

Autonomous Navigation in Unknown Environments with Sparse Bayesian Kernel-based Occupancy Mapping

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Abstract—This paper focuses on online occupancy mapping and real-time collision checking onboard an autonomous robot navigating in a large unknown environment. Commonly used voxel and octree map representations can be easily maintained in a small environment but have increasing memory requirements as the environment grows. We propose a fundamentally different approach for occupancy mapping, in which the boundary between occupied and free space is viewed as the decision boundary of a machine learning classifier. This work generalizes a kernel perceptron model which maintains a very sparse set of support vectors to represent the environment boundaries efficiently. We develop a probabilistic formulation based on Relevance Vector Machines, allowing robustness to measurement noise and localization errors as well as probabilistic occupancy classification, supporting autonomous navigation. We provide an online training algorithm, updating the sparse Bayesian map incrementally from streaming range data, and an efficient collision-checking method for general curves, representing potential robot trajectories. The effectiveness of our mapping and collision checking algorithms is evaluated in tasks requiring autonomous robot navigation in unknown environments.

Index Terms—Sparse Bayesian Classification, Kernel-based Occupancy Mapping, Relevance Vector Machine, Autonomous Navigation, Collision Avoidance.

SUPPLEMENTARY MATERIAL

Software and videos supplementing this paper:

<https://thaipduong.github.io/sbkm>

I. INTRODUCTION

Autonomous navigation in robotics involves online localization, mapping, motion planning, and control in partially known environments perceived through streaming data from onboard sensors [1], [2]. This paper focuses on the occupancy mapping problem and, specifically, on enabling large-scale, yet compact, representations and efficient collision checking to support autonomous navigation. Occupancy mapping is a well established and widely studied problem in robotics and a variety of explicit and implicit map representations have been proposed. Explicit maps model the obstacle surfaces

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directly, e.g., via surfels [3]–[7], geometric primitives [8]–[12], or polygonal meshes [13]–[15]. Implicit maps model the obstacle surfaces as the level set of an occupancy [16]–[21] or signed distance [22]–[26] or spatial function encoded via voxels [25]–[27] or octrees [19]–[21]. Many techniques, however, view accurate mapping as a final goal rather than as an integral component of autonomous navigation. A unique goal of this work is to generate *sparse probabilistic* maps online, enabling large-scale environment modeling, map uncertainty quantification, and efficient collision checking.

Our preliminary work [28] develops a kernel perceptron model for online occupancy mapping. The model uses support vectors and a kernel function to represent obstacle boundaries in configuration space. The number of support vectors scales with the complexity of the obstacle boundaries rather than the environment size. We develop an online training algorithm to update the support vectors incrementally as new range observations of the local surroundings are provided by the robot’s sensors. To enable motion planning in the new occupancy representation, we develop efficient collision checking algorithms for piecewise-linear and piecewise-polynomial trajectories in configuration space. Our kernel perceptron model, however, provides occupancy labels without a probability distribution, making the classification accuracy susceptible to measurement noise and localization errors. Since unknown regions are frequently assumed free for motion planning purposes, the lack of probabilistic information also does not allow us to distinguish between well-observed and unseen regions. This is especially important in active exploration problems, where the robot autonomously chooses the unknown regions to explore.

This paper develops a sparse Bayesian formulation of the occupancy mapping problem and introduces an incremental Relevance Vector Machine training algorithm to probabilistically model the environment. To make our sparse Bayesian kernel-based map compatible with motion planning algorithms, we derive collision checking algorithms for linear and general trajectories.

Contributions. This paper introduces a sparse Bayesian kernel-based mapping method that:

- represents continuous-space probabilistic occupancy using a sparse set of relevance vectors stored in an R^* -tree data structure (Sec. V and VIII-A),
- allows online map updates from streaming partial observations using an incremental Relevance Vector Machine training algorithm with the predictive distribution modeled by a probit function. (Sec. V-B), and



Fig. 1: A ground robot in an unknown environment relying on lidar scan data (red) for online occupancy mapping and collision-free trajectory planning. Our mapping algorithm learns a sparse set of occupied and free relevance vectors (light red and green dots, respectively) that represents the environment based on the lidar scans.

- provides efficient and complete (without sampling) collision checking for general robot trajectories with safety guarantees (Sec. VI and VIII-B).

II. RELATED WORK

Occupancy grid mapping is a commonly used approach for modeling the free and occupied space of an environment. The space is discretized into a collection of cells, whose occupancy probabilities are estimated online using the robot’s sensory data. While early work [29], [30] assumes that the cells are independent, Gaussian process (GP) occupancy mapping [31]–[33] uses a kernel function to capture the correlation among grid cells and predict the occupancy of unobserved cells. Online training of a Gaussian process model, however, does not scale well as its computational complexity grows cubically with the number of data points. Ramos et al. [34] improve on this by projecting the data points into Hilbert space and training a logistic regression model. Senanayake and Ramos [35] propose a Bayesian treatment of Hilbert maps, called Sequential Bayesian Hilbert Map (SBHM), that updates the map from sequential observations of the environment. They achieve sparseness by calculating feature vectors based on a sparse set of hinged points, e.g., on a coarse grid. Instead of a fixed set of hinged points, Relevance Vector Machine (RVM) [36]–[38] learns a sparse set of relevance vectors from the training dataset. The original RVM work [36] initially assumes that all data points are relevance vectors and prunes them down, incurring high computation cost. Tipping and Faul [38] derive a fast training algorithm that starts from an empty set of relevance vectors and adds points to the set gradually. Meanwhile, Lopez and How [39] propose an efficient deterministic alternative, which builds a k-d tree from point clouds and queries the nearest obstacles for collision checking. Using spatial partitioning similar to a k-d tree, octree-based maps [19], [40] offer efficient map storage by performing octree compression, while AtomMap [41] stores a collection of spheres in a k-d tree as a way to avoid grid cell discretization of the map. Instead of storing occupancy information, Voxblox [25] stores distance to obstacles in each

cell and builds an Euclidean Signed Distance Field, as a map representation, online from streaming sensor data.

Navigation, in an unknown environment, requires the safety of potential robot trajectories to be evaluated through a huge amount of collision checks with respect to the map representation [42]–[44]. Many works rely on sampling-based collision checking, simplifying the safety verification of continuous-time trajectories by evaluating only a finite set of samples along the trajectory [43], [45]. This may be undesirable in safety critical applications. Bialkowski et al. [42] propose an efficient collision checking method using safety certificates with respect to the nearest obstacles. Using a different perspective, learning-based collision checking methods [46]–[48] sample data from the environment and train machine learning models to approximate the obstacle boundaries. Pan et al. [47] propose an incremental support vector machine model for pairs of obstacles but train the models offline. Closely related to our work, Das et al. [46], [49] develop an online training algorithm, called Fastron, to train a kernel perceptron collision classifier. To handle dynamic environments, Fastron actively resamples the environment and updates the model globally. Geometry-based collision checking methods, such as the Flexible Collision Library (FCL) [50], are also related but rely on mesh representations of the environment which may be inefficient to generate from local observations.

Our preliminary work [28], summarized in Sec. IV, provides an approach to online occupancy mapping that supports efficient collision checking with guarantees. However, to achieve robustness to noisy measurements and localization errors and probabilistically model well-observed and unknown regions, we introduce a probabilistic formulation based on RVM inference that enables online sparse Bayesian kernel-based occupancy mapping. Inspired by GP mapping techniques, we utilize a kernel function to capture occupancy correlations but focus on a compact representation of obstacle boundaries by building an RVM model, i.e. a sparse set of relevance vectors, incrementally from streaming local sensor data. Specifically, only a local subset of the relevance vectors is updated each time using our incremental RVM training algorithm. Furthermore, motivated by the safety certificates in [42], we derive our own safety guarantees for efficient collision checking algorithms. We develop an “inflated boundary” of the obstacle boundary that enables closed-form conditions for safe line segments and safe ellipsoids. These key conditions allow us to check potential robot trajectories for motion planning purposes.

III. PROBLEM FORMULATION

Consider a robot with state $s \in \mathcal{S}$, consisting of the robot’s position $\mathbf{p} \in [0, 1]^d$ and other variables such as orientation, velocity, etc., navigating in an unknown environment (Fig. 1). Let $\mathcal{O} \subset [0, 1]^d$ be a closed set representing occupied space and let \mathcal{F} be its complement, representing free space. Assume that the robot can be enclosed by a sphere of radius $r \in \mathbb{R}_{>0}$ centered at \mathbf{p} . In configuration space (C-space), the robot body becomes a point \mathbf{p} , while the obstacle space and free space are transformed as $\bar{\mathcal{O}} = \cup_{\mathbf{x} \in \mathcal{O}} \mathcal{B}(\mathbf{x}, r)$, where $\mathcal{B}(\mathbf{x}, r) = \{\mathbf{x}' \in$

$[0, 1]^d : \|\mathbf{x} - \mathbf{x}'\|_2 \leq r\}$, and $\bar{\mathcal{F}} = [0, 1]^d \setminus \bar{\mathcal{O}}$. Let $\bar{\mathcal{S}}$ be the subset of the robot state space that corresponds to the collision-free robot positions $\bar{\mathcal{F}}$.

Let $\dot{\mathbf{s}}(t) = f(\mathbf{s}(t), \mathbf{a}(t))$ characterize the continuous-time robot dynamics with control input trajectory $\mathbf{a}(t) \in \mathcal{A}$. We consider constant control inputs (zero-order hold) applied at discrete time steps t_k for $k = 0, 1, \dots, N$ so that $\mathbf{a}(t) \equiv \mathbf{a}_k$ for $[t_k, t_{k+1})$. We assume that the state $\mathbf{s}(t)$ is known or estimated by a localization algorithm and let $\mathbf{s}_k := \mathbf{s}(t_k)$.

The robot is equipped with a sensor, such as lidar or depth camera, that provides distance measurements \mathbf{z}_k at time t_k to the obstacle space \mathcal{O} within its field of view. Our objective is to construct an occupancy map $\hat{m}_k : [0, 1]^d \rightarrow \{-1, 1\}$ of the C-space based on accumulated observations $\mathbf{z}_{0:k}$, where “−1” and “1” mean “free” and “occupied”, respectively. As the robot is navigating, new sensor data are used to update the map as a function, $\hat{m}_{k+1} = g(\hat{m}_k, \mathbf{z}_k)$, of the previous estimate \hat{m}_k and a newly received range observation \mathbf{z}_k . Assuming unobserved regions are free, we rely on \hat{m}_k to plan a robot trajectory to a goal region $\mathcal{G} \subseteq \bar{\mathcal{S}}$. Applying control action \mathbf{a} at \mathbf{s} incurs a motion cost $c(\mathbf{s}, \mathbf{a})$, e.g., based on traveled distance or energy expenditure, and we aim to minimize the cumulative cost of navigating safely to the goal \mathcal{G} .

Problem 1. Given a start state $\mathbf{s}_0 \in \bar{\mathcal{S}}$ and a goal region $\mathcal{G} \subseteq \bar{\mathcal{S}}$, find a sequence of control actions that leads the robot to \mathcal{G} safely, while minimizing the motion cost:

$$\begin{aligned} \min_{N, \mathbf{a}_0, \dots, \mathbf{a}_N} & \sum_{k=0}^{N-1} c(\mathbf{s}_k, \mathbf{a}_k) \\ \text{s.t. } & \dot{\mathbf{s}} = f(\mathbf{s}, \mathbf{a}), \mathbf{a}(t) = \mathbf{a}_k \text{ for } t \in [t_k, t_{k+1}), \\ & \mathbf{s}(t_0) = \mathbf{s}_0, \mathbf{s}_N \in \mathcal{G}, \hat{m}_{k+1} = g(\hat{m}_k, \mathbf{z}_k), \\ & \hat{m}_k(\mathbf{s}(t)) = -1 \text{ for } t \in [t_k, t_{k+1}), k = 0, \dots, N. \end{aligned} \quad (1)$$

In the remainder of the paper, we develop a sparse Bayesian kernel-based map representation, offering efficient collision checking for robot trajectories, and propose a complete solution to Problem 1.

IV. A SPARSE KERNEL-BASED CLASSIFIER FOR OCCUPANCY MAPPING

Our preliminary work [28] develops a sparse kernel perceptron model for online classification of occupied and free space in the environment. The model uses a set of support vectors and a kernel function to represent the obstacle boundaries in configuration space. The number of support vectors necessary for accurate classification scales with the complexity of the obstacle boundaries rather than the environment size. Our approach extends the Fastron algorithm [46], [47], which efficiently trains a kernel perceptron model using a training dataset collected globally from the environment. We develop an online training procedure (Alg. 1) that updates the support vectors incrementally as new range observations \mathbf{z}_k of the local surroundings arrive. Given a training dataset $\mathcal{D} = \{(\mathbf{x}_l, y_l)\}$ generated from \mathbf{z}_k (e.g. see Sec. VIII-A for details), Alg. 1 prioritizes updating misclassified points’ weight based on their margins (lines 6 and 7) and remove the redundant support vectors (line 8) without affecting the model. When the next

Algorithm 1 Incremental Kernel Perceptron Training [28]

Input: Support vectors $\Lambda^+ = \{(\mathbf{x}_i^+, \alpha_i^+)\}_i$ and $\Lambda^- = \{(\mathbf{x}_j^-, \alpha_j^-)\}_j$ stored in an R^* -tree; Local dataset $\mathcal{D} = \{(\mathbf{x}_l, y_l)\}; \xi^+, \xi^- > 0; N_{max}$.
Output: Updated Λ^+, Λ^- .

- 1: Query K^+, K^- nearest negative and positive support vectors from an R^* -tree data structure.
- 2: **for** (\mathbf{x}_l, y_l) in \mathcal{D} **do**
- 3: Calculate $F_l = \sum_{i=1}^{K^+} \alpha_i^+ k(\mathbf{x}_i^+, \mathbf{x}_l) - \sum_{j=1}^{K^-} \alpha_j^- k(\mathbf{x}_j^-, \mathbf{x}_l)$
- 4: **for** $t = 1$ to N_{max} **do**
- 5: **if** $y_l F_l > 0 \ \forall l$ **then return** Λ^+, Λ^-
- 6: $m = \operatorname{argmin}_l y_l F_l$
- 7: WEIGHT CORRECTION($F_m, y_m, \Lambda^+, \Lambda^-, \xi^+, \xi^-$)
- 8: REDUNDANCY REMOVAL($\Lambda^+, \Lambda^-, \mathcal{D}$)
- 9: **return** Λ^+, Λ^-
- 10: **function** WEIGHT CORRECTION($F_m, y_m, \Lambda^+, \Lambda^-, \xi^+, \xi^-$)
- 11: $\xi = \xi^+$ if $y_m > 0$; and $\xi = \xi^-$, otherwise.
- 12: Calculate $\Delta_\alpha = \xi y_m - F_m$.
- 13: **if** $\exists (\mathbf{x}_m, \alpha_m) \in \Lambda^+ \cup \Lambda^-$ **then**
- 14: Update weights: $\alpha_m \leftarrow y_m \Delta_\alpha$, $F_l \leftarrow k(\mathbf{x}_l, \mathbf{x}_m) y_m \Delta_\alpha, \forall l$
- 15: **else**
- 16: Calculate $\alpha_m = y_m \Delta_\alpha$
- 17: Add (\mathbf{x}_m, α_m) to Λ^+ if $y_m > 0$ and Λ^- , otherwise.
- 18: **function** REDUNDANCY REMOVAL($\Lambda^+, \Lambda^-, \mathcal{D}$)
- 19: **for** (\mathbf{x}_l, y_l) in \mathcal{D} **do**
- 20: **if** $\exists (\mathbf{x}_l, \alpha_l) \in \Lambda^+ \cup \Lambda^-$ and $y_l(F_l - \alpha_l^+) > 0$ **then**
- 21: Remove (\mathbf{x}_l, α_l) from Λ^+ or Λ^-
- 22: Update $F_n \leftarrow k(\mathbf{x}_l, \mathbf{x}_n) \alpha_l^+, \forall (\mathbf{x}_n, \cdot) \in \mathcal{D}$

local dataset arrives, it looks for new misclassified points and incrementally adds them to the set of support vectors. Alg. 1 returns a set of M^+ positive support vectors and their weight $\Lambda^+ = \{(\mathbf{x}_i^+, \alpha_i^+)\}_i$ and a set of M^- negative support vectors and their weight $\Lambda^- = \{(\mathbf{x}_j^-, \alpha_j^-)\}_j$. The classifier decision boundary is characterized by a score function:

$$F(\mathbf{x}) = \sum_{i=1}^{M^+} \alpha_i^+ k(\mathbf{x}_i^+, \mathbf{x}) - \sum_{j=1}^{M^-} \alpha_j^- k(\mathbf{x}_j^-, \mathbf{x}), \quad (2)$$

where $k(\cdot, \cdot)$ is a kernel function and $\alpha_j^-, \alpha_i^+ > 0$. The occupancy of a query point \mathbf{x} can be checked by evaluating the score function $F(\mathbf{x})$ in Eq. (2). Specifically, $\hat{m}_t(\mathbf{x}) = -1$ if $F(\mathbf{x}) < 0$ and $\hat{m}_t(\mathbf{x}) = 1$ if $F(\mathbf{x}) \geq 0$. The score calculation becomes slower when the number of support vectors increases. We improve on this by storing the support vectors in an R^* -tree data structure and efficiently query K^+ and K^- nearest positive and negative support vectors (line 1 in Alg. 1) from the R^* -tree to approximate $F(\mathbf{x})$.

Motivated by the use of piecewise-linear and piecewise-polynomial trajectories in many robot motion planning and control algorithms [51]–[53], we derive conditions to classify lines and curves, i.e., to check if every point on the curve is free using the trained model. Checking that a curve $\mathbf{p}(t)$ is classified as free is equivalent to verifying that $F(\mathbf{p}(t)) < 0$, $\forall t \geq 0$. It is not possible to express this condition for t explicitly due to the nonlinearity of F . In Prop. 1, we show that an accurate upper bound $\bar{F}(\mathbf{p}(t))$ on the score $F(\mathbf{p}(t))$ exists and can be used to evaluate the condition $\bar{F}(\mathbf{p}(t)) < 0$ explicitly in t . The upper bound provides a conservative but fairly accurate “inflated boundary” and allows efficient classifications of curves $\mathbf{p}(t)$, assuming a radial basis function kernel $k(\mathbf{x}, \mathbf{x}') = \eta \exp(-\gamma \|\mathbf{x} - \mathbf{x}'\|^2)$ is used.

Proposition 1 ([28]). *For any $(\mathbf{x}_j^-, \alpha_j^-) \in \Lambda^-$, the score $F(\mathbf{x})$ is bounded above by $\bar{F}(\mathbf{x}) = k(\mathbf{x}, \mathbf{x}_*) \sum_{i=1}^{M^+} \alpha_i^+ - k(\mathbf{x}, \mathbf{x}_j^-) \alpha_j^-$ where \mathbf{x}_*^+ is the closest positive support vector to \mathbf{x} .*

To check if a line $\mathbf{p}(t)$ collides with the inflated boundary, we find the first time t_u such that $\bar{F}(\mathbf{p}(t_u)) \geq 0$. This means that $\mathbf{p}(t)$ is classified as free for $t \in [0, t_u]$.

Proposition 2 ([28]). *Consider a ray $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$, $t \geq 0$ such that \mathbf{p}_0 is classified as free, i.e., $\bar{F}(\mathbf{p}_0) < 0$, and \mathbf{v} is constant. Let \mathbf{x}_i^+ and \mathbf{x}_j^- be arbitrary positive and negative support vectors. Then, any point $\mathbf{p}(t)$ with $t \in [0, t_u] \subseteq [0, t_u^*]$ is free for*

$$t_u := \min_{i \in \{1, \dots, M^+\}} \rho(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-), \quad (3)$$

$$t_u^* := \min_{i \in \{1, \dots, M^+\}} \max_{j \in \{1, \dots, M^-\}} \rho(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-), \quad (4)$$

where $\beta = \frac{1}{\gamma} (\log(\alpha_j^-) - \log(\sum_{i=1}^{M^+} \alpha_i^+))$ and

$$\rho(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-) = \begin{cases} +\infty, & \text{if } \mathbf{v}^T(\mathbf{x}_i^+ - \mathbf{x}_j^-) \leq 0 \\ \frac{\beta - \|\mathbf{p}_0 - \mathbf{x}_j^-\|^2 - \|\mathbf{p}_0 + \mathbf{x}_i^+\|^2}{2\mathbf{v}^T(\mathbf{x}_j^- - \mathbf{x}_i^+)}, & \text{if } \mathbf{v}^T(\mathbf{x}_i^+ - \mathbf{x}_j^-) > 0 \end{cases}.$$

For a line segment $(\mathbf{p}_A, \mathbf{p}_B)$, all points on the segment can be expressed as $\mathbf{p}(t_A) = \mathbf{p}_A + t_A \mathbf{v}_A$, $\mathbf{v}_A = \mathbf{p}_B - \mathbf{p}_A$, $0 \leq t_A \leq 1$ or $\mathbf{p}(t_B) = \mathbf{p}_B + t_B \mathbf{v}_B$, $\mathbf{v}_B = \mathbf{p}_A - \mathbf{p}_B$, $0 \leq t_B \leq 1$. Using the upper bound provided by Eq. (3) or Eq. (4), we find the free regions $[0, t_{uA}]$ and $[0, t_{uB}]$ starting from \mathbf{p}_A and \mathbf{p}_B , respectively. If the free regions overlap, the segment is classified as free and vice versa.

We extend line segment classification to general curves by finding a Euclidean ball $\mathcal{B}(\mathbf{p}_0, r)$ around \mathbf{p}_0 whose interior is free of obstacles.

Corollary 1 ([28]). *Let $\mathbf{p}_0 \in \mathcal{C}$ be such that $\bar{F}(\mathbf{p}_0) < 0$ and let \mathbf{x}_i^+ and \mathbf{x}_j^- be arbitrary positive and negative support vectors. Then, every point inside the Euclidean balls $\mathcal{B}(\mathbf{p}_0, r_u) \subseteq \mathcal{B}(\mathbf{p}_0, r_u^*)$ is free for:*

$$r_u := \min_{i \in \{1, \dots, M^+\}} \bar{\rho}(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-), \quad (5)$$

$$r_u^* := \min_{i \in \{1, \dots, M^+\}} \max_{j \in \{1, \dots, M^-\}} \bar{\rho}(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-) \quad (6)$$

where $\bar{\rho}(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-) = \frac{\beta - \|\mathbf{p}_0 - \mathbf{x}_j^-\|^2 + \|\mathbf{p}_0 + \mathbf{x}_i^+\|^2}{2\|\mathbf{x}_j^- - \mathbf{x}_i^+\|}$ and $\beta = \frac{1}{\gamma} (\log(\alpha_j^-) - \log(\sum_{i=1}^{M^+} \alpha_i^+))$.

Consider a polynomial $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{a}_1 t + \mathbf{a}_2 t^2 + \dots + \mathbf{a}_d t^d$, $t \in [0, t_f]$ from \mathbf{p}_0 to $\mathbf{p}_f := \mathbf{p}(t_f)$. Corollary 1 shows that all points inside $\mathcal{B}(\mathbf{p}_0, r)$ are free for $r = r_u$ or r_u^* . If we can find the smallest positive t_1 such that $\|\mathbf{p}(t_1) - \mathbf{p}_0\| = r$, then all points on the curve $\mathbf{p}(t)$ for $t \in [0, t_1]$ are free. We classify the polynomial curve by iteratively covering it by Euclidean balls. If any ball's radius is smaller than a threshold ε , the curve is considered colliding. Otherwise, it is considered free.

The sparse kernel-based model [28] is accurate, updates efficiently from streaming range data, and evaluates curves $\mathbf{p}(t)$ for collisions without sampling. However, the model does not provide occupancy probability, which is desirable in autonomous navigation applications for distinguishing between unknown and well-observed free regions and for identifying

map areas with large uncertainty. This observation motivates us to develop a sparse probabilistic model for online occupancy classification and efficient collision checking.

V. ONLINE PROBIT RVM TRAINING

In this section, we develop an online probit relevance vector machine (RVM) training algorithm that builds a sparse probabilistic model for online occupancy mapping from streaming range observations.

A. Relevance Vector Machine Preliminaries

A relevance vector machine [38] is a sparse Bayesian approach for classification. Given a training dataset of N binary-labeled samples $\mathcal{D} = (\mathbf{X}, \mathbf{y}) = \{(\mathbf{x}_l, y_l)\}_l$, where $y_l \in \{-1, 1\}$, an RVM model maintains a sparse set of relevance vectors \mathbf{x}_m for $m = 1, \dots, M$. The relevance vectors map a point \mathbf{x} to a feature vector $\Phi_{\mathbf{x}} = [k_1(\mathbf{x}), k_2(\mathbf{x}), \dots, k_M(\mathbf{x})]^T \in \mathbb{R}^M$ via a kernel function $k_m(\mathbf{x}) := k(\mathbf{x}, \mathbf{x}_m)$. The likelihood of label y at point \mathbf{x} is modeled by squashing a linear feature function:

$$F(\mathbf{x}) := \Phi_{\mathbf{x}}^T \mathbf{w} + b, \quad (7)$$

with weights $\mathbf{w} \in \mathbb{R}^M$ and bias $b \in \mathbb{R}$ through a function $\sigma : \mathbb{R} \mapsto [0, 1]$:

$$\mathbb{P}(y = 1 | \mathbf{x}, \mathbf{w}) = \sigma(F(\mathbf{x})), \mathbb{P}(y = -1 | \mathbf{x}, \mathbf{w}) = 1 - \sigma(F(\mathbf{x})).$$

Note that Eq. (2) is a special case of (7) with $b = 0$. Examples of σ are the logistic function $\sigma(f) := \frac{1}{1 + \exp(-f)}$ and the probit function $\sigma(f) := \int_{-\infty}^f \varphi(z) dz$, where $\varphi(z) := \frac{1}{\sqrt{2\pi}} \exp(-z^2/2)$ is the standard normal probability density. The data likelihood of the whole training set is:

$$p(\mathbf{y} | \mathbf{X}, \mathbf{w}) = \prod_{l=1}^N \sigma(F(\mathbf{x}_l))^{\frac{1+y_l}{2}} (1 - \sigma(F(\mathbf{x}_l)))^{\frac{1-y_l}{2}}. \quad (8)$$

An RVM model imposes a Gaussian prior on each weight w_m with zero mean and precision ξ_m (i.e., variance $1/\xi_m$):

$$p(\mathbf{w} | \boldsymbol{\xi}) = (2\pi)^{M/2} \prod_{m=1}^M \xi_m^{1/2} \exp\left(-\frac{\xi_m w_m^2}{2}\right). \quad (9)$$

The weight posterior is obtained via Bayes' rule:

$$p(\mathbf{w} | \mathbf{y}, \mathbf{X}, \boldsymbol{\xi}) = \frac{p(\mathbf{y} | \mathbf{X}, \mathbf{w}) p(\mathbf{w} | \boldsymbol{\xi})}{p(\mathbf{y} | \mathbf{X}, \boldsymbol{\xi})}. \quad (10)$$

The precision $\boldsymbol{\xi}$ is determined via type-II maximum likelihood estimation, i.e., by maximizing the marginal likelihood:

$$\mathcal{L}(\boldsymbol{\xi}) = \log p(\mathbf{y} | \mathbf{X}, \boldsymbol{\xi}) = \log \int p(\mathbf{y} | \mathbf{X}, \mathbf{w}) p(\mathbf{w} | \boldsymbol{\xi}) d\mathbf{w}. \quad (11)$$

Given a maximizer $\boldsymbol{\xi}$, the posterior $p(\mathbf{w} | \mathbf{y}, \mathbf{X}, \boldsymbol{\xi})$ is generally intractable and approximated by a Gaussian distribution $p(\mathbf{w} | \mathbf{y}, \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\Sigma})$ with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$ using Laplace approximation [54]. Training consists in determining $\boldsymbol{\xi}, \boldsymbol{\mu}, \boldsymbol{\Sigma}$.

At test time, due to the Laplace approximation, the predictive distribution of a query point \mathbf{x} becomes:

$$p(y | \mathbf{x}, \boldsymbol{\xi}) \approx \int p(y | \mathbf{x}, \mathbf{w}) p(\mathbf{w} | \mathbf{y}, \mathbf{X}, \boldsymbol{\mu}, \boldsymbol{\Sigma}) d\mathbf{w}. \quad (12)$$

The usual formulation of RVM [38] uses a logistic function for σ , requiring additional approximations to the integral in (12). We emphasize that using a probit function, instead, enables a closed-form for the predictive distribution:

$$\begin{aligned} p(y|\mathbf{x}, \xi) &\approx \int \sigma(y(\Phi_{\mathbf{x}}^T \mathbf{w} + b)) p(\mathbf{w}|y, \mathbf{X}, \mu, \Sigma) d\mathbf{w} \\ &= \sigma\left(\frac{y(\Phi_{\mathbf{x}}^T \mu + b)}{\sqrt{1 + \Phi_{\mathbf{x}}^T \Sigma \Phi_{\mathbf{x}}}}\right). \end{aligned} \quad (13)$$

This expression enables our results on closed-form classification of curves in Sec. VI.

We review the details of RVM training and then propose an online training algorithm that handles streaming training data.

1) *Laplace approximation*: Approximation of the weight posterior $p(\mathbf{w}|y, \mathbf{X}, \xi)$ is performed by fitting a Gaussian density function around its mode μ , the maximizer of

$$L(\mathbf{w}) := \log(p(y|\mathbf{X}, \mathbf{w})p(\mathbf{w}|\xi)). \quad (14)$$

Substituting (8) and (9) in (14), we can obtain the gradient and Hessian of $L(\mathbf{w})$ for the probit function σ :

$$\nabla L(\mathbf{w}) = \Phi^T \delta - \mathbf{A}\mathbf{w}, \quad \nabla^2 L(\mathbf{w}) = -\Phi^T \mathbf{B} \Phi - \mathbf{A}, \quad (15)$$

where $\Phi \in \mathbb{R}^{N \times M}$ is the feature matrix with entries $\Phi_{i,j} := k_j(\mathbf{x}_i)$, $\delta \in \mathbb{R}^N$ is a vector with entries $\delta_l := \frac{\varphi(y_l F(\mathbf{x}_l))}{\sigma(y_l F(\mathbf{x}_l))} y_l$, $\mathbf{A} := \text{diag}(\xi) \in \mathbb{R}^{M \times M}$, $\mathbf{B} := \text{diag}(\mathbf{D}\Phi^T \mathbf{w} + \mathbf{b}\delta + \mathbf{D}\delta) \in \mathbb{R}^{N \times N}$, and $\mathbf{D} := \text{diag}(\delta) \in \mathbb{R}^{N \times N}$. The Hessian is negative semi-definite and, hence, $L(\mathbf{w})$ is concave. Setting $L(\mathbf{w}) = 0$, we obtain a Gaussian approximation $p(\mathbf{w}|y, \mathbf{X}, \mu, \Sigma)$ with:

$$\Sigma = (\Phi^T \mathbf{B} \Phi + \mathbf{A})^{-1}, \quad (16)$$

$$\mu = \Sigma \Phi^T \mathbf{B} (\Phi \mu + \mathbf{B}^{-1} \delta), \quad (17)$$

where μ is defined implicitly and is obtained via first- or second-order ascent in practice [55].

2) *Sequential RVM training*: To determine the precision ξ of the weight prior in (9), Tipping and Faul[38] proposed a sequential training algorithm that starts from an empty set of relevance vectors, i.e., $\xi_l = \infty$, and incrementally introduces new vectors to maximize the marginal likelihood in (11):

$$\mathcal{L}(\xi) \approx -\frac{1}{2} \left(N \log 2\pi + \log \det \mathbf{C} + \hat{\mathbf{t}}^T \mathbf{C}^{-1} \hat{\mathbf{t}} \right) \quad (18)$$

where $\hat{\mathbf{t}} := \Phi \mu + \mathbf{B}^{-1} \delta$ and $\mathbf{C} := \mathbf{B} + \Phi \mathbf{A}^{-1} \Phi^T$. For each (\mathbf{x}_l, y_l) in the training set \mathcal{D} , define $\theta_l = q_l^2 - s_l$ as follows:

$$s_l := \begin{cases} \frac{\xi_l S_l}{\xi_l - S_l}, & \text{if } \xi_l < \infty \\ S_l, & \text{else} \end{cases} \quad q_l := \begin{cases} \frac{\xi_l Q_l}{\xi_l - S_l}, & \text{if } \xi_l < \infty \\ Q_l, & \text{else} \end{cases} \quad (19)$$

where $S_l = \Phi_l^T \mathbf{C}^{-1} \Phi_l$, $Q_l = \Phi_l^T \mathbf{C}^{-1} \hat{\mathbf{t}}$, and Φ_l is the l -th row of Φ . If $\theta_l > 0$, the point \mathbf{x}_l is updated (if $\xi_l < \infty$) or added (if $\xi_l = \infty$) as a relevance vector with $\xi_l = \frac{s_l^2}{q_l^2 - s_l}$. If $\theta_l \leq 0$ and $\xi_l < \infty$, the point \mathbf{x}_l is removed from the RVM model. These steps are shown in lines 8-12 of Alg. 2.

Algorithm 2 Online Probit RVM Training.

```

Input: Relevance vectors  $\Lambda_k = \{(\mathbf{x}_i^{(k)}, y_i^{(k)}, \xi_i^{(k)})\}$ ; training set  $\mathcal{D}_{k+1} = \{(\mathbf{x}_l, y_l)\}_l$ ; number of nearest relevance vectors to use  $K$  (optional)
Output: Relevance vectors  $\Lambda_{k+1} = \{(\mathbf{x}_i^{(k+1)}, y_i^{(k+1)}, \xi_i^{(k+1)})\}$ ; weight posterior mean  $\mu$  and covariance  $\Sigma$ 

1: Initialize  $\Lambda_{k+1} = \Lambda_k$ .
2: if  $K$  is defined then  $\Lambda_{local} = K$  nearest relevance vectors from  $\Lambda_k$ 
3: else  $\Lambda_{local} = \Lambda_k$ .
4:  $\Phi = \text{FEATUREMATRIX}(\Lambda_{local}, \mathcal{D}_{k+1})$ 
5:  $\xi_l = \infty$  for each  $(\mathbf{x}_l, y_l)$  in  $\mathcal{D}_{k+1}$ 
6:  $\Sigma, \mu = \text{LAPLACEAPPROXIMATION}(\Lambda_{local}, \mathcal{D}_{k+1})$ .
7: while not converged and max number iterations not reached do
8:   Pick a candidate  $(\mathbf{x}_m, y_m)$  from  $\mathcal{D}_{k+1}$ .
9:   Calculate  $S_m, Q_m, s_m, q_m, \theta_m$ .
10:  If  $\theta_m > 0$  and  $\xi_m = \infty$ , add  $(\mathbf{x}_m, y_m, \xi_m)$  to  $\Lambda_{local}$ .
11:  If  $\theta_m \leq 0$  and  $\xi_m < \infty$ , remove  $(\mathbf{x}_m, y_m, \xi_m)$  from  $\Lambda_{local}$ .
12:  If  $\theta_m > 0$  and  $\xi_m < \infty$ , re-estimate  $\xi_m = \frac{s_m^2}{q_m^2 - s_m}$  in  $\Lambda_{local}$ .
13:   $\Sigma, \mu = \text{LAPLACEAPPROXIMATION}(\Lambda_{local}, \mathcal{D}_{k+1})$ .
14:  $\Lambda_{k+1} = \Lambda_{k+1} \cup \Lambda_{local}$ .
15:  $\Sigma, \mu = \text{GLOBALPOSTERIORAPPROXIMATION}(\Lambda_{k+1})$ 
16: return  $\Lambda_{k+1}, \Sigma, \mu$ 
17:
18: function  $\text{FEATUREMATRIX}(\Lambda, \mathcal{D})$ 
19:   Calculate  $\Phi_{i,j} = k(\mathbf{x}_i, \mathbf{x}_j)$  for all  $\mathbf{x}_j \in \Lambda$  and all  $\mathbf{x}_i \in \mathcal{D}$ 
20:   return  $\Phi$ 
21: function  $\text{LAPLACEAPPROXIMATION}(\Lambda, \mathcal{D})$ 
22:   Calculate  $\Sigma, \mu$  for relevance vectors  $\Lambda$  using  $\mathcal{D}$  (Eq. (16) and (17)).
23:   return  $\Sigma, \mu$ .
24: function  $\text{GLOBALPOSTERIORAPPROXIMATION}(\Lambda)$ 
25:   return  $\text{LAPLACEAPPROXIMATION}(\Lambda, \Lambda)$ .

```

B. Online RVM Training using Streaming Data

Existing techniques for RVM training assume that all data is available a priori. In this section, we develop an online RVM training algorithm that updates the set of relevance vectors $\Lambda_k = \{\mathbf{x}_i^{(k)}, y_i^{(k)}, \xi_i^{(k)}\}_i$ incrementally using streaming data. Suppose that Λ_k has been obtained based on prior data $\mathcal{D}_0, \dots, \mathcal{D}_k$. At time $k+1$, a new training set \mathcal{D}_{k+1} is received. The training set generation depends on the application. We construct \mathcal{D}_{k+1} using a lidar scan \mathbf{z}_{k+1} of an unknown environment as detailed in Sec. VIII-A. New relevance vectors are added to Λ_k to correctly classify the latest training set \mathcal{D}_{k+1} without affecting the accuracy of the classification on the prior data and maintaining the sparsity of the model.

Alg. 2 presents our online probit RVM training approach. The algorithm starts with the existing set of relevance vectors Λ_k and adds new relevance vectors based on the samples in \mathcal{D}_{k+1} using the sequential training approach in Sec. V-A. Instead of using the feature matrix Φ (line 4) associated with all prior relevance vectors, we use a feature matrix approximation based on a local set Λ_{local} of K nearest relevance vectors (line 2). Sec. VII provides a discussion on the computational improvements and assumptions of the score function approximation resulting from using Λ_{local} instead of Λ_k . For test time classification, we compute the mean μ and covariance Σ of the Laplace approximation to the weight posterior according to Eq. (17) and (16). Laplace approximation requires all data $\mathcal{D} = \bigcup_{i=1}^{k+1} \mathcal{D}_i$, used for training up to time $k+1$ but only the local dataset \mathcal{D}_{k+1} is available. Interestingly, the set Λ_{k+1} of relevance vectors itself globally and sparsely represents all the data used for training and, therefore, can be used for Laplace approximation (line 15).

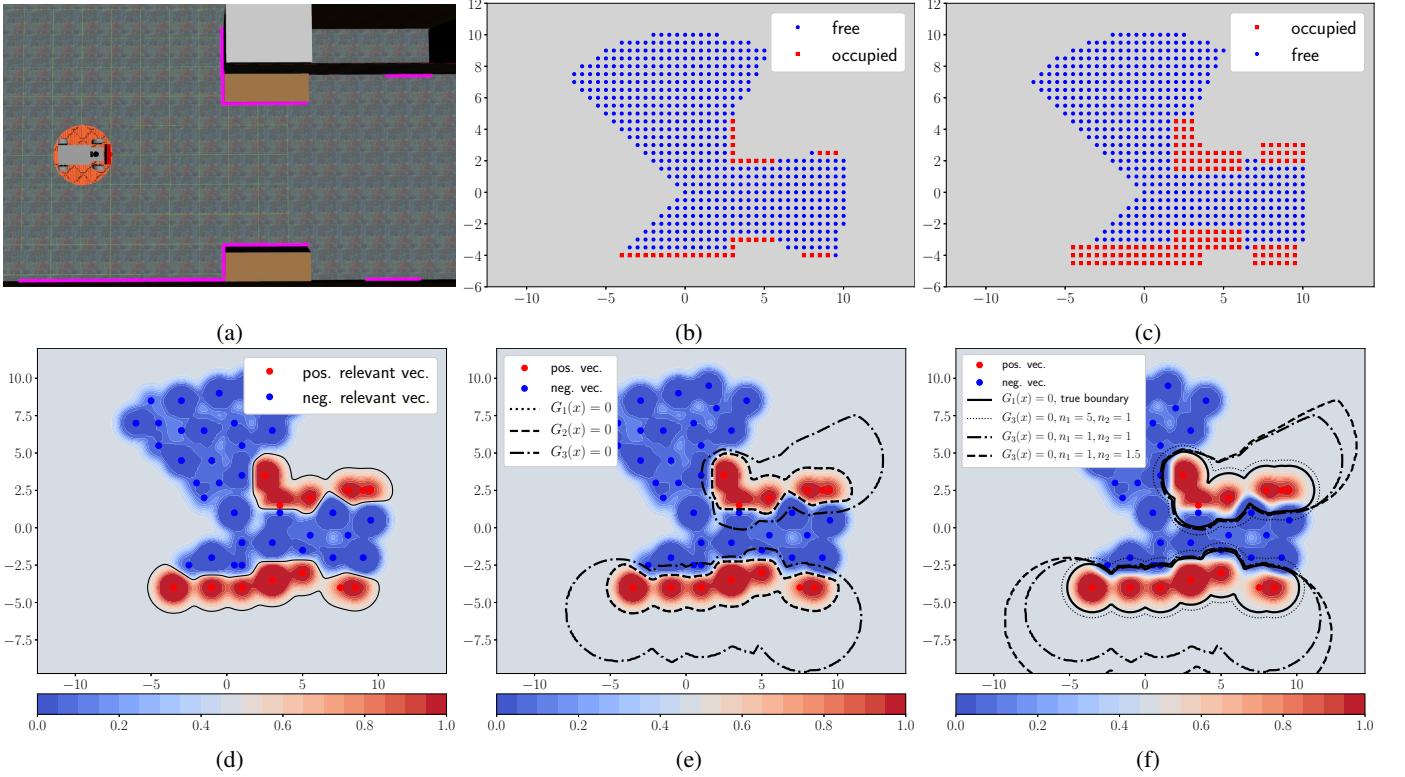


Fig. 2: Example of our mapping method: (a) a robot observing the environment via a laser scan (magenta); (b) work-space samples generated from the laser scan \mathbf{z}_k ; (c) configuration-space samples used as a training set at time k ; (d) exact decision boundary with bias $b = -0.05$ and classification threshold $e = -0.01$, $\bar{e} = 0.494$; (e) inflated boundaries (c.f. Sec. VI) generated by $G_1(\mathbf{x})$, $G_2(\mathbf{x})$, $G_3(\mathbf{x})$ with $n_1 = n_2 = 1$; (f) inflated boundary $G_3(\mathbf{x}) = 0$ with various n_1, n_2 .

If additional computation for Laplace approximation is not feasible, one might directly store the weight mean μ and covariance Σ (line 15) over time. The memory requirements for either case are discussed in Sec. VII.

Fig. 2a depicts a ground robot equipped with a lidar scanner whose goal is to build an occupancy map of the environment. Fig. 2c plots the training set \mathcal{D}_{k+1} generated from the lidar scan \mathbf{z}_{k+1} , assuming the current set of relevance vectors Λ_k is empty. Fig. 2d shows the trained RVM model as a sparse set of relevance vectors, serving as a sparse probabilistic occupancy map of the environment, incrementally updated via the streaming lidar scans. A map representation is useful for autonomous navigation (Problem 1) only if it allows checking potential robot trajectories $\mathbf{s}(t)$ for collisions. We propose classification methods for points, line segments, and general curves next.

VI. RVM CLASSIFICATION OF POINTS, LINES, AND CURVES

This section discusses classification using the predictive distribution in Eq. (13) and makes a connection with our preliminaries results in [28]. Commonly, machine learning models are only able to classify point queries but applications, such as robot trajectory planning, may require classification of general curves. This can be done by successively checking a dense set of points, sampled along the curve. However, we show that under certain assumptions on the kernel function and

the decision threshold, line and general curve classification based on the RVM decision boundary can be performed directly and efficiently, based on the closed-form of the predictive distribution in Eq. (13), without the need to sample.

A. RVM Classification of Points

Consider a set Λ of M relevance vectors with prior weight precision ξ and mean μ and covariance Σ of the approximate weight posterior $p(\mathbf{w}|\mathbf{y}, \mathbf{X}, \mu, \Sigma)$. To classify a query point \mathbf{x} using the RVM model, we place a threshold \bar{e} on the probability $\mathbb{P}(y = 1|\mathbf{x}, \xi)$ (Def. 1). Fig. 2d illustrates the decision boundary defined by Def. 1 with $\bar{e} = \sigma(e) = 0.494$, i.e., $e = -0.01$.

Definition 1. Let $\bar{e} \in [0, 1]$ and $e := \sigma^{-1}(\bar{e})$. A point \mathbf{x} is classified as “-1” if

$$\mathbb{P}(y = 1|\mathbf{x}, \xi) = \sigma\left(\frac{\Phi_{\mathbf{x}}^T \mu + b}{\sqrt{1 + \Phi_{\mathbf{x}}^T \Sigma \Phi_{\mathbf{x}}}}\right) \leq \bar{e}, \quad (20)$$

or, equivalently, if

$$G_1(\mathbf{x}) := \Phi_{\mathbf{x}}^T \mu + b - e \sqrt{1 + \Phi_{\mathbf{x}}^T \Sigma \Phi_{\mathbf{x}}} \leq 0. \quad (21)$$

The condition in Eq. (21) can be verified for a given point but it is challenging to obtain an explicit expression in terms of \mathbf{x} . If, instead of a point \mathbf{x} , we consider a time-parameterized curve $\mathbf{p}(t)$, then Eq. (21) becomes a nonlinear programming feasibility problem in t . To avoid nonlinear programming, we

develop a series of upper bounds for $G_1(\mathbf{x})$ that make the condition for classifying a point as free (i.e., $y = -1$) more conservative but with a simpler dependence on \mathbf{x} .

Proposition 3. *For a non-negative kernel function $k_m(\mathbf{x}) := k(\mathbf{x}, \mathbf{x}_m)$, a point \mathbf{x} is classified as “-1” if*

$$G_2(\mathbf{x}) := \sum_{m=1}^M (\mu_m - e \mathbb{1}_{\{e>0\}} \sqrt{\lambda_{\max}}) k_m(\mathbf{x}) + b - e \leq 0, \quad (22)$$

where $\lambda_{\max} \geq 0$ is the largest eigenvalue of the covariance Σ , μ_m is the m th element of the mean μ , and $\mathbb{1}_{\{e>0\}}$ is an indicator function which equals 1 if $e > 0$ and 0, otherwise.

Proof. Please refer to Appendix A. \square

The relaxed condition in Eq. (22) adjusts the weights of the relevance vectors by an amount of $\delta\mu = -e \mathbb{1}_{\{e>0\}} \lambda_{\max} \geq 0$. Intuitively, this increases the effect of the positive relevance vectors, leading to a more conservative condition than Def. 1. Prop. 3 also allows us to use only the largest eigenvalue λ_{\max} of Σ for point classification, which is easier to obtain and store than the whole covariance matrix Σ . Methods for computing λ_{\max} are discussed in Sec. VII-A.

To simplify the notation, let $\nu_m := \mu_m - e \mathbb{1}_{\{e>0\}} \lambda_{\max}$ be the corrected relevance vector weights and split Λ into M^+ positive relevance vectors $\Lambda^+ = \{(\mathbf{x}_m^+, \nu_m^+)\}$ and M^- negative relevance vectors $\Lambda^- = \{(\mathbf{x}_m^-, \nu_m^-)\}$, where $\nu_m^+ = \nu_m$ if $\nu_m > 0$ and $\nu_m^- = -\nu_m$ if $\nu_m < 0$. Now, Eq. (22) can be re-written as:

$$G_2(\mathbf{x}) = \sum_{i=1}^{M^+} \nu_i^+ k(\mathbf{x}, \mathbf{x}_i^+) - \sum_{j=1}^{M^-} \nu_j^- k(\mathbf{x}, \mathbf{x}_j^-) + b - e \leq 0. \quad (23)$$

Hence, Prop. 3 allows us to make an important connection between sparse kernel classification with a ‘hard’ decision threshold (Sec. IV) and its Bayesian counterpart (Sec. V-B). Specifically, after the relevance vector weight correction, Eq. (22) is equivalent to the kernel perceptron score in Eq. (2) except for the bias term $b - e$.

1) *The role of the bias term:* One of the motivations for developing a Bayesian map representation is to distinguish between observed and unobserved regions in the environment. Intuitively, as a query point \mathbf{x} is chosen further away from “observed” regions, where training data has been obtained, its correlation with existing relevance vectors, measured by $k(\mathbf{x}, \mathbf{x}_m)$, decreases. To capture and exploit this property, we assume that the kernel has a common radial basis function structure that depends only on a quadratic norm $\|\Gamma(\mathbf{x} - \mathbf{x}_m)\|$.

Assumption 1. Let $k(\mathbf{x}, \mathbf{x}_m) := \eta \exp(-\|\Gamma(\mathbf{x} - \mathbf{x}_m)\|^2)$ with parameters $\eta > 0$ and $\Gamma \in \mathbb{R}^{d \times d}$.

In our application, the kernel parameters η and Γ may be optimized offline via automatic relevance determination [56] using training data from known occupancy maps. Under this assumption, the feature vector Φ_x tends to 0 as \mathbf{x} goes towards unobserved regions and the occupancy probability $\mathbb{P}(y = 1 | \mathbf{x}, \xi)$ tends to $\sigma(b)$ in Eq. (20). Therefore, the value of $\sigma(b)$ represents the occupancy probability of points in the unknown regions. In other words, $\sigma(b)$ specifies how much

we trust that unknown regions are occupied and should be a constant. For this reason, the bias b is fixed in our online RVM training algorithm. A common assumption in motion planning [57] is to treat unknown regions as free in order to allow trajectory planning to goals in the unknown space. In the context of this paper, this means that the occupancy probability of points in unknown regions, $\sigma(b)$, should be lower than or equal to the decision threshold $\bar{e} = \sigma(e)$ in Def. 1.

Assumption 2. Assume that $e \geq b$ and, hence, $\bar{e} \geq \sigma(b)$.

2) *RVM Classification with $e = b$:* A natural choice for the occupancy probability of unknown regions, $\sigma(b)$, is to set it exactly equal to the decision threshold between free and occupied space, i.e., $e = b$. In this case, Eq. (23) becomes exactly equivalent to Eq. (2) and all results in Sec. IV for classification of points, lines, and curves can be reused.

Corollary 2. *If the bias b in Eq. (7) is used as the decision threshold for classification in Def. 1, i.e., $e = b$, then, according to Prop. 3 and Eq. (23), \mathbf{x} is classified as “-1” if:*

$$\sum_{i=1}^{M^+} \nu_i^+ k(\mathbf{x}, \mathbf{x}_i^+) - \sum_{j=1}^{M^-} \nu_j^- k(\mathbf{x}, \mathbf{x}_j^-) \leq 0. \quad (24)$$

Hence, Prop. 2 and Corollary 1 hold for line and curve classification using a Relevance Vector Machine model.

3) *RVM Classification with $e \geq b$:* For a general decision threshold, $e \geq b$, and a kernel function $k_m(\mathbf{x})$ satisfying Assumption 1, we develop an explicit condition for classifying a point \mathbf{x} as free.

Proposition 4. *For integers $n_1, n_2 \geq 1$, define $\rho(a, b) := (n_1 + n_2) \left(\frac{a}{n_1} \right)^{\frac{n_1}{n_1+n_2}} \left(\frac{b}{n_2} \right)^{\frac{n_2}{n_1+n_2}}$. A point \mathbf{x} is classified as “-1” if*

$$G_3(\mathbf{x}) := \left(\sum_{i=1}^{M^+} \nu_i^+ \right) k(\mathbf{x}, \mathbf{x}_*^+) - \rho(e - b, \nu_j^- k(\mathbf{x}, \mathbf{x}_j^-)) \leq 0, \quad (25)$$

where \mathbf{x}_*^+ is the closest positive relevance vector to \mathbf{x} and \mathbf{x}_j^- is any negative relevance vector.

Proof. Please refer to Appendix B. \square

Fig. 2e illustrates the exact RVM decision boundary from Eq. (21), $G_1(\mathbf{x}) = 0$, and the boundaries $G_2(\mathbf{x}) = 0$ and $G_3(\mathbf{x}) = 0$ resulting from the upper bounds in Prop. 3 and Prop. 4. Note that the boundary generated by $G_2(\mathbf{x})$ is very close to the true boundary from $G_1(\mathbf{x})$. The upper bound $G_3(\mathbf{x})$ provides a conservative “inflated boundary”, whose accuracy can be controlled via the integers n_1, n_2 in Prop. 4. Note that $G_3(\mathbf{x})$ is inaccurate mainly in the unknown regions because the Arithmetic Mean-Geometric Mean inequality used in Prop. 4’s proof (Appendix B) effectively replaces the kernel function $k(\mathbf{x}, \mathbf{x}_j^-)$ by a slower decaying one $k(\mathbf{x}, \mathbf{x}_j^-)^{\frac{n_2}{n_1+n_2}}$. This suits the intuition that unknown regions should be categorized as free more cautiously. Fig. 2f shows that increasing the ratio n_2/n_1 makes the “inflated boundary” closer to the true decision boundary in the unknown regions but slightly looser in the well-observed regions and vice versa. Next, based on

Algorithm 3 RVM Line Classification

Input: Line segment $(\mathbf{p}_A, \mathbf{p}_B)$; relevance vectors $\Lambda = \{(\mathbf{x}_i, y_i, \xi_i)\}$; weight posterior mean μ and max covariance eigenvalue λ_{max}

- 1: $\mathbf{v}_A = \mathbf{p}_B - \mathbf{p}_A$, $\mathbf{v}_B = \mathbf{p}_A - \mathbf{p}_B$
- 2: Calculate t_{uA} and t_{uB} using Eq. (26) or Eq. (27).
- 3: **if** $t_{uA} + t_{uB} > 1$ **then return** True (Free).
- 4: **else return** False (Colliding)

Prop. 4, we develop conditions for classification of lines and curves when $e \geq b$ without the need for sampling.

B. RVM Classification of Lines

Consider a linear trajectory described by a ray $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$, $t \geq 0$ such that \mathbf{p}_0 is obstacle-free according to Prop. 4, i.e., $G_3(\mathbf{p}_0) \leq 0$, and \mathbf{v} is a constant. To check if $\mathbf{p}(t)$ collides with the inflated boundary $G_3(\mathbf{x}) = 0$, we find a time t_u such that any point $\mathbf{p}(t)$ is classified free for $t \in [0, t_u]$.

Proposition 5. Consider a ray $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$, $t \geq 0$. Let \mathbf{x}_i^+ and \mathbf{x}_j^- be arbitrary positive and negative relevance vectors. Then, any point $\mathbf{p}(t)$ with $t \in [0, t_u] \subseteq [0, t_u^*]$ is free for:

$$t_u := \min_{i=1,\dots,M^+} \tau(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-) \quad (26)$$

$$t_u^* := \min_{i=1,\dots,M^+} \max_{j=1,\dots,M^-} \tau(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-), \quad (27)$$

where $\tau(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-)$

$$= \begin{cases} +\infty, & \text{if } V(t, \mathbf{x}_i^+, \mathbf{x}_j^-) \text{ has less than 2 roots} \\ +\infty, & \text{if } V(t, \mathbf{x}_i^+, \mathbf{x}_j^-) \text{ has 2 roots } t_1 < t_2 \leq 0 \\ t_1 & \text{if } V(t, \mathbf{x}_i^+, \mathbf{x}_j^-) \text{ has 2 roots } 0 \leq t_1 < t_2 \\ 0 & \text{if } V(t, \mathbf{x}_i^+, \mathbf{x}_j^-) \text{ has 2 roots } t_1 \leq 0 \leq t_2 \end{cases}.$$

and $V(t, \mathbf{x}_i^+, \mathbf{x}_j^-) = at^2 + b(\mathbf{x}_i^+, \mathbf{x}_j^-)t + c(\mathbf{x}_i^+, \mathbf{x}_j^-)$ with

$$a := -n_1 \|\Gamma \mathbf{v}\|^2,$$

$$b(\mathbf{x}_i^+, \mathbf{x}_j^-) := -2\mathbf{v}^\top \Gamma^\top \Gamma (n_1 \mathbf{p}_0 - (n_1 + n_2) \mathbf{x}_i^+ + n_2 \mathbf{x}_j^-),$$

$$c(\mathbf{x}_i^+, \mathbf{x}_j^-) := -(n_1 + n_2) \|\Gamma(\mathbf{p}_0 - \mathbf{x}_i^+)\|^2 + n_2 \|\Gamma(\mathbf{p}_0 - \mathbf{x}_j^-)\|^2 - (n_1 + n_2) \log \frac{\rho(e - b, \nu_j^-)}{\eta^{\frac{n_1}{n_1+n_2}} \sum_{i=1}^{M^+} \nu_i^+}.$$

Proof. Please refer to Appendix C. \square

For a line segment $(\mathbf{p}_A, \mathbf{p}_B)$, all points on the segment can be expressed as $\mathbf{p}(t_A) = \mathbf{p}_A + t_A \mathbf{v}_A$, $\mathbf{v}_A = \mathbf{p}_B - \mathbf{p}_A$, $0 \leq t_A \leq 1$ or $\mathbf{p}(t_B) = \mathbf{p}_B + t_B \mathbf{v}_B$, $\mathbf{v}_B = \mathbf{p}_A - \mathbf{p}_B$, $0 \leq t_B \leq 1$. Using the upper bound t_{uA} on t_A provided by Eq. (26) or Eq. (27), we find the free region on $(\mathbf{p}_A, \mathbf{p}_B)$ starting from \mathbf{p}_A . Likewise, we calculate t_{uB} which specifies the free region from \mathbf{p}_B . If $t_{uA} + t_{uB} > 1$, the entire line segment is free, otherwise the segment is considered colliding. The proposed approach is summarized in Alg. 3 and illustrated in Fig. 3a for the trained RVM model in Fig. 2.

C. RVM Classification of Curves

Instead of a constant velocity \mathbf{v} representing the direction of motion, we can define a general curve $\mathbf{p}(t)$ by considering a time-varying term $\mathbf{v}(t)$. We extend the collision checking conditions in Prop. 5 by finding an ellipsoid $\mathcal{E}(\mathbf{p}_0, r) :=$

Algorithm 4 RVM Curve Classification

Input: Curve $\mathbf{p}(t)$, $t \in [0, t_f]$; threshold ε ; relevance vectors $\Lambda = \{(\mathbf{x}_i, y_i, \xi_i)\}$; weight posterior mean μ and max covariance eigenvalue λ_{max}

while True **do**

- Calculate r_k using Eq. (28) or Eq. (29).
- if** $r_k < \varepsilon$ **then return** False (Colliding)
- Solve $\|\Gamma(\mathbf{p}(t) - \mathbf{p}(t_k))\| = r_k$ for $t_{k+1} \geq t_k$
- if** $t_{k+1} \geq t_f$ **then return** True (Free)

$\{\mathbf{x} : \|\Gamma(\mathbf{x} - \mathbf{p}_0)\| \leq r\}$ around \mathbf{p}_0 whose interior is free of obstacles.

Proposition 6. Let \mathbf{p}_0 be such that $G_3(\mathbf{p}_0) < 0$ and let \mathbf{x}_i^+ and \mathbf{x}_j^- be arbitrary positive and negative support vectors. Then, every point inside the ellipsoids $\mathcal{E}(\mathbf{p}_0, r_u) \subseteq \mathcal{E}(\mathbf{p}_0, r_u^*)$ is free for:

$$r_u = \min_{i=1,\dots,M^+} r(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-) \quad (28)$$

$$r_u^* = \min_{i=1,\dots,M^+} \max_{j=1,\dots,M^-} r(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-). \quad (29)$$

where $r(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-)$

$$= \begin{cases} +\infty, & \text{if } \bar{V}(t, \mathbf{x}_i^+, \mathbf{x}_j^-) \text{ has less than 2 roots} \\ +\infty, & \text{if } \bar{V}(t, \mathbf{x}_i^+, \mathbf{x}_j^-) \text{ has 2 roots } t_1 < t_2 \leq 0 \\ t_1 & \text{if } \bar{V}(t, \mathbf{x}_i^+, \mathbf{x}_j^-) \text{ has 2 roots } 0 \leq t_1 < t_2 \\ 0 & \text{if } \bar{V}(t, \mathbf{x}_i^+, \mathbf{x}_j^-) \text{ has 2 roots } t_1 \leq 0 \leq t_2 \end{cases},$$

and $\bar{V}(t, \mathbf{x}_i^+, \mathbf{x}_j^-) = \bar{a}t^2 + \bar{b}(\mathbf{x}_i^+, \mathbf{x}_j^-)t + \bar{c}(\mathbf{x}_i^+, \mathbf{x}_j^-)$ with

$$\bar{a} := -n_1,$$

$$\bar{b}(\mathbf{x}_i^+, \mathbf{x}_j^-) := 2\|\Gamma(n_1 \mathbf{p}_0 - (n_1 + n_2) \mathbf{x}_i^+ + n_2 \mathbf{x}_j^-)\|,$$

$$\bar{c}(\mathbf{x}_i^+, \mathbf{x}_j^-) := c(\mathbf{x}_i^+, \mathbf{x}_j^-).$$

Proof. Please refer to Appendix D. \square

Consider a general time-parameterized curve $\mathbf{p}(t)$, $t \in [0, t_f]$ from $\mathbf{p}_0 := \mathbf{p}(0)$ to $\mathbf{p}_f := \mathbf{p}(t_f)$. Prop. 6 shows that all points inside the ellipsoid $\mathcal{E}(\mathbf{p}_0, r)$ are free for $r = r_u \leq r_u^*$. If we can find the smallest positive t_1 such that

$$\|\Gamma(\mathbf{p}(t_1) - \mathbf{p}_0)\| = r, \quad (30)$$

then all points on the curve $\mathbf{p}(t)$ for $t \in [0, t_1]$ are free. This is equivalent to finding the smallest positive solution of Eq. (30). We perform curve classification by iteratively covering the curve by safe ellipsoids. If the value of r is smaller than a threshold ε , the curve is considered colliding. Otherwise, it is considered free. The classification process for curves is shown in Alg. 4 and illustrated in Fig. 3b and 3c for the trained RVM model in Fig. 2 for a colliding curve and a free curve, respectively.

In Prop. 5 and 6, calculating t_u and r_u takes $O(M)$ time, while the computational complexity of calculating t_u^* and r_u^* are $O(M^2)$, where $M = M^+ + M^-$. If the line segments or curves are limited to the neighborhood of the starting point \mathbf{p}_0 , the bound t_u and r_u can reasonably approximate t_u^* and r_u^* , respectively, if \mathbf{x}_j^- is chosen as the negative support vector, closest to \mathbf{p}_0 . Calculation of t_u and r_u in Prop. 5 and 6 is efficient in the sense that it has the same complexity as classifying a point, yet it can classify an entire line segment for $t \in [0, t_u]$ and an entire ellipsoid $\mathcal{E}(\mathbf{p}_0, r_u)$, respectively.

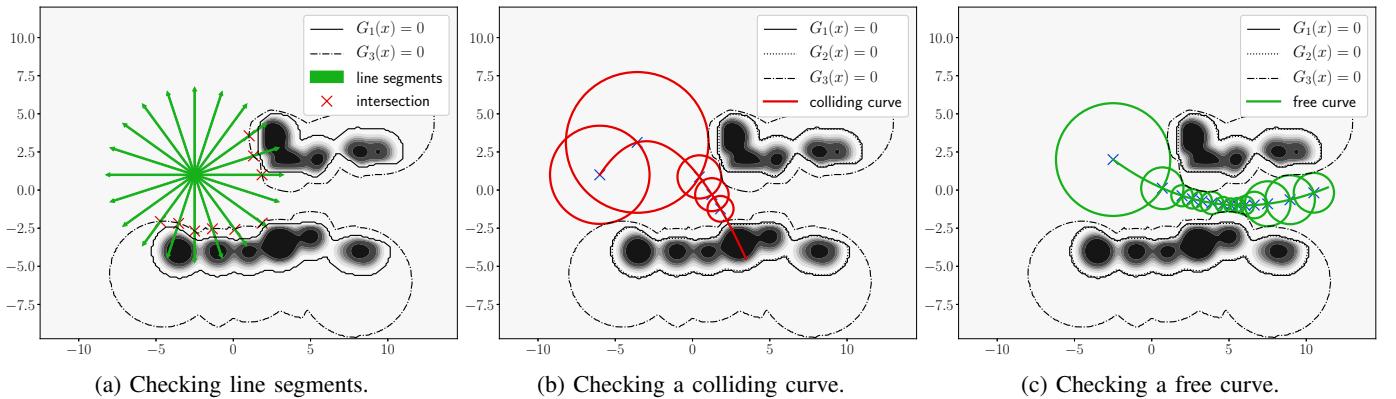


Fig. 3: Illustration of our classification algorithms for the trained RVM model in Fig. 2 with $b = -0.05$, $e = -0.01$, and $n_1 = n_2 = 1$.

VII. COMPUTATIONAL AND STORAGE IMPROVEMENTS

A. Computational Improvements

In the context of autonomous navigation, as a robot explores new regions of its environment, the number of relevance vectors required to represent the obstacle boundaries increases. Since the score function in Eq. (7) depends on all relevance vectors, the training time (Alg. 2) and the classification time (Def. 1 for points, Alg. 3 for lines, and Alg. 4 for curves) increase as well. We propose an approximation to the score function $F(\mathbf{x})$ for the radial basis kernel in Assumption 1. Since $k(\mathbf{x}, \mathbf{x}_m)$ approaches zero rapidly as the distance between \mathbf{x} and \mathbf{x}_m increases, the value of $F(\mathbf{x})$ is not affected significantly by relevance vectors far from \mathbf{x} . We use R^* -tree data structures constructed from the relevance vectors Λ^+ , Λ^- to allow efficient lookup of the nearest K^+ and K^- positive and negative relevance vectors. Approximating the score function $F(\mathbf{x})$ using the nearest K^+ and K^- relevance vectors improves its computational complexity from $O(M)$ to $O(\log M)$. Similarly, to classify a point \mathbf{x} , the M -dimensional feature vector Φ_x , may be approximated by a K -dimensional one using the K relevance vectors closest to \mathbf{x} . Classification of a line segment or a curve in Prop. 5 and 6 can be approximated by using the K^+ and K^- nearest positive and negative relevance vectors. The computational complexities of Eq. (26), (27), (28), and (29) improve from $O(M)$ and $O(M^2)$ to $O(\log M)$.

The line and curve classification algorithms depend on Prop. 3 which requires the largest eigenvalue λ_{max} of the weight posterior covariance matrix Σ . Obtaining λ_{max} from Σ can be expensive as the number of relevance vectors grows. Under Assumption 1, the entries in the feature matrix Φ for relevance vectors that are far from each other go to 0 quickly and can be set to zero, e.g., using a cut-off threshold for the kernel values or only keeping the kernel values for the K nearest relevance vectors. This leads to a sparse matrix Φ and, in turn, the inverse covariance matrix Σ^{-1} in Eq. (16) is sparse and its smallest eigenvalue $1/\lambda_{max}$ can be approximated efficiently (e.g. [58]).

B. Storage Improvements

Alg. 2 returns a set of M relevance vectors \mathbf{x}_m with labels y_m and weight prior precision ξ_m . This set represents the RVM

model parameters and its memory requirements are linear in M . However, the predictive distribution in Eq. (13) needs to be obtained via Laplace approximation (Eq. (17) and (16)) when the RVM model is used for classification. If additional computation for Laplace approximation is not feasible during test time, the weight posterior mean μ and covariance Σ may be stored also but Σ requires $O(M^2)$ storage. Fortunately, the approximate decision boundary $G_2(\mathbf{x}) = 0$ in Prop. 3 used for point, line, and curve classification only requires the largest eigenvalue λ_{max} of Σ . Hence, only the value of λ_{max} needs to be stored in addition to the relevance vectors x_m , labels y_m , and weight mean μ_m . In this case, line 15 in Alg. 2 should return the weight posterior mean μ and λ_{max} instead of μ and Σ .

VIII. APPLICATION TO OCCUPANCY MAPPING AND AUTONOMOUS NAVIGATION

A. Online Mapping

We consider a robot placed in an unknown environment at time t_k as illustrated in Fig. 1. It is equipped with a lidar scanner measuring distances to nearby obstacles. Samples generated from the lidar range scan \mathbf{z}_k are shown in Fig. 2b. Since the robot body is bounded by a sphere of radius r , each laser ray end point in configuration space becomes a ball-shaped obstacle, while the robot body becomes a point. To generate local training data, the occupied and free C-space areas observed by the lidar are sampled (e.g., on a regular grid). As shown in Fig. 2c, this generates a set $\bar{\mathcal{D}}_k$ of points with label “1” (occupied) in the ball-shaped occupied areas and with label “-1” (free) between the robot position and each laser end point. To accelerate training, only the difference between two consecutive local datasets $\mathcal{D}_k = \bar{\mathcal{D}}_k \setminus \bar{\mathcal{D}}_{k-1}$ is used in our online RVM training algorithm (Alg. 2). Storing the sets of relevance vectors Λ_k over time requires significantly less memory than storing the training data $\cup_k \mathcal{D}_k$. The occupancy of a query point \mathbf{x} can be estimated from the relevance vectors by evaluating the function $G_1(\mathbf{x})$ in Eq. (21). Specifically, $\hat{m}_k(\mathbf{x}) = -1$ if $G_1(\mathbf{x}) \leq 0$ and $\hat{m}_k(\mathbf{x}) = 1$ if $G_1(\mathbf{x}) > 0$. Fig. 2d illustrates the boundaries generated by Alg. 2.

Algorithm 5 GETSUCCESSORS and OBSTACLEFREE subroutines in A^* [59] and RRT^* [60], respectively

Input: Current position \mathbf{p}_k ; set of relevance vectors $\Lambda = \{(\mathbf{x}_i, y_i, \xi_i)\}$ with posterior weight mean μ and covariance Σ ; set $\mathcal{N}(\mathbf{p}_k)$ of potential reference trajectories $\mathbf{p}(t - t_k)$ with $\mathbf{p}(t_k) = \mathbf{p}_k$.

Output: Set of collision-free trajectories S .

```

 $S \leftarrow \emptyset;$ 
 $\text{for } \mathbf{p}' \text{ in } \mathcal{N}(\mathbf{p}_k) \text{ do}$ 
     $\quad \text{if } \mathbf{p}' \text{ is a line and } \text{CHECKLINE}(\mathbf{p}_k, \mathbf{p}', \Lambda) \text{ then} \quad \triangleright \text{Alg. 3}$ 
         $S \leftarrow S \cup \{\mathbf{p}'\}$ 
     $\quad \text{if } \mathbf{p}' \text{ is a curve and } \text{CHECKCURVE}(\mathbf{p}_k, \mathbf{p}', \Lambda) \text{ then} \quad \triangleright \text{Alg. 4}$ 
         $S \leftarrow S \cup \{\mathbf{p}'\}$ 
 $\text{return } S$ 
```

B. Autonomous Navigation

Finally, we present a complete online mapping and navigation approach that solves Problem 1. Given the sparse Bayesian kernel-based map \hat{m}_k proposed in Sec. VIII-A, a motion planning algorithm such as A^* [59] or RRT^* [60] may be used with our collision-checking algorithms to generate a path that solves the autonomous navigation problem (Alg. 5). The robot follows the path for some time and updates the map estimate \hat{m}_{k+1} with new observations. Using the updated map, the robot re-plans the path and follows the new path instead. This process is repeated until the goal is reached or a time limit is exceeded (Alg. 6).

The use of the “inflated boundary” $G_3(\mathbf{x}) = 0$ from Prop. 4 for collision checking might block the motion planning task if it is not tight enough in certain regions of the environment (e.g., unobserved regions as discussed in Sec. VI-A). For such regions, a different ratio of n_2/n_1 can be used in Prop. 4 to achieve a tighter bound $G_3(\mathbf{x})$. Increasing the decision threshold \bar{e} (Def. 1) can also improve the accuracy of $G_3(\mathbf{x})$ if a trade-off with robot safety is allowed. Another resort is to use sampling-based collision checking, selecting points along the curve $\mathbf{p}(t)$ and using Def. 1.

We consider robots with two different motion models. In simulation, we use a first-order fully actuated robot, $\dot{\mathbf{p}} = \mathbf{v}$, where the state \mathbf{s} is the robot position $\mathbf{p} \in [0, 1]^3$, with piecewise-constant velocity $\mathbf{v}(t) \equiv \mathbf{v}_k \in \mathcal{V}$ for $t \in [t_k, t_{k+1}]$, leading to piecewise-linear trajectories:

$$\mathbf{p}(t) = \mathbf{p}_k + (t - t_k)\mathbf{v}_k, \quad t \in [t_k, t_{k+1}], \quad (31)$$

where $\mathbf{p}_k := \mathbf{p}(t_k)$. In this case, the classification algorithm for line segments (Alg. 3) is used during motion planning.

In the real experiments, we consider a ground wheeled Ackermann-drive robot with dynamics model:

$$\dot{\mathbf{p}} = v \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}, \quad \dot{\theta} = \frac{v}{\ell} \tan \phi, \quad (32)$$

where the state \mathbf{s} consists of the position $\mathbf{p} \in \mathbb{R}^2$ and orientation $\theta \in \mathbb{R}$, the control input \mathbf{a} consists of the linear velocity $v \in \mathbb{R}$ and the steering angle $\phi \in \mathbb{R}$, and ℓ is the distance between the front and back wheels. The nonlinear car dynamics can be transformed into a 2nd-order fully actuated system $\ddot{\mathbf{p}} = \mathbf{a}$ via feedback linearization [52], [53].

Algorithm 6 Autonomous Mapping and Navigation with a Sparse Bayesian Kernel-based Map

Input: Initial state $\mathbf{s}_0 \in \bar{\mathcal{S}}$; goal region \mathcal{G} ; prior relevance vectors Λ_0 .

```

1: for  $k = 0, 1, \dots$  do
2:   if  $\mathbf{s}_k \in \mathcal{G}$  then break
3:    $\mathbf{z}_k \leftarrow$  new range sensor observation
4:    $\mathcal{D}_k \leftarrow$  Training Data Generation( $\mathbf{z}_k, \mathbf{s}_k$ ) ▷ Sec. VIII-A
5:    $\Lambda_{k+1} \leftarrow$  Online RVM Training( $\Lambda_k, \mathcal{D}_k$ ) ▷ Alg. 2
6:   Path Planning( $\Lambda_{k+1}, \mathbf{s}_k, \mathcal{G}$ ) ▷ Alg. 5
7:   Move to the first state  $\mathbf{s}_{k+1}$  along the path
```

Using piecewise-constant acceleration $\mathbf{a}(t) \equiv \mathbf{a}_k \in \mathcal{A}$ for $t \in [t_k, t_{k+1}]$ leads to piecewise-polynomial trajectories:

$$\mathbf{p}(t) = \mathbf{p}_k + (t - t_k)\mathbf{v}_k \begin{bmatrix} \cos(\theta_k) \\ \sin(\theta_k) \end{bmatrix} + \frac{(t - t_k)^2}{2} \mathbf{a}_k, \quad (33)$$

where $\mathbf{p}_k := \mathbf{p}(t_k)$, $\theta_k := \theta(t_k)$, $\mathbf{v}_k := \mathbf{v}(t_k)$. In our experiments, the input set \mathcal{A} is finite and the classification algorithm for curves (Alg. 4) is used to get successor nodes in an A^* motion planning algorithm.

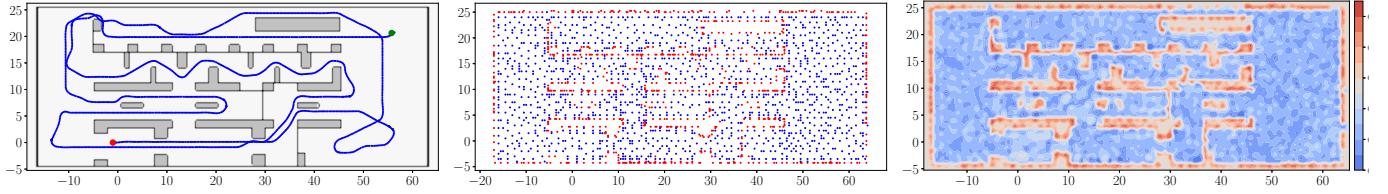
IX. EXPERIMENTAL RESULTS

This section presents an evaluation of our autonomous mapping and navigation method using a fully actuated robot (31) in a simulated environment (Sec. IX-A), the Intel Research Lab dataset [61] (Sec. IX-B), and a car-like robot (Fig. 1) with Ackermann-drive dynamics (32) in real experiments (Sec. IX-C). We used a radial basis function (RBF) kernel with parameters $\eta = 1$ and $\Gamma = \sqrt{\gamma}\mathbf{I}$. Timing results are reported from an Intel i9 3.1 GHz CPU with 32GB RAM.

A. Comparison with non-Bayesian map representations

In this section, we compared the accuracy and storage requirements of our sparse Bayesian kernel-based map (SBKM) with those of the non-Bayesian sparse kernel-based map (SKM) from our preliminary work [28] and the popular occupancy mapping algorithm OctoMap [19]. As the ground-truth map (Fig. 4a) represents the work space instead of C-space, a point robot ($r = 0$) was used for an accurate comparison. Lidar scans were simulated along the robot trajectory shown in Fig. 4a and used to build our sparse Bayesian kernel-based map (SBKM), the non-Bayesian sparse kernel-based map (SKM) [28] and OctoMap. An R^* -tree approximation of the score $F(\mathbf{x})$ was used with $K^+ + K^- = 200$ nearest support vectors around the robot location \mathbf{p}_k for map updating and with $K^+ + K^- = 10$ nearest support vectors for collision checking. OctoMap’s resolution was set to $0.25m$ to match that of the grid used to sample our training data from.

Table I compares the accuracy and the storage requirements of our SBKM and SKM maps versus those of OctoMap’s binary and probabilistic map. The SBKM map and its sparse set of relevance vectors are shown in Fig. 4. To calculate map accuracy, we used different thresholds \bar{e} to generate binary versions of our map and compare with the ground truth. The ground truth map was sampled on a grid with the same resolution $0.25m$ and the accuracy was calculated as the number of correct predictions divided by the total number of



(a) Ground truth map and robot trajectory. (b) Set of 2141 relevance vectors. (c) The final SBKM map.

Fig. 4: Sparse map representation (with $\eta = 1, \Gamma = \sqrt{\gamma}I, \gamma = 3.0$) built from local streaming laser scans along the robot trajectory.

TABLE I: Comparison among our sparse Bayesian kernel-based map (SBKM), our sparse kernel-based map (SKM) [28], and OctoMap (OM) [19]. An RBF kernel with $\eta = 1, \Gamma = \sqrt{\gamma}I$ was used for our SBKM and SKM maps. [†]The storage requirements for SBKM are calculated for two storing approaches mentioned in Sec. VII: 1) with Laplace approximation at test time, i.e., storing the relevance vectors' location with their label and precision ξ ; 2) without Laplace approximation at test time, i.e., storing the relevance vectors' location with their label and weight's mean μ , and the largest eigenvalue of the covariance matrix λ_{max} . As we only need an extra float to store λ_{max} , both approaches offer similar storage requirements.

Methods	SBKM $\gamma = 0.5$ $\bar{e} = 0.5$	SBKM $\gamma = 1.0$ $\bar{e} = 0.5$	SBKM $\gamma = 2.0$ $\bar{e} = 0.5$	SBKM $\gamma = 3.0$ $\bar{e} = 0.45$	SBKM $\gamma = 3.0$ $\bar{e} = 0.5$	SBKM $\gamma = 3.0$ $\bar{e} = 0.55$	SKM $\gamma = 3.0$ -	OM -
Accuracy	95.6%	97.8%	99.0%	99.2%	99.3%	99.5%	99.9%	99.9%
Recall	96.8%	97.9%	99.3%	99.7%	99.4%	98.7%	99.0%	99.7%
Vectors/Nodes	848	1115	1642	2141			2463	12432 non-leafs & 34756 leafs
Storage	7kB [†]	9kB [†]	13kB [†]	17kB [†]			20kB	25kB(binary)/236kB (full)

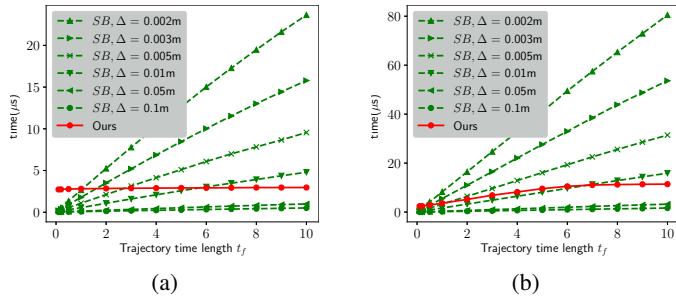


Fig. 5: Collision checking time comparison between our methods and sampling-based (SB) ones with different sampling interval Δ for (a) line segments $p(t) = p_0 + vt$ and (b) 2nd-order polynomial curves $p(t) = p_0 + vt + at^2$ for $t \in [0, t_f]$ with various values of t_f .

TABLE II: Comparison between our sparse Bayesian kernel-based map (SBKM) and Sequential Bayesian Hilbert Map (SBHM) [35] on the Intel Research Lab dataset [61]. An RBK kernel with $\eta = 1, \Gamma = \sqrt{\gamma}I$ are used for our SBKM maps. The metrics are the area under the receiver operating characteristic curve (AUC) and the negative log-likelihood loss (NLL).

Methods	SBHM $\gamma = 6.71$ full Σ	SBKM $\gamma = 6.71$ full Σ	SBKM $\gamma = 6.71$ λ_{max} only
AUC	0.98	0.96	0.95
NLL	0.24	0.36	0.52
Points	5600	3492	3492
Average time/scan	11.8s	0.76s	0.43s

samples. Note that the interior (gray regions in the ground-truth map) of the obstacles were considered occupied for our map - since it was surrounded by positive relevance vectors - but were considered free in SKM and OctoMap maps.

Table I shows that SBKM (with $\Gamma = \sqrt{\gamma}I, \gamma = 3.0$, and threshold $\bar{e} = 0.5$), SKM and OctoMap's binary map led to a similar accuracy of $\sim 99\%$ compared to the ground truth map. As we decreased the decision threshold \bar{e} , the accuracy de-

creased, as more free cells were classified as "occupied", while the recall increased, as more occupied cells were classified as "occupied", and vice versa. When the parameter γ decreased, the support of the kernel expanded, leading to fewer relevance vectors, i.e., less storage but lower accuracy and recalls. This illustrates the trade-off between storage gains and accuracy when the details of the obstacles' boundaries can be reduced via a lower value of γ to achieve higher compression rate.

We also compared the storage requirements for our SBKM and SKM representations and OctoMap. OctoMap's binary map required a compressed octree with 12432 non-leaf nodes with 2 bytes per node, leading to a storage requirement of $\sim 25kB$. Its fully probabilistic map required to store 47188 leaf and non-leaf nodes with 5 bytes per node, leading to a storage requirement of $\sim 236kB$. As the space consumption depends on the computer architecture and how the relevance vector information is compressed, we provide only a rough estimate of storage requirements for our maps. For the SKM map, each support vector required 8 bytes, including an integer for the support vector's location on the underlying grid and a float for its weight. As a result, $\sim 20kB$ were needed to store the 2463 resulting support vectors. As discussed in Sec. VII-B, the SBKM map could be stored in two ways: 1) the relevance vectors' location, their label and their weight prior precision if Laplace approximation was allowed at test time; 2) the relevance vectors' location, their label, their weight mean and the largest eigenvalue λ_{max} of the covariance matrix Σ if Laplace approximation was not allowed at test time and our collision checking methods were used. The former stored an integer representing a relevance vector's location on the underlying grid and a float representing its weight prior's precision and its label (using the float sign). This required 8 bytes on a 32-bit architecture per relevance vector. Our SBKM map with $\Gamma = \sqrt{3.0}I$ contained 2141 relevance vectors, leading to storage requirements of $\sim 17kB$. The latter also

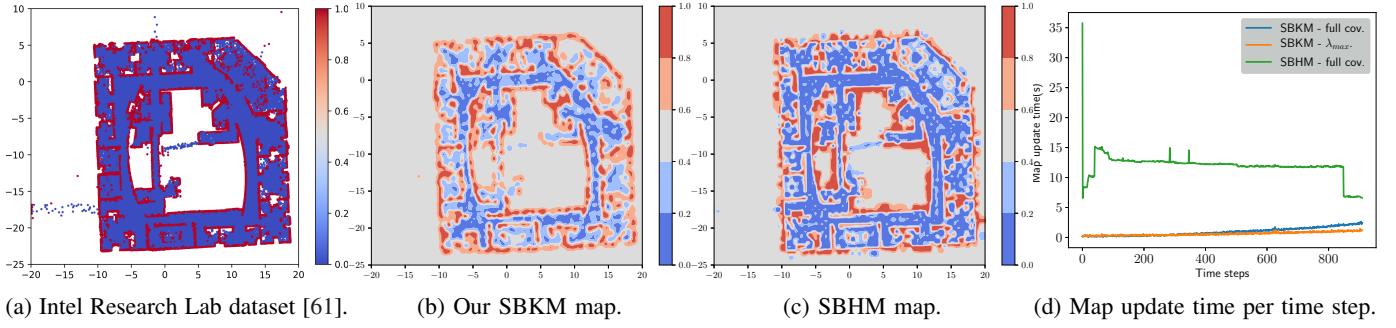


Fig. 6: Comparison between our sparse Bayesian kernel-based map (SBKM) and Sequential Bayesian Hilbert Map (SBHM) [35] with $\eta = 1$, $\Gamma = \sqrt{6.71I}$ built online using lidar scans from the Intel Research Lab dataset [61].

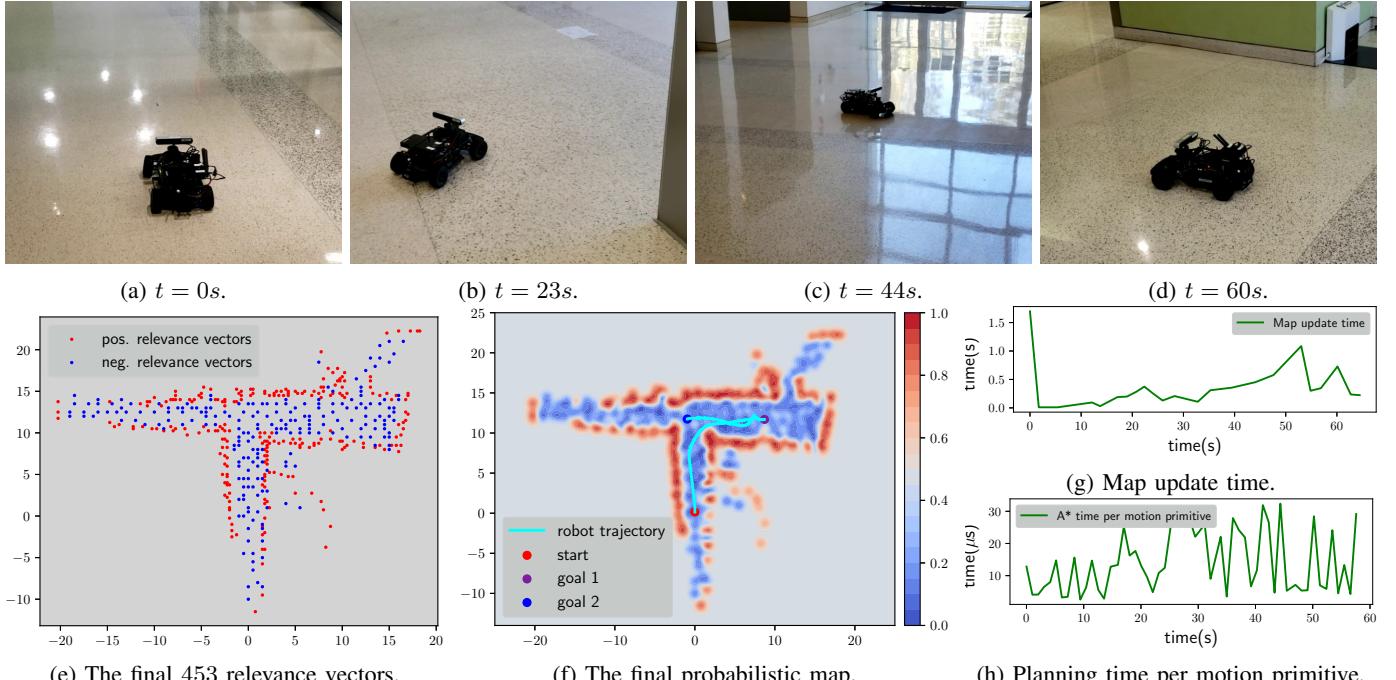


Fig. 7: Real experiment with an autonomous Racecar robot navigating in an unknown hallway environment: (a)-(d) Onboard camera and third person views of the robot and the environment; (e) The final 453 relevance vectors; (f) The final probabilistic map; (g) Map update time; (h) Planning time per motion primitive. Please refer to our website <https://thaipduong.github.io/sbkm> for the experiment video.

needed $17kB$ to store the relevance vectors' location, their weight's mean and label. Besides, an extra float (4 bytes) is needed to store λ_{max} leading to a similar total storage requirement of $17kB$. These requirements were noticeably better than those of OctoMap and our non-Bayesian SKM map.

We also compared the average collision checking time over one million random line segments $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{v}t$ and one million random second order polynomial curves $\mathbf{p}(t) = \mathbf{p}_0 + \mathbf{v}t + \mathbf{a}t^2$ for $t \in [0, t_f]$ using our complete method (Alg. 3 with Eq. (27) for line segments, Alg. 4 with Eq. (29) for curves, and $K^+ + K^- = 10$ for score approximation) and sampling-based methods with different sampling resolutions using the ground truth map. Fig. 5a and 5b show that the time for sampling-based collision checking increased as the time length t_f increased or the sampling resolution decreased. Meanwhile, our method's time was stable at $\sim 3\mu s$ for checking line segments and at $\sim 11\mu s$ for checking second-

order polynomial curves suggesting our collision checking algorithms' suitability for real-time applications.

B. Comparison with Sequential Bayesian Hilbert Map

In this section, we compared our sparse Bayesian kernel-based map (SBKM) approach with Sequential Bayesian Hilbert Map (SBHM) [35], a similar approach for Bayesian probabilistic mapping method from streaming local observation. We tried our best to match the parameters for a fair comparison, e.g. using the same kernel parameter $\Gamma = \sqrt{\gamma I}$ with $\gamma = 6.71$ as provided by SBHM code [35]. The Intel Research Lab dataset [61] was used with both methods to build the map of the environment in an online manner. Our online training data (Sec. V-B) were generated from a grid with resolution $0.2m$. Fig. 6a visualizes the data points with their labels accumulated from all time steps in the dataset. Fig. 6b and 6c show similar final maps from our SBKM method and the SBHM approach, respectively. The dataset

was split into a training set (90%) and a test set (10%). The metrics for comparison were the area under the receiver operating characteristic curve (AUC) and the negative log-likelihood loss (NLL) of a point \mathbf{x} , defined as $NLL(y|\mathbf{x}, \xi) = -\log p(y|\mathbf{x}, \xi)$, where $y \in \{-1, 1\}$ is the true label and $p(y|\mathbf{x}, \xi)$ is the predictive distribution in Eq. (13). The AUC score and NLL loss were calculated over the test set.

Table II presents the metrics for both mapping methods. The SBHM map used a fixed grid of 5600 hinged points with resolution $0.5m$. Meanwhile, our map incrementally learned a sparse set of relevance vectors from the training dataset, not requiring a set of fixed hinged points which is hard to predetermine for unknown environments. Our final map's AUC score and NLL loss were slightly worse than those of SBHM while maintaining $\sim 35\%$ fewer points to represent the environment and having faster map updates with less than $1s$ per scan, on average, as shown in Table II and Fig. 6d. Our training algorithm incrementally built the set of relevance vectors and only updated the weights of the local vectors due to the use of K nearest relevance vectors in Alg. 2. Consequently, it did not have a fixed global set of points to optimize over as done by SBHM, leading to suboptimality in trade-off for sparseness. Note that both our map and the SBHM map estimated the mean μ and the full covariance matrix Σ of the weights' posterior for test time. If our collision checking algorithms are used for planning, only the largest eigenvalue λ_{max} of Σ is needed and can be calculated efficiently using the sparsified inverse covariance matrix as shown in Sec. VII-B. In this case, Table II shows that our map update time was reduced by half to about $\sim 0.43s$ per scan (Table II and Fig. 6d) while offering similar AUC score to that of our SBKM map with full covariance matrix. The higher NLL loss was due to the upper bound used in Prop. 3 for point classification instead of the true occupancy probability.

C. Real Experiments

Real experiments were carried out on an 1/10th scale Racecar robot (Fig. 7a, 7b, 7c, and 7d) equipped with a Hokuyo UST-10LX Lidar and Nvidia TX2 computer. The robot body was modeled by a ball of radius $r = 0.25m$. The online training data (Sec. V-B) were generated from a grid with resolution $0.25m$. We used an RBF kernel parameter $\Gamma = \sqrt{\gamma I}$ with $\gamma = 3.0$ and an R^* -tree approximation of the score $F(\mathbf{x})$ with $K^+ + K^- = 20$ nearest support vectors around the robot location \mathbf{p}_k for map updating. For motion planning, second-order polynomial motion primitives were generated with time discretization of $\tau = 1s$ as described in Sec. VIII-B. The motion cost was defined as $c(\mathbf{s}, \mathbf{a}) := (\|\mathbf{a}\|^2 + 2)\tau$ to encourage both smooth and fast motion [51]. Alg. 4 with Eq. (29), $\varepsilon = 0.1$, and score approximation with $K^+ = K^- = 2$ was used for collision checking in Alg. 5. The trajectory generated by an A^* motion planner was tracked using a closed-loop controller [62]. The robot navigated in an unknown hallway to two destinations consequently chosen by a human operator. Fig. 7e shows the learned relevance vectors representing the environment. Fig. 7f shows the probabilistic map recovered from the relevance vectors together with the robot trajectory and the two chosen destinations.

The time taken by Alg. 2 to update the relevance vectors from one lidar scan and the A^* replanning time per motion primitive are shown in Fig. 7g and 7h. Map updates implemented in Python took $0.4s$ on average. It took a longer time ($\sim 1s$) to update the map when the robot observed new large parts of the environment, e.g., at the beginning and toward the end of our experiment. To evaluate collision checking time, the A^* replanning time was normalized by the number of motion primitives being checked to account for differences in planning to nearby and far goals. The planning time per motion primitive was $\sim 15\mu s$ on average and $\sim 30\mu s$ at most, suggesting our collision checking algorithms' suitability for real-time applications.

X. CONCLUSION

This paper proposes a sparse Bayesian kernel-based mapping method for efficient online generation of large occupancy maps, supporting autonomous robot navigation in unknown environments. Our map representation, as a sparse set of relevance vectors learned from streaming range observations of the environment, is efficient to store. It supports efficient and complete collision checking for general curves modeling potential robot trajectories. Our experiments demonstrate the potential of this model at generating compressed, yet accurate, probabilistic environment models. Our results offer a promising venue for quantifying safety and uncertainty and enabling real-time long-term autonomous navigation in unpredictable environments. Future work will explore active exploration and map uncertainty reduction as well as simultaneous localization and mapping using the proposed map representations.

APPENDIX A PROOF OF PROPOSITION 3

Proof. A point \mathbf{x} is considered free if:

$$\Phi_x^\top \mu + b < e\sqrt{1 + \Phi_x^\top \Sigma \Phi_x}, \quad (34)$$

where Φ_x is the feature vector $\Phi_x = [k_1(\mathbf{x}), k_2(\mathbf{x}), \dots, k_M(\mathbf{x})]^\top$. We use the following lower bound and upper bound on $\Phi_x^\top \Sigma \Phi_x$: $0 \leq \Phi_x^\top \Sigma \Phi_x \leq \lambda_{max} \sum_{m=1}^M (k_m(\mathbf{x}))^2$ where $\lambda_{max} \geq 0$ is the largest eigenvalue of the covariance matrix Σ . Since $k_m(\mathbf{x}) > 0$ for all m , we have:

$$1 \leq \sqrt{1 + \Phi_x^\top \Sigma \Phi_x} \leq 1 + \sqrt{\lambda_{max}} \sum_{m=1}^M (k_m(\mathbf{x})). \quad (35)$$

Therefore, the point \mathbf{x} is still free if

$$\Phi_x^\top \mu + b \leq e(1 + \mathbb{1}_{\{e \geq 0\}} \sqrt{\lambda_{max}} \sum_{m=1}^M (k_m(\mathbf{x}))), \quad (36)$$

or $\sum_{m=1}^M (\mu_m - e \mathbb{1}_{\{e \geq 0\}} \sqrt{\lambda_{max}}) k_m(\mathbf{x}) + b - e \leq 0$. \square

APPENDIX B PROOF OF PROPOSITION 4

Proof. A point \mathbf{x} is free if Eq. (23) holds. Let \mathbf{x}_*^+ be the closest positive relevance vector to \mathbf{x} and \mathbf{x}_j^- be any negative relevance vector. We have:

$$\begin{aligned} \sum_{i=1}^{M^+} \nu_i^+ k(\mathbf{x}, \mathbf{x}_i^+) - \sum_{j=1}^{M^-} \nu_j^- k(\mathbf{x}, \mathbf{x}_j^-) + b - e &\leq \\ \leq (\sum_{i=1}^{M^+} \nu_i^+) k(\mathbf{x}, \mathbf{x}_*^+) - \nu_j^- k(\mathbf{x}, \mathbf{x}_j^-) + b - e \end{aligned}$$

Under Assumptions 1 and 2, both terms $\nu_j^- k_j(\mathbf{x})$ and $e - b$ are non-negative. By the arithmetic mean-geometric mean inequality, we have:

$$\begin{aligned} \nu_j^- k(\mathbf{x}, \mathbf{x}_j^-) + e - b &= n_2 \frac{\nu_j^- k(\mathbf{x}, \mathbf{x}_j^-)}{n_2} + n_1 \frac{e - b}{n_1} \\ &\geq (n_1 + n_2) \left(\frac{\nu_j^- k(\mathbf{x}, \mathbf{x}_j^-)}{n_2} \right)^{\frac{n_2}{n_1+n_2}} \left(\frac{e - b}{n_1} \right)^{\frac{n_1}{n_1+n_2}} \\ &= \rho(e - b, \nu_j^- k(\mathbf{x}, \mathbf{x}_j^-)). \end{aligned}$$

Therefore, a point \mathbf{x} is free if

$$(\sum_{i=1}^{M^+} \nu_i^+) k(\mathbf{x}, \mathbf{x}_i^+) - \rho(e - b, \nu_j^- k(\mathbf{x}, \mathbf{x}_j^-)) \leq 0. \quad (37)$$

APPENDIX C PROOF OF PROPOSITION 5

Proof. By plugging $k(\mathbf{x}, \mathbf{x}_*^+) = \eta e^{-\|\Gamma(\mathbf{x}-\mathbf{x}_*^+)\|^2}$, and $k(\mathbf{x}, \mathbf{x}_j^-) = \eta e^{-\gamma\|\Gamma(\mathbf{x}-\mathbf{x}_j^-)\|^2}$ into Eq. (37), a point \mathbf{x} is free if

$$e^{-\|\Gamma(\mathbf{x}-\mathbf{x}_*^+)\|^2 + \frac{n_2}{n_1+n_2}\|\Gamma(\mathbf{x}-\mathbf{x}_j^-)\|^2} \leq \frac{\rho(e - b, \nu_j^-)}{\eta^{\frac{n_1}{n_1+n_2}} \sum_{i=1}^{M^+} \nu_i^+} \quad (38)$$

Substituting the test point \mathbf{x} by $\mathbf{p}(t) = \mathbf{p}_0 + t\mathbf{v}$ in Eq. (38), the point $\mathbf{p}(t)$ is free if:

$$\begin{aligned} V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) &= -(n_1 + n_2)\|\Gamma(\mathbf{p}_0 + t\mathbf{v} - \mathbf{x}_*^+)\|^2 \\ &\quad + n_2\|\Gamma(\mathbf{p}_0 + t\mathbf{v} - \mathbf{x}_j^-)\|^2 - (n_1 + n_2)\beta \leq 0, \end{aligned}$$

where $\beta = \log \frac{\rho(e - b, \nu_j^-)}{\eta^{\frac{n_1}{n_1+n_2}} \sum_{i=1}^{M^+} \nu_i^+}$. By expanding the quadratic norms in $V(t, \mathbf{x}_*^+, \mathbf{x}_j^-)$, the point $\mathbf{p}(t)$ is free if:

$$\begin{aligned} V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) &= at^2 + b(\mathbf{x}_*^+, \mathbf{x}_j^-)t + c(\mathbf{x}_*^+, \mathbf{x}_j^-) \leq 0 \quad (39) \\ \text{where } a &= -n_1\|\Gamma\mathbf{v}\|^2, \\ b(\mathbf{x}_*^+, \mathbf{x}_j^-) &= -2\mathbf{v}^\top \Gamma^\top \Gamma(n_1\mathbf{p}_0 - (n_1 + n_2)\mathbf{x}_*^+ + n_2\mathbf{x}_j^-), \\ c(\mathbf{x}_*^+, \mathbf{x}_j^-) &= -(n_1 + n_2)\|\Gamma(\mathbf{p}_0 - \mathbf{x}_*^+)\|^2 \\ &\quad + n_2\|\Gamma(\mathbf{p}_0 - \mathbf{x}_j^-)\|^2 - (n_1 + n_2)\beta. \end{aligned}$$

Note that $V(t, \mathbf{x}_*^+, \mathbf{x}_j^-)$ is a quadratic polynomial in t and the point $\mathbf{p}(t)$ is free if $V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) \leq 0$.

- 1) If it has less than 2 roots, Eq. (39) is satisfied for all t .
- 2) If it has 2 roots $t_1 < t_2$, then $V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) \leq 0$ for $t \geq t_2$ or $t \leq t_1$. There are three cases:
 - a) $t_1 < t_2 \leq 0$: $V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) \leq 0$ for all $t \geq 0$ or the entire ray $s(t)$ is free;

- b) $0 \leq t_1 < t_2$: $V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) \leq 0$ for $t \in [0, t_1]$ or the ray $s(t)$ is free for $t \in [0, t_1]$.
- c) $t_1 \leq 0 \leq t_2$: $V(0, \mathbf{x}_*^+, \mathbf{x}_j^-) \geq 0$ or the ray $s(t)$ is colliding.

Let $\tau(\mathbf{p}_0, \mathbf{x}_*^+, \mathbf{x}_j^-)$

$$= \begin{cases} +\infty, & \text{if } V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) \text{ has less than 2 roots} \\ +\infty, & \text{if } V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) \text{ has 2 roots } t_1 < t_2 \leq 0 \\ t_1 & \text{if } V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) \text{ has 2 roots } 0 \leq t_1 < t_2 \\ 0 & \text{if } V(t, \mathbf{x}_*^+, \mathbf{x}_j^-) \text{ has 2 roots } t_1 \leq 0 \leq t_2 \end{cases}.$$

Note that \mathbf{x}_*^+ varies with t but belongs to a finite set, we can calculate $\tau(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-)$ for all positive relevance vectors \mathbf{x}_i^+ and take the minimum value. Therefore, $\mathbf{p}(t)$ is free as long as:

$$t \leq t_u = \min_{i=1, \dots, M^+} \tau(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-) \quad (40)$$

Note that Eq. (40) holds for any negative relevance vector \mathbf{x}_j^- . Therefore, the point $\mathbf{p}(t)$ is free as long as $t \leq t_u^* = \max_{j=1, \dots, M^-} \min_{i=1, \dots, M^+} \tau(\mathbf{p}_0, \mathbf{x}_i^+, \mathbf{x}_j^-)$. \square

APPENDIX D PROOF OF PROPOSITION 6

Proof. Consider an arbitrary vector v such that $\|\Gamma v\| = 1$ in Eq. (39). By the Cauchy-Schwarz inequality, we have:

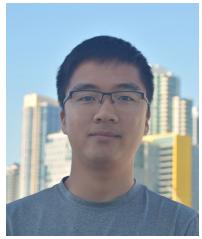
$$\begin{aligned} &-2t\mathbf{v}^\top \Gamma^\top \Gamma(n_1\mathbf{p}_0 - (n_1 + n_2)\mathbf{x}_*^+ + n_2\mathbf{x}_j^-) \\ &\leq 2t\|\Gamma(n_1\mathbf{p}_0 - (n_1 + n_2)\mathbf{x}_*^+ + n_2\mathbf{x}_j^-)\| \end{aligned}$$

Therefore, the point $\mathbf{p}(t)$ is free if $\bar{V}(t, \mathbf{x}_*^+, \mathbf{x}_j^-) \leq 0$. The proposition follows the same reasoning as Prop. 5. \square

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