alternative 3: expand as polynomials in z^{-1}:

$$X(z) = \frac{-3z}{2z^2 - 5z + 2} = \frac{-3z^{-1}}{2 - 5z^{-1} + 2z^{-2}}$$
$$= \frac{2}{2 - z^{-1}} - \frac{1}{1 - 2z^{-1}} = \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - 2z^{-1}}$$

Now

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + 2^n u[-1-n]$$

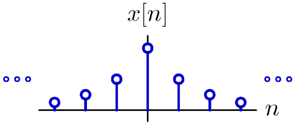


Check Yourself

Find the inverse transform of

$$X(z) = \frac{-3z}{2z^2 - 5z + 2}$$

given that the ROC includes the unit circle.



Solving Difference Equations with Z Transforms

Start with difference equation:

$$y[n] - \frac{1}{2}y[n-1] = \delta[n]$$

Take the Z transform of this equation:

$$Y(z) - \frac{1}{2}z^{-1}Y(z) = 1$$

Solve for $Y(z)$:

$$Y(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

Take the inverse Z transform (by recognizing the form of the transform):

$$y[n] = \left(\frac{1}{2}\right)^n u[n]$$

Inverse Z transform

The inverse Z transform is defined by an integral that is not particularly easy to solve.

Formally,

$$x[n] = \frac{1}{2\pi j} \int_C X(z) z^{n-1} ds$$

were C represents a closed contour that circles the origin by running in a counterclockwise direction through the region of convergence. This integral is not generally easy to compute.

This equation can be useful to prove theorems.

There are better ways (e.g., partial fractions) to compute inverse transforms for the kinds of systems that we frequently encounter.

Properties of Z Transforms

The use of Z Transforms to solve differential equations depends on several important properties.

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
Delay	$x[n-1]$	$z^{-1}X(z)$	R
Multiply by n	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R
Convolve in n	$\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$	$X_1(z)X_2(z)$	$\supset (R_1 \cap R_2)$

Check Yourself

Find the inverse transform of $Y(z) = \left(\frac{z}{z-1}\right)^2 \;; \quad |z| > 1.$

Check Yourself

Find the inverse transform of $Y(z) = \left(\frac{z}{z-1}\right)^2 \; ; \quad |z| > 1.$

$y[n]$ corresponds to unit-sample response of the **right-sided** system

$$\frac{Y}{X} = \left(\frac{z}{z-1}\right)^2 = \left(\frac{1}{1-z^{-1}}\right)^2 = \left(\frac{1}{1-\mathcal{R}}\right)^2$$

$$= \left(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots\right) \times \left(1+\mathcal{R}+\mathcal{R}^2+\mathcal{R}^3+\cdots\right)$$

	1	$\mathcal{R}$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\cdots$
1	1	$\mathcal{R}$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\cdots$
$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\mathcal{R}^4$	$\cdots$
$\mathcal{R}^2$	$\mathcal{R}^2$	$\mathcal{R}^3$	$\mathcal{R}^4$	$\mathcal{R}^5$	$\cdots$
$\mathcal{R}^3$	$\mathcal{R}^3$	$\mathcal{R}^4$	$\mathcal{R}^5$	$\mathcal{R}^6$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

$$\frac{Y}{X} = 1 + 2\mathcal{R} + 3\mathcal{R}^2 + 4\mathcal{R}^3 + \cdots = \sum_{n=0}^{\infty} (n+1)\mathcal{R}^n$$

$$y[n] = h[n] = (n+1)u[n]$$

Check Yourself

Table lookup method.

$$Y(z) = \left(\frac{z}{z-1}\right)^2 \quad \leftrightarrow \quad y[n] = ?$$

$$\frac{z}{z-1} \quad \leftrightarrow \quad u[n]$$

Properties of Z Transforms

The use of Z Transforms to solve differential equations depends on several important properties.

Property	$x[n]$	$X(z)$	ROC
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	$\supset (R_1 \cap R_2)$
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Multiply by n	$nx[n]$	$-z\dfrac{dX(z)}{dz}$	R
Convolve in n	$\displaystyle\sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$	$X_1(z)X_2(z)$	$\supset (R_1 \cap R_2)$

Check Yourself

Table lookup method.

$$Y(z)=\left(\frac{z}{z-1}\right)^2 \quad \leftrightarrow \quad y[n]=?$$

$$\frac{z}{z-1} \quad \leftrightarrow \quad u[n]$$

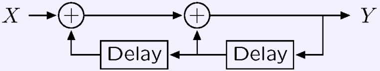
$$-z\frac{d}{dz}\left(\frac{z}{z-1}\right)=z\left(\frac{1}{z-1}\right)^2 \quad \leftrightarrow \quad nu[n]$$

$$z\times\left(-z\frac{d}{dz}\left(\frac{z}{z-1}\right)\right)=\left(\frac{z}{z-1}\right)^2 \quad \leftrightarrow \quad (n+1)u[n+1]=(n+1)u[n]$$

Concept Map: Discrete-Time Systems

Relations among representations.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1-\mathcal{R}-\mathcal{R}^2}$$

Unit-Sample Response

$$h[n]: \quad 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Difference Equation

$$y[n] = x[n] + y[n-1] + y[n-2]$$

System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1-z-z^2}$$

Assignments

- Reading Assignment: Chap. 10.1, 10.2, 10.3, 10.5
- Homework 3