Lecture 5: Classification and Naive Bayes classifier

Statistical Methods for Data Science

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Today

1. Classification
   - Modeling for classification
   - Training, validation and test
   - Performance evaluation
2. Naive Bayes classifier
   - Multinomial naive Bayes classifier
   - Gaussian naive Bayes classifier
3. Summary

Learning outcome

- Be able to explain classification related terminology: classification, binary/multi-class classification, annotation, training, validation and testing, etc
- Be able to calculate and interpret TP, TN, FP, FN, accuracy, precision, recall, specificity, F1 score
- Understand basic concepts of performance evaluation and comparison of different classifiers
- Be able to explain the Bayes' rule for both multinomial and Gaussian naive Bayes classifiers
- Be able to explain the differences between the multinomial naive Bayes classifier and the Gaussian naive Bayes classifier
- For a given problem, be able to formulate and implement a naive Bayes classifier

Today

Modeling for classification

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Note: if you are a "bottom-up" kind of thinker and have a hard time interpreting g, just imagine it as

$$y = g(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}, \quad \text{for two classes 0 and 1}$$

Parameter estimation

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- In supervised learning, the parameters are estimated from a data set called the **training data set**
- The training data set contains paired data $\{(\boldsymbol{x}_1, y_1), \cdots, (\boldsymbol{x}_N, y_N)\}$, e.g.

  $\{($  $, scoter), ($  $, goldeneye), \cdots, ($  $, scoter)\}$

  where $\boldsymbol{x}_i =$ pixel values in a picture, $y_i \in \{scoter, goldeneye\}$ is called the **ground truth labels**; the data set is called a **labeled data set**

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  where $\boldsymbol{x}_i$ = pixel values in a picture, $y_i \in \{scoter, goldeneye\}$ is called the **ground truth labels**; the data set is called a **labeled data set**
- The targets y_i's in the training set are typically created by humans. The process of creating the ground truth labels is called **annotation** or **labeling**

Training, validation and test

Problem solving in data science

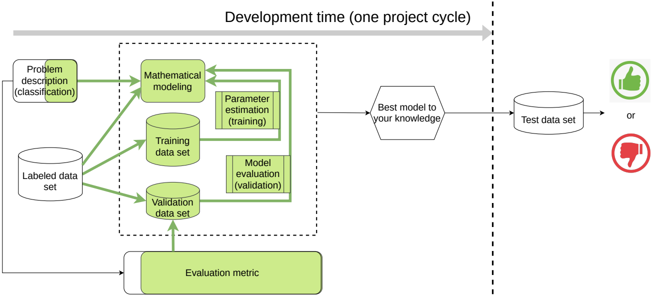

Figure: The (green) shadowed boxes are the actions performed by the data scientist.

During development: training and validation

During development: split the available data set into a training data set and a validation data set

- **Training data set**: to estimate the parameters
- **Validation data**: to evaluate the performance of one or more classifiers

  What is being evaluated:

  - g (the selection of a family of models with the functional form g)

    Philosophies:

    - **Occam's razor**: if multiple models are showing similar performances, the simplest model is preferred
    - **All models are wrong, but some are useful**
  - $\hat{\theta}$ (parameter estimation method)
  - h (hyperparameter tuning)

After development: testing

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- **Test data set**: the real deal - you DO NOT have access to it during development

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For the project, do not use the "test data set" for development! Don't even look at it before testing your model!!! 😡 Because in reality, you do not have access to it!

Performance evaluation

How to split the data

Given a labeled data set $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)\}$, split the data set into a **training data set** and a **validation data set**

- Training-validation split
  - $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$, $\{(x_5, y_5), (x_6, y_6)\}$
- K-fold cross validation, e.g. 3-fold
  - $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$, $\{(x_5, y_5), (x_6, y_6)\}$
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  - $\{(x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)\}$, $\{(x_1, y_1), (x_2, y_2)\}$
- Leave-one-out cross validation
  - $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5)\}$, $\{(x_6, y_6)\}$
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  - $\{(x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)\}$, $\{(x_1, y_1)\}$

Validation: four outcomes

- Given a **binary classification** problem, a trained classifier $g(x; \hat{\theta} \mid h)$ and a validation data set containing pairs (x, y)
- Positive: $y = 1$; negative: $y = 0$

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  - **True Positive (TP):** count(ground truth $y = 1$, classifier output $\hat{y} = 1$)
  - **False Positive (FP):** count(ground truth $y = 0$; classifier output $\hat{y} = 1$)
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Confusion matrix (contingency table)

	$y = 1$	$y = 0$
$\hat{y} = 1$	**TP**	**FP** (Type I error)
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- What is count$(y = 1)$? (15 sec) **TP+FN**

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- What is count$(y = 0)$? (15 sec)

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- What is count($y = 1$)? (15 sec) **TP**+**FN**
- What is count($y = 0$)? (15 sec) **TN**+**FP**
- What is size of the entire data set, i.e. count($y = 1$)+count($y = 0$)? (15 sec)

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- What is count$(y = 0)$? (15 sec) **TN+FP**
- What is size of the entire data set, i.e. count$(y = 1)$+count$(y = 0)$? (15 sec) **TP+FN+TN+FP**

Evaluation metric

- Accuracy:

$$accuracy = \frac{\mathbf{TP}+\mathbf{TN}}{count(y=1)+count(y=0)} = \frac{\mathbf{TP}+\mathbf{TN}}{\mathbf{TP}+\mathbf{TN}+\mathbf{FP}+\mathbf{FN}}$$

- True Positive Rate (recall, sensitivity):

$$TPR = \frac{\mathbf{TP}}{count(y=1)} = \frac{\mathbf{TP}}{\mathbf{TP}+\mathbf{FN}}$$

- True Negative Rate (specificity):

$$TNR = \frac{\mathbf{TN}}{count(y=0)} = \frac{\mathbf{TN}}{\mathbf{TN}+\mathbf{FP}}$$

- Precision:

$$precision = \frac{\mathbf{TP}}{count(\hat{y}=1)} = \frac{\mathbf{TP}}{\mathbf{TP}+\mathbf{FP}}$$

- F1 score:

$$F = 2 \times \frac{precision \times recall}{precision + recall}$$

- More from scikit-learn:
  `www.scikit-learn.org/stable/modules/classes.html#module-sklearn.metrics`

Evaluation metric (cont.)

- **Use with caution** especially for **imbalanced** data set, e.g. data from medical measurements typically contains more negative data (90% healthy volunteers) than positive data (10% patients); the data set is then imbalanced
  - Terrible metric: accuracy
  - Okay metric:
    - Precision vs recall
    - Sensitivity vs specificity
    - F1 score

  Reference: read the data science design manual, section 7.4.1
- In this lecture, we only consider **binary classification** $c \in \{0, 1\}$
- In the multi-class case $c \in \{1, \cdots, C\}$:
  - **Macro**: the metrics are computed for each class c and then the average is calculated
  - **Micro**: the metrics are computed globally for all classes

Finding a good classifier

- **Given**: a labeled data set with $y_i \in \{0, 1\}$

$$\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)\}$$

- **Task**: use this data set to build a good classifier
- **Method**:
  1) Come up with M models g_m, $m = 1, \cdots, M$. Let $\hat{y}_i^m = g_m(x_i; \theta \mid h)$

     For the sake of simplicity (not to overwhelm you with complex notations), we neglect the different choices of h. In practice, when you choose a different set of hyperparameters h, the model needs to be evaluated as a different model.
  2) Split the data set (cf. page 13) into a **training data set** and a **validation data set**
     - Training-validation split
     - K-fold cross validation
     - Leave-one-out cross validation
  3) Estimate parameters $\hat{\theta}$ using the **training data set**
     - Similar to the choice of h, different parameter estimation techniques will give a different $\hat{\theta}$, which in turn needs to be evaluated.
  4) **Evaluate** $g_m(x; \hat{\theta} \mid h)$ on the **validation data set**
     - $M = 1$: evaluation of one classifier
     - $M > 1$: comparison of multiple classifiers

How to evaluate a classifier?

Given a classifier $g(x; \hat{\theta} \mid h)$,

- **Goal:** to evaluate the performance of the classifier $g(x; \hat{\theta} \mid h)$
- **Steps:**
  - Step 1: compute $\hat{y}_i = g(x_i; \hat{\theta} \mid h)$ on the validation data set $\{(x_1, y_1), \cdots, (x_n, y_n)\}$
  - Step 2: compute an evaluation metric (e.g. page 16), denoted by s
- **Interpretation:**
  - Case 1: one validation data set (e.g. training-validation split, leave-one-out cross validation) - only one s, e.g. $s = 0.92$
  - Case 2: multiple validation data sets (e.g. K-fold cross validation) - a set of s, e.g. for 3-fold cross validation, we have 3 validation datasets, which gives us 3 results $\{s_1, s_2, s_3\}$. We can then compute sample statistics (e.g. sample mean, sample standard deviation) from this set.

How to compare two classifiers?

Given two classifiers $g_1(x; \hat{\theta} \mid h)$ and $g_2(x; \hat{\xi} \mid t)$ (note: I use different symbols $\hat{\theta}$ and $\hat{\xi}$ to indicate that they might have different parameters; likewise, h and t for different hyperparameters)

- **Goal:** to check which classifier is better, $g_1(x; \hat{\theta} \mid h)$ or $g_2(x; \hat{\xi} \mid t)$
- **Steps:**
  - Step 1: compute $\hat{y}_i^1 = g_1(x_i; \hat{\theta} \mid h)$ and $\hat{y}_i^2 = g_2(x_i; \hat{\xi} \mid t)$ on the validation data set $\{(x_1, y_1), \cdots, (x_n, y_n)\}$
  - Step 2: compute an evaluation metric (e.g. page 16) for each classifier, denoted by s^j for $j = 1, 2$
  - Step 3: compare s^1 and s^2 to choose which classier is better, $g_1(x; \hat{\theta} \mid h)$ or $g_2(x; \hat{\xi} \mid t)$

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    Example: say we choose the accuracy as the metric

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    - Still $g_1(x; \hat{\theta} \mid h)$ with $s^1 \approx s^2$?

  We can use **hypothesis testing** (lecture 7) to determine which classifier is better.
    - Case 1: one validation data set (e.g. training-validation split, leave-one-out cross validation) - **McNemar's test**
    - Case 2: multiple validation data sets (e.g. K-fold cross validation) - **paired t-test**

Summary

- Classification, binary/multi-class classification
- Training, validation and test data set
- TP, TN, FP, FN
- Evaluation metrics
- Basic concepts of performance evaluation and comparison of different classifiers

Today

Naive Bayes classifier

- Multinomial naive Bayes classifier (categorical y, categorical x)
- Gaussian naive Bayes classifier (categorical y, continuous x)

Naive Bayes classifier

- Multinomial naive Bayes classifier (categorical y, categorical x)
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Bayes' rule and MAP for parameter estimation

- In parameter estimation:

$$f_{\Theta|data}(\theta \mid data) = \frac{\overbrace{f_{data|\Theta}(data \mid \theta)}^{\text{likelihood}}\ \overbrace{f_{\Theta}(\theta)}^{\text{prior}}}{\underbrace{f_{data}(data)}_{\text{normalization constant}}}$$

$$\hat{\theta}_{MAP} = \arg\max_{\theta}\ f_{data|\Theta}(data \mid \theta) f_{\Theta}(\theta)$$

Bayes' rule and MAP for naive Bayes classifier

Let x be the input variables and y the target,

- In multinomial naive Bayes classifier (categorical y, categorical x):

$$P(Y = y \mid X = x) = \frac{\overbrace{P(X = x \mid Y = y)}^{\text{likelihood}} \overbrace{P(Y = y)}^{\text{prior}}}{\underbrace{P(X = x)}_{\text{normalization constant}}}$$

$$\hat{y}_{MAP} = \arg\max_{y} P(X = x \mid Y = y)P(Y = y)$$

- In Gaussian naive Bayes classifier (categorical y, continuous x):

$$P(Y = y \mid X = x) = \frac{\overbrace{f_{X|Y=y}(x \mid Y = y)}^{\text{likelihood}} \overbrace{P(Y = y)}^{\text{prior}}}{\underbrace{f_X(x)}_{\text{normalization constant}}}$$

$$\hat{y}_{MAP} = \arg\max_{y} f_{X|Y=y}(x \mid Y = y)P(Y = y)$$

Multinomial naive Bayes classifier

Example 1: spam filter

An email server would like to build a spam filter for its clients

- **Input variables $\mathbf{x}$**: the body of an email
- **Training data**: there are 1000 emails labeled either "spam" or "not spam"
- **Prediction task**: for a new email, the server would like to identify if it is a spam

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- **Model g**: **Multinomial naive Bayes classifier**

Modeling for spam filter

- **Prediction y:** spam or not spam
- **Variables x_i, $i = 1, \cdots, n$:** the body of an email with n words
  - **Assumptions:**
    - the words are independent - a bag of words - **NAIVE!**
    - the words are generated from a categorical distribution; each category is a word from a vocabulary
- **Model g: multinomial naive Bayes classifier**

$$\hat{y} = g(x_1, \cdots, x_n) = \underset{c \in \{spam,\ not\ spam\}}{\arg\max} P(c) \prod_{i=1}^{n} P(x_i \mid c)$$

  where $P(c)$ is the prior and $\prod_{i=1}^{n} P(x_i \mid c)$ is the likelihood under the assumptions
  Note: it is the maximum a posteriori estimation of the label (spam or not spam)
- **Hyperparameters h:** smoothing factor α (explained soon)
- **Parameters θ:** $P(c)$, a vocabulary (if not given) and $P(word \mid c)$ for all words in the vocabulary

Parameter estimation (training)

In this demo, we only consider 7 training emails for illustration purposes

- **Training data**: there are 7 emails labeled either "spam" or "not spam"
    - Email 1 (not spam): "Hi see you at dinner."
    - Email 2 (spam): "Buy lottery!"
    - Email 3 (not spam): "Hi, wanna have dinner?"
    - Email 4 (not spam): "Hi you, nice dinner today!"
    - Email 5 (spam): "Wanna get rich today?"
    - Email 6 (not spam): "Lottery dinner?"
    - Email 7 (spam): "Win lottery; get rich today!"

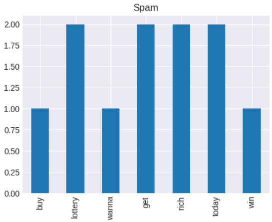
Parameter estimation (training) (cont.)

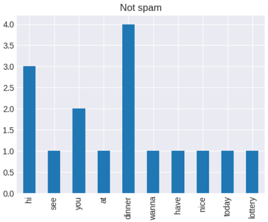
- Step 1: Estimate the likelihood $P(word_i \mid spam)$ and $P(word_i \mid not\ spam)$
  1.1 Build a vocabulary containing all unique words from the 7 emails:

$$V = \{\text{buy, lottery, wanna, get, rich, today, win, hi, see, you, at, dinner, have, nice}\}$$

  1.2 Count how many times each word appears in both spams and not spams:

	buy	lottery	wanna	...	dinner	have	nice
Spam	1	2	1	...	0	0	0
Not spam	0	1	1	...	4	1	1





Parameter estimation (training) (cont.)

- Step 1 (cont.):
  1.3 Count how many words in total for each class:
    - Spam: 11 words
    - Not spam: 16 words

  1.4 Estimate the **likelihood** $P(word_i \mid spam)$ **and** $P(word_i \mid not\ spam)$ for all $word_i \in V$:

	buy	lottery	...	you	at	dinner	have	nice
Spam $P(word_i \mid spam)$	$\frac{1}{11}$	$\frac{2}{11}$	...	$\frac{0}{11}$	$\frac{0}{11}$	$\frac{0}{11}$	$\frac{0}{11}$	$\frac{0}{11}$
Not spam $P(word_i \mid not\ spam)$	$\frac{0}{16}$	$\frac{1}{16}$	...	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

- Step 2: Estimate the **prior** $P(spam)$ **and** $P(not\ spam)$
  - Spam $P(spam)$: $\frac{\text{\# of spams}}{\text{\# of total emails}} = \frac{3}{7}$
  - Not spam $P(not\ spam)$: $\frac{\text{\# of not spams}}{\text{\# of total emails}} = \frac{4}{7}$

Classify a new email

- Construct the multinomial naive Bayes classifier
  The naive Bayes classifier is a function of a given email. Let s_{spam} and $s_{not\ spam}$ be the posterior without the normalization constant

$$s_{spam} = P(spam) \prod_{\forall word \in email} P(word \mid spam)$$

$$s_{not\ spam} = P(not\ spam) \prod_{\forall word \in email} P(word \mid not\ spam)$$

  - If $s_{spam} > s_{not\ spam}$: the email is spam
  - If $s_{spam} \leq s_{not\ spam}$: the email is not spam

Classify a new email (cont.)

- Compute the posterior of this email
  - Spam: $P(spam \mid an\ email) = \frac{s_{spam}}{s_{spam}+s_{not\ \ spam}}$
  - Not spam: $P(not\ spam \mid an\ email) = \frac{s_{not\ \ spam}}{s_{spam}+s_{not\ \ spam}}$

  These are the probability of the email being a spam and not a spam, respectively.

One problemo

Say, the email is "You! Lottery! Lottery! Lottery!!" and it is clearly a spam. But when we compute the likelihood (cf. page 31),

$$s_{spam} = P(spam)P(''you'' \mid spam)P(''lottery'' \mid spam)^3$$

$$s_{not\ spam} = P(not\ spam)P(''you'' \mid not\ spam)P(''lottery'' \mid not\ spam)^3$$

	buy	lottery	...	you	at	dinner	have	nice
Spam $P(word_i \mid spam)$	$\frac{1}{11}$	$\frac{2}{11}$	...	$\frac{0}{11}$	$\frac{0}{11}$	$\frac{0}{11}$	$\frac{0}{11}$	$\frac{0}{11}$
Not spam $P(word_i \mid not\ spam)$	$\frac{0}{16}$	$\frac{1}{16}$	...	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

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Spam $P(word_i \mid spam)$	$\frac{1}{11}$	$\frac{2}{11}$	...	$\frac{0}{11}$	$\frac{0}{11}$	$\frac{0}{11}$	$\frac{0}{11}$	$\frac{0}{11}$
Not spam $P(word_i \mid not\ spam)$	$\frac{0}{16}$	$\frac{1}{16}$	...	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

- $s_{spam} = \frac{3}{7} \times \frac{0}{11} \times \frac{2}{11}^3 = \mathbf{0}$

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Not spam $P(word_i \mid not\ spam)$	$\frac{0}{16}$	$\frac{1}{16}$	...	$\frac{2}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	$\frac{1}{16}$

- $s_{spam} = \frac{3}{7} \times \frac{0}{11} \times \frac{2}{11}^3 = \mathbf{0}$
- $s_{not\ spam} = \frac{4}{7} \times \frac{2}{16} \times \frac{1}{16}^3 > s_{spam}$

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- $s_{not\ spam} = \frac{4}{7} \times \frac{2}{16} \times \frac{1}{16}^3 > s_{spam}$

This email will be classified as not a spam simply because the word "you" has never appeared in spam emails.

Solution to the problemo

Smoothing or discounting with **hyperparameter** α: we need to alter Step 1.4

Let $\mid V \mid$ be the size of the vocabulary

	buy	lottery	...	you	at	dinner	have	nice
Spam $P(word_i \mid spam)$	$\frac{1+\alpha}{11+\alpha\lvert V\rvert}$	$\frac{2+\alpha}{11+\alpha\lvert V\rvert}$	...	$\frac{0+\alpha}{11+\alpha\lvert V\rvert}$	$\frac{0+\alpha}{11+\alpha\lvert V\rvert}$	$\frac{0+\alpha}{11+\alpha\lvert V\rvert}$	$\frac{0+\alpha}{11+\alpha\lvert V\rvert}$	$\frac{0+\alpha}{11+\alpha\lvert V\rvert}$
Not spam $P(word_i \mid not\ spam)$	$\frac{0+\alpha}{16+\alpha\lvert V\rvert}$	$\frac{1+\alpha}{16+\alpha\lvert V\rvert}$	...	$\frac{2+\alpha}{16+\alpha\lvert V\rvert}$	$\frac{1+\alpha}{16+\alpha\lvert V\rvert}$	$\frac{4+\alpha}{16+\alpha\lvert V\rvert}$	$\frac{1+\alpha}{16+\alpha\lvert V\rvert}$	$\frac{1+\alpha}{16+\alpha\lvert V\rvert}$

Let $\alpha = 1$,

- $s_{spam} = \frac{3}{7} \times \frac{1}{25} \times \frac{3}{25}^3 = 0.0000296$
- $s_{not\ spam} = \frac{4}{7} \times \frac{3}{30} \times \frac{2}{30}^3 = 0.0000169 < s_{spam}$

Note: these are very small values due to the product of small values. Typically, we apply the logarithm function to avoid underflow as in MAP and MLE (cf. lecture 4).

Summary: Bayes' rule for multinomial naive Bayes classifier

- **Data**: categorical y, categorical x
- **Random variable**: discrete Y, discrete X

$$P(Y = y \mid X = x) = \frac{P(X = x \mid Y = y)P(Y = y)}{P(X = x)}$$

$$\hat{y}_{MAP} = \arg\max_{y} P(X = x \mid Y = y)P(Y = y)$$

Summary: multinomial naive Bayes classifier

- **Prediction y:** categorical data $y \in \{1, \cdots, C\}$
- **Variables x_i, $i = 1, \cdots, n$:** categorical data $x_i \in V$, where V is the vocabulary $V = \{w_1, \cdots, w_K\}$ given K unique categories
  - **Assumptions:**
    - x_i's are independent - **NAIVE!**
    - x_i follows a categorical distribution

  Note: here n is the size of the input data, e.g. the length of a document
- **Model g:**

$$\hat{y} = g(x_1, \cdots, x_n) = \underset{c \in \{1, \cdots, C\}}{\arg\max}\, P(c) \prod_{i=1}^{n} P(x_i \mid c)$$

  where $P(c)$ is the prior and $\prod_{i=1}^{n} P(x_i \mid c)$ is the likelihood under the assumptions
- **Hyperparameters h:** smoothing factor α
- **Parameters θ:** $P(c)$, V (if not given) and $P(w_i \mid c)$ for all $w_i \in V$

Summary: multinomial naive Bayes classifier (cont.)

- **Parameter estimation (training)**:
  Given the vocabulary $V = \{w_k\}_{k=1}^{K}$ and a training data set $\{(b_1, y_1), \cdots, (b_N, y_N)\}$, where each b_j contains a list of words. Let $N_c = count(y_j = c)$.
  - Likelihood $P(w_i \mid c)$ for each w_i:

$$P(w_i \mid c) = \frac{count(occurrences\ of\ \ w_i\ \ in\ all\ \ b_j\ \ for\ \ y_j = c) + \alpha}{count(all\ \ words\ \ from\ \ class\ \ c) + \alpha K}$$

  - Prior $P(c)$:

$$P(c) = \frac{N_c}{N}$$

Gaussian naive Bayes classifier

Example 2: real-time customer insight

An online shop is selling a new gaming computer

- **Prediction task** y: for a customer browsing this computer, the shop would like to predict if the customer will complete the transaction. If the prediction says no, the shop will perform certain actions, such as
  - proposing a discount to the customer
  - threatening the customer by showing an irritating message, e.g. "there are 20 people looking at this item right now"
  - offering a free item to encourage the transaction

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  - offering a free item to encourage the transaction
- **Input variables** x: the shop has the following information about the customers who are browsing this computer:
  - All kinds of personal information from different sources (Google, Facebook, via e.g. cookies, IP address, the version of your browser, etc)
  - In this example, they choose the following features as the input variables: **1)** average time they stay on Facebook everyday; **2)** how much money they spend on games (yes they have access to their steam account); **3)** daily active time on average (and yes they have access to their smart watch)

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- **Training data**:
  - The aforementioned personal information recorded from 1000 customers
  - If they have completed the transaction of purchasing the computer or not

Modeling for real-time customer insight

- **Prediction** y**:** complete transaction or drop out before paying
- **Variables** $\boldsymbol{x} = [x_1, x_2, x_3]$**:** x_1 = duration (hour) on Facebook, x_2 = money (k dollar) spent on games, x_3 = active time (hour)
  - **Assumptions:**
    - x_1, x_2, x_3 are independent - **NAIVE!**
    - given data from class c, x_i is generated from a Gaussian distribution with PDF

$$f_i(x_i \mid c) = \frac{1}{\sqrt{2\pi\sigma_{c,i}^2}} e^{-\frac{(x_i - \mu_{c,i})^2}{2\sigma_{c,i}^2}}$$

- **Model** g**: Gaussian naive Bayes classifier**

$$\hat{y} = \underset{c \in \{complete,\ drop\ \ out\}}{\arg\max}\ P(c) \prod_{i=1}^{d} f_i(x_i \mid c)$$

  where $P(c)$ is the prior and $\prod_{i=1}^{d} f_i(x_i \mid c)$ is the likelihood under the assumptions
  Note: it is the maximum a posteriori estimation of the label (complete or drop out).
- **Parameters** θ**:** $P(c)$ and $\mu_{c,i}$, $\sigma_{c,i}$ in the likelihood $f_i(x_i \mid c)$ for all variable i and all classes c

Parameter estimation (training)

In this demo, we only consider 5 customers in the training data for illustration purposes

- **Training data**: there are 5 customers:
  - $\boldsymbol{x}_1 = [x_1^1, x_2^1, x_3^1] = [2.44, 2.48, 2.64]$, $y_1 = 1$ drop out
  - $\boldsymbol{x}_2 = [x_1^2, x_2^2, x_3^2] = [9.77, 6.82, 0.55]$, $y_2 = 0$ complete
  - $\boldsymbol{x}_3 = [x_1^3, x_2^3, x_3^3] = [2.15, 8.05, 3.11]$, $y_3 = 1$ drop out
  - $\boldsymbol{x}_4 = [x_1^4, x_2^4, x_3^4] = [1.96, 3.78, 3.75]$, $y_4 = 1$ drop out
  - $\boldsymbol{x}_5 = [x_1^5, x_2^5, x_3^5] = [8.31, 7.93, 0.16]$, $y_5 = 0$ complete

Parameter estimation (training) (cont.)

- Step 1: Estimate $\mu_{c,i}$, $\sigma_{c,i}$ in the likelihood

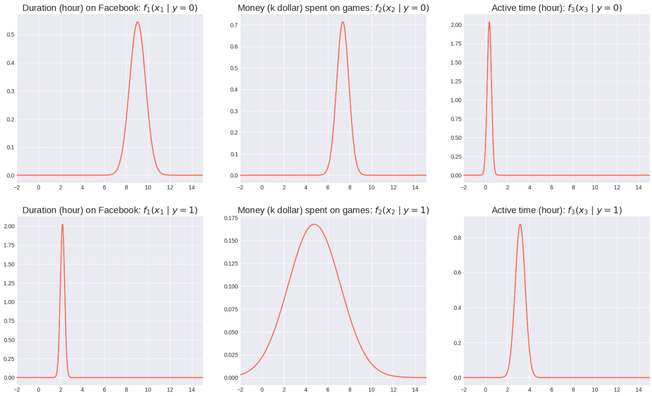
$$f_i(x_i \mid c) = \frac{1}{\sqrt{2\pi\sigma_{c,i}^2}} e^{-\frac{(x_i-\mu_{c,i})^2}{2\sigma_{c,i}^2}}$$

for all variable i and $c \in \{complete, drop\ out\}$

  - For each $i = 1, 2, 3$, collect all x_i^j for $y_j = drop\ out$. Compute the sample mean $\hat{\mu}_{drop\ out,i}$ and the sample standard deviation $\hat{\sigma}_{drop\ out,i}$.
  - For each $i = 1, 2, 3$, collect all x_i^j for $y_j = complete$. Compute the sample mean $\hat{\mu}_{complete,i}$ and the sample standard deviation $\hat{\sigma}_{complete,i}$.

Parameter estimation (training) (cont.)

Estimated likelihood



Parameter estimation (training) (cont.)

- Step 2: Estimate the **prior** $P(complete)$ **and** $P(drop\ out)$
  - Customers who have completed transaction $P(complete)$:

$$P(complete) = \frac{\#\text{ of complete}}{\#\text{ of customers}} = \frac{2}{5}$$

  - Customers who have dropped out before paying $P(drop\ out)$:

$$P(drop\ out) = \frac{\#\text{ of drop out}}{\#\text{ of customers}} = \frac{3}{5}$$

Classify a new customer

- Construct the Gaussian naive Bayes classifier
  The Gaussian naive Bayes classifier is a function of a given customer, i.e. $\boldsymbol{x} = [x_1, x_2, x_3]$. Let $s_{complete}$ and $s_{drop\ out}$ be the posterior without the normalization constant

$$s_{complete} = P(complete) \prod_{i=1}^{3} f_i(x_i \mid complete)$$

$$s_{drop\ out} = P(drop\ out) \prod_{i=1}^{3} f_i(x_i \mid drop\ out)$$

  - If $s_{complete} > s_{drop\ out}$: the customer will complete the transaction
  - If $s_{complete} \leq s_{drop\ out}$: the customer will drop out

Classify a new customer (cont.)

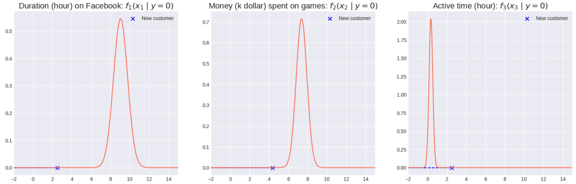
- Compute the posterior of this customer
  - Complete: $P(complete \mid a\ customer) = \frac{s_{complete}}{s_{complete} + s_{drop\ \ out}}$
  - Drop out: $P(drop\ out \mid a\ customer) = \frac{s_{drop\ \ out}}{s_{complete} + s_{drop\ \ out}}$

  These are the probability of the customer completing the transaction and dropping out, respectively.

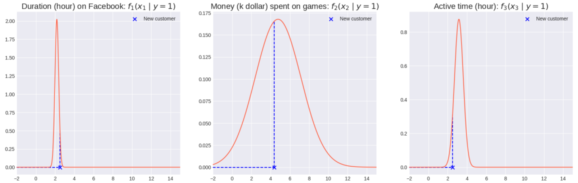
Classify a new customer (cont.)

For a new customer: hours spent on Facebook $x_1 = 2.51$; money spent on games $x_2 = 4.38$; active time $x_3 = 2.51$

- The likelihood of this customer completing a transaction:



- The likelihood of this customer dropping out before paying:



Classify a new customer (cont.)

- Compute the scores:

$$
\begin{aligned}
s_{complete} &= P(complete)\prod_{i=1}^{3} f_i(x_i \mid complete) \\
&= \frac{2}{5} f_1(2.51 \mid complete) f_2(4.38 \mid complete) f_3(2.51 \mid complete) \\
&\approx 0
\end{aligned}
$$

$$
\begin{aligned}
s_{drop\ out} &= P(drop\ out)\prod_{i=1}^{3} f_i(x_i \mid drop\ out) \\
&= \frac{3}{5} f_1(2.51 \mid drop\ out) f_2(4.38 \mid drop\ out) f_3(2.51 \mid drop\ out) \\
&= 0.016
\end{aligned}
$$

Classify a new customer (cont.)

- Compute the scores:

$$\begin{aligned} s_{complete} &= P(complete)\prod_{i=1}^{3} f_i(x_i \mid complete) \\ &= \frac{2}{5} f_1(2.51 \mid complete) f_2(4.38 \mid complete) f_3(2.51 \mid complete) \\ &\approx 0 \end{aligned}$$

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- $s_{complete} < s_{drop\ out}$: the customer will drop out before paying

Classify a new customer (cont.)

- Compute the scores:

$$
\begin{aligned}
s_{complete} &= P(complete)\prod_{i=1}^{3} f_i(x_i \mid complete) \\
&= \frac{2}{5} f_1(2.51 \mid complete) f_2(4.38 \mid complete) f_3(2.51 \mid complete) \\
&\approx 0
\end{aligned}
$$

$$
\begin{aligned}
s_{drop\ out} &= P(drop\ out)\prod_{i=1}^{3} f_i(x_i \mid drop\ out) \\
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&= 0.016
\end{aligned}
$$

- $s_{complete} < s_{drop\ out}$: the customer will drop out before paying
- Therefore, the online shop will send a message to threaten this customer.

Classify a new customer (cont.)

- Compute the scores:

$$\begin{aligned}
s_{complete} &= P(complete)\prod_{i=1}^{3} f_i(x_i \mid complete) \\
&= \frac{2}{5} f_1(2.51 \mid complete) f_2(4.38 \mid complete) f_3(2.51 \mid complete) \\
&\approx 0
\end{aligned}$$

$$\begin{aligned}
s_{drop\ out} &= P(drop\ out)\prod_{i=1}^{3} f_i(x_i \mid drop\ out) \\
&= \frac{3}{5} f_1(2.51 \mid drop\ out) f_2(4.38 \mid drop\ out) f_3(2.51 \mid drop\ out) \\
&= 0.016
\end{aligned}$$

- $s_{complete} < s_{drop\ out}$: the customer will drop out before paying
- Therefore, the online shop will send a message to threaten this customer.

$\mathcal{THE} \;\; \mathcal{END}$

Summary: Bayes' rule for Gaussian naive Bayes classifier

- **Data**: categorical y, continuous x
- **Random variable**: discrete Y, continuous X

$$P(Y = y \mid X = x) = \frac{f_{X|Y=y}(x \mid Y = y)P(Y = y)}{f_X(x)}$$

$$\hat{y}_{MAP} = \arg\max_y f_{X|Y=y}(x \mid Y = y)P(Y = y)$$

Summary: Gaussian naive Bayes classifier

- **Prediction** y**:** categorical data $y \in \{1, \cdots, C\}$
- **Variables** x_i**,** $i = 1, \cdots, d$**:** continuous numerical data $x_i \in \mathbb{R}$
  - **Assumption:**
    - x_i's are independent - **NAIVE!**
    - x_i follows a Gaussian distribution
- **Model** g**:**

$$\begin{aligned} \hat{y} &= g(x_1, \cdots, x_d) \\ &= \underset{c \in \{1, \cdots, C\}}{\arg\max}\, P(c) \prod_{i=1}^{d} f_i(x_i \mid y = c) \end{aligned}$$

where $P(c)$ is the prior and $\prod_{i=1}^{d} f_i(x_i \mid y = c)$ is the likelihood under the assumptions with $f_i(x_i \mid y = c) = \frac{1}{\sqrt{2\pi\sigma_{c,i}^2}} e^{-\frac{(x_i - \mu_{c,i})^2}{2\sigma_{c,i}^2}}$

- **Parameters** θ**:** $P(c)$, $\mu_{c,i}$, $\sigma_{c,i}$ in $f_i(x_i \mid y = c)$ for all c and i

Summary: Gaussian naive Bayes classifier (cont.)

- **Parameter estimation (training)**:
  Given a training data set $\{(\boldsymbol{x}_1, y_1), \cdots, (\boldsymbol{x}_N, y_N)\}$, where each $\boldsymbol{x}_j = [x_1^j, \cdots, x_d^j]$ is a vector containing all the features for one data point. Let $N_c = count(y_j = c)$.
  - $\mu_{c,i}$, $\sigma_{c,i}$ in the likelihood $f_i(x_i \mid y = c)$ for all variable i and all classes c:

$$\hat{\mu}_{c,i} = \frac{1}{N_c} \sum_{t=1}^{N_c} x_i^t$$

$$\hat{\sigma}_{c,i} = \sqrt{\frac{1}{N_c - 1} \sum_{t=1}^{N_c} (x_i^t - \hat{\mu}_{c,i})^2}$$

    for all $t \in$ class c
  - Prior $P(c)$:

$$P(c) = \frac{N_c}{N}$$

Naive Bayes: pros and cons

- Pros:
  - Highly scalable
  - Simple
  - Interpretable
  - Easy to implement
  - Working fine for some use cases (e.g. spam filter)
- Cons:
  - Too simple for most use cases
  - Assumptions are too naive

A word on model complexity

- Models with high complexity:
  - Smart-ass models: they usually suffer from overfitting when the training data set is "small", i.e. working well on the training data set, but generalizing poorly on unseen data
  - Low bias (good)
  - High variance (bad)
  - Regularization is needed during training (cf. lecture 4 MLE vs MAP)
- Simple models:
  - They usually suffer less from overfitting, i.e. they might not work very well on the training data set; they are not performing much worse on unseen data
  - High bias (bad)
  - Low variance (good)

Today

Summary

So far:

- Data types and data containers
- Descriptive data analysis: descriptive statistics, visualization
- Probability distributions, events, random variables, PMF, PDF, parameters
- CDF, Q-Q plot, how to compare two distributions (data vs theoretical, data vs data)
- Modeling
- Parameter estimation: maximum likelihood estimation (MLE) and maximum a posteriori estimation (MAP)
- Classification, multinomial naive Bayes classifier, Gaussian naive Bayes classifier

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Next:

- Interval estimation, confidence interval

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Before next lecture:

- Gaussian distribution, sample mean, CDF, quantiles

Only in this one lecture! Sorry!