Signals and Systems

Lecture 11: Feedback and Control

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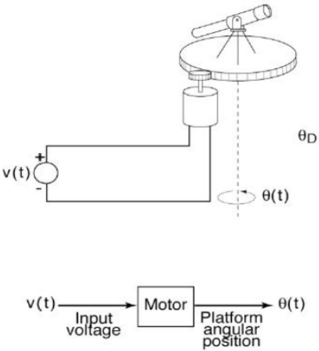
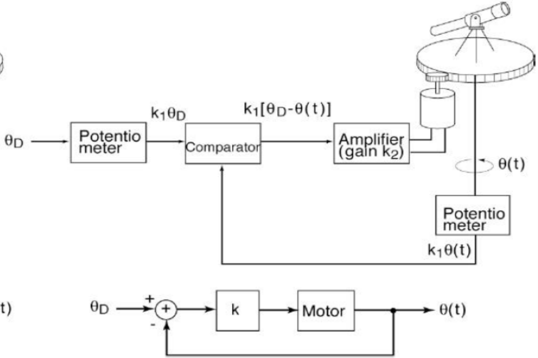
Partly adapted from the materials provided on the MIT OpenCourseWare

Feedback and Control

One Motivating Example — pointing a telescope

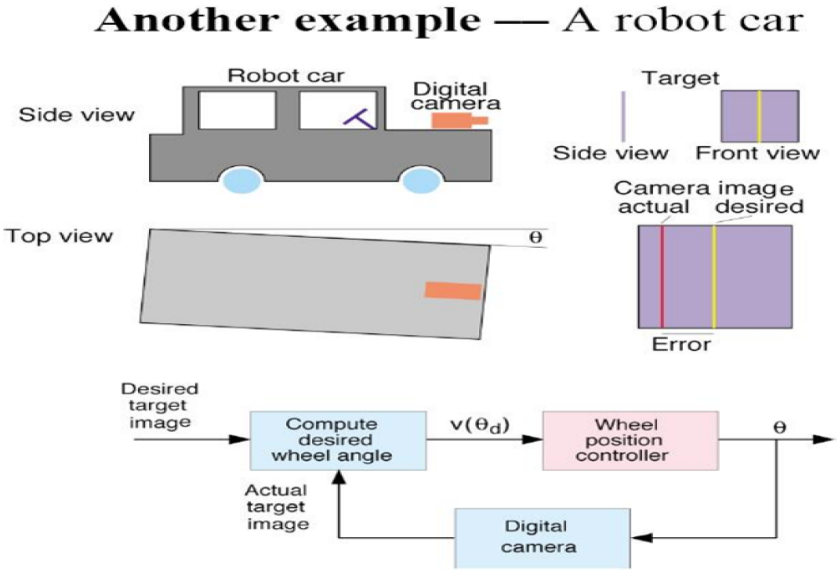




Open-loop System
Aim — shoot

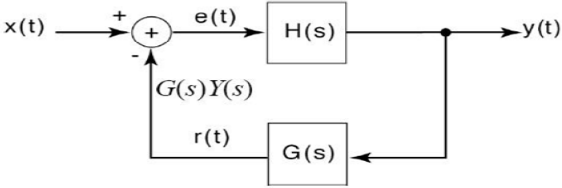
Closed-loop Feedback System
Not done until it is pointed

Feedback and Control



System Function of a Closed-loop System

Example: A basic Feedback System — By its nature, we are dealing with real physical systems. ⇒ They are all *causal*.



$$E(s) = X(s) - R(s) = X(s) - G(s)Y(s)$$

$$Y(s) = H(s)E(S) = H(s)[X(s) - G(s)Y(s)]$$

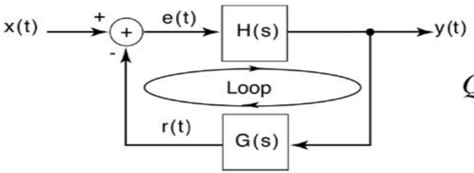
$$\Downarrow$$

$$Q(s) = \frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)}$$

— System function of the close-loop

Feedback and Control

General formula for a closed-loop system: Black's Formula



$$Q(s) = \frac{Y(s)}{X(s)} = \frac{H(s)}{1 + G(s)H(s)}$$

Can show for any closed-loop systems, the system function is given by Black's formula (H. S. Black in the 1920's, along with Nyquist and Bode):

$$\boxed{\text{Closed - loop system function} = \frac{\text{forward gain}}{1 - \text{loop gain}}}$$

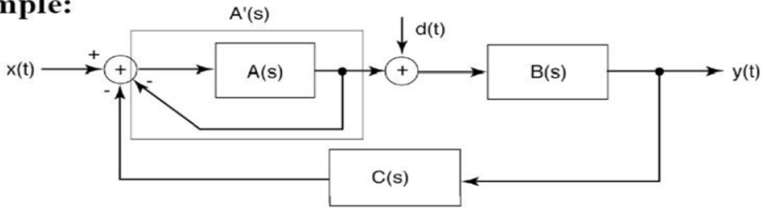
Forward gain — total gain along the forward path from the *input* to the *output*
the gain of an adder is = 1

Loop gain — total gain along the closed loop — shared by all the system functions

Feedback and Control

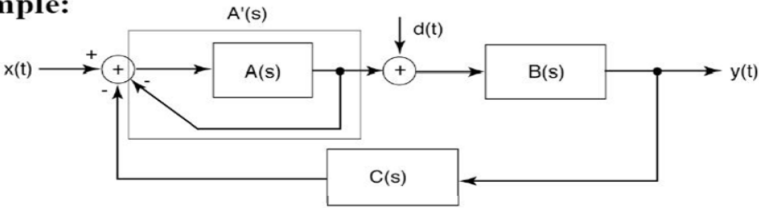
Applications of Black's Formula

Example:



Applications of Black's Formula

Example:



1) $$\frac{Y(s)}{X(s)} = ? = \frac{\text{Forward gain}}{1 - \text{loop gain}} = \frac{A'B}{1 + A'BC}$$

$$A' = \frac{A}{1+A} \quad \Rightarrow \quad \frac{Y(s)}{X(s)} = \frac{AB}{1 + A + ABC}$$

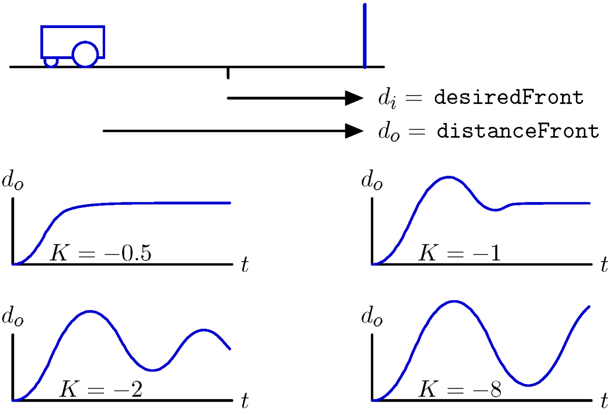
2) $$\frac{Y(s)}{D(s)} = ? = \frac{\text{Forward gain}}{1 - \text{loop gain}} = \frac{B}{1 + A'BC} = \frac{B(1+A)}{1 + A + ABC}$$

Today's goal

Use systems theory to gain insight into how to control a system.

Example: wallFinder System

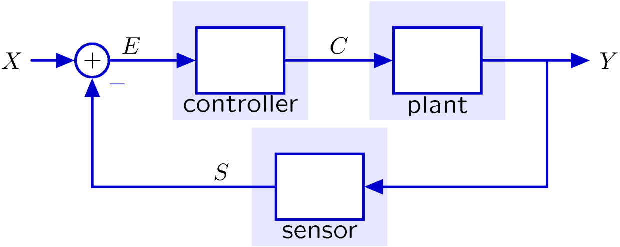
Approach a wall, stopping a desired distance d_i in front of it.



What causes these different types of responses?

Structure of a Control Problem

(Simple) Control systems have three parts.



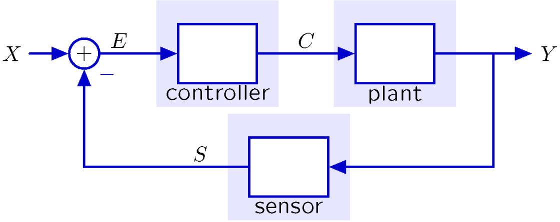
The **plant** is the system to be controlled.

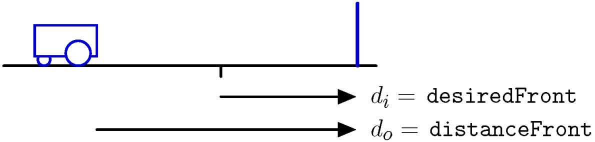
The **sensor** measures the output of the plant.

The **controller** specifies a command C to the plant based on the *difference* between the input X and sensor output S.

Analysis of wallFinder System

Cast wallFinder problem into control structure.





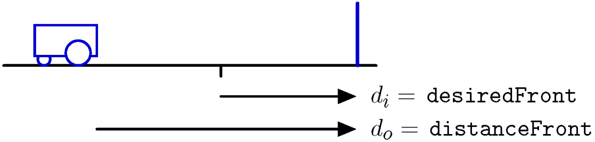
proportional controller: $v[n] = Ke[n] = K\big(d_i[n] - d_s[n]\big)$

locomotion: $d_o[n] = d_o[n-1] - Tv[n-1]$

sensor with no delay: $d_s[n] = d_o[n]$

Analysis of wallFinder System: Block Diagram

Visualize as block diagram.



proportional controller: $v[n] = Ke[n] = K\big(d_i[n] - d_s[n]\big)$

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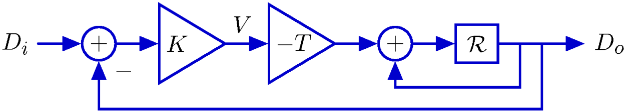
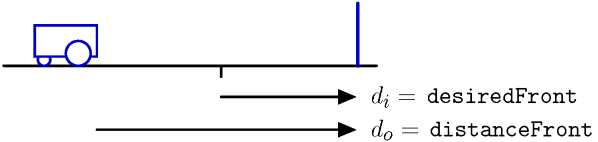
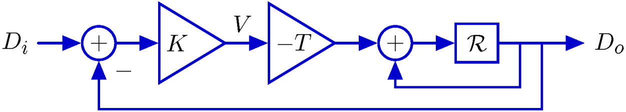


Analysis of wallFinder System: System Function

Solve.





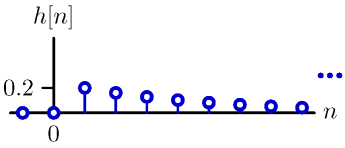
$$\frac{D_o}{D_i} = \frac{\frac{-KT\mathcal{R}}{1-\mathcal{R}}}{1+\frac{-KT\mathcal{R}}{1-\mathcal{R}}} = \frac{-KT\mathcal{R}}{1-\mathcal{R}-KT\mathcal{R}} = \frac{-KT\mathcal{R}}{1-(1+KT)\mathcal{R}}$$

Analysis of wallFinder System: Poles

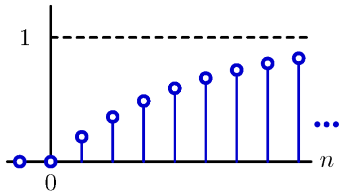
The system function contains a single **pole** at $z = 1 + KT$.

$$\frac{D_o}{D_i} = \frac{-KT\mathcal{R}}{1-(1+KT)\mathcal{R}}$$

Unit-sample response for $KT = -0.2$:



Unit-step response $s[n]$ for $KT = -0.2$:



What determines the speed of the response? Could it be faster?

Check Yourself

Find KT for fastest convergence of unit-sample response.

$$\frac{D_o}{D_i} = \frac{-KT\mathcal{R}}{1-(1+KT)\mathcal{R}}$$

1. $KT = -2$
2. $KT = -1$
3. $KT = 0$
4. $KT = 1$
5. $KT = 2$

0. none of the above

Check Yourself

Find KT for fastest convergence of unit-sample response.

$$\frac{D_o}{D_i} = \frac{-KT\mathcal{R}}{1-(1+KT)\mathcal{R}}$$

If $KT = -1$ then the pole is at $z = 0$.

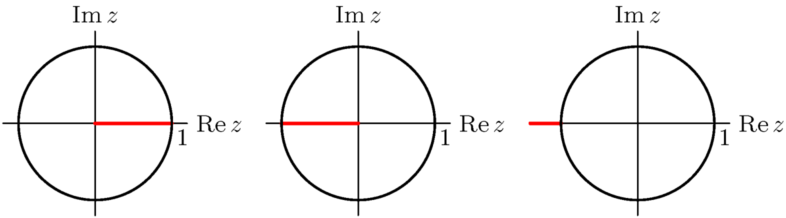
$$\frac{D_o}{D_i} = \frac{-KT\mathcal{R}}{1-(1+KT)\mathcal{R}} = \mathcal{R}$$

Unit-sample response has a single non-zero output sample, at $n = 1$.

Analysis of wallFinder System: Poles

The poles of the system function provide insight for choosing K.

$$\frac{D_o}{D_i} = \frac{-KT\mathcal{R}}{1-(1+KT)\mathcal{R}} = \frac{(1-p_o)\mathcal{R}}{1-p_o\mathcal{R}} \;; \qquad p_0 = 1+KT$$



$0 < p_0 < 1$	$-1 < p_0 < 0$	$p_0 < -1$
$-1 < KT < 0$	$-2 < KT < -1$	$KT < -2$
monotonic converging	alternating converging	alternating diverging

Check Yourself

Find KT for fastest convergence of unit-sample response.

$$\frac{D_o}{D_i} = \frac{-KT\mathcal{R}}{1-(1+KT)\mathcal{R}}$$

1. $KT = -2$
2. $KT = -1$
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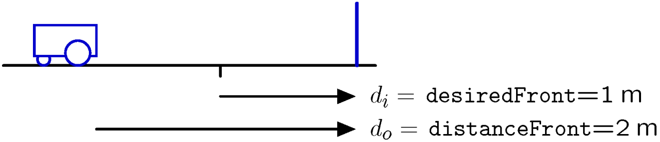
0. none of the above

Analysis of wallFinder System

The optimum gain K moves robot to desired position in **one** step.



$$KT = -1$$

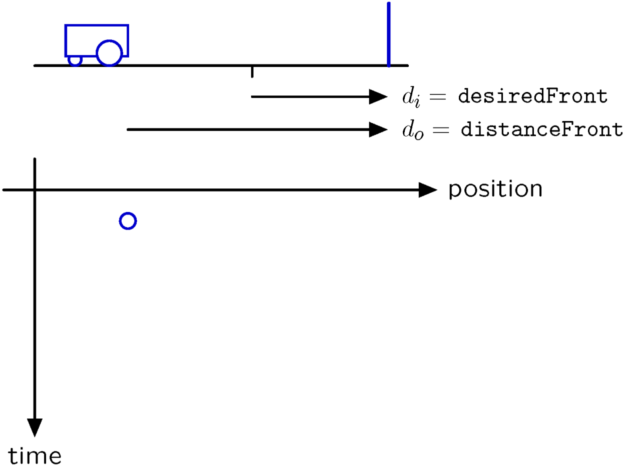
$$K = -\frac{1}{T} = -\frac{1}{1/10} = -10$$

$$v[n] = K\big(d_i[n] - d_o[n]\big) = -10\big(1-2\big) = 10 \text{ m/s}$$

exactly the right speed to get there in one step!

Analyzing wallFinder: Space-Time Diagram

The optimum gain K moves robot to desired position in **one** step.



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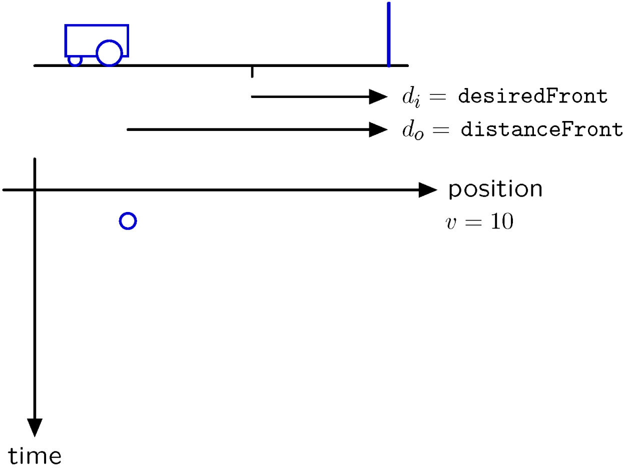

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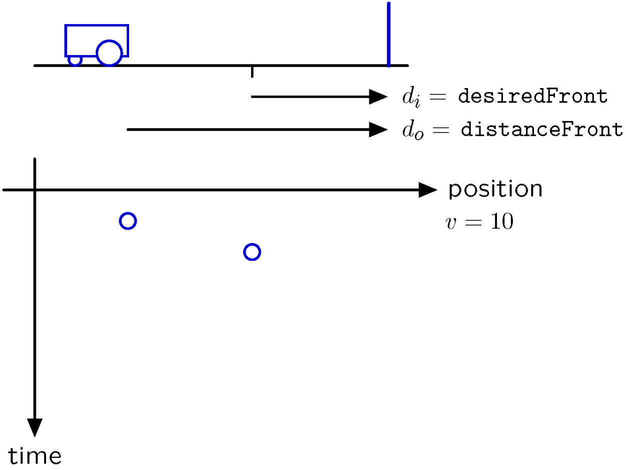

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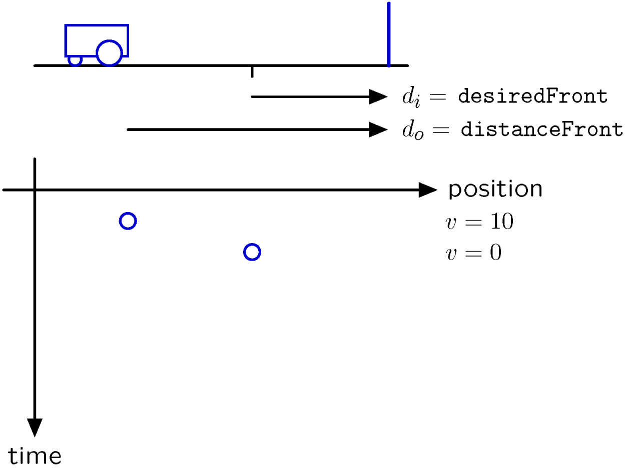

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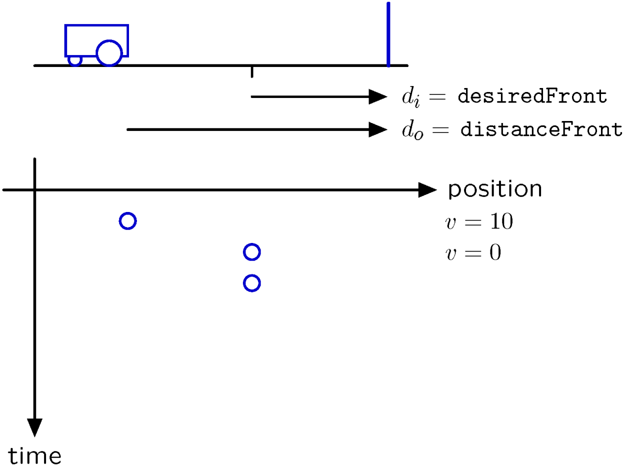

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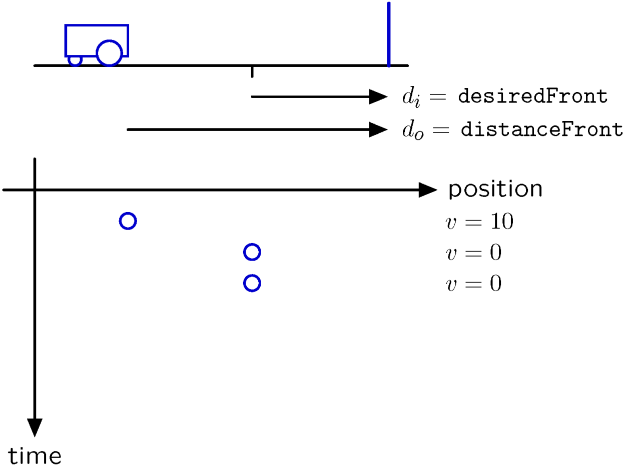

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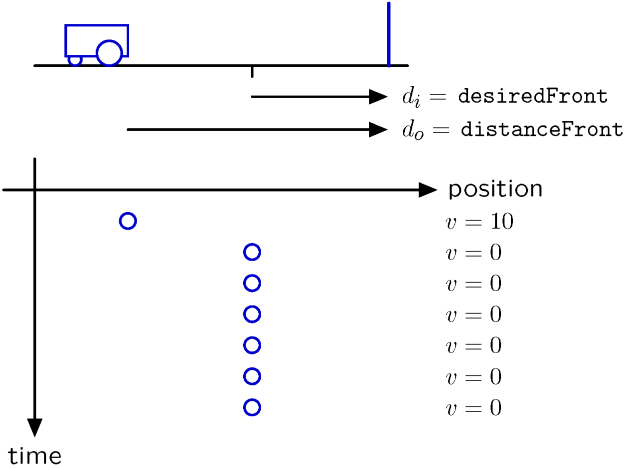
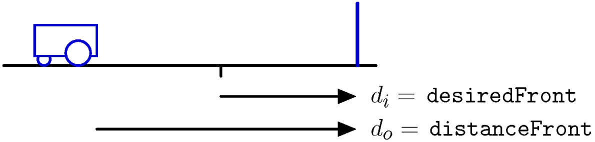


Analysis of wallFinder System: Adding Sensor Delay

Adding delay tends to destabilize control systems.



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sensor **with delay**: $d_s[n] = d_o[\mathbf{n-1}]$

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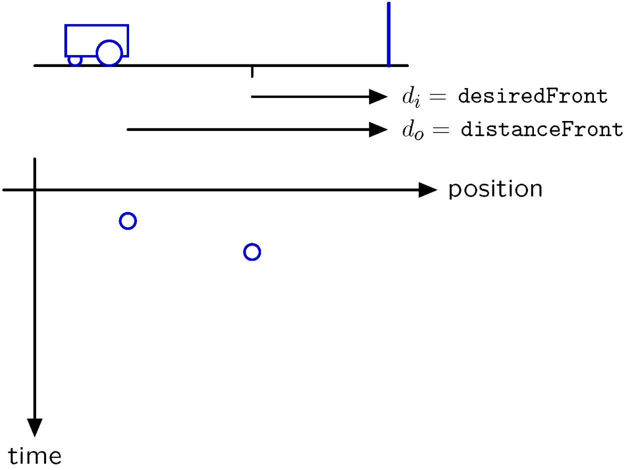

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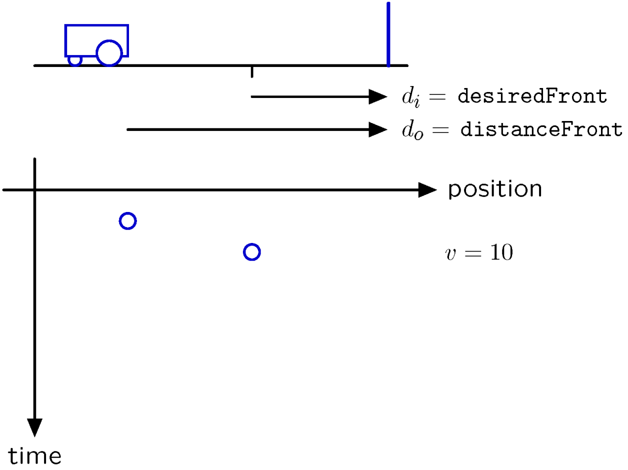

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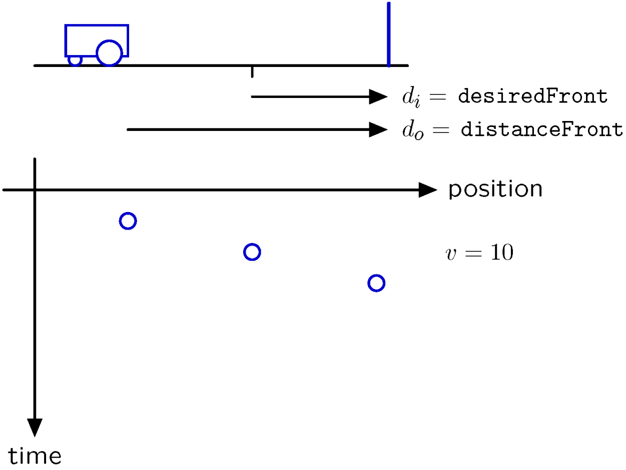

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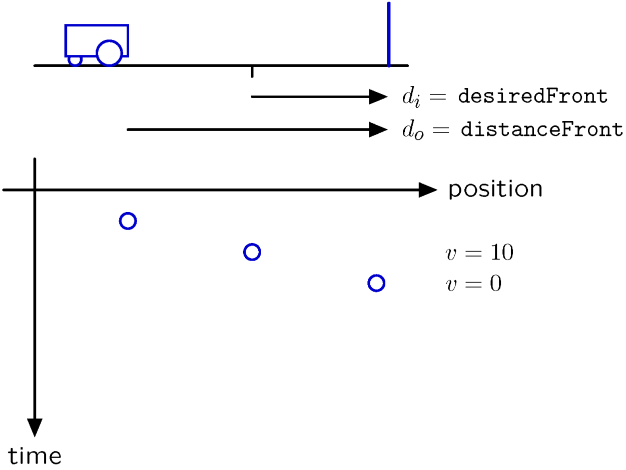

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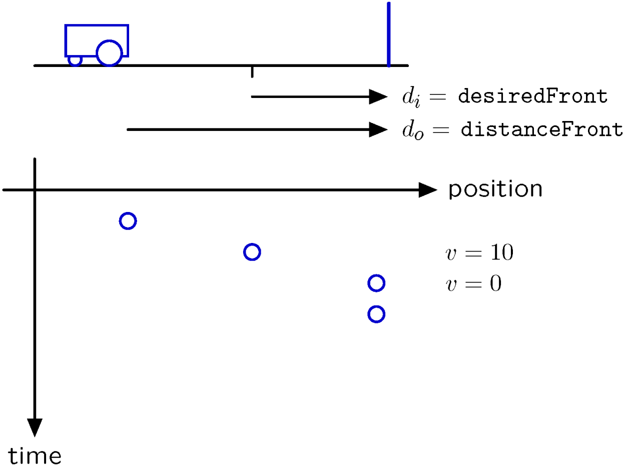

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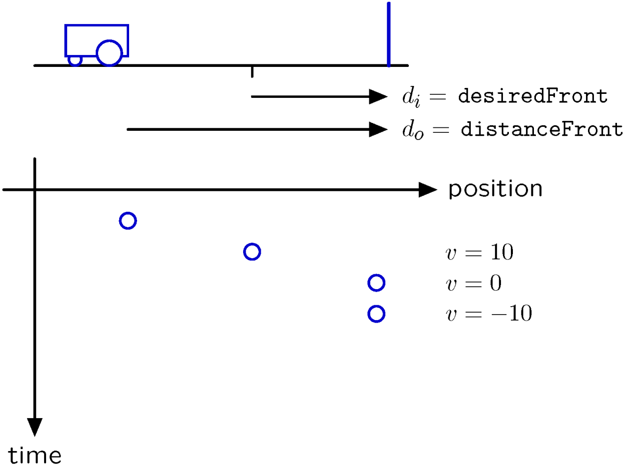

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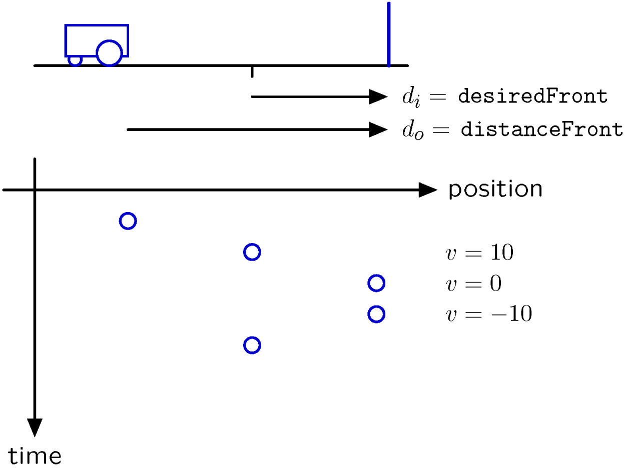

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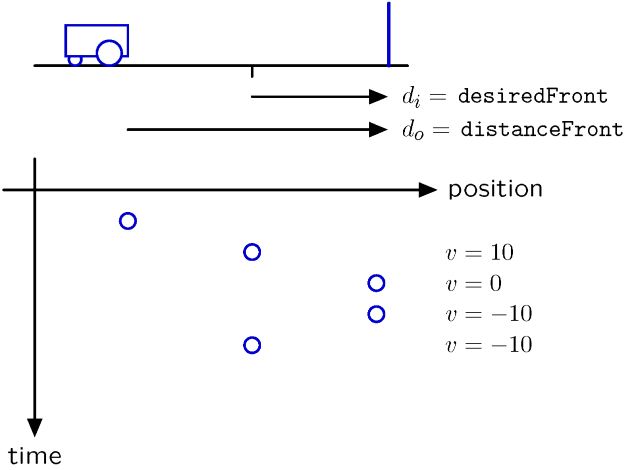

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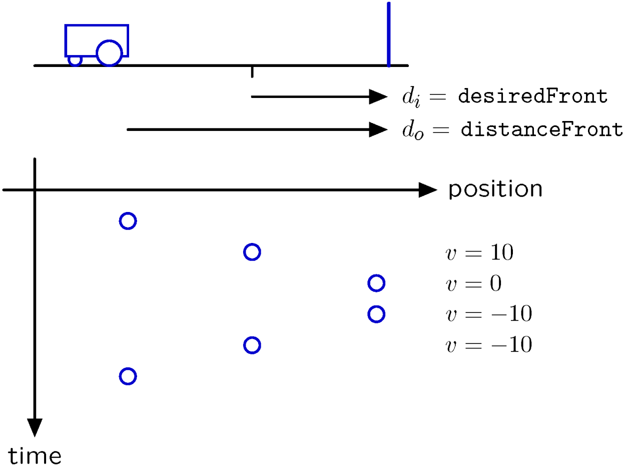

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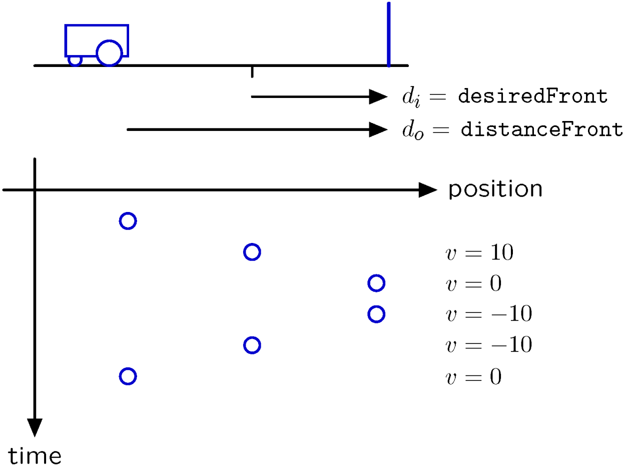

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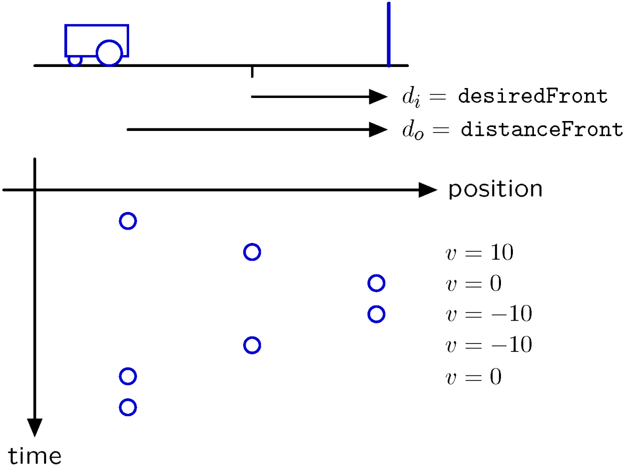
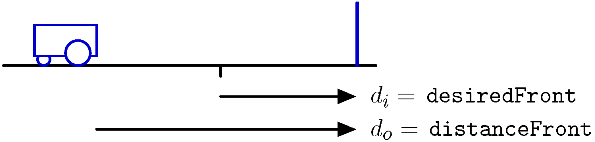


Analysis of wallFinder System: Block Diagram

Incorporating sensor delay in block diagram.



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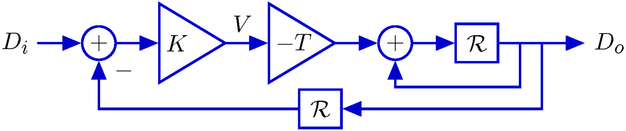
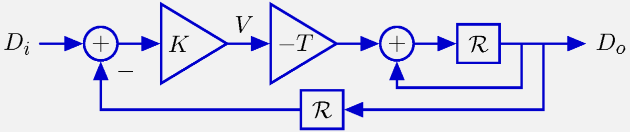


Check Yourself

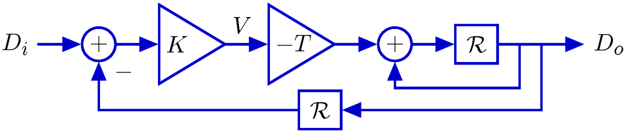
Find the system function $H = \dfrac{D_o}{D_i}$.



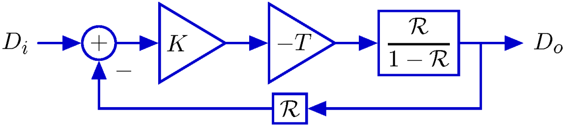
1. $\dfrac{KT\mathcal{R}}{1-\mathcal{R}}$
2. $\dfrac{-KT\mathcal{R}}{1+\mathcal{R}-KT\mathcal{R}^2}$
3. $\dfrac{KT\mathcal{R}}{1-\mathcal{R}} - KT\mathcal{R}$
4. $\dfrac{-KT\mathcal{R}}{1-\mathcal{R}-KT\mathcal{R}^2}$
5. none of the above

Check Yourself

Find the system function $H = \dfrac{D_o}{D_i}$.



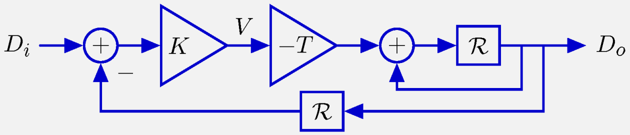
Replace accumulator with equivalent block diagram.



$$\frac{D_o}{D_i} = \frac{\dfrac{-KT\mathcal{R}}{1-\mathcal{R}}}{1+\dfrac{-KT\mathcal{R}^2}{1-\mathcal{R}}} = \frac{-KT\mathcal{R}}{1-\mathcal{R}-KT\mathcal{R}^2}$$

Check Yourself

Find the system function $H = \dfrac{D_o}{D_i}$.



1. $\dfrac{KT\mathcal{R}}{1-\mathcal{R}}$
2. $\dfrac{-KT\mathcal{R}}{1+\mathcal{R}-KT\mathcal{R}^2}$
3. $\dfrac{KT\mathcal{R}}{1-\mathcal{R}} - KT\mathcal{R}$
4. $\dfrac{-KT\mathcal{R}}{1-\mathcal{R}-KT\mathcal{R}^2}$
5. none of the above

Analyzing wallFinder: Poles

Substitute $\mathcal{R} \rightarrow \dfrac{1}{z}$ in the system functional to find the poles.

$$\frac{D_o}{D_i} = \frac{-KT\mathcal{R}}{1 - \mathcal{R} - KT\mathcal{R}^2} = \frac{-KT\frac{1}{z}}{1 - \frac{1}{z} - KT\frac{1}{z^2}} = \frac{-KTz}{z^2 - z - KT}$$

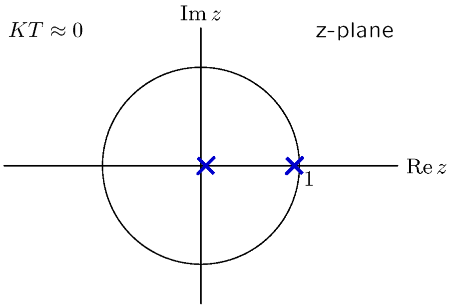
The poles are then the roots of the denominator.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT}$$

Feedback and Control: Poles

If KT is small, the poles are at $z \approx -KT$ and $z \approx 1 + KT$.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT} \approx \frac{1}{2} \pm \sqrt{\left(\frac{1}{2} + KT\right)^2} = 1 + KT, \, -KT$$



Pole near 0 generates fast response.

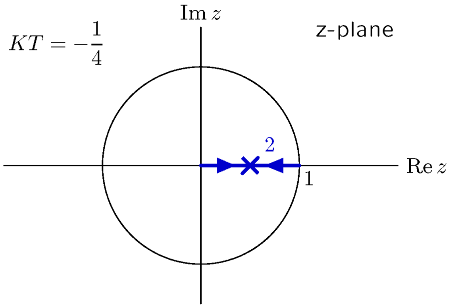
Pole near 1 generates slow response.

Slow mode (pole near 1) dominates the response.

Feedback and Control: Poles

As KT becomes more negative, the poles move toward each other and collide at $z = \frac{1}{2}$ when $KT = -\frac{1}{4}$.

$$z = \tfrac{1}{2} \pm \sqrt{\left(\tfrac{1}{2}\right)^2 + KT} = \tfrac{1}{2} \pm \sqrt{\left(\tfrac{1}{2}\right)^2 - \tfrac{1}{4}} = \tfrac{1}{2}, \tfrac{1}{2}$$

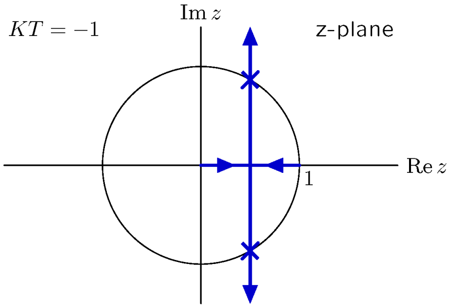


Persistent responses decay. The system is stable.

Feedback and Control: Poles

If $KT < -1/4$, the poles are complex.

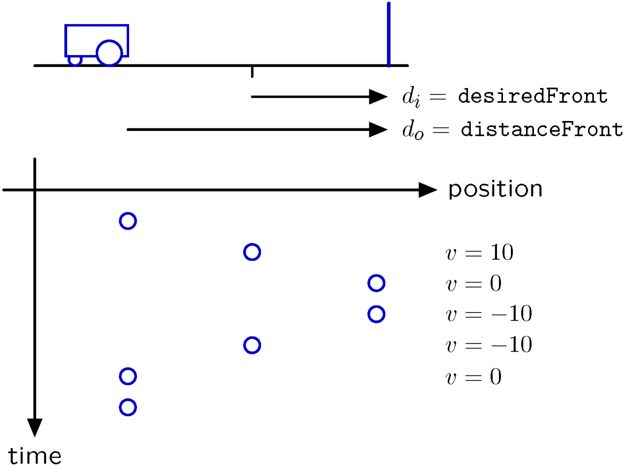
$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT} = \frac{1}{2} \pm j\sqrt{-KT - \left(\frac{1}{2}\right)^2}$$



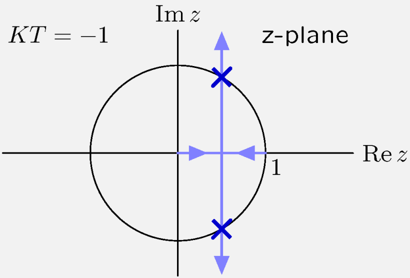
Complex poles $\rightarrow$ oscillations.

Same oscillation we saw earlier!

Adding delay tends to destabilize control systems.



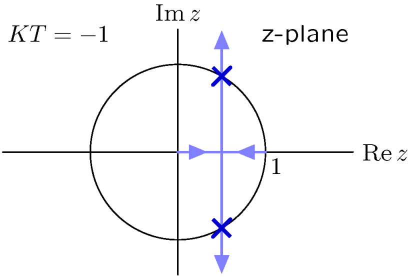
Check Yourself



What is the period of the oscillation?

1. 1   2. 2   3. 3
4. 4   5. 6   0. none of above

Check Yourself

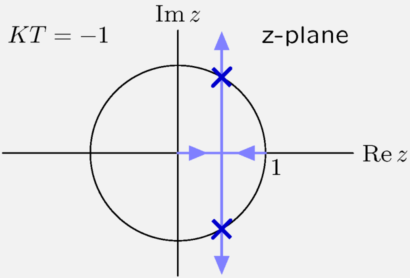


$$p_0 = \frac{1}{2} \pm j\frac{\sqrt{3}}{2} = e^{\pm j\pi/3}$$

$$p_0^n = e^{\pm j\pi n/3}$$

$$\underbrace{e^{\pm j0\pi/3}}_{1},\ e^{\pm j\pi/3},\ e^{\pm j2\pi/3},\ e^{\pm j3\pi/3},\ e^{\pm j4\pi/3},\ e^{\pm j5\pi/3},\ \underbrace{e^{\pm j6\pi/3}}_{e^{\pm j2\pi}=1}$$

Check Yourself

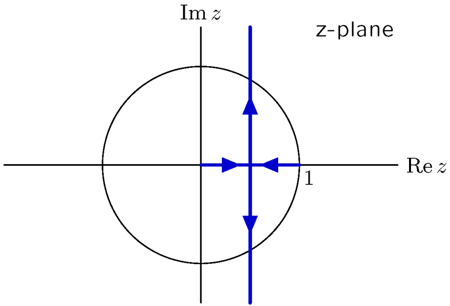


What is the period of the oscillation?

1. 1 2. 2 3. 3
4. 4 5. 6 0. none of above

Feedback and Control: Poles

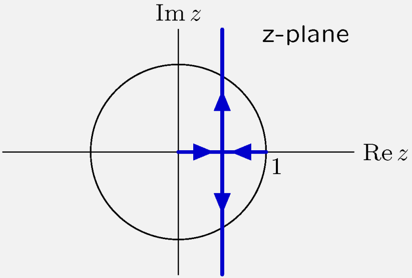
The closed loop poles depend on the gain.



If $KT : 0 \to -\infty$: then $z_1, z_2 : 0, 1 \to \frac{1}{2}, \frac{1}{2} \to \frac{1}{2} \pm j\infty$

Check Yourself

Find KT for fastest response.



closed-loop poles

$$\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT}$$

1. 0
2. $-\frac{1}{4}$
3. $-\frac{1}{2}$
4. -1
5. $-\infty$
0. none of above

Check Yourself

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT}$$

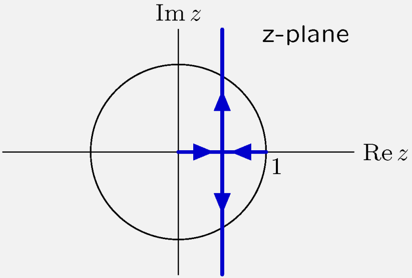
The dominant pole always has a magnitude that is $\geq \frac{1}{2}$.

It is smallest when there is a double pole at $z = \frac{1}{2}$.

Therefore, $KT = -\frac{1}{4}$.

Check Yourself

Find KT for fastest response.



closed-loop poles

$$\frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT}$$

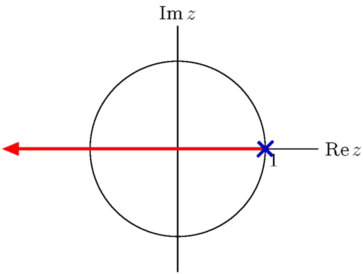
1. 0
2. $-\frac{1}{4}$
3. $-\frac{1}{2}$
4. -1
5. $-\infty$
0. none of above

Destabilizing Effect of Delay

Adding delay in the feedback loop makes it more difficult to stabilize.

Ideal sensor: $d_s[n] = d_o[n]$

More realistic sensor (with delay): $d_s[n] = d_o[n-1]$



Fastest response without delay: single pole at $z = 0$.

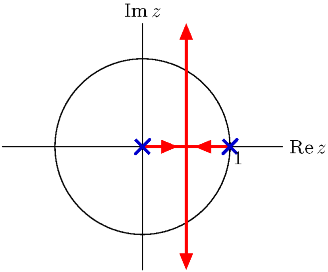
Fastest response with delay: double pole at $z = \frac{1}{2}$. **much slower!**

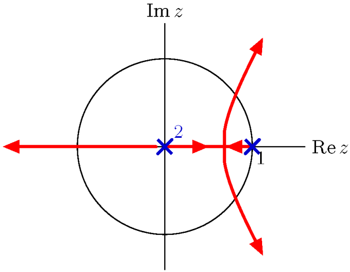
Destabilizing Effect of Delay

Adding more delay in the feedback loop is even worse.

More realistic sensor (with delay): $d_s[n] = d_o[n-1]$

Even more delay: $d_s[n] = d_o[n-2]$





Fastest response with delay: double pole at $z = \frac{1}{2}$.

Fastest response with more delay: double pole at $z = 0.682$.

→ **even slower**

Feedback and Control: Summary

Feedback is an elegant way to design a control system.

Stability of a feedback system is determined by its dominant pole.

Delays tend to decrease the stability of a feedback system.

Assignments

- Reading Assignment: Chap. 11.0-11.2