

Lecture 11: Decision Tree and Random Forest

Statistical Methods for Data Science

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Today

- 1 Decision trees
 - Classification tree
 - Regression tree
 - Pruning
 - ID3, C4.5, C5.0, CART
- 2 Ensemble methods
 - Bias-variance trade-off
 - Bagging
 - Random forest
- 3 Summary



Learning outcome

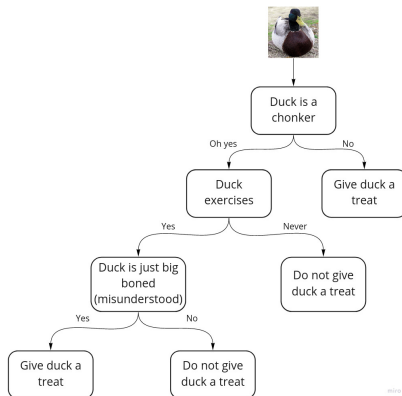
- Understand classification and regression decision trees
- Be able to compute the Gini score
- Be able to explain how to compute the split in the CART algorithm
- Be able to explain the random forest algorithm

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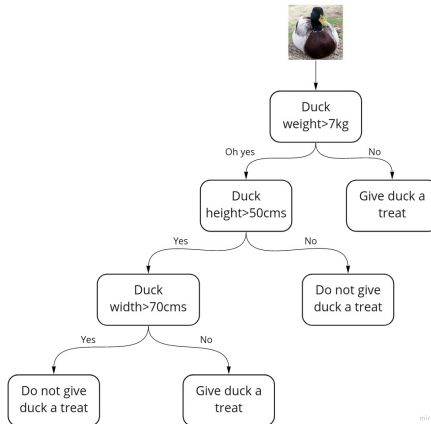
Classification tree with categorical variables

- **Variable (categorical):** $\mathbf{x} = [x_1, x_2, x_3]$,
 $x_1 \in \{\text{chonker, slim}\}$, $x_2 \in \{\text{exercises, never exercises}\}$, $x_3 \in \{\text{big-boned, small-boned}\}$
- **Target (categorical):** $y \in \{\text{give duck a treat, do not give duck a treat}\}$



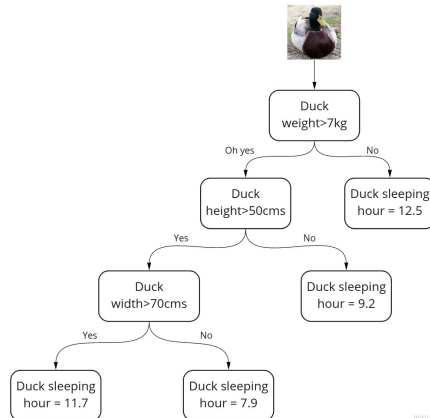
Classification tree with numerical variables

- **Variable (numerical):** $x = [\text{weight, height, width}]$
- **Target (categorical):** $y \in \{\text{give duck a treat, do not give duck a treat}\}$



Regression tree with numerical variables

- **Variable (numerical):** $x = [\text{weight, height, width}]$
- **Target (numerical):** $y = \text{sleep hours per day}$



Classification tree

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where I_{R_m} is the indicator function:

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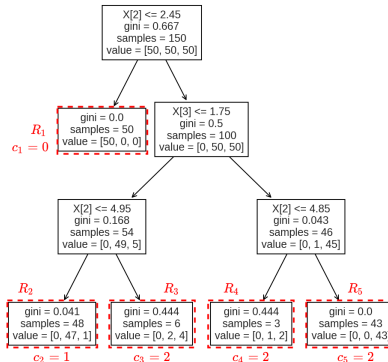
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- **Hyperparameter**: stopping criteria (e.g. maximum depth, minimum number of data points assigned to each region), hyperparameters for pruning, etc



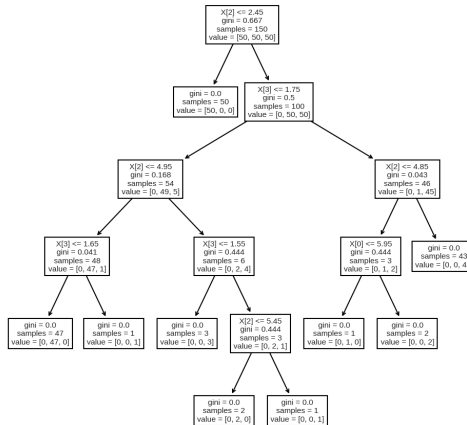
Visualization - maximum depth 3

Example visualization using scikit-learn on the data set iris

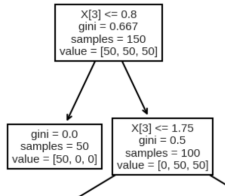


R_1, \dots, R_5 are the decision regions (leaf nodes)

Visualization - maximum depth 5

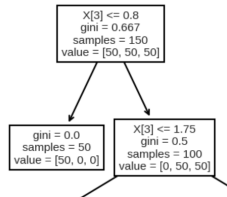


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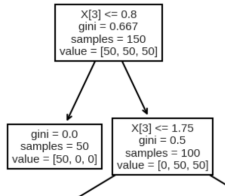
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- Once a tree is **trained**, an unseen data point \mathbf{x} will be assigned to a region R_m (leaf node) by the decision tree to identify the class \hat{y}
- To **train a tree**, we need to estimate the parameters, i.e. all the nodes and a class c_m associated with each leaf node R_m for $m = 1, \dots, M$

Parameter estimation (CART) (cont.)

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 - Yes, but they are in general more complex. We focus on binary splits in this lecture

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Impurity

- There are alternative impurity measures
 - **Gini score (used by CART):** $\sum_{b=1}^C \hat{p}_b(1 - \hat{p}_b)$
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- Previous example: left=[20, 15, 5] and right=[0, 35, 0]; in total, $N_{\text{left}} = 20 + 15 + 5 = 40$ and $N_{\text{right}} = 0 + 35 + 0 = 35$
 - $G_{\text{left}} = \frac{20}{40}(1 - \frac{20}{40}) + \frac{15}{40}(1 - \frac{15}{40}) + \frac{5}{40}(1 - \frac{5}{40}) = 0.59$
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- Impurity of the split: $G = \frac{N_{\text{left}}}{N_{\text{left}} + N_{\text{right}}} G_{\text{left}} + \frac{N_{\text{right}}}{N_{\text{left}} + N_{\text{right}}} G_{\text{right}}$

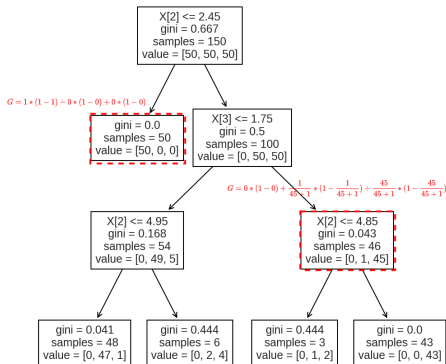
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 - $G = \frac{40}{40+35} G_{\text{left}} + \frac{35}{40+35} G_{\text{right}} = 0.53 * 0.59 + 0.467 * 0 = 0.3127$

Exercise: compute the Gini score for each node



$$\text{Gini: } G = \sum_{c=1}^C \hat{p}_{mc}(1 - \hat{p}_{mc})$$

Choose the best feature i and category c

Use the impurity score to choose the best i and c

- Loop over all $i = 1, \dots, d$
- For each i , loop over all categories $c = 0, \dots, C_i - 1$

For each choice in the loop, compute the impurity score and choose the i and c that minimizes the impurity.

Pseudocode for categorical variables (CART)

- For each node:
 - Choose the best feature index i
 - Choose the best category c

```
1: for  $i$  in  $[1, \dots, d]$  do
2:   for  $c$  in  $[0, \dots, C_i - 1]$  do
3:     Compute the Gini score
4:   end for
5: end for
6: Find the best  $i$  and  $c$  corresponding to the smallest Gini score
```

where C is the number of classes and C_i is the number of categories for feature x_i

- After the tree is constructed,

$$\hat{c}_m = \arg \max_c N_m^c, \forall c \in \{0, \dots, C - 1\}$$

where N_m^c is the number of data points that are 1) from class c and 2) assigned to region (leaf node) R_m

Example ($C = 3$): in region 2, if $[N_2^0, N_2^1, N_2^2] = [10, 25, 0]$, then $\hat{c}_m = 1$ since 25 is the largest count.

Parameter estimation (CART) for numerical variables

Similar to categorical variables with minor **modifications**

- Start with the root node that contains the entire training set
- **Grow the tree** by splitting each node until a stopping criterion is met, e.g. maximum depth, minimum number of data assigned to a leaf node, etc; this is a **recursive procedure**
- This **split** will create two child nodes (groups): left and right
- Each child node will contain a (sub) training set
 - $\mathcal{I}_{\text{left}} \leftarrow \{l \mid x_l^i \leq t\}$: all indices assigned to the left child node
 - $\mathcal{I}_{\text{right}} \leftarrow \{r \mid x_r^i > t\}$: all indices assigned to the right child node
- In order to **split** a node into the left and right child nodes, we need to estimate
 - The best feature x_i for the split
 - The splitting threshold t (**modification**)by minimizing the **impurity**
- To estimate x_i and t (**modification**)
 - Loop over all $i = 1, \dots, d$
 - For each feature index i , let $t \leftarrow x_i^j$, loop over all data in the (sub) training sample in the node

Pseudocode for numerical variables (CART)

- For each node:
 - Choose the best feature index i
 - Choose the best threshold t

```
1: for  $i$  in  $[1, \dots, d]$  do
2:   for  $j$  in  $[1, \dots, N_{\text{node}}]$  do
3:      $t \leftarrow x_i^j$ 
4:     Compute the Gini score
5:   end for
6: end for
7: Find the best  $i$  and  $t$  corresponding to the smallest Gini score
```

where N_{node} is the number of training data points in the current node

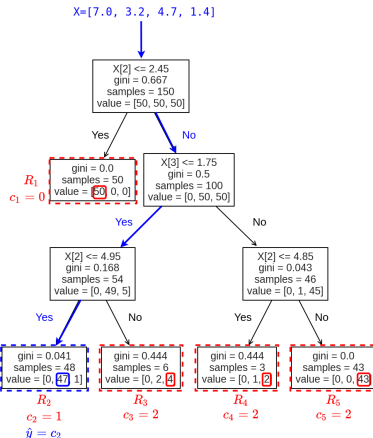
- After the tree is constructed,

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where N_m^c is the number of data points that are 1) from class c and 2) assigned to region (leaf node) R_m

Prediction with a trained decision tree

For a new data point $X = [7.0, 3.2, 4.7, 1.4]$, $\hat{y} = \sum_{m=1}^5 c_m I_{R_i}(x)$



Regression tree

Mathematical modeling for regression

- Modeling for regression

$$y = g(x; \theta | h)$$

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- Parameter estimation: supervised learning

Regression decision tree

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$$y = \sum_{m=1}^M a_m I_{R_i}(\mathbf{x})$$

where I_{R_i} is the indicator function:

$$I_{R_i}(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in R_i \\ 0, & \text{otherwise} \end{cases}$$

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- **Parameter estimation**: the classification and regression tree (CART) algorithm

Parameter estimation

- Similar to parameter estimation for classification trees
- The difference is the score being calculated
- Instead of using the impurity (e.g. Gini score), which is defined for classification, we use the sum of squared error instead

Pseudocode for regression tree (CART)

- For each node:
 - Choose the best feature i
 - Choose the best threshold t
- ```
1: for i in $[1, \dots, d]$ do
2: for j in $[1, \dots, N_{\text{node}}]$ do
3: $t \leftarrow x_i^j$
4: $\mathcal{I}_{\text{left}} \leftarrow \{l \mid x_i^l \leq t\}$: all indices assigned to the left child node
5: $\mathcal{I}_{\text{right}} \leftarrow \{r \mid x_i^r > t\}$: all indices assigned to the right child node
6: Let $a_{\text{left}} \leftarrow \bar{y}_l$ and $a_{\text{right}} \leftarrow \bar{y}_r$, where \bar{y}_l and \bar{y}_r are the sample means of the left and right child nodes, respectively.
7: Compute $s = \sum_{l \in \mathcal{I}_{\text{left}}} (y_l - a_{\text{left}})^2 + \sum_{r \in \mathcal{I}_{\text{right}}} (y_r - a_{\text{right}})^2$
8: end for
9: end for
```
- 10: Find the best  $i$  and  $t$  corresponding to the lowest score  $s$

where  $N_{\text{node}}$  is the number of training data points in the current node

- After the tree is constructed,

$$\hat{a}_m = \frac{1}{N_m} \sum_{i=1}^{N_m} y_{m_i}$$

where  $N_m$  is the number of data points assigned to region (leaf node)  $R_m$  and  $y_{m_i}$  denotes labels of data points assigned to  $R_m$



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Typically, there are additional hyperparameters (e.g.  $\alpha$ ) to tune using, e.g. cross-validation; a commonly used technique is **cost complexity pruning**, where complexity is defined as the number of leaf nodes.



# ID3, C4.5, C5.0, CART

# Comparison of parameter estimation algorithms

They are parameter estimation algorithms in the literature.

| Algorithm                | ID3              | C4.5                       | C5.0                       | CART                       |
|--------------------------|------------------|----------------------------|----------------------------|----------------------------|
| Variable type            | Categorical      | Continuous and categorical | Continuous and categorical | Continuous and categorical |
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- For example, the scikit-learn documentation (December 2021):  
<https://scikit-learn.org/stable/modules/tree.html>
- The most important thing is to understand the different components in the algorithms, e.g. splitting score, splitting criteria, maximum tree depth, etc

# Summary

- **Pros:**
  - Interpretable, visualizable, intuitive
  - Do not require preprocessing
  - Handle categorical and continuous variables
  - Fairly simple to implement
- **Cons:**
  - High variance, overfitting
  - Not robust

# Today

- 1 Decision trees
  - Classification tree
  - Regression tree
  - Pruning
  - ID3, C4.5, C5.0, CART
- 2 Ensemble methods
  - Bias-variance trade-off
  - Bagging
  - Random forest
- 3 Summary

## Bias-variance trade-off



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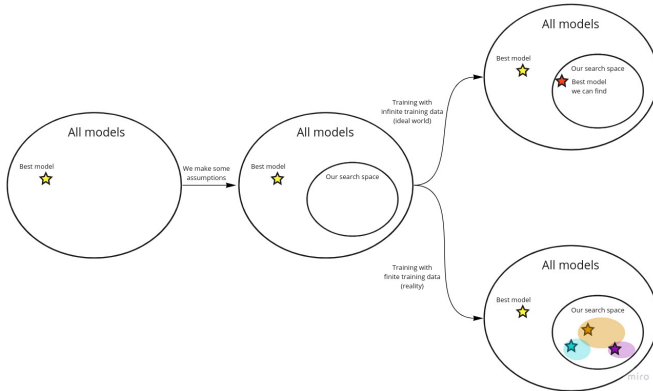
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## Bias-variance trade-off (cont.)

- Complex models tend to have low bias and high variance
- Simple models tend to have high bias and low variance
- Ensemble methods combine multiple models to produce a better result - with reduced bias or variance



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  - Note: this variance cannot be reduced to zero

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- Step 2: Build a tree based on the bootstrapped sample
- Step 3: Repeat step 1 and 2  $B$  times, which results in  $B$  trees

## Bagging: prediction

Aggregate the predictions from all  $B$  trees

- Regression tree (sample mean):

$$\hat{y} = \frac{1}{B} \sum_{b=1}^B \hat{y}_b^*$$

- Classification tree (majority vote):

$$\hat{y} = \arg \max_c \text{count}(\hat{y}_b^* == c)$$

# Random forest

# Random forest

Similar to bagging with one addition +

- Step 1: Create a bootstrapped sample
- Step 2: Build a tree based on the bootstrapped sample
  - + Randomly choose  $\tilde{d} \approx \sqrt{d}$  at each node and only use  $\tilde{d}$  features for the split (**decorrelation**)
- Step 3: Repeat step 1 and 2  $B$  times, which results in  $B$  trees

Compared to decision trees, random forest results in significantly lower variance with slightly increased bias; less interpretable

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# Summary

- Data types and data containers
- Descriptive data analysis: descriptive statistics, visualization
- Probability distributions, events, random variables, PMF, PDF, parameters
- CDF, Q-Q plot, how to compare two distributions (data vs theoretical, data vs data)
- Modeling
- Parameter estimation: maximum likelihood estimation (MLE) and maximum a posteriori estimation (MAP)
- Classification, multinomial naive Bayes classifier, Gaussian naive Bayes classifier
- Central limit theorem, interval estimation
- Hypothesis tests, comparison of two classifiers
- Clustering, cluster tendency, k-means
- SSE and Silhouette score for cluster evaluation, one dimensional Gaussian Mixture Models, AIC/BIC, the EM algorithm, clustering validation
- Decision tree for classification and regression, random forest

Stay safe, happy, curious and enthusiastic!

