Signals and Systems

Lecture 9: Frequency Response

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Partly adapted from the materials provided on the MIT OpenCourseWare

Review

Last time, we saw how a linear, time-invariant (LTI) system can be characterized by its unit-sample/impulse response.

$$\text{DT:}\quad y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$\text{CT:}\quad y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Characterizing a system by its unit-sample/impulse response is especially insightful for some systems.

Frequency Response

Today we will investigate a different way to characterize a system: the **frequency response**.

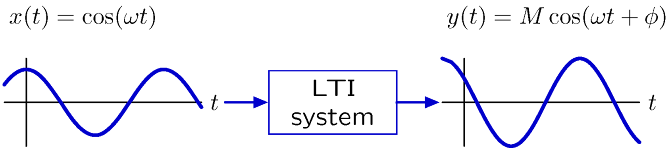
Many systems are naturally described by their responses to sinusoids.

Example: audio systems

Frequency Response Preview

If the input to a linear, time-invariant system is an eternal sinusoid, then the output is also an eternal sinusoid:

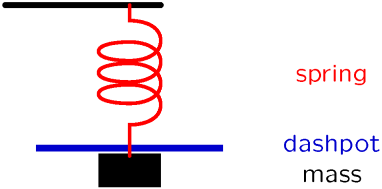
- same frequency
- possibly different amplitude, and
- possibly different phase angle.



The **frequency response** is a plot of the magnitude M and angle ϕ as a function of frequency ω.

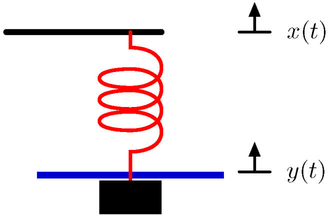
Example

Mass, spring, and dashpot system.



Demonstration

Measure the frequency response of a mass, spring, dashpot system.



Frequency Response

Calculate the frequency response.

Methods

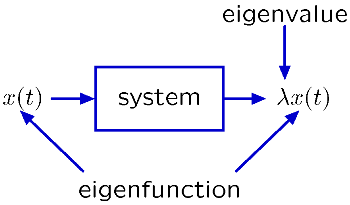
- solve differential equation
  $\rightarrow$ find particular solution for $x(t) = \cos\omega_0 t$
- find impulse response of system
  $\rightarrow$ convolve with $x(t) = \cos\omega_0 t$

New method

- use eigenfunctions and eigenvalues

Eigenfunctions

If the output signal is a scalar multiple of the input signal, we refer to the signal as an eigenfunction and the multiplier as the eigenvalue.



Check Yourself: Eigenfunctions

Consider the system described by

$$\dot{y}(t) + 2y(t) = x(t).$$

Determine if each of the following functions is an eigenfunction of this system. If it is, find its eigenvalue.

1. e^{-t} for all time
2. e^{t} for all time
3. e^{jt} for all time
4. $\cos(t)$ for all time
5. $u(t)$ for all time

Check Yourself: Eigenfunctions

$$\dot{y}(t) + 2y(t) = x(t)$$

1. $e^{-t}: \quad -\lambda e^{-t} + 2\lambda e^{-t} = e^{-t} \rightarrow \lambda = 1$

2. $e^{t}: \quad \lambda e^{t} + 2\lambda e^{t} = e^{t} \rightarrow \lambda = \frac{1}{3}$

3. $e^{jt}: \quad j\lambda e^{jt} + 2\lambda e^{jt} = e^{jt} \rightarrow \lambda = \frac{1}{j+2}$

4. $\cos t: \quad -\lambda \sin t + 2\lambda \cos t = \cos t \rightarrow$ not possible!

5. $u(t): \quad \lambda \delta(t) + 2\lambda u(t) = u(t) \rightarrow$ not possible!

Check Yourself: Eigenfunctions

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1. e^{-t} for all time $\surd$ $\lambda = 1$
2. e^{t} for all time $\surd$ $\lambda = \frac{1}{3}$
3. e^{jt} for all time $\surd$ $\lambda = \frac{1}{j+2}$
4. $\cos(t)$ for all time X
5. $u(t)$ for all time X

Complex Exponentials

Complex exponentials are eigenfunctions of LTI systems.

If $x(t) = e^{st}$ and $h(t)$ is the impulse response then

$$y(t) = (h * x)(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s)\, e^{st}$$

$$e^{st} \longrightarrow \boxed{\begin{array}{c}\text{LTI}\\ h(t)\end{array}} \longrightarrow H(s)\, e^{st}$$

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$$e^{st} \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ h(t)\end{matrix}} \longrightarrow H(s)\, e^{st}$$

Eternal sinusoids are sums of complex exponentials.

$$\cos \omega_0 t = \frac{1}{2} \left(e^{j\omega_0 t} + e^{-j\omega_0 t} \right)$$

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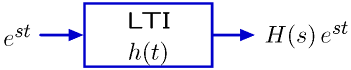


Eternal sinusoids are sums of complex exponentials.

$$\cos \omega_0 t = \frac{1}{2} \left(e^{j\omega_0 t} + e^{-j\omega_0 t} \right)$$

Furthermore, the eigenvalue associated with e^{st} is $H(s)$!

Rational System Functions

Eigenvalues are particularly easy to evaluate for systems represented by linear differential equations with constant coefficients.

Then the system function is a ratio of polynomials in s.

Example:

$$\ddot{y}(t) + 3\dot{y}(t) + 4y(t) = 2\ddot{x}(t) + 7\dot{x}(t) + 8x(t)$$

Then

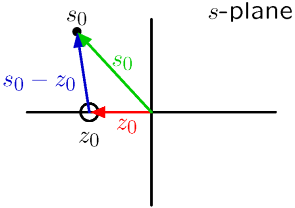
$$H(s) = \frac{2s^2 + 7s + 8}{s^2 + 3s + 4} \equiv \frac{N(s)}{D(s)}$$

Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined graphically using vectorial analysis.

Factor the numerator and denominator of the system function to make poles and zeros explicit.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$



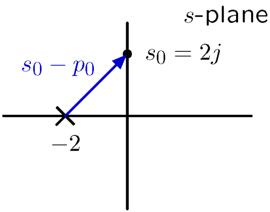
Each factor in the numerator/denominator corresponds to a vector from a zero/pole (here z_0) to s_0, the point of interest in the s-plane.

Vector Diagrams

Example: Find the response of the system described by

$$H(s) = \frac{1}{s+2}$$

to the input $x(t) = e^{2jt}$ (for all time).



The denominator of $H(s)|_{s=2j}$ is $2j+2$, a vector with length $2\sqrt{2}$ and angle $\pi/4$. Therefore, the response of the system is

$$y(t) = H(2j)e^{2jt} = \frac{1}{2\sqrt{2}}e^{-\frac{j\pi}{4}}e^{2jt}.$$

Vector Diagrams

The value of $H(s)$ at a point $s = s_0$ can be determined by combining the contributions of the vectors associated with each of the poles and zeros.

$$H(s_0) = K\frac{(s_0 - z_0)(s_0 - z_1)(s_0 - z_2)\cdots}{(s_0 - p_0)(s_0 - p_1)(s_0 - p_2)\cdots}$$

The magnitude is determined by the product of the magnitudes.

$$|H(s_0)| = |K|\frac{|(s_0 - z_0)||(s_0 - z_1)||(s_0 - z_2)|\cdots}{|(s_0 - p_0)||(s_0 - p_1)||(s_0 - p_2)|\cdots}$$

The angle is determined by the sum of the angles.

$$\angle H(s_0) = \angle K + \angle(s_0 - z_0) + \angle(s_0 - z_1) + \cdots - \angle(s_0 - p_0) - \angle(s_0 - p_1) - \cdots$$

Frequency Response

Response to eternal sinusoids.

Let $x(t) = \cos \omega_0 t$ (for all time). Then

$$x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

and the response to a sum is the sum of the responses.

$$y(t) = \frac{1}{2}\left(H(j\omega_0)\, e^{j\omega_0 t} + H(-j\omega_0)\, e^{-j\omega_0 t}\right)$$

Conjugate Symmetry

The complex conjugate of $H(j\omega)$ is $H(-j\omega)$.

The system function is the Laplace transform of the impulse response:

$$H(s) = \int_{-\infty}^{\infty} h(t)e^{-st}dt$$

where $h(t)$ is a real-valued function of t for physical systems.

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t}dt$$
$$H(-j\omega) = \int_{-\infty}^{\infty} h(t)e^{j\omega t}dt \equiv \big(H(j\omega)\big)^*$$

Frequency Response

Response to eternal sinusoids.

Let $x(t) = \cos \omega_0 t$ (for all time), which can be written as

$$x(t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

The response to a sum is the sum of the responses,

$$\begin{aligned} y(t) &= \frac{1}{2}\left(H(j\omega_0)e^{j\omega_0 t} + H(-j\omega_0)e^{-j\omega_0 t}\right) \\ &= \mathrm{Re}\left\{H(j\omega_0)e^{j\omega_0 t}\right\} \\ &= \mathrm{Re}\left\{|H(j\omega_0)|e^{j\angle H(j\omega_0)}e^{j\omega_0 t}\right\} \\ &= |H(j\omega_0)|\mathrm{Re}\left\{e^{j\omega_0 t + j\angle H(j\omega_0)}\right\} \end{aligned}$$

$$y(t) = |H(j\omega_0)| \cos\left(\omega_0 t + \angle\left(H(j\omega_0)\right)\right).$$

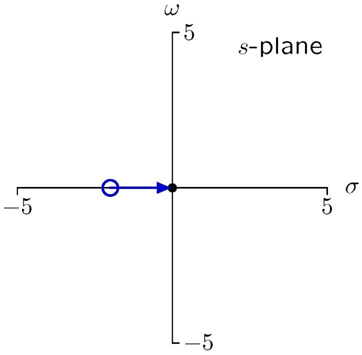
Frequency Response

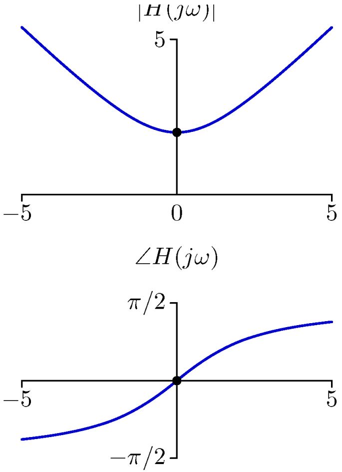
The magnitude and phase of the response of a system to an eternal cosine signal is the magnitude and phase of the system function evaluated at $s = j\omega$.

$$\cos(\omega t) \longrightarrow \boxed{H(s)} \longrightarrow |H(j\omega)| \cos\big(\omega t + \angle H(j\omega)\big)$$

Vector Diagrams

$$H(s) = s - z_1$$





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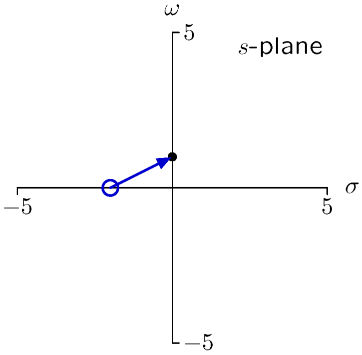

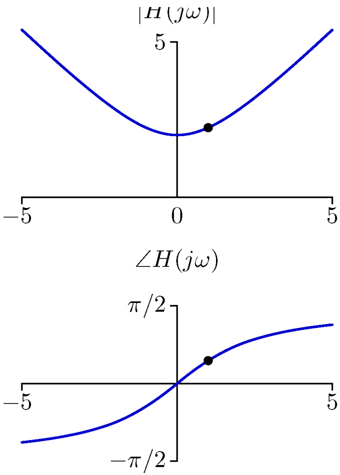

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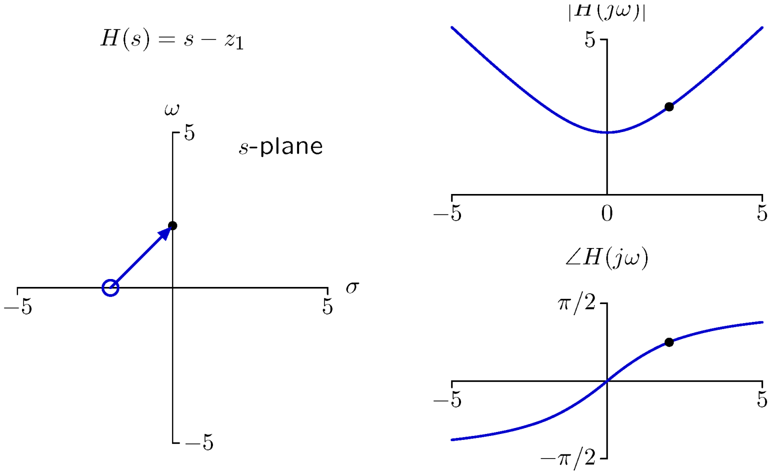

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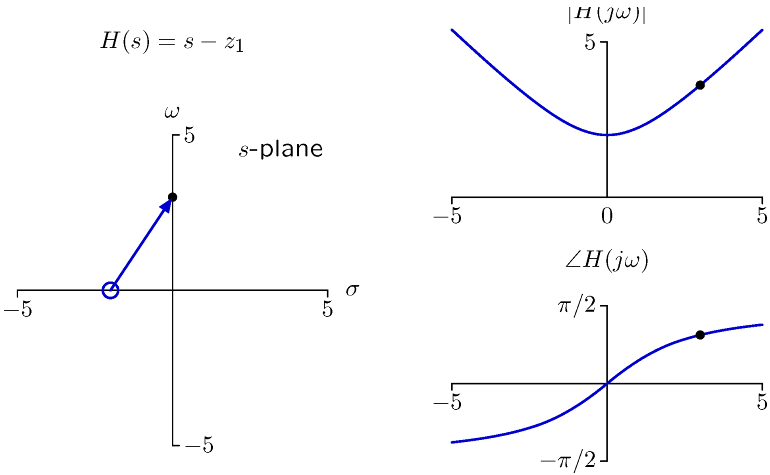

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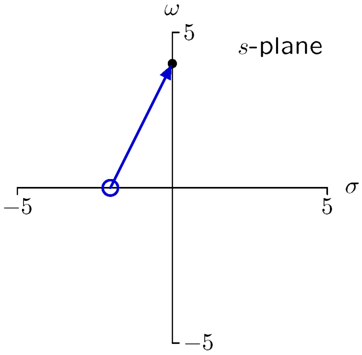

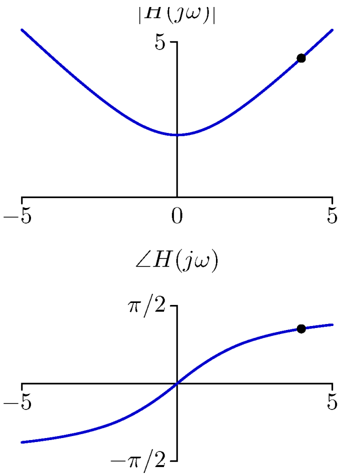

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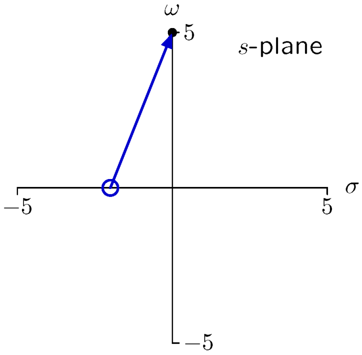

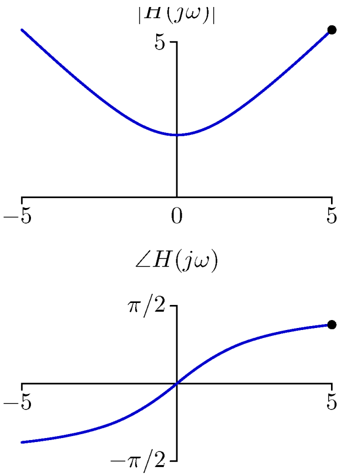

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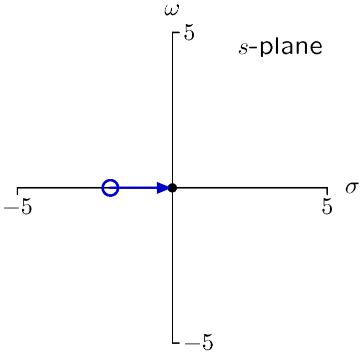

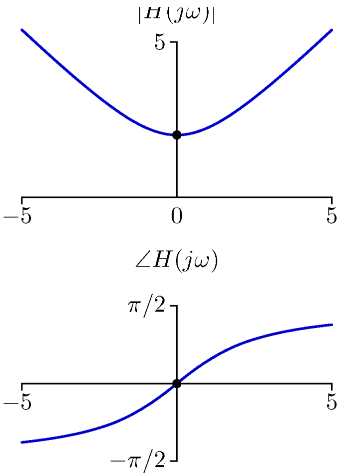

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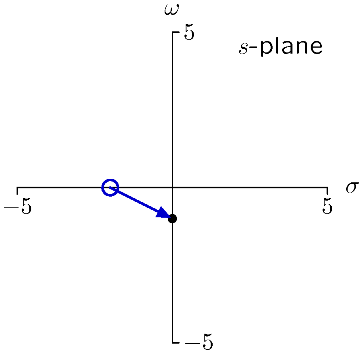

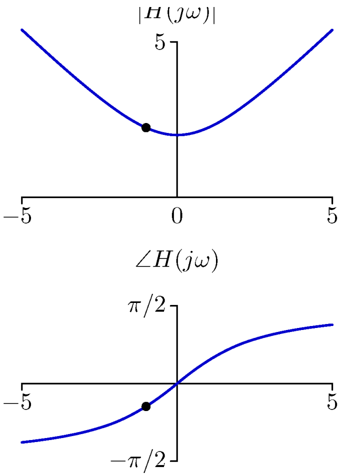

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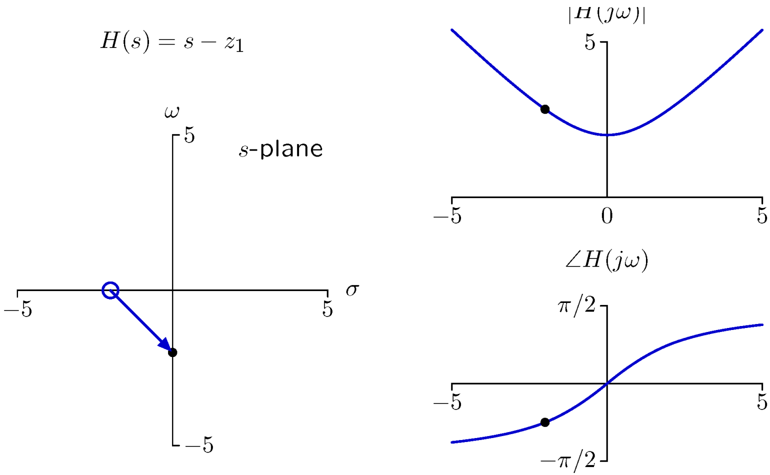

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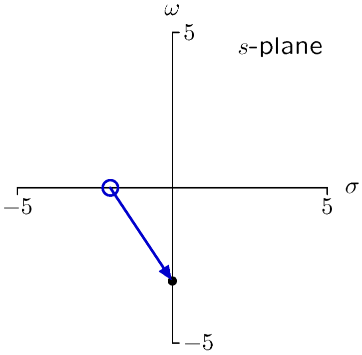

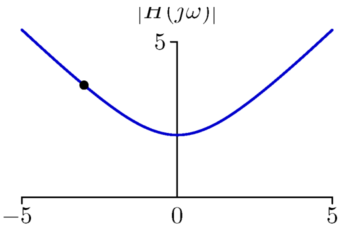

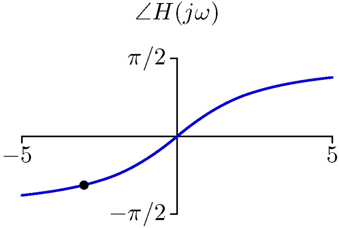

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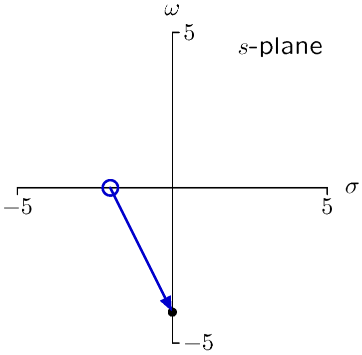

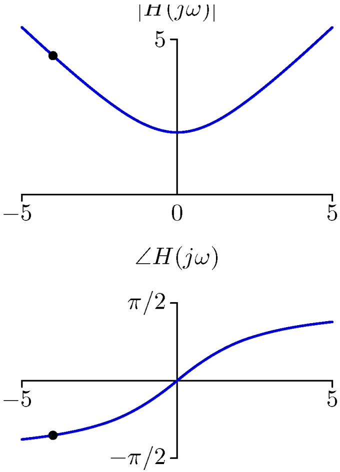

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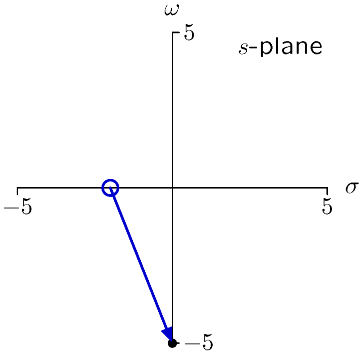
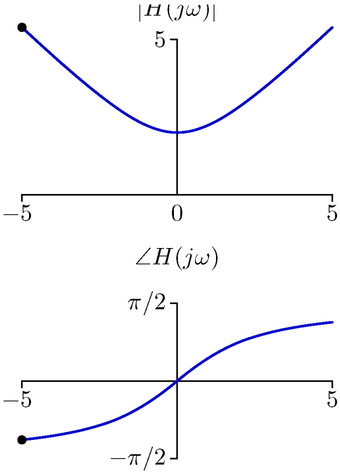
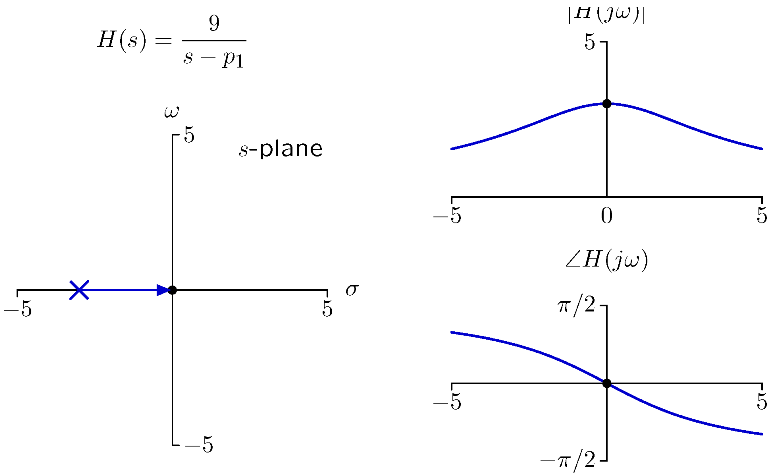




Vector Diagrams

$$H(s) = \frac{9}{s - p_1}$$



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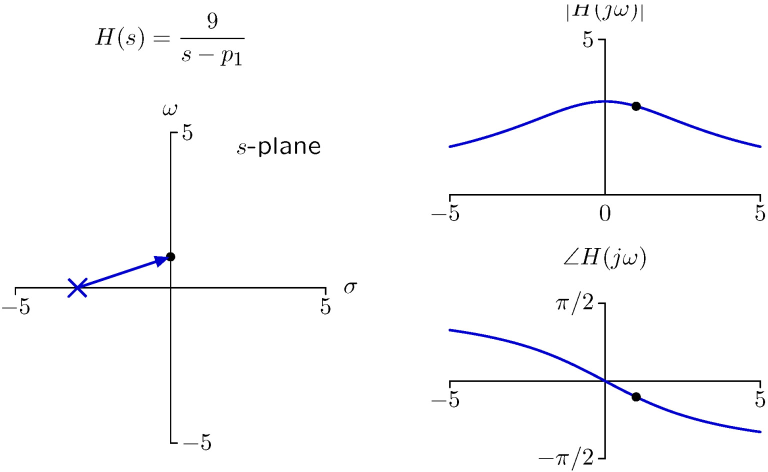

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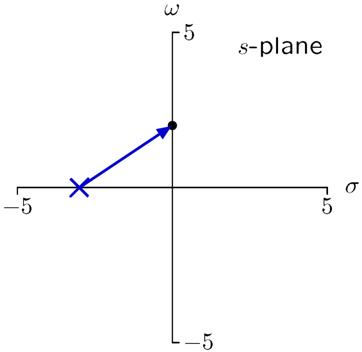

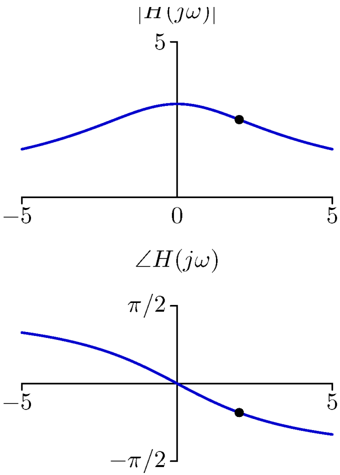

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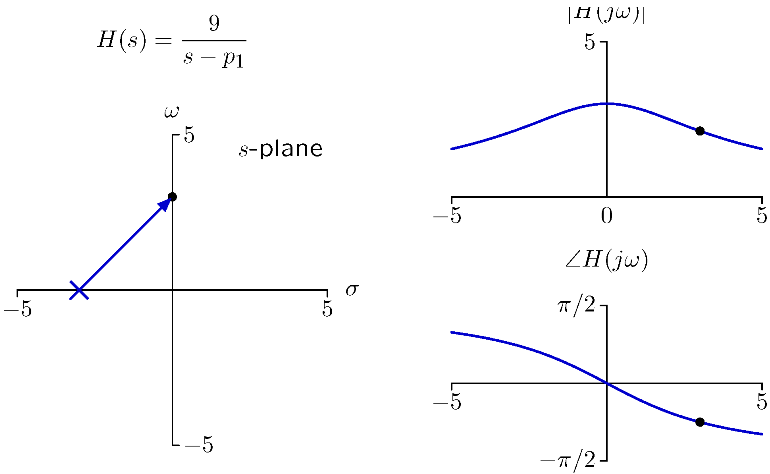

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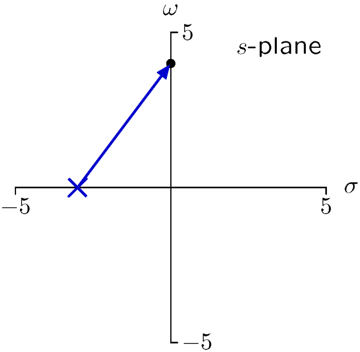

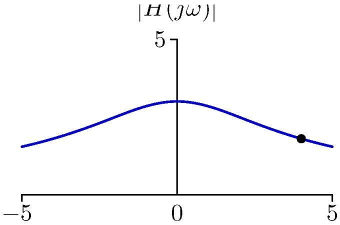

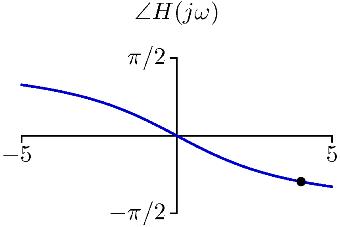

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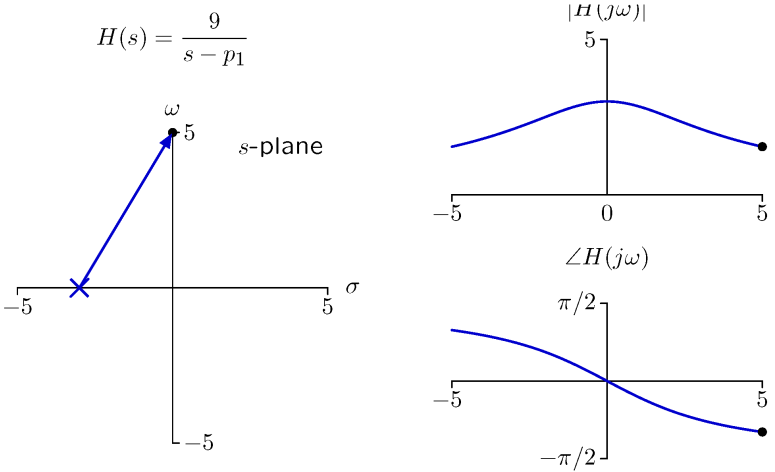
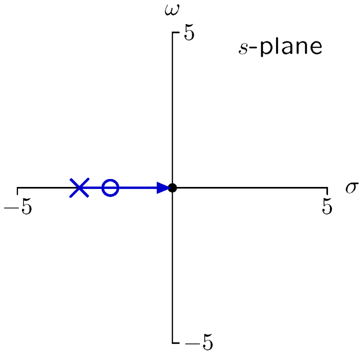
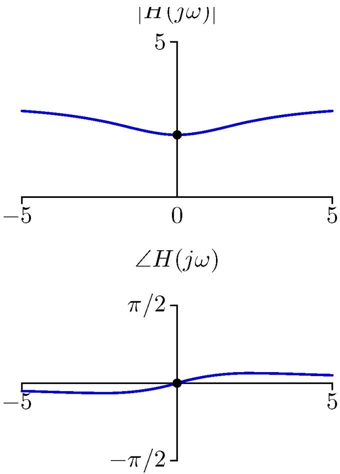


Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p1}$$





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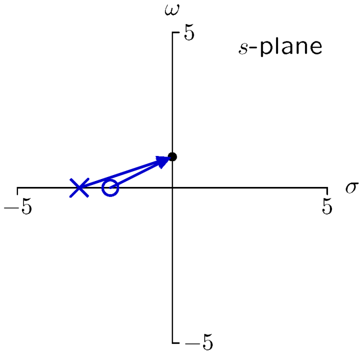

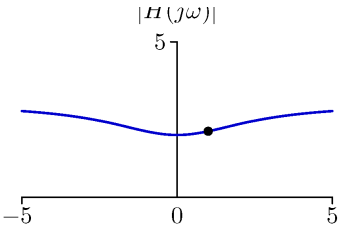

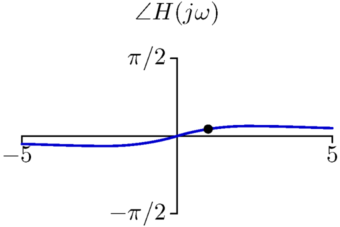

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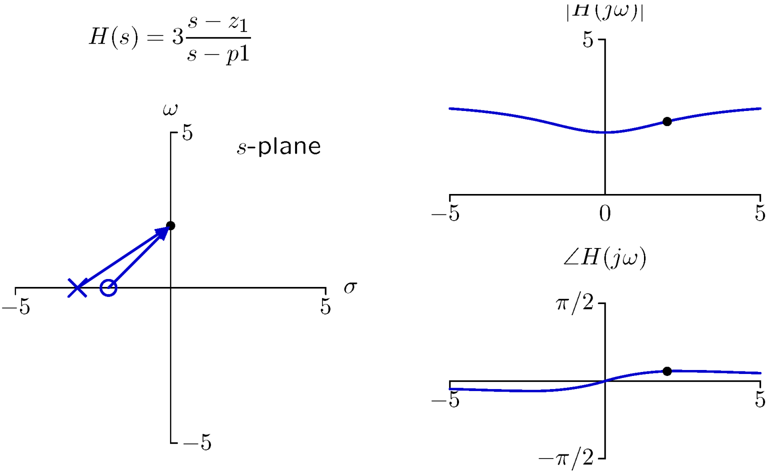

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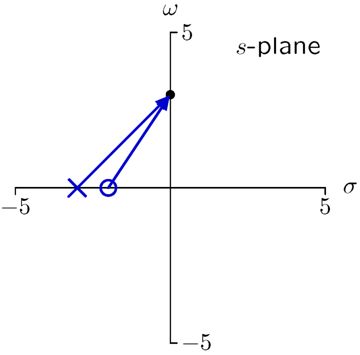
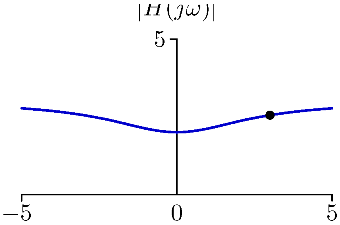
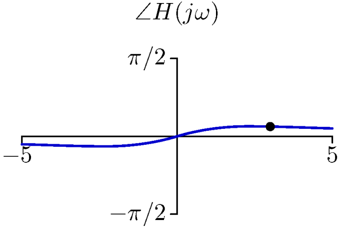
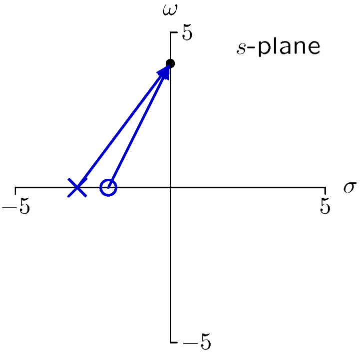


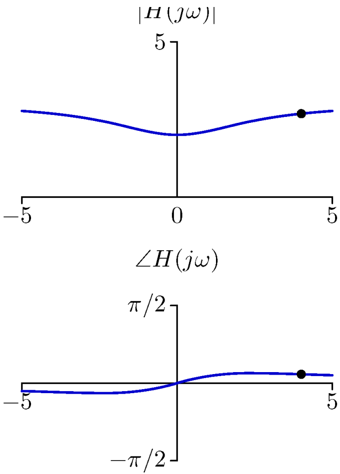




Vector Diagrams

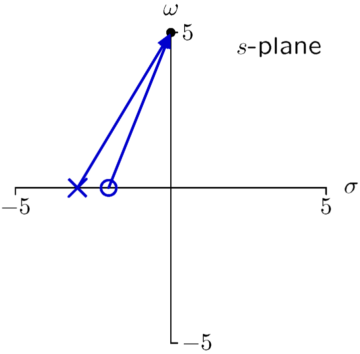
$$H(s) = 3\frac{s - z_1}{s - p1}$$

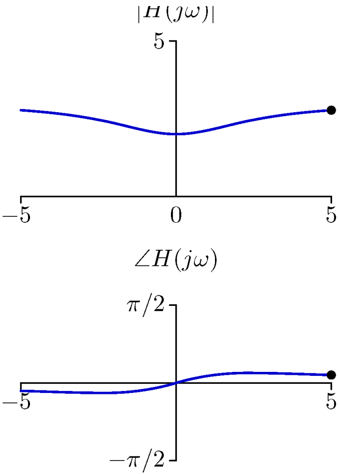




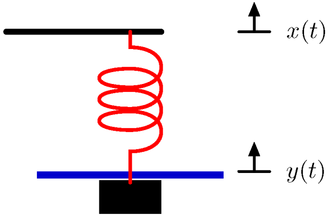
Vector Diagrams

$$H(s) = 3\frac{s - z_1}{s - p1}$$





Example: Mass, Spring, and Dashpot



$$F = Ma = M\ddot{y}(t) = K(x(t) - y(t)) - B\dot{y}(t)$$

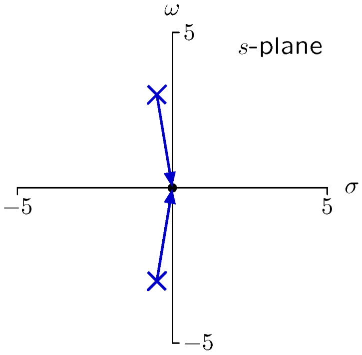
$$M\ddot{y}(t) + B\dot{y}(t) + Ky(t) = Kx(t)$$

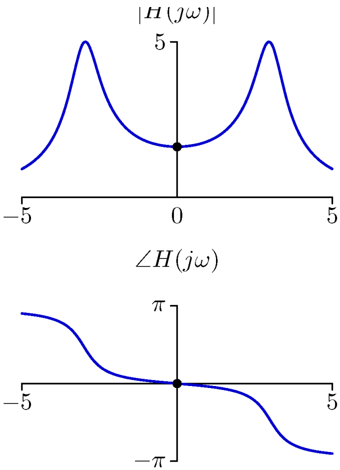
$$(s^2M + sB + K)\; Y(s) = KX(s)$$

$$H(s) = \frac{K}{s^2M + sB + K}$$

Vector Diagrams

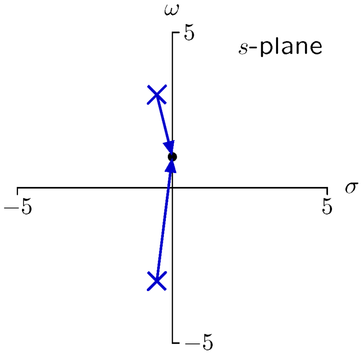
$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$

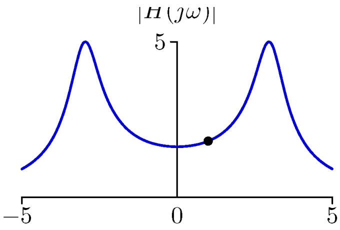


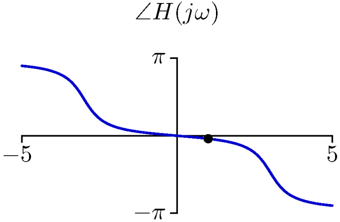


Vector Diagrams

$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$

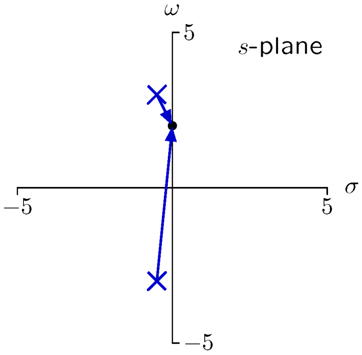


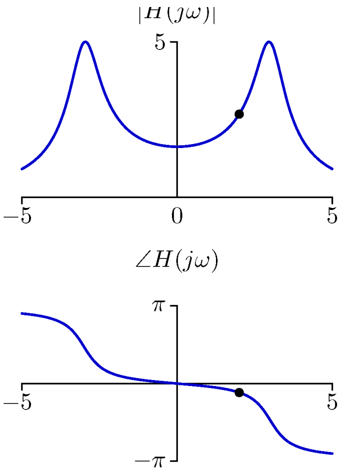




Vector Diagrams

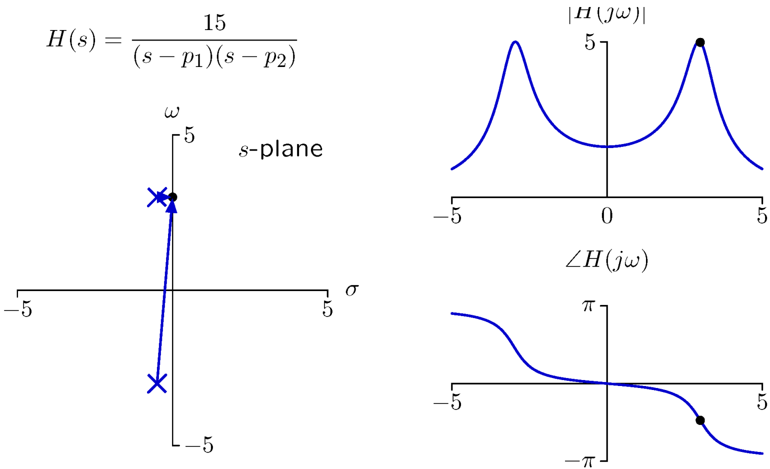
$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$





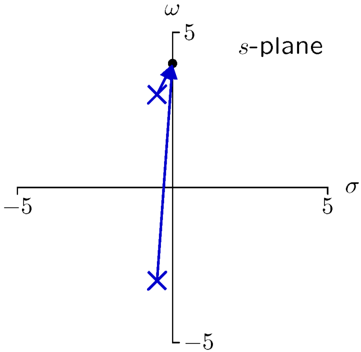
Vector Diagrams

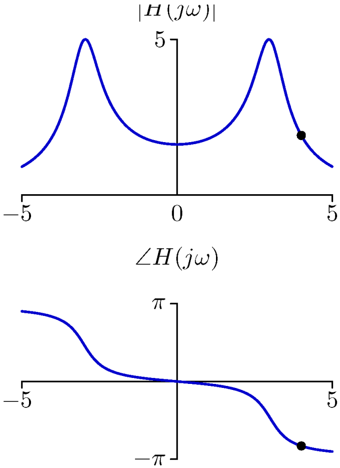
$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$



Vector Diagrams

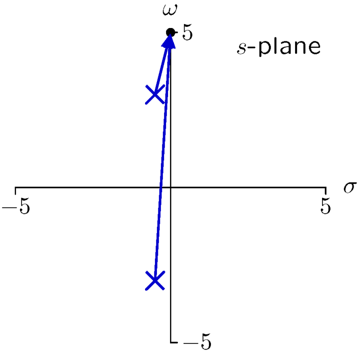
$$H(s) = \frac{15}{(s-p_1)(s-p_2)}$$

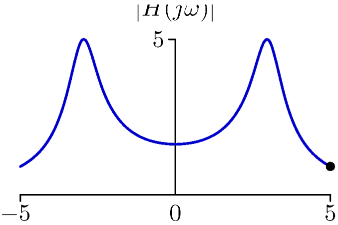


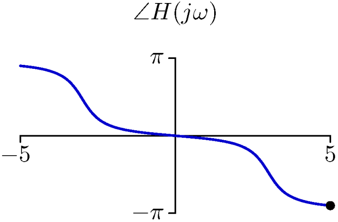


Vector Diagrams

$$H(s)=\frac{15}{(s-p_1)(s-p_2)}$$

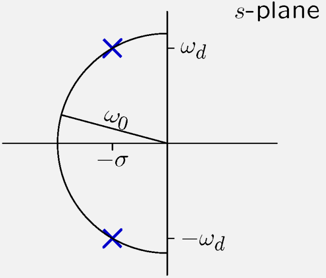






Check Yourself

Consider the system represented by the following poles.

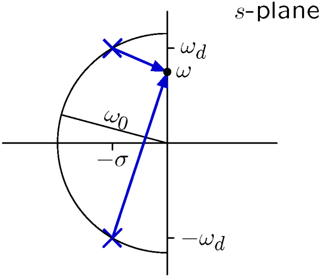


Find the frequency ω at which the magnitude of the response $y(t)$ is greatest if $x(t) = \cos \omega t$.

1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $0 < \omega < \omega_d$
4. none of the above

Check Yourself: Frequency Response

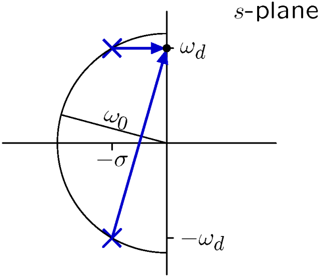
Analyze with vectors.



The product of the lengths is $\left(\sqrt{(\omega+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega-\omega_d)^2+\sigma^2}\right)$.

Check Yourself: Frequency Response

Analyze with vectors.

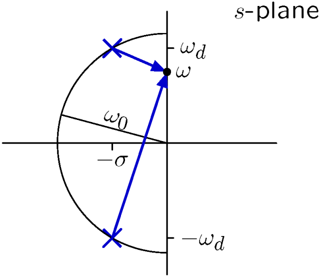


The product of the lengths is $\left(\sqrt{(\omega+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega-\omega_d)^2+\sigma^2}\right)$.

Decreasing ω from ω_d to $\omega_d-\epsilon$ decreases the product since length of bottom vector decreases as ϵ while length of top vector increases only ϵ^2.

Check Yourself: Frequency Response

More mathematically ...



The product of the lengths is $\left(\sqrt{(\omega+\omega_d)^2+\sigma^2}\right)\left(\sqrt{(\omega-\omega_d)^2+\sigma^2}\right)$.

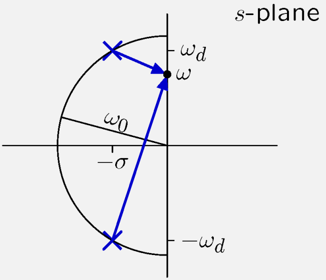
Maximum occurs where derivative of squared lengths is zero.

$$\frac{d}{d\omega}\left((\omega+\omega_d)^2+\sigma^2\right)\left((\omega-\omega_d)^2+\sigma^2\right)=0$$

$$\rightarrow \quad \omega^2=\omega_d^2-\sigma^2=\omega_0^2-2\sigma^2\,.$$

Check Yourself

Consider the system represented by the following poles.

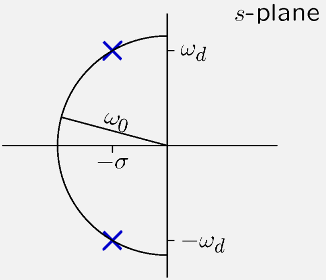


Find the frequency ω at which the magnitude of the response $y(t)$ is greatest if $x(t) = \cos \omega t$. 3

1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $0 < \omega < \omega_d$
4. none of the above

Check Yourself

Consider the system represented by the following poles.

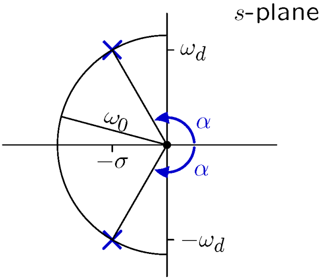


Find the frequency ω at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos \omega t$.

0. $0 < \omega < \omega_d$
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $\omega = \omega_0$
4. $\omega > \omega_0$
5. none

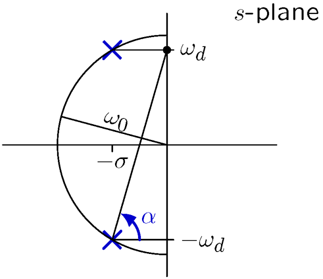
Check Yourself

The phase is 0 when $\omega = 0$.



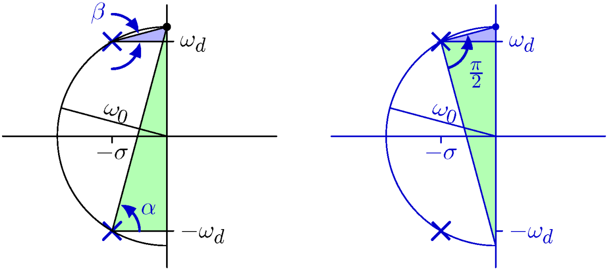
Check Yourself

The phase is less than $\pi/2$ when $\omega = \omega_d$.



Check Yourself

The phase at $\omega = \omega_0$ is $-\pi/2$.



Check Yourself

Check result by evaluating the system function.

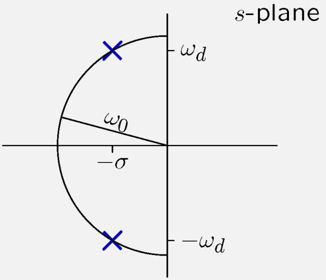
Substitute $s = j\omega_0 = j\sqrt{\frac{K}{M}}$ into

$$H(s) = \frac{K}{s^2M + sB + K} = \frac{K}{-\frac{K}{M}M + j\omega_0 B + K} = \frac{K}{j\omega_0 B}$$

The phase is $-\frac{\pi}{2}$.

Check Yourself

Consider the system represented by the following poles.



Find the frequency ω at which the phase of the response $y(t)$ is $-\pi/2$ if $x(t) = \cos \omega t$. 3

0. $0 < \omega < \omega_d$
1. $\omega = \omega_d$
2. $\omega_d < \omega < \omega_0$
3. $\omega = \omega_0$
4. $\omega > \omega_0$
5. none

Frequency Response: Summary

LTI systems can be characterized by responses to eternal sinusoids.

Many systems are naturally described by their frequency response.
– audio systems
– mass, spring, dashpot system

Frequency response is easy to calculate from the system function.

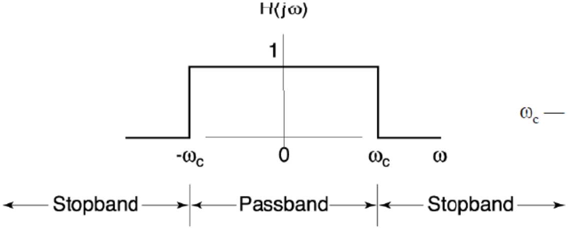
Frequency response lives on the $j\omega$ axis of the Laplace transform.

Filtering

Ideal*ized* Filters

Lowpass filter

CT

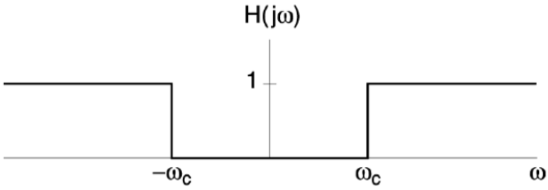


ω_c — cutoff frequency

Filtering

Highpass

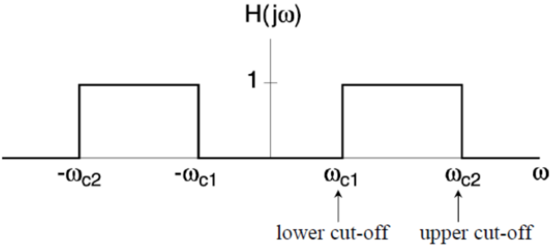
CT



Filtering

Bandpass

CT



Assignments

- Reading Assignment: Chap. 9.4, 10.4, 3.8-3.11, 6.0-6.2, 6.5
- Project 1