
CSE 3400/ CSE 5850 - Introduction to Computer & Network
Security
/ Introduction to Cybersecurity

Lecture 6
Hash Functions – Part I

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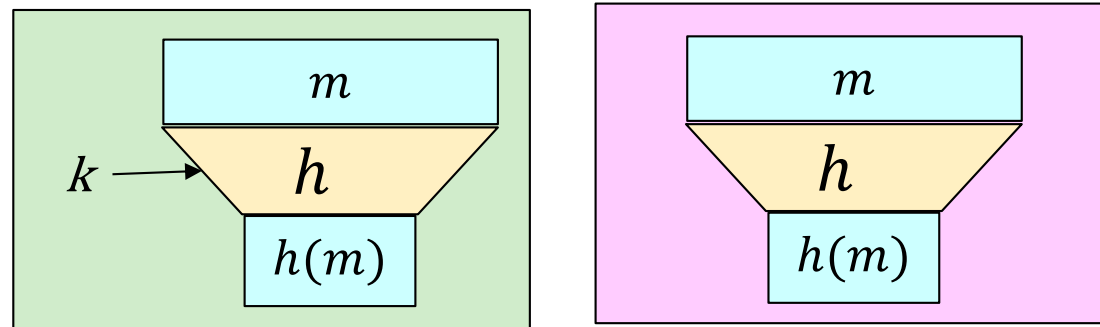
Adapted from textbook slides by Prof. Amir Herzberg
UConn

Outline

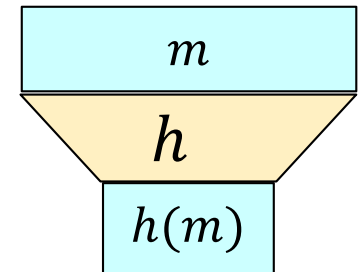
- Introduction and motivation.
- Collision resistant hash functions (CRHF).
- CRHF applications.
- Other notions of security.
 - TCR, SPR, OWF.
- Randomness extraction.
- The random oracle model.

Hash Functions

- Input m : binary strings
- Output $h(m)$:
 - 'Short' (n-bit) binary strings
 - Aka **message digest**
- Efficiently computable
- Applications: cryptography, security, efficiency
- Keyed $h_k(m)$, where the key is public, or unkeyed $h(m)$



Hash functions: simple examples

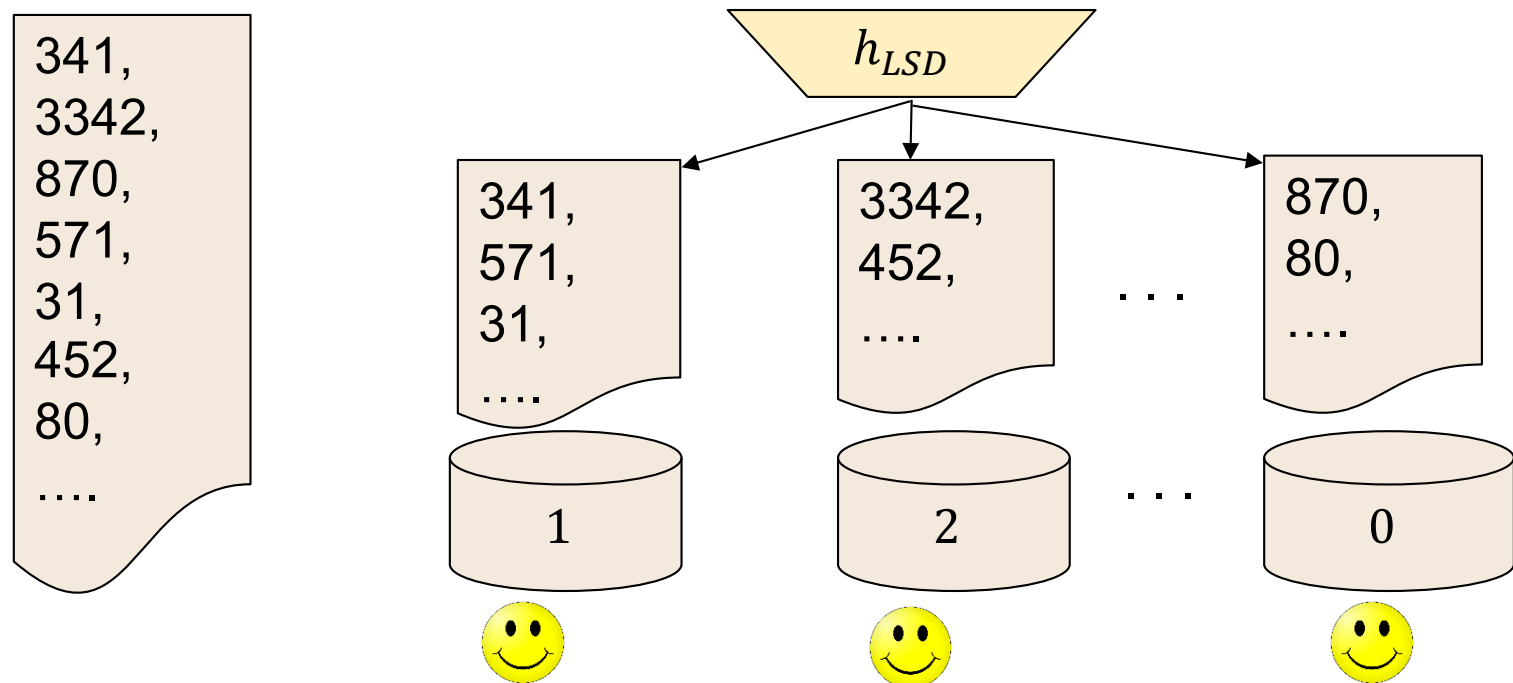


- For simplicity: input m is decimal integer
 - View as string of (three) digits
 - For example, $m = 127 \rightarrow m_1 = 1, m_2 = 2, m_3 = 7$
- Least Significant Digit hash:
$$h_{LSD}(m) = m_3$$
- Sum hash:
$$h_{Sum}(m) = (m_1 + m_2 + m_3) \bmod 10$$
- Exercise:
$$h_{LSD}(117) = \underline{7}$$
$$h_{Sum}(117) = \underline{9}$$

Note: the above are insecure hash functions, these are just toy examples to grasp the concept of hashing.

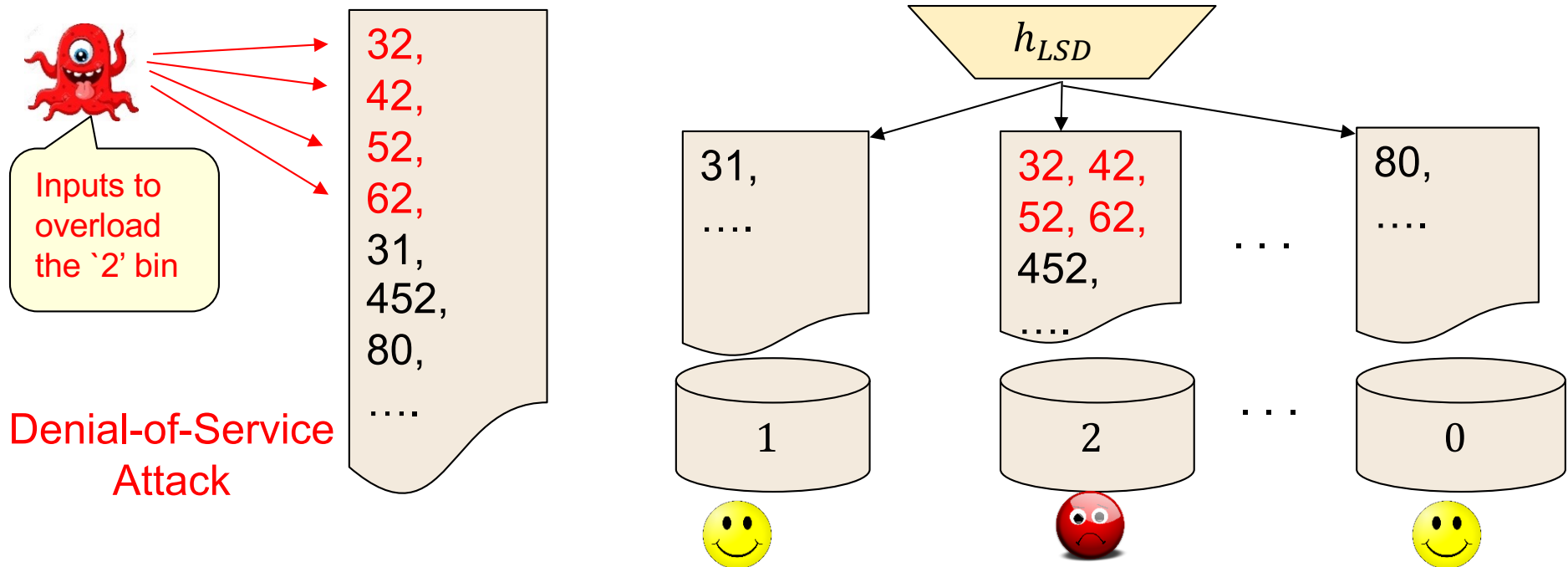
Motivation: Hashing for efficiency

- Input: large set (e.g., integers or strings)
- Goal: map 'randomly' to few bins
 - E.g., to ensure efficiency – load balancing, etc.



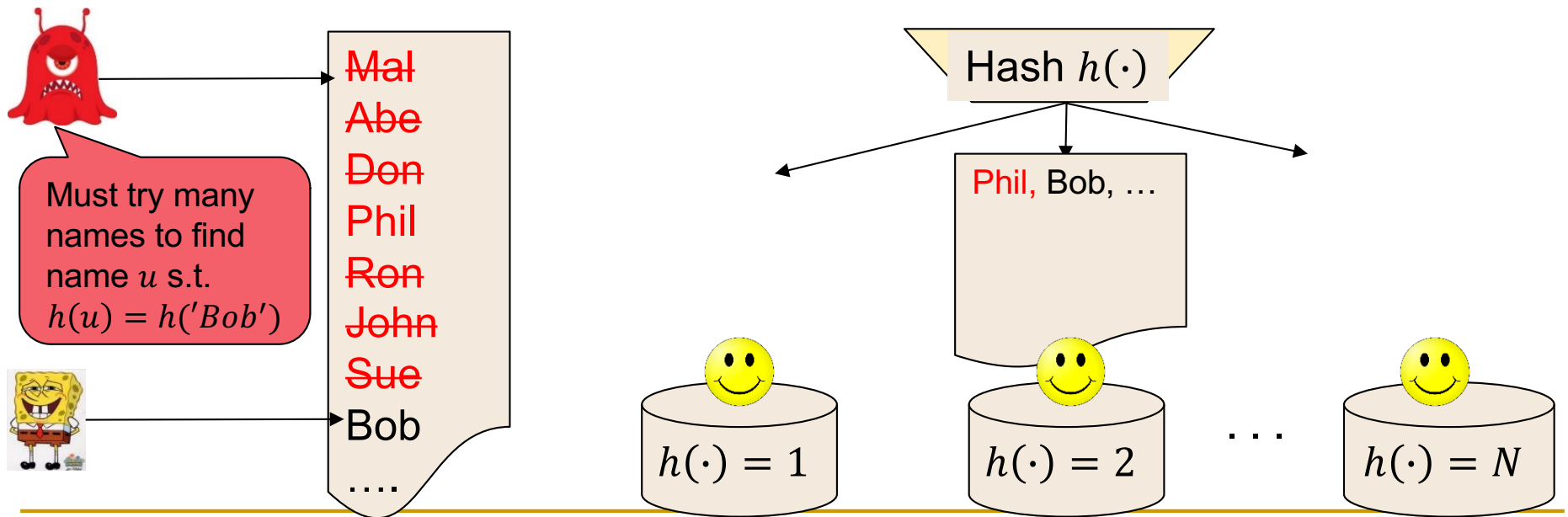
Collisions?

- Input: large set (e.g., integers or strings)
- Goal: map 'randomly' to few bins
 - E.g., to ensure efficiency – load balancing, etc.
 - Adversary chooses inputs that hash to same bin



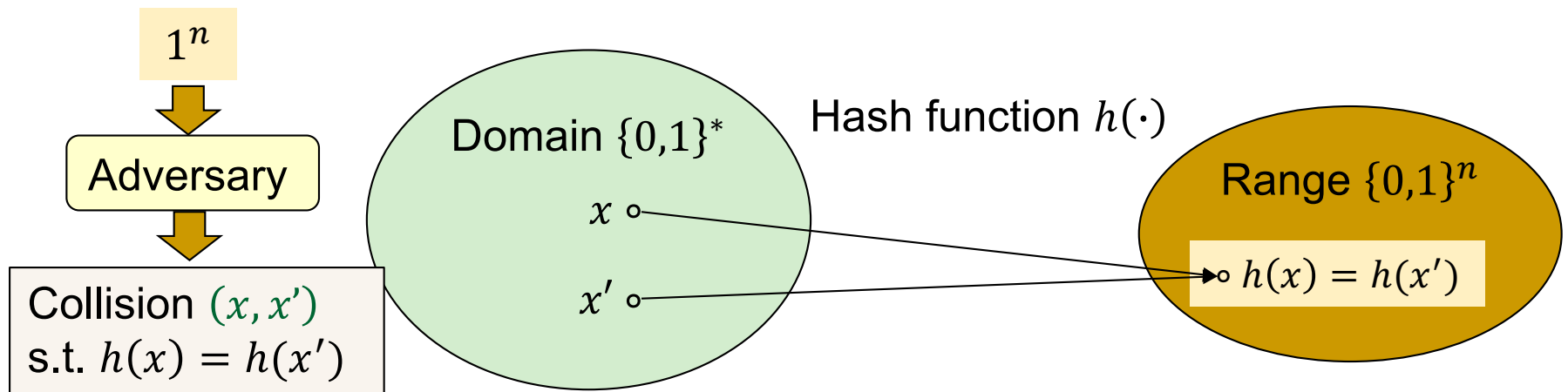
Security Goal: Collision Resistance

- A **collision**: two inputs (names) with same hash:
 $h('Bob') = h('Phil')$
- Every hash has collisions, since $|\text{input}| \gg |\text{output}|$!
- Collision resistance: hard to **find** these collisions
 - ❑ Note: attacker can always try names randomly until a collision is found
 - ❑ But this should be ineffective: must try about (on average) N names (number of bins)



Collision Resistant Hash Function (CRHF)

- h is CRHF if it is hard to **find** collisions $h(x) = h(x')$
 - Note: attacker can always try inputs randomly till finding collisions
 - But this should be ineffective: must try about $|Range|$ values
- Hard means that the probability that the attacker succeeds in finding a collision is negligible.



Collision Resistant Hash Function (CRHF)

- h is CRHF if it is hard to **find** collisions $h(x)=h(x')$
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- Hard means that the probability that the attacker succeeds in finding a collision is negligible.

Definition . . . (Keyless Collision Resistant Hash Function (CRHF)). *A keyless hash function $h^{(n)}(\cdot) : \{0, 1\}^* \rightarrow \{0, 1\}^n$ is collision-resistant if for every efficient (PPT) algorithm \mathcal{A} , the advantage $\varepsilon_{h,\mathcal{A}}^{CRHF}(n)$ is negligible in n , i.e., smaller than any positive polynomial for sufficiently large n (as $n \rightarrow \infty$), where:*

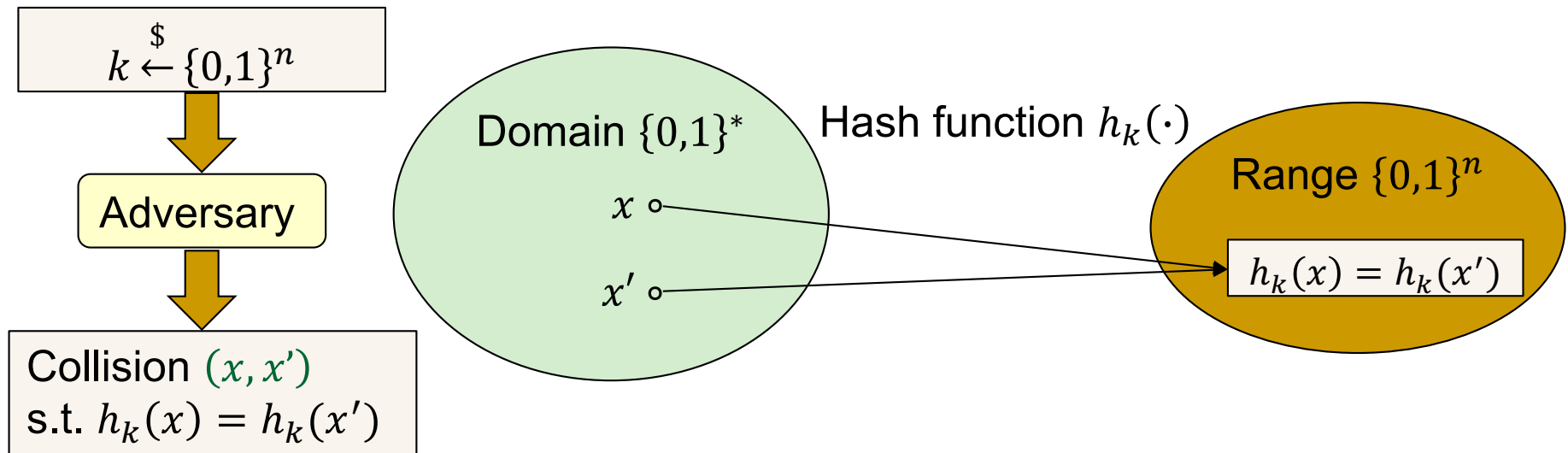
$$\varepsilon_{h,\mathcal{A}}^{CRHF}(n) \equiv \Pr \left[(x, x') \leftarrow \mathcal{A}(1^n) \text{ s.t. } (x \neq x') \wedge (h^{(n)}(x) = h^{(n)}(x')) \right]$$

Where the probability is taken over the random coin tosses of \mathcal{A} .

Keyless CRHF **Do Not Exist!**

- $|\text{Range}| \ll |\text{Domain}|$ so there is a collision where $h(x') = h(x)$, $x \neq x'$
- For a keyless CRHF there is a PPT algorithm A that can always output a collision: $A(1^n) = \{\text{return } x, x'\}$
 - Proof: in textbook.
 - Intuitively, since the function is fixed (same input-output mapping), a collision instance can be hardcoded in the attacker algorithm and just output that collision and win the security game.
- **Solutions:**
 - keyed CRHF,
 - use weaker collision notions,
 - or ignore! (more like asking if the collision is useful for the attacker?)

Keyed CRHF



Adversary knows k but **not in advance** – cannot `know` a collision

Often referred to as **ACR**-hash (**ANY**-collision resistance)

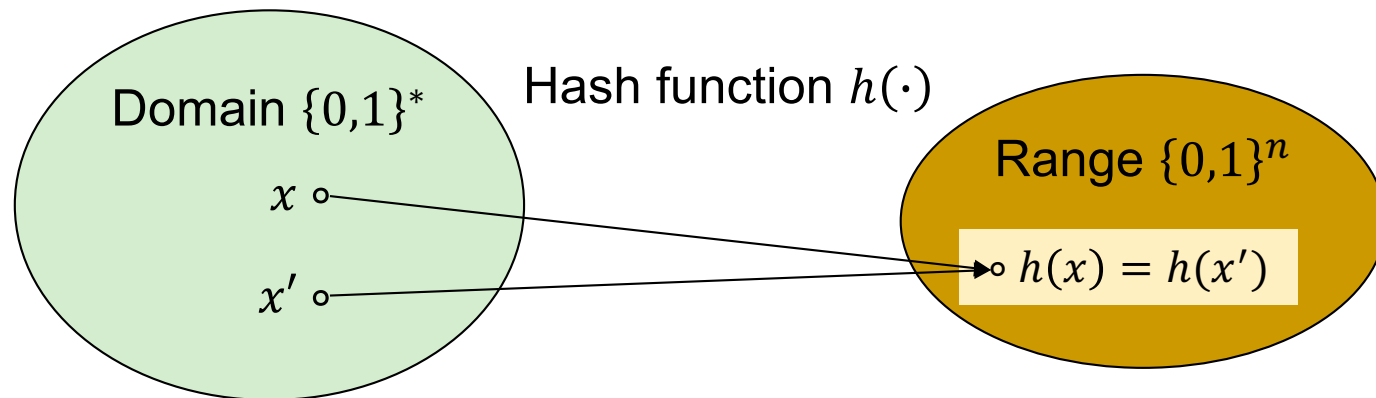
Keyed CRHF - Definition

Definition (Keyed Collision Resistant Hash Function (CRHF)). Consider a keyed hash function $h_k(\cdot) : \{0, 1\}^n \times \{0, 1\}^* \rightarrow \{0, 1\}^n$, defined for any $n \in \mathbb{N}$. We say that h is collision-resistant if for every efficient (PPT) algorithm \mathcal{A} , the advantage $\varepsilon_{h, \mathcal{A}}^{CRHF}(n)$ is negligible in n , i.e., $\varepsilon_{h, \mathcal{A}}^{CRHF}(n) \in \text{NEGL}(n)$, where:

$$\varepsilon_{h, \mathcal{A}}^{CRHF}(n) \equiv \Pr_{k \leftarrow \{0, 1\}^n} [(x, x') \leftarrow \mathcal{A}(k) \text{ s.t. } (x \neq x') \wedge (h_k(x) = h_k(x'))]$$

Where the probability is taken over the random coin tosses of the adversary \mathcal{A} and the random choice of k .

Generic Collision Attacks



- An attacker that runs in exponential time can always find a collision (i.e., non PPT attacker)
 - Easy: find collisions in 2^n time by trying $2^n + 1$ distinct inputs (compute their hash and locate a collision).
- An attacker finds a collision with 2^{-n} probability (negligible probability).
 - Choose x and x' at random and check if they produce a collision.

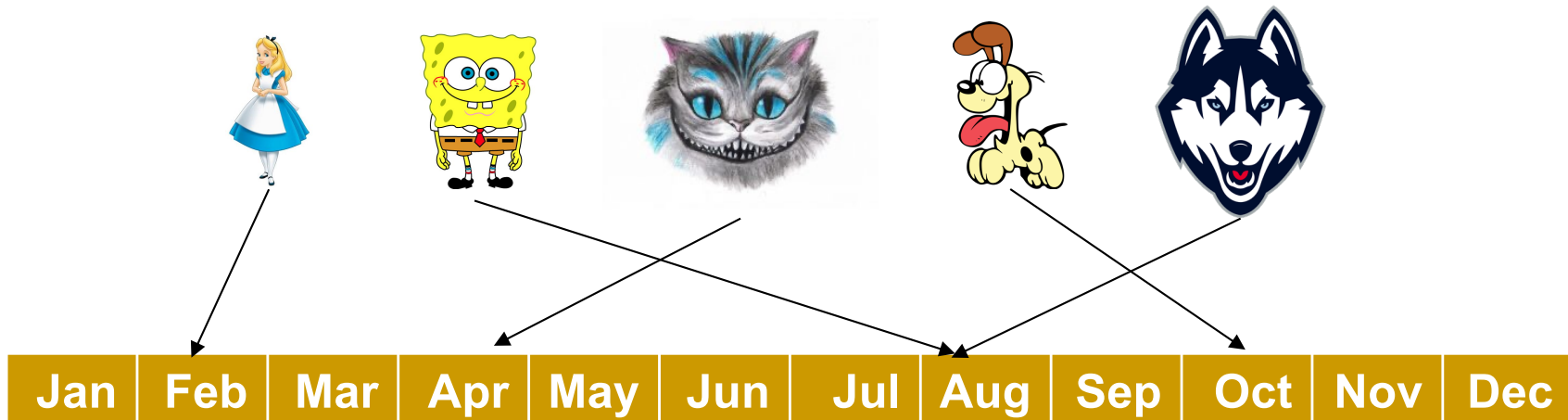
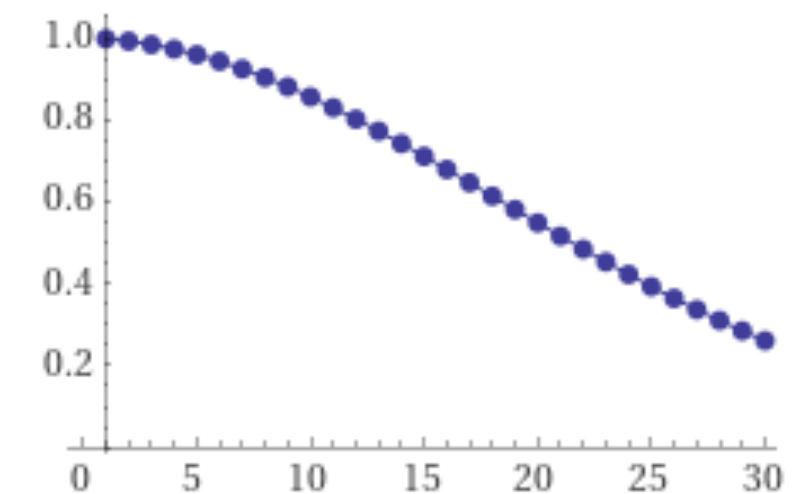
The Birthday Paradox

- **The birthday paradox** states that the expected number q of hashes until a collision is found in $O(2^{n/2})$ not $O(2^n)$.
 - It is $q \lesssim 2^{n/2} \cdot \sqrt{\frac{\pi}{2}} \lesssim 1.254 \cdot 2^{n/2}$
- For 80 bit of effective security, use $n=160$!
 - So to defend against an attacker who can perform 2^{80} hashes set the digest length to be at least 160 bits.
 - So the range has a size of 2^{160} digests.
- Why? Intuition?

The Birthday Attack ('Paradox')

■ Probability of NO birthday-collision:

- Two persons: $(364/365)$
- Three persons: $(364/365)*(363/365)$
- ...
- n persons: $\prod_{i=1}^{n-1} \frac{365-i}{365}$
- So for 23 individuals, the probability of birthday collision is about 0.5

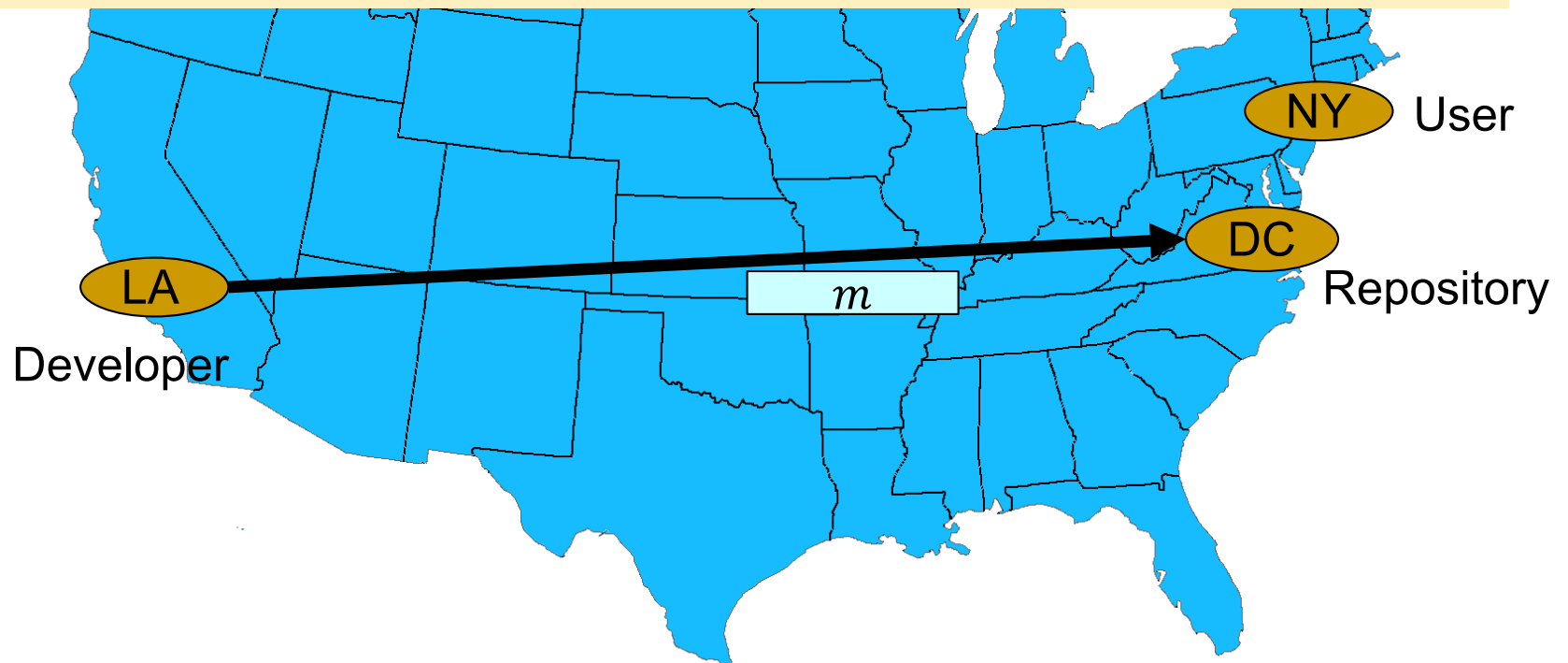


Collision-Resistance: Applications

- Integrity (of object / file / message)
 - Send $hash(m)$ securely to validate m
 - Later we will see how a hash function can be used to construct a MAC (called HMAC).
- Hash-then-Sign
 - Instead of signing m sign $hash(m)$
 - More efficient!
 - We will explore this in detail once we study digital signatures.
- Blockchains
 - Later

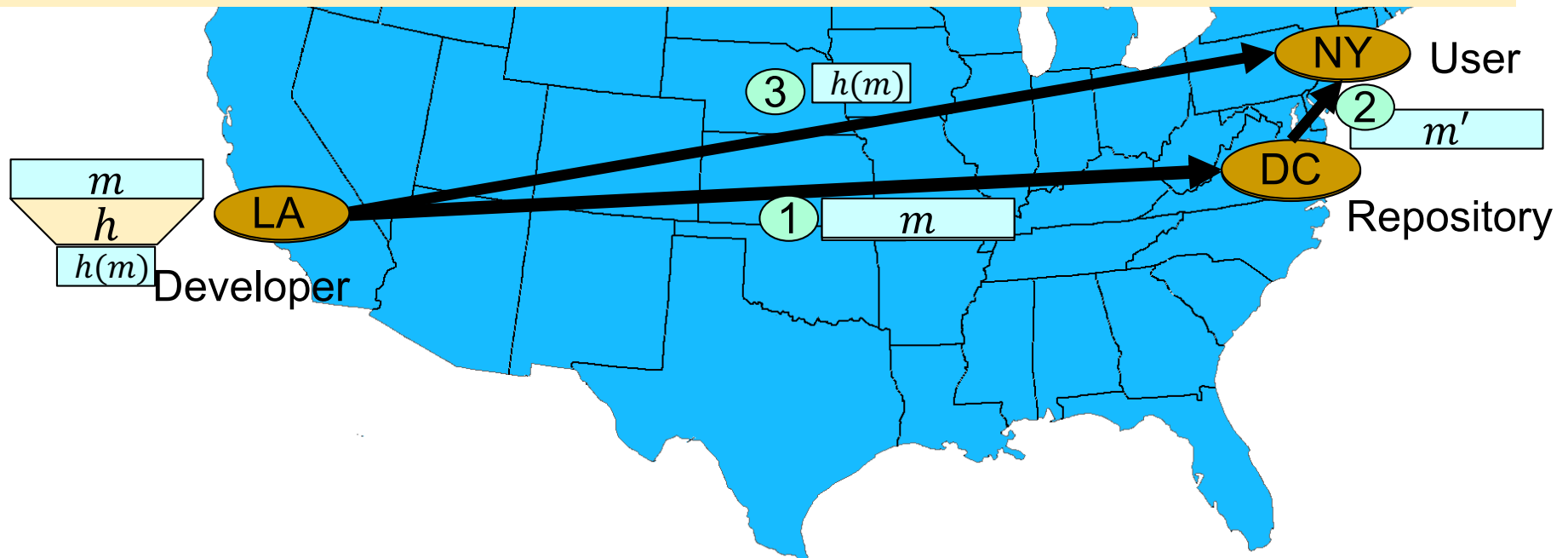
CRHF and Software Distribution

- ❑ Developer in LA develops a large software m
- ❑ Repository in DC obtains a copy of m
- ❑ User in NY wants to obtain m – securely and efficiently
 - Don't send m from LA to **both** NY and DC
- ❑ How?



CRHF: secure, efficient SW distribution

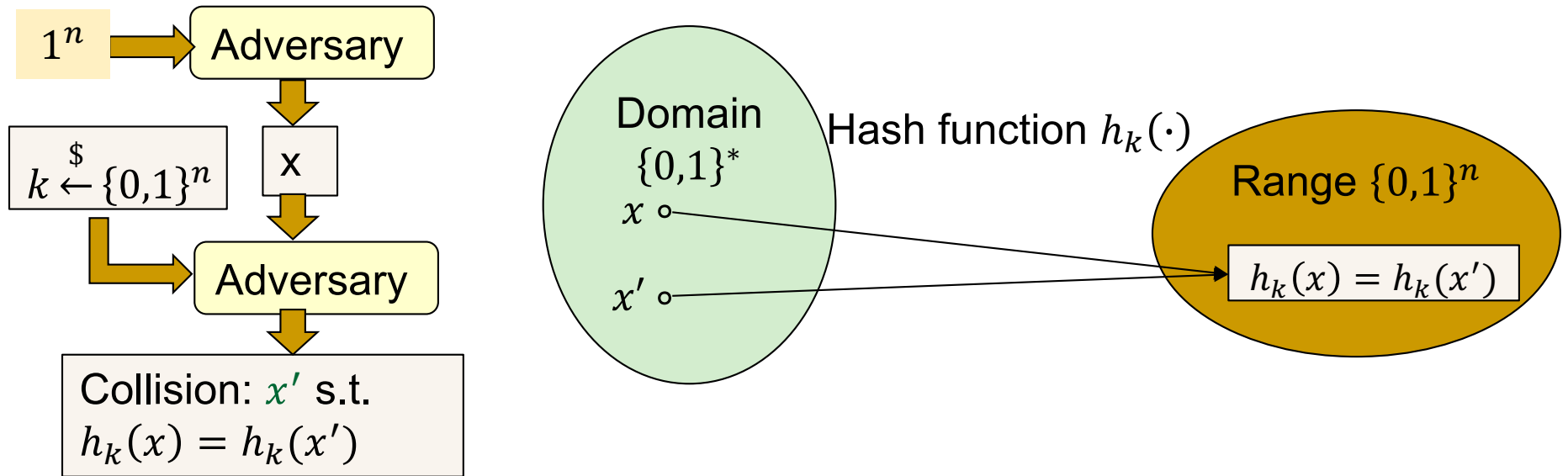
1. Repository in DC downloads software m from developer in LA
2. User downloads from (nearby) repository; receives m'
 - Is $m' = m$? User should validate! How?
3. User securely downloads $h(m)$ directly from developer
 - Digest $h(m)$ is short – much less overhead than downloading m
4. User validates: $h(m) = h(m') \rightarrow m = m'$



Other Notions of Security

- Collision resistance provides the strongest guarantee.
 - ❑ Gives more freedom to the adversary; the adversary wins if it finds any two inputs with the same digest.
 - No conditions on these two inputs other than being in the domain of the hash function.
 - Other security notions (but sufficient for many applications):
 - ❑ Target collision resistance (TCR).
 - ❑ Second preimage resistance.
 - ❑ First preimage resistance.
 - Birthday paradox (or attack) does not work against these weaker notions.
 - ❑ It is for collision resistance; find **any** two inputs that collide!
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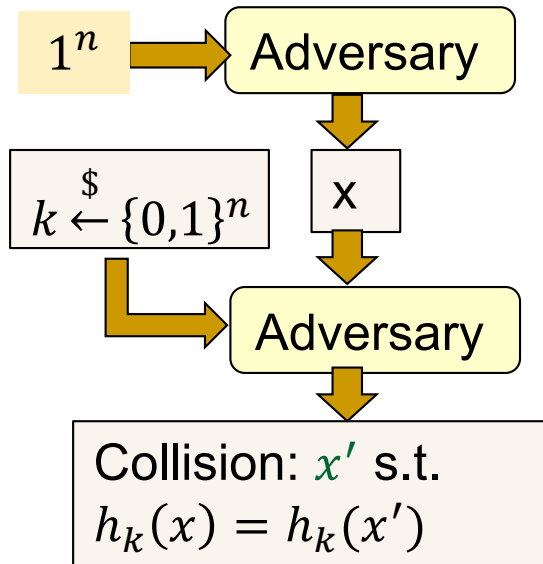
Target CRHF (TCR Hash Function)



Adversary has to select target **before** knowing key

$$\varepsilon_{h,\mathcal{A}}^{TCR}(n) \equiv \Pr_{k \leftarrow \{0,1\}^n} \left[\left\{ \begin{array}{l} x \leftarrow A(1^n); \\ x' \leftarrow A(x, k) \end{array} \right\} \text{ s.t. } (x \neq x') \wedge (h_k(x) = h_k(x')) \right]$$

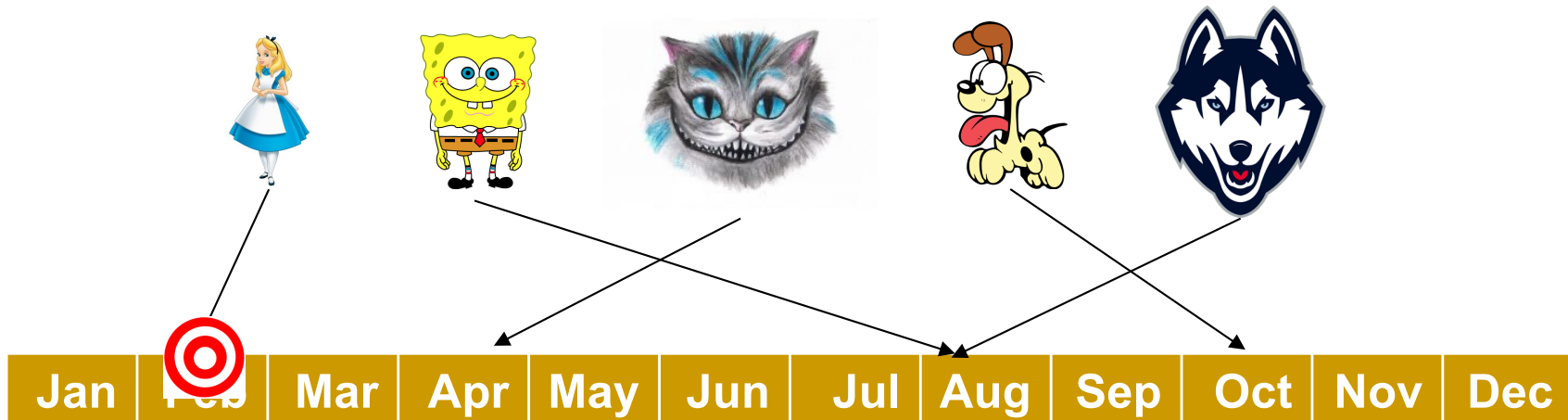
TCR and Birthday Paradox?



- **First:** adversary selects x
- Probability for NO birthday-collision with x :

- Two persons: $(364/365)$
- Three persons: $(364/365) \cdot (364/365)$
- ...

- n persons: $\prod_{i=1}^{n-1} \frac{364}{365} = \left(\frac{364}{365}\right)^{n-1}$

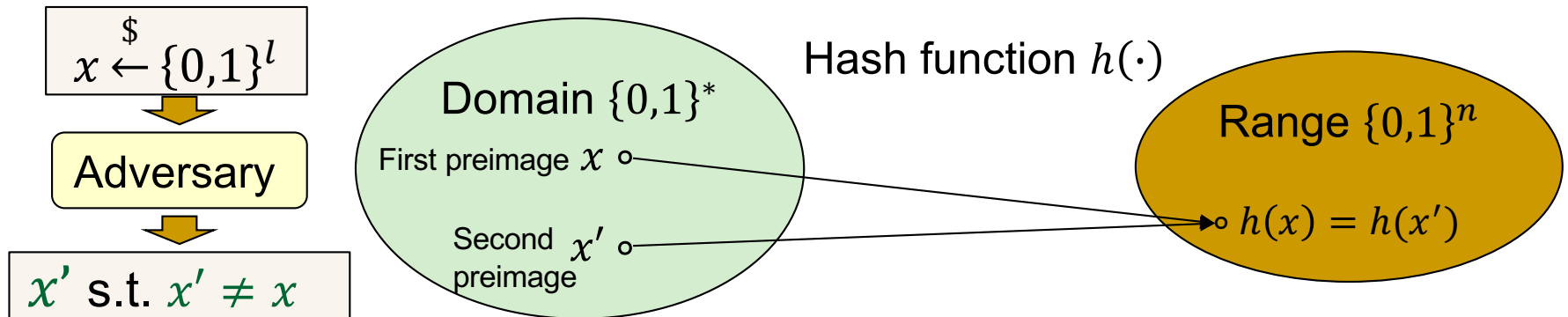


We (mostly) focus on keyless hash...

- Although there are no CRHFs
- And theory papers focus on keyed hash
- But...
 - It's a bit less complicated and easier to work with.
 - No need to consider both ACR (this stands for Any Collision Resistance, which is the same as CRHF) and TCR
 - Why?
 - Modifying to CRHF is quite trivial
 - Just make it keyed!
 - Usually used in practice: libraries, standards, ...

2nd-Preimage-Resistant Hash (SPR)

- Hard to find collision with a specific random x .



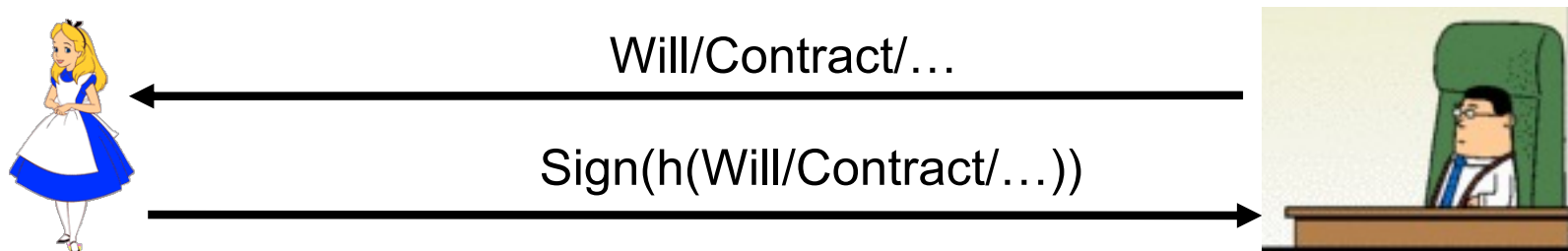
$$\varepsilon_{h,\mathcal{A}}^{SPR}(n) \equiv \Pr_{x \leftarrow \{0,1\}^{A(1^n)}} [x' \leftarrow \mathcal{A}(x) \text{ s.t. } x \neq x' \wedge h(x) = h(x')]$$

Use with care!

(think carefully about the security you want to achieve and see if SPR suffices)

CRHF/SPR vs. Applications

- CRHF secure for signing, SW-distribution
- How about SPR hash (weak-CRHF)?
 - SW-distribution? **YES**
 - Hash-then-sign? **NO**
- Why?
 - Attacker can't impact SW to be distributed
 - But... attacker may be able to impact signed msg!



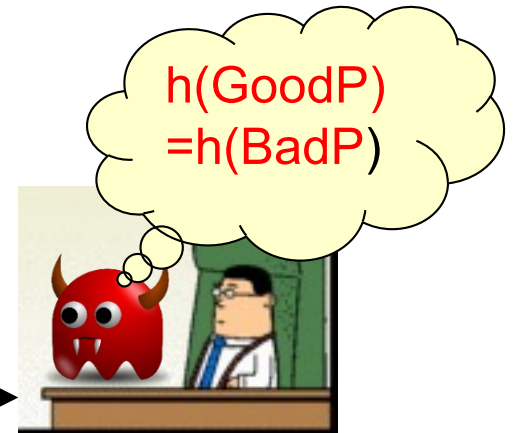
SPR: Collisions to Chosen Messages

- Or: Alice and Mal, the corrupt lawyer
- Mal finds two `colliding wills', GoodW and BadW:
 - GoodW: contents agreeable to Alice
 - $h(\text{GoodW}) = h(\text{BadW})$
 - Alice Signs good will: $\text{Sign}_A(h(\text{GoodW}))$



GoodW: 'I leave all to Bob'

$\text{Sign}_A(h(\text{GoodW}))$

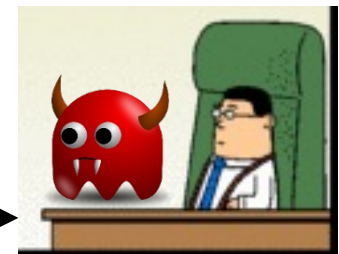


- Later... Mal presents to the court:



BadW: 'I leave all to Mal', $\text{Sign}_A(h(\text{BadW}))$

\$\$\$\$



SPR: collisions to **chosen** message

- Or: Alice and Mal, the corrupt lawyer
- Mal finds two 'colliding wills', GoodW and BadW:
 - GoodW: contents agreeable to Alice

Is such attack realistic?
Or SPR is enough 'in practice'?



SPR & Chosen-prefix Vulnerability

- Chosen-prefix vulnerability:
 - Mal selects 'prefix string' p
 - Efficient algorithm finds:
$$x \neq x' \text{ s.t. } h(p||x) = h(p||x')$$
 - Or, also for any suffix: $(\forall s) h(p||x||s) = h(p||x'||s)$
- Hash may be SPR yet allow chosen-prefix attacks
- Such attacks found for several proposed, standard cryptographic hash function, e.g., MD5 and SHA1
- We show chosen prefix attack on HtS
 - Example of possible attack on HtS with SPR

Chosen-prefix Attack

- Let $x < x'$ be collision for prefix: $p = \text{'Pay Mal \$'}$
- Mal tricks Alice into signing him an IOU for $\$x$
- Alice signs, sends $s = S_s^h(m)$ where $m = \text{'Pay Mal \$'} || x$
 - $S_s^h(m) = S_s(h(p || x)) = S_s(h(p || x')) = S_s^h(m')$
 - $m' = \text{'Pay Mal \$'} || x'$
- Mal sends s, m' to Alice's bank
 - Bank validates " Ok " = $Verify_{Alice.v}(m', s)$
- Bank gives $\$x'$ of Alice to Mal!!
- This attack is simplified:
 - Mal has to find 'good' collision (high profit, convince Alice to sign)
 - People sign (PDF) files, not plain text...
- In reality, attacker also chooses suffix → stronger attack

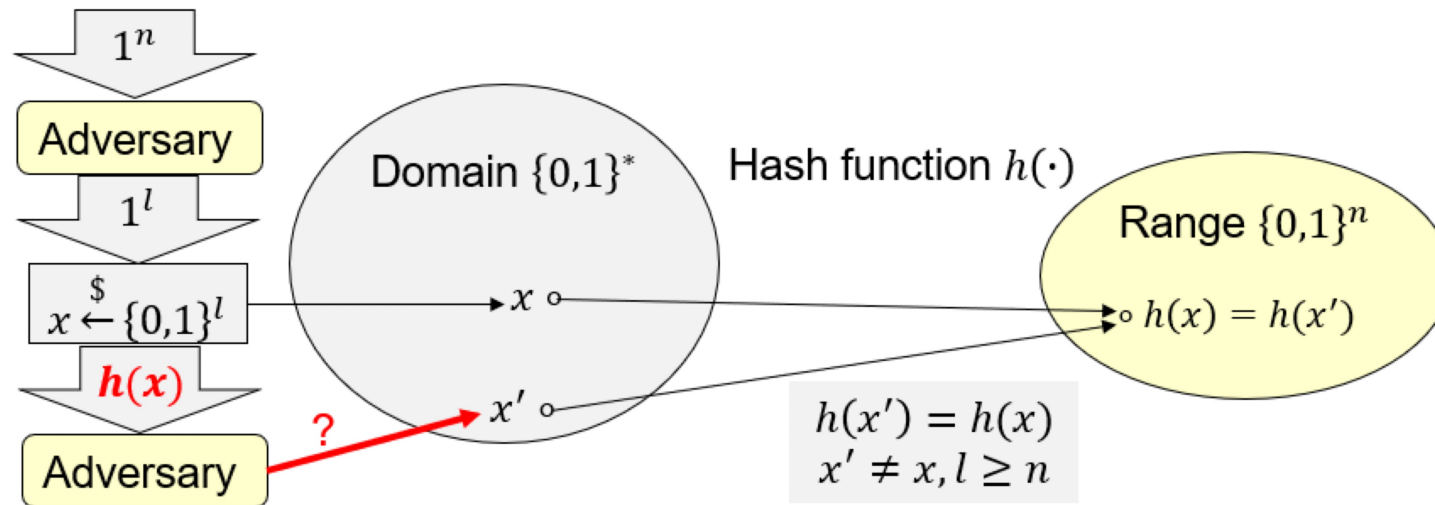
Examples

- 1) Let h_k be a keyed CRHF. Is $h_k' = h_k(h_k(x))$ a CRHF? Why?
- 2) For x parsed as $x = x_1 || x_2 || x_3$, let $h(x_1 || x_2 || x_3) = x_1 + x_2 + x_3 \bmod p$, is h a CRHF? Why? Is it an SPR? Why?
- 3) Let $h_k(m)$ be a TCR function. Construct $h_k'(m) = 0^n$ if $m[1: |k|] = k$ and $h_k(m)$ otherwise (recall m , the hash function input, can be of any length).
 - Is h_k' a CRHF? Why?
 - Is h_k' a TCR? Why?





One-Way Function (OWF)



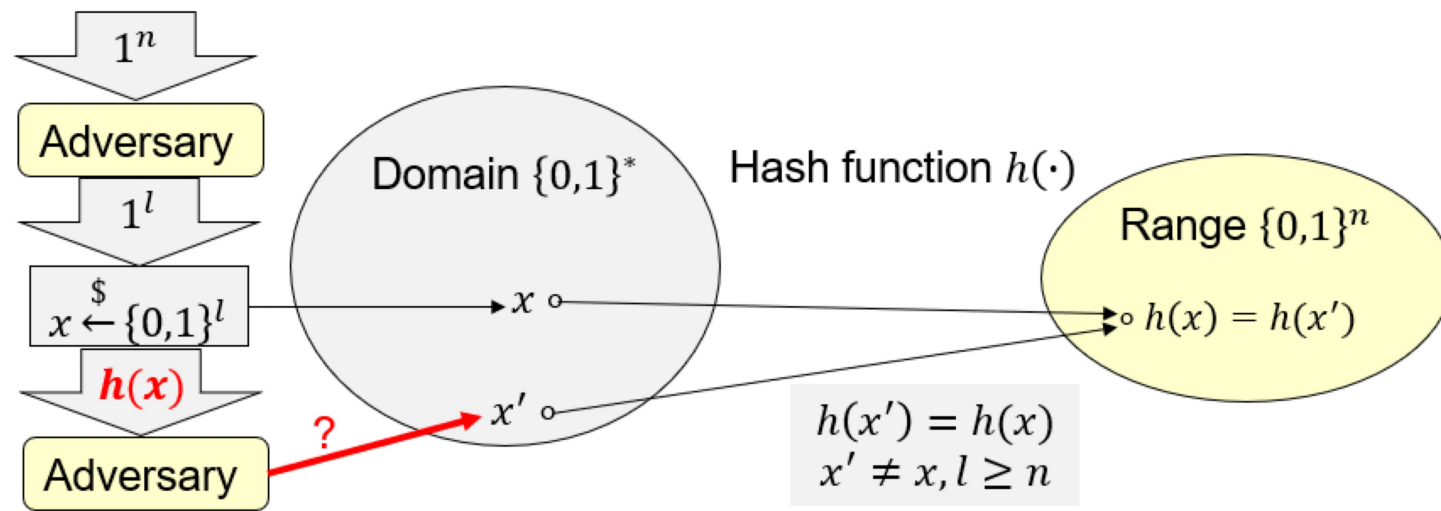
- ❑ **One-way function or first preimage resistance:** given $h(x)$ for a random x , it is hard to find x , or any x' s.t. $h(x') = h(x)$

Compare to:

- ❑ **Collision-Resistance (CRHF):** hard to find any collision, i.e., any (x, x') s.t. $h(x') = h(x)$, $x \neq x'$
- ❑ **Second-preimage resistance (SPR):** hard to find a collision with random x , i.e., x' s.t. $h(x') = h(x)$, $x \neq x'$

Note: the textbook has a typo in formula 3.17, page 201. The attacker A takes $h(x)$ as input not x .

Application: One-time Password Authentication



❑ One-time password authentication:

- Select random x : 'one-time password' (keep secret!)
- Validate using non-secret 'one-time validation token': $h(x)$

Exercise

- Let h_1, h_2 be both CRHF and OWF
- Use them to construct:
 - h_{CRHF} - CRHF but not OWF
 - h_{OWF} - OWF but not CRHF

Exercise

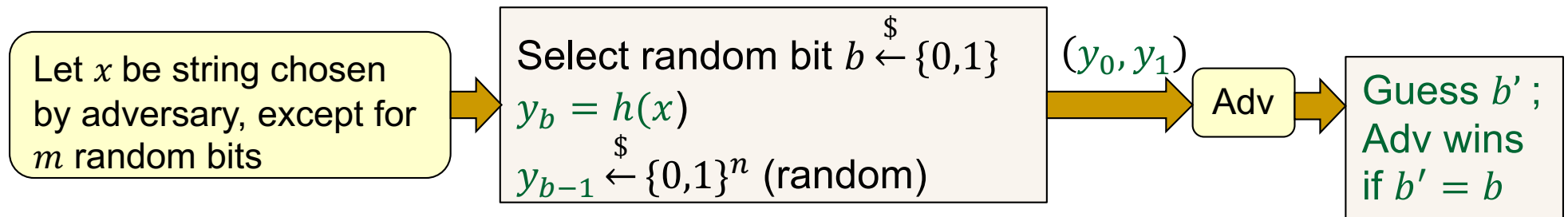
- Let h_1, h_2 be both CRHF and OWF
- Use them to construct:
 - h_{CRHF} - CRHF but not OWF
 - h_{OWF} - OWF but not CRHF
- One possible solution:
 - $$h_{CRHF}(m) = \begin{cases} m & \text{if } |m| = n \\ h_1(m) & \text{otherwise} \end{cases}$$
 - $$h_{OWF}(m) = \begin{cases} h_1(m) & \text{if } |m| = n \\ h_1(m_{1..n} \oplus h_2(m')) & \text{if } m = m_{1..n} || m' \end{cases}$$



What does all of that tell you about ...

- ... *the relation between CRHF, TCR, SPR, and OWF?*
 - CRHF vs. TCR?
 - CRHF vs. SPR?
 - TCR vs. SPR?
 - CRHF vs. OWF?
 - SPR vs. OWF?

Randomness Extraction



- ❑ 'If input is sufficiently random, then output is random'
- ❑ Multiple 'sufficiently random' models
- ❑ **Randomness extraction:** if any m input bits are random \rightarrow all n output bits are pseudorandom
 - For sufficiently large m
 - **Pseudorandom:** it is not computationally-feasible to distinguish between these bits and truly random bits

Random Oracle Model (ROM)

- Use a fixed, keyless hash function h
- Analyse security as if hash $h()$ is a *random function*
 - An invalid assumption: $h()$ is fixed!
 - Whenever $h()$ is used, use oracle (black box) for random function
- Good for screening insecure solutions
 - Random oracle security \rightarrow many attacks fail
- In practice: assume random oracle and use a standard hash function
 - It was shown that in some cases the construction will become insecure.
- Better to have security with standard assumption than the non-standard ROM.

Covered Material From the Textbook

- ❑ Chapter 3
 - ❑ Section 3.1,
 - ❑ Section 3.2 (except 3.2.6 and 3.2.7),
 - ❑ Section 3.3,
 - ❑ Section 3.4 (except 3.4.2),
 - ❑ Section 3.5 (only its introduction, i.e., all text before 3.5.1),
 - ❑ Section 3.6 (only what was covered in class)

Thank You!

