Problem 1

(3) $a_k = \int_{t_0}^{t_0+T} x(t)\, e^{-j\frac{\pi}{3}kt} = \int_{t_0}^{t_0+T} x(t)\, e^{-j\frac{\pi}{3}(k+6)t} = a_{k+6} = a_{k-6} = 0$ for $k>2$

$\Rightarrow a_k = 0 \,,\ k \le -3$

(2) $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{j\frac{\pi}{3}kt} = a_{-2} e^{j\frac{\pi}{3}(-2)t} + a_{-1} e^{j\frac{\pi}{3}(-1)t} + a_1 e^{j\frac{\pi}{3}t} + a_2 e^{j\frac{\pi}{3}\cdot 2t}$

(4) $x(t) = -x(t-3) \Rightarrow \sum_{k=-\infty}^{+\infty} a_k e^{j\frac{\pi}{3}kt} = -\sum_{k=-\infty}^{+\infty} a_k e^{j\frac{\pi}{3}k(t-3)}$

$\Leftrightarrow \sum_{k=-\infty}^{+\infty} a_k e^{j\frac{\pi}{3}kt} = -e^{-j\pi k} \sum_{k=-\infty}^{+\infty} a_k e^{j\frac{\pi}{3}kt}$

$\Rightarrow a_k = 0$ for k is even $\Rightarrow a_2 = a_{-2} = 0$

(1) $x(t)$ is real value $\Rightarrow a_{-2} = a_2 \,,\ a_{-1} = a_1$

$\Rightarrow x(t) = a_1 \left(e^{-j\frac{\pi}{3}t} + e^{j\frac{\pi}{3}t} \right) = 2a_1 \cos\left(\frac{\pi}{3}t\right)$

(5)(6) $\frac{1}{6}\int_{-3}^{3} |x(t)|^2\, dt = \frac{1}{2} \Rightarrow \frac{1}{6}\int_{-3}^{3} 4a_1^2 \cos^2\left(\frac{\pi}{3}t\right) dt = \frac{1}{2}$

$\Rightarrow a_1 = \frac{1}{2}$

$\Rightarrow x(t) = \cos\left(\frac{\pi}{3}t\right) \quad A = 1,\ B = \frac{\pi}{3},\ C = 0$

Problem 2.

(a) $x[n-n_0] = \sum_{k=\langle N\rangle} a_k e^{jk\cdot\frac{2\pi}{N}(n-n_0)} = \sum_{k=\langle N\rangle} e^{-jk\cdot\frac{2\pi}{N}n_0} a_k e^{jk\frac{2\pi}{N}\cdot n}$

(b) $x[n] - x[n-1] = \sum_{k=\langle N\rangle} a_k e^{jk\cdot\frac{2\pi}{N}n} - \sum_{k=\langle N\rangle} a_k e^{jk\cdot\frac{2\pi}{N}(n-1)}$

$= \sum_{k=\langle N\rangle} \left(1 - e^{-jk\frac{2\pi}{N}}\right) a_k e^{jk\frac{2\pi}{N}\cdot n}$

(c) $x[n] - x[n-\frac{N}{2}] = \sum_{k=\langle N\rangle} a_k e^{jk\cdot\frac{2\pi}{N}n} - \sum_{k=\langle N\rangle} a_k e^{jk\cdot\frac{2\pi}{N}(n-\frac{N}{2})}$

$= \sum_{k=\langle N\rangle} \left(1 - e^{-jk\frac{2\pi}{N}\cdot\frac{N}{2}}\right) a_k e^{jk\frac{2\pi}{N}\cdot n}$

$= \begin{cases} 0, & k \text{ is even} \\ \sum_{k=\langle N\rangle} 2a_k e^{jk\frac{2\pi}{N}n}, & k \text{ is odd} \end{cases}$

(d) $x[n] + x[n+\frac{N}{2}] = \sum_{k=\langle N\rangle} a_k e^{jk\cdot\frac{2\pi}{N}n} + \sum_{k=\langle N\rangle} a_k e^{jk\cdot\frac{2\pi}{N}(n+\frac{N}{2})}$

$= \sum_{k=\langle N\rangle} \left(1 + e^{jk\frac{2\pi}{N}\cdot\frac{N}{2}}\right) a_k e^{jk\frac{2\pi}{N}\cdot n}$

$= \begin{cases} 0, & k \text{ is odd} \\ \sum_{k=\langle N\rangle} 2a_k e^{jk\frac{2\pi}{N}n}, & k \text{ is even} \end{cases}$

(e) $x^*[-n] = \left(\sum_{k=\langle N \rangle} a_k e^{-jk\frac{2\pi}{N} n} \right)^* = \sum_{k=\langle N \rangle} a_k^* \, e^{jk\frac{2\pi}{N}\cdot n}$

(f) $(-1)^n x[n] = \left(e^{j\frac{2\pi}{N}\cdot\frac{N}{2}} \right)^n \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}\cdot n} = \sum_{k=\langle N \rangle} a_k e^{j(k+\frac{N}{2})\frac{2\pi}{N}\cdot n}$

$= \sum_{k=\langle N \rangle} a_{k-\frac{N}{2}} \, e^{jk\frac{2\pi}{N} n}$

(g) $a_{ck} = \frac{1}{2N} \sum_{n=0}^{2N-1} (-1)^n x[n] e^{-jk(\frac{2\pi}{2N}) n} = \frac{1}{2N} \sum_{n=0}^{2N-1} x[n] \, e^{j\frac{2\pi}{2N}\cdot N \cdot n} \, e^{-jk(\frac{2\pi}{2N}) n}$

$= \frac{1}{2N} \sum_{n=0}^{2N-1} x[n] \, e^{-(k-N)\cdot j \cdot \frac{2\pi}{2N}\cdot n}$

$= \frac{1}{2N} \sum_{n=0}^{N-1} x[n] \, e^{\frac{-(k-N)}{2}\cdot j \cdot \frac{2\pi}{N}\cdot n} + \frac{1}{2N} \sum_{n=N}^{2N-1} x[n] \, e^{\frac{-(k-N)}{2}\cdot j\cdot\frac{2\pi}{N}\cdot n}$

$= \frac{1}{2N} \sum_{n=0}^{N-1} x[n] \, e^{\frac{-(k-N)}{2}\cdot j \cdot \frac{2\pi}{N}\cdot n} + \frac{1}{2N} \sum_{n=0}^{N-1} x[n] \, e^{\frac{-(k-N)}{2}\cdot j\cdot\frac{2\pi}{N}(n+N)}$

$= \frac{1}{2} a_{\frac{k-N}{2}} + \frac{1}{2} e^{-j\pi(k-N)} a_{\frac{k-N}{2}}$

$\Rightarrow (-1)^n x[n] = \sum_{k=\langle N \rangle} \frac{1}{2}(1 - e^{-j\pi k}) a_k e^{jk\frac{2\pi}{2N}\cdot n}$

(h) $y[n] = \frac{1}{2}\left(x[n] + (-1)^n x[n] \right)$

$$= \begin{cases} \sum_{k=\langle N \rangle} \frac{1}{2}(a_k + a_{k-\frac{N}{2}}) e^{jk\frac{2\pi}{N}\cdot n} & N \text{ is even} \\ \sum_{k=\langle n \rangle} \frac{1}{2}\left(a_k + \frac{1}{2}(1 - e^{-j\pi k}) a_{\frac{k-N}{2}}\right) \, e^{jk(\frac{2\pi}{2N})\cdot n} \end{cases}$$

Problem 3.

(a) $z[n+N] = \sum_{r=\langle N \rangle} x[r] y[n+N-r] = \sum_{r=\langle N \rangle} x[r] y[n-r] = z[n]$

$\Rightarrow z[n]$ is cycled and $T = N$

(b) $c_k = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{r=\langle N \rangle} x[r] y[n-r] e^{-j\frac{2\pi}{N}\cdot kn}$

$= \frac{1}{N} \sum_{r=\langle N \rangle} x[r] \sum_{n=0}^{N-1} y[n-r] \, e^{-j\cdot\frac{2\pi}{N}\cdot k(n-r+r)}$

$= \sum_{r=\langle N \rangle} x[r] \cdot e^{-j\frac{2\pi}{N}\cdot kr} \cdot b_k$

$= N a_k b_k$

(c) $x[n] = \sin\left(\frac{3\pi n}{4}\right) = \frac{1}{2i}\left(e^{j\frac{3\pi}{4}n} - e^{-j\frac{3\pi}{4}n}\right)$

$= \frac{1}{2i}\left(e^{j\cdot\frac{2\pi}{8}3n} - e^{j\cdot\frac{2\pi}{8}(-3)n}\right)$

$\Rightarrow a_k = \begin{cases} \frac{1}{2i}, & k=3 \\ -\frac{1}{2i}, & k=-3 \\ 0, & \text{else} \end{cases}$

$b_k = \frac{1}{8}\sum_{n=0}^{7} y[n]\, e^{-j\frac{2\pi}{8}\cdot kn} = \frac{1}{8}\left(1+e^{-j\frac{\pi}{4}k} + e^{-j\frac{\pi}{2}k} + e^{-j\frac{3\pi}{4}k}\right)$

$\Rightarrow c_k = 8a_k b_k = \begin{cases} \frac{1}{2i}\left(1+e^{-j\frac{3\pi}{4}} + e^{-j\frac{3\pi}{2}} + e^{-j\frac{9\pi}{4}}\right), & k=3 \\ -\frac{1}{2i}\left(1+e^{j\frac{3\pi}{4}} + e^{j\frac{3\pi}{2}} + e^{j\frac{9\pi}{4}}\right), & k=-3 \\ 0, & \text{else} \end{cases}$

$\Rightarrow z[n] = \sum_{k=\langle r\rangle} c_k e^{j\frac{2\pi}{8}\cdot kn}$

(d) $a_k = \frac{1}{8}\sum_{n=0}^{7} x[n] \cdot e^{-j\frac{2\pi}{8}kn} = \frac{1}{8}\sum_{n=0}^{3} \sin\left(\frac{3\pi n}{4}\right) e^{-j\frac{2\pi}{8}kn}$

$= \frac{1}{8}\left(\frac{\sqrt{2}}{2}e^{-j\frac{\pi}{4}k} - e^{-j\frac{\pi}{2}k} + \frac{\sqrt{2}}{2}e^{-j\frac{3\pi}{4}k}\right)$

$b_k = \frac{1}{8}\sum_{n=0}^{7}\left(\frac{1}{2}\right)^n e^{-j\frac{2\pi}{8}k\cdot n} = \frac{1}{8}\sum_{n=0}^{7}\left(\frac{1}{2}e^{-j\frac{2\pi}{8}k}\right)^n$

$= \frac{1}{8}\,\frac{1-\left(\frac{1}{2}e^{-j\frac{2\pi}{8}k}\right)^7}{1-\frac{1}{2}e^{-j\frac{2\pi}{8}k}}$

$c_k = 8a_k b_k \qquad z[n] = \sum_{k=\langle r\rangle} c_k e^{j\frac{2\pi}{8}\cdot kn}$

Problem 4

(a) $K\int v(t)\,dt = f_s(t)$

$$\Rightarrow \frac{B}{K}\frac{df_s(t)}{dt} + f_s(t) = f(t)$$

$$\Rightarrow \frac{B}{K} jw\, F_s(w) + F_s(w) = F(w)$$

$$\Rightarrow H(w) = \frac{F_s(w)}{F(w)} = \frac{1}{1+j\frac{B}{K}w}$$

$\Rightarrow$ Low-pass filter

(b) $B\,v(t) = f_d(t)$

$$\Rightarrow f_d(t) + \frac{K}{B}\int f_d(t)\,dt = f(t)$$

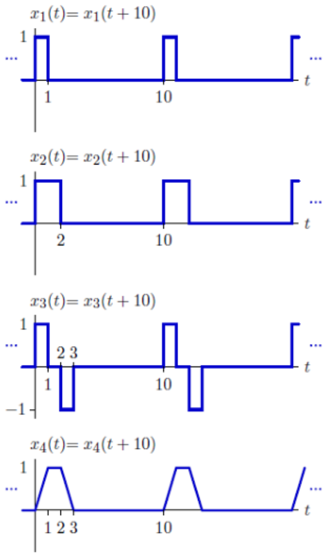
$$\Rightarrow F_d(w) + \frac{K}{B}\cdot\frac{1}{jw} F_d(w) = F(w)$$

$$\Rightarrow H(w) = \frac{F_d(w)}{F(w)} = \frac{1}{1+\frac{K}{B}\cdot\frac{1}{jw}} = \frac{1}{1-j\cdot\frac{K}{Bw}}$$

$\Rightarrow$ High-pass filter

Problem 5:

Determine the Fourier series coefficients for each of the following periodic CT signals.



(1) $a_k = \frac{1}{T}\int_0^T x(t)e^{-j\frac{2\pi}{T}kt}dt = \frac{1}{10}\int_0^1 e^{-j\frac{2\pi}{10}kt}dt$

$= \begin{cases} \frac{1}{10}, & k=0 \\ \frac{1}{j2\pi k}(1-e^{-j\frac{\pi}{5}k}), & k\neq 0 \end{cases}$

(2) $a_k = \frac{1}{10}\int_0^2 e^{-j\frac{2\pi}{10}kt}dt$

$= \begin{cases} \frac{1}{5}, & k=0 \\ \frac{1}{j2\pi k}(1-e^{-j\frac{2\pi}{5}k}), & k\neq 0 \end{cases}$

(3) $a_k = \frac{1}{10}\left(\int_0^1 e^{-j\frac{2\pi}{10}kt}dt - \int_2^3 e^{-j\frac{2\pi}{10}kt}dt\right)$

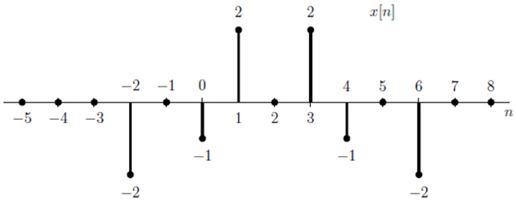
$= \begin{cases} 0, & k=0 \\ \frac{1}{j2\pi k}(1-e^{-j\frac{\pi}{5}k} - e^{-j\frac{2\pi}{5}k} + e^{-j\frac{3\pi}{5}k}), & k\neq 0 \end{cases}$

(4) $x_3(t) = \frac{d}{dt}x_4(t)$

$\Rightarrow a_k = \begin{cases} \frac{1}{5}, & k=0 \\ \frac{5}{2\pi^2k^2}(1-e^{-j\frac{\pi}{5}k} - e^{-j\frac{2\pi}{5}k} + e^{-j\frac{3\pi}{5}k}), & k\neq 0 \end{cases}$

Problem 6:

Let $X(e^{j\omega})$ denote the Fourier transform of the signal $x[n]$ depicted below.



(a) Find $X(1) = X(e^{j0})$.

(b) Find α such that $e^{j\alpha\omega}X(e^{j\omega})$ is real.

(c) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.

(d) Find $X(e^{j\pi})$.

(e) Determine and sketch the signal whose Fourier transform is $\Re e\{X(e^{j\omega})\}$.

(f) Evaluate the following integrals:

$$\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

$X(e^{jw}) = \sum_{k=-\infty}^{+\infty} x[n]e^{-jwn}$

(a) $X(1) = \sum_{k=-\infty}^{+\infty} x[n] = -2$

(b) $X(e^{jw}) = 2(e^{-jw} + e^{-j3w} - e^{j2w} - e^{-j6w}) - (e^{-j4w}+1)$

$\Rightarrow a = 2$

$e^{j2w}X(e^{jw}) = 2(e^{jw} + e^{-jw} - e^{j4w} - e^{-j4w}) - (e^{-j2w} + e^{j2w})$

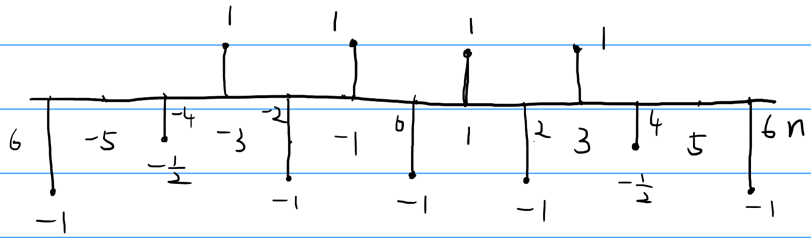
it is real

(c) $\int_{-\pi}^{\pi} X(e^{jw})dw = \int_{-\pi}^{\pi} 2(e^{-jw} + e^{-j3w} - e^{j2w} - e^{-j6w}) - (e^{-j4w}+1)\,dw$

$= -2\pi$

(d) $X(e^{j\pi}) = 2(-1-1-1-1) - (1+1) = -10$

(e) the signal is $\frac{x[n]+x[-n]}{2}$



(f) $\int_{-\pi}^{\pi} |X(e^{jw})|^2 \, dw = 2\pi \sum_{n=-\infty}^{+\infty} |x[n]|^2 = 36\pi$

Problem 7:

Find the Fourier transforms of the following signals.

a. $x_1(t) = e^{-|t|}\cos(2t)$

b. $x_2(t) = \frac{\sin(2\pi t)}{\pi(t-1)}$

c. $x_3(t) = \begin{cases} t^2 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$

d. $x_4(t) = (1-|t|)\,u(t+1)u(1-t)$

a. $\int_{-\infty}^{+\infty} e^{-|t|} e^{-jwt} \, dt = \int_{0}^{\infty} e^{-t} e^{-jwt} \, dt + \int_{-\infty}^{0} e^{t} e^{-jwt} \, dt$

$= \frac{1}{1+jw} + \frac{1}{1-jw} = \frac{2}{1+w^2}$

$x_1(t) = e^{-|t|}\cos 2t = e^{-|t|}\left(\frac{e^{j2t}+e^{-j2t}}{2}\right)$

$\Rightarrow X_1(w) = \frac{1}{1+(w-2)^2} + \frac{1}{1+(w+2)^2}$

b. $x_2(t) = \frac{\sin(2\pi t)}{\pi(t-1)} = \frac{\sin(2\pi(t-1))}{\pi(t-1)}$

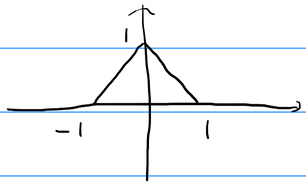
$y(t) = \frac{\sin 2\pi t}{\pi t} \Rightarrow Y(w) = \mathrm{rect}\left(\frac{w}{4\pi}\right)$

$\Rightarrow X_2(w) = \mathrm{rect}\left(\frac{w}{4\pi}\right) e^{-jw}$

c. $X_3(w) = \int_0^1 t^2 e^{-jwt} \, dt = \left(-\frac{1}{jw}t^2 + \frac{2}{w^2}t + \frac{2}{jw^3}\right) e^{-jwt} \Big|_0^1$

$= \frac{(-w^2+2jw+2)e^{-jwt} - 2}{jw^3}$

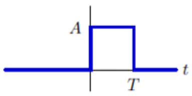
(4)



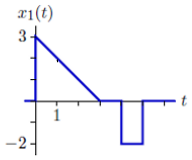
$X_4(w) = \mathrm{sinc}^2\left(\frac{w}{2}\right)$

Problem 8:

We are given that the impulse response of a CT LTI system is of the form



where A and T are unknown. When the system is subjected to the input



the output $y_1(t)$ is zero at $t = 5$. When the input is

$$x_2(t) = \sin\left(\frac{\pi t}{3}\right) u(t),$$

the output $y_2(t)$ is equal to 9 at $t = 9$. Determine A and T. Also determine $y_2(t)$ for all t.

$y_1(t) = x_1(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau)\, x_1(t-\tau)\, d\tau$

$y_1(5) = 0 \Rightarrow \int_0^T h(\tau) x_1(5-\tau)\, d\tau = 0 \Rightarrow \int_0^T A\, x_1(5-\tau)\, d\tau = 0$

$\Rightarrow T = 4$

$y_2(t) = x_2(t) * h(t) = \int_{-\infty}^{+\infty} h(\tau)\, x_2(t-\tau)\, d\tau$

$y_2(9) = 9 \Rightarrow \int_0^T h(\tau)\, x_2(9-\tau)\, d\tau = 9 \Rightarrow \int_0^4 A \sin\left(3\pi - \frac{\pi\tau}{3}\right) d\tau = 9$

$\Rightarrow \int_0^4 A \sin\frac{\pi\tau}{3}\, d\tau = 9 \Rightarrow A = 2\pi$

$\Rightarrow y_2(t) = \int_0^4 2\pi \cdot \sin\frac{\pi}{3}(t-\tau)\, d\tau = 6\cos\frac{\pi}{3}(t-4) - 6$

Above all, $T = 4$, $A = 2\pi$, $y_2(t) = 6\cos\frac{\pi}{3}(t-4) - 6$