Optimized Luenberger Observer for Linear Dynamical Systems

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1 Dynamical System and Observer Design

Consider the LTI system given by the equation

$$\begin{aligned} \dot{\xi} &= A\xi + Bu \quad \xi(t_0) = \xi_0 \\ y &= C\xi \end{aligned} \tag{1}$$

where $\xi(t) \in \mathbb{R}^n$ is the state of the system at time t, $A \in \mathbb{R}^{n\times n}$ is the state transition matrix, $B \in \mathbb{R}^{n\times m}$ is the input gain matrix, $u(t) \in \mathbb{R}^m$ is the control input to the dynamical system, $C \in \mathbb{R}^{p\times n}$ is the measurement/output matrix and $y \in \mathbb{R}^p$ is the output of the system. The objective is to design an observer for the system in (1) to reconstruct the state, when the initial x_0 is unknown. We can design a Luenberger observer of the form

$$\dot{\hat{\xi}} = A\hat{\xi} + Bu + L\left(y - C\hat{\xi}\right) \quad \hat{\xi}(t_0) = \hat{\xi}_0 \tag{2}$$

where $\hat{\xi}(t)$ is the estimate of $\xi(t)$, $L\left(y - C\hat{\xi}\right)$ is the innovation term with the observer gain matrix $L \in \mathbb{R}^{n\times p}$ which drives the observer estimate to the true state. Consider the estimation error defined by

$$e(t) = \xi(t) - \hat{\xi}(t) \tag{3}$$

Then the estimation error dynamics are given by $\dot{e}(t) = \dot{\xi}(t) - \dot{\hat{\xi}}(t)$. Substituting the equations (1) and (2), we can simplify the estimation error dynamics as

$$\begin{aligned} \dot{e} &= A\xi + Bu - A\hat{\xi} - Bu - L\left(y - C\hat{\xi}\right) \\ &= (A - LC)\, e \end{aligned} \tag{4}$$

The objective is to make the estimation error dynamics stable. To achieve this, the observer gain matrix L should be designed such that the matrix $A - LC$ must be Hurwitz. We first assume that the pair (A, C) is observable such that

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \tag{5}$$

To design the observer gain and analyze the stability of the observer, consider a candidate Lyapunov function $V(e) : \mathbb{R}^n \to \mathbb{R}^+$

$$V(e) = e^T P e$$

where $P \in \mathbb{R}^{n\times n}$, $P = P^T$ and $P > 0$. The candidate Lyapunov function can be bounded using the Rayleigh-Ritz inequality as $\lambda_{\min}\{P\}\|e\|^2 \le V(e) \le \lambda_{\max}\{P\}\|e\|^2$ where $\lambda_{\min}\{\cdot\}$ and $\lambda_{\max}\{\cdot\}$ are the minimum and maximum eigenvalues of the matrix. Taking the time derivative of V and substituting the error dynamics in (4), we obtain the following

$$\begin{aligned}\dot{V} &= e^T P\dot{e} + \dot{e}^T Pe \\ &= e^T P(A-LC)e + e^T (A-LC)^T Pe \\ &= e^T \left(P(A-LC) + (A-LC)^T P\right) e \\ &= e^T \left(A^T P + PA - PLC - C^T L^T P\right) e \end{aligned} \tag{6}$$

If the the matrix $A^T P + PA - PLC - C^T L^T P < 0$, we can claim global asymptotic stability of the error dynamics in (4). For the error to decay with fixed decay rate α, we consider the inequality,

$$A^T P + PA - PLC - C^T L^T P \leq -\alpha P \tag{7}$$

Substituting, (7) in (6), we get the following,

$$\begin{aligned}\dot{V} &\leq -\alpha e^T Pe \\ &\leq -\alpha V\end{aligned} \tag{8}$$

Then, the solution to V and $\|e(t)\|$ can be given as

$$\begin{aligned}V(t) &\leq V(t_0)\exp\left(-\alpha(t-t_0)\right) \\ \|e(t)\| &\leq \sqrt{\frac{\lambda_{\max}\{P\}}{\lambda_{\min}\{P\}}}\|e(t_0)\|\exp\left(-\frac{\alpha}{2}(t-t_0)\right)\end{aligned} \tag{9}$$

However, the equation (7) is not a Linear Matrix Inequality (LMI). We now use the substitution $PL = W$ and hence, $L = P^{-1}W$. Then (7) can be simplified to

$$A^T P + PA - WC - C^T W^T + \alpha P \leq 0 \tag{10}$$

Solving this LMI, we can optimize for the observer gain L and the the matrix P.

2 Numerical Simulation : State Estimation of an Inverted Pendulum

We consider the problem of estimate the state of and inverted pendulum on cart. Figure 1 represents the dynamical system considered.

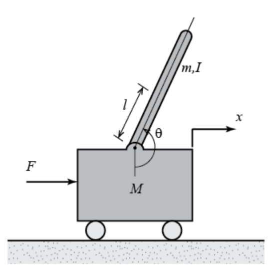

Figure 1: Dynamical system containing the cart and the pendulum.

The state of the dynamical system is given by $\xi(t) = \begin{bmatrix} x & \dot{x} & \phi & \dot{\phi} \end{bmatrix}^T$, where x and ϕ are the position of the cart and the pendulum respectively, $\dot{x}$ and $\dot{\phi}$ are the velocity of the cart and the pendulum respectively. The linearized state space model of the dynamical system is given by

$$
\begin{gathered}
\dot{\xi} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2g^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{1(M+m)+Mml^2} & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ \frac{I+mP^2}{T(M+m)+Mml^2} \\ 0 \\ \frac{ml}{T(M+m)+Mml^2} \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u
\end{gathered}
\tag{11}
$$

where $M = 0.5$kg is the mass of the cart, $m = 0.2$kg is the mass of the pendulum, $b = 0.1$N/m/sec is the coefficient of friction of the cart, $l = 0.3$m is the length of the pendulum, $I = 0.006\text{kg/m}^2$ is the moment of inertia of the pendulum and $F = u$ is the force applied to the cart. A standard LQR controller is implemented to stabilize the system and the estimated state is not used in the LQR. The state estimation algorithm is run in parallel to demonstrate the efficiency. The LMI is solved using the CVX package for MATLAB 2020a and the following values are computed

$$
P = \begin{bmatrix} 25.4097 & -6.9774 & -0.6989 & 0.5481 \\ 6.9774 & -25.7638 & 0.8308 & -2.1053 \\ -0.6989 & 0.8308 & 12.4667 & -1.0942 \\ 0.5481 & -2.1053 & -1.0942 & 0.7134 \end{bmatrix} \quad W = \begin{bmatrix} 15.6809 & -0.4237 \\ 25.0233 & 1.9538 \\ -0.4237 & -6.4630 \\ -0.6860 & 22.8851 \end{bmatrix} \quad L = \begin{bmatrix} 0.9611 & 0.1221 \\ 1.4613 & 4.0226 \\ 0.1754 & 3.5466 \\ 2.8814 & 49.2930 \end{bmatrix}
$$

The results of the simulation are summarized in the following plots

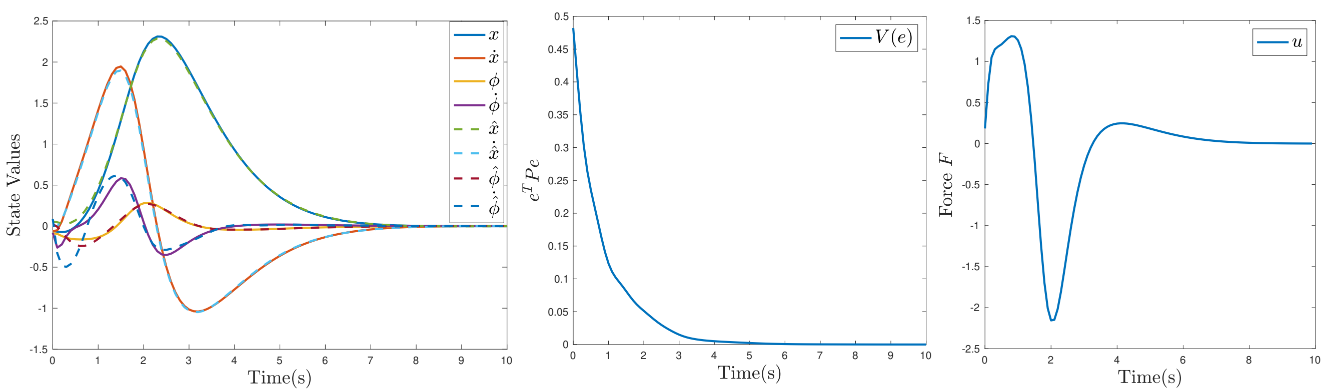

Figure 2: Results of the numerical simulation. (a) True state (solid lines) and estimated state (dashed lines) evolution with time. (b) Lyapunov function computed using the P matrix. (c) Control input to the system generated by the LQR controller.