Lecture 8: Hypothesis testing part II

Statistical Methods for Data Science

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Today

1. Test statistics and hypothesis tests
   - z-test
   - One-sample t-test
   - Two-sample t-test
   - Paired t-test

Learning outcome

- Be able to explain the following hypothesis tests
  - One-sample and two-sample z-test
  - One-sample and two-sample t-test
  - Paired t-test

  For each of these tests, be able to describe the typical set up for the experiment, the general purpose of the test, data produced by the experiment, random variables, parameter of interest, null hypothesis, alternative hypothesis, test statistic, null distribution, the computation of *p*-value
- ... more to come (to be updated)

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- For the **exact binomial test with $p \neq 0.5$**, the null distribution is not symmetric; in this case, the computation of the two-tailed p-value is not uniquely defined; in this lecture, we will not go into details for these cases; we will only look at the one-tailed tests for asymmetric binomial null distributions

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- For each hypothesis test, the purpose of the Python code snippet is to provide a better understanding of the calculation; in practice, there are alternative libraries and built-in functions for these tests that might result in a more compact implementation

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- **Data**: symbolic description of the data produced by the experiment
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  - PDF/PMF: description of the PDF/PMF
  - Python: code snippet of the PDF/PMF
- p-**value**
  - Definition: an expression of p-value in terms of a probability
  - Python: code snippet to illustrate the computation of the p-value (see page 5)

z-test

One-sample z-test

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$$\begin{aligned} H_0: &\quad \mu = c \\ H_A: &\quad \mu \neq c \end{aligned}$$

Note: only two-tailed H_A is illustrated here.

One-sample z-test (cont.)

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$$z_0 = \frac{\bar{x} - c}{\sigma/\sqrt{N}}$$

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  - PDF: $f(z \mid H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
  - Python: stats.norm.pdf(z, 0, 1)

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- ***p*-value**
  - Definition: $p = 2 \min \left(P(Z \leq z_0 \mid H_0), P(Z \geq z_0 \mid H_0) \right)$
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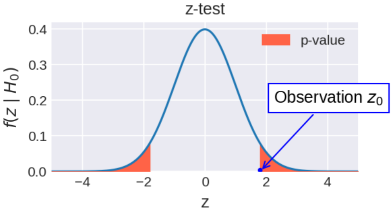

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- Random variable and assumption: $X_1, \cdots, X_{N_X}$, $Y_1, \cdots, Y_{N_Y}$
  - X_i and Y_j independent
  - X_i i.i.d.; Y_j i.i.d.
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- Parameter of interest: μ_X, μ_Y

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- **Parameter of interest**: μ_X, μ_Y
- **Parameter estimate**: $\bar{x}$, $\bar{y}$

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- **Parameter of interest**: μ_X, μ_Y
- **Parameter estimate**: $\bar{x}$, $\bar{y}$
- **Hypotheses H_0 and H_A**: given c a constant (typically $c = 0$)

$$
\begin{aligned}
H_0 &: \quad \mu_X - \mu_Y = c \\
H_A &: \quad \mu_X - \mu_Y \neq c
\end{aligned}
$$

Two-sample z-test (cont.)

- **Test statistic**:

$$z_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{\sigma_X^2}{N_X} + \frac{\sigma_Y^2}{N_Y}}}$$

Hint: $\bar{X} - \bar{Y} \sim \mathcal{N}\left(\mu_X - \mu_Y, \sigma_X^2/N_X + \sigma_Y^2/N_Y\right)$

Two-sample z-test (cont.)

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  - PDF: $f(z \mid H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$
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Two-sample z-test (cont.)

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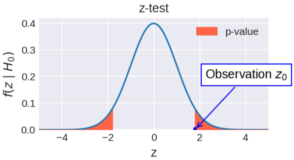

One-sample t-test

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- **Typical set up for the experiment** (same as the one-sample z-test):
  - One sample of independent test subjects, e.g. a sample of patients, a sample of customers, etc
  - Run the same experiment on each subject and collect the outcomes, e.g. give a new drug to a sample of patients and measure the effect on each individual patient; test a new web design on a sample of customers and record the time they spend on the web page, etc
  - The result contains one i.i.d. sample with **continuous numerical values**
- **Purpose**: to test if the mean of the result differs from a predefined constant
- **Data**: $x_1, \cdots, x_N$, e.g. blood pressure after taking a new drug
- **Random variable** and **assumption**: $X_1, \cdots, X_N$
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  - X_i standard deviation σ **unknown**
- **Parameter of interest**: μ
- **Parameter estimate**: $\bar{x}$
- **Hypotheses H_0 and H_A**: given c a constant

$$
\begin{aligned}
H_0: &\quad \mu = c \\
H_A: &\quad \mu \neq c
\end{aligned}
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One-sample t-test (cont.)

- **Test statistic**:

$$t_0 = \frac{\bar{x} - c}{s/\sqrt{N}}$$

where $s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$ is the sample standard deviation

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  - Python: stats.t.pdf$(t, df = N - 1)$

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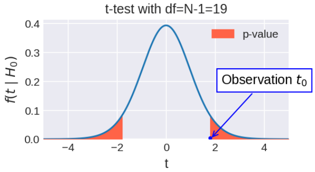

Two-sample t-test

Two-sample t-test

- **Typical set up for the experiment** (same as the two-sample z-test):
  - Two samples of independent test subjects, where the two samples $\mathcal{X}$ and $\mathcal{Y}$ are independent from one another, e.g. two samples of independent patients, two samples of independent customers, etc
  - Run two sets of experiments A and B on the test subjects from the two samples $\mathcal{X}$ and $\mathcal{Y}$, respectively, and collect the outcomes, e.g. give different drugs to the two samples of patients and measure the effect on each individual patient; test two web designs on two samples of customers and record the time they spend on the web page, etc
  - The result contains two i.i.d. samples with **continuous numerical values**
- **Purpose**: to test if two alternative options have different effects by testing if the mean of the result from one sample differs from the mean of the other sample
- **Data**: $x_1, \cdots, x_{N_X}$ and $y_1, \cdots, y_{N_Y}$, e.g. blood pressure measured after taking two different drugs
- **Random variable** and **assumption**: $X_1, \cdots, X_{N_X}$, $Y_1, \cdots, Y_{N_Y}$
  - X_i and Y_j independent
  - X_i i.i.d.; Y_j i.i.d.
  - X_i Gaussian or large N_X; Y_j Gaussian or large N_Y
  - X_i and Y_j have **unknown** standard deviation σ_X and σ_Y, respectively
- **Parameter of interest**: μ_X, μ_Y
- **Parameter estimate**: $\bar{x}$, $\bar{y}$
- **Hypotheses** H_0 and H_A: given c a constant

$$
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H_0 : \quad & \mu_X - \mu_Y = c \\
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Two-sample t-test (cont.)

- **Test statistic**:

$$t_0 = \frac{\bar{x} - \bar{y} - c}{\sqrt{\frac{s_X^2}{N_X} + \frac{s_Y^2}{N_Y}}}$$

with degrees of freedom $df = \frac{(s_X^2/N_X + s_Y^2/N_Y)^2}{(\frac{s_X^2}{N_X})^2/(N_X - 1) + (\frac{s_Y^2}{N_Y})^2/(N_Y - 1)}$

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  - Python: stats.t.pdf($t, df = df$)

Two-sample t-test (cont.)

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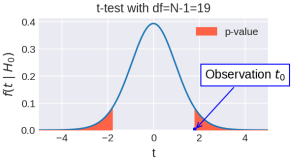

Paired t-test

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- **Random variable** and **assumption**: $X_1, \cdots, X_N$, $Y_1, \cdots, Y_N$
  - $X_i - Y_i$ i.i.d.
  - $X_i - Y_i \sim \mathcal{N}\left(\mu_{X-Y}, \sigma^2_{X-Y}\right)$ or large N (CLT)
  - standard deviation unknown

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- **Parameter of interest**: μ_{X-Y}

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- **Parameter of interest**: μ_{X-Y}
- **Parameter estimate**: $m_{X-Y} = \frac{1}{N}\sum_{i=1}^{N}(x_i - y_i)$

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H_0: &\quad \mu_{X-Y} = c \\
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Paired t-test

- **Test statistic**:

$$t_0 = \frac{m_{X-Y} - c}{s_{X-Y}/\sqrt{N}}$$

  where $s_{X-Y} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - y_i - m_{X-Y})^2}$
- **Null distribution**:
  - Student's-t distribution with degrees of freedom $N-1$
  - Python: stats.t.pdf($t, df = N-1$)
- ***p*-value**:
  - Definition: $p = 2\min\left(P(T \leq t_0 \mid H_0), P(T \geq t_0 \mid H_0)\right)$
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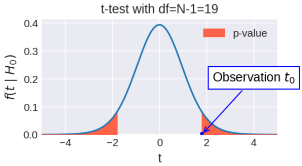

Exercise 1

- A company claims that a new drug E they have developed can increase the average sleeping hours of people with insomnia. Design three different hypothesis tests to test this statement.

Exercise 2

- One of the tests you have designed is a two-sample test. After the experiments, you realized the test subjects being selected in the second group are parents or siblings of the first group. Would that be a problem? Can you still use the result somehow?