Signals and Systems

Lecture 13: Continuous-time Fourier Series

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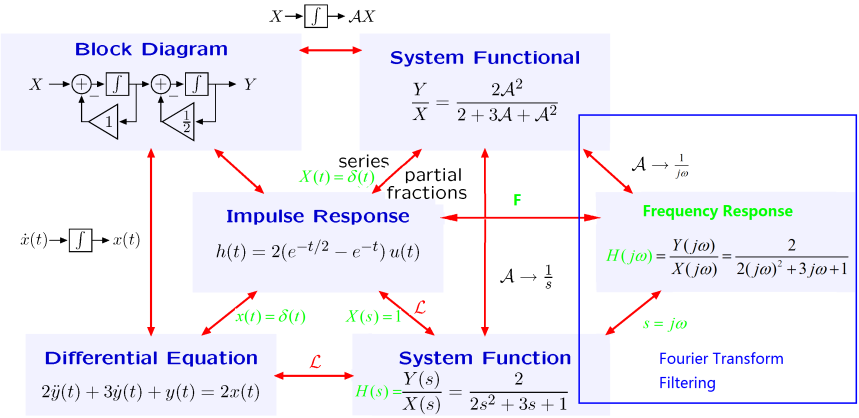
Adapted from the materials provided on the MIT OpenCourseWare

Review: System Representations

Relations among representations.



Fourier Representations

Fourier series represent **signals** in terms of **sinusoids**.

$\rightarrow$ leads to a new representation for **systems** as **filters**.

Harmonics

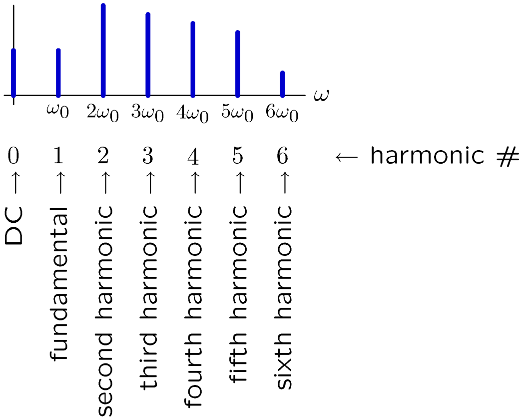
A harmonically related set of complex exponentials is a set of periodic exponentials with fundamental frequencies that are all multiples of a single positive frequency ω_0:

$$\phi_k(t) = e^{jk\omega_0 t}, \quad k = 0, \pm 1, \pm 2, \ldots$$

- $k = 0$, DC component;
- $k = \pm 1$, the first harmonic components;
- $k = \pm 2$, the second harmonic components;
- ...

Fourier Series

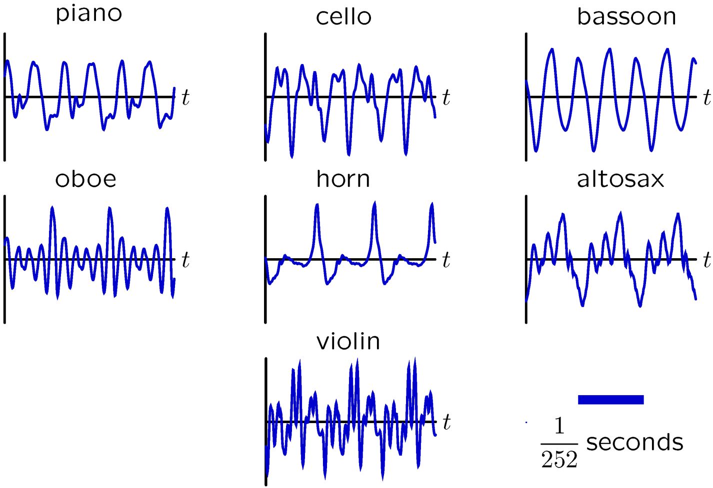
Representing signals by their harmonic components.



Musical Instruments

Harmonic content is natural way to describe some kinds of signals.

Ex: musical instruments (http://theremin.music.uiowa.edu/MIS)



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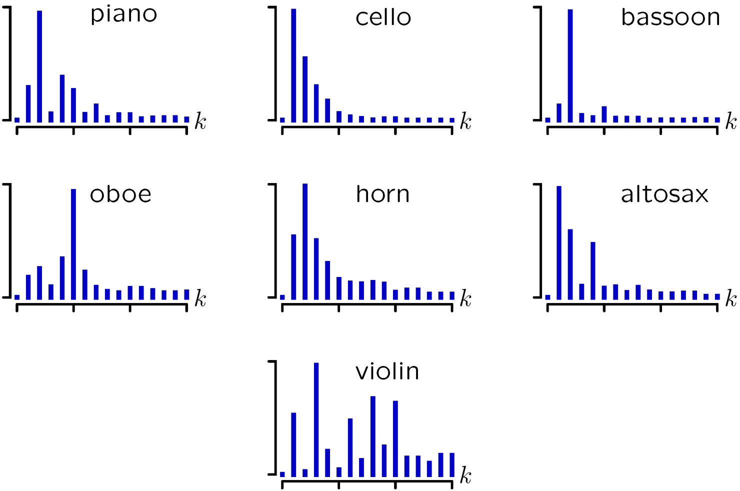

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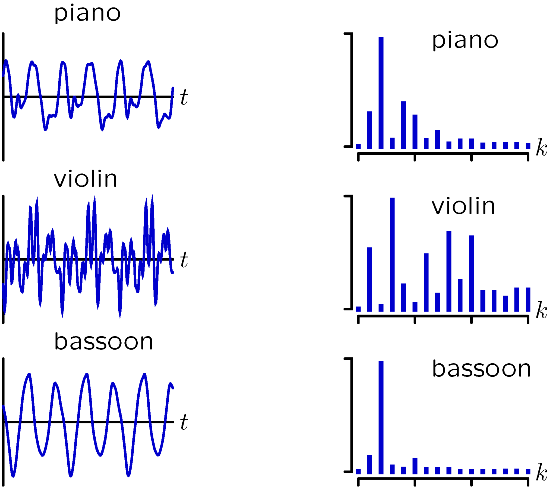
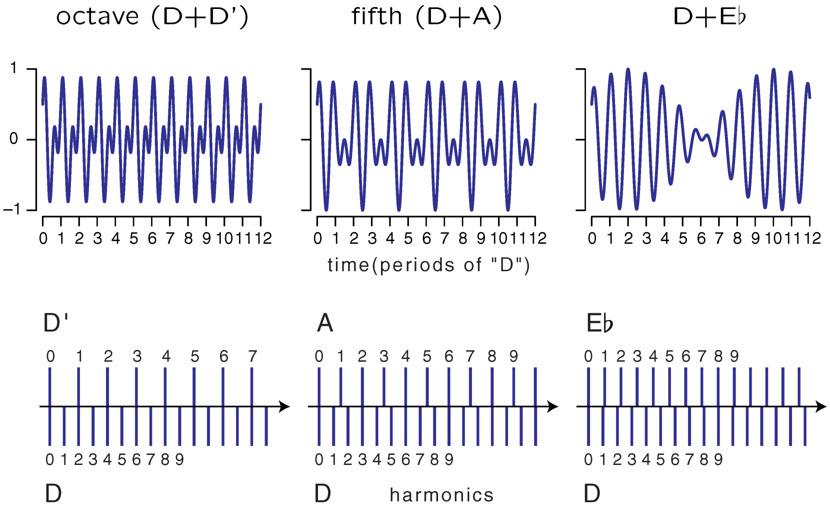


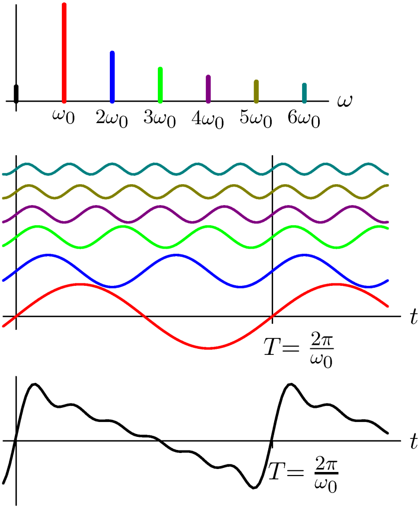
Harmonics

Harmonic structure determines consonance and dissonance.



Harmonic Representations

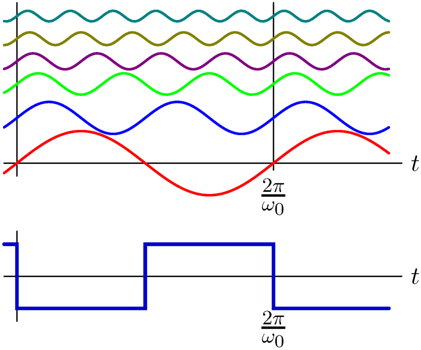
What signals can be represented by sums of harmonic components?



Only periodic signals: all harmonics of ω_0 are periodic in $T = 2\pi/\omega_0$.

Harmonic Representations

Is it possible to represent ALL periodic signals with harmonics? What about discontinuous signals?



Fourier claimed YES — even though all harmonics are continuous!

Lagrange ridiculed the idea that a discontinuous signal could be written as a sum of continuous signals.

We will assume the answer is YES and see if the answer makes sense.

Separating harmonic components

Underlying properties.

1. Multiplying two harmonics produces a new harmonic with the same fundamental frequency:

$$e^{jk\omega_0 t} \times e^{jl\omega_0 t} = e^{j(k+l)\omega_0 t}.$$

2. The integral of a harmonic over any time interval with length equal to a period T is zero unless the harmonic is at DC:

$$\int_{t_0}^{t_0+T} e^{jk\omega_0 t} dt \equiv \int_T e^{jk\omega_0 t} dt = \begin{cases} 0, & k \neq 0 \\ T, & k = 0 \end{cases}$$
$$= T\delta[k]$$

Separating harmonic components

Assume that $x(t)$ is periodic in T and is composed of a weighted sum of harmonics of $\omega_0 = 2\pi/T$.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

Then

$$\begin{aligned}
\int_T x(t) e^{-jl\omega_0 t} dt &= \int_T \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} e^{-j\omega_0 lt} dt \\
&= \sum_{k=-\infty}^{\infty} a_k \int_T e^{j\omega_0 (k-l)t} dt \\
&= \sum_{k=-\infty}^{\infty} a_k T \delta[k-l] = T a_l
\end{aligned}$$

Therefore

$$a_k = \frac{1}{T} \int_T x(t) e^{-j\omega_0 kt} dt \qquad = \frac{1}{T} \int_T x(t) e^{-j\frac{2\pi}{T}kt} dt$$

Fourier Series

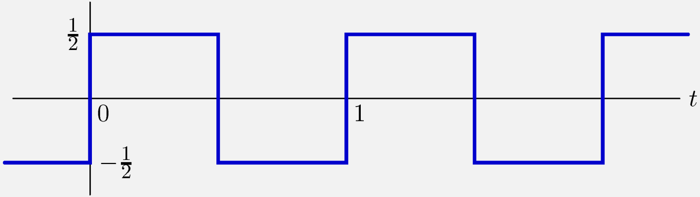
Determining harmonic components of a periodic signal.

$$a_k = \frac{1}{T}\int_T x(t)e^{-j\frac{2\pi}{T}kt}dt$$ ("analysis" equation)

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$ ("synthesis" equation)

Check Yourself

Let a_k represent the Fourier series coefficients of the following square wave.

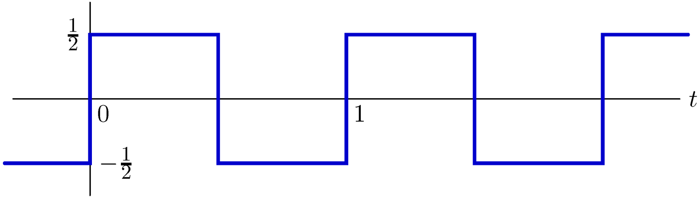


How many of the following statements are true?

1. $a_k = 0$ if k is even
2. a_k is real-valued
3. $|a_k|$ decreases with k^2
4. there are an infinite number of non-zero a_k
5. all of the above

Check Yourself

Let a_k represent the Fourier series coefficients of the following square wave.



$$a_k = \int_T x(t)e^{-j\frac{2\pi}{T}kt}dt = -\frac{1}{2}\int_{-\frac{1}{2}}^{0} e^{-j2\pi kt}dt + \frac{1}{2}\int_{0}^{\frac{1}{2}} e^{-j2\pi kt}dt$$

$$= \frac{1}{j4\pi k}\left(2 - e^{j\pi k} - e^{-j\pi k}\right)$$

$$= \begin{cases} \dfrac{1}{j\pi k}\ ; & \text{if } k \text{ is odd} \\ 0\ ; & \text{otherwise} \end{cases}$$

Check Yourself

Let a_k represent the Fourier series coefficients of the following square wave.

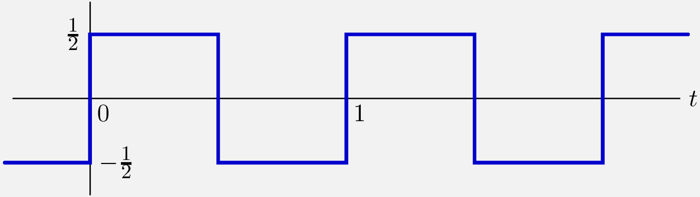
$$a_k = \begin{cases} \dfrac{1}{j\pi k}\ ; & \text{if } k \text{ is odd} \\ 0\ ; & \text{otherwise} \end{cases}$$

How many of the following statements are true?

1. $a_k = 0$ if k is even $\surd$
2. a_k is real-valued $\times$
3. $|a_k|$ decreases with k^2 $\times$
4. there are an infinite number of non-zero a_k $\surd$
5. all of the above $\times$

Check Yourself

Let a_k represent the Fourier series coefficients of the following square wave.



How many of the following statements are true? 2

1. $a_k = 0$ if k is even $\surd$
2. a_k is real-valued X
3. $|a_k|$ decreases with k^2 X
4. there are an infinite number of non-zero a_k $\surd$
5. all of the above X

Check Yourself

Determine the Fourier series representation for

$$x(t) = \cos 4\pi t + 2\sin 8\pi t$$

Check Yourself

Determine the Fourier series representation for

$$x(t) = \cos 4\pi t + 2 \sin 8\pi t$$

Euler's relation (memorize!)

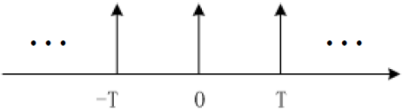
$$= \frac{1}{2}\left[e^{j4\pi t} + e^{-j4\pi t}\right] + \frac{2}{2j}\left[e^{j8\pi t} - e^{-j8\pi t}\right]$$

$$\omega_0 = 4\pi \qquad T = \frac{2\pi}{\omega_0} = \frac{2\pi}{4\pi} = \frac{1}{2}$$

$$\begin{aligned}
a_0 &= 0 \\
a_1 &= \frac{1}{2} \\
a_{-1} &= \frac{1}{2} \\
a_2 &= \frac{1}{j} \\
a_{-2} &= -\frac{1}{j} \\
a_3 &= 0 \\
a_{-3} &= 0
\end{aligned}$$

Check Yourself

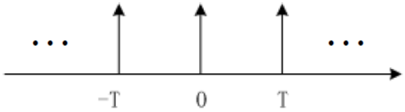
Determine the Fourier series representation for



$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Check Yourself

Determine the Fourier series representation for



$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$\begin{aligned} a_k &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jk\omega_0 t}dt \\ &= \frac{1}{T} \quad \text{for } \mathit{all}\ k\ ! \end{aligned}$$

$$\Downarrow$$

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

— All components have:
(1) the same amplitude,
&
(2) the same phase.

Fourier Series Properties

If a signal is differentiated in time, its Fourier coefficients are multiplied by $j\frac{2\pi}{T}k$.

Proof: Let

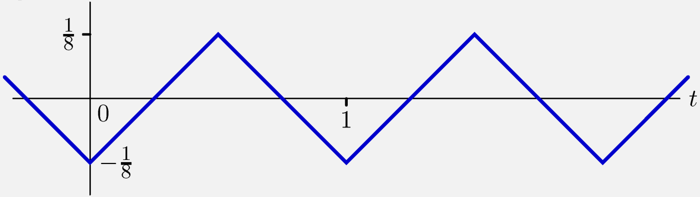
$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

then

$$\dot{x}(t) = \dot{x}(t+T) = \sum_{k=-\infty}^{\infty} \left(j\frac{2\pi}{T}k\, a_k \right) e^{j\frac{2\pi}{T}kt}$$

Check Yourself

Let b_k represent the Fourier series coefficients of the following triangle wave.

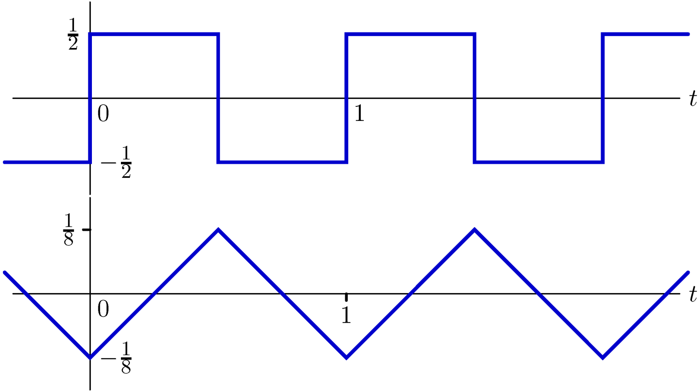


How many of the following statements are true?

1. $b_k = 0$ if k is even
2. b_k is real-valued
3. $|b_k|$ decreases with k^2
4. there are an infinite number of non-zero b_k
5. all of the above

Check Yourself

The triangle waveform is the integral of the square wave.



Therefore the Fourier coefficients of the triangle waveform are $\dfrac{1}{j2\pi k}$ times those of the square wave.

$$b_k = \frac{1}{jk\pi} \times \frac{1}{j2\pi k} = \frac{-1}{2k^2\pi^2} \ ; \ k \text{ odd}$$

Check Yourself

Let b_k represent the Fourier series coefficients of the following triangle wave.

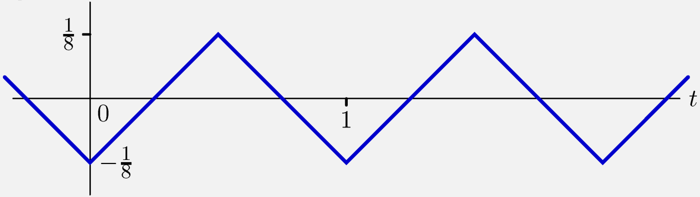
$$b_k = \frac{-1}{2k^2\pi^2} \;;\; k \text{ odd}$$

How many of the following statements are true?

1. $b_k = 0$ if k is even $\surd$
2. b_k is real-valued $\surd$
3. $|b_k|$ decreases with k^2 $\surd$
4. there are an infinite number of non-zero b_k $\surd$
5. all of the above $\surd$

Check Yourself

Let b_k represent the Fourier series coefficients of the following triangle wave.



How many of the following statements are true? 5

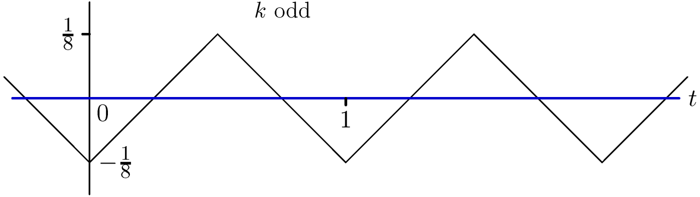
1. $b_k = 0$ if k is even $\surd$
2. b_k is real-valued $\surd$
3. $|b_k|$ decreases with k^2 $\surd$
4. there are an infinite number of non-zero b_k $\surd$
5. all of the above $\surd$

Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-0 \\ k \text{ odd}}}^{0} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

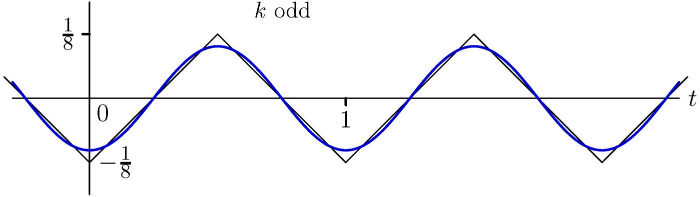


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-1 \\ k \text{ odd}}}^{1} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

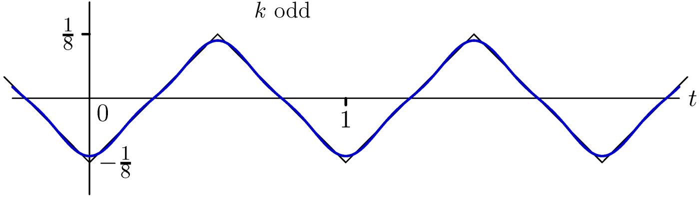


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-3\\k\text{ odd}}}^{3} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

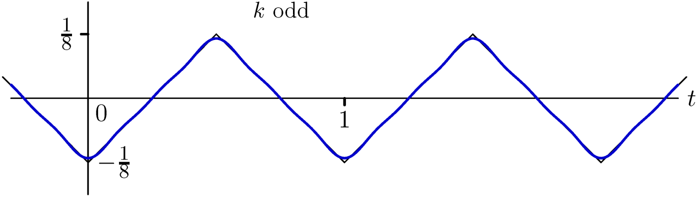


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

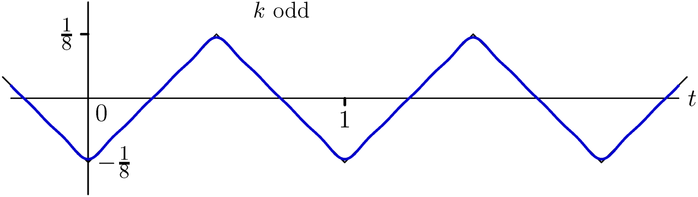


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-7 \\ k \text{ odd}}}^{7} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

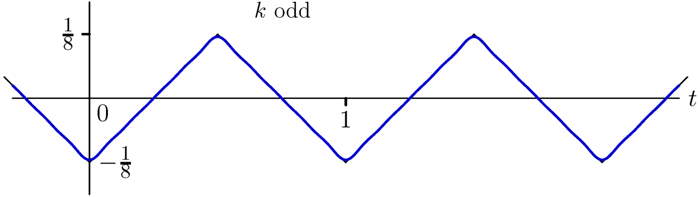


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-9 \\ k \text{ odd}}}^{9} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

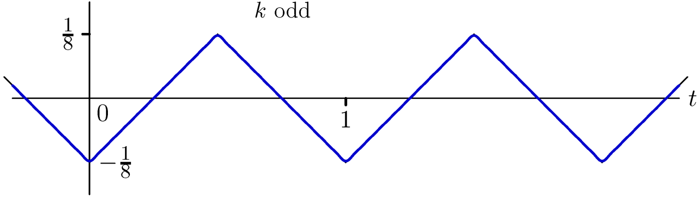


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-19 \\ k \text{ odd}}}^{19} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

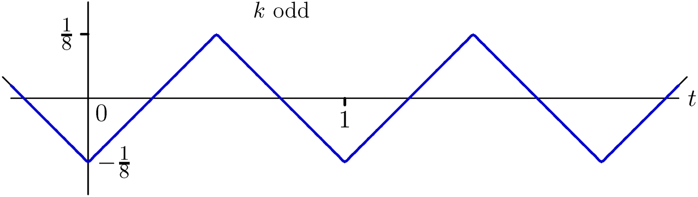


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-29 \\ k \text{ odd}}}^{29} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$

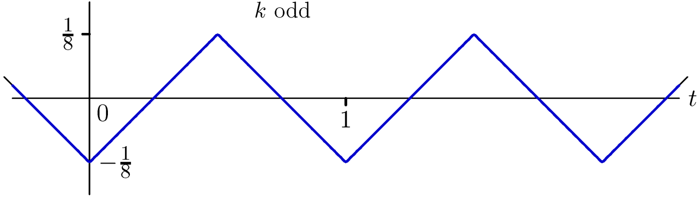


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: triangle waveform

$$\sum_{\substack{k=-39 \\ k \text{ odd}}}^{39} \frac{-1}{2k^2\pi^2} e^{j2\pi kt}$$



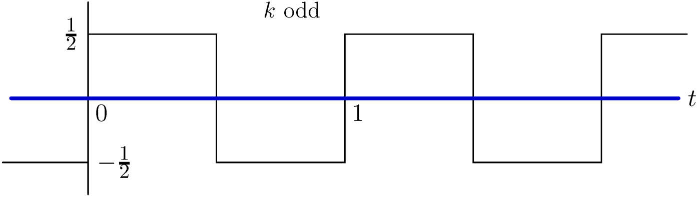
Fourier series representations of functions with discontinuous slopes converge toward functions with discontinuous slopes.

Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-0 \\ k \text{ odd}}}^{0} \frac{1}{jk\pi} e^{j2\pi kt}$$

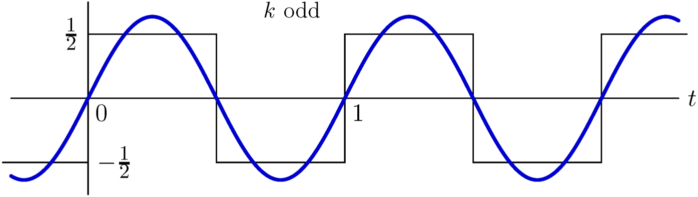


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-1 \\ k \text{ odd}}}^{1} \frac{1}{jk\pi} e^{j2\pi kt}$$

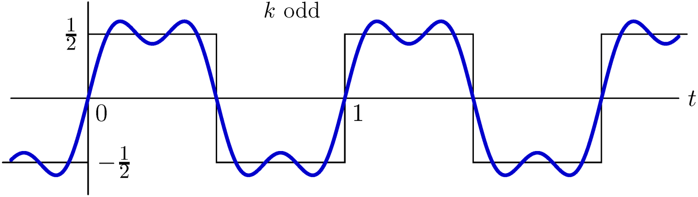


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-3 \\ k \text{ odd}}}^{3} \frac{1}{jk\pi} e^{j2\pi kt}$$

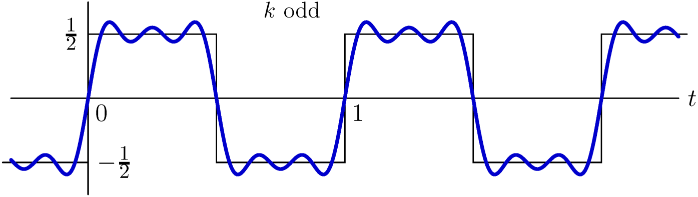


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-5 \\ k \text{ odd}}}^{5} \frac{1}{jk\pi} e^{j2\pi kt}$$

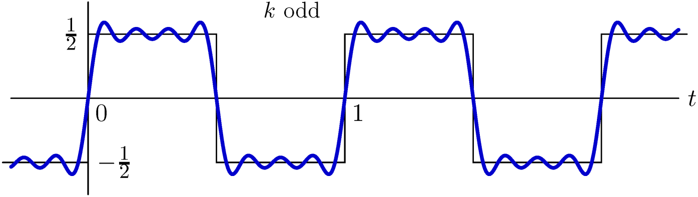


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-7 \\ k \text{ odd}}}^{7} \frac{1}{jk\pi} e^{j2\pi kt}$$

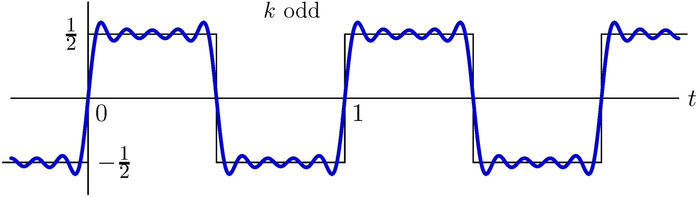


Fourier Series

One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-9 \\ k \text{ odd}}}^{9} \frac{1}{jk\pi} e^{j2\pi kt}$$



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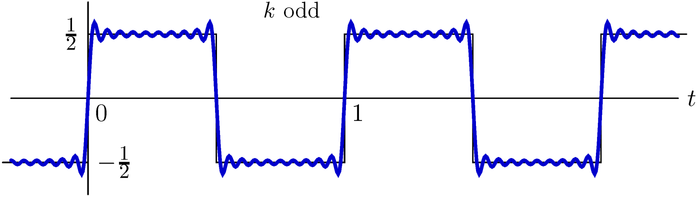
One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave

$$\sum_{\substack{k=-19 \\ k \text{ odd}}}^{19} \frac{1}{jk\pi} e^{j2\pi kt}$$



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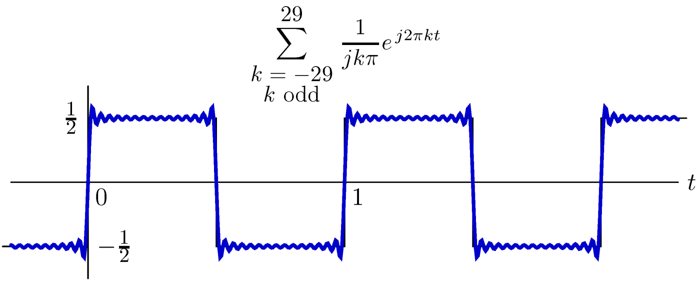
One can visualize convergence of the Fourier Series by incrementally adding terms.

Example: square wave



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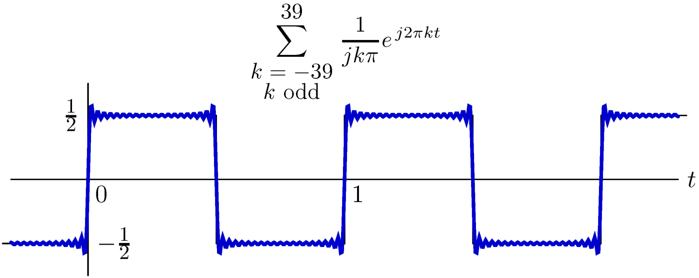
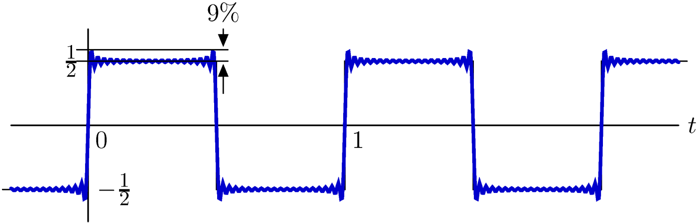
Example: square wave



Fourier Series

Partial sums of Fourier series of discontinuous functions "ring" near discontinuities: Gibb's phenomenon.



This ringing results because the magnitude of the Fourier coefficients is only decreasing as $\frac{1}{k}$ (while they decreased as $\frac{1}{k^2}$ for the triangle).

You can decrease (and even eliminate the ringing) by decreasing the magnitudes of the Fourier coefficients at higher frequencies.

Gibbs Phenomenon

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- As $N \to \infty$, $x_N(t)$ exhibits Gibb's phenomenon at points of discontinuity;
- As N increases, the ripples in the partial sums become compressed toward the discontuity;
- For any finite value of N, the peak amplitude of the ripples remains constant.

Fourier Series: Convergence

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \Leftarrow \quad x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$$\Rightarrow \quad e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$$\Rightarrow \quad E_N = \int_T |e_N(t)|^2 dt$$

$$\Rightarrow \quad \left\{ \begin{array}{l} E_N \to 0, \quad N \to \infty \\ a_k = \dfrac{1}{T}\displaystyle\int_T x(t) e^{-jk\omega_0 t} dt \\ \boxed{\displaystyle\int_T |x(t)|^2 dt < \infty} \end{array} \right.$$

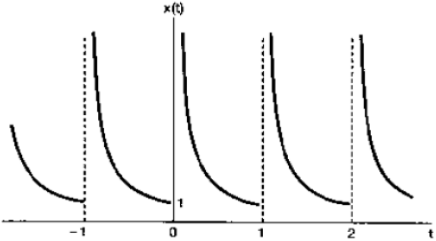
Fourier Series: The Dirichlet Conditions

- **Condition 1:**

  Over any period, x(t) must be absolutely integrable

$$\int_T |x(t)|dt < \infty$$

Counter example



$$x(t) = \frac{1}{t}, \quad 0 < t \leq 1$$

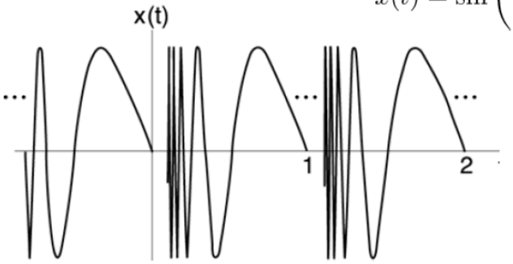
Fourier Series: The Dirichlet Conditions

- **Condition 2:**

  In any finite interval of time, x(t) is of bounded variation; i.e. there are no more than a finite number of maxima and minima during any single period of the signal.

  Counter example

$$x(t) = \sin\left(\frac{2\pi}{t}\right) \quad 0 < t \leq 1$$

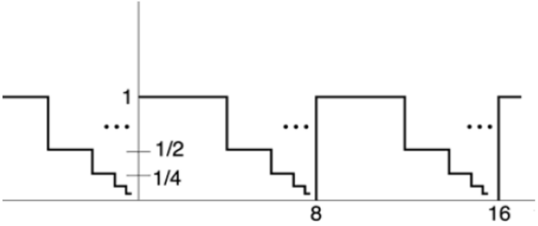


Fourier Series: The Dirichlet Conditions

- **Condition 3:**

  In any finite interval of time, there are only a finite number of discontinuities. Each of these discontinuities is finite.

  Counter example



Fourier Series: Summary

Fourier series represent periodic signals as sums of sinusoids.

- valid for an extremely large class of periodic signals
- valid even for discontinuous signals such as square wave

However, convergence as # harmonics increases can be complicated.

Filtering

The output of an LTI system is a “filtered” version of the input.

Input: Fourier series $\rightarrow$ sum of complex exponentials.

$$x(t) = x(t+T) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}$$

Complex exponentials: eigenfunctions of LTI systems.

$$e^{j\frac{2\pi}{T}kt} \rightarrow H(j\frac{2\pi}{T}k)e^{j\frac{2\pi}{T}kt}$$

Output: same eigenfunctions, amplitudes/phases set by system.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\frac{2\pi}{T}k)e^{j\frac{2\pi}{T}kt}$$

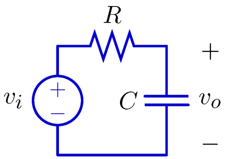
Filtering

Notion of a filter.

LTI systems

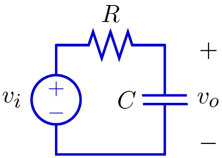
- cannot create new frequencies.
- can scale magnitudes and shift phases of existing components.

Example: Low-Pass Filtering with an RC circuit

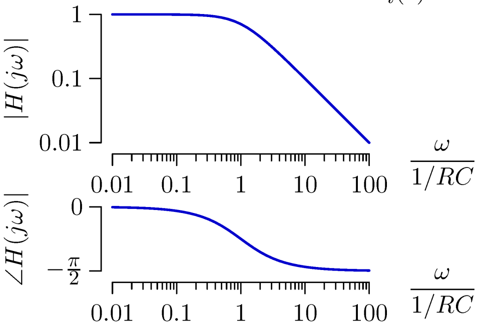


Lowpass Filter

Calculate the frequency response of an RC circuit.

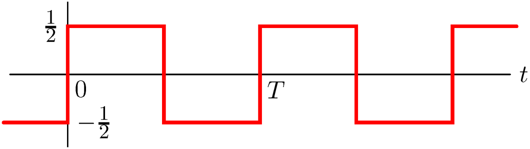


$$
\begin{aligned}
&\text{KVL:} && v_i(t) &&= Ri(t) + v_o(t)\\
&\text{C:} && i(t) &&= C\dot{v}_o(t)\\
&\text{Solving:} && v_i(t) &&= RC\dot{v}_o(t) + v_o(t)\\
& && V_i(s) &&= (1 + sRC)V_o(s)\\
& && H(s) &&= \frac{V_o(s)}{V_i(s)} = \frac{1}{1 + sRC}
\end{aligned}
$$

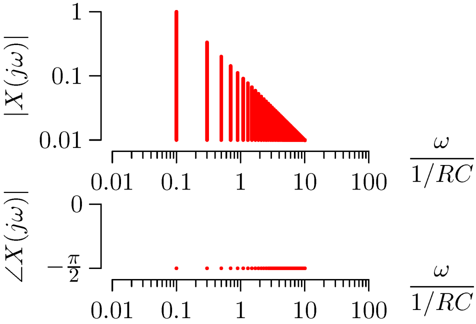


Lowpass Filtering

Let the input be a square wave.

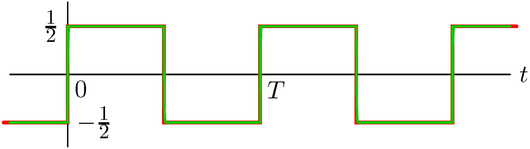


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

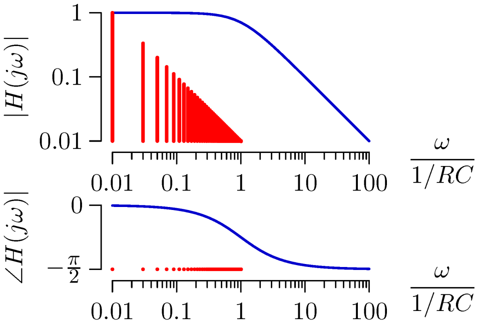


Lowpass Filtering

Low frequency square wave: $\omega_0 << 1/RC$.

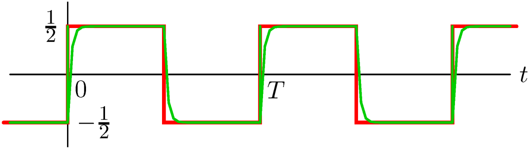


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

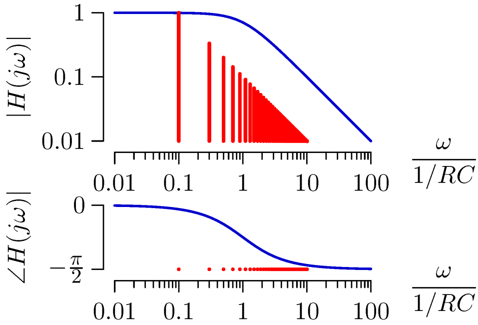


Lowpass Filtering

Higher frequency square wave: $\omega_0 < 1/RC$.

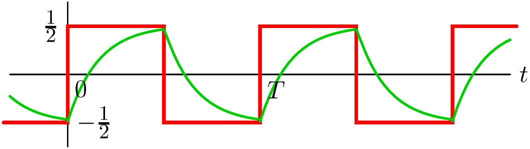


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \quad \omega_0 = \frac{2\pi}{T}$$

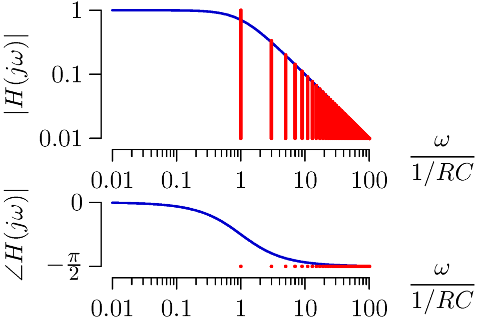


Lowpass Filtering

Still higher frequency square wave: $\omega_0 = 1/RC$.

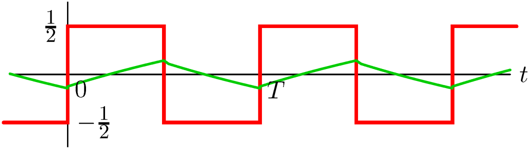


$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 kt} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$

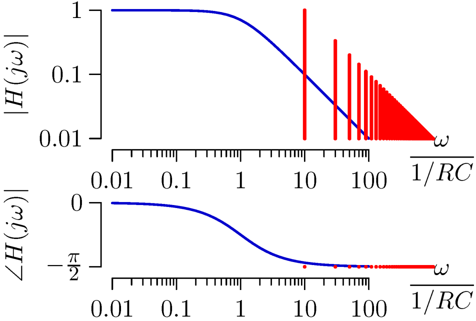


Lowpass Filtering

High frequency square wave: $\omega_0 > 1/RC$.



$$x(t) = \sum_{k \text{ odd}} \frac{1}{j\pi k} e^{j\omega_0 k t} \quad ; \qquad \omega_0 = \frac{2\pi}{T}$$



Fourier Series: Summary

Fourier series represent signals by their frequency content.

Representing a signal by its frequency content is useful for many signals, e.g., music.

Fourier series motivate a new representation of a system as a filter.

Assignments

- Reading Assignment: Ch. 1.3, Ch. 3.0 - 3.5, Ch. 3.9
- Homework 7