Reference book: *Numerical Methods, S. Iyengar, R. Jain*

W1L1.
Intro to equations.

Polynomials

A problem of great importance in science and engineering is that of determining the roots/ zeros of an equation of the form

$$f(x) = 0, \tag{1.1}$$

A polynomial equation of the form

$$f(x) = P_n(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0 \tag{1.2}$$

is called an *algebraic equation*. An equation which contains polynomials, exponential functions, logarithmic functions, trigonometric functions etc. is called a *transcendental equation*.

For example,

$$3x^3 - 2x^2 - x - 5 = 0, \qquad x^4 - 3x^2 + 1 = 0, \qquad x^2 - 3x + 1 = 0,$$

are algebraic (polynomial) equations, and

$$xe^{2x} - 1 = 0, \qquad \cos x - xe^x = 0, \qquad \tan x = x$$

are transcendental equations.

Root

We assume that the function $f(x)$ is continuous in the required interval.

We define the following.

Root/zero A number α, for which $f(\alpha) \equiv 0$ is called a root of the equation $f(x) = 0$, or a zero of $f(x)$. Geometrically, a root of an equation $f(x) = 0$ is the value of x at which the graph of the equation $y = f(x)$ intersects the x-axis (see Fig. 1.1).

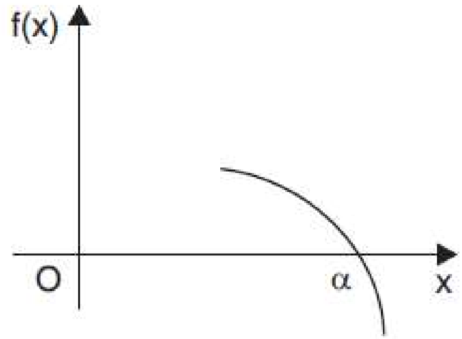

Fig. 1.1 'Root of $f(x) = 0$'

Simple root

Simple root A number α is a simple root of $f(x) = 0$, if $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Then, we can write $f(x)$ as

$$f(x) = (x - \alpha)\, g(x),\ g(\alpha) \neq 0. \tag{1.3}$$

For example, since $(x - 1)$ is a factor of $f(x) = x^3 + x - 2 = 0$, we can write

$$f(x) = (x - 1)(x^2 + x + 2) = (x - 1)\, g(x),\ g(1) \neq 0.$$

Alternately, we find $f(1) = 0$, $f'(x) = 3x^2 + 1$, $f'(1) = 4 \neq 0$. Hence, $x = 1$ is a simple root of $f(x) = x^3 + x - 2 = 0$.

Multiple roots

Multiple root A number α is a multiple root, of multiplicity m, of $f(x) = 0$, if

$$f(\alpha) = 0, f'(\alpha) = 0, \ldots, f^{(m-1)}(\alpha) = 0, \text{ and } f^{(m)}(\alpha) \neq 0. \tag{1.4}$$

Then, we can write $f(x)$ as

$$f(x) = (x - \alpha)^m g(x), g(\alpha) \neq 0.$$

For example, consider the equation $f(x) = x^3 - 3x^2 + 4 = 0$. We find

$$f(2) = 8 - 12 + 4 = 0, f'(x) = 3x^2 - 6x, f'(2) = 12 - 12 = 0,$$

$$f''(x) = 6x - 6, f''(2) = 6 \neq 0.$$

Hence, $x = 2$ is a multiple root of multiplicity 2 (double root) of $f(x) = x^3 - 3x^2 + 4 = 0$.

We can write $f(x) = (x - 2)^2 (x + 1) = (x - 2)^2 g(x), g(2) = 3 \neq 0.$

Number of roots

Remark 1 A polynomial equation of degree n has exactly n roots, real or complex, simple or multiple, where as a transcendental equation may have one root, infinite number of roots or no root.

We shall derive methods for finding only the real roots.

The methods for finding the roots are classified as (i) direct methods, and (ii) iterative methods.

Direct method

Direct methods These methods give the exact values of all the roots in a finite number of steps (disregarding the round-off errors). Therefore, for any direct method, we can give the total number of operations (additions, subtractions, divisions and multiplications). This number is called the *operational count* of the method.

For example, the roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, can be obtained using the method

$$x = \frac{1}{2a}\left[-b \pm \sqrt{b^2 - 4ac}\right].$$

For this method, we can give the count of the total number of operations.

There are direct methods for finding all the roots of cubic and fourth degree polynomials. However, these methods are difficult to use.

Direct methods for finding the roots of polynomial equations of degree greater than 4 or transcendental equations are not available in literature.

Iterative method

Iterative methods These methods are based on the idea of successive approximations. We start with one or two initial approximations to the root and obtain a sequence of approximations $x_0, x_1, \ldots, x_k, \ldots$, which in the limit as $k \to \infty$, converge to the exact root α. An iterative method for finding a root of the equation $f(x) = 0$ can be obtained as

$$x_{k+1} = \phi(x_k), \quad k = 0, 1, 2, \ldots.. \tag{1.5}$$

This method uses one initial approximation to the root x_0. The sequence of approximations is given by

$$x_1 = \phi(x_0), \quad x_2 = \phi(x_1), \quad x_3 = \phi(x_2), \ldots..$$

The function ϕ is called an *iteration function* and x_0 is called an *initial approximation.*

If a method uses two initial approximations x_0, x_1, to the root, then we can write the method as

$$x_{k+1} = \phi(x_{k-1}, x_k), \quad k = 1, 2, \ldots.. \tag{1.6}$$

Convergence

Convergence of iterative methods The sequence of iterates, $\{x_k\}$, is said to converge to the exact root α, if

$$\lim_{k \to \infty} x_k = \alpha, \quad \text{or} \quad \lim_{k \to \infty} | x_k - \alpha | = 0. \tag{1.7}$$

The error of approximation at the kth iterate is defined as $\varepsilon_k = x_k - \alpha$. Then, we can write (1.7) as

$$\lim_{k \to \infty} | \text{error of approximation} | = \lim_{k \to \infty} | x_k - \alpha | = \lim_{k \to \infty} | \varepsilon_k | = 0.$$

Criterion to stop

Criterion to terminate iteration procedure Since, we cannot perform infinite number of iterations, we need a criterion to stop the iterations. We use one or both of the following criterion:

(i) The equation $f(x) = 0$ is satisfied to a given accuracy or $f(x_k)$ is bounded by an *error tolerance* ε.

$$| f(x_k) | \leq \varepsilon. \tag{1.8}$$

(ii) The magnitude of the difference between two successive iterates is smaller than a given accuracy or an error bound ε.

$$| x_{k+1} - x_k | \leq \varepsilon. \tag{1.9}$$

Generally, we use the second criterion. In some very special problems, we require to use both the criteria.

For example, if we require two decimal place accuracy, then we iterate until $| x_{k+1} - x_k | < 0.005$. If we require three decimal place accuracy, then we iterate until $| x_{k+1} - x_k | < 0.0005$.

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