Signals and Systems

Lecture3: Feedback, Poles, and Fundamental Modes

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Partly adapted from the materials provided on the MIT OpenCourseWare

Review: Time-invariant Systems

Ex.#1 $y(t) = x^2(t+1)$

Ex.#2 $y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$

Ex.#3 $y(t) = \sin[x(t)]$

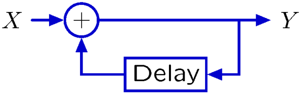
Ex.#4 $y[n] = nx[n]$

Ex.#5 $y(t) = x(2t)$

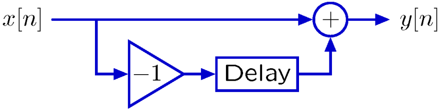
Feedback

Systems with signals that depend on previous values of the same signal are said to have **feedback**.

Example: The accumulator system has feedback.



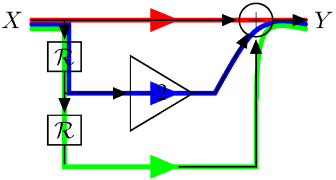
By contrast, the difference machine does not have feedback.



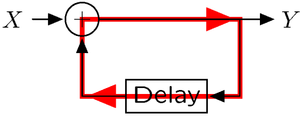
Cyclic Signal Paths, Feedback, and Modes

Block diagrams help visualize feedback.

Feedback occurs when there is a cyclic signal flow path.



acyclic



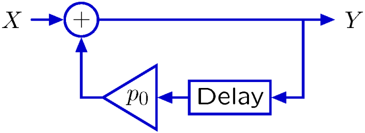
cyclic

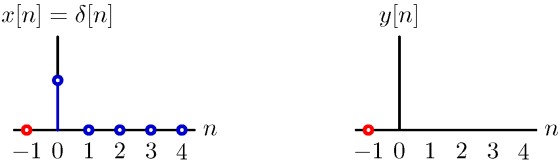
Acyclic: all paths through system go from input to output with no cycles.

Cyclic: at least one cycle.

Feedback, Cyclic Signal Paths, and Modes

The effect of feedback can be visualized by tracing each cycle through the cyclic signal paths.

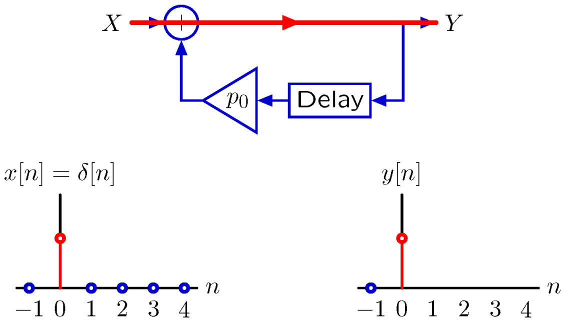




Each cycle creates another sample in the output.

Feedback, Cyclic Signal Paths, and Modes

The effect of feedback can be visualized by tracing each cycle through the cyclic signal paths.



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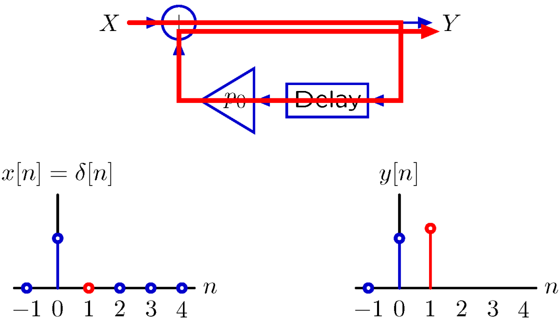

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Feedback, Cyclic Signal Paths, and Modes

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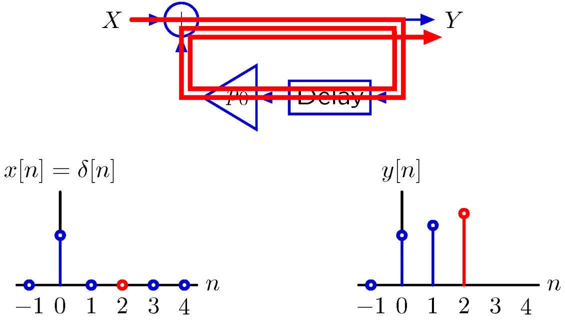

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Feedback, Cyclic Signal Paths, and Modes

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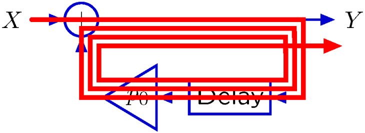
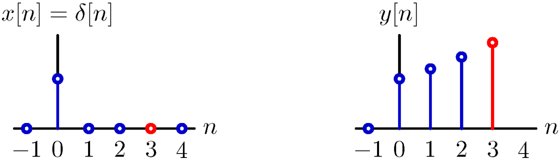
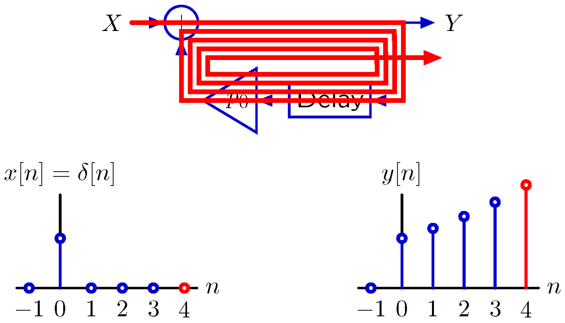




Each cycle creates another sample in the output.

Feedback, Cyclic Signal Paths, and Modes

The effect of feedback can be visualized by tracing each cycle through the cyclic signal paths.

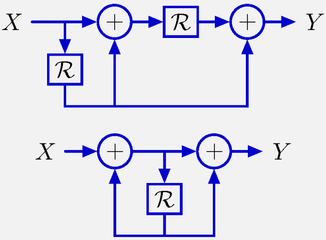


Each cycle creates another sample in the output.

The response will persist even though the input is transient.

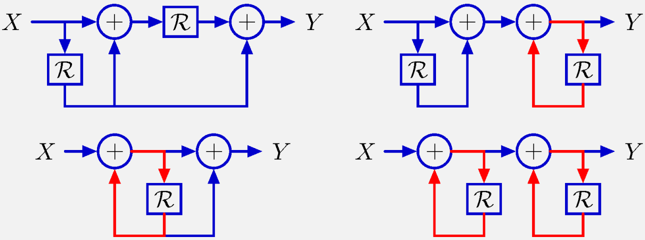
Check Yourself

How many of the following systems have cyclic signal paths?



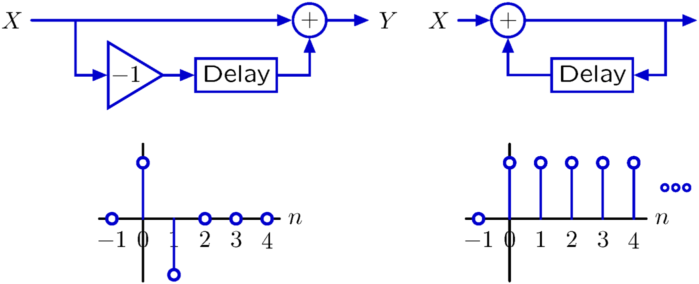
Check Yourself

How many of the following systems have cyclic signal paths? 3



Finite and Infinite Impulse Responses

The impulse response of an acyclic system has finite duration, while that of a cyclic system can have infinite duration.



Analysis of Cyclic Systems: Geometric Growth

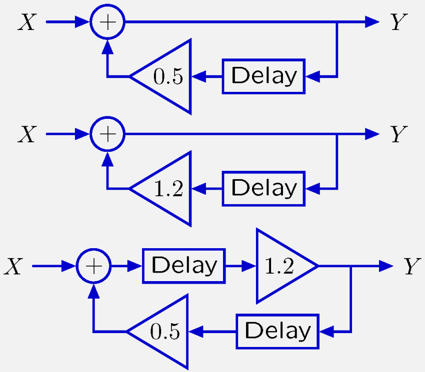
If traversing the cycle decreases or increases the magnitude of the signal, then the fundamental mode will decay or grow, respectively.

If the response decays toward zero, then we say that it **converges**.

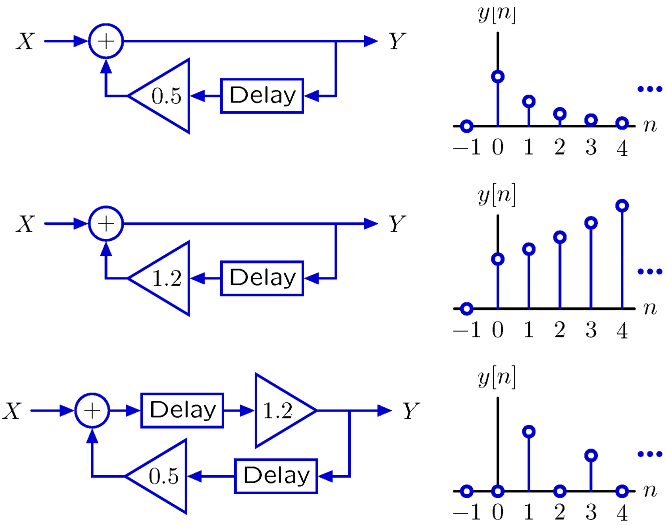
Otherwise, we it **diverges**.

Check Yourself

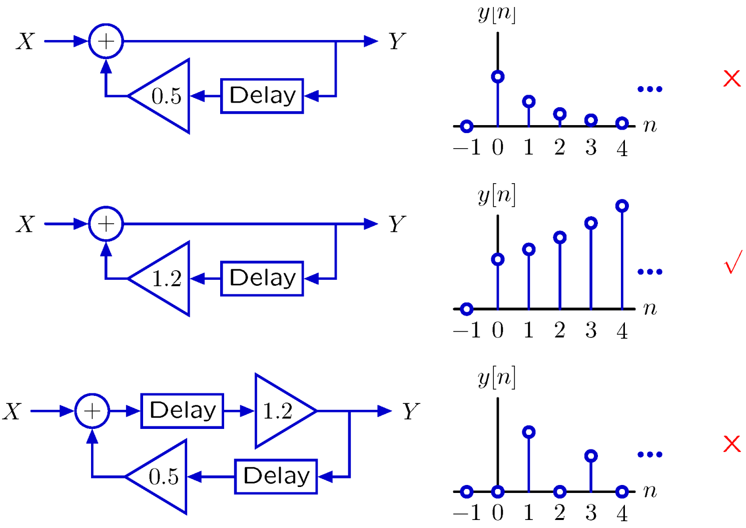
How many of these systems have divergent unit-sample responses?



Check Yourself

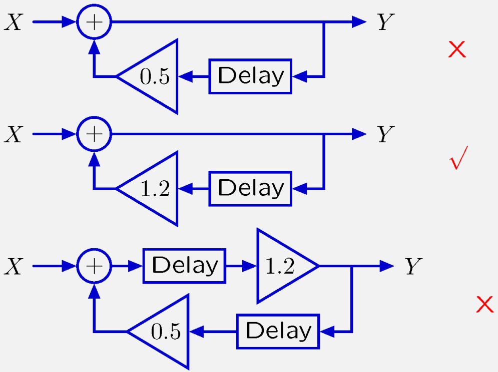


Check Yourself



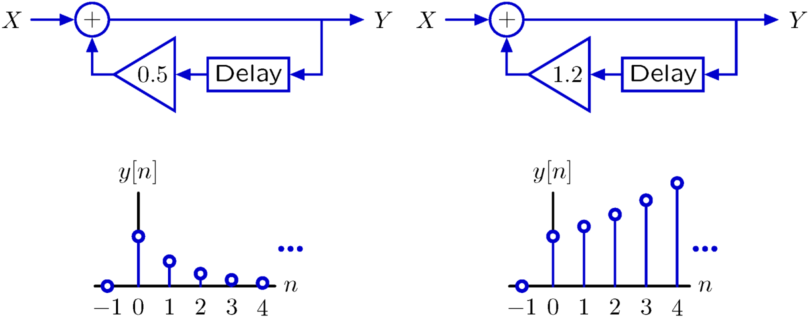
Check Yourself

How many of these systems have divergent unit-sample responses? 1



Cyclic Systems: Geometric Growth

If traversing the cycle decreases or increases the magnitude of the signal, then the fundamental mode will decay or grow, respectively.



These are geometric sequences: $y[n] = (0.5)^n$ and $(1.2)^n$ for $n \geq 0$.

These geometric sequences are called **fundamental modes**.

Representations of Discrete-Time Systems

Now you know four representations of discrete-time systems.

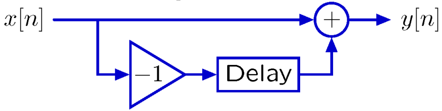
Verbal descriptions: preserve the rationale.

> "To reduce the number of bits needed to store a sequence of large numbers that are nearly equal, record the first number, and then record successive differences."

Difference equations: mathematically compact.

$$y[n] = x[n] - x[n-1]$$

Block diagrams: illustrate signal flow paths.

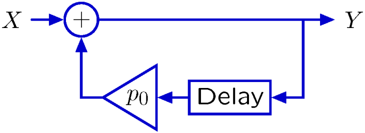


Operator representations: analyze systems as polynomials.

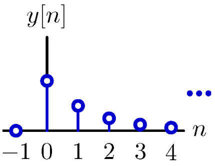
$$Y = (1 - \mathcal{R})\, X$$

Geometric Growth: Poles

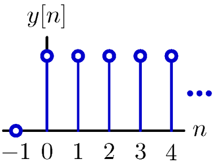
These unit-sample responses can be characterized by a single number — the **pole** — which is the base of the geometric sequence.



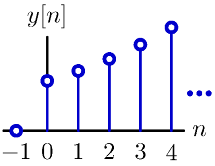
$$y[n] = \begin{cases} p_0^n, & \text{if } n >= 0; \\ 0, & \text{otherwise.} \end{cases}$$



$p_0 = 0.5$



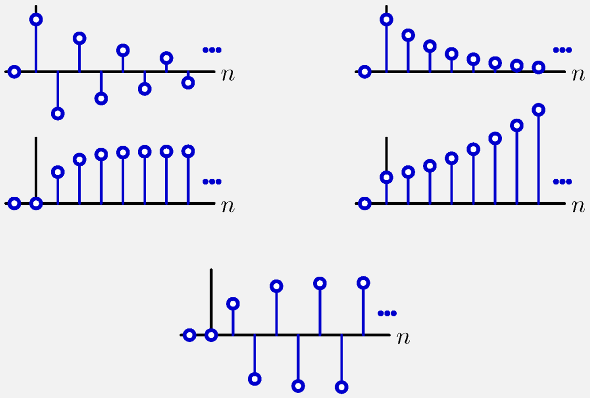
$p_0 = 1$



$p_0 = 1.2$

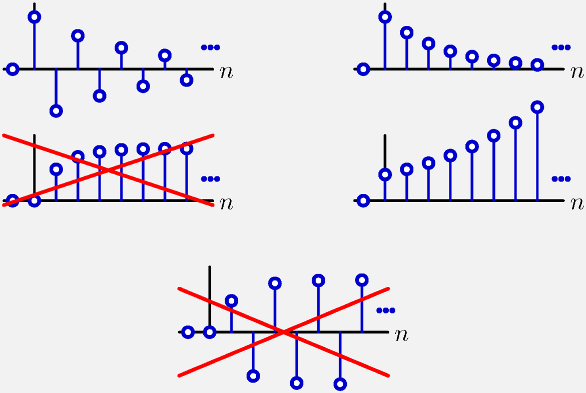
Check Yourself

How many of the following unit-sample responses can be represented by a single pole?



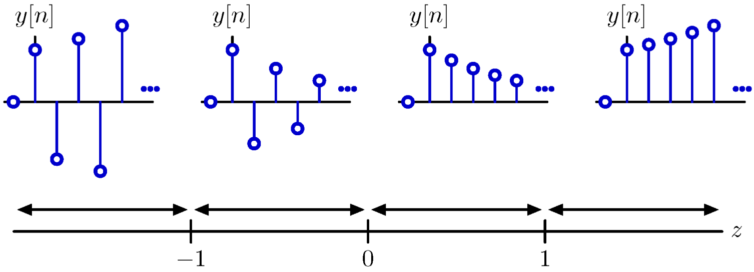
Check Yourself

How many of the following unit-sample responses can be represented by a single pole? 3



Geometric Growth

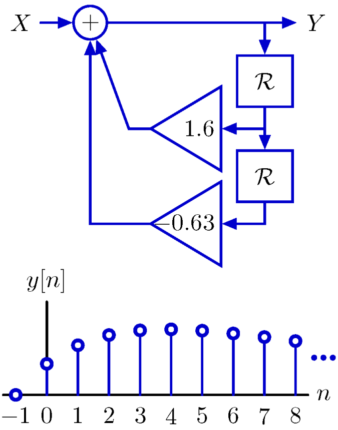
The value of p_0 determines the rate of growth.



$p_0 < -1$:	magnitude diverges, alternating sign
$-1 < p_0 < 0$:	magnitude converges, alternating sign
$0 < p_0 < 1$:	magnitude converges monotonically
$p_0 > 1$:	magnitude diverges monotonically

Second-Order Systems

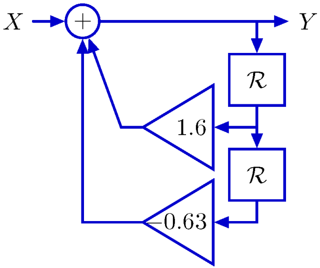
The unit-sample responses of more complicated cyclic systems are more complicated.



Not geometric. This response grows then decays.

Factoring Second-Order Systems

Factor the operator expression to break the system into two simpler systems (divide and conquer).



$$Y = X + 1.6\mathcal{R}Y - 0.63\mathcal{R}^2Y$$

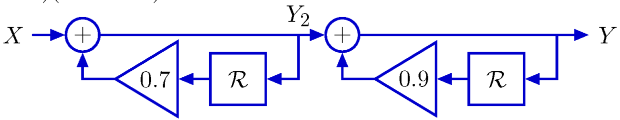
$$(1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2)\,Y = X$$

$$(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})\,Y = X$$

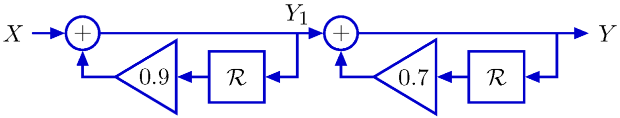
Factoring Second-Order Systems

The factored form corresponds to a cascade of simpler systems.

$$(1 - 0.7\mathcal{R})(1 - 0.9\mathcal{R})\,Y = X$$



$$(1 - 0.7\mathcal{R})\,Y_2 = X \qquad (1 - 0.9\mathcal{R})\,Y = Y_2$$



$$(1 - 0.9\mathcal{R})\,Y_1 = X \qquad (1 - 0.7\mathcal{R})\,Y = Y_1$$

The order doesn't matter (if systems are initially at rest).

Factoring Second-Order Systems

The unit-sample response of the cascaded system can be found by multiplying the polynomial representations of the subsystems.

$$\frac{Y}{X} = \frac{1}{(1-0.7\mathcal{R})(1-0.9\mathcal{R})} = \underbrace{\frac{1}{(1-0.7\mathcal{R})}} \times \underbrace{\frac{1}{(1-0.9\mathcal{R})}}$$

$$= \overbrace{(1+0.7\mathcal{R}+0.7^2\mathcal{R}^2+0.7^3\mathcal{R}^3+\cdots)} \times \overbrace{(1+0.9\mathcal{R}+0.9^2\mathcal{R}^2+0.9^3\mathcal{R}^3+\cdots}$$

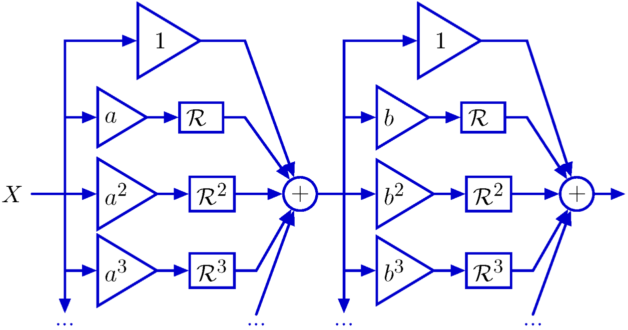
Multiply, then collect terms of equal order:

$$\begin{aligned}\frac{Y}{X} &= 1+(0.7+0.9)\mathcal{R}+(0.7^2+0.7\times 0.9+0.9^2)\mathcal{R}^2 \\ &\quad +(0.7^3+0.7^2\times 0.9+0.7\times 0.9^2+0.9^3)\mathcal{R}^3+\cdots\end{aligned}$$

Multiplying Polynomial

Graphical representation of polynomial multiplication.

$$\frac{Y}{X} = (1 + a\mathcal{R} + a^2\mathcal{R}^2 + a^3\mathcal{R}^3 + \cdots) \times (1 + b\mathcal{R} + b^2\mathcal{R}^2 + b^3\mathcal{R}^3 + \cdots)$$



Collect terms of equal order:

$$\frac{Y}{X} = 1 + (a+b)\mathcal{R} + (a^2 + ab + b^2)\mathcal{R}^2 + (a^3 + a^2b + ab^2 + b^3)\mathcal{R}^3 + \cdots$$

Multiplying Polynomial

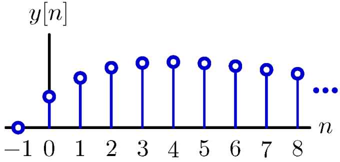
Tabular representation of polynomial multiplication.

$$(1 + a\mathcal{R} + a^2\mathcal{R}^2 + a^3\mathcal{R}^3 + \cdots) \times (1 + b\mathcal{R} + b^2\mathcal{R}^2 + b^3\mathcal{R}^3 + \cdots)$$

	1	$b\mathcal{R}$	$b^2\mathcal{R}^2$	$b^3\mathcal{R}^3$	$\cdots$
1	1	$b\mathcal{R}$	$b^2\mathcal{R}^2$	$b^3\mathcal{R}^3$	$\cdots$
$a\mathcal{R}$	$a\mathcal{R}$	$ab\mathcal{R}^2$	$ab^2\mathcal{R}^3$	$ab^3\mathcal{R}^4$	$\cdots$
$a^2\mathcal{R}^2$	$a^2\mathcal{R}^2$	$a^2b\mathcal{R}^3$	$a^2b^2\mathcal{R}^4$	$a^2b^3\mathcal{R}^5$	$\cdots$
$a^3\mathcal{R}^3$	$a^3\mathcal{R}^3$	$a^3b\mathcal{R}^4$	$a^3b^2\mathcal{R}^5$	$a^3b^3\mathcal{R}^6$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

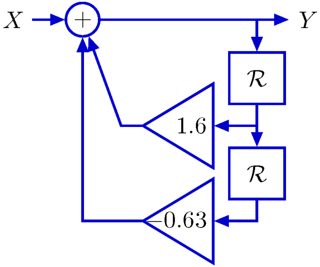
Group same powers of $\mathcal{R}$ by following reverse diagonals:

$$\frac{Y}{X} = 1 + (a+b)\mathcal{R} + (a^2 + ab + b^2)\mathcal{R}^2 + (a^3 + a^2b + ab^2 + b^3)\mathcal{R}^3 + \cdots$$



Partial Fractions

Use partial fractions to rewrite as a **sum** of simpler parts.

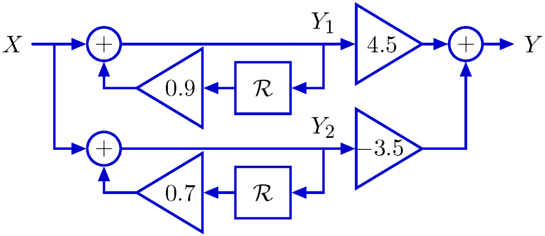


$$\frac{Y}{X} = \frac{1}{1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2} = \frac{1}{(1 - 0.9\mathcal{R})(1 - 0.7\mathcal{R})} = \frac{4.5}{1 - 0.9\mathcal{R}} - \frac{3.5}{1 - 0.7\mathcal{R}}$$

Second-Order Systems: Equivalent Forms

The sum of simpler parts suggests a parallel implementation.

$$\frac{Y}{X} = \frac{4.5}{1 - 0.9\mathcal{R}} - \frac{3.5}{1 - 0.7\mathcal{R}}$$

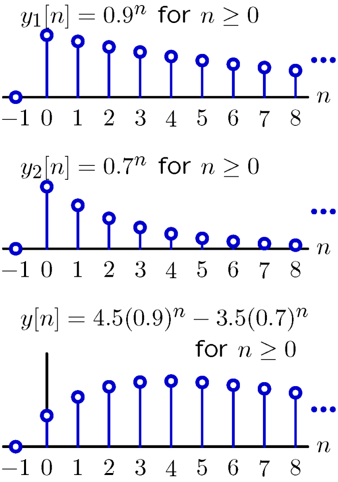


If $x[n] = \delta[n]$ then $y_1[n] = 0.9^n$ and $y_2[n] = 0.7^n$ for $n \geq 0$.

Thus, $y[n] = 4.5(0.9)^n - 3.5(0.7)^n$ for $n \geq 0$.

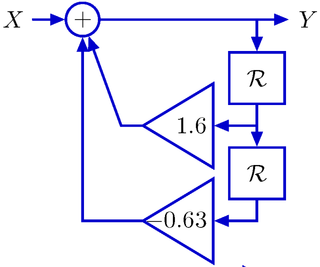
Partial Fractions

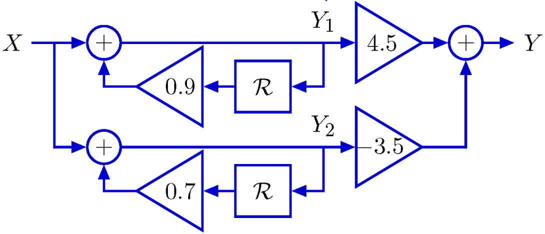
Graphical representation of the **sum** of geometric sequences.



Partial Fractions

Partial fractions provides a remarkable equivalence.





→ follows from thinking about system as polynomial (factoring).

Poles

The key to simplifying a higher-order system is identifying its **poles**.

Poles are the roots of the denominator of the system functional when $\mathcal{R} \to \frac{1}{z}$.

Start with system functional:

$$\frac{Y}{X} = \frac{1}{1 - 1.6\mathcal{R} + 0.63\mathcal{R}^2} = \frac{1}{(1 - p_0\mathcal{R})(1 - p_1\mathcal{R})} = \frac{1}{\underbrace{(1 - 0.7\mathcal{R})}_{p_0 = 0.7}\underbrace{(1 - 0.9\mathcal{R})}_{p_1 = 0.9}}$$

Substitute $\mathcal{R} \to \dfrac{1}{z}$ and find roots of denominator:

$$\frac{Y}{X} = \frac{1}{1 - \frac{1.6}{z} + \frac{0.63}{z^2}} = \frac{z^2}{z^2 - 1.6z + 0.63} = \frac{z^2}{\underbrace{(z - 0.7)}_{z_0 = 0.7}\underbrace{(z - 0.9)}_{z_1 = 0.9}}$$

The poles are at 0.7 and 0.9.

Check Yourself

Consider the system described by

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

How many of the following are true?

1. The unit sample response converges to zero.
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$.
3. There is a pole at $z = \frac{1}{2}$.
4. There are two poles.
5. None of the above

Check Yourself

$$y[n] = -\frac{1}{4}y[n-1] + \frac{1}{8}y[n-2] + x[n-1] - \frac{1}{2}x[n-2]$$

$$(1 + \frac{1}{4}\mathcal{R} - \frac{1}{8}\mathcal{R}^2)Y = (\mathcal{R} - \frac{1}{2}\mathcal{R}^2)X$$

$$H(\mathcal{R}) = \frac{Y}{X} = \frac{\mathcal{R} - \frac{1}{2}\mathcal{R}^2}{1 + \frac{1}{4}\mathcal{R} - \frac{1}{8}\mathcal{R}^2} = \frac{\frac{1}{z} - \frac{1}{2}\frac{1}{z^2}}{1 + \frac{1}{4}\frac{1}{z} - \frac{1}{8}\frac{1}{z^2}} = \frac{z - \frac{1}{2}}{z^2 + \frac{1}{4}z - \frac{1}{8}}$$

$$= \frac{z - \frac{1}{2}}{\left(z + \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

1. The unit sample response converges to zero. $\surd$
2. There are poles at $z = \frac{1}{2}$ and $z = \frac{1}{4}$. X
3. There is a pole at $z = \frac{1}{2}$. X
4. There are two poles. $\surd$
5. None of the above X

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Consider the system described by

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How many of the following are true? 2

1. The unit sample response converges to zero.
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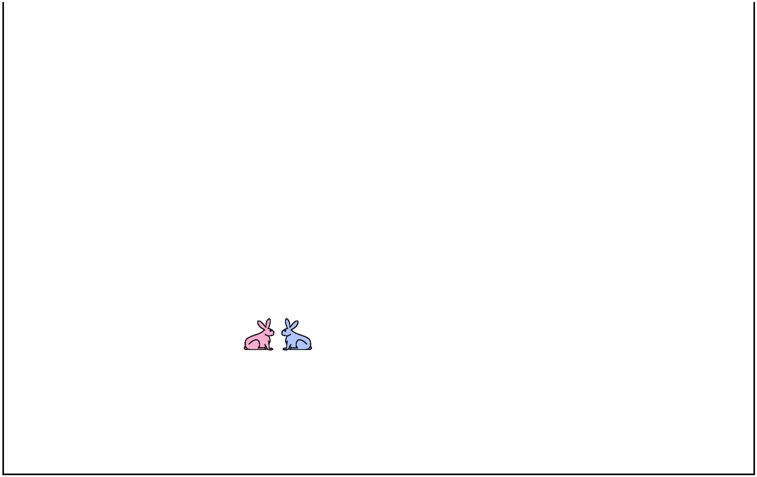
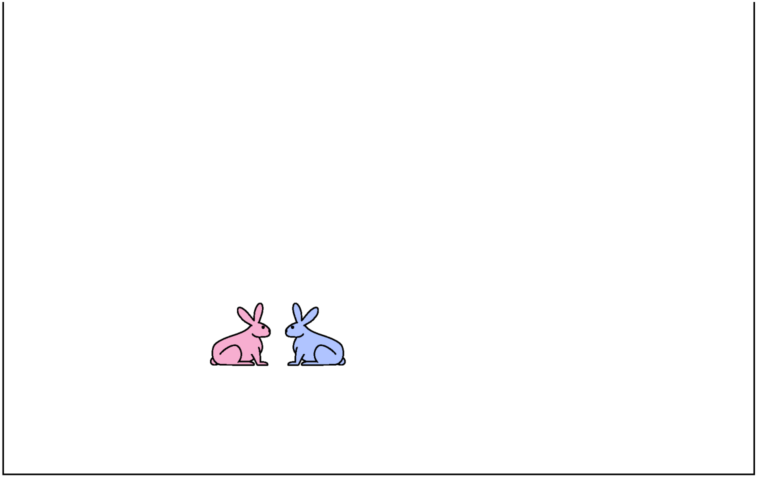
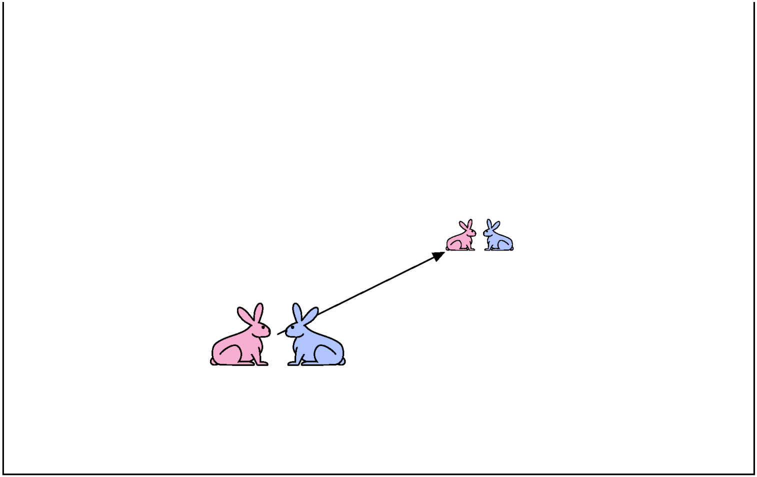
Population Growth



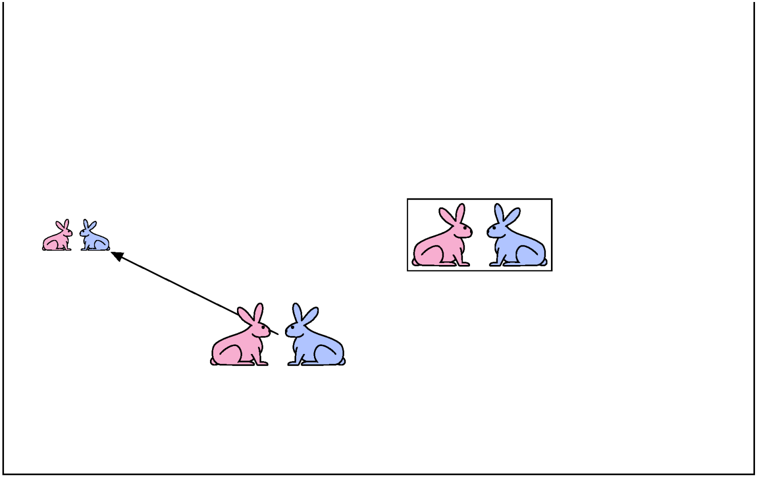
Population Growth



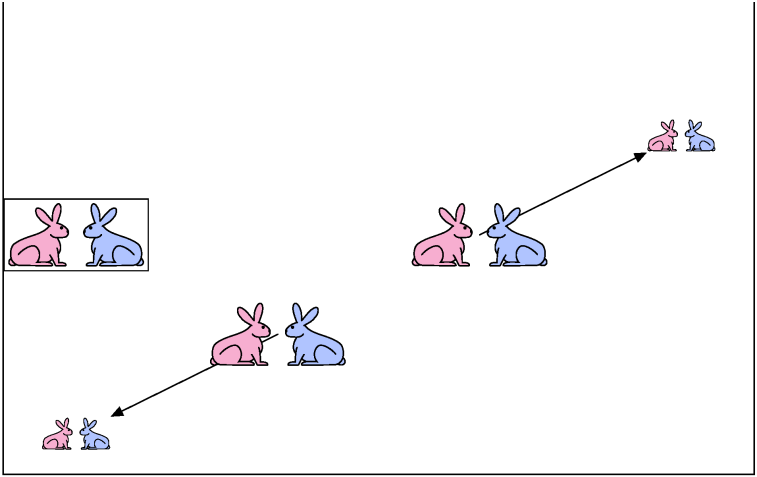
Population Growth



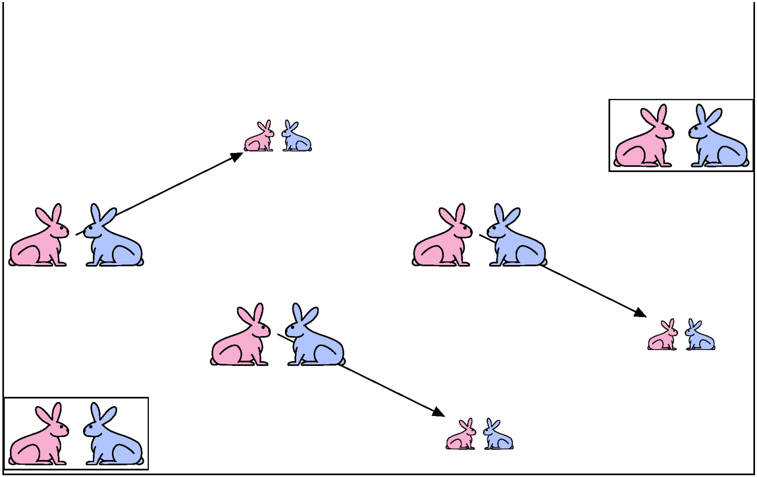
Population Growth



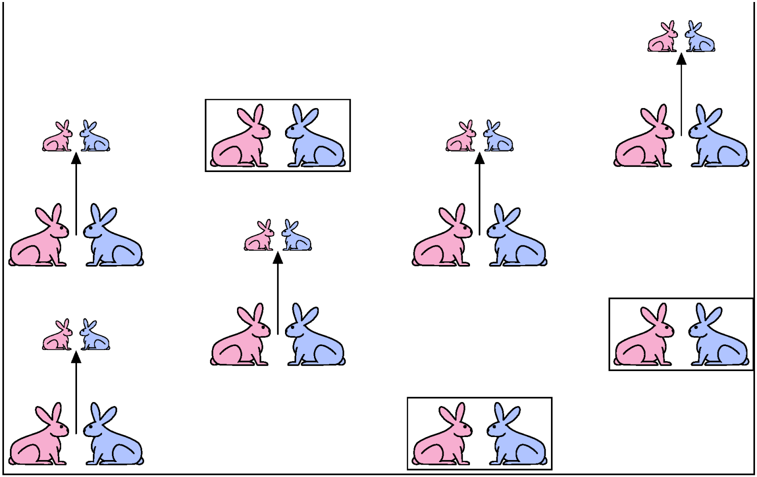
Population Growth



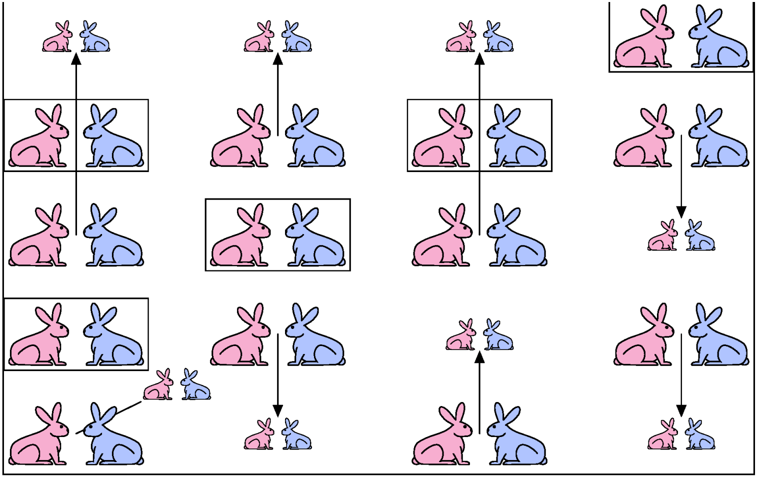
Population Growth



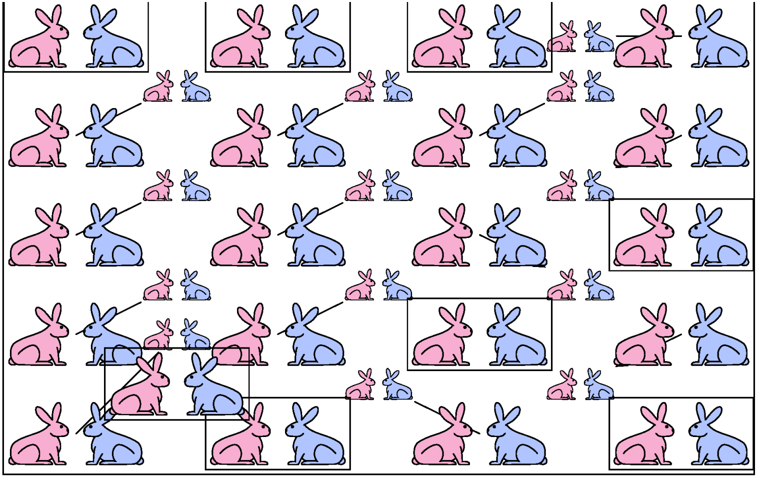
Population Growth



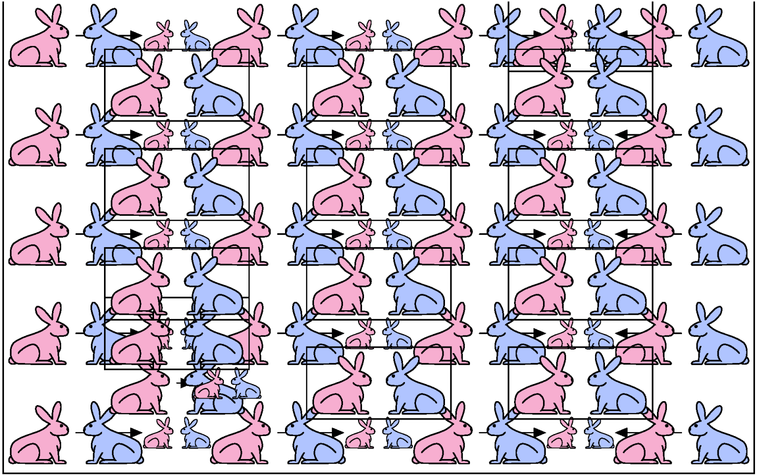
Population Growth



Population Growth



Population Growth



Check Yourself

What are the pole(s) of the Fibonacci system?

1. 1
2. 1 and -1
3. -1 and -2
4. $1.618\ldots$ and $-0.618\ldots$
5. none of the above

Check Yourself

What are the pole(s) of the Fibonacci system?

Difference equation for Fibonacci system:

$$y[n] = x[n] + y[n-1] + y[n-2]$$

System functional:

$$H = \frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2}$$

Denominator is second order $\rightarrow$ 2 poles.

Check Yourself

Find the poles by substituting $\mathcal{R} \to 1/z$ in system functional.

$$H = \frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2} \to \frac{1}{1 - \frac{1}{z} - \frac{1}{z^2}} = \frac{z^2}{z^2 - z - 1}$$

Poles are at

$$z = \frac{1 \pm \sqrt{5}}{2} = \phi, -\frac{1}{\phi}$$

where ϕ represents the "golden ratio"

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

The two poles are at

$$z_0 = \phi \approx 1.618 \quad \text{and} \quad z_1 = -\frac{1}{\phi} \approx -0.618$$

Check Yourself

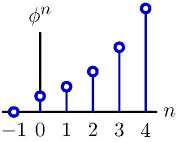
What are the pole(s) of the Fibonacci system? 4

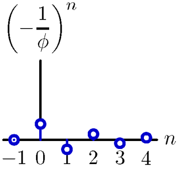
1. 1
2. 1 and -1
3. -1 and -2
4. $1.618\ldots$ and $-0.618\ldots$
5. none of the above

Example: Fibonacci's Bunnies

Each pole corresponds to a fundamental mode.

$$\phi \approx 1.618 \quad \text{and} \quad -\frac{1}{\phi} \approx -0.618$$





One mode diverges, one mode oscillates!

Example: Fibonacci's Bunnies

The unit-sample response of the Fibonacci system can be written as a weighted sum of fundamental modes.

$$H = \frac{Y}{X} = \frac{1}{1 - \mathcal{R} - \mathcal{R}^2} = \frac{\frac{\phi}{\sqrt{5}}}{1 - \phi\mathcal{R}} + \frac{\frac{1}{\phi\sqrt{5}}}{1 + \frac{1}{\phi}\mathcal{R}}$$

$$h[n] = \frac{\phi}{\sqrt{5}}\phi^n + \frac{1}{\phi\sqrt{5}}(-\phi)^{-n}; \qquad n \geq 0$$

But we already know that $h[n]$ is the Fibonacci sequence f:

$$f : 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

Therefore we can calculate $f[n]$ without knowing $f[n-1]$ or $f[n-2]$!

Complex Poles

What if a pole has a non-zero imaginary part?

Example:

$$\frac{Y}{X} = \frac{1}{1 - \mathcal{R} + \mathcal{R}^2}$$

$$= \frac{1}{1 - \frac{1}{z} + \frac{1}{z^2}} = \frac{z^2}{z^2 - z + 1}$$

Poles are $z = \frac{1}{2} \pm \frac{\sqrt{3}}{2} j = e^{\pm j\pi/3}$.

What are the implications of complex poles?

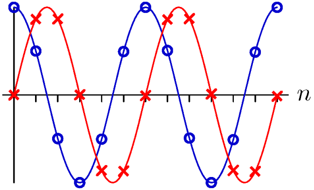
Complex Poles

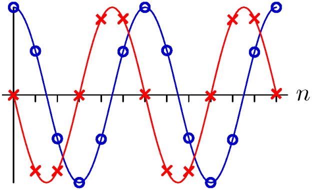
Partial fractions work even when the poles are complex.

$$\frac{Y}{X}=\frac{1}{1-e^{j\pi/3}\mathcal{R}}\times\frac{1}{1-e^{-j\pi/3}\mathcal{R}}=\frac{1}{j\sqrt{3}}\left(\frac{e^{j\pi/3}}{1-e^{j\pi/3}\mathcal{R}}-\frac{e^{-j\pi/3}}{1-e^{-j\pi/3}\mathcal{R}}\right)$$

There are two fundamental modes (both geometric sequences):

$$e^{jn\pi/3}=\cos(n\pi/3)+j\sin(n\pi/3) \quad \text{and} \quad e^{-jn\pi/3}=\cos(n\pi/3)-j\sin(n\pi/3)$$

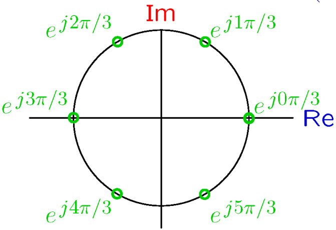


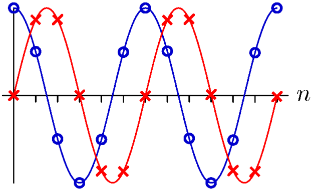


Complex Poles

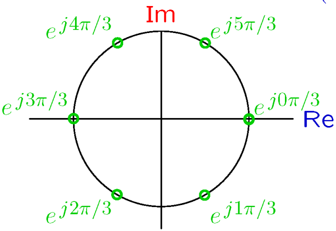
Complex modes are easier to visualize in the complex plane.

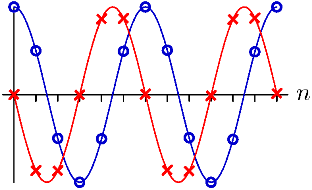
$$e^{jn\pi/3} = \cos(n\pi/3) + j\sin(n\pi/3)$$





$$e^{-jn\pi/3} = \cos(n\pi/3) - j\sin(n\pi/3)$$





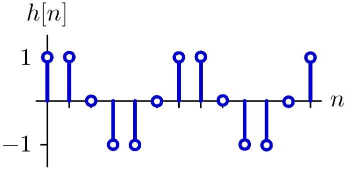
Complex Poles

The output of a "real" system has real values.

$$y[n] = x[n] + y[n-1] - y[n-2]$$

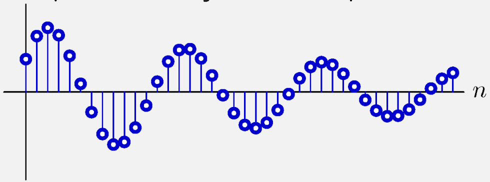
$$\begin{aligned} H = \frac{Y}{X} &= \frac{1}{1-\mathcal{R}+\mathcal{R}^2} \\ &= \frac{1}{1-e^{j\pi/3}\mathcal{R}} \times \frac{1}{1-e^{-j\pi/3}\mathcal{R}} \\ &= \frac{1}{j\sqrt{3}}\left(\frac{e^{j\pi/3}}{1-e^{j\pi/3}\mathcal{R}} - \frac{e^{-j\pi/3}}{1-e^{-j\pi/3}\mathcal{R}}\right) \end{aligned}$$

$$h[n] = \frac{1}{j\sqrt{3}}\left(e^{j(n+1)\pi/3} - e^{-j(n+1)\pi/3}\right) = \frac{2}{\sqrt{3}} \sin\frac{(n+1)\pi}{3}$$



Check Yourself

Unit-sample response of a system with poles at $z = re^{\pm j\Omega}$.

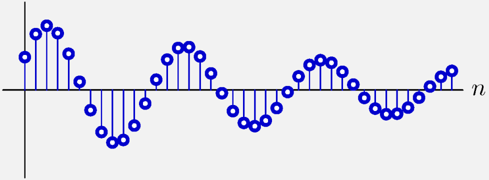


Which of the following is/are true?

1. $r < 0.5$ and $\Omega \approx 0.5$
2. $0.5 < r < 1$ and $\Omega \approx 0.5$
3. $r < 0.5$ and $\Omega \approx 0.08$
4. $0.5 < r < 1$ and $\Omega \approx 0.08$
5. none of the above

Check Yourself

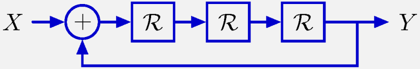
Unit-sample response of a system with poles at $z = re^{\pm j\Omega}$.



Which of the following is/are true? 2

1. $r < 0.5$ and $\Omega \approx 0.5$
2. $0.5 < r < 1$ and $\Omega \approx 0.5$
3. $r < 0.5$ and $\Omega \approx 0.08$
4. $0.5 < r < 1$ and $\Omega \approx 0.08$
5. none of the above

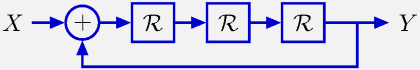
Check Yourself



How many of the following statements are true?

1. This system has 3 fundamental modes.
2. All of the fundamental modes can be written as geometrics.
3. Unit-sample response is $y[n] : 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
4. Unit-sample response is $y[n] : 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
5. One of the fundamental modes of this system is the unit step.

Check Yourself



How many of the following statements are true? 4

1. This system has 3 fundamental modes.
2. All of the fundamental modes can be written as geometrics.
3. Unit-sample response is $y[n] : 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
4. Unit-sample response is $y[n] : 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots$
5. One of the fundamental modes of this system is the unit step.

Summary

Systems composed of adders, gains, and delays can be characterized by their poles.

The poles of a system determine its fundamental modes.

The unit-sample response of a system can be expressed as a weighted sum of fundamental modes.

These properties follow from a polynomial interpretation of the system functional.

Assignments

- Reading Assignment: Supplementary notes