

Lecture 7: Hypothesis testing part I

Statistical Methods for Data Science

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Today

1 Terminology

- Experiment and the parameter of interest
- Null hypothesis and alternative hypothesis
- Test statistic
- Null distribution $f(s | H_0)$
- Significance level α , power and p -value

2 Example

3 Summary

Learning outcome

- Be able to explain the following terminology
 - Null hypothesis H_0 and alternative hypothesis H_A
 - Test statistic s
 - Null distribution $f(s | H_0)$
 - Significance level α and power
 - p -value
- Be able to design and interpret the one-sample z-test

Today

1 Terminology

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2 Example

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Example

If you control the diet of your ducks, they lose 2.1 kg after one month on average

- Company A has developed a drug D to help ducks lose weight. They claim that **on average** the drug works better than diet control
- Company B has developed a drug E and they claim that drug E is more effective than drug D **on average**

You need to help your chonker ducks lose weight. Which drug do you buy? Or should you just control their diet?

- If company A tested drug D on 30 ducks and the average weight loss after one month is 2.2 kg, would you buy drug D instead of regular diet control?
- What if company A tested drug D on 30 ducks and the average weight loss after one month is 2.3 kg? Would you buy drug D instead of regular diet control in this case?
- What if company A tested drug D on 100 ducks and the average weight loss after one month is 2.3 kg?
- Now company B tested drug E on 30 ducks and the average weight loss after one month is 2.5 kg, while drug D results in 2.3 kg weight loss with the same setup, would you buy drug E instead of drug D?

How would you make your decision?

Hypothesis

- **Hypothesis:** a hypothesis is a proposed explanation for a phenomenon (wikipedia)
- **Statistical hypothesis:** a proposed distribution that explains a set of random variables
- **Hypothesis testing in statistics:** we want to decide if it is likely that the random variables follow the distribution proposed by the statistical hypothesis
 - The test is based on sample statistics, which are computed from data
 - Hypothesis + data \rightarrow decision on rejecting/not rejecting the hypothesis

Hypothesis testing: a list to go through

- A “boring” statement
- Experiment
- Data x , random variable X
- Parameter of interest θ
- Parameter estimate $\hat{\theta}$
- Null hypothesis H_0
- Alternative hypothesis H_A
- Test statistic s
- Null distribution $f(s \mid H_0)$
- Significance level α
- p -value

Experiment and the parameter of interest

Experiment design

- Before formulating the statistical hypothesis, we need a **“boring” statement**: a claim that we would like to test against, e.g. drug D is not more effective than regular diet on average; drug E works the same as drug D on average
- How do we test the “boring” statement? We design and run **experiments** to collect evidence (**data**)
- Example 1: recall if you control the diet of your ducks, they lose 2.1 kg after one month on average
 - **A “boring” statement**: drug D is not more effective than regular diet on average
 - **Experiment** (5 sec): test drug D on N chonker ducks and record the average weight loss after one month
 - **Data** and **random variable** (5 sec): data - x_i weight loss after one month for $i = 1, \dots, N$; random variable - X_i i.i.d.
 - **Parameter of interest** (5 sec): the average weight loss μ_D
 - **Parameter estimate** (5 sec): $\hat{\mu}_D = \bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$

Then we can use \bar{x} to approximate μ_D and check if it is greater than diet control (2.1 kg)

Experiment design (cont.)

- Example 2:
 - **A “boring” statement**: drug E and drug D work the same on average
 - **Experiment** (5 sec): test drug D on N_D chonker ducks and record the average weight loss after one month; test drug E on another N_E chonker ducks and record the average weight loss after one month
 - **Data** and **random variable** (5 sec): data - x_i weight loss for drug D after one month for $i = 1, \dots, N_D$; random variable - X_i i.i.d.; likewise, we have data y_j and random variable Y_j for drug E
 - **Parameter of interest** (5 secs): the average weight loss μ_D and μ_E for drug D and E, respectively
 - **Parameter estimate** (5 secs): $\hat{\mu}_D = \bar{x} = \frac{1}{N_D} \sum_{i=1}^{N_D} x_i$ and $\hat{\mu}_E = \bar{y} = \frac{1}{N_E} \sum_{j=1}^{N_E} y_j$

Then we use \bar{x} and \bar{y} to approximate μ_D and μ_E to see if they are the same

Experiment design (cont.)

- We make our decision by observing data; if the evidence does not support the “boring” statement, we **reject the statement**; otherwise, we **do not reject the statement**
- But we can never prove or accept the statement - we can only **reject** a statement by showing counterexamples
- The logic here is: if the statement is true, then the evidence must support the statement \iff if the evidence does not support the statement, the statement must be false \niff if the evidence supports the statement, the statement must be true

Null hypothesis and alternative hypothesis

Hypotheses H_0 and H_A

- **Statistical hypothesis**: a proposed distribution - a statement in terms of the **parameter of interest**
- **Null hypothesis H_0** : the “boring” statement translated into a mathematical expression
 - Example 1: drug D is not more effective than regular diet on average

$$H_0 : \mu_D = 2.1$$

- Example 2: drug E and drug D work the same on average (5 sec)

$$H_0 : \mu_D = \mu_E$$

- **Alternative hypothesis H_A** : a complementary alternative explanation to the “boring” statement
 - Example 1: drug D is more effective than regular diet on average (5 sec)

$$H_A : \mu_D > 2.1$$

- Example 2: drug E and drug D do not work the same on average (5 sec)

$$H_A : \mu_D \neq \mu_E$$

Hypotheses H_0 and H_A (cont.)

Questions:

- Question 1: why are $H_A : \mu_D > 2.1$ and $H_0 : \mu_D = 2.1$ complementary to each other? What about $H_A : \mu_D < 2.1$?

Answer: an implicit assumption here is that μ_D will not be smaller than 2.1

- Question 2: can H_0 and H_A be ANYTHING I want? Like a magic mirror!?

Answer: no

- Follow up question: what are the choices for H_0 and H_A ?

Choices for H_0

- In this course, we only deal with null hypotheses **with an equal sign** in them - only one fixed choice for the distribution proposed by H_0
- **Null hypothesis H_0** : two cases
 - **One-sample test**: to test a data distribution against a theoretical probability distribution, i.e. for a given constant c

$$H_0 : \theta = c$$

For example, is a binary classifier more accurate than random? $H_0 : p = 50\%$

- **Two-sample test**: to test a data distribution against another data distribution, i.e.

$$H_0 : \theta_1 = \theta_2$$

For example, is classifier A better than classifier B? $H_0 : p_A = p_B$

- We have seen one-sample test and two-sample test in the Q-Q plot lecture
- In practice, you can narrow down your choice of hypotheses by making a Q-Q plot

Choices for H_A

Given

$$H_0 : \theta = \beta$$

where β can be either a constant (one-sample test) or a parameter from another data distribution (two-sample test)

- **Alternative hypothesis H_A :** H_A can be **one-tailed** or **two-tailed**
 - **One-tailed:**

$$H_A : \theta > \beta$$

or

$$H_A : \theta < \beta$$

- **Two-tailed:**

$$H_A : \theta \neq \beta \iff \theta < \beta \text{ or } \theta > \beta$$

Summary: choices for H_0 and H_A

Putting everything together,

	One-sample test	Two-sample test
Two-tailed	$H_0 : \theta = c, H_A : \theta \neq c$	$H_0 : \theta_1 = \theta_2, H_A : \theta_1 \neq \theta_2$
One-tailed	$H_0 : \theta = c, H_A : \theta > c$	$H_0 : \theta_1 = \theta_2, H_A : \theta_1 > \theta_2$
	$H_0 : \theta = c, H_A : \theta < c$	$H_0 : \theta_1 = \theta_2, H_A : \theta_1 < \theta_2$

where $\theta, \theta_1, \theta_2$ are the parameters of interest and c is a constant

Test statistic

Test statistic

- **Test statistic s , random variable S :** the statistic used for testing the hypothesis
 - s is the **observation**
 - Given a set of parameters of interest and a set of estimates, s is typically a **standardized statistic** computed from the estimates
 - **Purpose:** to compare s with a **standard distribution**, e.g. the standard Gaussian distribution $\mathcal{N}(0, 1)$, to see if it is likely that the standard distribution is the underlying distribution of S , i.e. if the null hypothesis is plausible
- What is needed for computing the test statistic?
 - Assumptions on random variables X_i
 - We only need the null hypothesis H_0 (not H_A) to choose the test statistic

Note: in this course, we only deal with null hypothesis where we are able to express the PDF/PMF $f(s | H_0)$, i.e. H_0 with an equal sign in them

Test statistic (cont.)

- Example 1. one-sample test

- **Data:** x_1, \dots, x_N
- **Random variable:** X_1, \dots, X_N i.i.d. **Gaussian with known σ**
- **Parameter of interest:** μ_D
- **Parameter estimate:** \bar{x}
- **Null hypothesis:** $H_0 : \mu_D = 2.1$
- **Test statistic:** standardized \bar{x} assuming the null hypothesis (15 sec)

$$z = \frac{\bar{x} - 2.1}{\sigma/\sqrt{N}}$$

- Example 2. two-sample test

- **Data:** x_1, \dots, x_{N_D} and y_1, \dots, y_{N_E}
- **Random variable:** X_1, \dots, X_{N_D} i.i.d. **Gaussian with known σ_D** ; Y_1, \dots, Y_{N_E} i.i.d. **Gaussian with known σ_E** ; X_i and Y_j independent
- **Parameter of interest:** μ_D, μ_E
- **Parameter estimate:** \bar{x}, \bar{y}
- **Null hypothesis:** $H_0 : \mu_D = \mu_E \iff H_0 : \mu_D - \mu_E = 0$
- **Test statistic:** standardized $\bar{x} - \bar{y}$ assuming the null hypothesis

$$z = \frac{\bar{x} - \bar{y}}{\sqrt{\sigma_D^2/N_D + \sigma_E^2/N_E}} \text{ (explained later)}$$

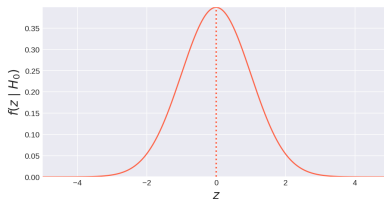
Null distribution $f(s | H_0)$

Null distribution

- **Null distribution $f(s | H_0)$:** the distribution of the test statistic given the null hypothesis
- Example:
 - **Data:** x_1, \dots, x_N
 - **Random variable:** X_1, \dots, X_N i.i.d. Gaussian with known σ
 - **Parameter of interest:** μ
 - **Parameter estimate:** \bar{x}
 - **Null hypothesis:** $H_0 : \mu = \mu_0$
 - **Test statistic:**

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{N}}$$

- **Null distribution:** standard Gaussian distribution



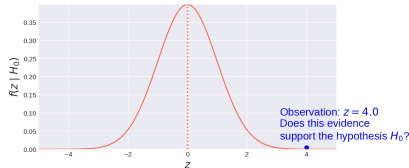
Significance level α , power and p -value

Significance level

Given a null hypothesis $H_0 : \mu = 2.1$ and the null distribution $f(s | H_0)$, we decide if we reject the hypothesis or not by observing data

- Run some experiments and collect data x_1, \dots, x_N
- Estimate the parameter of interest $\hat{\theta}$, e.g. $\hat{\mu} = \bar{x} = \frac{1}{N} \sum_{i=1}^N x_i$
- Standardize $\hat{\theta}$ assuming H_0 to compute the test statistic, e.g.

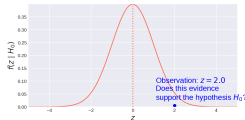
$$z = \frac{\bar{x} - 2.1}{\sigma/\sqrt{N}} = 4.0$$



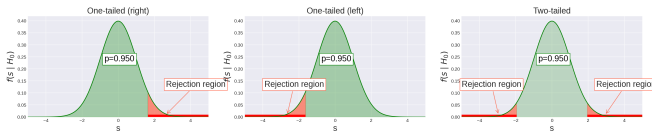
- Does this evidence support the hypothesis H_0 ? Probably not since it's so far away from the center?

Significance level (cont.)

- What about this observation?



- To be able to answer the question, you need to decide where you draw the line - define a **rejection region** by choosing a significance level
- Significance level α** : red area under the curve



In these three images, $\alpha = 0.05$

More conservative \Rightarrow less probable to reject H_0 , which indicates a smaller rejection region

Two-tailed H_A is more conservative

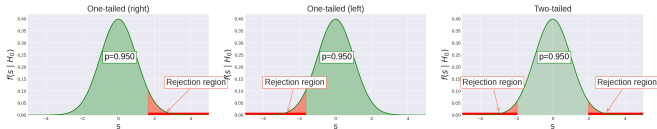
Significance level (cont.)

What is needed for choosing a meaningful α ?

- Null distribution
- H_A one-tailed or two-tailed

Interpretation of α

- $\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true})$ - the probability of making such a mistake



- The rejection region indicates that H_0 is **unlikely**, but the probability is not zero
- It is possible that H_0 is true, but our observation happens to fall in the rejection region
- If H_0 is true and our observation falls in the rejection region, we will **mistakenly** reject H_0
- The probability of making this type of mistakes is α
- Similar to the confidence interval, $1 - \alpha$ is called the **confidence level** - “with 95% confidence, rejecting H_0 is the right thing to do”
- Define the significance level **before you run the experiments** so that you can't cheat!

Significance level and power

- Contingency table:

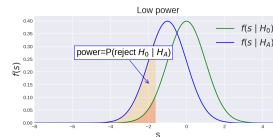
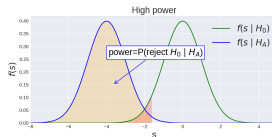
	$y = H_A$	$y = H_0$
$\hat{y} = \text{reject } H_0$	TP	FP (Type I error)
$\hat{y} = \text{do not reject } H_0$	FN (Type II error)	TN

- Significance level α : incorrectly rejecting H_0

$$\alpha = P(\text{type I error})$$

- Power: correctly rejecting H_0

$$\text{power} = P(\text{reject } H_0 | H_A) = 1 - P(\text{type II error})$$



- What is needed for computing the power? $f(s | H_0)$, $f(s | H_A)$

p -value

- p -value:

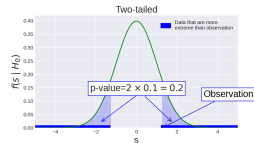
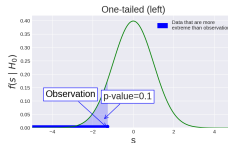
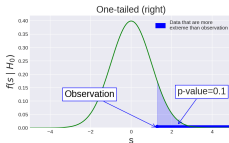
- One-tailed:

- Right tail: $p = P(S \geq s | H_0)$
- Left tail: $p = P(S \leq s | H_0)$

- Two-tailed: $p = 2 \min(P(S \leq s | H_0), P(S \geq s | H_0))$

Note: for example, if $f(s | H_0)$ is symmetric around zero and $s < 0$,

$$p = 2P(S \leq s | H_0)$$



- What is needed for computing the p -value? (10 sec)

- Null distribution
- Alternative hypothesis H_A to know one-tailed or two-tailed
- Observation - test statistic computed from data

Summary: steps for hypothesis testing

- Step 1 Make a “boring” statement
- Step 2 Design an **experiment**
- Step 3 Describe the **data** generated from the experiment and the corresponding random variables
- Step 4 Describe the parameter of interest and their estimates
- Step 5 Translate the “boring” statement into a statistical hypothesis and call it the **null hypothesis H_0**
- Step 6 Find the expression for the **test statistic s**
- Step 7 Find the expression for the **null distribution**
- Step 8 Define **an alternative hypothesis H_A** : one-tailed or two-tailed
- Step 9 Choose a **significance level α** (the tail), which defines the **rejection region**
- Step 10 Collect **data**
- Step 11 Compute the test statistic from data
- Step 12 Compute the p -value
- Step 13 If $p\text{-value} < \alpha$, i.e. the test statistic falls in the rejection region of the null distribution, then we reject the hypothesis H_0 ; otherwise, we fail to reject H_0 .

Today

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Example

Recall example: if you control the diet of your ducks, they lose 2.1 kg after one month on average. Company A has developed a drug D to help ducks lose weight. They claim that on average the drug works better than diet control. Here is the set up for the experiment.

- Step 1 Make a “boring” statement (5 secs): **drug D works the same as diet**
- Step 2 Design an **experiment** (choose $N = 30$) (10 secs): **let 30 chonker ducks take drug D and measure their weight loss after one month**
- Step 3 Describe the **data** and **random variables** with assumptions about their distributions (5 secs): **weight loss x_1, \dots, x_{30} ; X_1, \dots, X_{30} i.i.d. Gaussian random variables** - let's make an additional assumption to simplify the problem - the standard deviation of X_i $\sigma = 0.6$ is known
- Step 4 Describe the parameter of interest and their estimates (10 secs): **the mean value μ_D and $\hat{\mu}_D = \bar{x}$**
- Step 5 Translate the “boring” statement into a statistical hypothesis and call it the **null hypothesis H_0** (10 secs): **$H_0 : \mu_D = 2.1$**
- Step 6 Find the expression for the **test statistic s** (60 secs):

$$s = z = \frac{\bar{x} - 2.1}{\sigma\sqrt{30}}$$

- Step 7 Find the expression for the **null distribution $f(s | H_0)$** (10 secs):

$$f(z | H_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

Example (cont.)

Step 8 Define **an alternative hypothesis** H_A (10 secs):

$$H_A : \mu_D \neq 2.1 \text{ or } H_A : \mu_D > 2.1$$

One-tailed or two-tailed

- **Two-tailed (5 secs):** $H_A : \mu_D \neq 2.1$
- **One-tailed (5 secs):** $H_A : \mu_D > 2.1$

Step 9 Choose a **significance level** α (the tail), which defines the rejection region (5 secs): e.g. $\alpha = 0.05$

Step 10 Collect 30 ducks in 20 secs and feed them drugs - great job! Weights measured after one month x_1, \dots, x_{30}

$$\text{Say } \frac{1}{30} \sum_{i=1}^{30} x_i = 2.2$$

Step 11 Compute the test statistic from data (5 secs):

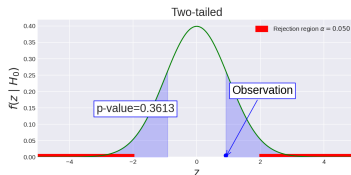
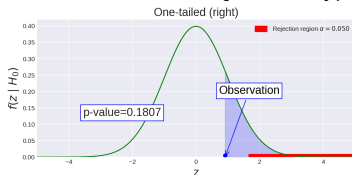
$$z_0 = \frac{2.2 - 2.1}{0.6/\sqrt{30}} = 0.91$$

Example (cont.)

Step 12 Compute the p -value (20 secs):

- For $H_A : \mu_D > 2.1$ (**one-tailed**): $p = P(Z \geq z_0 | H_0) = 0.1807 > \alpha$
- For $H_A : \mu_D \neq 2.1$ (**two-tailed**): $p = 2P(Z \geq z_0 | H_0) = 0.3613 > \alpha$

Step 13 If $p\text{-value} < \alpha$, i.e. the test statistic falls in the rejection region of the null distribution, then we reject the hypothesis H_0



Do not reject H_0 for both one-tailed and two-tailed H_A

Example (cont.)

What if $\bar{x} = 2.3$?

Step 11 Compute the test statistic from data (5 secs):

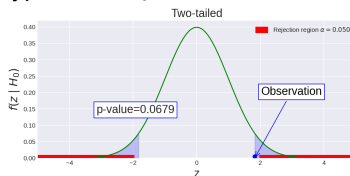
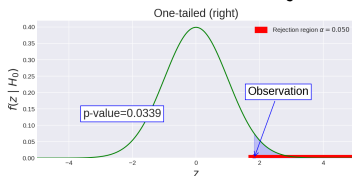
$$z_0 = \frac{2.3 - 2.1}{0.6/\sqrt{30}} = 1.826$$

Step 12 Compute the p -value (20 secs):

- **For $H_A : \mu_D > 2.1$ (one-tailed):** $p = P(Z \geq z_0 \mid H_0) = 0.0339 < \alpha$
- **For $H_A : \mu_D \neq 2.1$ (two-tailed):** $p = 2P(Z \geq z_0 \mid H_0) = 0.0679 > \alpha$

Example (cont.)

Step 13 If $p\text{-value} < \alpha$, i.e. the test statistic falls in the rejection region of the null distribution, then we reject the hypothesis H_0



Reject H_0 for one-tailed H_A ; do not reject H_0 for two-tailed H_A for the same confidence level $1 - \alpha = 5\%$

Note: the two-tailed test is more conservative - if the data passes a two-tailed test, it is more conclusive than one-tailed test for the same confidence level

Example (cont.)

What if $\bar{x} = 2.3$ with $N = 100$?

Step 11 Compute the test statistic from data (5 secs):

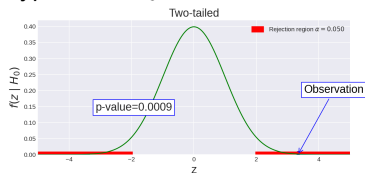
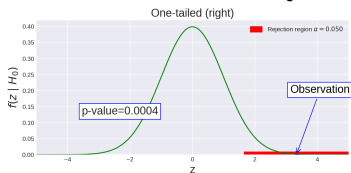
$$z_0 = \frac{2.3 - 2.1}{0.6/\sqrt{100}} = 3.33$$

Step 12 Compute the p -value (20 secs):

- **For $H_A : \mu_D > 2.1$ (one-tailed):** $p = P(Z \geq z_0 \mid H_0) = 0.0004 < \alpha$
- **For $H_A : \mu_D \neq 2.1$ (two-tailed):** $p = 2P(Z \geq z_0 \mid H_0) = 0.0009 < \alpha$

Example (cont.)

Step 13 If $p\text{-value} < \alpha$, i.e. the test statistic falls in the rejection region of the null distribution, then we reject the hypothesis H_0



Reject H_0 for both one-tailed and two-tailed H_A

Note:

- With more data, it becomes more certain that we should reject H_0 in favor of H_A given the observation $\bar{x} = 2.3$

This test is called **one-sample z-test**

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Summary

So far:

- Data types and data containers
- Descriptive data analysis: descriptive statistics, visualization
- Probability distributions, events, random variables, PMF, PDF, parameters
- CDF, Q-Q plot, how to compare two distributions (data vs theoretical, data vs data)
- Modeling
- Parameter estimation: maximum likelihood estimation (MLE) and maximum a posteriori estimation (MAP)
- Classification, multinomial naive Bayes classifier, Gaussian naive Bayes classifier
- Central limit theorem, interval estimation
- Hypothesis test

Next:

- More examples, test statistics; comparison of two classifiers

Before next lecture:

- Steps for hypothesis testing

