Lecture 6: Interval estimation

Statistical Methods for Data Science

Yinan Yu

Department of Computer Science and Engineering

November 19, 2020

Today

1. Central limit theorem
   - Terminology
   - Standardization
   - Central limit theorem
2. Interval estimation
   - Confidence interval
   - Credible interval
3. Summary

Learning outcome

- Be able to explain the following terminology:
  - Sample statistic, sampling distribution, sample mean, sample variance, standardization, z-table, t-table
  - Point estimation, interval estimation
  - Confidence interval, credible interval
- Be able to explain the central limit theorem (CLT)
- Be able to construct the following interval estimates:
  - Confidence interval for
    - sample mean of i.i.d. sample with unknown σ
    - unknown sampling distribution using bootstrap
  - Credible interval for a given posterior function

Today

1. Central limit theorem
   - Terminology
   - Standardization
   - Central limit theorem
2. Interval estimation
3. Summary

Terminology

Terminology

- **Sample**: a random data set $\{x_1, x_2, \cdots, x_N\}$; the corresponding random variables are denoted as $X_1, X_2 \cdots, X_N$.
- **i.i.d. sample**: $X_1, X_2 \cdots, X_N$ are i.i.d. random variables
- **Sample statistic**: a statistic computed from a sample. For example
  - **Sample mean**:

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i$$

  - **Sample variance**:

$$S^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2$$

- **Sampling distribution**: the probability distribution of a sample statistic that is computed from a random sample of size N
- Asymptotic: in this context, asymptotic means $N \to \infty$

Note: as usual, capital letters and small letters are used to denote random variables and the values, respectively.

Properties of Gaussian random variables

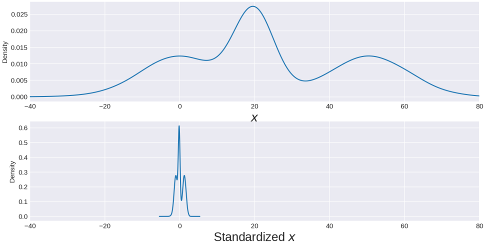
- Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ be a Gaussian random variable, then the following random variables are also Gaussian
  - Scaling: $tX \sim \mathcal{N}(t\mu_X, t^2\sigma_X^2)$, $t \neq 0$ is a constant
  - Translation: $X + c \sim \mathcal{N}(\mu_X + c, \sigma_X^2)$, c is a constant
  - $tX + c \sim \mathcal{N}(t\mu_X + c, t^2\sigma_X^2)$
- Let $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ be two **independent** Gaussian random variables, then the following random variables are also Gaussian
  - $X + Y \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$
  - $X - Y \sim \mathcal{N}(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2)$

Standardization

Standardization

- Why standardization? We want to translate and scale data into a standard "shape" so that we can use standard tools to compare and analyze them
- Let X be a random variable that follows **any probability distribution** with mean μ and standard deviation σ. The standardization of X is

$$Y = \frac{X - \mu}{\sigma}$$

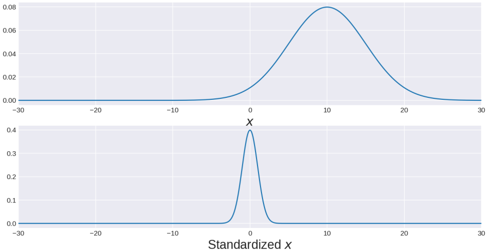


Now the new variable Y has mean 0 and standard deviation 1.

Standardization

- Let X be a random variable following a **Gaussian distribution** with mean μ and standard deviation σ, i.e. $X \sim \mathcal{N}(\mu, \sigma^2)$. The standardization of X is

$$Z = \frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1) \tag{1}$$



The distribution $\mathcal{N}(0, 1)$ is called a **standard Gaussian (normal) distribution**

Standard Gaussian distribution

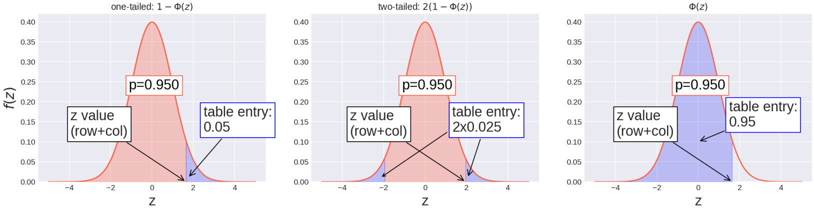
- Remember how much we love Gaussian distributions? **We love the standard Gaussian distribution even more!** We love it so much that we gave its CDF a special name: $\Phi(z)$.
- There is a table describing the quantiles of the standard Gaussian called the **z-table**.
  - Each row represents the integer and the first decimal of z
  - Each column represents the second decimal of z
  - Each cell is the

$$
\begin{aligned}
P(Z \leq \text{row} + \text{column}) &= \Phi(\text{row} + \text{column}) \\
&= \textbf{stats.norm.cdf(x=row + column, loc=0, scale=1)}
\end{aligned}
$$

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670

Standard Gaussian distribution (cont.)

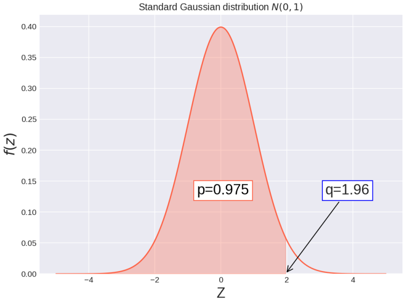
- There are different types for the z-table. The difference is what is inside each cell, e.g. $\Phi(\text{row} + \text{column})$, $2(1 - \Phi(\text{row} + \text{column}))$, $1 - \Phi(\text{row} + \text{column})$ or $\frac{1}{2}(1 - \Phi(\text{row} + \text{column}))$. But the principle is the same. We will come back to this later. For now we use the version with $\Phi(\text{row} + \text{column})$.



- Due to symmetry, there are only positive values for z in the z-table.

Standard Gaussian distribution (cont.)

Exercise:



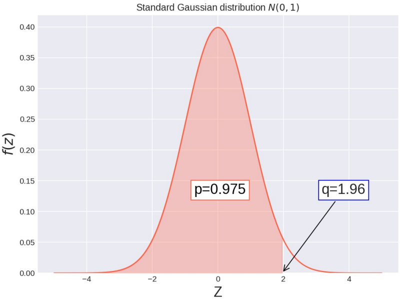
z-table

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670

Try to find the corresponding pair $(p, q) = (0.975, 1.96)$ in the z-table (60 secs).

Standard Gaussian distribution (cont.)

Answer:



z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670

q=1.9+0.06=1.96

p=0.9750

Central limit theorem

Distribution of the sample mean

- You have 1000 ducks.
- Now, you take 30 of them and measure the sample mean of their weights x_i:

$$\hat{\mu}_1 = \frac{1}{30} \sum_{i=1}^{30} x_i$$

.

- Then you take another 30 ducks to measure the sample mean of their weights y_i:

$$\hat{\mu}_2 = \frac{1}{30} \sum_{i=1}^{30} y_i$$

.

- You do this experiment 100 times and plot the histogram of these 100 sample means $\hat{\mu}_j$ for $j = 1, \cdots, 100$.
- Then you realize **these sample means $\hat{\mu}_j$ follow a Gaussian distribution**. 🤔

Distribution of the sample mean (cont.)

- The colors of your 1000 ducks can be either red $t_i = 0$ or blue $t_i = 1$.
- Now, you take 30 of them and measure the sample mean of their color t_i:

$$\hat{n}_1 = \frac{1}{30}\sum_{i=1}^{30} t_i = \frac{1}{30}(1 + 1 + 0 + 1 + \cdots \ldots 1 + 1)$$

  Note: here $t_i \in \{0, 1\}$ has discrete value.

- You take another 30 ducks and measure the sample mean of their color t_i:

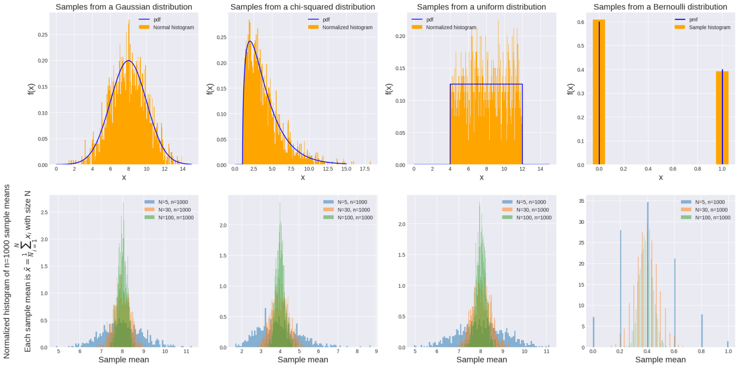
$$\hat{n}_2 = \frac{1}{30}\sum_{i=1}^{30} t_i = \frac{1}{30}(0 + 0 + 0 + 1 + \cdots \ldots 1 + 0)$$

  .

- You do this experiment 100 times and plot the histogram of these 100 sample means $\hat{n}_j$.
- Then you realize **these sample means n_j also follow a Gaussian distribution**. 🤯

Distribution of the sample mean (cont.)

- In fact, this is true for i.i.d. samples drawn from ANY probability distribution.



- The larger the sample size N (in the previous example $N = 30$), the more Gaussian it becomes
- A rule of thumb: $N \geq 30$
- If the data distribution is Gaussian-like (bell-shaped, symmetric), only a small sample size is needed for the sample mean to be Gaussian

Central limit theorem

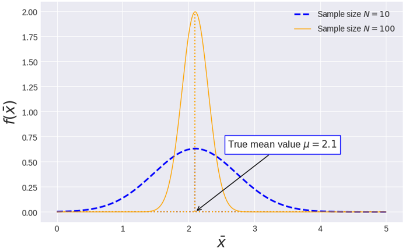
- One of the most important results in probability theory and statistics
- Given an i.i.d. sample $X_1, X_2, \cdots, X_N$ from **ANY probability distribution** with *finite mean* μ *and variance* σ^2 (most distributions satisfy this!), when the sample size N is sufficiently large, the **sample mean** approximately follows a Gaussian distribution with mean μ and variance $\frac{\sigma^2}{N}$, i.e.

$$\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{N}) \tag{2}$$

Central limit theorem (cont.)

$$\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{N})$$

- The estimate $\bar{X}$ is around the true value μ
- The "deviation" of $\bar{X}$ from μ is $\frac{\sigma^2}{N}$; the larger N, the smaller the deviation



Central limit theorem use cases

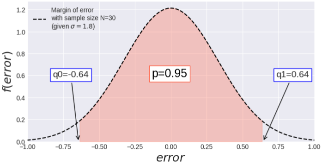
In what scenarios we care about the sample mean?

- All the time!
- Example: we want to test the effectiveness of a drug. A patient can be either cured by this drug ($X = 1$) or not cured ($X = 0$), i.e. we can model X using a (2 secs) *Bernoulli distribution* with parameter (2 second) p (cure rate) and the maximum likelihood estimation of p is the (4 secs) **sample mean**.
- Example: we want to compare the effectiveness of two drugs. Then we have two random variables X (for drug 1) and Y (for drug 2) tested on independent patients. The **sample mean** $\bar{X}$ are $\bar{Y}$ are the maximum likelihood of their cure rates for X and Y, respectively. Now, we want to compare these two sample means to see if they are sufficiently different and the difference $\bar{X} - \bar{Y}$ also follows a Gaussian distribution.
- In general, we are often interested in how things work "on average"

Estimation error $\bar{X} - \mu$

Random variable: $X_1, \cdots, X_N$
Assumption: i.i.d. with **known** standard deviation σ and **unknown** mean μ

- In many use cases, we want to estimate μ using the sample mean $\bar{X}$
- From CLT (cf. Eq. (20)), we know that for a large N, the sample mean approximately follows a Gaussian distribution $\bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{N})$ - $\bar{X}$ is around the true mean μ
- Let $\mathcal{E} = \bar{X} - \mu \sim \mathcal{N}(0, \frac{\sigma^2}{N})$ be the random **error** term of the estimate $\bar{X}$. Can we plot the PDF of $\mathcal{E}$? (5 secs) Yes! σ and N are both **known**



- Interpretation of the plot: (5 secs) **95% of the time, the error $\bar{X} - \mu$ is within $q0 = -0.64$ and $q1 = 0.64$**
- Now it's pretty cool because not only can we estimate the mean, but we can also give a margin of error!
- This **95%** is called the **confidence level**. For a given confidence level, we can find a corresponding **interval** $(q0, q1)$.

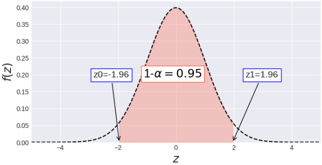
Calculate the margin of error

- For a given confidence level, denoted as $1 - \alpha$, how do we find this interval for the error in Python? We can use the function **ppf** from **scipy.stats**

```
std = 1.8
N = 30
alpha = 0.05
confidence_level = 1 - alpha # 95% confidence level
q0 = stats.norm.ppf(alpha/2,
                    0, std/math.sqrt(N))
q1 = stats.norm.ppf(confidence_level+alpha/2,
                    0, std/math.sqrt(N))
>> (-0.6441098917381766, 0.6441098917381766)
```

Find a standardized expression for the margin of error

- Standardize $\mathcal{E}$ by $\frac{\mathcal{E}}{\sigma/\sqrt{N}} = \frac{\bar{X}-\mu}{\sigma/\sqrt{N}} \sim \mathcal{N}(0,1)$
- We just learned that there is a special name for the standard Gaussian distributed random variable - $Z \sim \mathcal{N}(0,1)$ - let $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{N}}$
- Now we have an expression for the error term $\mathcal{E} = \bar{X} - \mu = Z\frac{\sigma}{\sqrt{N}}$
- The only random variable here is $Z \sim \mathcal{N}(0,1)$

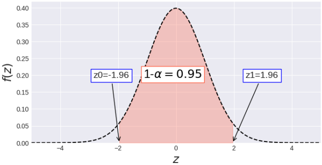


- In order to find an interval for $\mathcal{E}$, we just need to look at the distribution of Z and find the interval $(z0\frac{\sigma}{\sqrt{N}}, z1\frac{\sigma}{\sqrt{N}})$
- We can use a two-tailed z-table (cf. page 12) to find the values for $z0$ and $z1$

Find a standardized expression for the margin of error (cont.)

- For example, with $1-\alpha = 95\%$ confidence level, the error is within

$$\left(-1.96\frac{\sigma}{\sqrt{N}},\ 1.96\frac{\sigma}{\sqrt{N}}\right)$$



- Generally speaking, the value $z1$ (denoted by $z_{\alpha/2}$) is the quantile at $1-\alpha/2$. The value of $z_{\alpha/2}$ is called the **(right) critical value**; $\frac{\sigma}{\sqrt{N}}$ is called the **standard error**. In this example, we have $z_{\alpha/2} = z1 = -z0 = 1.96$.
- Why **two-tailed** z-table: there are two tails $z \leq -z_{\alpha/2}$ and $z \geq z_{\alpha/2}$.

Find a standardized expression for the margin of error (cont.)

- In Python

```
std = 1.8
N = 30
alpha = 0.05
confidence_level = 1 - alpha # 95% confidence level
z0 = stats.norm.ppf(alpha/2, 0, 1)
z1 = stats.norm.ppf(confidence_level+alpha/2, 0, 1)
print(z0*std/math.sqrt(N), z1*std/math.sqrt(N))
>> (-0.6441098917381766, 0.6441098917381766)
```

Find a standardized expression for the margin of error (cont.)

- For a given sample with an estimate $\bar{x}$ (note: here the small letter $\bar{x}$ denotes the value of the estimate itself instead of a random variable), it's more convenient to have this margin of error around $\bar{x}$ instead - so that we can say: the estimated mean is $\bar{x}$ with this uncertainty:

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{N}},\ \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right)$$

- This is called the **confidence interval**
- The confidence interval for the sample mean is exact when the data distribution is Gaussian, otherwise it is an approximation under the central limit theorem
- This calculation is called **interval estimation**, because it gives an interval estimate $\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{N}},\ \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right)$ instead of a single value $\bar{x}$

Today

1. Central limit theorem
2. Interval estimation
   - Confidence interval
   - Credible interval
3. Summary

Interval estimation

- MLE and MAP are **point estimation** techniques since they only return one single value, i.e. a point, for the parameter estimation.
- However, we are often interested in the **uncertainty** associated with the point estimate. A point estimate + uncertainty is called an **interval estimate** since they return an interval instead a single value.

Confidence interval

Confidence interval (CI)

- **Data**: $x_1, \cdots, x_N$
- **Random variable**: $X_1, \cdots, X_N$ with i.i.d. assumption
- **Parameter of interest**: θ, e.g. the mean μ
- **Estimate**: $\hat{\theta}$, e.g. the sample mean $\bar{x}$
- **Confidence interval** for a given confidence level $1 - \alpha$ (e.g. 95%)
  - Definition:

    confidence interval = ($\hat{\theta}$ - **margin of error**, $\hat{\theta}$ + **margin of error**)

    where

    margin of error = critical value $\times$ standard error of $\hat{\theta}$
  - Calculation:

Distribution of X_i	Scenario	θ	$\hat{\theta}$ (**sampling distribution**)	Critical value	Standard error	Confidence interval	Note
i.i.d. Gaussian	✅ σ known	mean	sample mean $\bar{x}$	$z_{\alpha/2}$	$\frac{\sigma}{\sqrt{N}}$	$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{N}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right)$	exact
	? σ unknown		(**Gaussian distribution**)	$t_{\alpha/2}$	$\frac{s}{\sqrt{N}}$	$\left(\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{N}}, \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{N}}\right)$	
i.i.d.	✅ σ known		sample mean $\bar{x}$	$z_{\alpha/2}$	$\frac{\sigma}{\sqrt{N}}$	$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{N}}, \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right)$	approximate
	? σ unknown		(**approximately Gaussian under CLT**)	$t_{\alpha/2}$	$\frac{s}{\sqrt{N}}$	$\left(\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{N}}, \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{N}}\right)$	for large N
i.i.d.	🙈 -	any	MLE (**asymptotically Gaussian**)	$z_{\alpha/2}$	$\frac{1}{\sqrt{NI_N(\hat{\theta})}}$	$\left(\hat{\theta} - z_{\alpha/2}\frac{1}{\sqrt{NI_N(\hat{\theta})}}, \hat{\theta} + z_{\alpha/2}\frac{1}{\sqrt{NI_N(\hat{\theta})}}\right)$	asymptotic
i.i.d.	? -	any	any statistic (**any distribution**)	bootstrap the error quantile		$\left(\hat{\theta} - \epsilon_{1-\alpha/2}, \hat{\theta} - \epsilon_{\alpha/2}\right)$	approximate

where σ is the standard deviation of the X_i and s the sample standard deviation

Calculation of the confidence interval

Data: $x_1, \cdots, x_N$

Random variable: $X_1, \cdots, X_N$ i.i.d. with standard deviation σ

- CI for Gaussian sampling distribution (exact, approximate, asymptotic):
  - **Parameter of interest:** mean value
    Estimation method: sample mean $\bar{x}$
    - ✅ σ known: $\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{N}},\ \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right)$ (cf. page 27)
    - ? σ unknown: $\left(\bar{x} - t_{\alpha/2}\frac{\sigma}{\sqrt{N}},\ \bar{x} + t_{\alpha/2}\frac{\sigma}{\sqrt{N}}\right)$
  - **Parameter of interest:** any statistic
    Estimation method: MLE (cf. lecture 3 properties of MLE)
    - 🙈 [not required] $\left(\bar{x} - z_{\alpha/2}\frac{1}{\sqrt{NI_N(\hat{\theta})}}, \bar{x} + z_{\alpha/2}\frac{1}{\sqrt{NI_N(\hat{\theta})}}\right)$
- CI for unknown sampling distribution
  - **Parameter of interest:** any parameter, e.g. median
    Estimation method: any method
    - ? Bootstrap $(\bar{x} - \epsilon_{1-\alpha/2}, \bar{x} - \epsilon_{\alpha/2})$

CI for unknown σ

- When the standard deviation σ is **known**, we have shown the standardization of the error term $\frac{\mathcal{E}}{\sigma/\sqrt{N}} = \frac{\bar{X}-\mu}{\sigma/\sqrt{N}} \sim \mathcal{N}(0,1)$ (cf. page. 24).
- When σ is **unknown**, which is the most common case, we replace σ by its estimate $\hat{\sigma}$ - the **sample standard deviation** S

$$\hat{\sigma} = S = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2}$$

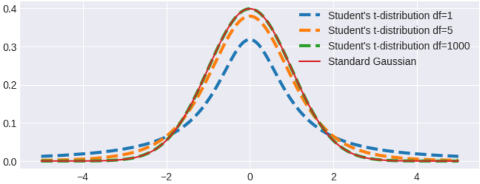
- Now the standardization becomes (**random**, **constant**):

$$\frac{\mathcal{E}}{\sigma/\sqrt{N}} \rightarrow \frac{\mathcal{E}}{S/\sqrt{N}} = \frac{\bar{X}-\mu}{S/\sqrt{N}} \sim t(N-1)$$

- Compared to the case with known σ, $\frac{\bar{X}-\mu}{\sigma/\sqrt{N}} \sim \mathcal{N}(0,1)$, the distribution of $\frac{\bar{X}-\mu}{S/\sqrt{N}}$ is no longer the standard Gaussian ($\frac{\mu}{S/\sqrt{N}}$ is no longer a constant because S is a random variable). Instead, it follows a **Student's t-distribution** t. The Student's t-distribution has one parameter $df = N-1$ (**degrees of freedom**).

CI for unknown σ (cont.)

- The Student's t-distribution is a function of the sample size: $df = N - 1$
- Think of it as a standard Gaussian compensated for the small sample size. For a large N, they become very similar.



CI for unknown σ (cont.)

- **t-table**: similar to the z-table for the standard Gaussian distribution, there is a t-table for the Student's t-distribution (image from `http://www.ttable.org/`).
- **each cell = stats.t.ppf(q=cum.prob, df=N-1, loc=0, scale=1)**
- α = **two-tails** and **confidence level** $= 1 - \alpha$

cum. prob	$t_{.50}$	$t_{.75}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$
one-tail	0.50	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
two-tails	1.00	0.50	0.40	0.30	0.20	0.10	0.05	0.02	0.01	0.002	0.001
df											
1	0.000	1.000	1.376	1.963	3.078	6.314	12.71	31.82	63.66	318.31	636.62
2	0.000	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.327	31.599
3	0.000	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.215	12.924
4	0.000	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.000	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.000	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.000	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.000	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.000	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.000	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.000	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.000	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.000	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.000	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.000	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.000	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.000	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.000	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.000	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.000	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.000	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.000	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.000	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.000	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.000	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.000	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.000	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.000	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.000	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.000	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	0.000	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.000	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
80	0.000	0.678	0.846	1.043	1.292	1.664	1.990	2.374	2.639	3.195	3.416
100	0.000	0.677	0.845	1.042	1.290	1.660	1.984	2.364	2.626	3.174	3.390
1000	0.000	0.675	0.842	1.037	1.282	1.646	1.962	2.330	2.581	3.098	3.300
z	0.000	0.674	0.842	1.036	1.282	1.645	1.960	2.326	2.576	3.090	3.291
	0%	50%	60%	70%	80%	90%	95%	98%	99%	99.8%	99.9%
	Confidence Level										

✅ Summary

Data: $x_1, \cdots, x_N$
Random variable: $X_1, \cdots, X_N$ i.i.d. with standard deviation σ
CI for unknown σ with Gaussian sampling distribution

$$\left(\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{N}}, \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{N}}\right)$$

CI for unknown sampling distribution

- Sample mean approximately follows a Gaussian distribution under the central limit theorem, but most other statistics do not have such luxury
- When the sampling distribution is unknown, we cannot use the t-table or z-table to find the critical values
- Recall the definition of CI:
  confidence interval = ($\hat{\theta}$ - **margin of error**, $\hat{\theta}$ + **margin of error**)
- One solution is to approximate the **margin of error** using **bootstrap**

Bootstrap

- **Data**: $x_1, \cdots, x_N$
- **Random variables**: $X_1, \cdots, X_N$ i.i.d. from any distribution
- **Parameter of interest**: any θ
- **Estimation method**: any method
- **Confidence interval**: $(\bar{x} - \epsilon_{1-\alpha/2}, \bar{x} - \epsilon_{\alpha/2})$, where ϵ_p denotes the quantile of the error term at p

The idea of bootstrap is to approximate the error ϵ_p directly from data

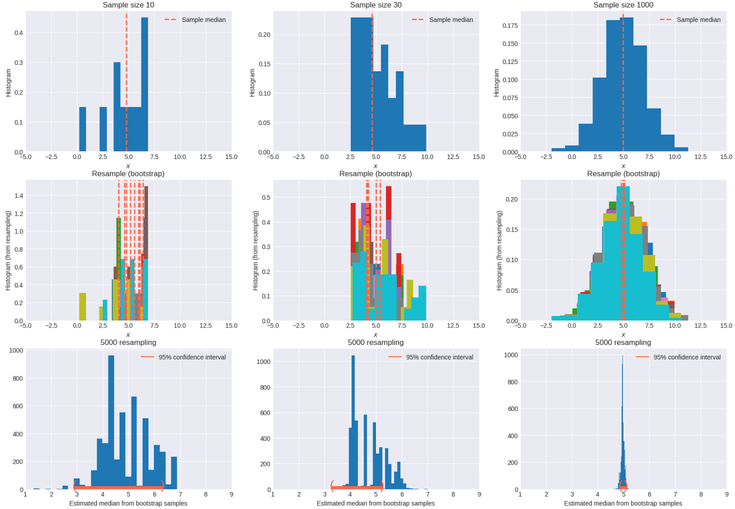
Bootstrap example

Given a data set $\mathcal{X} = \{1, 2, 3, 4, 5\}$ with size $N = 5$ and $\hat{\theta} = median(\mathcal{X}) = 3$ estimated from this data set, construct CI with 95% confidence level:

- **Sample with replacement**

  Step 1.1: Randomly choose 5 elements from $\mathcal{X}$: $\mathcal{X}_1^* = \{1, 2, 1, 1, 4\}$
  Step 1.2: Compute the median from $\mathcal{X}_1^*$: $m_1 = 1.0$
  Step 2.1: Randomly choose 5 elements from $\mathcal{X}$: $\mathcal{X}_2^* = \{2, 5, 2, 4, 4\}$
  Step 2.2: Compute the median from $\mathcal{X}_2^*$: $m_2 = 4.0$
  ...
- Repeat this 100 times and get the set $\{m_1, \cdots, m_{100}\}$
- Compute $\epsilon^i = m_i - 3$ for $i = 1, \cdots, 100$
- Compute 0.025-quantile $\epsilon_{0.025}$ and 0.975-quantile $\epsilon_{0.975}$ from the set $\{\epsilon^1, \cdots, \epsilon^{100}\}$
- The 95% CI is constructed as $(3 - \epsilon_{0.975}, 3 - \epsilon_{0.025})$
- **Intuition**:
  - $\hat{\theta} = 3$ is approximating the true median θ
  - m_i is approximating $\hat{\theta} = 3$
  - We can use $m_i - 3$ to approximate $3 - \theta$

Bootstrap example (cont.)

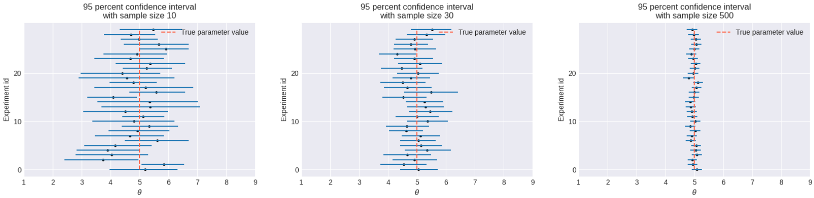


✅ CI for unknown sampling distribution using bootstrap

- **Steps** Given a data set $\mathcal{X}$ with size N and a statistic $\hat{\theta}$ computed from this data set, construct CI with $1-\alpha$ confidence level:
  - Choose a large n
  - For $i = 1, \cdots, n$, repeat
    - Sample N elements from $\mathcal{X}$ with replacement: $\mathcal{X}_i^*$
    - Estimate the parameter of interest from $\mathcal{X}_i^*$: $\hat{\theta}_i$
    - Compute $\epsilon^i = \hat{\theta}_i - \hat{\theta}$
  - Compute $\alpha/2$-quantile $\epsilon_{\alpha/2}$ and $1-\alpha/2$-quantile $\epsilon_{1-\alpha/2}$ from the set $\{\epsilon^1, \cdots, \epsilon^n\}$
  - The 95% CI is constructed as $\left(\bar{x} - \epsilon_{1-\alpha/2}, \bar{x} - \epsilon_{\alpha/2}\right)$
- **Intuition**:
  - $\hat{\theta}$ is approximating θ
  - $\hat{\theta}_i$ is approximating $\hat{\theta}$
  - We can use $\hat{\theta}_i - \hat{\theta}$ to approximate $\hat{\theta} - \theta$
- **Note**: there are many alternative methods for bootstrap; the exact method needs to be described when you talk about bootstrap

Confidence interval interpretation

- Confidence interval is **random** (data is random; statistic is random); the true parameter value θ is **not random** (illustrated in the image)
- A 95% confidence interval means that 95% of the time, the interval covers the true value θ



- Question 1: with the same problem setup, the larger the confidence level,

  A. the wider the confidence interval

  B. the narrower the confidence interval

  Answer: A
- Question 2: for a given confidence level, a good estimate has

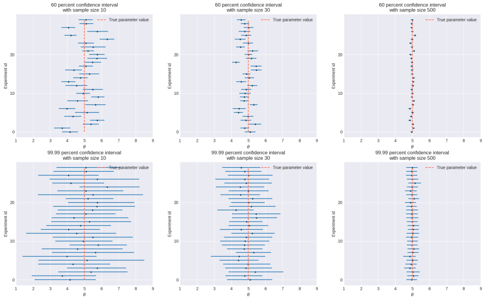
  A. a wide confidence interval

  B. a narrow confidence interval

  Answer: B

Confidence level interpretation

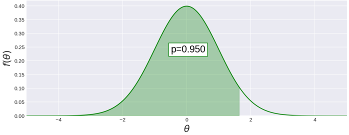
- If we compare 60% CI with 99.99% CI, the 60% CI does not always cover the true value $\theta = 5$ (it only covers it 60% of the time). On the other hand, the 99.99% CI covers the true value pretty much all the time. From this perspective, 99.99% CI is more meaningful to use as a quality measure.
- However, 99.99% CI can be very wide - of course - since it promises to cover the true value 99.99% of the time. A wide interval might not be meaningful sometimes, e.g. if you claim that you have estimated $\hat{\theta} = 4.3$ and you are 100% sure that the interval $(4.3 - \infty, 4.3 + \infty)$ contains the true value, your client might get mad.

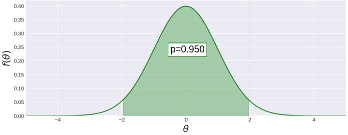


Credible interval

Credible interval for Bayesian approach

- In maximum a posteriori estimation, the parameter of interest θ is modeled as **a random variable** - θ is generated from an underlying probability distribution described by $f(\theta)$
- Technically, any interval (a, b) with $P(a \leq \Theta \leq b) = 0.95$ is a 95% credible interval, but not all of them make sense, e.g.





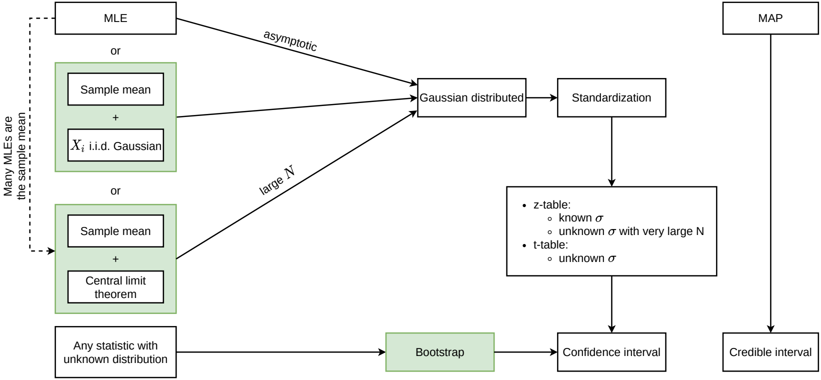
- There are different techniques for choosing this interval

Credible interval for Bayesian approach (cont.)

- In Python, for a given posterior (e.g. a standard Gaussian distribution $\mathcal{N}(0, 1)$), the .interval method computes the interval with equal areas around the median:

```
posterior = stats.norm(loc=0, scale=1)
credible_interval = posterior.interval(0.95)
```

Recap


MLE
or
Sample mean
+
X_i i.i.d. Gaussian
or
Sample mean
+
Central limit theorem
Many MLEs are the sample mean
Any statistic with unknown distribution
asymptotic
large N
Gaussian distributed
Standardization
z-table:
known σ
unknown σ with very large N
t-table:
unknown σ
Bootstrap
Confidence interval
MAP
Credible interval

Today

Summary

So far:

- Data types and data containers
- Descriptive data analysis: descriptive statistics, visualization
- Probability distributions, events, random variables, PMF, PDF, parameters
- CDF, Q-Q plot, how to compare two distributions (data vs theoretical, data vs data)
- Modeling
- Parameter estimation: maximum likelihood estimation (MLE) and maximum a posteriori estimation (MAP)
- Classification, multinomial naive Bayes classifier, Gaussian naive Bayes classifier
- Central limit theorem, interval estimation

Next:

- Hypothesis testing

Before next lecture:

- Standardization, confidence interval, z-table, t-table

See you next week!