

Lecture 8: Hypothesis testing part II

Statistical Methods for Data Science

Yinan Yu

Department of Computer Science and Engineering

Dec 1, 2021

Today

1 Test statistics and hypothesis tests

- z-test
- One-sample t-test
- Two-sample t-test
- Paired t-test
- Binomial test
- McNemar's test
- Summary

2 Compare two classifiers

3 Summary

Learning outcome

- Be able to explain the following hypothesis tests
 - One-sample and two-sample z-test
 - One-sample and two-sample t-test
 - Paired t-test
 - Binomial test (exact, approximate)
 - McNemar's test (exact, approximate)

For each of these tests, be able to describe the typical set up for the experiment, the general purpose of the test, data produced by the experiment, random variables, parameter of interest, null hypothesis, alternative hypothesis, test statistic, null distribution, the computation of p -value

- Be able to generalize the learning routine to new hypothesis tests
- Be able to compare two classifiers using the paired t-test and McNemar's test for different scenarios

Today

1 Test statistics and hypothesis tests

- z-test
- One-sample t-test
- Two-sample t-test
- Paired t-test
- Binomial test
- McNemar's test
- Summary

2 Compare two classifiers

3 Summary



Remark

- Recall that in this course, we only consider H_0 **with an equal sign in them**, i.e. the **null distribution is fully specified**; the description of H_0 is based on this assumption

Remark

- Recall that in this course, we only consider H_0 **with an equal sign in them**, i.e. the **null distribution is fully specified**; the description of H_0 is based on this assumption
- For **symmetric null distributions**, e.g. **standard Gaussian distribution**, **student's t distribution**, **binomial distribution with $p = 0.5$** , etc, we only illustrate examples with the two-tailed alternative hypothesis H_A in this lecture without loss of generality; the one-tailed version can be easily derived

Remark

- Recall that in this course, we only consider H_0 **with an equal sign in them**, i.e. the **null distribution is fully specified**; the description of H_0 is based on this assumption
- For **symmetric null distributions**, e.g. **standard Gaussian distribution**, **student's t distribution**, **binomial distribution with $p = 0.5$** , etc, we only illustrate examples with the two-tailed alternative hypothesis H_A in this lecture without loss of generality; the one-tailed version can be easily derived
- For the **exact binomial test with $p \neq 0.5$** , the null distribution is not symmetric; in this case, the computation of the two-tailed p -value is not uniquely defined; in this lecture, we will not go into details for these cases; we will only look at the one-tailed tests for asymmetric binomial null distributions

Remark

- Recall that in this course, we only consider H_0 **with an equal sign in them**, i.e. the **null distribution is fully specified**; the description of H_0 is based on this assumption
- For **symmetric null distributions**, e.g. **standard Gaussian distribution**, **student's t distribution**, **binomial distribution with $p = 0.5$** , etc, we only illustrate examples with the two-tailed alternative hypothesis H_A in this lecture without loss of generality; the one-tailed version can be easily derived
- For the **exact binomial test with $p \neq 0.5$** , the null distribution is not symmetric; in this case, the computation of the two-tailed p -value is not uniquely defined; in this lecture, we will not go into details for these cases; we will only look at the one-tailed tests for asymmetric binomial null distributions
- For each hypothesis test, the purpose of the Python code snippet is to provide a better understanding of the calculation; in practice, there are alternative libraries and built-in functions for these tests that might result in a more compact implementation

Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

- **Typical set up for the experiment**
 - **Test subjects**, e.g. the number of samples, the number of groups, etc
 - Description of the **experiment** and the **result**
 - Description of the **data type** produced in the result

Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

- **Typical set up for the experiment**
 - **Test subjects**, e.g. the number of samples, the number of groups, etc
 - Description of the **experiment** and the **result**
 - Description of the **data type** produced in the result
- **Purpose**: the general purpose of the test

Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

- **Typical set up for the experiment**
 - **Test subjects**, e.g. the number of samples, the number of groups, etc
 - Description of the **experiment** and the **result**
 - Description of the **data type** produced in the result
- **Purpose**: the general purpose of the test
- **Data**: symbolic description of the data produced by the experiment
- **Random variable** and **assumption** corresponding to the data

Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

- **Typical set up for the experiment**
 - **Test subjects**, e.g. the number of samples, the number of groups, etc
 - Description of the **experiment** and the **result**
 - Description of the **data type** produced in the result
- **Purpose**: the general purpose of the test
- **Data**: symbolic description of the data produced by the experiment
- **Random variable** and **assumption** corresponding to the data
- **Parameter of interest** and the **estimates**

Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

- **Typical set up for the experiment**
 - **Test subjects**, e.g. the number of samples, the number of groups, etc
 - Description of the **experiment** and the **result**
 - Description of the **data type** produced in the result
- **Purpose**: the general purpose of the test
- **Data**: symbolic description of the data produced by the experiment
- **Random variable** and **assumption** corresponding to the data
- **Parameter of interest** and the **estimates**
- **Hypotheses** H_0 and H_A

Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

- **Typical set up for the experiment**
 - **Test subjects**, e.g. the number of samples, the number of groups, etc
 - Description of the **experiment** and the **result**
 - Description of the **data type** produced in the result
- **Purpose**: the general purpose of the test
- **Data**: symbolic description of the data produced by the experiment
- **Random variable** and **assumption** corresponding to the data
- **Parameter of interest** and the **estimates**
- **Hypotheses** H_0 and H_A
- **Test statistic**

Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

- **Typical set up for the experiment**
 - **Test subjects**, e.g. the number of samples, the number of groups, etc
 - Description of the **experiment** and the **result**
 - Description of the **data type** produced in the result
- **Purpose**: the general purpose of the test
- **Data**: symbolic description of the data produced by the experiment
- **Random variable** and **assumption** corresponding to the data
- **Parameter of interest** and the **estimates**
- **Hypotheses** H_0 and H_A
- **Test statistic**
- **Null distribution**
 - PDF/PMF: description of the PDF/PMF
 - Python: code snippet of the PDF/PMF

Remark (cont.)

For each of the hypothesis tests we introduce, we present the following components:

- **Typical set up for the experiment**
 - **Test subjects**, e.g. the number of samples, the number of groups, etc
 - Description of the **experiment** and the **result**
 - Description of the **data type** produced in the result
- **Purpose**: the general purpose of the test
- **Data**: symbolic description of the data produced by the experiment
- **Random variable** and **assumption** corresponding to the data
- **Parameter of interest** and the **estimates**
- **Hypotheses** H_0 and H_A
- **Test statistic**
- **Null distribution**
 - PDF/PMF: description of the PDF/PMF
 - Python: code snippet of the PDF/PMF
- **p-value**
 - Definition: an expression of p -value in terms of a probability
 - Python: code snippet to illustrate the computation of the p -value (see page 5)

z-test

Test statistics and hypothesis tests
Compare two classifiers
Summary

z-test

One-sample t-test
Two-sample t-test
Paired t-test
Binomial test
McNemar's test
Summary

One-sample z-test

- Typical set up for the experiment:



One-sample z-test

- Typical set up for the experiment:
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc

One-sample z-test

- Typical set up for the experiment:
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc

One-sample z-test

- Typical set up for the experiment:
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
 - The result contains one i.i.d. sample with **continuous numerical values**

One-sample z-test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
 - The result contains one i.i.d. sample with **continuous numerical values**
- **Purpose:** to test if the mean of the result differs from a predefined constant

One-sample z-test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
 - The result contains one i.i.d. sample with **continuous numerical values**
- **Purpose:** to test if the mean of the result differs from a predefined constant
- **Data:** x_1, \dots, x_N , e.g. blood pressure after taking a new drug

One-sample z-test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
 - The result contains one i.i.d. sample with **continuous numerical values**
- **Purpose:** to test if the mean of the result differs from a predefined constant
- **Data:** x_1, \dots, x_N , e.g. blood pressure after taking a new drug
- **Random variable** and **assumption:** X_1, \dots, X_N
 - X_i i.i.d.
 - X_i Gaussian or large N (CLT)
 - X_i standard deviation σ known

One-sample z-test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
 - The result contains one i.i.d. sample with **continuous numerical values**
- **Purpose:** to test if the mean of the result differs from a predefined constant
- **Data:** x_1, \dots, x_N , e.g. blood pressure after taking a new drug
- **Random variable** and **assumption:** X_1, \dots, X_N
 - X_i i.i.d.
 - X_i Gaussian or large N (CLT)
 - X_i standard deviation σ known
- **Parameter of interest:** μ

One-sample z-test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
 - The result contains one i.i.d. sample with **continuous numerical values**
- **Purpose:** to test if the mean of the result differs from a predefined constant
- **Data:** x_1, \dots, x_N , e.g. blood pressure after taking a new drug
- **Random variable** and **assumption:** X_1, \dots, X_N
 - X_i i.i.d.
 - X_i Gaussian or large N (CLT)
 - X_i standard deviation σ known
- **Parameter of interest:** μ
- **Parameter estimate:** $\bar{x}, \bar{X} \sim \mathcal{N}(\mu, \sigma^2/N)$

One-sample z-test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
 - The result contains one i.i.d. sample with **continuous numerical values**
- **Purpose:** to test if the mean of the result differs from a predefined constant
- **Data:** x_1, \dots, x_N , e.g. blood pressure after taking a new drug
- **Random variable** and **assumption:** X_1, \dots, X_N
 - X_i i.i.d.
 - X_i Gaussian or large N (CLT)
 - X_i standard deviation σ known
- **Parameter of interest:** μ
- **Parameter estimate:** \bar{x} , $\bar{X} \sim \mathcal{N}(\mu, \sigma^2/N)$
- **Hypotheses** H_0 and H_A : given c a constant

$$H_0 : \mu = c$$

$$H_A : \mu \neq c$$

One-sample z-test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
 - The result contains one i.i.d. sample with **continuous numerical values**
- **Purpose:** to test if the mean of the result differs from a predefined constant
- **Data:** x_1, \dots, x_N , e.g. blood pressure after taking a new drug
- **Random variable** and **assumption:** X_1, \dots, X_N
 - X_i i.i.d.
 - X_i Gaussian or large N (CLT)
 - X_i standard deviation σ known
- **Parameter of interest:** μ
- **Parameter estimate:** $\bar{x}, \bar{X} \sim \mathcal{N}(\mu, \sigma^2/N)$
- **Hypotheses** H_0 and H_A : given c a constant

$$H_0 : \mu = c$$

$$H_A : \mu \neq c$$



Note: only two-tailed H_A is illustrated here.

CHALMERS



GÖTEBORGS UNIVERSITET

One-sample z-test (cont.)

- Test statistic:

$$z_0 = \frac{\bar{X} - c}{\sigma/\sqrt{N}}$$

One-sample z-test (cont.)

- **Test statistic:**

$$z_0 = \frac{\bar{x} - c}{\sigma / \sqrt{N}}$$

- **Null distribution:** standard normal distribution
 - PDF: $f(z | H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
 - Python: `stats.norm.pdf(z, 0, 1)`

One-sample z-test (cont.)

- **Test statistic:**

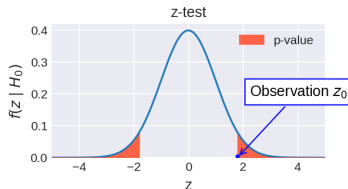
$$z_0 = \frac{\bar{X} - c}{\sigma / \sqrt{N}}$$

- **Null distribution:** standard normal distribution

- PDF: $f(z | H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
- Python: `stats.norm.pdf(z, 0, 1)`

- **p-value**

- Definition: $p = 2 \min(P(Z \leq z_0 | H_0), P(Z \geq z_0 | H_0))$
- Python: `2 * min(stats.norm.cdf(z_0, 0, 1), 1 - stats.norm.cdf(z_0, 0, 1))`



Two-sample z-test

- Typical set up for the experiment:

Two-sample z-test

- Typical set up for the experiment:
 - **Two samples** of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} letters with a calligraphic font are typically used to denote sets are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc

Two-sample z-test

- Typical set up for the experiment:

- **Two samples** of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} letters with a calligraphic font are typically used to denote sets are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
- Run two sets of experiments A and B on the test subjects from the two samples \mathcal{X} and \mathcal{Y} , respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc

Two-sample z-test

- Typical set up for the experiment:

- **Two samples** of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} letters with a calligraphic font are typically used to denote sets are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
- Run two sets of experiments A and B on the test subjects from the two samples \mathcal{X} and \mathcal{Y} , respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
- The result contains two i.i.d. samples with **continuous numerical values**

Two-sample z-test

- **Typical set up for the experiment:**
 - **Two samples** of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} letters with a calligraphic font are typically used to denote sets are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
 - Run two sets of experiments A and B on the test subjects from the two samples \mathcal{X} and \mathcal{Y} , respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
 - The result contains two i.i.d. samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the result from one sample differs from the mean of the other sample

Two-sample z-test

- **Typical set up for the experiment:**

- **Two samples** of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} *letters with a calligraphic font are typically used to denote sets* are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
- Run two sets of experiments A and B on the test subjects from the two samples \mathcal{X} and \mathcal{Y} , respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
- The result contains two i.i.d. samples with **continuous numerical values**

- **Purpose:** to test if two alternative options have different effects by testing if the mean of the result from one sample differs from the mean of the other sample
- **Data:** x_1, \dots, x_{N_X} and y_1, \dots, y_{N_Y} , e.g. blood pressure measured after taking two different drugs

Two-sample z-test

- **Typical set up for the experiment:**

- **Two samples** of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} *letters with a calligraphic font are typically used to denote sets* are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
- Run two sets of experiments A and B on the test subjects from the two samples \mathcal{X} and \mathcal{Y} , respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
- The result contains two i.i.d. samples with **continuous numerical values**

- **Purpose:** to test if two alternative options have different effects by testing if the mean of the result from one sample differs from the mean of the other sample

- **Data:** x_1, \dots, x_{N_X} and y_1, \dots, y_{N_Y} , e.g. blood pressure measured after taking two different drugs

- **Random variable** and **assumption:** X_1, \dots, X_{N_X} , Y_1, \dots, Y_{N_Y}
 - X_i and Y_j independent
 - X_i i.i.d.; Y_j i.i.d.
 - X_i Gaussian or large N_X ; Y_j Gaussian or large N_Y
 - X_i and Y_j have known standard deviation σ_X and σ_Y , respectively

Two-sample z-test

- **Typical set up for the experiment:**
 - **Two samples** of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} *letters with a calligraphic font are typically used to denote sets* are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
 - Run two sets of experiments A and B on the test subjects from the two samples \mathcal{X} and \mathcal{Y} , respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
 - The result contains two i.i.d. samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the result from one sample differs from the mean of the other sample
- **Data:** x_1, \dots, x_{N_X} and y_1, \dots, y_{N_Y} , e.g. blood pressure measured after taking two different drugs
- **Random variable** and **assumption:** X_1, \dots, X_{N_X} , Y_1, \dots, Y_{N_Y}
 - X_i and Y_j independent
 - X_i i.i.d.; Y_j i.i.d.
 - X_i Gaussian or large N_X ; Y_j Gaussian or large N_Y
 - X_i and Y_j have known standard deviation σ_X and σ_Y , respectively
- **Parameter of interest:** μ_X, μ_Y

Two-sample z-test

- **Typical set up for the experiment:**
 - **Two samples** of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} *letters with a calligraphic font are typically used to denote sets* are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
 - Run two sets of experiments A and B on the test subjects from the two samples \mathcal{X} and \mathcal{Y} , respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
 - The result contains two i.i.d. samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the result from one sample differs from the mean of the other sample
- **Data:** x_1, \dots, x_{N_X} and y_1, \dots, y_{N_Y} , e.g. blood pressure measured after taking two different drugs
- **Random variable** and **assumption:** X_1, \dots, X_{N_X} , Y_1, \dots, Y_{N_Y}
 - X_i and Y_j independent
 - X_i i.i.d.; Y_j i.i.d.
 - X_i Gaussian or large N_X ; Y_j Gaussian or large N_Y
 - X_i and Y_j have known standard deviation σ_X and σ_Y , respectively
- **Parameter of interest:** μ_X, μ_Y
- **Parameter estimate:** \bar{x}, \bar{y}

Two-sample z-test

- **Typical set up for the experiment:**
 - **Two samples** of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} *letters with a calligraphic font are typically used to denote sets* are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
 - Run two sets of experiments A and B on the test subjects from the two samples \mathcal{X} and \mathcal{Y} , respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
 - The result contains two i.i.d. samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the result from one sample differs from the mean of the other sample
- **Data:** x_1, \dots, x_{N_X} and y_1, \dots, y_{N_Y} , e.g. blood pressure measured after taking two different drugs
- **Random variable** and **assumption:** X_1, \dots, X_{N_X} , Y_1, \dots, Y_{N_Y}
 - X_i and Y_j independent
 - X_i i.i.d.; Y_j i.i.d.
 - X_i Gaussian or large N_X ; Y_j Gaussian or large N_Y
 - X_i and Y_j have known standard deviation σ_X and σ_Y , respectively
- **Parameter of interest:** μ_X, μ_Y
- **Parameter estimate:** \bar{x}, \bar{y}
- **Hypotheses** H_0 and H_A : given c a constant (typically $c = 0$)

$$H_0 : \mu_X - \mu_Y = c$$

$$H_A : \mu_X - \mu_Y \neq c$$

Two-sample z-test (cont.)

- Test statistic:

$$z_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{\sigma_X^2}{N_X} + \frac{\sigma_Y^2}{N_Y}}}$$

Hint: $\bar{X} - \bar{Y} \sim \mathcal{N}(\mu_X - \mu_Y, \sigma_X^2/N_X + \sigma_Y^2/N_Y)$

Two-sample z-test (cont.)

- **Test statistic:**

$$z_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{\sigma_X^2}{N_X} + \frac{\sigma_Y^2}{N_Y}}}$$

Hint: $\bar{X} - \bar{Y} \sim \mathcal{N}(\mu_X - \mu_Y, \sigma_X^2/N_X + \sigma_Y^2/N_Y)$

- **Null distribution:** standard normal distribution
 - PDF: $f(z | H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
 - Python: `stats.norm.pdf(z, 0, 1)`

Two-sample z-test (cont.)

- **Test statistic:**

$$z_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{\sigma_X^2}{N_X} + \frac{\sigma_Y^2}{N_Y}}}$$

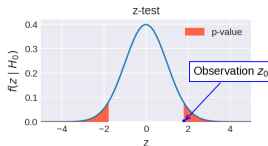
Hint: $\bar{X} - \bar{Y} \sim \mathcal{N}(\mu_X - \mu_Y, \sigma_X^2/N_X + \sigma_Y^2/N_Y)$

- **Null distribution:** standard normal distribution

- PDF: $f(z | H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
- Python: `stats.norm.pdf(z, 0, 1)`

- **p-value**

- Definition: $p = 2 \min(P(Z \leq z_0 | H_0), P(Z \geq z_0 | H_0))$
- Python: `2 * min(stats.norm.cdf(z_0, 0, 1), 1-stats.norm.cdf(z_0, 0, 1))`



One-sample t-test

One-sample t-test

- **Typical set up for the experiment** (same as the one-sample z-test):
 - One sample of independent test subjects, e.g. a sample of patients, a sample of customers, etc
 - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
 - The result contains one i.i.d. sample with **continuous numerical values**
- **Purpose**: to test if the mean of the result differs from a predefined constant
- **Data**: x_1, \dots, x_N , e.g. blood pressure after taking a new drug
- **Random variable** and **assumption**: X_1, \dots, X_N
 - X_i i.i.d.
 - X_i Gaussian or large N
 - X_i standard deviation σ **unknown**
- **Parameter of interest**: μ
- **Parameter estimate**: \bar{x}
- **Hypotheses** H_0 and H_A : given c a constant

$$H_0 : \mu = c$$

$$H_A : \mu \neq c$$

One-sample t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\bar{x} - c}{s/\sqrt{N}}$$

where $s = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2}$ is the sample standard deviation

One-sample t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\bar{x} - c}{s/\sqrt{N}}$$

where $s = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2}$ is the sample standard deviation

- **Null distribution:**

- Student's-t distribution with degrees of freedom $df = N - 1$
- Python: `stats.t.pdf(t, df = N - 1)`

One-sample t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\bar{x} - c}{s/\sqrt{N}}$$

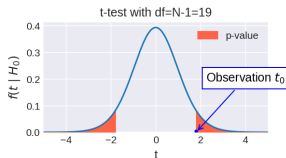
where $s = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - \bar{x})^2}$ is the sample standard deviation

- **Null distribution:**

- Student's-t distribution with degrees of freedom $df = N - 1$
- Python: `stats.t.pdf(t, df = N - 1)`

- **p-value:**

- Definition: $p = 2 \min(P(T \leq t_0 | H_0), P(T \geq t_0 | H_0))$
- Python: `2 * min(stats.t.cdf(t_0, df = N - 1), 1 - stats.t.cdf(t_0, df = N - 1))`



Two-sample t-test

Two-sample t-test

- **Typical set up for the experiment** (same as the two-sample z-test):
 - Two samples of independent test subjects, where the two samples \mathcal{X} and \mathcal{Y} are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
 - Run two sets of experiments A and B on the test subjects from the two samples \mathcal{X} and \mathcal{Y} , respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
 - The result contains two i.i.d. samples with **continuous numerical values**
- **Purpose**: to test if two alternative options have different effects by testing if the mean of the result from one sample differs from the mean of the other sample
- **Data**: x_1, \dots, x_{N_X} and y_1, \dots, y_{N_Y} , e.g. blood pressure measured after taking two different drugs
- **Random variable** and **assumption**: X_1, \dots, X_{N_X} , Y_1, \dots, Y_{N_Y}
 - X_i and Y_j independent
 - X_i i.i.d.; Y_j i.i.d.
 - X_i Gaussian or large N_X ; Y_j Gaussian or large N_Y
 - X_i and Y_j have **unknown** standard deviation σ_X and σ_Y , respectively
- **Parameter of interest**: μ_X, μ_Y
- **Parameter estimate**: \bar{x}, \bar{y}
- **Hypotheses** H_0 and H_A : given c a constant

$$H_0 : \mu_X - \mu_Y = c$$

$$H_A : \mu_X - \mu_Y \neq c$$

Two-sample t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{s_X^2}{N_X} + \frac{s_Y^2}{N_Y}}}$$

with degrees of freedom $df = \frac{(s_X^2/N_X + s_Y^2/N_Y)^2}{(\frac{s_X^2}{N_X})^2/(N_X-1) + (\frac{s_Y^2}{N_Y})^2/(N_Y-1)}$

Two-sample t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{s_X^2}{N_X} + \frac{s_Y^2}{N_Y}}}$$

with degrees of freedom $df = \frac{(s_X^2/N_X + s_Y^2/N_Y)^2}{(\frac{s_X^2}{N_X})^2/(N_X-1) + (\frac{s_Y^2}{N_Y})^2/(N_Y-1)}$

- **Null distribution:**

- Student's-t distribution with degrees of freedom df
- Python: `stats.t.pdf(t, df = df)`

Two-sample t-test (cont.)

- **Test statistic:**

$$t_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{s_X^2}{N_X} + \frac{s_Y^2}{N_Y}}}$$

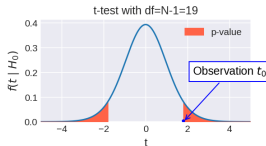
with degrees of freedom $df = \frac{(s_X^2/N_X + s_Y^2/N_Y)^2}{(\frac{s_X^2}{N_X})^2/(N_X-1) + (\frac{s_Y^2}{N_Y})^2/(N_Y-1)}$

- **Null distribution:**

- Student's-t distribution with degrees of freedom df
- Python: `stats.t.pdf(t, df = df)`

- **p-value:**

- Definition: $p = 2 \min(P(T \leq t_0 | H_0), P(T \geq t_0 | H_0))$
- Python: `2 * min(stats.t.cdf(t_0, df=df), 1-stats.t.cdf(t_0, df=df))`



Paired t-test

Paired t-test

- Typical set up for the experiment:

Paired t-test

- **Typical set up for the experiment:**
 - Typically one sample of independent test subjects, e.g. one sample of independent patients

Paired t-test

- **Typical set up for the experiment:**

- Typically one sample of independent test subjects, e.g. one sample of independent patients
- Run two sets of experiments A and B on all subjects from the sample and collect the outcomes, e.g. measure the blood pressure of the patients **before** giving them a new drug (experiment A); measure the blood pressure of the patients **after** giving them the new drug (experiment B)

Paired t-test

- **Typical set up for the experiment:**
 - Typically one sample of independent test subjects, e.g. one sample of independent patients
 - Run two sets of experiments A and B on all subjects from the sample and collect the outcomes, e.g. measure the blood pressure of the patients **before** giving them a new drug (experiment A); measure the blood pressure of the patients **after** giving them the new drug (experiment B)
 - The result contains two samples with **continuous numerical values**

Paired t-test

- **Typical set up for the experiment:**
 - Typically one sample of independent test subjects, e.g. one sample of independent patients
 - Run two sets of experiments A and B on all subjects from the sample and collect the outcomes, e.g. measure the blood pressure of the patients **before** giving them a new drug (experiment A); measure the blood pressure of the patients **after** giving them the new drug (experiment B)
 - The result contains two samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the difference between two results differs from a predefined constant

Paired t-test

- **Typical set up for the experiment:**
 - Typically one sample of independent test subjects, e.g. one sample of independent patients
 - Run two sets of experiments A and B on all subjects from the sample and collect the outcomes, e.g. measure the blood pressure of the patients **before** giving them a new drug (experiment A); measure the blood pressure of the patients **after** giving them the new drug (experiment B)
 - The result contains two samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the difference between two results differs from a predefined constant
- **Data:** $x_1, \dots, x_N, y_1, \dots, y_N$

Paired t-test

- **Typical set up for the experiment:**
 - Typically one sample of independent test subjects, e.g. one sample of independent patients
 - Run two sets of experiments A and B on all subjects from the sample and collect the outcomes, e.g. measure the blood pressure of the patients **before** giving them a new drug (experiment A); measure the blood pressure of the patients **after** giving them the new drug (experiment B)
 - The result contains two samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the difference between two results differs from a predefined constant
- **Data:** $x_1, \dots, x_N, y_1, \dots, y_N$
- **Random variable** and **assumption:** $X_1, \dots, X_N, Y_1, \dots, Y_N$
 - $X_i - Y_i$ i.i.d.
 - $X_i - Y_i \sim \mathcal{N}(\mu_{X-Y}, \sigma_{X-Y}^2)$ or large N (CLT)
 - standard deviation unknown

Paired t-test

- **Typical set up for the experiment:**
 - Typically one sample of independent test subjects, e.g. one sample of independent patients
 - Run two sets of experiments A and B on all subjects from the sample and collect the outcomes, e.g. measure the blood pressure of the patients **before** giving them a new drug (experiment A); measure the blood pressure of the patients **after** giving them the new drug (experiment B)
 - The result contains two samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the difference between two results differs from a predefined constant
- **Data:** $x_1, \dots, x_N, y_1, \dots, y_N$
- **Random variable** and **assumption:** $X_1, \dots, X_N, Y_1, \dots, Y_N$
 - $X_i - Y_i$ i.i.d.
 - $X_i - Y_i \sim \mathcal{N}(\mu_{X-Y}, \sigma_{X-Y}^2)$ or large N (CLT)
 - standard deviation unknown
- **Parameter of interest:** μ_{X-Y}

Paired t-test

- **Typical set up for the experiment:**
 - Typically one sample of independent test subjects, e.g. one sample of independent patients
 - Run two sets of experiments A and B on all subjects from the sample and collect the outcomes, e.g. measure the blood pressure of the patients **before** giving them a new drug (experiment A); measure the blood pressure of the patients **after** giving them the new drug (experiment B)
 - The result contains two samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the difference between two results differs from a predefined constant
- **Data:** $x_1, \dots, x_N, y_1, \dots, y_N$
- **Random variable** and **assumption:** $X_1, \dots, X_N, Y_1, \dots, Y_N$
 - $X_i - Y_i$ i.i.d.
 - $X_i - Y_i \sim \mathcal{N}(\mu_{X-Y}, \sigma_{X-Y}^2)$ or large N (CLT)
 - standard deviation unknown
- **Parameter of interest:** μ_{X-Y}
- **Parameter estimate:** $m_{X-Y} = \frac{1}{N} \sum_{i=1}^N (x_i - y_i)$

Paired t-test

- **Typical set up for the experiment:**
 - Typically one sample of independent test subjects, e.g. one sample of independent patients
 - Run two sets of experiments A and B on all subjects from the sample and collect the outcomes, e.g. measure the blood pressure of the patients **before** giving them a new drug (experiment A); measure the blood pressure of the patients **after** giving them the new drug (experiment B)
 - The result contains two samples with **continuous numerical values**
- **Purpose:** to test if two alternative options have different effects by testing if the mean of the difference between two results differs from a predefined constant
- **Data:** $x_1, \dots, x_N, y_1, \dots, y_N$
- **Random variable** and **assumption:** $X_1, \dots, X_N, Y_1, \dots, Y_N$
 - $X_i - Y_i$ i.i.d.
 - $X_i - Y_i \sim \mathcal{N}(\mu_{X-Y}, \sigma_{X-Y}^2)$ or large N (CLT)
 - standard deviation unknown
- **Parameter of interest:** μ_{X-Y}
- **Parameter estimate:** $m_{X-Y} = \frac{1}{N} \sum_{i=1}^N (x_i - y_i)$
- **Hypotheses** H_0 and H_A : given c a constant

$$H_0 : \quad \mu_{X-Y} = c$$

$$H_A : \quad \mu_{X-Y} \neq c$$

Paired t-test

- **Test statistic:**

$$t_0 = \frac{m_{X-Y} - c}{s_{X-Y} / \sqrt{N}}$$

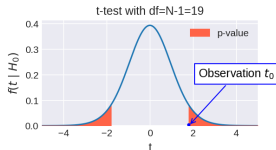
where $s_{X-Y} = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (x_i - y_i - m_{X-Y})^2}$

- **Null distribution:**

- Student's t distribution with degrees of freedom $N - 1$
- Python: `stats.t.pdf(t, df = N - 1)`

- **p-value:**

- Definition: $p = 2 \min (P(T \leq t_0 | H_0), P(T \geq t_0 | H_0))$
- Python: `2 * min (stats.t.cdf(t_0, df = N - 1), 1-stats.t.cdf(t_0, df = N - 1))`



Exercise 1

- A company claims that a new drug E they have developed can increase the average sleeping hours of people with insomnia. Design three different hypothesis tests to test this statement.

Exercise 2

- One of the tests you have designed is a two-sample test. After the experiments, you realized the test subjects being selected in the second group are parents or siblings of the first group. Would that be a problem? Can you still use the result somehow?

Binomial test

Binomial distribution

- Discrete distribution

Binomial distribution

- Discrete distribution
- Applies to discrete numerical data - the number of success from n independent Bernoulli trials with probability of success p

Binomial distribution

- Discrete distribution
- Applies to discrete numerical data - the number of success from n independent Bernoulli trials with probability of success p
- PMF:

Binomial distribution

- Discrete distribution
- Applies to discrete numerical data - the number of success from n independent Bernoulli trials with probability of success p
- PMF:
 - Equation

$$f_X(k | n, p) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad k = 0, \dots, n, \quad p \in [0, 1]$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient (n choose k)

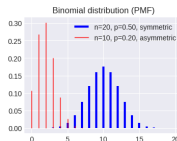
Binomial distribution

- Discrete distribution
- Applies to discrete numerical data - the number of success from n independent Bernoulli trials with probability of success p
- PMF:
 - Equation

$$f_X(k | n, p) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \dots, n, \quad p \in [0, 1]$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient (n choose k)

- Shape
 - When $p = 0.5$, the PMF is symmetric
 - When $p \neq 0.5$, the PMF is asymmetric



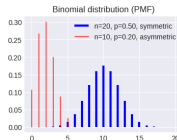
Binomial distribution

- Discrete distribution
- Applies to discrete numerical data - the number of success from n independent Bernoulli trials with probability of success p
- PMF:
 - Equation

$$f_X(k | n, p) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, \dots, n, \quad p \in [0, 1]$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient (n choose k)

- Shape
 - When $p = 0.5$, the PMF is symmetric
 - When $p \neq 0.5$, the PMF is asymmetric



- Parameters: p and n ; n is typically known

(exact) Binomial test

- Typical set up for the experiment:

(exact) Binomial test

- **Typical set up for the experiment:**
 - One sample of independent test subjects, e.g. one sample of independent patients

(exact) Binomial test

- **Typical set up for the experiment:**
 - One sample of independent test subjects, e.g. one sample of independent patients
 - Run the same experiment on all subjects from the sample and collect the outcomes, e.g. give a new drug to a sample of patients and measure how many patients are cured

(exact) Binomial test

- **Typical set up for the experiment:**
 - One sample of independent test subjects, e.g. one sample of independent patients
 - Run the same experiment on all subjects from the sample and collect the outcomes, e.g. give a new drug to a sample of patients and measure how many patients are cured
 - The result contains one sample with **nominal categorical values with two categories**, which is then summarized into one **discrete numerical value** - the number of "success"

(exact) Binomial test

- **Typical set up for the experiment:**
 - One sample of independent test subjects, e.g. one sample of independent patients
 - Run the same experiment on all subjects from the sample and collect the outcomes, e.g. give a new drug to a sample of patients and measure how many patients are cured
 - The result contains one sample with **nominal categorical values with two categories**, which is then summarized into one **discrete numerical value** - the number of "success"
- **Purpose:** to test if the proportion of "success" differs from a predefined constant

(exact) Binomial test

- **Typical set up for the experiment:**
 - One sample of independent test subjects, e.g. one sample of independent patients
 - Run the same experiment on all subjects from the sample and collect the outcomes, e.g. give a new drug to a sample of patients and measure how many patients are cured
 - The result contains one sample with **nominal categorical values with two categories**, which is then summarized into one **discrete numerical value** - the number of "success"
- **Purpose:** to test if the proportion of "success" differs from a predefined constant
- **Data:** N independent Bernoulli trials x_i with k_0 "success" outcomes, e.g. the number of cured patients within the sample of size N

(exact) Binomial test

- **Typical set up for the experiment:**
 - One sample of independent test subjects, e.g. one sample of independent patients
 - Run the same experiment on all subjects from the sample and collect the outcomes, e.g. give a new drug to a sample of patients and measure how many patients are cured
 - The result contains one sample with **nominal categorical values with two categories**, which is then summarized into one **discrete numerical value** - the number of "success"
- **Purpose:** to test if the proportion of "success" differs from a predefined constant
- **Data:** N independent Bernoulli trials x_i with k_0 "success" outcomes, e.g. the number of cured patients within the sample of size N
- **Random variable** and **assumption:** $X_i \sim \text{Bernoulli}(p)$, $K \sim \text{Binomial}(N, p)$ with known N and unknown success rate p

(exact) Binomial test

- **Typical set up for the experiment:**
 - One sample of independent test subjects, e.g. one sample of independent patients
 - Run the same experiment on all subjects from the sample and collect the outcomes, e.g. give a new drug to a sample of patients and measure how many patients are cured
 - The result contains one sample with **nominal categorical values with two categories**, which is then summarized into one **discrete numerical value** - the number of "success"
- **Purpose:** to test if the proportion of "success" differs from a predefined constant
- **Data:** N independent Bernoulli trials x_i with k_0 "success" outcomes, e.g. the number of cured patients within the sample of size N
- **Random variable** and **assumption:** $X_i \sim \text{Bernoulli}(p)$, $K \sim \text{Binomial}(N, p)$ with known N and unknown success rate p
- **Parameter of interest:** p

(exact) Binomial test

- **Typical set up for the experiment:**
 - One sample of independent test subjects, e.g. one sample of independent patients
 - Run the same experiment on all subjects from the sample and collect the outcomes, e.g. give a new drug to a sample of patients and measure how many patients are cured
 - The result contains one sample with **nominal categorical values with two categories**, which is then summarized into one **discrete numerical value** - the number of "success"
- **Purpose:** to test if the proportion of "success" differs from a predefined constant
- **Data:** N independent Bernoulli trials x_i with k_0 "success" outcomes, e.g. the number of cured patients within the sample of size N
- **Random variable** and **assumption:** $X_i \sim \text{Bernoulli}(p)$, $K \sim \text{Binomial}(N, p)$ with known N and unknown success rate p
- **Parameter of interest:** p
- **Parameter estimate:** $\hat{p} = \frac{k_0}{N}$

(exact) Binomial test

- **Typical set up for the experiment:**
 - One sample of independent test subjects, e.g. one sample of independent patients
 - Run the same experiment on all subjects from the sample and collect the outcomes, e.g. give a new drug to a sample of patients and measure how many patients are cured
 - The result contains one sample with **nominal categorical values with two categories**, which is then summarized into one **discrete numerical value** - the number of “success”
- **Purpose:** to test if the proportion of “success” differs from a predefined constant
- **Data:** N independent Bernoulli trials x_i with k_0 “success” outcomes, e.g. the number of cured patients within the sample of size N
- **Random variable** and **assumption:** $X_i \sim \text{Bernoulli}(p)$, $K \sim \text{Binomial}(N, p)$ with known N and unknown success rate p
- **Parameter of interest:** p
- **Parameter estimate:** $\hat{p} = \frac{k_0}{N}$
- **Null hypothesis:** given π a constant,

$$H_0 : p = \pi$$

(exact) Binomial test (cont.)

- Test statistic: k_0

(exact) Binomial test (cont.)

- **Test statistic:** k_0
- **Null distribution:**

$$P(X = k) = \binom{N}{k} \pi^k (1 - \pi)^{N-k}$$

(exact) Binomial test (cont.)

- **Test statistic:** k_0
- **Null distribution:**

$$P(X = k) = \binom{N}{k} \pi^k (1 - \pi)^{N-k}$$

- Binomial distribution with parameters N and π

(exact) Binomial test (cont.)

- **Test statistic:** k_0
- **Null distribution:**

$$P(X = k) = \binom{N}{k} \pi^k (1 - \pi)^{N-k}$$

- Binomial distribution with parameters N and π
- Python: `stats.binom.pmf(k, N, π)`

(exact) Binomial test (cont.)

- **Test statistic:** k_0
- **Null distribution:**

$$P(X = k) = \binom{N}{k} \pi^k (1 - \pi)^{N-k}$$

- Binomial distribution with parameters N and π
- Python: `stats.binom.pmf(k, N, π)`
- As discussed in the remarks (cf. page 5), we only introduce the following scenarios:

(exact) Binomial test (cont.)

- **Test statistic:** k_0
- **Null distribution:**

$$P(X = k) = \binom{N}{k} \pi^k (1 - \pi)^{N-k}$$

- Binomial distribution with parameters N and π
- Python: `stats.binom.pmf(k, N, π)`
- As discussed in the remarks (cf. page 5), we only introduce the following scenarios:
 - One-tailed (left) binomial test with any $\pi \in (0, 1)$

(exact) Binomial test (cont.)

- **Test statistic:** k_0
- **Null distribution:**

$$P(X = k) = \binom{N}{k} \pi^k (1 - \pi)^{N-k}$$

- Binomial distribution with parameters N and π
- Python: `stats.binom.pmf(k, N, π)`
- As discussed in the remarks (cf. page 5), we only introduce the following scenarios:
 - One-tailed (left) binomial test with any $\pi \in (0, 1)$
 - One-tailed (right) binomial test with any $\pi \in (0, 1)$

(exact) Binomial test (cont.)

- **Test statistic:** k_0
- **Null distribution:**

$$P(X = k) = \binom{N}{k} \pi^k (1 - \pi)^{N-k}$$

- Binomial distribution with parameters N and π
- Python: `stats.binom.pmf(k, N, π)`
- As discussed in the remarks (cf. page 5), we only introduce the following scenarios:
 - One-tailed (left) binomial test with any $\pi \in (0, 1)$
 - One-tailed (right) binomial test with any $\pi \in (0, 1)$
 - Two-tailed binomial test with $\pi = 0.5$, where the null distribution is symmetric

(exact) One-tailed (left) binomial test

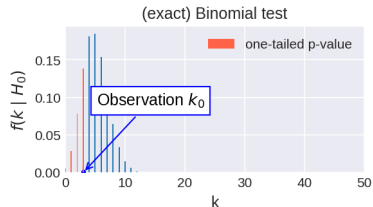
- Hypotheses H_0 and H_A :

$$H_0 : p = \pi$$

$$H_A : p < \pi$$

- p -value:

- Definition: $P(K \leq k_0 \mid H_0)$
- Python: `stats.binom.cdf(k0, n=N, p=π)`



(exact) One-tailed (right) binomial test

- Hypotheses H_0 and H_A :

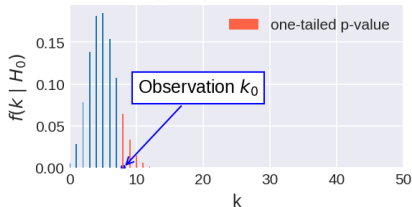
$$H_0 : p = \pi$$

$$H_A : p > \pi$$

- p -value:

- Definition: $P(K \geq k_0 | H_0)$
- Python: `1 - stats.binom.cdf(k0, n = N, p = π) + stats.binom.pmf(k0, n = N, p = π)`

(exact) Binomial test



(exact) Two-tailed binomial test

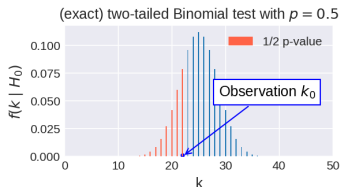
- Hypotheses H_0 and H_A :

$$H_0 : p = 0.5$$

$$H_A : p \neq 0.5$$

- p -value:

- Definition: $2 \min(P(K \leq k_0 | H_0), P(K \geq k_0 | H_0))$
- Python:
 - `c = stats.binom.cdf(k0, n = N, p = 0.5)`
 - `2 * min(c, 1 - c + stats.binom.pmf(k0, n = N, p = 0.5))`



(large N) Binomial test

Same set up as page 25, but with large N

- **Test statistic:**

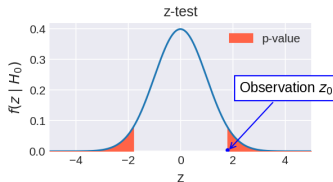
$$z_0 = \frac{k_0 - N\pi}{\sqrt{N\pi(1-\pi)}}$$

- **Null distribution:** standard normal distribution

- PDF: $f(z | H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
- Python: `stats.norm.pdf(z, 0, 1)`

- **p-value:**

- Definition: $p = 2 \min(P(Z \leq z_0 | H_0), P(Z \geq z_0 | H_0))$
- Python: `2 * min(stats.norm.cdf(z_0, 0, 1), 1-stats.norm.cdf(z_0, 0, 1))`



McNemar's test

McNemar's test

- Typical set up for the experiment:

McNemar's test

- Typical set up for the experiment:
 - One sample of independent test subjects, e.g. one sample of independent patients

McNemar's test

- Typical set up for the experiment:
 - **One sample** of independent test subjects, e.g. one sample of independent patients
 - Within the sample, there are two **groups**; each subject belongs to one and only one group, e.g. within the sample of patients, we have one group with high blood pressure and another group with normal blood pressure

McNemar's test

- Typical set up for the experiment:
 - One sample of independent test subjects, e.g. one sample of independent patients
 - Within the sample, there are two groups; each subject belongs to one and only one group, e.g. within the sample of patients, we have one group with high blood pressure and another group with normal blood pressure
 - Run two sets of experiments A and B on all test subjects from the sample and collect the outcomes, e.g. measure the blood pressure (high or normal) of the patients before giving them a new drug (experiment A); measure the blood pressure (high or normal) of the patients after giving them the new drug (experiment B)

McNemar's test

- Typical set up for the experiment:

- **One sample** of independent test subjects, e.g. one sample of independent patients
- Within the sample, there are two **groups**; each subject belongs to one and only one group, e.g. within the sample of patients, we have one group with high blood pressure and another group with normal blood pressure
- Run two sets of experiments A and B on all test subjects from the sample and collect the outcomes, e.g. measure the blood pressure (high or normal) of the patients **before** giving them a new drug (experiment A); measure the blood pressure (high or normal) of the patients **after** giving them the new drug (experiment B)
- The result contains one sample with **nominal categorical values with two categories** measured from each test subject, e.g. high blood pressure and normal blood pressure

McNemar's test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. one sample of independent patients
 - Within the sample, there are two **groups**; each subject belongs to one and only one group, e.g. within the sample of patients, we have one group with high blood pressure and another group with normal blood pressure
 - Run two sets of experiments A and B on all test subjects from the sample and collect the outcomes, e.g. measure the blood pressure (high or normal) of the patients **before** giving them a new drug (experiment A); measure the blood pressure (high or normal) of the patients **after** giving them the new drug (experiment B)
 - The result contains one sample with **nominal categorical values with two categories** measured from each test subject, e.g. high blood pressure and normal blood pressure
- **Purpose:** to test if an action have different effects on two different **groups**

McNemar's test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. one sample of independent patients
 - Within the sample, there are two **groups**; each subject belongs to one and only one group, e.g. within the sample of patients, we have one group with high blood pressure and another group with normal blood pressure
 - Run two sets of experiments A and B on all test subjects from the sample and collect the outcomes, e.g. measure the blood pressure (high or normal) of the patients **before** giving them a new drug (experiment A); measure the blood pressure (high or normal) of the patients **after** giving them the new drug (experiment B)
 - The result contains one sample with **nominal categorical values with two categories** measured from each test subject, e.g. high blood pressure and normal blood pressure
- **Purpose:** to test if an action have different effects on two different **groups**
- **Data:** N independent Bernoulli trials with outcomes x_1, \dots, x_N and y_1, \dots, y_N for the two experiments, respectively; $x_i, y_i \in \{0, 1\}$

	$x_i = 0$	$x_i = 1$	
$y_j = 0$	n_{00}	n_{10}	$n_{00} + n_{10}$
$y_j = 1$	n_{01}	n_{11}	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	N

where n_{mn} is the count of $x_i = m$ and $y_j = n$

McNemar's test

- **Typical set up for the experiment:**
 - **One sample** of independent test subjects, e.g. one sample of independent patients
 - Within the sample, there are two **groups**; each subject belongs to one and only one group, e.g. within the sample of patients, we have one group with high blood pressure and another group with normal blood pressure
 - Run two sets of experiments A and B on all test subjects from the sample and collect the outcomes, e.g. measure the blood pressure (high or normal) of the patients **before** giving them a new drug (experiment A); measure the blood pressure (high or normal) of the patients **after** giving them the new drug (experiment B)
 - The result contains one sample with **nominal categorical values with two categories** measured from each test subject, e.g. high blood pressure and normal blood pressure
- **Purpose:** to test if an action have different effects on two different **groups**
- **Data:** N independent Bernoulli trials with outcomes x_1, \dots, x_N and y_1, \dots, y_N for the two experiments, respectively; $x_i, y_i \in \{0, 1\}$

	$x_i = 0$	$x_i = 1$	
$y_j = 0$	n_{00}	n_{10}	$n_{00} + n_{10}$
$y_j = 1$	n_{01}	n_{11}	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	N

where n_{mn} is the count of $x_i = m$ and $y_j = n$

- **Random variable** and **assumption:** i.i.d. $X_i \sim \text{Bernoulli}(p_X)$ and i.i.d. $Y_i \sim \text{Bernoulli}(p_Y)$

McNemar's test (cont.)

Example

- A company is trying to determine the effectiveness of a drug on lowering blood pressure
- The company tested the drug on a sample of 229 **independent** patients
- There are **two groups** within this sample: a high blood pressure group (105 patients) and a normal blood pressure group (124 patients); each patient belongs to one and only one of these two groups
- The blood pressure of each patient is measured **before** (to determine the group) and **after** (to determine the effect) taking the drug
- The data is summarized as follows:

	Before (high blood pressure)	Before (normal blood pressure)	
After (high blood pressure)	90	15	105
After (normal blood pressure)	22	102	124
	112	117	229

(small discordance $n_{01} + n_{10}$) McNemar's test (cont.)

	$x_i = 0$	$x_i = 1$	
$y_j = 0$	n_{00}	n_{10}	$n_{00} + n_{10}$
$y_j = 1$	n_{01}	n_{11}	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	N

- Parameter of interest: **discordance**

$$p = \min(P(X_i = 0, Y_i = 1 \mid X_i \neq Y_i), P(X_i = 1, Y_i = 0 \mid X_i \neq Y_i))$$

- Parameter estimate: $\hat{p} = \frac{\min(n_{01}, n_{10})}{n_{01} + n_{10}}$
- Hypotheses H_0 and H_A :

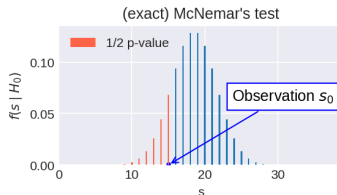
$$H_0 : p = 0.5$$

$$H_A : p \neq 0.5$$

- H_0 : the drug does not have any effect on blood pressure control
- H_1 : the drug has effect on blood pressure control

(small discordance $n_{01} + n_{10}$) McNemar's test (cont.)

- **Test statistic:** $s_0 = \min(n_{01}, n_{10})$
- **Null distribution:**
 - Binomial distribution with parameters $(n_{01} + n_{10}, 0.5)$
 - Python: `stats.binom.pmf(s, $n_{01} + n_{10}$, 0.5)`
- **p-value:**
 - Definition: $p = 2P(S \leq s_0 \mid H_0)$
 - Python: `2 * stats.binom.cdf(s_0, $n_{01} + n_{10}$, 0.5)`



(large discordance $n_{01} + n_{10}$) McNemar's test

	$x_i = 0$	$x_i = 1$	
$y_j = 0$	n_{00}	n_{10}	$n_{00} + n_{10}$
$y_j = 1$	n_{01}	n_{11}	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	N

Same set up as page 32, but with large $n_{01} + n_{10}$, e.g. $n_{01} + n_{10} > 25$

- **Parameter of interest:** **discordance** (note: it is different from the previous definition (cf. page 34)) $p_{01} = P(X = 0, Y = 1)$ and $p_{10} = P(X = 1, Y = 0)$
- **Parameter estimate:** $\hat{p}_{01} = \frac{n_{01}}{N}$ and $\hat{p}_{10} = \frac{n_{10}}{N}$
- **Hypotheses** H_0 and H_A :

$$H_0 : p_{01} = p_{10}$$

$$H_A : p_{01} \neq p_{10}$$

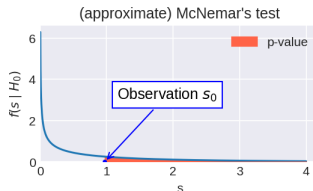
- **Test statistic:**

$$s_0 = \frac{(|n_{01} - n_{10}| - 1)^2}{n_{01} + n_{10}}$$

Note: "-1" is called the **continuity correction** (https://en.wikipedia.org/wiki/Continuity_correction)

(large discordance $n_{01} + n_{10}$) McNemar's test (cont.)

- **Null distribution:**
 - Chi-squared distribution with $df = 1$
 - Python: `stats.chi2.pdf(s, df = 1)`
- **p-value:**
 - Definition: $P(S \geq s_0 \mid H_0)$
 - Python: `1-stats.chi2.cdf(s_0, df = 1)`



Exercise 3

- Run both the exact McNemar's test and the approximate McNemar's test on the data set provided on page 33

Exercise 4

- To test equality of variances given two samples from Gaussian distributions, we can use the F-test, where the test statistic is $F = \frac{S_X^2}{S_Y^2}$. What is the null and alternative hypothesis? What is the null distribution? How to compute the p-value?

Summary

Comparison

	Data discrete/continuous	# of samples	Remark	Test statistic	Null distribution
One-sample z-test	Continuous	1	σ known	$\frac{\bar{x} - c}{\sigma / \sqrt{N}}$	Standard Gaussian
One-sample t-test	Continuous	1	σ unknown	$\frac{\bar{x} - c}{s / \sqrt{N}}$	Student's-t distribution
Two-sample z-test	Continuous	2	σ_X, σ_Y known	$\frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{\sigma_X^2}{N_X} + \frac{\sigma_Y^2}{N_Y}}}$	Standard Gaussian
Two-sample t-test	Continuous	2	σ_X, σ_Y unknown	$\frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{s_X^2}{N_X} + \frac{s_Y^2}{N_Y}}}$	Student's-t distribution
Paired t-test	Continuous	1 or 2 (paired)	σ_X, σ_Y unknown	$\frac{\bar{m}_X - \bar{y} - c}{s_{X-Y} / \sqrt{N}}$	Student's-t distribution
Binomial test (exact)	Discrete	1	Small N	k_0	Binomial distribution
Binomial test (approximate)	Discrete	1	Large N	$\frac{k_0 - N\pi}{\sqrt{N\pi(1-\pi)}}$	Standard Gaussian
McNemar's test (exact)	Discrete	1 (2 groups)	Small $n_{01} + n_{10}$	$\min(n_{01}, n_{10})$	Binomial distribution
McNemar's test (approximate)	Discrete	1 (2 groups)	Large $n_{01} + n_{10}$	$\frac{(n_{01} - n_{10} - 1)^2}{n_{01} + n_{10}}$	Chi-squared distribution

Today

- 1 Test statistics and hypothesis tests
- 2 Compare two classifiers
- 3 Summary

K-fold cross validation

- **Classifiers:** A and B
- **Data:** evaluation metric; **continuous numerical data**, e.g. accuracies p_1^A, \dots, p_K^A and p_1^B, \dots, p_K^B on the K validation sets

	fold 1	fold 2	...	fold K
classifier A	p_1^A	p_2^A	...	p_K^A
classifier B	p_1^B	p_2^B	...	p_K^B
$p_i^A - p_i^B$	$p_1^A - p_1^B$	$p_2^A - p_2^B$...	$p_K^A - p_K^B$

- **Random variable** and **assumption:** $p_1^A, \dots, p_K^A, p_1^B, \dots, p_K^B$
 - $p_i^A - p_i^B$ i.i.d.
 - $p_i^A - p_i^B \sim \mathcal{N}(\mu_{A-B}, \sigma_{A-B}^2)$ with unknown standard deviation
- **Parameter of interest:** μ_{A-B}
- **Parameter estimate:** $m_{A-B} = \frac{1}{K} \sum_{i=1}^K (p_i^A - p_i^B)$
- **Hypotheses** H_0 and H_A :

$$H_0 : \mu_{A-B} = 0$$

$$H_A : \mu_{A-B} \neq 0$$

- **Test statistic:** $t = \frac{m_{A-B}}{s_{A-B}/\sqrt{K}}$, where $s_{A-B} = \sqrt{\frac{1}{K-1} \sum_{i=1}^K (p_i^A - p_i^B - m_{A-B})^2}$
- **Hypothesis test:** paired t-test

Training-validation split and leave-one-out cross validation

- **Classifiers:** A and B
- **Data:** classifiers A and B tested on the validation data; the outcome x_i^A and x_i^B can be either correct (0) or incorrect (1); **nominal categorical data** with two categories correct or incorrect

	classifier A correct	classifier A incorrect
classifier B correct	$n_{00} = \text{count}(\text{A correct, B correct})$	$n_{10} = \text{count}(\text{A incorrect, B correct})$
classifier B incorrect	$n_{01} = \text{count}(\text{A correct, B incorrect})$	$n_{11} = \text{count}(\text{A incorrect, B incorrect})$

- **Random variable** and **assumption:** $X_i^A \sim \text{Bernoulli}(p_A)$, $X_i^B \sim \text{Bernoulli}(p_B)$
- **Test statistic:**
 - Small discordance (e.g. $n_{01} + n_{10} < 25$): $s_0 = \min(n_{01}, n_{10})$
 - Large discordance: $s_0 = \frac{(|n_{01} - n_{10}| - 1)^2}{n_{01} + n_{10}}$
- **Hypothesis test:** McNemar's test

Exercise 5

- You have a labeled data set $\mathcal{D} = \{(x_1, y_1), \dots, (x_N, y_N)\}$
- You developed two classifiers using the 10-fold validation
- Construct a table of the resulting F1 scores for these two classifiers
- Design a hypothesis test to compare these two classifiers

Exercise 6

- You have a labeled data set $\mathcal{D} = \{(x_1, y_1), \dots, (x_N, y_N)\}$
- You developed two classifiers using the training-validation split
- Construct a table of the results for these two classifiers
- Design a hypothesis test to compare these two classifiers

Today

- 1 Test statistics and hypothesis tests
- 2 Compare two classifiers
- 3 Summary**

Summary

So far:

- Data types and data containers
- Descriptive data analysis: descriptive statistics, visualization
- Probability distributions, events, random variables, PMF, PDF, parameters
- CDF, Q-Q plot, how to compare two distributions (data vs theoretical, data vs data)
- Modeling
- Parameter estimation: maximum likelihood estimation (MLE) and maximum a posteriori estimation (MAP)
- Classification, multinomial naive Bayes classifier, Gaussian naive Bayes classifier
- Central limit theorem, interval estimation
- Hypothesis tests, comparison of two classifiers

Summary

So far:

- Data types and data containers
- Descriptive data analysis: descriptive statistics, visualization
- Probability distributions, events, random variables, PMF, PDF, parameters
- CDF, Q-Q plot, how to compare two distributions (data vs theoretical, data vs data)
- Modeling
- Parameter estimation: maximum likelihood estimation (MLE) and maximum a posteriori estimation (MAP)
- Classification, multinomial naive Bayes classifier, Gaussian naive Bayes classifier
- Central limit theorem, interval estimation
- Hypothesis tests, comparison of two classifiers

Next:

- Unsupervised learning, clustering, k-means, Gaussian Mixture Models

Summary

So far:

- Data types and data containers
- Descriptive data analysis: descriptive statistics, visualization
- Probability distributions, events, random variables, PMF, PDF, parameters
- CDF, Q-Q plot, how to compare two distributions (data vs theoretical, data vs data)
- Modeling
- Parameter estimation: maximum likelihood estimation (MLE) and maximum a posteriori estimation (MAP)
- Classification, multinomial naive Bayes classifier, Gaussian naive Bayes classifier
- Central limit theorem, interval estimation
- Hypothesis tests, comparison of two classifiers

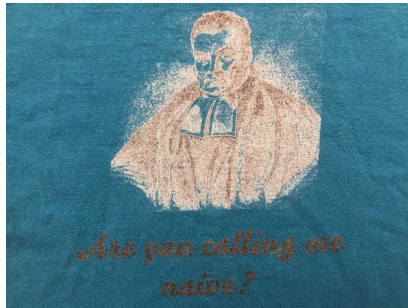
Next:

- Unsupervised learning, clustering, k-means, Gaussian Mixture Models

Before next lecture:

- Gaussian distribution
- The Bayes' rule





I heard my name again!