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*Robotics 2*

# **Robots with kinematic redundancy**

## **Part 1: Fundamentals**

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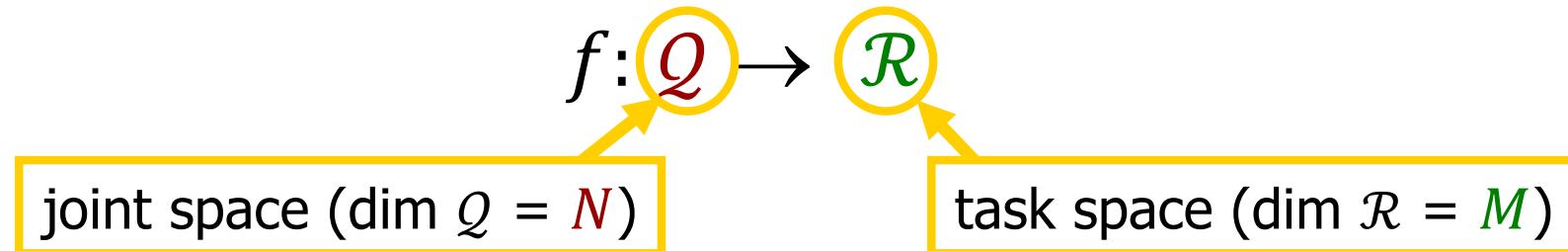
DIPARTIMENTO DI INGEGNERIA INFORMATICA  
AUTOMATICA E GESTIONALE ANTONIO RUBERTI





# Redundant robots

- direct kinematics of the task  $r = f(q)$



- a robot is (kinematically) **redundant** for the task if  $N > M$   
(more degrees of freedom than strictly needed for executing the task)
- $r$  may contain the position and/or the orientation of the end-effector or, more in general, be any parameterization of the task (even not in the Cartesian workspace)
- “redundancy” of a robot is thus a relative concept, i.e., it holds **with respect to a given task**



# Some E-E tasks and their dimensions

TASKS [for the robot end-effector (E-E)]	dimension $M$
■ position in the plane	→ 2
■ position in 3D space	→ 3
■ orientation in the plane	→ 1
■ pointing in 3D space	→ 2
■ position and orientation in 3D space	→ 6

a planar robot with  $N = 3$  joints is redundant for the task of positioning its E-E in the plane ( $M = 2$ ), but NOT for the task of positioning AND orienting the E-E in the plane ( $M = 3$ )



# Typical cases of redundant robots

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- 6R robot mounted on a linear track/rail
  - 7 dofs for positioning and orienting its end-effector in 3D space
- 6-dof robot used for arc welding tasks
  - task does not prescribe the final roll angle of the welding gun
- dexterous robotic hands
- multiple cooperating manipulators
- manipulator on a mobile (wheeled or legged) base
- humanoid robots, team of mobile robots ...
- “kinematic” redundancy is not the only type...
  - redundancy of components (actuators, sensors)
  - redundancy in the control/supervision architecture



# Uses of robot redundancy

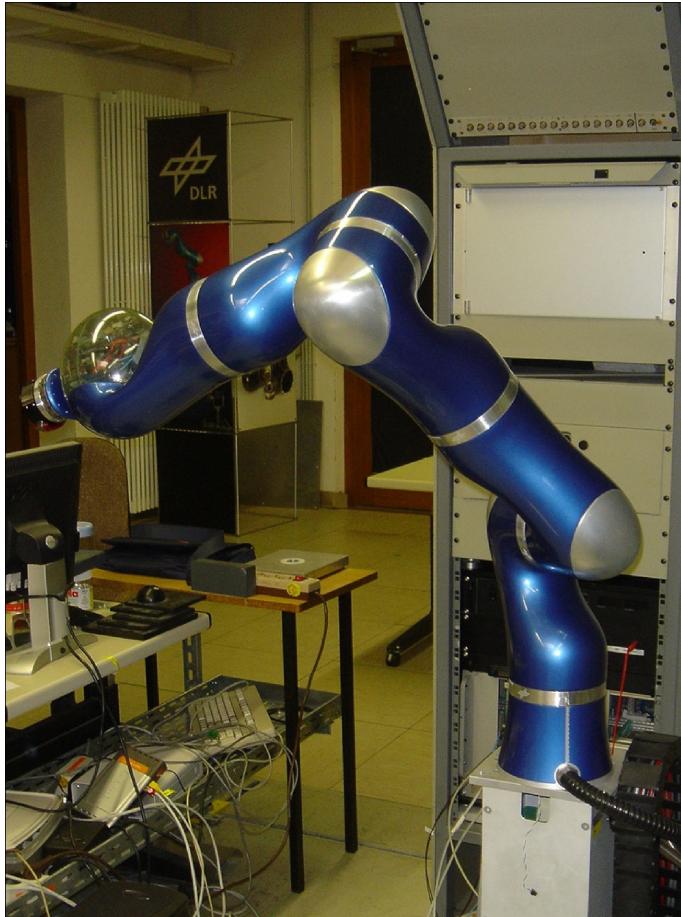
- avoid collision with obstacles (in **Cartesian** space) ...
- ... or kinematic singularities (in **joint** space)
- stay within the admissible joint ranges
- increase manipulability in specified directions
- uniformly distribute/limit joint velocities and/or accelerations
- minimize energy consumption or needed motion torques
- optimize execution time (on a given Cartesian path)
- increase dependability with respect to actuation faults
- ...



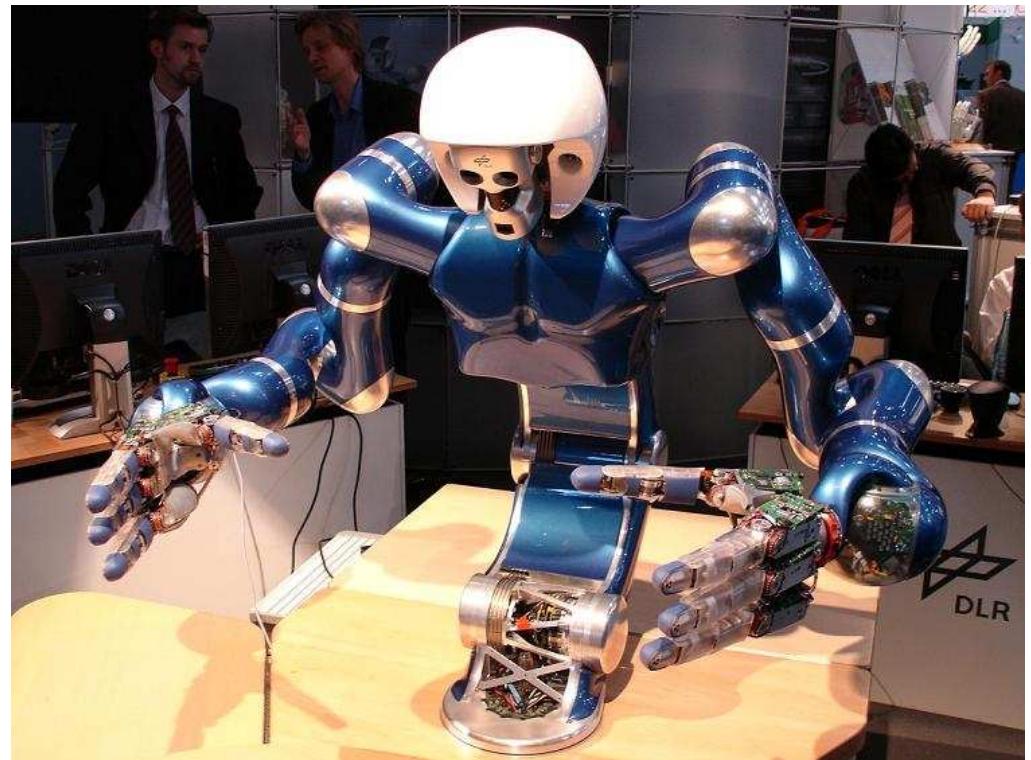
all objectives should be  
quantitatively “measurable”



# DLR robots: LWR-III and Justin



**7R LWR-III** lightweight manipulator:  
elastic joints (HD), joint torque sensing,  
13.5 kg weight = payload



**Justin** two-arm upper-body humanoid:  
**43R** actuated =

two arms ( $2 \times 7$ ) + torso (3\*)  
+ head (2) + two hands ( $2 \times 12$ ),  
45 kg weight

\* = one joint is dependent on the motion of the other two



# Justin carrying a trailer

[video](#)



LAAS-CNRS



SIXTH FRAMEWORK PROGRAMME



motion planning for **DLR Justin robot** in the configuration space,  
avoiding Cartesian obstacles and using robot redundancy



# Dual-arm redundancy



video

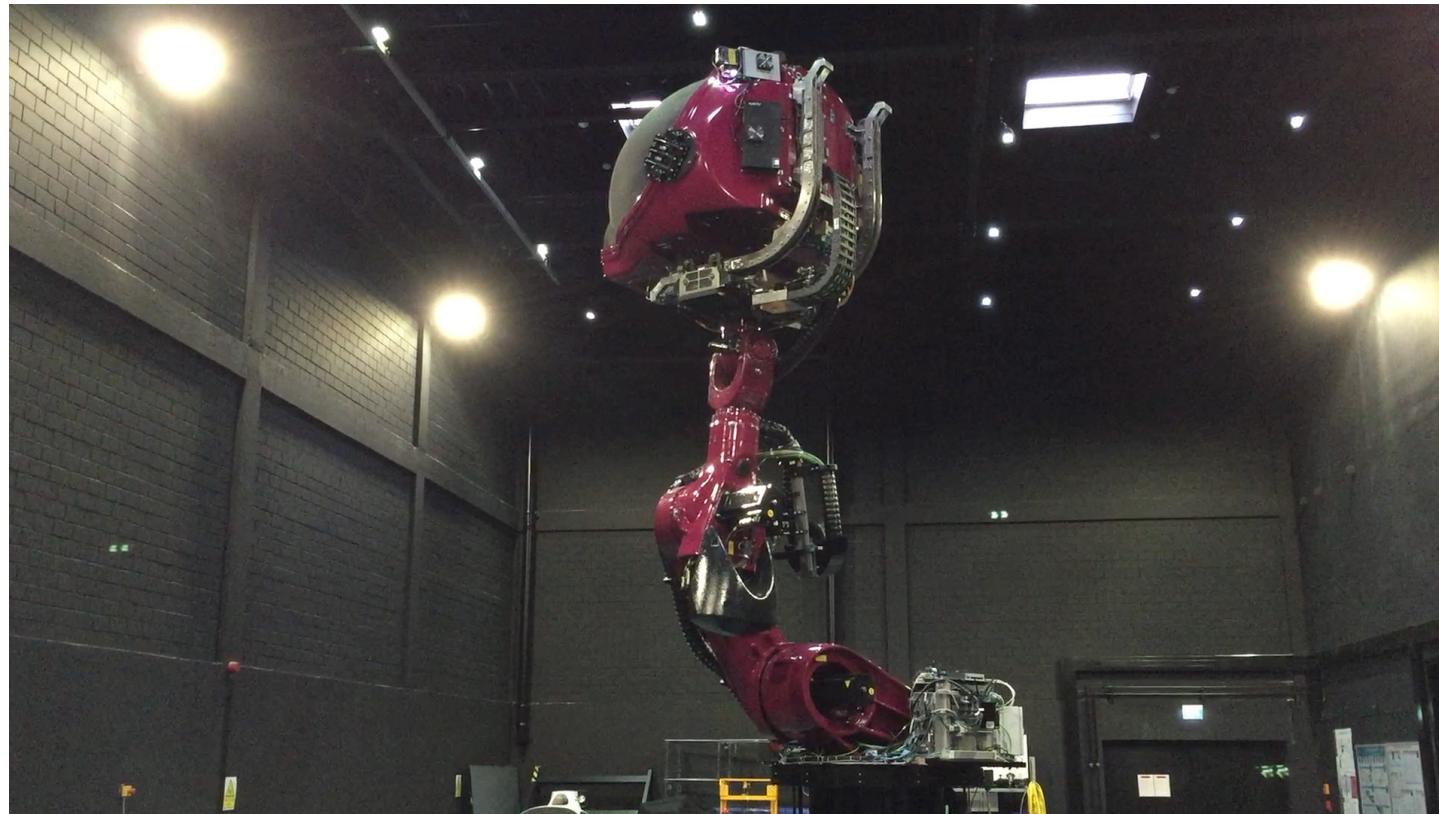
DIS, Uni Napoli

two 6R Comau robots, one mounted on a linear track (+1P)  
coordinated 6D motion using the null-space of the right-side robot ( $N - M = 1$ )



# Motion cueing from redundancy

[video](#)



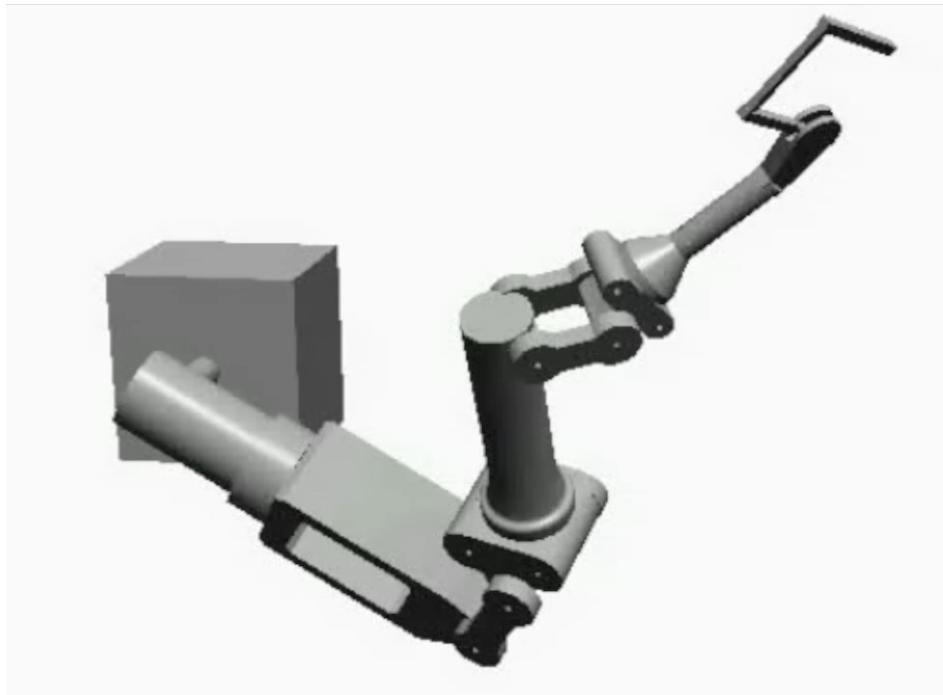
Max Planck Institute for Biological Cybernetics, Tübingen

a **6R KUKA KR500** mounted on a linear track (**+1P**) with a sliding cabin (**+1R**),  
used as a dynamic emulation platform for human perception ( $N - M = 2$ )



# Self-motion

video



8R Dexter: self-motion with  
constant 6D pose of E-E ( $N - M = 2$ )

video

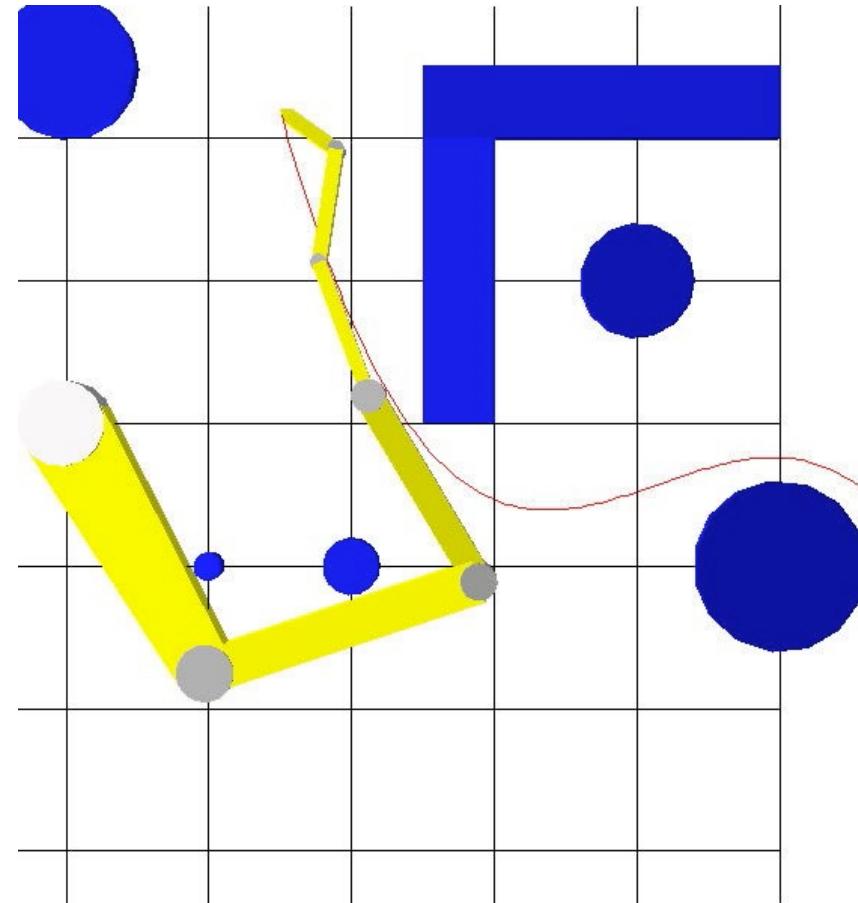


Nakamura's Lab, Uni Tokyo

6R robot with spherical shoulder  
in compliant tasks for the  
Cartesian E-E position ( $N - M = 3$ )



# Obstacle avoidance



video

6R planar arm moving on a given geometric path for the E-E ( $N - M = 4$ )



# An Echord++ industrial experiment

**Results:**

- > simulation of a 6+3 DoF CLOOS robot system
- > Cartesian + nullspace control with all axes included

**Robot system**

- 9 degrees of freedom
- treated as one kinematic chain

Portal  
- 3 prismatic joints (x,y,z)

Qirox QRC350  
- 6 revolute joints (A1 - A6)

Selected keypos\_endeff (1.0=1)  
translation = (2.02064 , 0.00000 , 1.34383 )  
quaternion = (0.38258 , 0.00000 , 0.92388 , -0.00000 )

**Results:**

- > easy and user-friendly demonstration of redundancy resolutions
- > automatic bootstrapping of training data
- > on-line neural-network based learning of redundancy resolutions

**Results:**

- > implementation in a real-world prototype
- > qualitative user study with experienced CLOOS programmers

**3 clips  
of a single video**

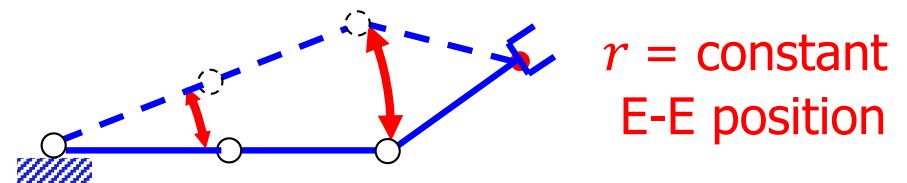


# Inverse kinematics problem

- find  $q(t)$  that realizes the task:  $f(q(t)) = r(t)$  (at all times  $t$ )

- **infinite solutions** exist when the robot is redundant  
(even for  $r(t) = r = \text{constant}$ )

$$N = 3 > 2 = M$$



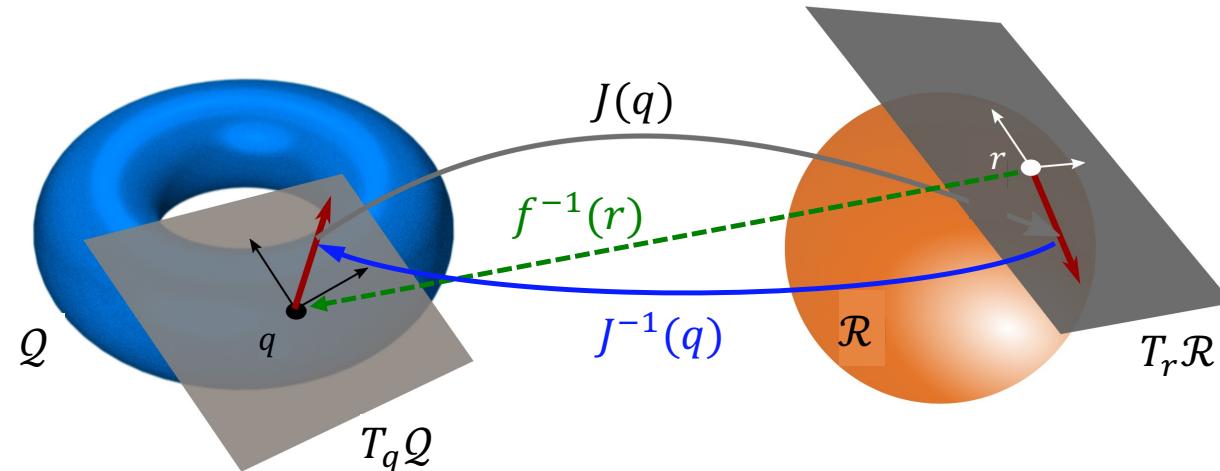
- the robot arm may have “**internal displacements**” that are **unobservable** at the task level (e.g., not contributing to E-E motion)
  - these joint displacements can be chosen so as to **improve/optimize** in some way the behavior of the robotic system
- **self-motion**: an arm reconfiguration in the joint space that does not change/affect the value of the task variables  $r$
- solutions are mainly sought at **differential level** (e.g., **velocity**)



# Inverse kinematics at velocity level

working with tangent spaces to manifolds

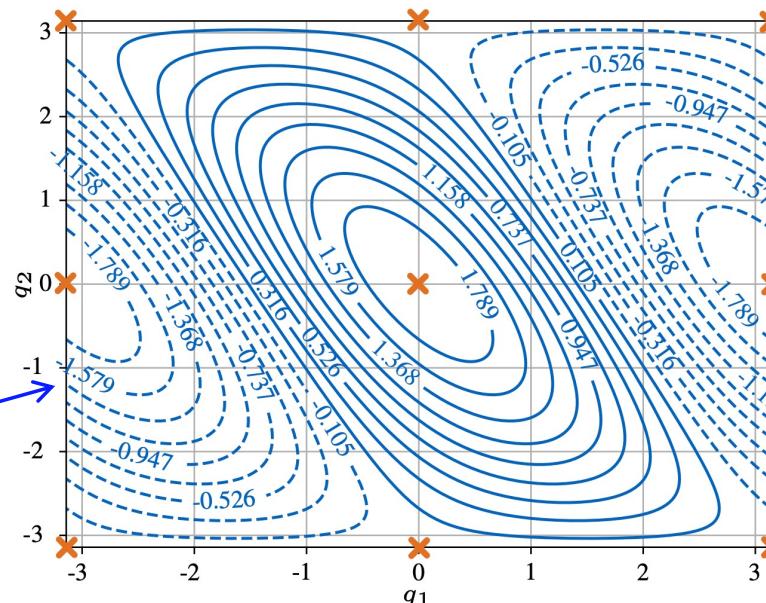
non-redundant case:  
2R planar robot  
in E-E position task  
( $N = M = 2$ )



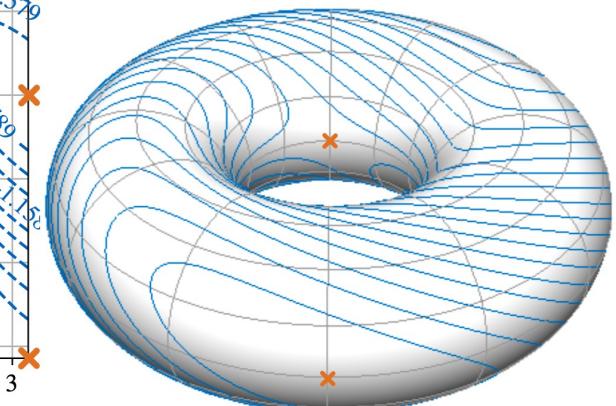
the map between the tangent spaces is linear!

redundant case:  
2R planar robot  
in  $x$ -position only task  
( $N = 2, M = 1$ )

iso-level curves  
correspond to  
 $q \in Q: f_x(q) = x$



local ( $\mathbb{R}^2$ ) vs global (torus)  
representation of the  
configuration manifold  $Q$

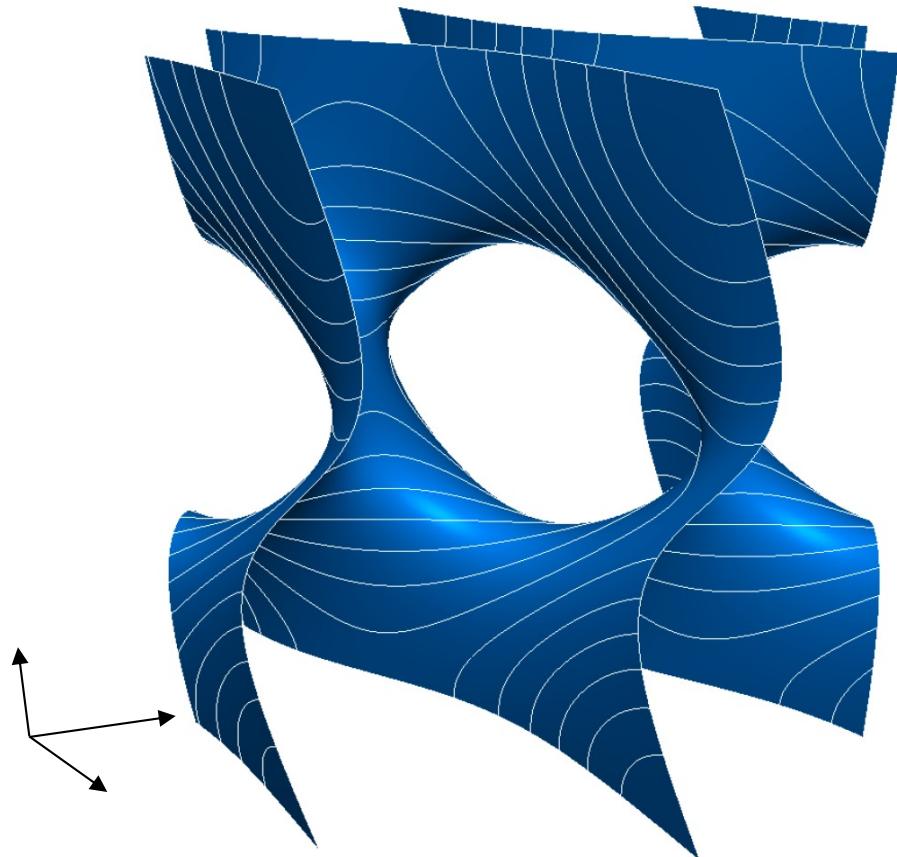




# Self-motion manifold

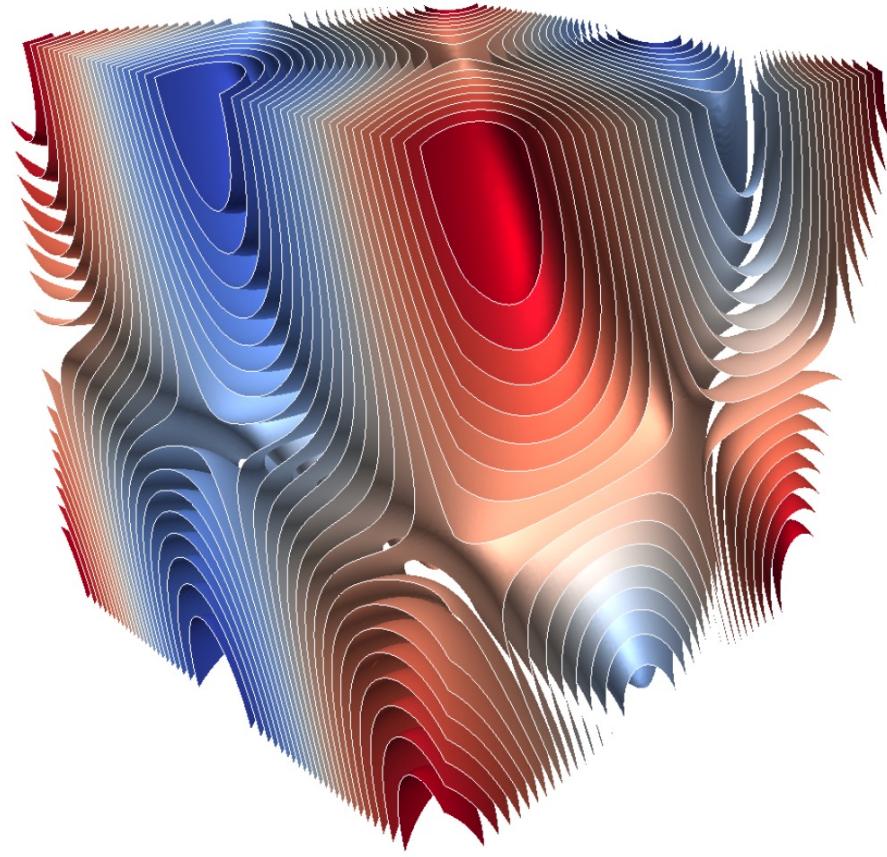
## for a 3R planar robot ( $N = 3$ ) in $x$ -position task ( $M = 1$ )

one leaf in the 3-dimensional  
configuration space  $Q$



using the local representation in  $\mathbb{R}^3$  of manifold  $Q$   
(boundaries at  $q_i = \pm\pi$ ,  $i = 1,2,3$ , coincide!)

multiple  
leaves



pictures courtesy of A. Albu-Schäffer, A. Sachtler:  
"Redundancy resolution at position level,"  
*IEEE Transactions on Robotics*, 2023



# Redundancy resolution

## via optimization of an objective function

### Local methods

given  $\dot{r}(t)$  and  $q(t)$ ,  $t = kT_s$

optimization of  $H(q, \dot{q})$

$\dot{q}(kT_s) \leftarrow$  ON-LINE



$$q((k+1)T_s) = q(kT_s) + T_s \dot{q}(kT_s)$$

discrete-time form

### Global methods

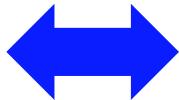
given  $r(t)$ ,  $t \in [t_0, t_0 + T]$ ,  $q(t_0)$

optimization of  $\int_{t_0}^{t_0+T} H(q, \dot{q}) dt$

$$q(t), t \in [t_0, t_0 + T]$$

often, expressed  
in parametric form

↑  
OFF-LINE

relatively EASY  
(an LQ problem)  quite DIFFICULT  
(nonlinear TPBV problems arise)



# Local resolution methods

three classes of methods for solving  $\dot{r} = J(q)\dot{q}$

## 1 Jacobian-based methods (here, analytic Jacobian in general!)

among the infinite solutions, one is chosen, e.g., that minimizes a suitable (possibly weighted) norm

## 2 null-space methods

a term is added to the previous solution so as not to affect execution of the task trajectory, i.e., belonging to the null-space  $\mathcal{N}(J(q))$

## 3 task augmentation methods

redundancy is reduced/eliminated by adding  $S \leq N - M$  further auxiliary tasks (when  $S = N - M$ , the problem has been “squared”)

$$r = f(q) \rightarrow \dot{r} = J(q)\dot{q}$$



# 1

# Jacobian-based methods

we look for a solution to  $\dot{r} = J(q)\dot{q}$  in the form

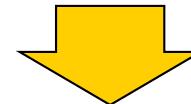
$$J = \boxed{\quad} \underbrace{M}_{N}$$

$$\dot{q} = K(q)\dot{r}$$

$$K = \boxed{\quad} \underbrace{N}_{M}$$

minimum requirement for  $K$ :  $J(q)K(q)J(q) = J(q)$

(  $\Rightarrow$   $K$  = generalized inverse of  $J$  )



$$\forall \dot{r} \in \mathcal{R}(J(q)) \Rightarrow J(q)[K(q)\dot{r}] = J(q)K(q)J(q)\dot{q} = J(q)\dot{q} = \dot{r}$$

example:

if  $J = [J_a \ J_b]$ ,  $\det(J_a) \neq 0$ , one such generalized inverse of  $J$  is  $K_r = \begin{pmatrix} J_a^{-1} \\ 0 \end{pmatrix}$   
(actually, this is a **stronger** right-inverse)



# Pseudoinverse

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$$\dot{q} = J^\#(q)\dot{r} \quad \dots \text{a very common choice: } K = J^\#$$

- $J^\#$  always **exists**, and is the **unique** matrix satisfying

$$\begin{array}{ll} JJ^\#J = J & J^\#JJ^\# = J^\# \\ (JJ^\#)^T = JJ^\# & (J^\#J)^T = J^\#J \end{array}$$

- if  $J$  is **full (row) rank**,  $J^\# = J^T(JJ^T)^{-1}$ ; else, it is computed numerically using the SVD (Singular Value Decomposition) of  $J$  (**pinv** of Matlab)
- the pseudo-inverse solution is the only joint velocity that **minimizes the norm**  $\|\dot{q}\|^2 = \dot{q}^T\dot{q}$  among all joint velocities that **minimize the task error norm**  $\|\dot{r} - J(q)\dot{q}\|^2$
- if the task is feasible ( $\dot{r} \in \mathcal{R}(J(q))$ ), there will be **no task error**



# Weighted pseudoinverse

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$$\dot{q} = J_W^\#(q)\dot{r} \quad \text{another choice: } K = J_W^\#$$

- the solution  $\dot{q}$  minimizes the weighted norm  
$$\|\dot{q}\|_W^2 = \dot{q}^T W \dot{q} \quad W > 0, \text{ symmetric}$$
  
(often diagonal)
- if  $J$  is full (row) rank,  $J_W^\# = W^{-1}J^T(JW^{-1}J^T)^{-1}$
- large weight  $W_i \Rightarrow$  small  $\dot{q}_i$ 
  - larger weights for proximity joints (carrying/moving more “mass”)
  - weights chosen proportionally to the inverse of the joint ranges
- it is NOT a “pseudoinverse” (4th relation does **not** hold),  
but it shares similar properties



# Singular Value Decomposition (SVD)

- the **SVD** routine of Matlab applied to  $J$  provides two orthonormal matrices  $U_{M \times M}$  and  $V_{N \times N}$ , and a matrix  $\Sigma_{M \times N}$  of the form

$$\Sigma = \begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_M \end{pmatrix} \quad \left| \begin{array}{l} \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_\rho > 0 \\ \sigma_{\rho+1} = \cdots = \sigma_M = 0 \\ \text{singular values of } J \end{array} \right.$$

where  $\rho = \text{rank}(J) \leq M$ , so that their product is

$$J = U\Sigma V^T$$

- the columns of  $U$  are eigenvectors of  $J J^T$  (associated to its non-negative eigenvalues  $\sigma_i^2$ ), the columns of  $V$  are eigenvectors of  $J^T J$
- the last  $N - \rho$  columns of  $V$  are a basis for the **null space** of  $J$

$$Jv_i = \sigma_i u_i \quad (i = 1, \dots, \rho)$$

$$Jv_i = 0 \quad (i = \rho + 1, \dots, N)$$



# Computation of pseudoinverses

- **show** that the pseudoinverse of  $J$  is equal to

$$J = U\Sigma V^T \quad \Rightarrow \quad J^\# = V\Sigma^\# U^T \quad \Sigma^\# = \begin{pmatrix} \frac{1}{\sigma_1} & & & \\ & \ddots & & \\ & & \frac{1}{\sigma_\rho} & \\ \hline & & & 0_{(M-\rho) \times (M-\rho)} \\ & & & 0_{(N-M) \times M} \end{pmatrix}$$

for any rank  $\rho$  of  $J$

- **show** that matrix  $J_W^\#$  appears when solving the constrained linear-quadratic (LQ) optimization problem (with  $W > 0$ , symmetric, and assuming  $J$  of full rank)

$$\min \frac{1}{2} \|\dot{q}\|_W^2 \quad \text{s.t.} \quad J(q)\dot{q} - \dot{r} = 0$$

**and** that the pseudoinverse is a particular case for  $W = I$

- **show** that a weighted pseudoinverse of  $J$  can be computed by SVD/pinv as

$$J_{aux} = JW^{-1/2} \quad J_W^\# = W^{-1/2} \text{pinv}(J_{aux})$$

applies **equally** to  
square and non-square  
matrices



# Singularity robustness

## Damped Least Squares (DLS)

unconstrained  
minimization  
of a **suitable**  
objective function

$$\min_{\dot{q}} H(\dot{q}) = \frac{\mu^2}{2} \|\dot{q}\|^2 + \frac{1}{2} \|\dot{r} - J\dot{q}\|^2$$

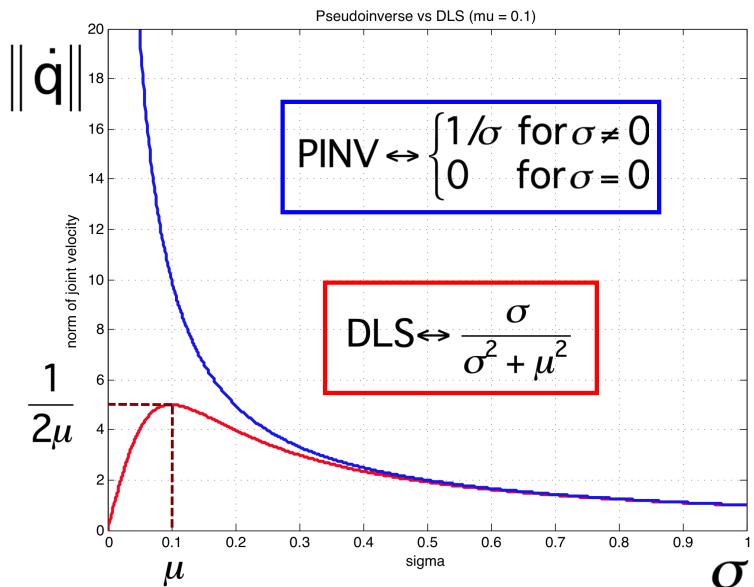
**compromise** between  
large joint velocity  
and task accuracy

**SOLUTION**  $\dot{q} = J_{DLS}(q)\dot{r} = J^T(JJ^T + \mu^2 I_M)^{-1}\dot{r}$

- induces a **robust behavior** when crossing **singularities**, but in its basic version gives always a **task error**  $\dot{e} = \mu^2(JJ^T + \mu^2 I_M)^{-1}\dot{r}$  (as for  $N = M$ )
  - $J_{DLS}$  is **not** a generalized inverse  $K$
  - using SVD:  $J = U \Sigma V^T \Rightarrow J_{DLS} = V \Sigma_{DLS} U^T$ ,  $\Sigma_{DLS} = \begin{pmatrix} \text{diag} \left\{ \frac{\sigma_i}{\sigma_i^2 + \mu^2} \right\} & \\ \rho \times \rho & 0_{(M-\rho) \times (M-\rho)} \\ \hline 0_{(N-M) \times \rho} & 0_{(N-M) \times (M-\rho)} \end{pmatrix}$
  - choice of a **variable damping factor**  $\mu^2(q) \geq 0$ , function of the minimum singular value  $\sigma_\rho(q) > 0$  of  $J \cong$  a measure of distance from a singularity (if  $\rho = M$ ) or of further loss of rank (when  $\rho < M$ )
  - **numerical filtering**: introduces damping **only/mostly** in non-feasible directions for the task (see Maciejewski and Klein, *J of Rob Syst*, 1988)

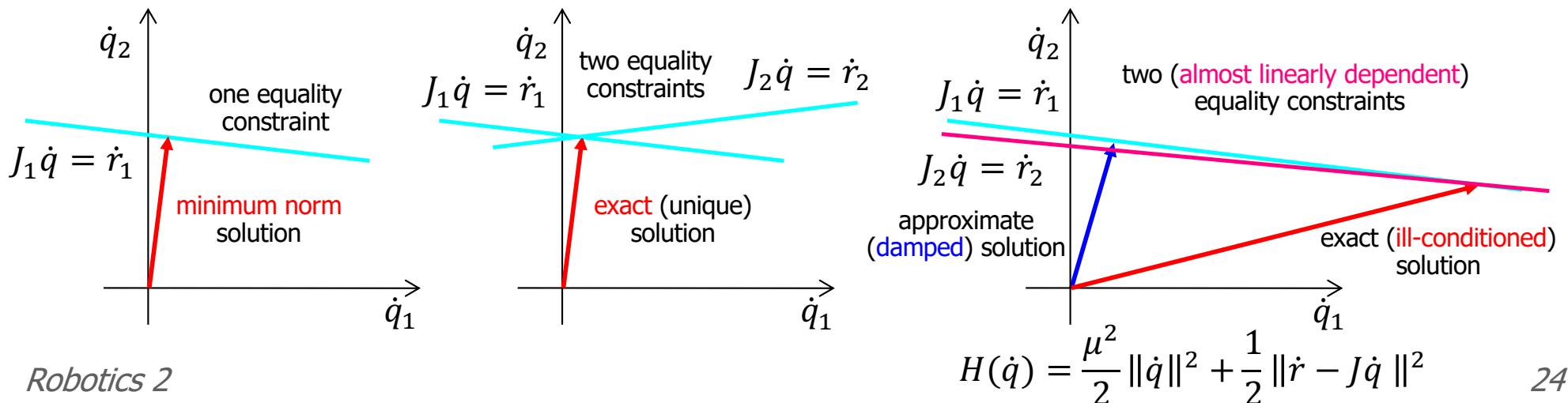


# Behavior of DLS solution



- a. comparison of joint velocity norm with **PINV** (pseudoinverse) or **DLS** solutions
- in a task direction along a vector  $u$  of  $U$ , when the associated singular value  $\sigma \rightarrow 0$
  - PINV** goes to infinity (and then is 0 at  $\sigma = 0$ )
  - DLS** peaks a value of  $1/2\mu$  at  $\sigma = \mu$  (and then smoothly goes to 0...)

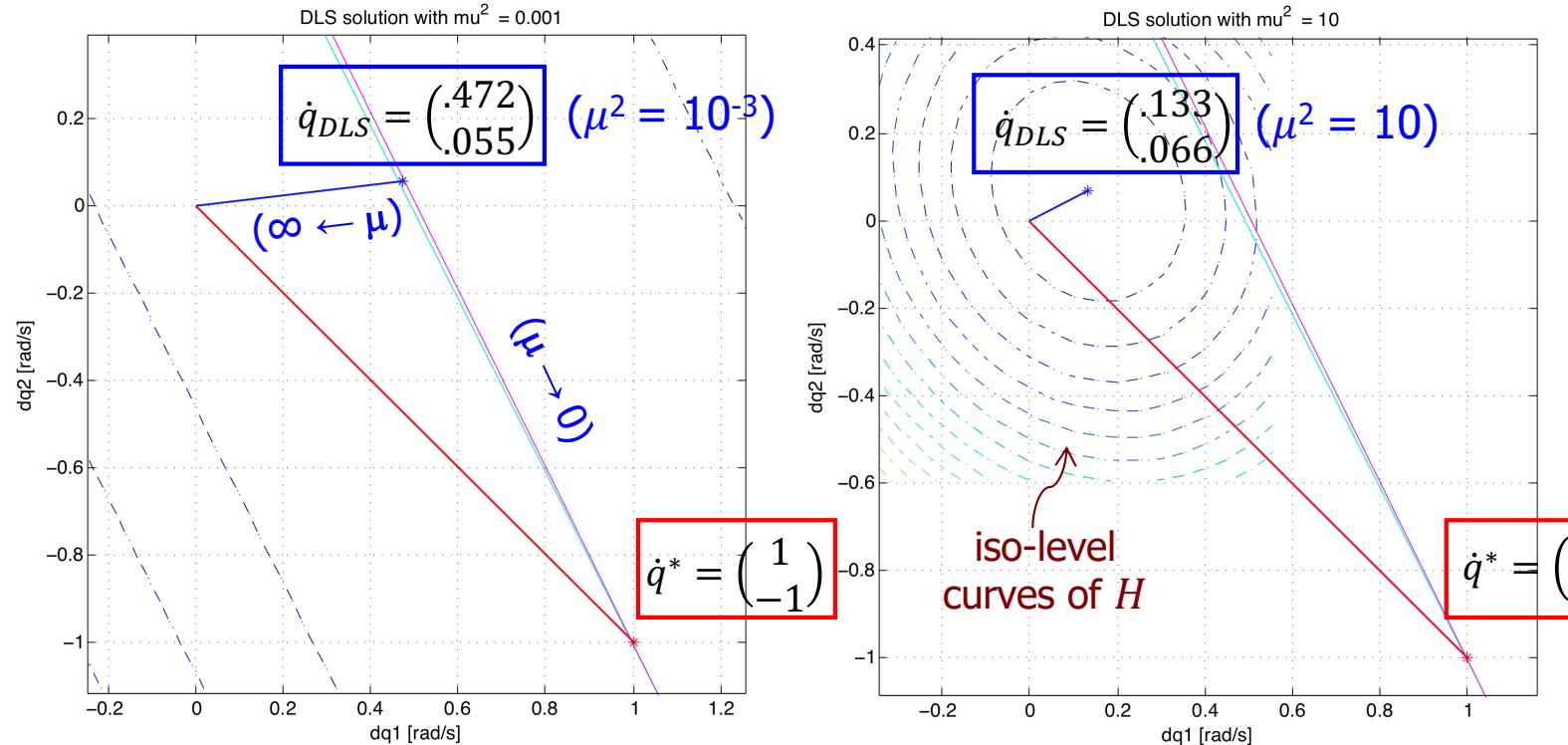
b. graphical interpretation of “damping” effect (here  $M = N = 2$ , for simplicity)





# Numerical example of DLS solution

planar 2R arm, unit links, close to (stretched) singular configuration  $q_1 = 45^\circ, q_2 = 1.5^\circ$



$$\dot{r} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$\in \mathcal{R}(J)$  even @singularity!

exact solution ( $\mu=0$ )

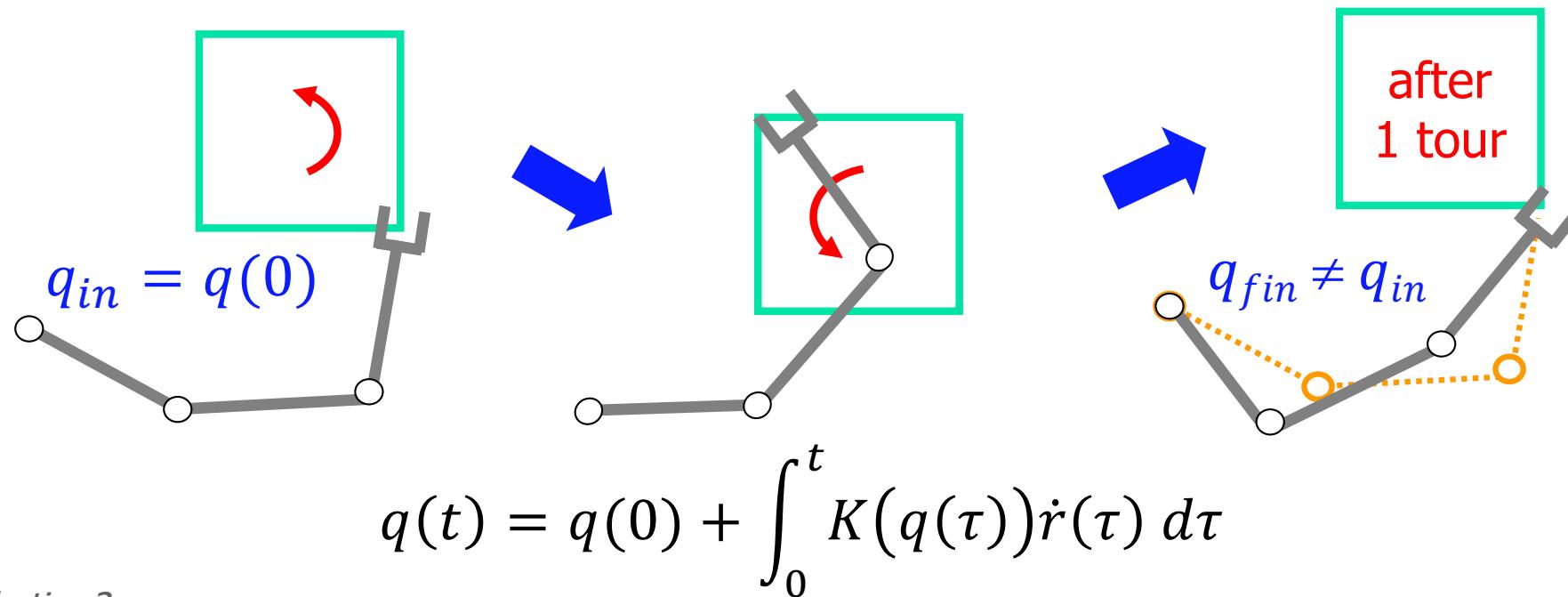
$$H = \frac{\mu^2}{2} \|\dot{q}\|^2 + \frac{1}{2} \|\dot{r} - J\dot{q}\|^2$$

$\mu^2$	0	10 <sup>-4</sup>	10 <sup>-3</sup>	10 <sup>-2</sup>	10
$\ \dot{q}\ $	$\sqrt{2}$	.8954	.4755	.4467	.1490
$\ \dot{e}\ $	0	$6.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$	.6668
$H_{min}$	0	$7.7 \cdot 10^{-5}$	$2.2 \cdot 10^{-4}$	$1.2 \cdot 10^{-3}$	$3.4 \cdot 10^{-1}$



# Limits of Jacobian-based methods

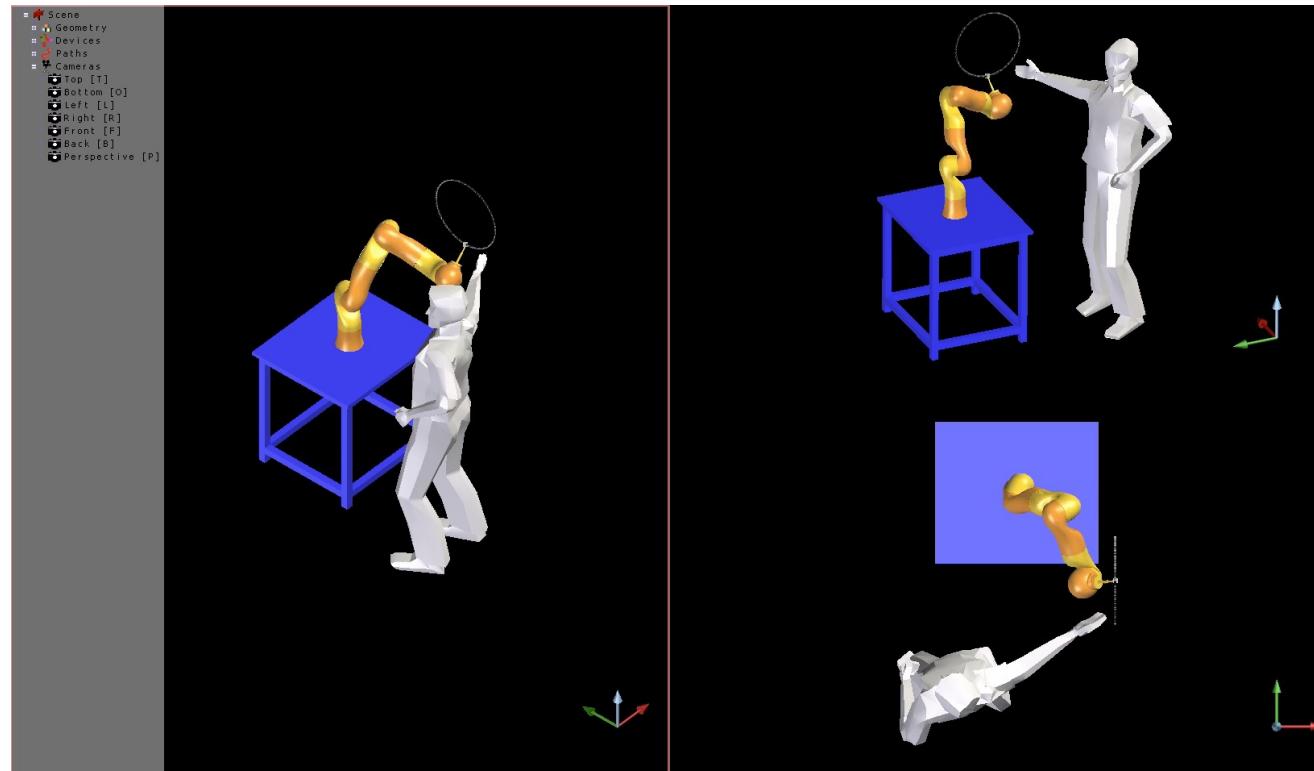
- no guarantee that **singularities** are globally avoided during task execution
  - despite joint velocities are kept to a minimum, this is only a local property and “avalanche” phenomena may occur
- typically lead to **non-repeatable** motion in the joint space
  - cyclic motions in **task space** do not map to cyclic motions in **joint space**





# Drift with Jacobian pseudoinverse

- a 7R KUKA LWR4 robot moves in the vicinity of a **human** operator
- we command a cyclic Cartesian path (only in position,  $M = 3$ ), to be repeated **several** times using the **pseudoinverse** solution
- (**unexpected**) collision of a link occurs during the **third** cycle ...



video



## 2

# Null-space methods

general solution of  $J\dot{q} = \dot{r}$

$$\dot{q} = J^\# \dot{r} + (I - J^\# J) \dot{q}_0$$

a particular solution  
(here, the pseudoinverse)  
in  $\mathcal{R}(J^T)$

“orthogonal” projection  
of  $\dot{q}_0$  in  $\mathcal{N}(J)$

all solutions of the associated  
homogeneous equation  $J\dot{q} = 0$   
(self-motions)

- symmetric
- idempotent:  $[I - J^\# J]^2 = [I - J^\# J]$
- $[I - J^\# J]^\# = [I - J^\# J]$
- $J^\# \dot{r}$  is orthogonal to  $[I - J^\# J] \dot{q}_0$

properties of  
projector  $[I - J^\# J]$

even more in general...

$$\dot{q} = K_1 \dot{r} + (I - K_2 J) \dot{q}_0$$

... but with less nice properties!

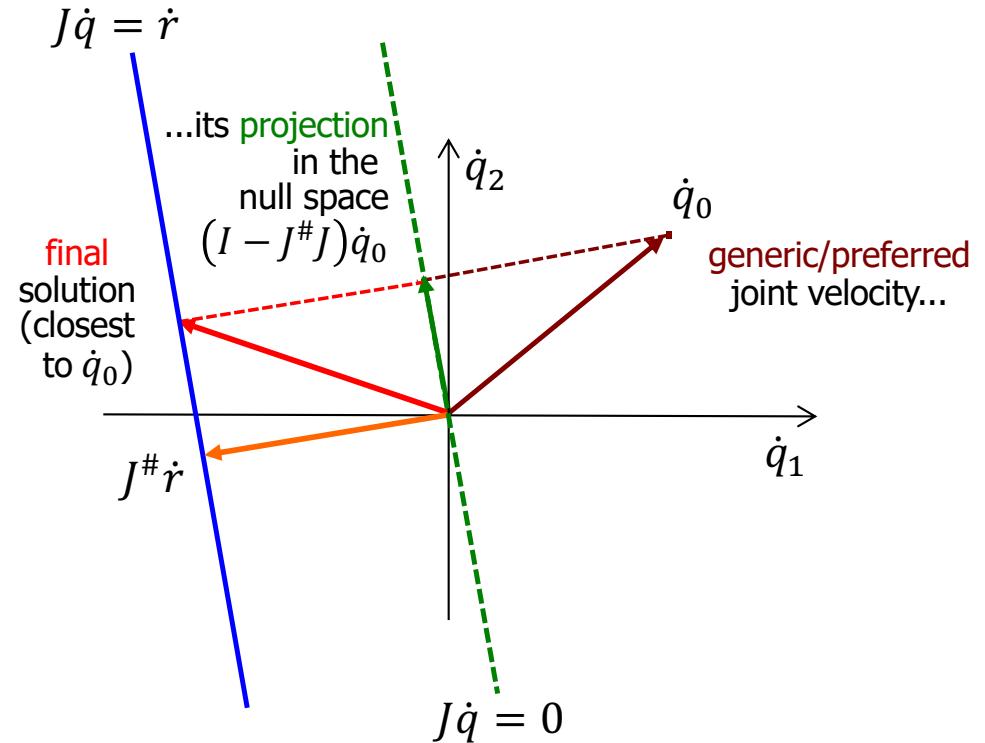
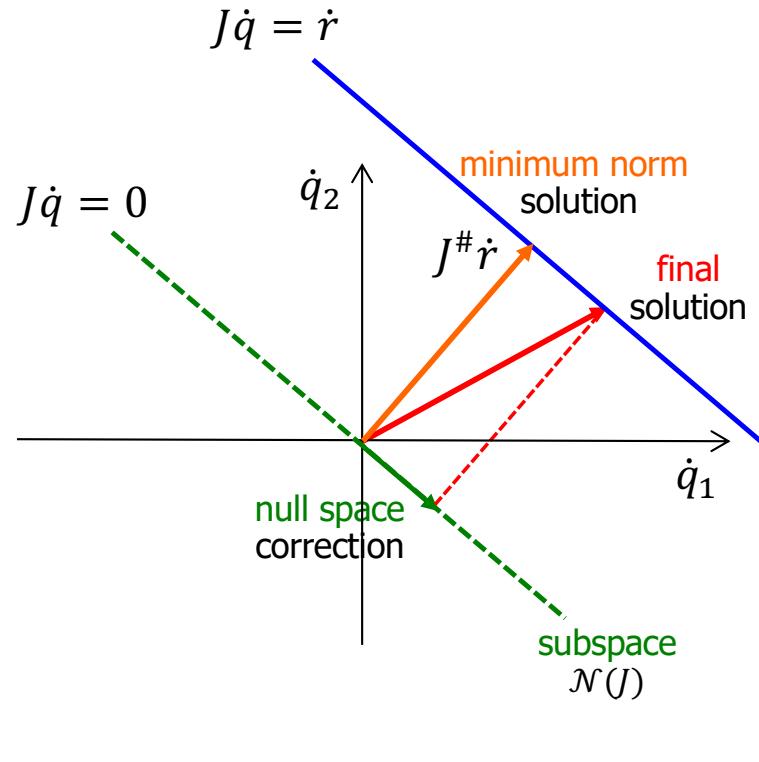
$K_1, K_2$  generalized  
inverses of  $J$   
( $J K_i J = J$ )

how do we choose  $\dot{q}_0$ ?



# Geometric view on Jacobian null space

in the space of velocity commands



a correction is added to the original pseudoinverse (minimum norm) solution

- which is in the **null space** of the Jacobian
- and possibly satisfies **additional criteria** or objectives



# Linear-Quadratic Optimization

## generalities

$$\min_x H(x) = \frac{1}{2} (x - x_0)^T W (x - x_0)$$

s.t.  $Jx = y$

$M \times N$

$$x \in \mathbb{R}^N$$

$$W > 0 \text{ (symmetric)}$$

$$y \in \mathbb{R}^M$$

$$\text{rank}(J) = \rho(J) = M$$

$$L(x, \lambda) = H(x) + \lambda^T (Jx - y) \leftarrow \text{Lagrangian (with multipliers } \lambda\text{)}$$

necessary conditions

$$\begin{cases} \nabla_x L = \left( \frac{\partial L}{\partial x} \right)^T = W(x - x_0) + J^T \lambda = 0 \\ \nabla_\lambda L = \left( \frac{\partial L}{\partial \lambda} \right)^T = Jx - y = 0 \end{cases} \rightarrow \boxed{x = x_0 - W^{-1} J^T \lambda}$$

+ sufficient condition for a minimum

$$\nabla_x^2 L = W > 0 \rightarrow \boxed{Jx_0 - JW^{-1}J^T\lambda - y = 0}$$

$\lambda = (JW^{-1}J^T)^{-1}(Jx_0 - y)$   $M \times M$  invertible  $\rightarrow x = x_0 + W^{-1}J^T(JW^{-1}J^T)^{-1}(y - Jx_0)$



# Linear-Quadratic Optimization

## application to robot redundancy resolution

PROBLEM

$$\begin{aligned} \min_{\dot{q}} H(\dot{q}) &= \frac{1}{2} (\dot{q} - \dot{q}_0)^T W (\dot{q} - \dot{q}_0) \\ \text{s.t. } J\dot{q} &= \dot{r} \end{aligned}$$

$\dot{q}_0$  is a  
“privileged”  
joint velocity

SOLUTION

$$\dot{q} = \dot{q}_0 + W^{-1} J^T (J W^{-1} J^T)^{-1} (\dot{r} - J \dot{q}_0)$$

$$J_W^\#$$

$$\dot{q} = J_W^\# \dot{r} + (I - J_W^\# J) \dot{q}_0$$

minimum weighted norm  
solution (for  $\dot{q}_0 = 0$ )

“projection” matrix in  
the null-space  $\mathcal{N}(J)$

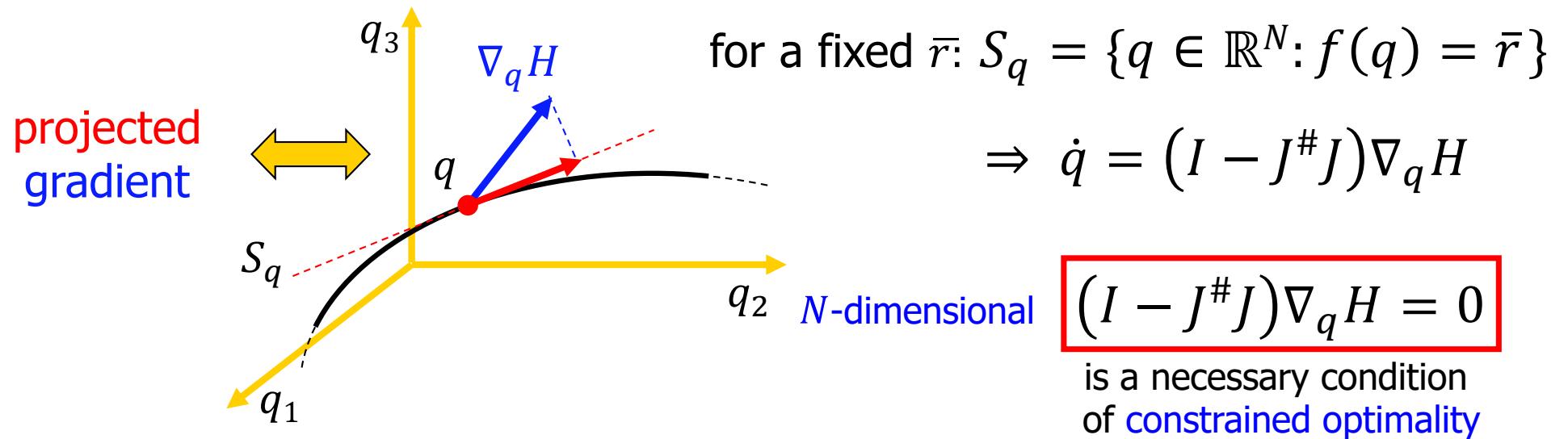


# Projected Gradient (PG)

$$\dot{q} = J^\# \dot{r} + (I - J^\# J) \dot{q}_0$$

the choice  $\dot{q}_0 = \nabla_q H(q)$  → differentiable objective function  
 realizes one step of a constrained optimization algorithm

while executing the time-varying task  $r(t)$   
 the robot tries to increase the value of  $H(q)$





# Typical objective functions $H(q)$

- **manipulability** (maximize the “distance” from singularities)

$$H_{\text{man}}(q) = \sqrt{\det[J(q)J^T(q)]}$$

- **joint range** (minimize the “distance” from the mid points of the joint ranges)

$$q_i \in [q_{m,i}, q_{M,i}]$$
$$\bar{q}_i = \frac{q_{M,i} + q_{m,i}}{2}$$

$$H_{\text{range}}(q) = \frac{1}{2N} \sum_{i=1}^N \left( \frac{q_i - \bar{q}_i}{q_{M,i} - q_{m,i}} \right)^2$$

$$\dot{q}_0 = -\nabla_q H(q)$$

- **obstacle avoidance** (maximize the minimum distance to Cartesian obstacles)

also known as  
“clearance”

$$H_{\text{obs}}(q) = \min_{\substack{a \in \text{robot} \\ b \in \text{obstacles}}} \|a(q) - b\|^2$$

potential difficulties due  
to non-differentiability  
(this is a max-min problem)



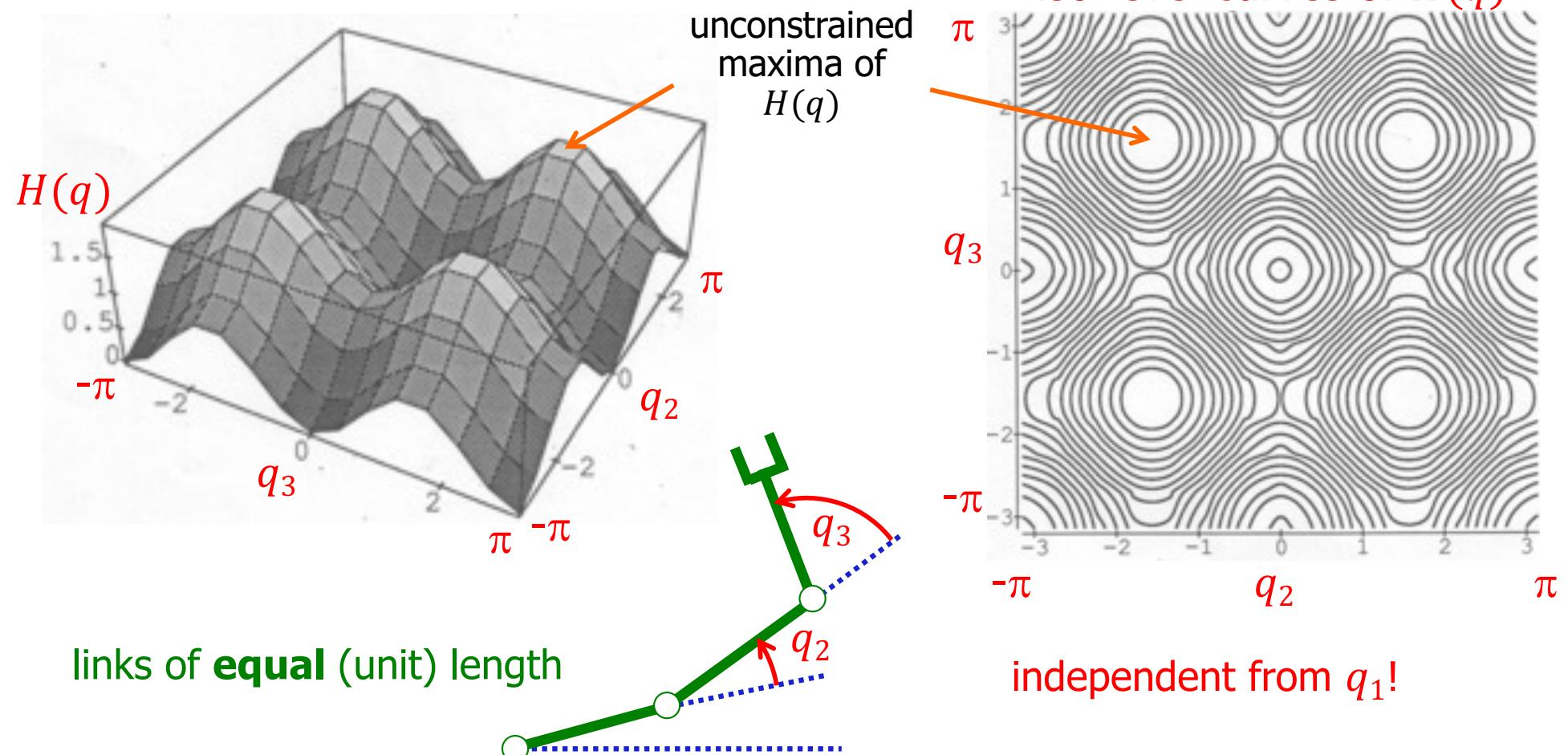
# Singularities of planar 3R arm

the robot is redundant  
for a positioning task  
in the plane ( $M = 2$ )

$$H(q) = \sin^2 q_2 + \sin^2 q_3$$

this  $H$  is **not**  $H_{\text{man}}$   
but has the **same** minima

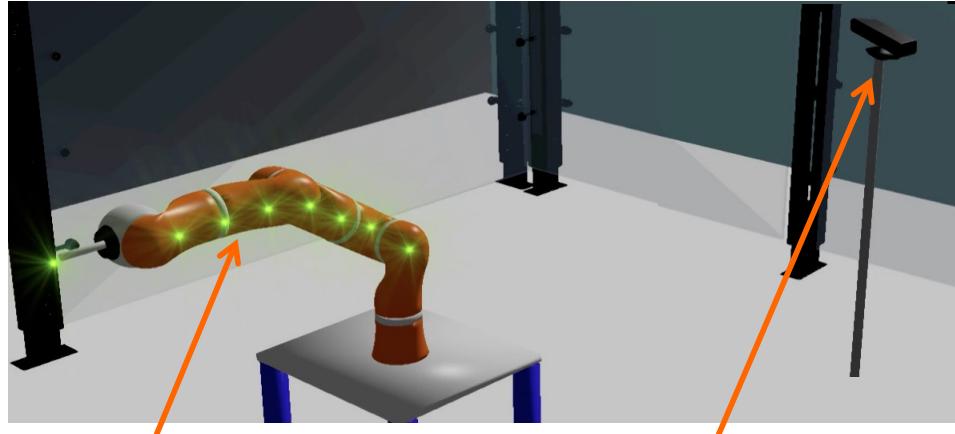
iso-level curves of  $H(q)$





# Minimum distance computation in human-robot interaction

video

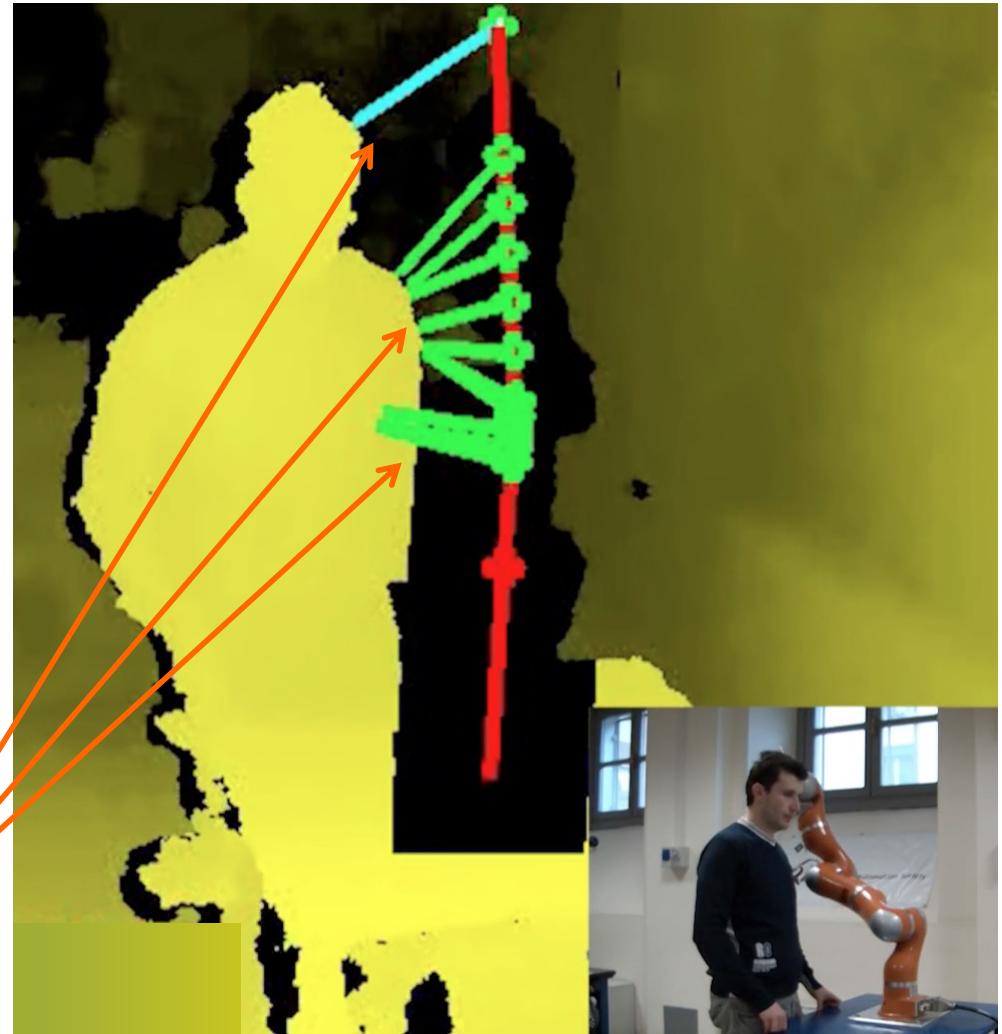


LWR4 robot with  
a finite number of  
control points  $a(q)$   
(8, including the E-E)

a Kinect sensor monitors  
the workspace giving the  
3D position of points  $b$   
on obstacles that are  
fixed or moving  
(like humans)

distances in 3D (and then the clearance)  
are computed in this case as

$$\min_{\substack{a \in \{\text{control points}\} \\ b \in \text{human body}}} \|a(q) - b\|^2$$





# IIT Centauro robot

## Execution of multiple tasks - Self-motions

<https://spectrum.ieee.org/centauro-a-new-disaster-response-robot-from-iit>

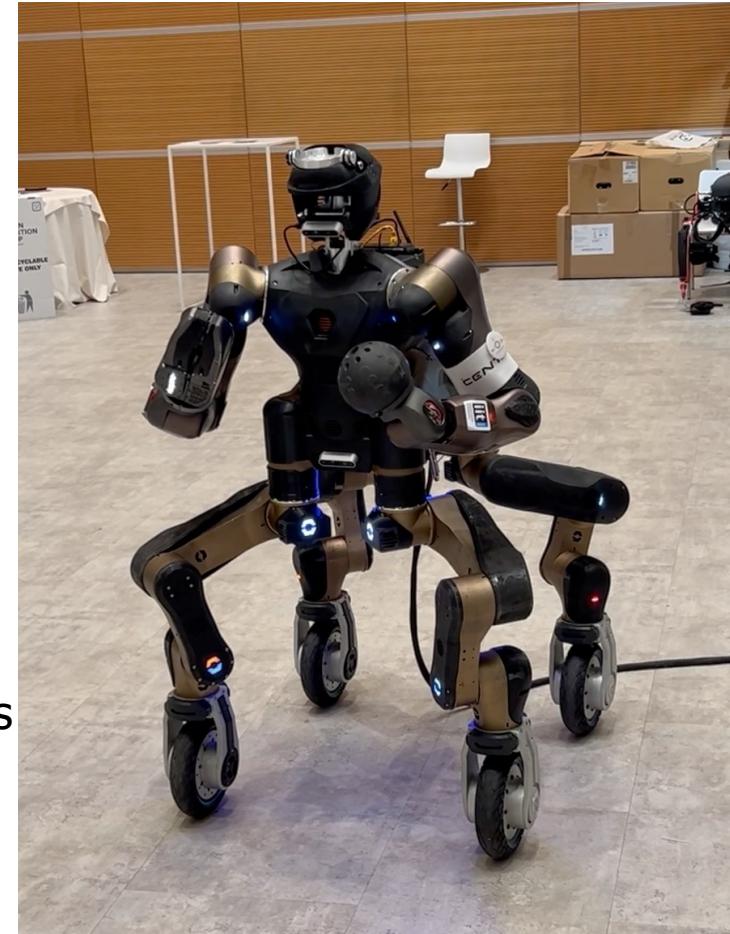
video



1.5 m high  
93 kg  
each arm  
10.5 kg  
(with 11 kg  
payload!)  
2.5 h  
autonomy

cameras  
RGBD sensors  
laser Lidar  
scanner

video



@European Robotics Forum (ERF) in Rimini, March 2024



# Comments on null-space methods

---

- the projection matrix  $(I - J^{\#}J)$  has dimension  $N \times N$ , but only rank  $N - M$  (if  $J$  is full rank  $M$ ), with some **waste of information**
- actual (efficient) evaluation of the solution

$$\dot{q} = J^{\#}\dot{r} + (I - J^{\#}J)\dot{q}_0 = \dot{q}_0 + J^{\#}(\dot{r} - J\dot{q}_0)$$

but the pseudoinverse is needed anyway, and this is **computationally intensive** (SVD in the general case)

- in principle, the additional complexity of a redundancy resolution method should depend only on the **redundancy degree  $N - M$**
- a constrained optimization method is available, which is known to be more efficient than the projected gradient (PG) —at least when the **Jacobian has full rank ...**



# Decomposition of joint space

- if  $\rho(J(q)) = M$ , there exists a **decomposition** of the set of joints (possibly, after a reordering)

$$q = \begin{pmatrix} q_a \\ q_b \end{pmatrix} \quad \begin{matrix} M \\ N - M \end{matrix}$$

$M \times M$

such that  $J_a(q) = \frac{\partial f}{\partial q_a}$  is nonsingular

- from the **implicit function theorem**, there exists an inverse function  $g$

$$f(q_a, qb) = r \quad \rightarrow \quad q_a = g(r, qb)$$

$$\text{with } \frac{\partial g}{\partial q_b} = - \left( \frac{\partial f}{\partial q_a} \right)^{-1} \frac{\partial f}{\partial q_b} = -J_a^{-1}(q)J_b(q)$$

- the  **$N - M$  variables  $q_b$**  can be selected **independently** (e.g., they are used for optimizing an objective function  $H(q)$ , “reduced” via the use of  $g$  to a **function of  $q_b$  only**)
- $q_a = g(r, qb)$  is then chosen so as to correctly execute the task



# Reduced Gradient (RG)

- $H(q) = H(q_a, q_b) = H(g(r, q_b), q_b) = H'(q_b)$ , with  $r$  at **current** value
- the **Reduced Gradient** (w.r.t.  $q_b$  only, but still keeping the effects of this choice into account) is

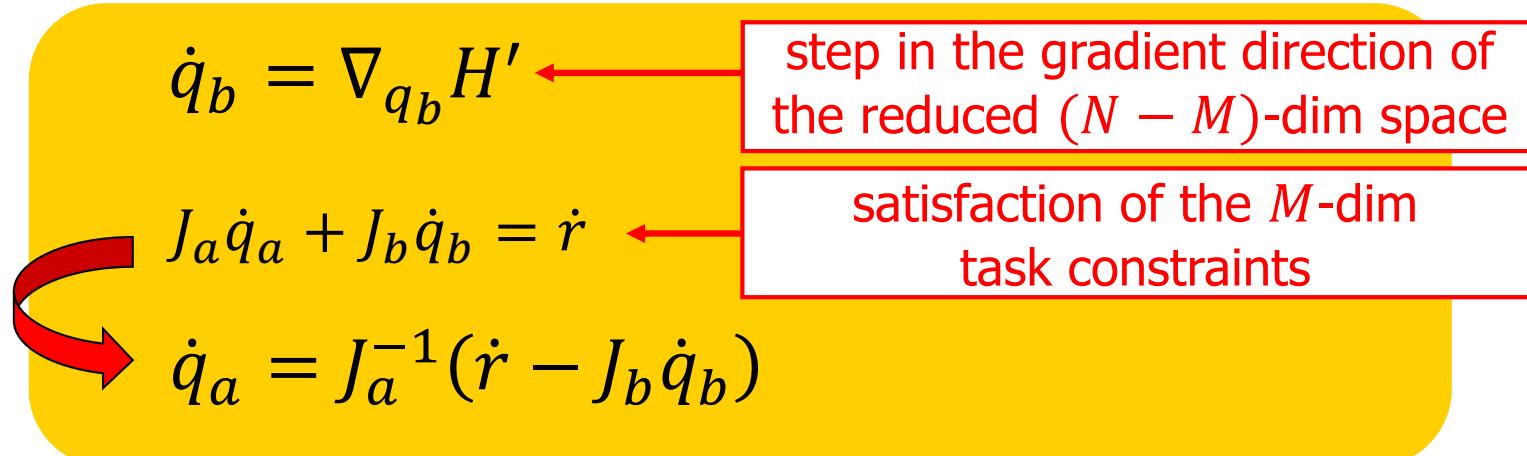
$$\nabla_{q_b} H' = [-(J_a^{-1} J_b)^T \quad I_{N-M}] \nabla_q H$$

$(\neq \nabla_{q_b} H \text{ only}!!)$

- algorithm

$$\boxed{\nabla_{q_b} H' = 0}$$

is a "compact"  
(i.e., **N – M** dimensional)  
**necessary** condition  
of **constrained optimality**





# Comparison between PG and RG

---

- Projected Gradient (PG)

$$\dot{q} = J^\# \dot{r} + (I - J^\# J) \nabla_q H$$

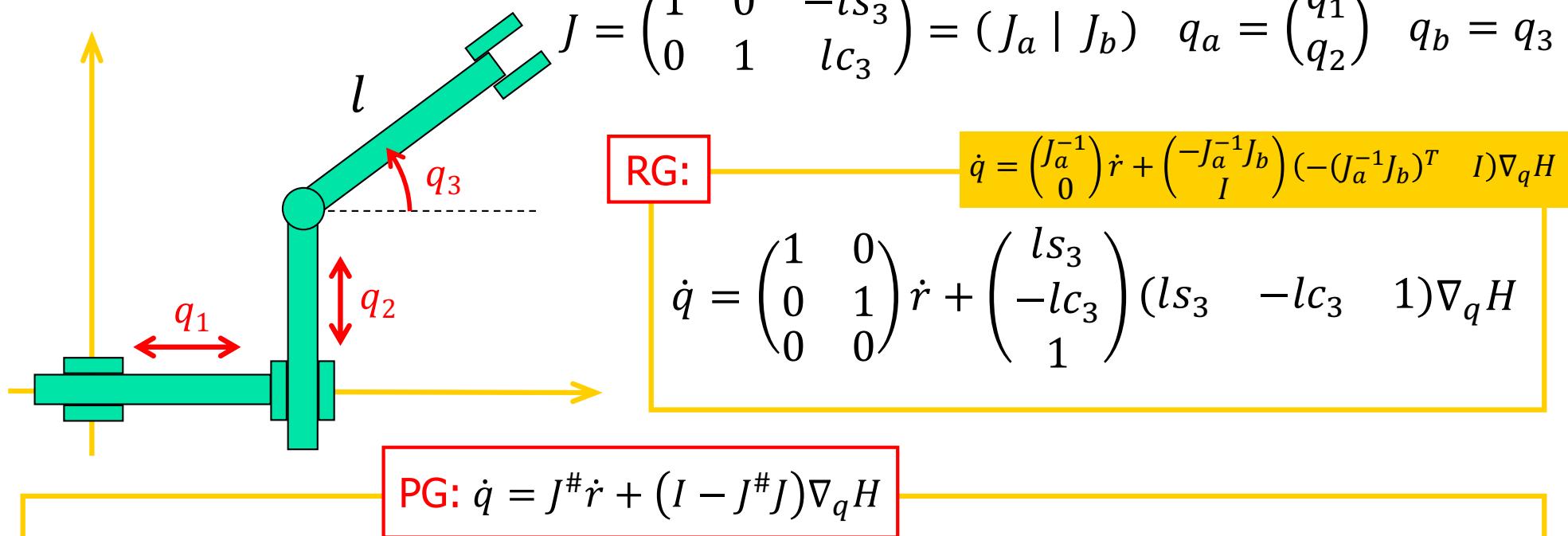
- Reduced Gradient (RG)

$$\dot{q} = \begin{pmatrix} \dot{q}_a \\ \dot{q}_b \end{pmatrix} = \begin{pmatrix} J_a^{-1} \\ 0 \end{pmatrix} \dot{r} + \begin{pmatrix} -J_a^{-1} J_b \\ I \end{pmatrix} (-J_a^{-1} J_b)^T \begin{pmatrix} -J_a^{-1} J_b \\ I \end{pmatrix} \nabla_q H$$

- RG is **analytically** simpler and **numerically** faster than PG, but requires the search for a non-singular minor ( $J_a$ ) of the robot Jacobian
- if  $r = \text{cost}$  &  $N - M = 1 \Rightarrow$  same (unique) direction for  $\dot{q}$ , but RG has automatically a **larger** optimization step size
- else  $\Rightarrow$  RG and PG methods provide always **different evolutions**



# Analytic comparison PPR robot



$$J^\# = \frac{1}{1+l^2} \begin{pmatrix} 1+l^2 c_3^2 & l^2 s_3 c_3 \\ l^2 s_3 c_3 & 1+l^2 s_3^2 \\ -ls_3 & lc_3 \end{pmatrix} \quad I - J^\# J = \frac{1}{1+l^2} \begin{pmatrix} l^2 s_3^2 & l^2 s_3 c_3 & ls_3 \\ l^2 s_3 c_3 & l^2 c_3^2 & -lc_3 \\ ls_3 & -lc_3 & 1 \end{pmatrix}$$

always  $< 1!!$



# Joint range limits

$$q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \theta = T\theta$$

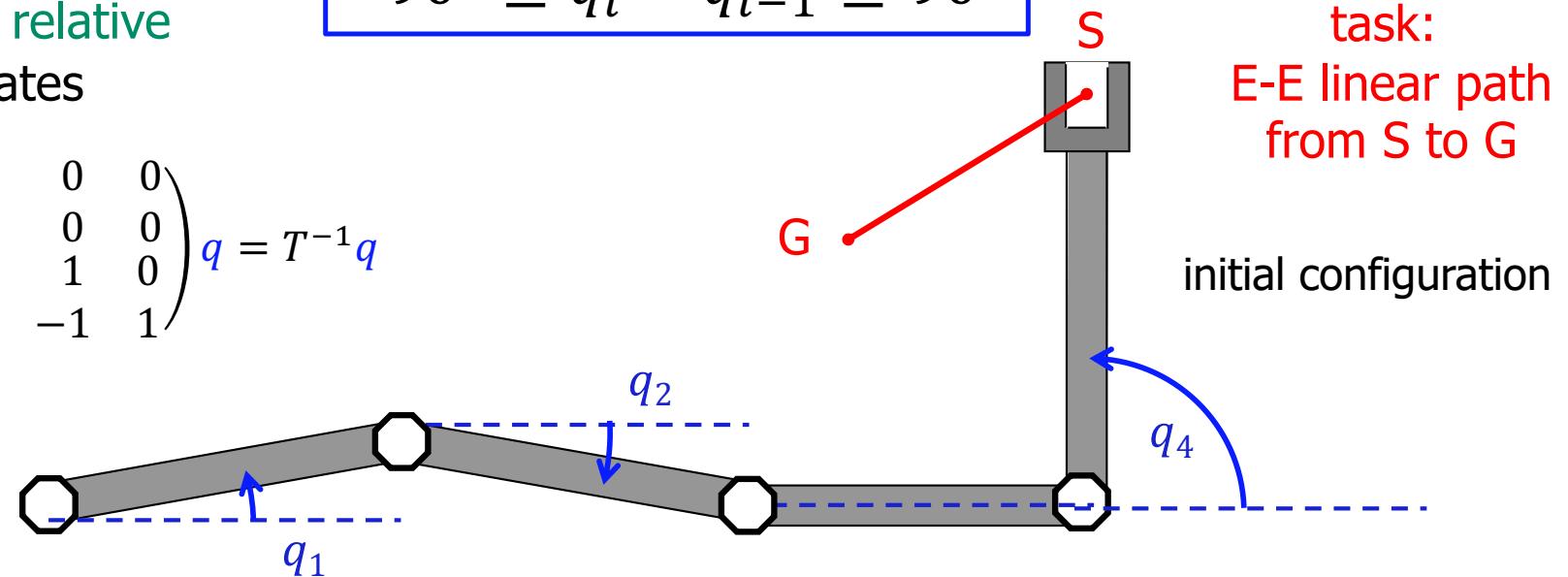
absolute  $\Leftrightarrow$  relative  
coordinates

$$\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} q = T^{-1}q$$

$$-90^\circ \leq \theta_i \leq 90^\circ$$

$\Updownarrow$

$$-90^\circ \leq q_i - q_{i-1} \leq 90^\circ$$

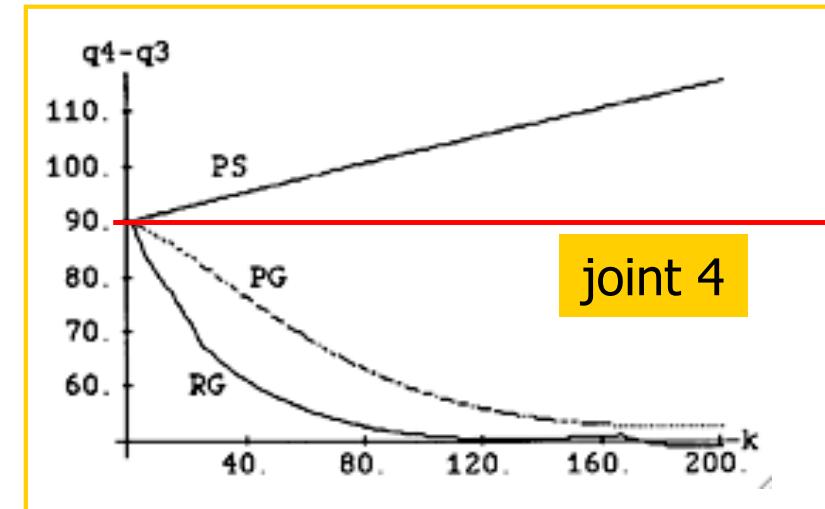
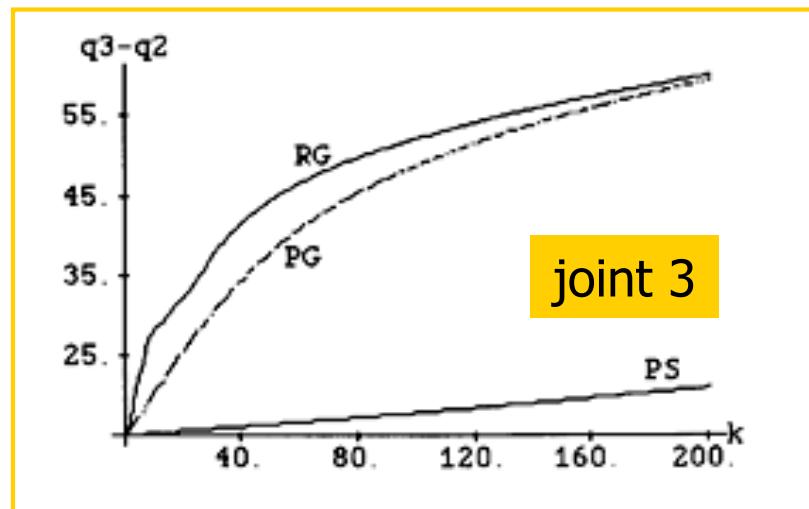
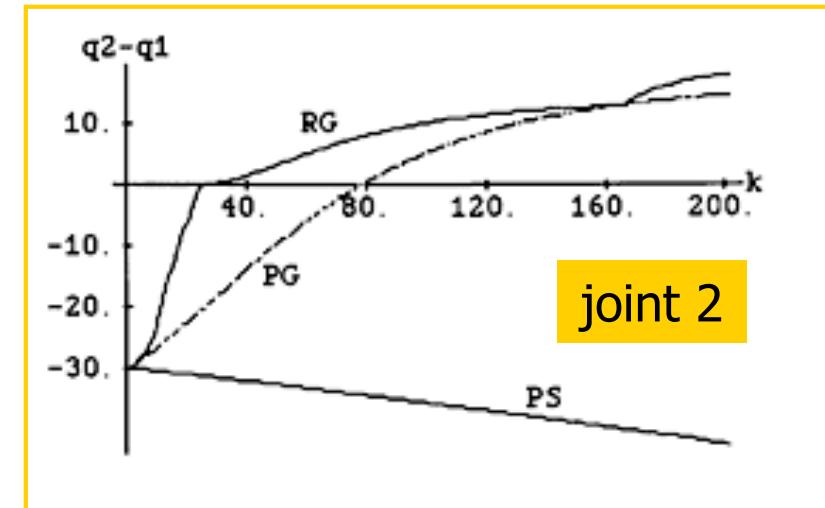
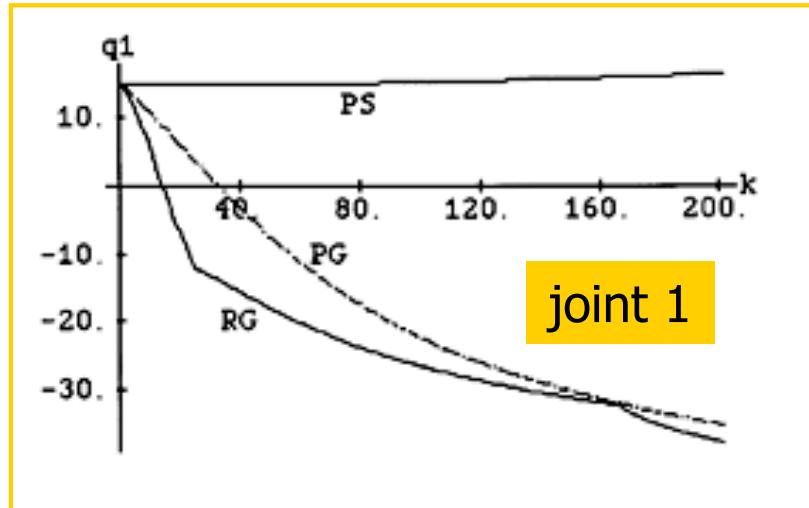


numerical comparison among pseudoinverse (**PS**),  
projected gradient (**PG**), and reduced gradient (**RG**) methods



# Numerical results

## minimizing distance from mid joint range

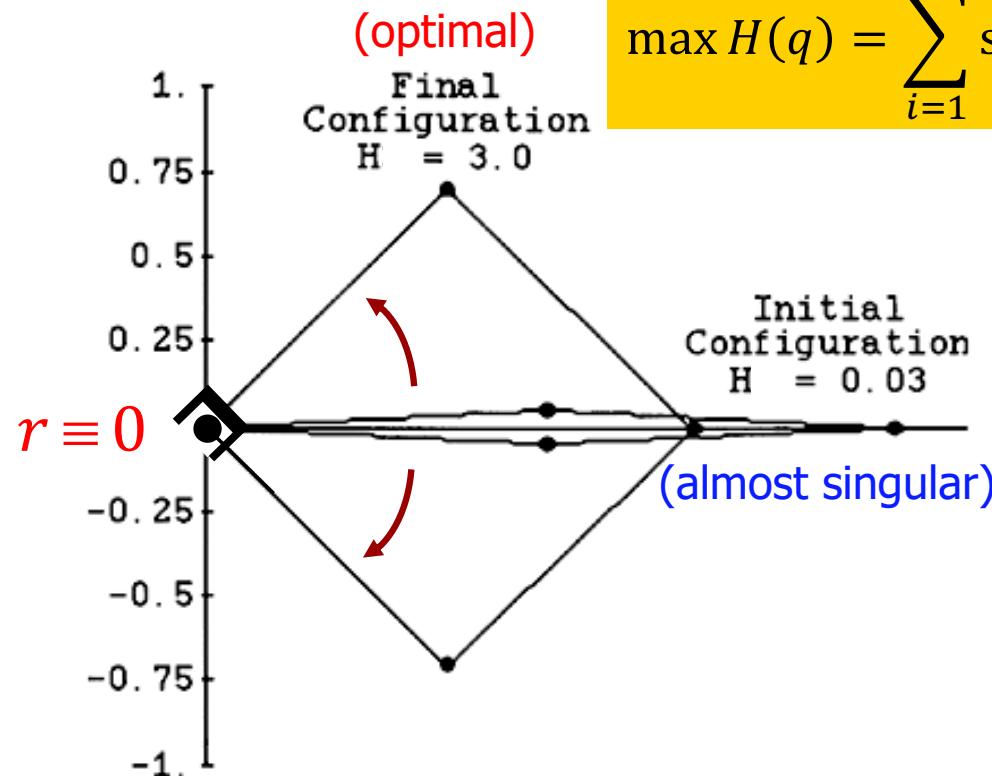


steps of numerical simulation



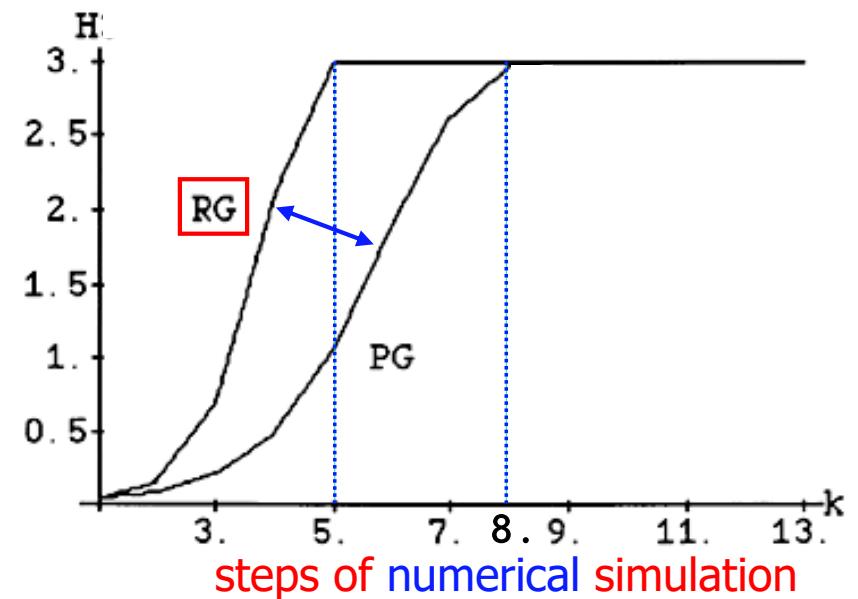
# Numerical results

## self-motion for escaping singularities



$$\max H(q) = \sum_{i=1}^3 \sin^2(q_{i+1} - q_i)$$

this function is **NOT**  
the manipulability index,  
but has the same minima ( $= 0$ )



**RG is faster than PG**  
(keeping the same accuracy on  $r$ )



## 3

## Task augmentation methods

- an **auxiliary task** is added (task augmentation)

$$S \updownarrow f_y(q) = y \quad S \leq N - M$$

corresponding to some desirable feature for the solution

$$r_A = \begin{pmatrix} r \\ y \end{pmatrix} = \begin{pmatrix} f(q) \\ f_y(q) \end{pmatrix} \rightarrow \dot{r}_A = \begin{pmatrix} J(q) \\ J_y(q) \end{pmatrix} \dot{q} = J_A(q) \dot{q}$$

$\boxed{J_A}$  }  $M + S$   
 $\underbrace{\phantom{J_A}}$   $N$

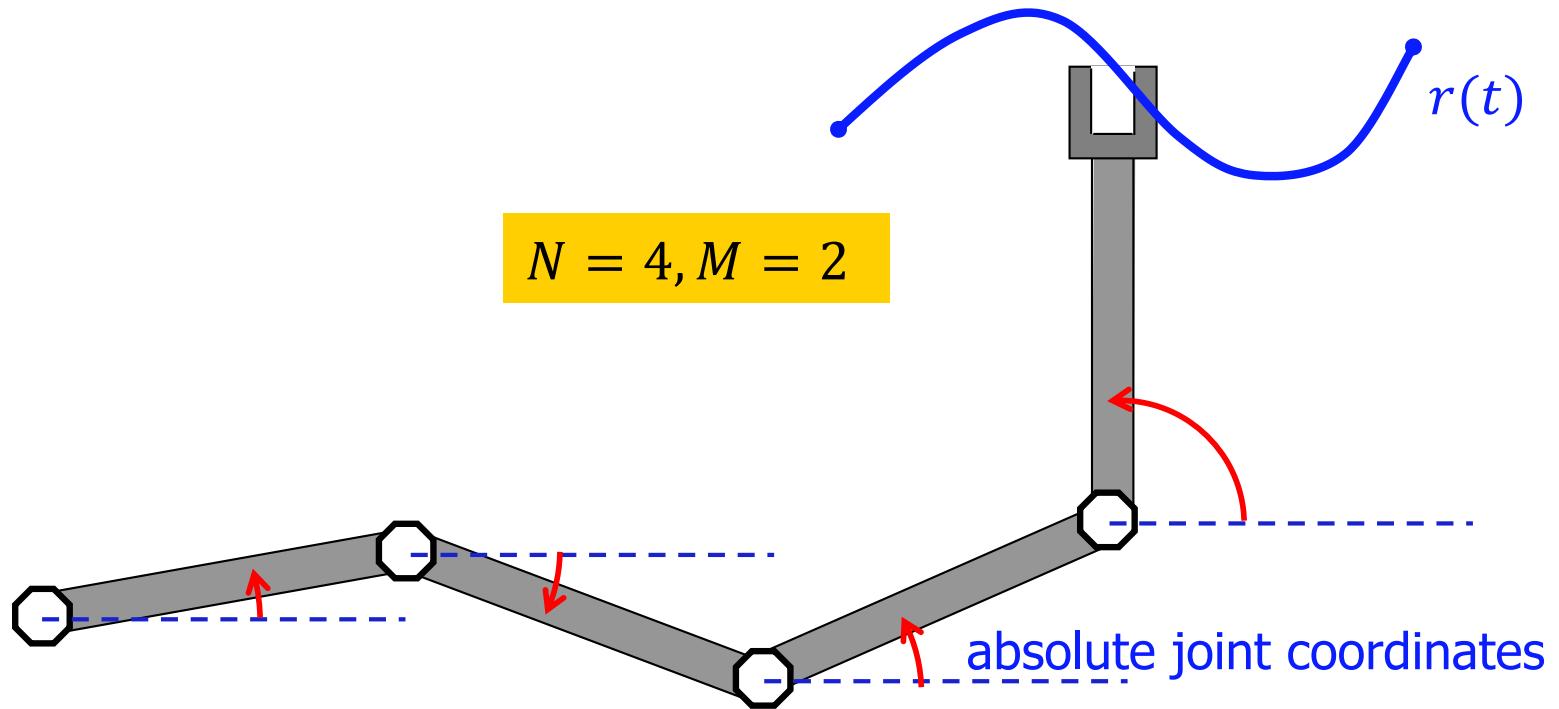
- a **solution** is chosen still in the form of a generalized inverse

$$\dot{q} = K_A(q) \dot{r}_A$$

or by adding a term in the null space of the **augmented Jacobian** matrix  $J_A$



# Augmented task example



$$f_y(q) = q_4 = \pi/2 \quad (S = 1)$$

last link is to be held vertical...



# Augmenting the task ...

- **advantage:** better shaping of the inverse kinematic solution
- **disadvantage:** algorithmic singularities are introduced when

$$\rho(J) = M \quad \rho(J_y) = S \quad \text{but} \quad \rho(J_A) < M + S$$

to avoid this, it should be always

$$\mathcal{R}(J^T) \cap \mathcal{R}(J_y^T) = \emptyset$$

difficult to be obtained globally!

rows of  $J$  AND rows of  $J_y$   
are all together linearly independent





# Extended Jacobian ( $S = N - M$ )

---

- square  $J_A$ : in the absence of **algorithmic** singularities, we can choose

$$\dot{q} = J_A^{-1}(q)\dot{r}_A$$

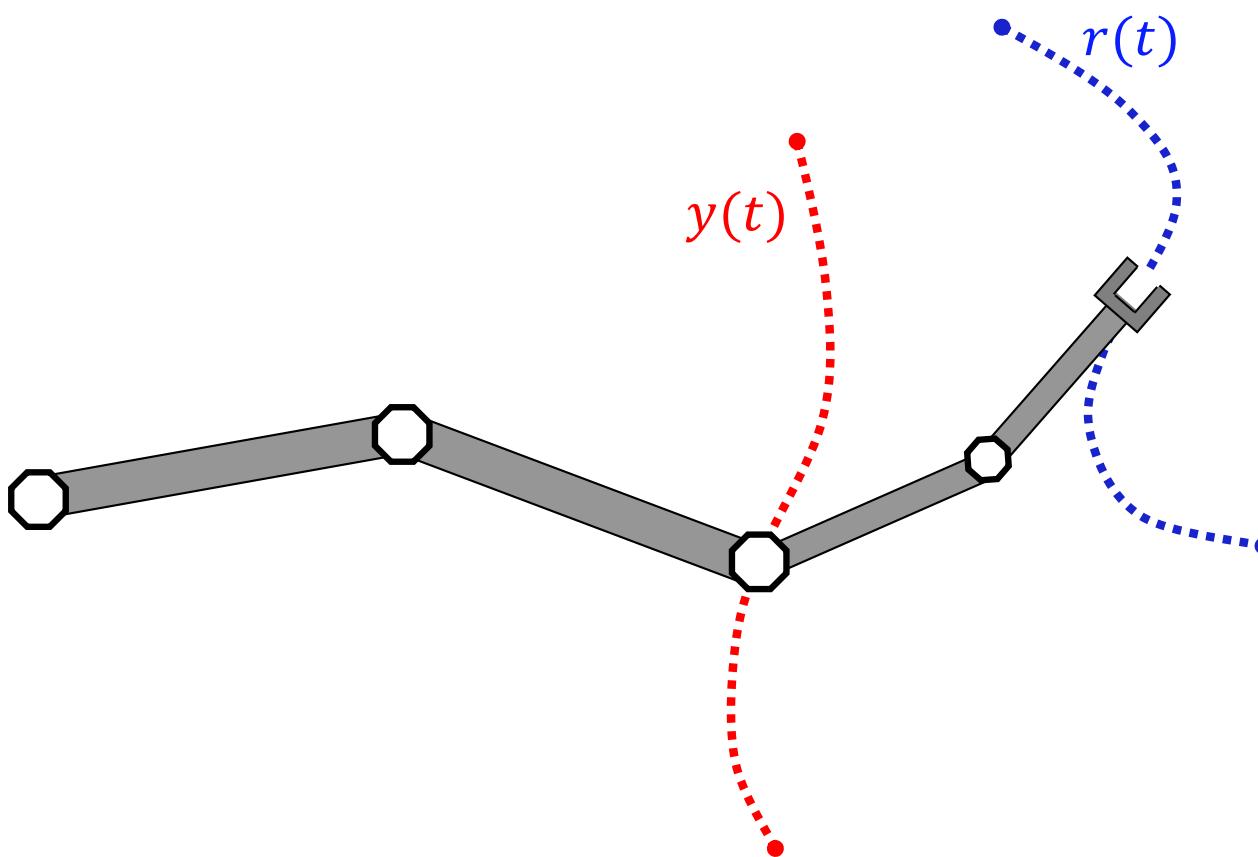
- the scheme is then **repeatable**
  - provided no singularities are encountered during a complete task cycle\*
- when the  $N - M$  conditions  $f_y(q) = 0$  correspond to necessary (and sufficient) conditions for constrained optimality of a given objective function  $H(q)$  (see RG method, slide #36), this scheme guarantees that constrained **optimality** is locally **preserved** during task execution
- in the vicinity of algorithmic singularities, for the simultaneous execution of the **original task** and the **auxiliary task(s)**, one can use the DLS method; however, **both** tasks will be affected by **errors**

\* there exists an unexpected phenomenon in some 3R manipulators having “generic” kinematics: the robot may sometimes perform a pose change after a full cycle, even if no singularity has been encountered during motion (see J. Burdick, *Mech. Mach. Theory*, 30(1), 1995)



# Extended Jacobian example

MACRO-MICRO manipulator



$$N = 4, M = 2$$

$$\dot{r} = J(q_1, \dots, q_4)\dot{q}$$

$$\dot{y} = J_y(q_1, q_2)\dot{q}$$



$$J_A = \left( \begin{array}{c|c} * & * \\ * & 0 \end{array} \right) \quad 4 \times 4$$



# Task Priority

if the original (primary) task  $\dot{r}_1 = J_1(q)\dot{q}$  has **higher priority** than the auxiliary (secondary) task  $\dot{r}_2 = J_2(q)\dot{q}$

- we **first** address the task with highest priority

$$\dot{q} = J_1^\# \dot{r}_1 + (I - J_1^\# J_1) v_1$$

- and **then** choose  $v_1$  so as to satisfy, if possible, also the secondary (lower priority) task

$$\dot{r}_2 = J_2 \dot{q} = J_2 J_1^\# \dot{r}_1 + J_2 (I - J_1^\# J_1) v_1 = J_2 J_1^\# \dot{r}_1 + J_2 P_1 v_1$$

the general solution for  $v_1$  takes the usual form

$$v_1 = (J_2 P_1)^\# (\dot{r}_2 - J_2 J_1^\# \dot{r}_1) + (I - (J_2 P_1)^\# (J_2 P_1)) v_2$$

$v_2$  is available for the execution of further tasks of lower (ordered) priorities



# Task Priority (continue)

- substituting the expression of  $v_1$  in  $\dot{q}$

$$\dot{q} = J_1^\# \dot{r}_1 + P_1 (J_2 P_1)^\# (\dot{r}_2 - J_2 J_1^\# \dot{r}_1) + P_1 (I - (J_2 P_1)^\# (J_2 P_1)) v_2$$

$P(BP)^\# = (BP)^\#$   
when matrix  $P$  is  
idempotent and symmetric

$$= (J_2 P_1)^\#$$

possibly = 0

- main advantage: **highest priority task** is ideally **no longer affected** by **algorithmic singularities** (error is restricted only to secondary task)

