Signals and Systems

Lecture 7: Relations between CT and DT

Instructor: Prof. Yunlong Cai
Zhejiang University

03/13/2025

Partly adapted from the materials provided on the MIT OpenCourseWare

LTI Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Repeated use of differentiation property: $\frac{d}{dt} \leftrightarrow s, \quad \frac{d^k}{dt^k} \leftrightarrow s^k$

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$\Downarrow$$

$$Y(s) = H(s)X(s)$$

where $H(s) = \underbrace{\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}_{\text{Rational}}$ ⟵ roots of numerator ⇒ *zeros*; ⟵ roots of denominator ⇒ *poles*

Rational Transforms

- Many (but by no means all) Laplace transforms of interest to us are rational functions of s
  (in general, LTIs described by LCCDEs), i. e.

  $$X(s) = \frac{\mathrm{N}(s)}{\mathrm{D}(s)} \quad , \quad \mathrm{N}(s), \mathrm{D}(s) \text{ — polynomials in } s$$

- Roots of $\mathrm{N}(s)$ = *zeros* of $X(s)$

- Roots of $\mathrm{D}(s)$ = *poles* of $X(s)$

- Any $x(t)$ consisting of a linear combination of complex exponentials for $t > 0$ and for $t < 0$
  has a rational Laplace transform.

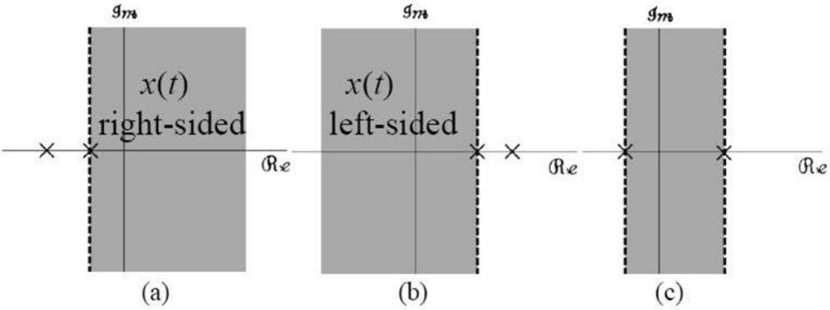
ROC for Rational Transforms

If $X(s)$ is rational, then its ROC is bounded by poles or extends to infinity. In addition, no poles of $X(s)$ are contained in the ROC.

Suppose $X(s)$ is rational, then

(a) If $x(t)$ is right-sided, the ROC is to the right of the rightmost pole.

(b) If $x(t)$ is left-sided, the ROC is to the left of the leftmost pole.

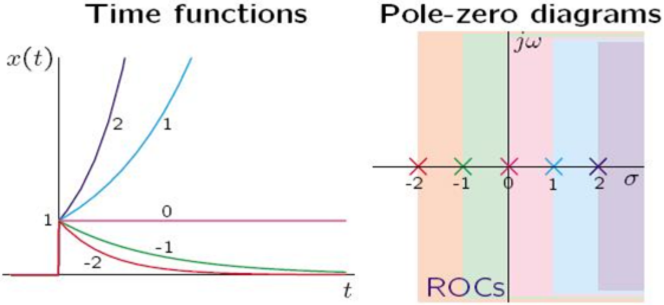


Relation Between Time Functions and Pole-zero Diagrams

Consider the causal exponential time function and its Laplace transform

$$x(t) = e^{\alpha t} u(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s) = \frac{1}{s - \alpha} \text{ for } \sigma > \alpha$$

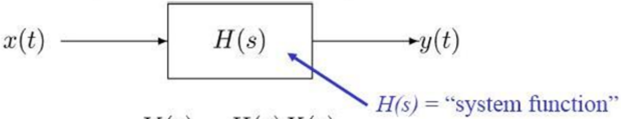
The following shows both the time functions and the pole-zero diagrams for 5 different values of α.



Relation Between Time Functions and Pole-zero Diagrams

Pole characteristics	Time function
On real axis	Exponential
On imaginary axis	Sinusoid
In complex s plane	Exponentially modulated sinusoid
Negative real part	Bounded
Positive real parts	Unbounded
Far from origin of s plane	Rapid time course

CT System Function Properties



$$Y(s) = H(s)X(s)$$

1) System is stable $\Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty \Leftrightarrow$ ROC of $H(s)$ includes $j\omega$ axis

2) Causality $\Rightarrow h(t)$ right-sided signal $\Rightarrow$ ROC of $H(s)$ is a right-half plane

Question:

If the ROC of $H(s)$ is a right-half plane, is the system causal?

Ex.

$$\begin{aligned} H(s) &= \frac{e^{sT}}{s+1}, \quad \Re e\{s\} > -1 \Rightarrow h(t) \text{ right-sided} \\ h(t) &= \mathcal{L}^{-1}\left\{\frac{e^{sT}}{s+1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}_{t \to t+T} = e^{-t}u(t)|_{t \to t+T} \\ &= e^{-(t+T)}u(t+T) \neq 0 \quad \text{at} \quad t < 0 \end{aligned}$$

Non-causal

Properties of CT Rational System Function

a) However, if $H(s)$ is ***rational***, then

The system is causal $\Leftrightarrow$ The ROC of $H(s)$ is to the right of the rightmost pole

b) If $H(s)$ is rational and is the system function of a causal system, then

The system is stable $\Leftrightarrow$ $j\omega$-axis is in ROC $\Leftrightarrow$ all poles are in LHP

The z-Transform - DT Laplace Transform

Recall:

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \overset{t=n\Delta t}{=} \lim_{\Delta t \to 0} \sum_{n=-\infty}^{\infty} \underbrace{x(n\Delta t)\cdot \Delta t}_{x[n]} \, (\underbrace{e^{s\Delta t}}_{z})^{-n}$$

$\Downarrow$ DT, Δt is now finite

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}}$$

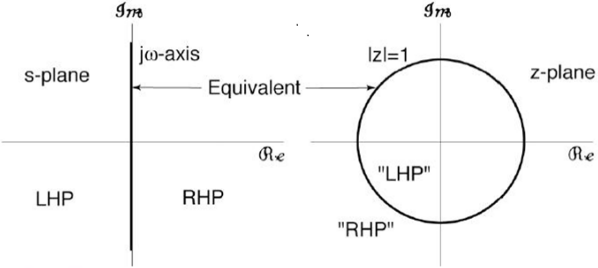
where $$z = e^{s\Delta t}$$

$$\Downarrow$$

The Relations Between z-Transform and Laplace Transform

$$e^{s\Delta t} = z$$

$j\omega$-axis in s-plane ($s = j\omega$) $\Leftrightarrow$ $|z| = |e^{j\omega\Delta t}| = 1$ — a unit circle in z-plane



- A vertical line in s-plane, $Re(s)$ = constant $\Leftrightarrow$ $|e^{s\Delta t}|$ = constant, a circle in z-plane.
- LHP in s-plane, $Re(s) < 0 \Rightarrow |z| = |e^{s\Delta t}| < 1$, inside the $|z| = 1$ circle, special case, $Re(s) = -\infty \Leftrightarrow |z| = 0$.
- RHP in s-plane, $Re(s) > 0 \Rightarrow |z| = |e^{s\Delta t}| > 1$, outside the $|z| = 1$ circle, special case, $Re(s) = +\infty \Leftrightarrow |z| = \infty$.

Relation Between Causal DT Time Functions and Pole-zero Diagrams

Pole characteristics	Time function
On positive real axis	Geometric
On negative real axis	Alternating sign geometric
On unit circle	Sinusoidal
In complex z plane	Geometrically modulated sinusoid
Inside unit circle	Bounded
Outside unit circle	Unbounded
Far from $z = 1$	Rapid time course

DT System Function Properties - Causality

$h[n]$ right-sided $\Rightarrow$ ROC is the exterior of a circle *possibly* including $z = \infty$:

$$H(z) = \sum_{n=N_1}^{\infty} h[n] z^{-n}$$

If $N_1 < 0$, then the term $h[N_1]z^{-N_1} \to \infty$ at $z = \infty$
$\Rightarrow$ ROC outside a circle, but does *not* include ∞.

Causal $\Leftrightarrow N_1 \geq 0$

No z^m terms with $m > 0$
$\Rightarrow z = \infty \in$ ROC

$\Downarrow$

A DT LTI system with system function $H(z)$ is causal $\Leftrightarrow$ the ROC of $H(z)$ is the exterior of a circle *including* $z = \infty$

DT LTI Systems Described by LCCDEs

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Use the time-shift property

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$\Downarrow$$

$$Y(z) = H(z)X(z)$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad \text{— Rational}$$

Causality for Systems with Rational System Functions

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

$$\Downarrow \text{ No poles at } \infty, \text{ if } M \leq N$$

A DT LTI system with rational system function $H(z)$ is causal

$\Leftrightarrow$ (a) the ROC is the exterior of a circle outside the outermost pole;

and (b) if we write $H(z)$ as a ratio of polynomials

$$H(z) = \frac{N(z)}{D(z)}$$

then

$$\text{degree } N(z) \leq \text{ degree } D(z)$$

DT System Function Properties - Stability

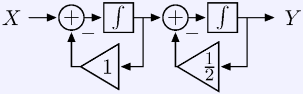
- LTI System Stable $\Leftrightarrow \sum_{n=-\infty}^{\infty} |h[n]| < \infty \Leftrightarrow$ ROC of $H(z)$ includes the unit circle $|z| = 1$

- A causal LTI system with rational system function is stable $\Leftrightarrow$ all poles are inside the unit circle, i.e. have magnitudes < 1

Concept Map: Continuous-Time Systems

Relations among CT representations.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{2\mathcal{A}^2}{2 + 3\mathcal{A} + \mathcal{A}^2}$$

Impulse Response

$$h(t) = 2(e^{-t/2} - e^{-t})\, u(t)$$

Differential Equation

$$2\ddot{y}(t) + 3\dot{y}(t) + y(t) = 2x(t)$$

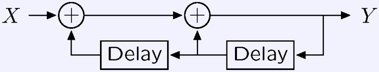
System Function

$$\frac{Y(s)}{X(s)} = \frac{2}{2s^2 + 3s + 1}$$

Concept Map: Discrete-Time Systems

Relations among DT representations.

Block Diagram



System Functional

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Unit-Sample Response

$$h[n]:\ \ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Difference Equation

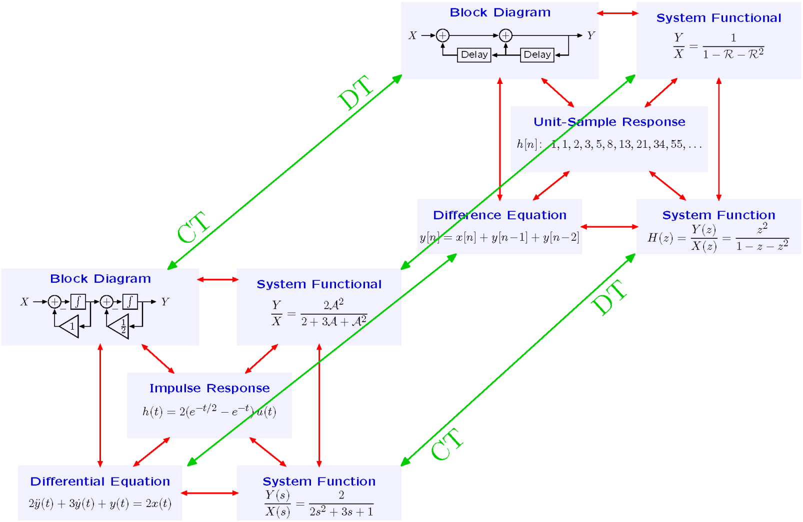
$$y[n] = x[n] + y[n-1] + y[n-2]$$

System Function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{1 - z - z^2}$$

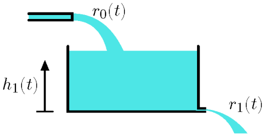
Concept Map

Relations between CT and DT representations.

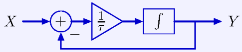


First-Order CT System

Example: leaky tank.



Block Diagram



System Functional

$$\frac{Y}{X} = \frac{\mathcal{A}}{\mathcal{A}+\tau}$$

Impulse Response

$$h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$$

Differential Equation

$$\tau\dot{r}_1(t) = r_0(t) - r_1(t)$$

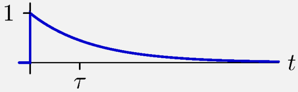
System Function

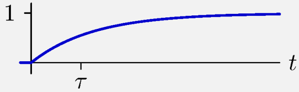
$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{1+\tau s}$$

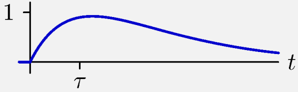
Check Yourself

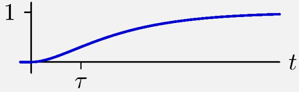
What is the "step response" of the leaky tank system?

$$u(t) \longrightarrow \boxed{\text{Leaky Tank}} \longrightarrow s(t) = ?$$

1.


2.


3.


4.


5. none of the above

Check Yourself

What is the "step response" of the leaky tank system?

$$\delta(t) \longrightarrow \boxed{H(s)} \longrightarrow h(t) = \tfrac{1}{\tau}e^{-t/\tau}u(t)$$

$$u(t) \longrightarrow \boxed{H(s)} \longrightarrow s(t) = ?$$

$$\delta(t) \longrightarrow \boxed{\tfrac{1}{s}} \xrightarrow{u(t)} \boxed{H(s)} \longrightarrow s(t) = ?$$

$$\delta(t) \longrightarrow \boxed{H(s)} \xrightarrow{h(t)} \boxed{\tfrac{1}{s}} \longrightarrow s(t) = \int_{-\infty}^{t} h(t')dt'$$

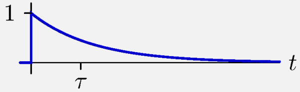
$$s(t) = \int_{-\infty}^{t} \frac{1}{\tau}e^{-t'/\tau}u(t')dt' = \int_{0}^{t} \frac{1}{\tau}e^{-t'/\tau}dt' = (1 - e^{-t/\tau})\,u(t)$$

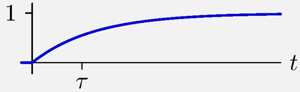
Reasoning with systems.

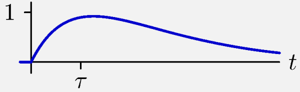
Check Yourself

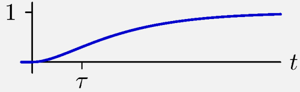
What is the "step response" of the leaky tank system? 2

$$u(t) \longrightarrow \boxed{\text{Leaky Tank}} \longrightarrow s(t) = ?$$

1. 

2. 

3. 

4. 

5. none of the above

Forward Euler Approximation

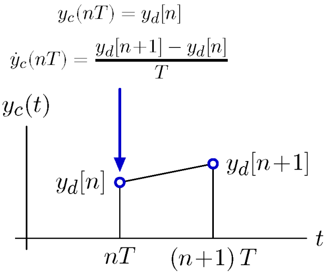
Approximate leaky-tank response using forward Euler approach.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big((n+1)T\big) - y_c\big(nT\big)}{T} = \frac{y_d[n+1] - y_d[n]}{T}$$



Forward Euler Approximation

Approximate leaky-tank response using forward Euler approach.

Substitute

$$x_d[n] = x_c(nT)$$
$$y_d[n] = y_c(nT)$$
$$\dot{y}_c(nT) \approx \frac{y_c\big((n+1)T\big) - y_c\big(nT\big)}{T} = \frac{y_d[n+1] - y_d[n]}{T}$$

into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

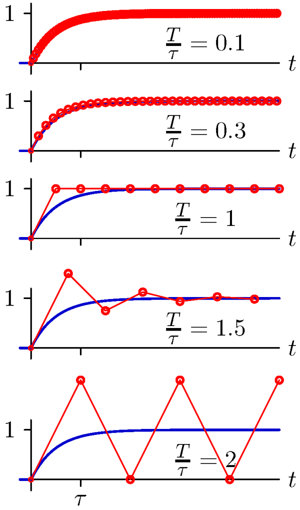
$$\frac{\tau}{T}\Big(y_d[n+1] - y_d[n]\Big) = x_d[n] - y_d[n]\,.$$

Solve:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Forward Euler Approximation

Plot.



Why is this approximation badly behaved?

Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole.

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

Check Yourself

DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Take the Z transform:

$$zY_d(z) - \left(1 - \frac{T}{\tau}\right) Y_d(z) = \frac{T}{\tau} X_d(z)$$

Solve for the system function:

$$H(z) = \frac{Y_d(z)}{X_d(z)} = \frac{\frac{T}{\tau}}{z - \left(1 - \frac{T}{\tau}\right)}$$

Pole at $z = 1 - \dfrac{T}{\tau}$.

Check Yourself

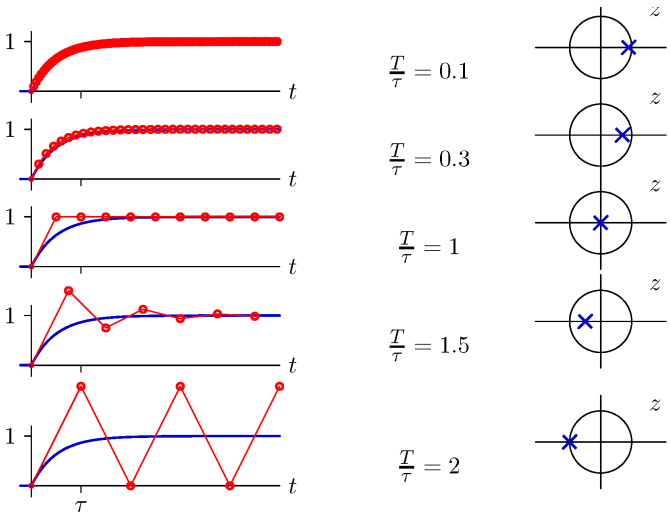
DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 2

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

Dependence of DT pole on Stepsize



The CT pole was fixed ($s = -\frac{1}{\tau}$). Why is the DT pole changing?

Dependence of DT pole on Stepsize

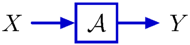
Change in DT pole: problem specific or property of forward Euler?

Dependence of DT pole on Stepsize

Change in DT pole: problem specific or property of forward Euler?

Approach: make a systems model of forward Euler method.

CT block diagrams: adders, gains, and integrators:

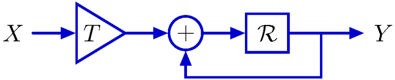


$$\dot{y}(t) = x(t)$$

Forward Euler approximation:

$$\frac{y[n+1] - y[n]}{T} = x[n]$$

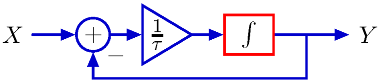
Equivalent system:



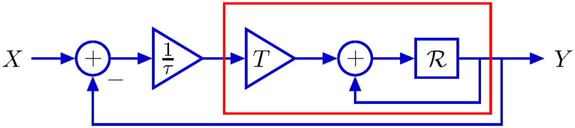
Forward Euler: substitute equivalent system for all integrators.

Example: leaky tank system

Started with leaky tank system:



Replace integrator with forward Euler rule:



Write system functional:

$$\frac{Y}{X} = \frac{\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}}{1+\frac{T}{\tau}\frac{\mathcal{R}}{1-\mathcal{R}}} = \frac{\frac{T}{\tau}\mathcal{R}}{1-\mathcal{R}+\frac{T}{\tau}\mathcal{R}} = \frac{\frac{T}{\tau}\mathcal{R}}{1-\left(1-\frac{T}{\tau}\right)\mathcal{R}}$$

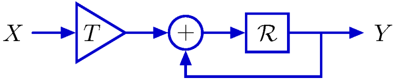
Equivalent to system we previously developed:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Model of Forward Euler Method

Replace every integrator in the CT system

with the forward Euler model:



Substitute the DT operator for $\mathcal{A}$:

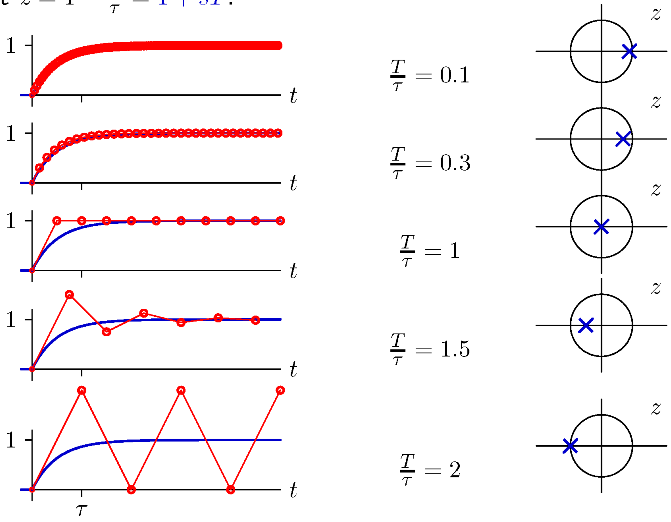
$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T\mathcal{R}}{1-\mathcal{R}} = \frac{\frac{T}{z}}{1-\frac{1}{z}} = \frac{T}{z-1}$$

Forward Euler maps $s \rightarrow \dfrac{z-1}{T}$.

Or equivalently: $z = 1 + sT$.

Dependence of DT pole on Stepsize

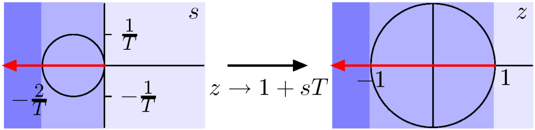
Pole at $z = 1 - \frac{T}{\tau} = 1 + sT$.



Forward Euler: Mapping CT poles to DT poles

Forward Euler Map:

s	$\rightarrow$	$z = 1 + sT$
0		1
$-\frac{1}{T}$		0
$-\frac{2}{T}$		-1



DT stability: CT pole must be inside circle of radius $\frac{1}{T}$ at $s = -\frac{1}{T}$.

$$-\frac{2}{T} < -\frac{1}{\tau} < 0 \quad \rightarrow \quad \frac{T}{\tau} < 2$$

Backward Euler Approximation

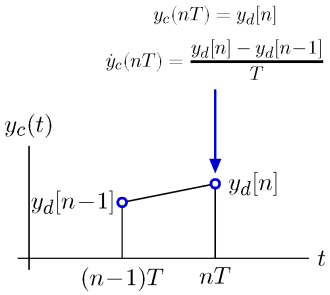
We can do a similar analysis of the backward Euler method.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c(nT) - y_c\big((n-1)T\big)}{T} = \frac{y_d[n] - y_d[n-1]}{T}$$



Backward Euler Approximation

We can do a similar analysis of the backward Euler method.

Substitute

$$x_d[n] = x_c(nT)$$

$$y_d[n] = y_c(nT)$$

$$\dot{y}_c(nT) \approx \frac{y_c\big(nT\big) - y_c\big((n-1)T\big)}{T} = \frac{y_d[n] - y_d[n-1]}{T}$$

into the differential equation

$$\tau \dot{y}_c(t) = x_c(t) - y_c(t)$$

to obtain

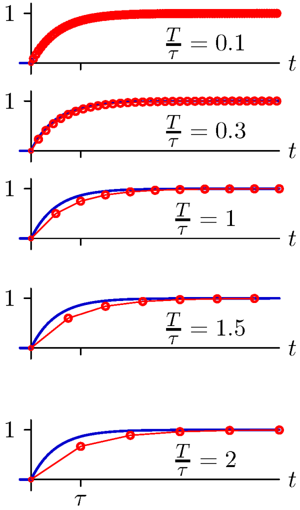
$$\frac{\tau}{T}\Big(y_d[n] - y_d[n-1]\Big) = x_d[n] - y_d[n]\,.$$

Solve:

$$\left(1 + \frac{T}{\tau}\right) y_d[n] - y_d[n-1] = \frac{T}{\tau} x_d[n]$$

Backward Euler Approximation

Plot.



This approximation is better behaved. Why?

Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Find the DT pole.

1. $z=\dfrac{T}{\tau}$
2. $z=1-\dfrac{T}{\tau}$
3. $z=\dfrac{\tau}{T}$
4. $z=-\dfrac{\tau}{T}$
5. $z=\dfrac{1}{1+\frac{T}{\tau}}$

Check Yourself

DT approximation:

$$\left(1+\frac{T}{\tau}\right)y_d[n]-y_d[n-1]=\frac{T}{\tau}x_d[n]$$

Take the Z transform:

$$\left(1+\frac{T}{\tau}\right)Y_d(z)-z^{-1}Y_d(z)=\frac{T}{\tau}X_d(z)$$

Find the system function:

$$H(z)=\frac{Y_d(z)}{X_d(z)}=\frac{\frac{T}{\tau}z}{\left(1+\frac{T}{\tau}\right)z-1}$$

Pole at $z=\dfrac{1}{1+\frac{T}{\tau}}$.

Check Yourself

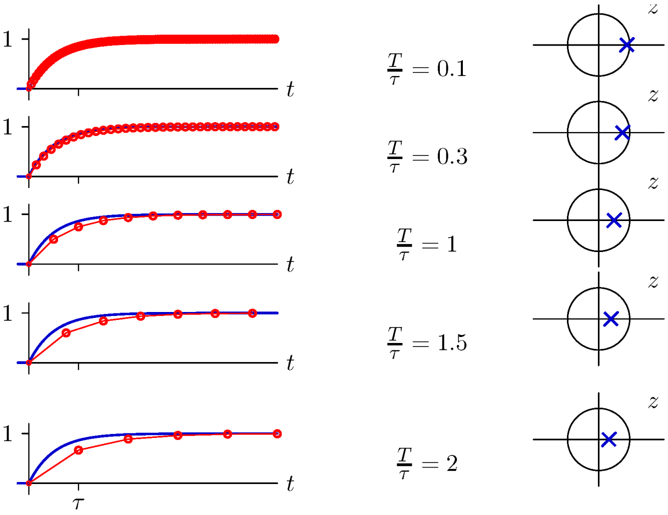
DT approximation:

$$y_d[n+1] - \left(1 - \frac{T}{\tau}\right) y_d[n] = \frac{T}{\tau} x_d[n]$$

Find the DT pole. 5

1. $z = \frac{T}{\tau}$
2. $z = 1 - \frac{T}{\tau}$
3. $z = \frac{\tau}{T}$
4. $z = -\frac{\tau}{T}$
5. $z = \frac{1}{1 + \frac{T}{\tau}}$

Dependence of DT pole on Stepsize



Why is this approximation better behaved?

Dependence of DT pole on Stepsize

Make a systems model of backward Euler method.

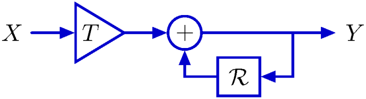
CT block diagrams: adders, gains, and integrators:

$$\dot{y}(t) = x(t)$$

Backward Euler approximation:

$$\frac{y[n] - y[n-1]}{T} = x[n]$$

Equivalent system:

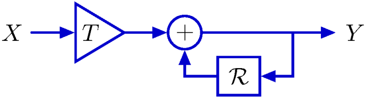


Backward Euler: substitute equivalent system for all integrators.

Model of Backward Euler Method

Replace every integrator in the CT system

with the backward Euler model:



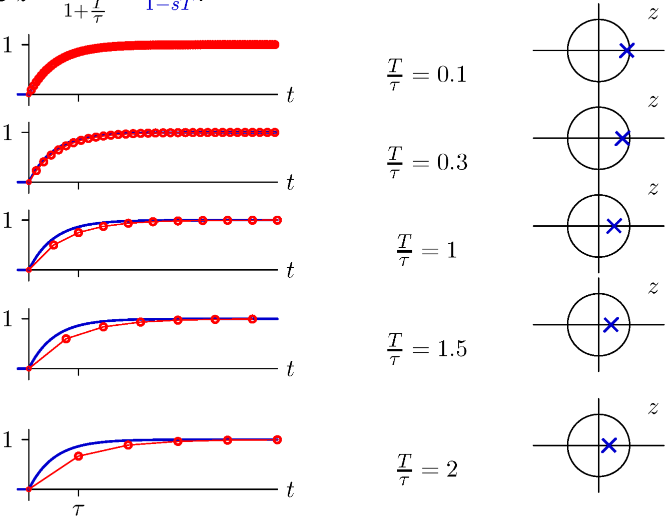
Substitute the DT operator for $\mathcal{A}$:

$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T}{1-\mathcal{R}} = \frac{T}{1-\frac{1}{z}}$$

Backward Euler maps $z \rightarrow \dfrac{1}{1-sT}$.

Dependence of DT pole on Stepsize

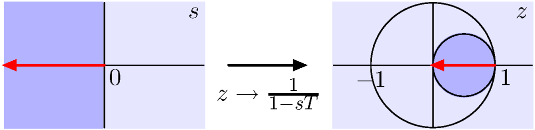
Pole at $z = \frac{1}{1+\frac{T}{\tau}} = \frac{1}{1-sT}$.



Backward Euler: Mapping CT poles to DT poles

Backward Euler Map:

s	$\rightarrow$	$z = \frac{1}{1-sT}$
0		1
$-\frac{1}{T}$		$\frac{1}{2}$
$-\frac{2}{T}$		$\frac{1}{3}$



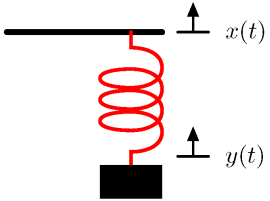
The entire left half-plane maps inside a circle with radius $\frac{1}{2}$ at $z = \frac{1}{2}$.

If CT system is stable, then DT system is also stable.

Masses and Springs, Forwards and Backwards

In Homework 2, you investigated three numerical approximations to a mass and spring system:

- forward Euler
- backward Euler
- centered method



Trapezoidal Rule

The trapezoidal rule uses centered differences.

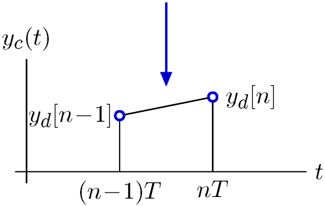
$$\dot{y}(t) = x(t)$$

Trapezoidal rule:

$$\frac{y[n] - y[n-1]}{T} = \frac{x[n] + x[n-1]}{2}$$

$$y_c\Big((n+\tfrac{1}{2})T\Big) = \frac{y_d[n] + y_d[n-1]}{2}$$

$$\dot{y}_c\Big((n+\tfrac{1}{2})T\Big) = \frac{y_d[n] - y_d[n-1]}{T}$$



Trapezoidal Rule

The trapezoidal rule uses centered differences.

$$\dot{y}(t) = x(t)$$

Trapezoidal rule:

$$\frac{y[n] - y[n-1]}{T} = \frac{x[n] + x[n-1]}{2}$$

Z transform:

$$H(z) = \frac{Y(s)}{X(s)} = \frac{T}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right) = \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Map:

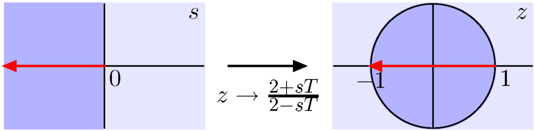
$$\mathcal{A} = \frac{1}{s} \rightarrow \frac{T}{2}\left(\frac{z+1}{z-1}\right)$$

Trapezoidal rule maps $z \rightarrow \dfrac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$.

Trapezoidal Rule: Mapping CT poles to DT poles

Trapezoidal Map:

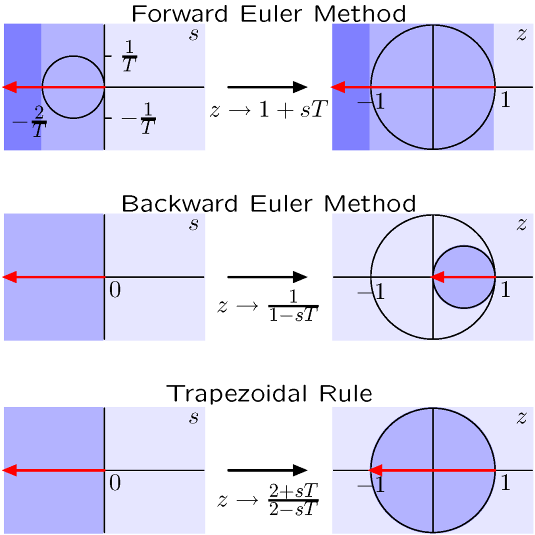
s	$\rightarrow$	$z = \frac{1+\frac{sT}{2}}{1-\frac{sT}{2}}$
0		1
$-\frac{1}{T}$		$\frac{1}{3}$
$-\frac{2}{T}$		0
$-\infty$		-1
$j\omega$		$\frac{2+j\omega T}{2-j\omega T}$



The entire left-half plane maps inside the unit circle.

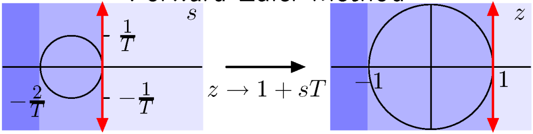
The $j\omega$ axis maps onto the unit circle

Mapping s to z: Leaky-Tank System

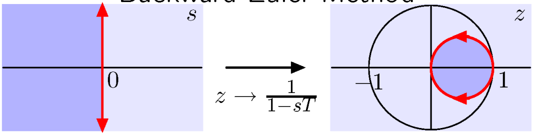


Mapping s to z: Leaky-Tank System

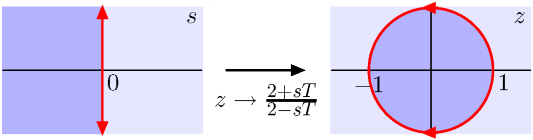
Forward Euler Method



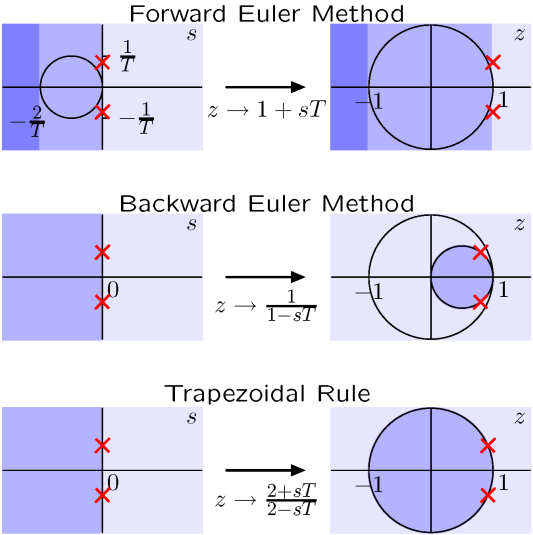
Backward Euler Method



Trapezoidal Rule

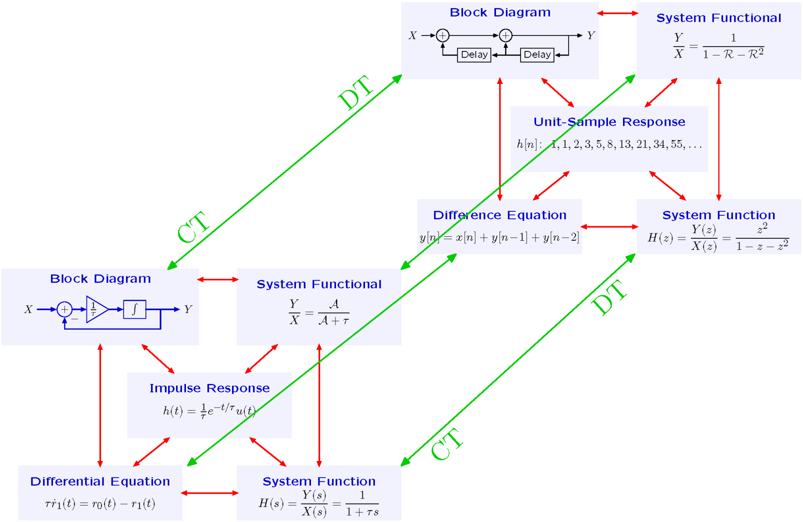


Mapping s to z: Leaky-Tank System



Concept Map

Relations between CT and DT representations.



Assignments

- Reading Assignment: Chap. 9.6-9.9, 10.6-10.9