**Robotics 1**

# Minimal representations of orientation
## (Euler and roll-pitch-yaw angles)
# Homogeneous transformations

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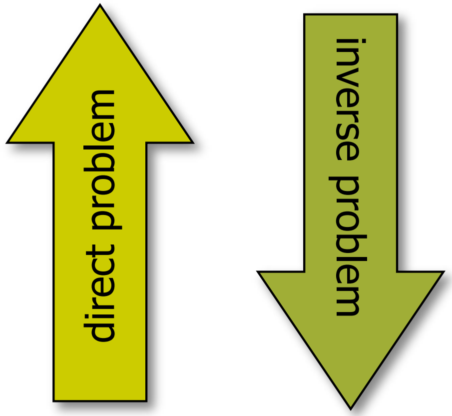
# “Minimal” representations

- rotation matrices in $SO(3)$:



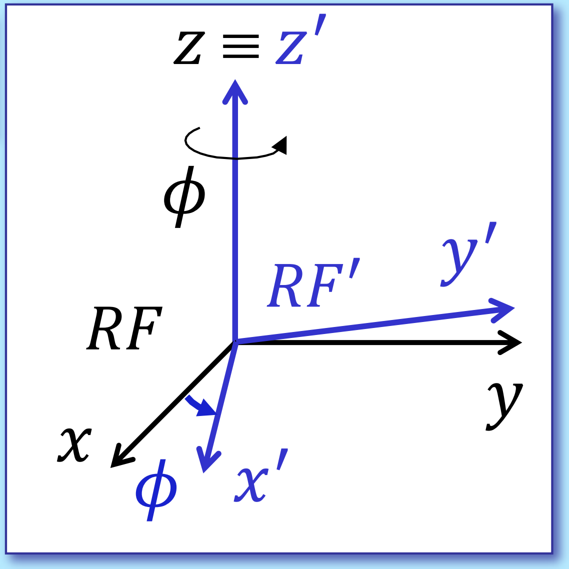

$$\begin{array}{rl} & 9 \text{ elements} \\ - & 3 \text{ orthogonality relationships} \\ - & 3 \text{ unitary relationships} \\ \hline = & 3 \text{ independent variables} \end{array}$$

- sequence of 3 rotations w.r.t. independent axes
  - by angles $\alpha_i, i = 1,2,3$, around fixed ($a_i$) or moving/current ($a_i'$) axes
    - generically called Roll-Pitch-Yaw (fixed axes) or Euler (moving axes) angles
  - 12 + 12 possible different sequences (e.g., XYX)
    - without contiguous repetitions of axes (e.g., no XXZ nor YZ′Z′)
  - however, only 12 sequences are different since we shall see that

$$\{(a_1,\alpha_1),(a_2,\alpha_2),(a_3,\alpha_3)\} \equiv \{(a_3',\alpha_3),(a_2',\alpha_2),(a_1',\alpha_1)\}$$

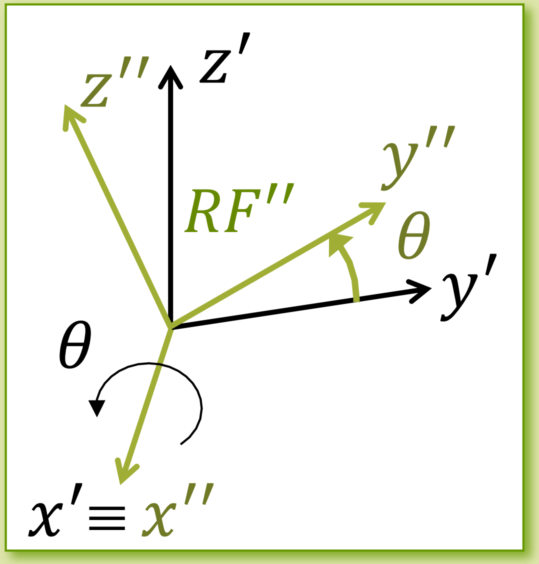
# $ZX'Z''$ Euler angles

**1**



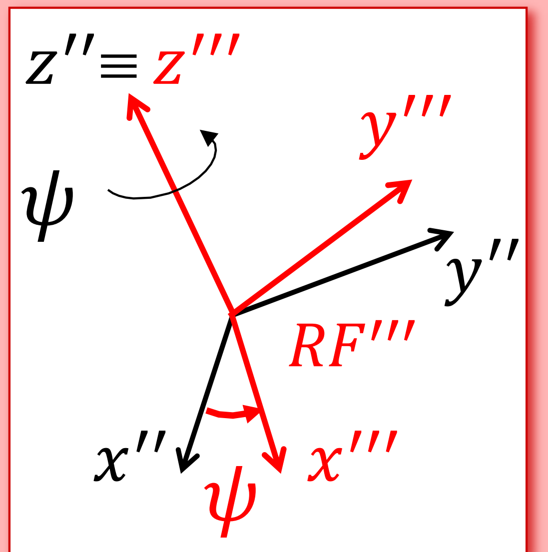

$$R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**2**




$$R_{X'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

**3**




$$R_{Z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

# $ZX'Z''$ Euler angles

- direct problem: given $\phi, \theta, \psi$, find $R$

$$R_{ZX'Z''}(\phi, \theta, \psi) = R_Z(\phi) R_{X'}(\theta) R_{Z''}(\psi)$$

order of definition in concatenation

$$= \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & -c\phi s\psi - s\phi c\theta c\psi & s\phi s\theta \\ s\phi c\psi + c\phi c\theta s\psi & -s\phi s\psi + c\phi c\theta c\psi & -c\phi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix}$$

- given a vector $v''' = (x''', y''', z''')$ expressed in $RF'''$, its expression in the coordinates of $RF$ is

$$v = R_{ZX'Z''}(\phi, \theta, \psi) v'''$$

- the orientation of $RF'''$ is the same that would be obtained with the sequence of rotations

$\psi$ around $z$, $\theta$ around $x$ (fixed), $\phi$ around $z$ (fixed)

# $ZX'Z''$ Euler angles

- inverse problem: given $R = \{r_{ij}\}$, find $\phi, \theta, \psi$

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\phi c\psi - s\phi c\theta s\psi & -c\phi s\psi - s\phi c\theta c\psi & s\phi s\theta \\ s\phi c\psi + c\phi c\theta s\psi & -s\phi s\psi + c\phi c\theta c\psi & -c\phi s\theta \\ s\theta s\psi & s\theta c\psi & c\theta \end{bmatrix}$$

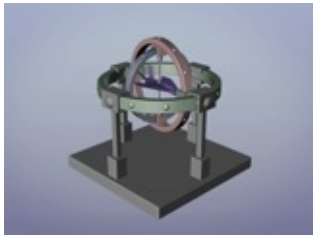
- $r_{13}^2 + r_{23}^2 = s^2\theta,\ r_{33} = c\theta \ \Rightarrow$ $\boxed{\theta = \text{atan2}\left\{\pm\sqrt{r_{13}^2 + r_{23}^2},\ r_{33}\right\}}$

  two values differing just for the sign

- if $r_{13}^2 + r_{23}^2 \neq 0$ (i.e., $s\theta \neq 0$)

  $r_{31}/s\theta = s\psi,\ r_{32}/s\theta = c\psi \Rightarrow$ $\boxed{\psi = \text{atan2}\{r_{31}/s\theta, r_{32}/s\theta\}}$

- similarly... $\boxed{\phi = \text{atan2}\{r_{13}/s\theta, -r_{23}/s\theta\}}$
- there is always a pair of solutions in the regular case
- there are always singularities (here $\theta = 0$ or $\pm\pi$) $\Rightarrow$ only the sum $\phi + \psi$ or the difference $\phi - \psi$ can be determined



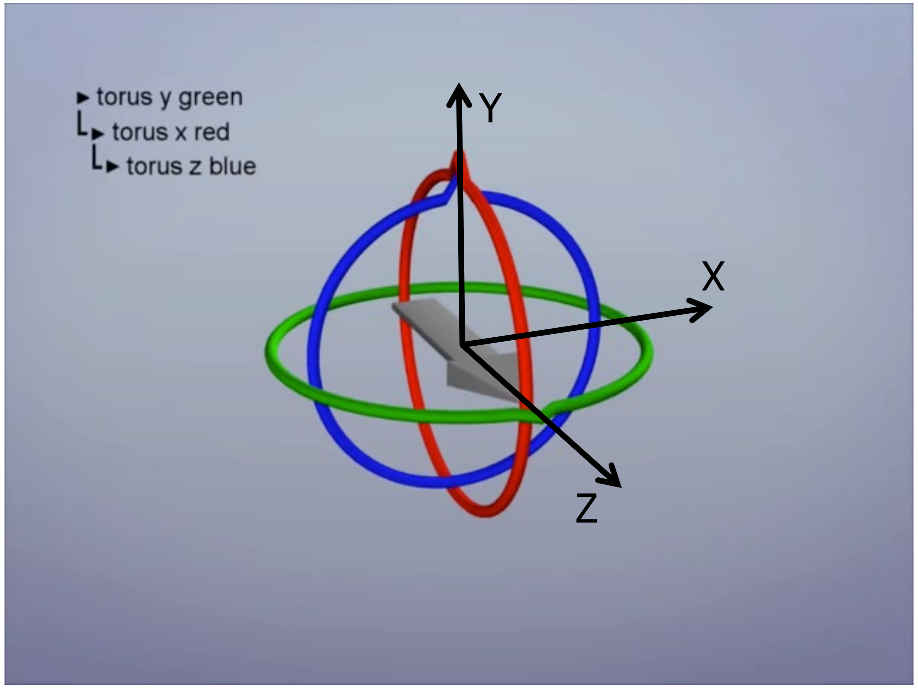
# Gimbal lock

## singularities of minimal representations

in a singularity: instantaneous rotational motion is forbidden around some axis

Euler YX′Z″

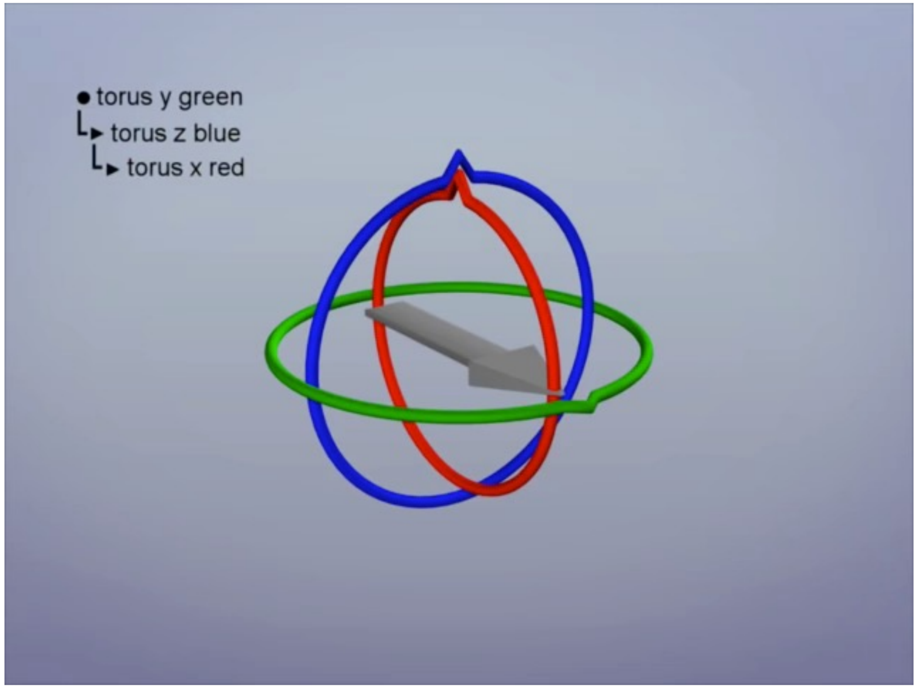
when $\theta = \pm\pi/2 \implies$ no $\boldsymbol{\omega}_{X'}$




(arrow is initially oriented as Z)

Euler YZ′X″

when $\theta = \pm\pi/2 \implies$ no $\boldsymbol{\omega}_{Z'}$



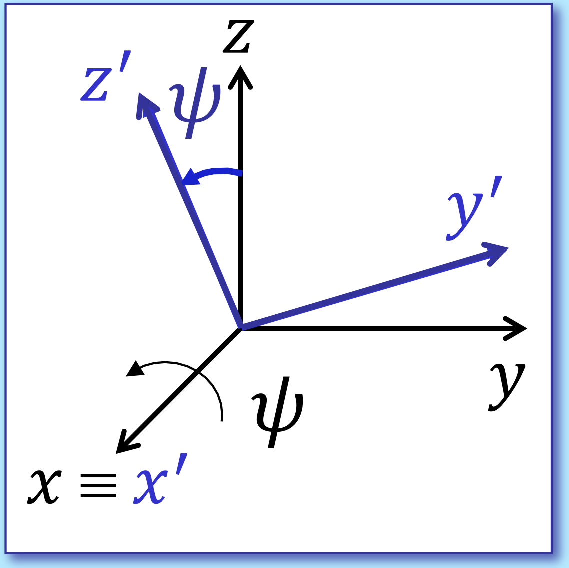

Euler ZY′X″

when $\theta = \pm\pi/2 \implies$ no $\boldsymbol{\omega}_{Y'}$

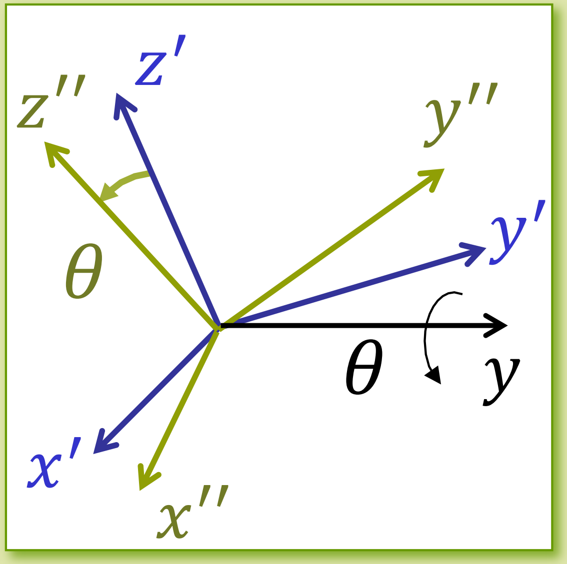
https://youtu.be/zc8b2Jo7mno

# Roll-Pitch-Yaw angles (fixed $XYZ$)

**1** **ROLL**




$$R_X(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$



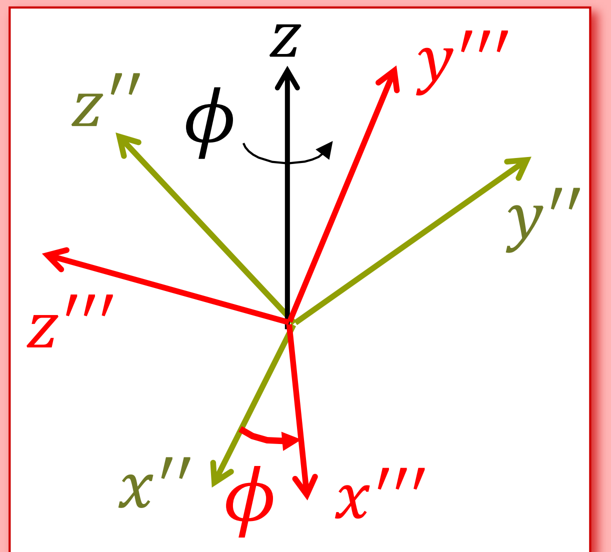

**PITCH** **2**

$C_1 R_Y(\theta) C_1^T$

with $R_Y(\theta) =$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

**3** **YAW**




$C_2 R_Z(\phi) C_2^T$

with $R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

# Roll-Pitch-Yaw angles (fixed $XYZ$)

- direct problem: given $\psi, \theta, \phi$, find $R$

$$R_{RPY}(\psi,\theta,\phi) = R_Z(\phi)R_Y(\theta)R_X(\psi)$$ ⇐ note the order of products!

order of definition

$$= \begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi - s\phi c\psi & c\phi s\theta c\psi + s\phi s\psi \\ s\phi c\theta & s\phi s\theta s\psi + c\phi c\psi & s\phi s\theta c\psi - c\phi s\psi \\ -s\theta & c\theta s\psi & c\theta c\psi \end{bmatrix}$$

- inverse problem: given $R = \{r_{ij}\}$, find $\psi, \theta, \phi$
- $r_{32}^2 + r_{33}^2 = c^2\theta,\ r_{31} = -s\theta \Rightarrow$ $\boxed{\theta = \text{atan2}\left\{-r_{31}, \pm\sqrt{r_{32}^2 + r_{33}^2}\right\}}$

for $r_{31} < 0$, two symmetric values w.r.t. $\pi/2$

- if $r_{32}^2 + r_{33}^2 \neq 0$ (i.e., $c\theta \neq 0$)

$r_{32}/c\theta = s\psi,\ r_{33}/c\theta = c\psi \Rightarrow$ $\boxed{\psi = \text{atan2}\{r_{32}/c\theta, r_{33}/c\theta\}}$

- similarly ... $\boxed{\phi = \text{atan2}\{r_{21}/c\theta, r_{11}/c\theta\}}$
- singularities for $\theta = \pm\pi/2 \Rightarrow$ only $\phi + \psi$ or $\phi - \psi$ are defined

# …why this order in the product?

$$R_{RPY}(\psi,\theta,\phi) = R_Z(\phi)R_Y(\theta)R_X(\psi)$$

order of definition

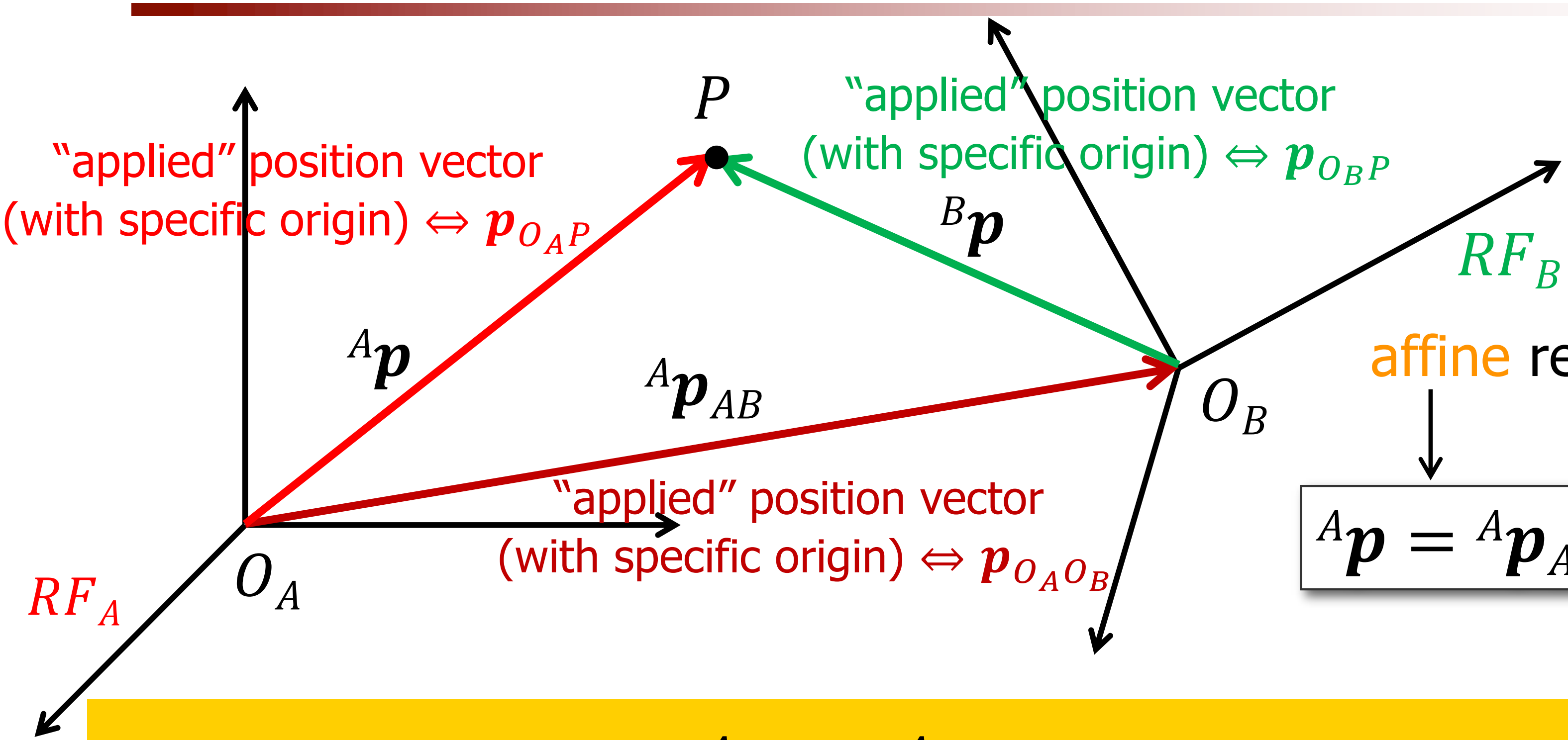
"reverse" order in the product (pre-multiplication…)

- need to refer each rotation in the sequence to one of the original fixed axes
  - use the angle/axis technique for each rotation in the sequence: $C\ R(\alpha)\ C^T$, with $C$ being the rotation matrix reverting the previously made rotations (= "go back" to the original axes)

concatenating three rotations: [ ] [ ] [ ] (post-multiplication…)

$$\begin{aligned} R_{RPY}(\psi,\theta,\phi) = &\ [R_X(\psi)][R_X^T(\psi)\ R_Y(\theta)\ R_X(\psi)] \\ &\ [R_X^T(\psi)R_Y^T(\theta)\ R_Z(\phi)\ R_Y(\theta)R_X(\psi)] \\ = &\ R_Z(\phi)R_Y(\theta)R_X(\psi) \end{aligned}$$

# Homogeneous transformations




affine relationship

$$\boxed{{}^A\boldsymbol{p} = {}^A\boldsymbol{p}_{AB} + {}^A R_B\, {}^B\boldsymbol{p}}$$

$${}^A\boldsymbol{p}_{\text{hom}} = \begin{bmatrix} {}^A\boldsymbol{p} \\ \hdashline 1 \end{bmatrix} = \left[\begin{array}{c:c} {}^A R_B & {}^A\boldsymbol{p}_{AB} \\ \hdashline \boldsymbol{0}^T & 1 \end{array}\right] \begin{bmatrix} {}^B\boldsymbol{p} \\ \hdashline 1 \end{bmatrix} = {}^A T_B\, {}^B\boldsymbol{p}_{\text{hom}}$$

linear relationship

vector in homogeneous coordinates

$4 \times 4$ matrix of homogeneous transformation

# Use of homogeneous transformation $T$

- describes the relation between two reference frames (relative pose = position & orientation)
- transforms the representation of a position vector (applied vector starting from the origin of the frame) from one frame to another frame
- it is a roto-translation operator on vectors in the three-dimensional space
- it is always invertible $\left({}^{A}T_{B}\right)^{-1} = {}^{B}T_{A}$
- can be composed, i.e., ${}^{A}T_{B}\,{}^{B}T_{C} = {}^{A}T_{C}$ ← note: it does not commute in general!

# Affine vs linear computations

## whiteboard…

$$^1p = {}^1p_{01} + {}^1R_0\,{}^0p$$

$$^2p = {}^2p_{12} + {}^2R_1\,{}^1p = {}^2p_{12} + {}^2R_1\,{}^1p_{01} + {}^2R_1\,{}^1R_0\,{}^0p$$

$$^3p = {}^3p_{23} + {}^3R_2\,{}^2p = \cdots = {}^2p_{23} + {}^3R_2\,{}^2p_{12} + {}^3R_2\,{}^2R_1\,{}^1p_{01} + {}^3R_2\,{}^2R_1\,{}^1R_0\,{}^0p$$

$$^4p = {}^4p_{34} + {}^4R_3\,{}^3p = \cdots$$

heavy on notation (and not only!)

$$^1T_0 = \begin{bmatrix} {}^1R_0 & {}^1p_{01} \\ 0^T & 1 \end{bmatrix} \quad \Rightarrow \quad {}^1p_{hom} = {}^1T_0\,{}^0p_{hom}$$

$$^2T_1 = \begin{bmatrix} {}^2R_1 & {}^2p_{12} \\ 0^T & 1 \end{bmatrix} \quad \Rightarrow \quad {}^2p_{hom} = {}^2T_1\,{}^1T_0\,{}^0p_{hom} = {}^2T_0\,{}^0p_{hom}$$

$$^3T_2 = \begin{bmatrix} {}^3R_1 & {}^3p_{23} \\ 0^T & 1 \end{bmatrix} \quad \Rightarrow \quad {}^3p_{hom} = {}^3T_2\,{}^2T_1\,{}^1T_0\,{}^0p_{hom} = {}^3T_0\,{}^0p_{hom}$$

$$^4T_3 = \begin{bmatrix} {}^4R_3 & {}^4p_{34} \\ 0^T & 1 \end{bmatrix} \quad \Rightarrow \quad {}^4p_{hom} = {}^4T_3\,{}^3T_2\,{}^2T_1\,{}^1T_0\,{}^0p_{hom} = {}^4T_0\,{}^0p_{hom}$$

# Inverse of a homogeneous transformation

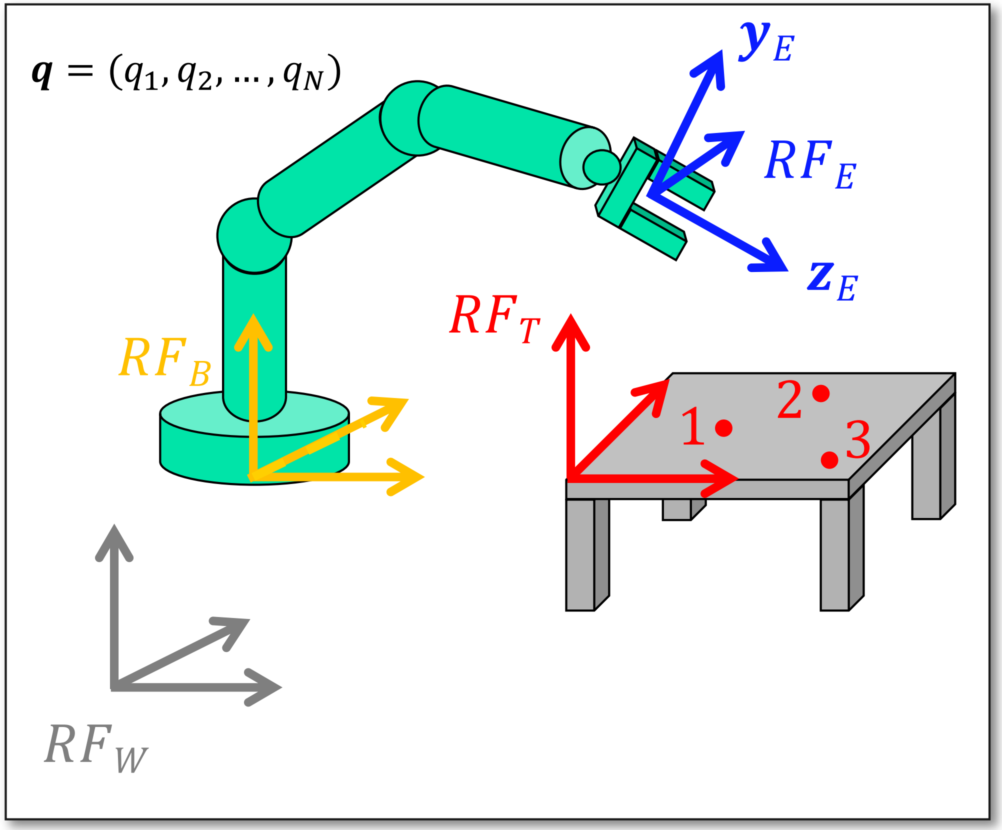
exchange $A \rightleftarrows B$      rewrite using the original vectors/matrices ...

$$ {}^{A}\boldsymbol{p} = {}^{A}\boldsymbol{p}_{AB} + {}^{A}R_B\, {}^{B}\boldsymbol{p} \qquad {}^{B}\boldsymbol{p} = {}^{B}\boldsymbol{p}_{BA} + {}^{B}R_A\, {}^{A}\boldsymbol{p} = -\, {}^{A}R_B^T\, {}^{A}\boldsymbol{p}_{AB} + {}^{A}R_B^T\, {}^{A}\boldsymbol{p} $$

$$ \left[\begin{array}{ccc|c} & {}^{A}R_B & & {}^{A}\boldsymbol{p}_{AB} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} & {}^{B}R_A & & {}^{B}\boldsymbol{p}_{BA} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & {}^{A}R_B^T & & -\, {}^{A}R_B^T\, {}^{A}\boldsymbol{p}_{AB} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^{A}T_B \qquad {}^{B}T_A \qquad \left({}^{A}T_B\right)^{-1} $$

# Defining a robotic task




absolute definition of task

task definition relative to the robot end-effector

$$^{W}T_{T} = {}^{W}T_{B}\,{}^{B}T_{E}\,{}^{E}T_{T}$$

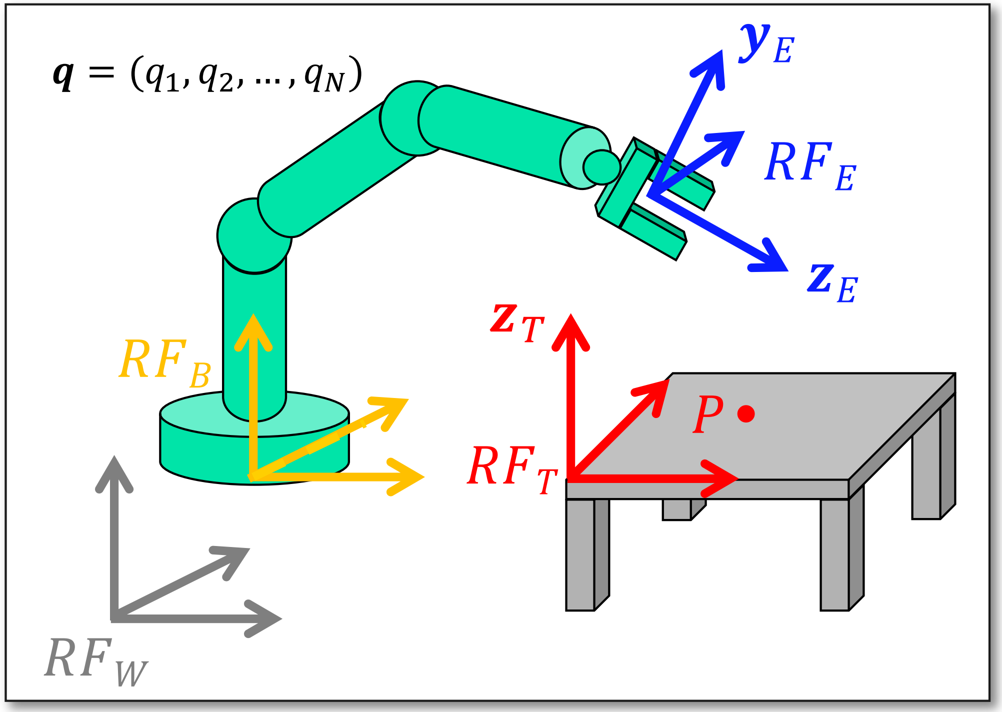
known, once the robot is placed

direct kinematics of the robot arm (function of $\boldsymbol{q}$)

solve for $\boldsymbol{q}$ (inverse kinematics)

$$^{B}T_{E}(\boldsymbol{q}) = {}^{W}T_{B}^{-1}\,{}^{W}T_{T}\,{}^{E}T_{T}^{-1} = \text{constant}$$

# Example of task definition




- the robot carries a depth camera (e.g., a Kinect) on the end-effector
- the end-effector should go to a pose above the point $P$ on the table, pointing its approach axis $\boldsymbol{z}_E$ downward and being aligned with the table sides

$$ {}^E R_T = \begin{bmatrix} {}^E\boldsymbol{x}_T & {}^E\boldsymbol{y}_T & {}^E\boldsymbol{z}_T \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} $$

- point $P$ is known in the table frame $RF_T$

$$ {}^T\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ 0 \end{bmatrix} $$

- the robot proceeds by centering point $P$ in its camera image until it senses a depth $h$ from the table (in $RF_E$)

$$ {}^E\boldsymbol{p} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix} $$

**Q:** where is the EE frame w.r.t. the table frame?

$$ {}^T T_E = \begin{bmatrix} {}^T R_E & {}^T\boldsymbol{p}_{TE} \\ 0^T & 1 \end{bmatrix} = {}^E T_T^{-1} $$

with $\quad {}^T R_E = \left({}^E R_T\right)^T = {}^E R_T$

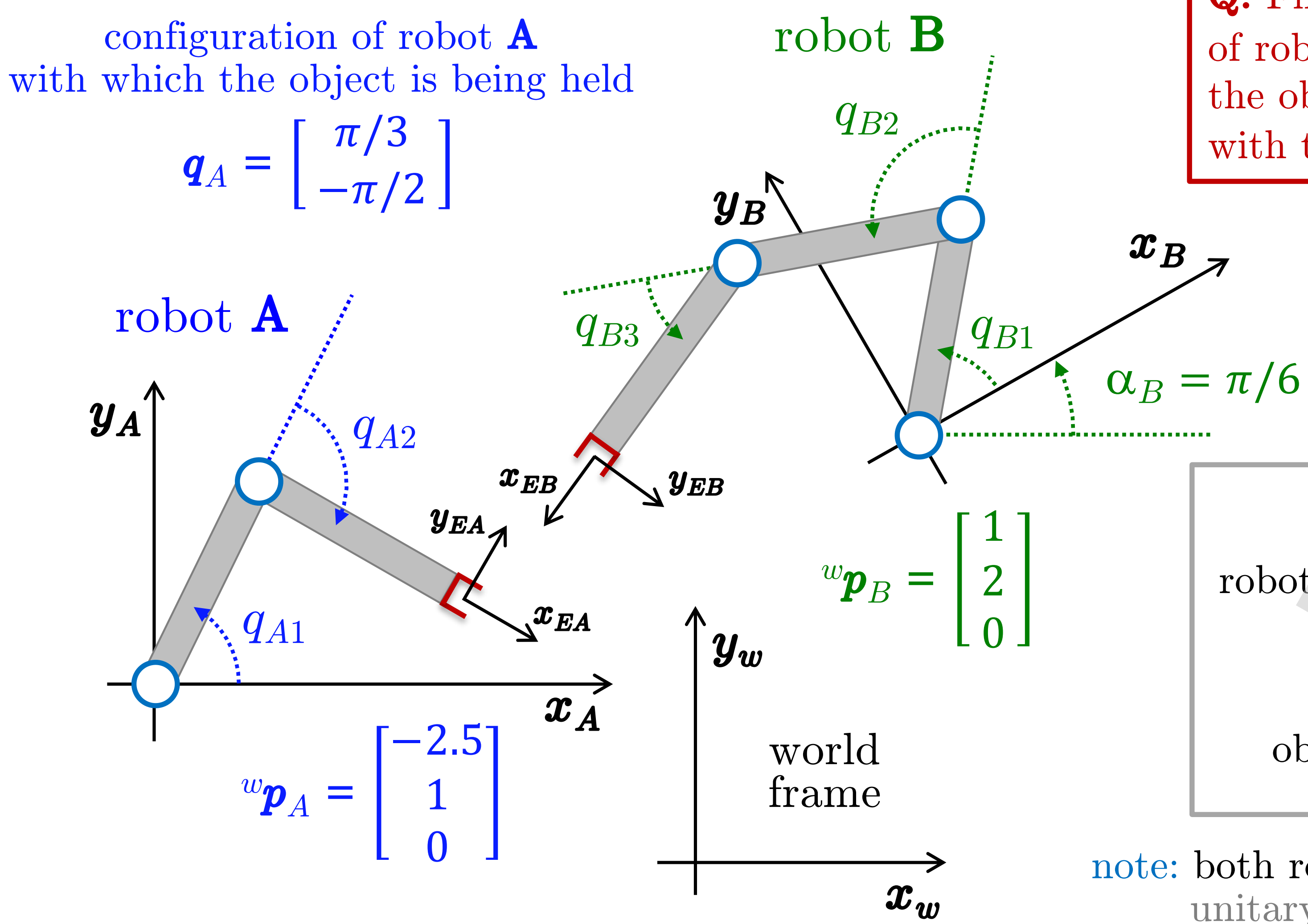
$$ {}^T\boldsymbol{p}_{TE} = {}^T\boldsymbol{p} - {}^T R_E\, {}^E p = \begin{bmatrix} p_x \\ p_y \\ h \end{bmatrix} $$

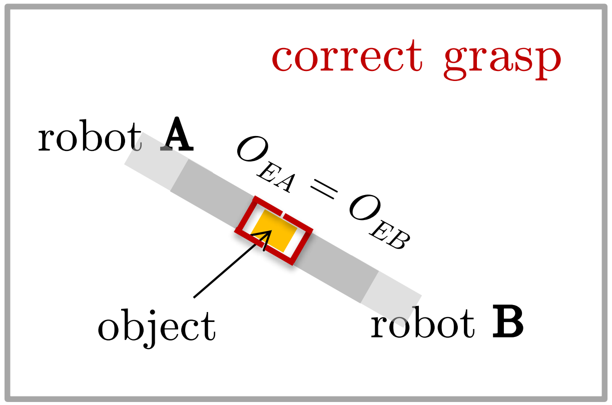
# A robotic problem with $T$ matrices

**Task:** 2R planar robot **A** should hand over an object at a given location to 3R planar robot **B**

**Q:** Find a configuration $\boldsymbol{q}_B$ of robot **B** so as to grasp the object held by robot **A** with the right orientation

Ex #3, Robotics 1
exam of Sep 11, 2020




note: both robots have unitary link lengths

# Solution procedure

$$ {}^{w}\boldsymbol{T}_A = \begin{pmatrix} {}^{w}\boldsymbol{R}_A & {}^{w}\boldsymbol{p}_A \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I}_{3\times 3} & \begin{matrix} -2.5 \\ 1 \\ 0 \end{matrix} \\ \boldsymbol{0}^T & 1 \end{pmatrix} $$

base frame of robot **A** w.r.t. world

$$ {}^{w}\boldsymbol{T}_B = \begin{pmatrix} {}^{w}\boldsymbol{R}_B & {}^{w}\boldsymbol{p}_B \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha_B & -\sin\alpha_B & 0 & 1 \\ \sin\alpha_B & \cos\alpha_B & 0 & 2 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} = \begin{pmatrix} 0.8660 & -0.5 & 0 & 1 \\ 0.5 & 0.8660 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} $$

base frame of robot **B** w.r.t. world

$$ \begin{aligned} {}^{A}\boldsymbol{T}_{EA} &= \begin{pmatrix} {}^{A}\boldsymbol{R}_{EA} & {}^{A}\boldsymbol{p}_{EA} \\ \boldsymbol{0}^T & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos(q_{A1}+q_{A2}) & -\sin(q_{A1}+q_{A2}) & 0 & \cos q_{A1} + \cos(q_{A1}+q_{A2}) \\ \sin(q_{A1}+q_{A2}) & \cos(q_{A1}+q_{A2}) & 0 & \sin q_{A1} + \sin(q_{A1}+q_{A2}) \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0.8660 & 0.5 & 0 & 1.3660 \\ -0.5 & 0.8660 & 0 & 0.3660 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} \end{aligned} $$

end-effector frame of robot **A** w.r.t. its base frame (uses $\boldsymbol{q}_A$) $=$ direct kinematics of robot **A**!

$$ {}^{EA}\boldsymbol{T}_{EB} = \begin{pmatrix} {}^{EA}\boldsymbol{R}_{EB} & {}^{EA}\boldsymbol{p}_{EB} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} $$

end-effector frame of robot **B** w.r.t. end-effector frame of robot **A** to realize the right grasp for correct handover

# Solution procedure

$$\boxed{{}^{w}\boldsymbol{T}_A\,{}^{A}\boldsymbol{T}_{EA}\,{}^{EA}\boldsymbol{T}_{EB} = {}^{w}\boldsymbol{T}_B\,{}^{B}\boldsymbol{T}_{EB}}$$

kinematic equation defining the task

| | | |
|---|---|---|
| end-effector frame of robot **B** w.r.t. world passing via the given configuration of robot **A** | = | end-effector frame of robot **B** w.r.t. world passing via its base frame |

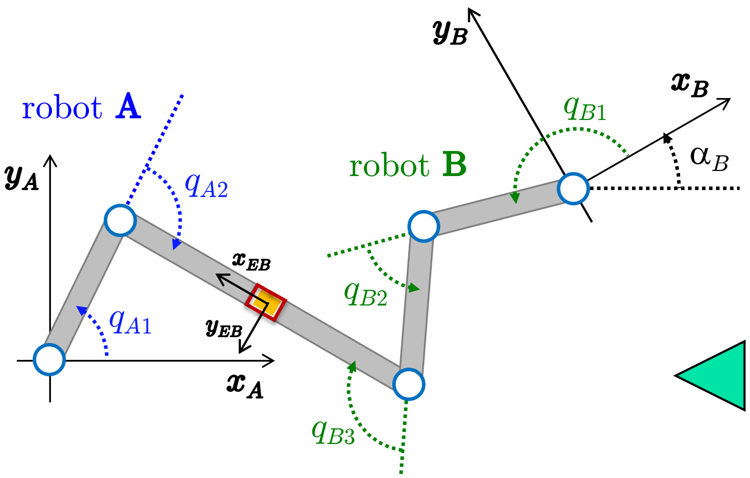
$$ {}^{B}\boldsymbol{T}_{EB,d} = \begin{pmatrix} {}^{B}\boldsymbol{R}_{EB,d} & {}^{B}\boldsymbol{p}_{EB,d} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \left({}^{w}\boldsymbol{T}_B\right)^{-1}\,{}^{w}\boldsymbol{T}_A\,{}^{A}\boldsymbol{T}_{EA}\,{}^{EA}\boldsymbol{T}_{EB} $$

$$ = \begin{pmatrix} -0.5 & -0.8660 & 0 & -2.1651 \\ 0.8660 & -0.5 & 0 & 0.5179 \\ 0 & 0 & 1 & 0 \\ & \boldsymbol{0}^T & & 1 \end{pmatrix} = $$

desired end-effector frame of robot **B** w.r.t. its base = input for the inverse kinematics of robot **B**!

one solution $\boldsymbol{q}_B$ (out of 2!) of the inverse kinematics of robot **B**

$$ q_B = \begin{bmatrix} q_{B1} \\ q_{B2} \\ q_{B3} \end{bmatrix} = \begin{bmatrix} 2.7939 \\ 1.1076 \\ -1.8071 \end{bmatrix} [\text{rad}] = \begin{bmatrix} 160.08° \\ 63.46° \\ -103.54° \end{bmatrix} $$



# Remarks on homogeneous matrices

- the main tool used for computing the direct kinematics of robot manipulators
- relevant in many other applications (in robotics and beyond)
  - in positioning/orienting a vision camera (matrix ${}^bT_c$ with extrinsic parameters of the camera pose)
  - in computer graphics, for the real-time visualization of 3D solid objects when changing the observation point

$$ {}^AT_B = \begin{bmatrix} \multicolumn{3}{c}{{}^AR_B} & {}^Ap_{AB} \\ \alpha_x & \alpha_y & \alpha_z & \sigma \end{bmatrix} $$

all zero in robotics

coefficients of perspective deformation

scaling coefficient

always unitary in robotics