Robotics 1

Statics and force transformations

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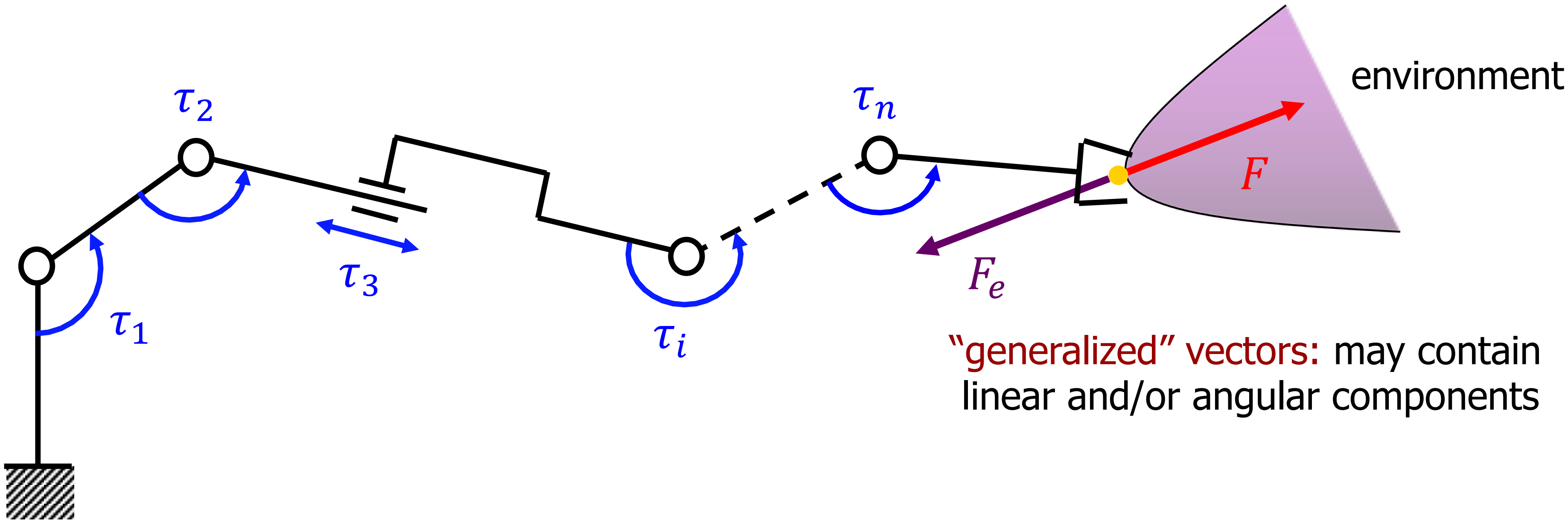
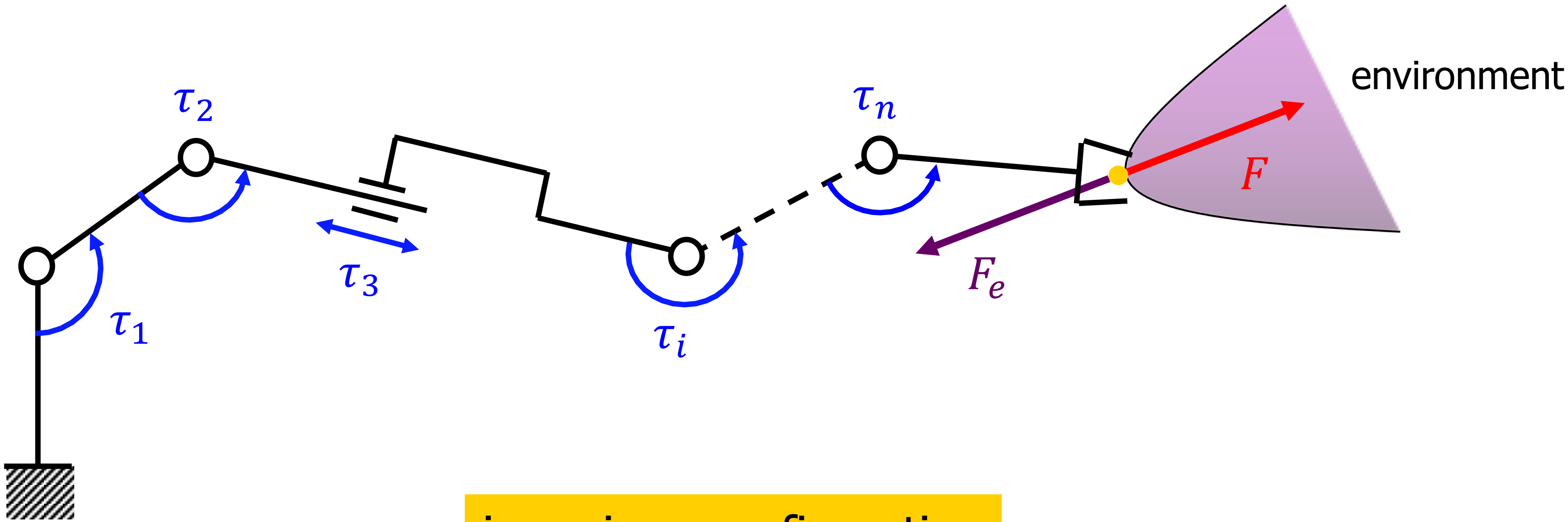
Generalized forces and torques



- τ = forces/torques exerted by the motors at the robot joints
- F = equivalent forces/torques exerted by the robot end-effector
- F_e = forces/torques exerted by the environment at the end-effector
- principle of action and reaction: $\boxed{F_e = -F}$

reaction from environment is equal and opposite to the robot action on it

Transformation of forces – Statics



in a given configuration

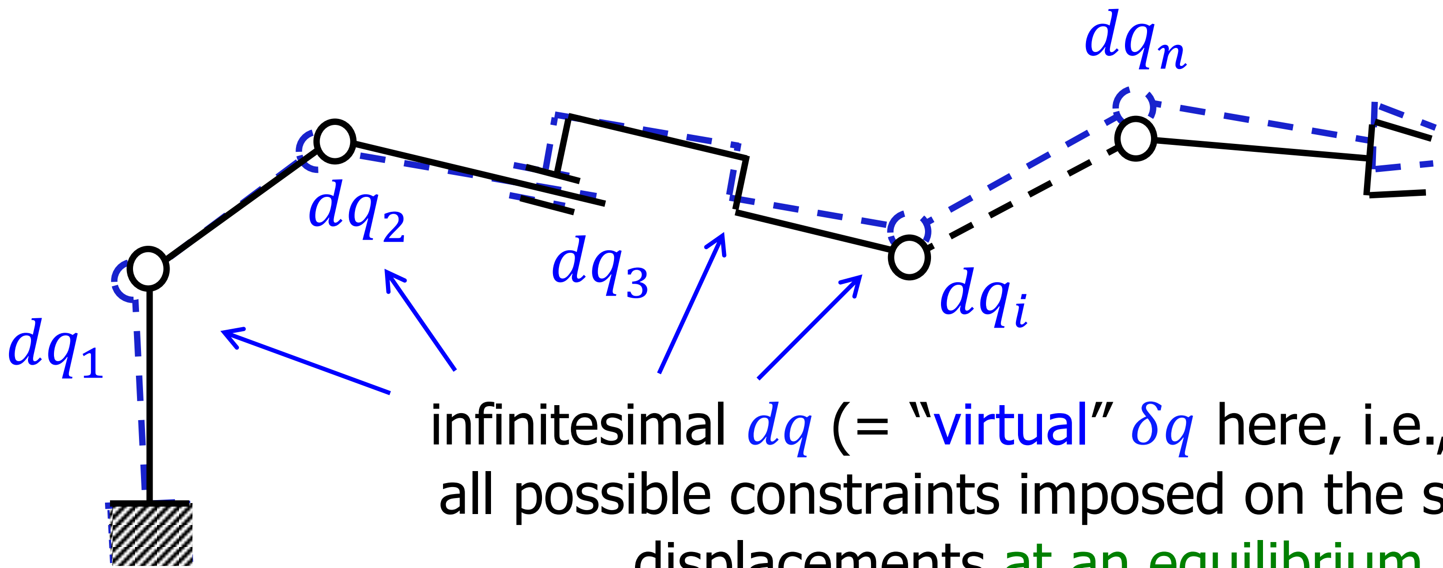
- what is the transformation between F at robot end-effector and τ at joints?

in **static equilibrium** conditions (i.e., **no motion**):

- what F will be exerted on environment by a τ applied at the robot joints?
- what τ at the joints will balance a F_e ($= -F$) exerted by the environment?

all equivalent formulations

Virtual displacements and works



$$\begin{pmatrix} dp \\ \omega\, dt \end{pmatrix} = J\, dq$$

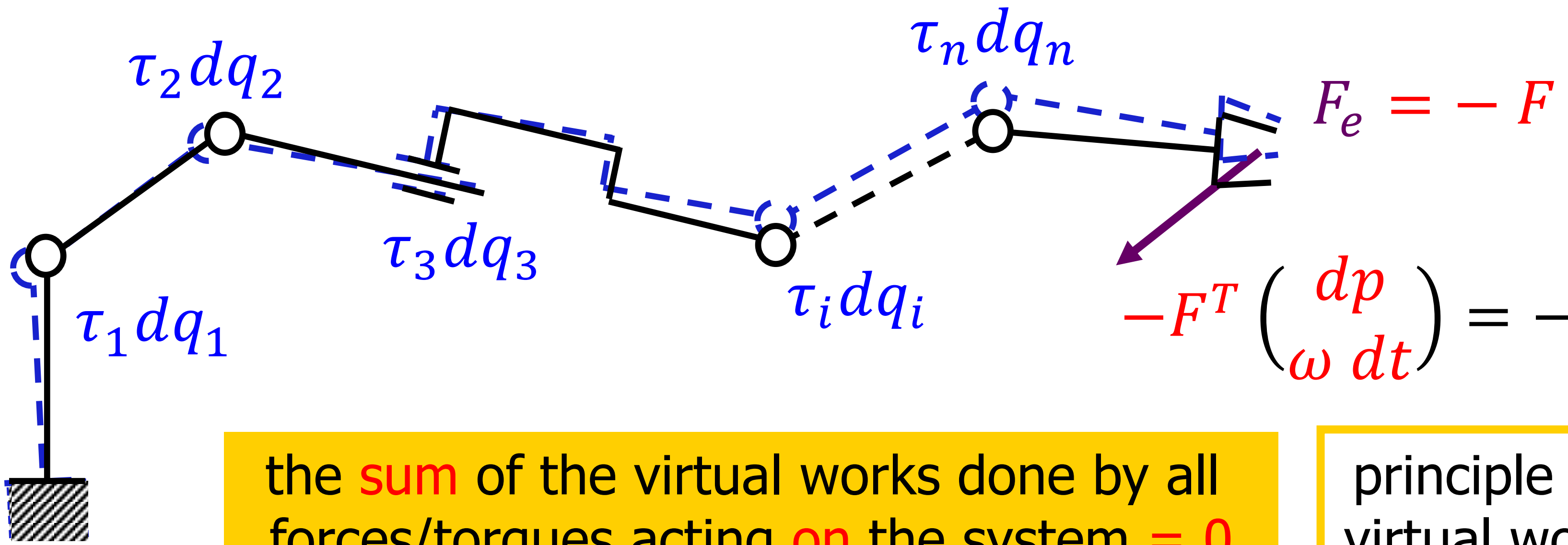
infinitesimal dq (= "virtual" δq here, i.e., satisfy all possible constraints imposed on the system) displacements at an equilibrium

- without kinetic energy variation (zero acceleration)
- without dissipative effects (zero velocity)

the virtual work is the work done by all forces/torques acting on the system for a given virtual displacement

Hint: one of the advantages of working with (D-H) joint variables is that they are already free of equality constraints (= generalized coordinates)

Principle of virtual work



$$-F^T \begin{pmatrix} dp \\ \omega\, dt \end{pmatrix} = -F^T J\, dq$$

the sum of the virtual works done by all forces/torques acting on the system = 0

principle of virtual work

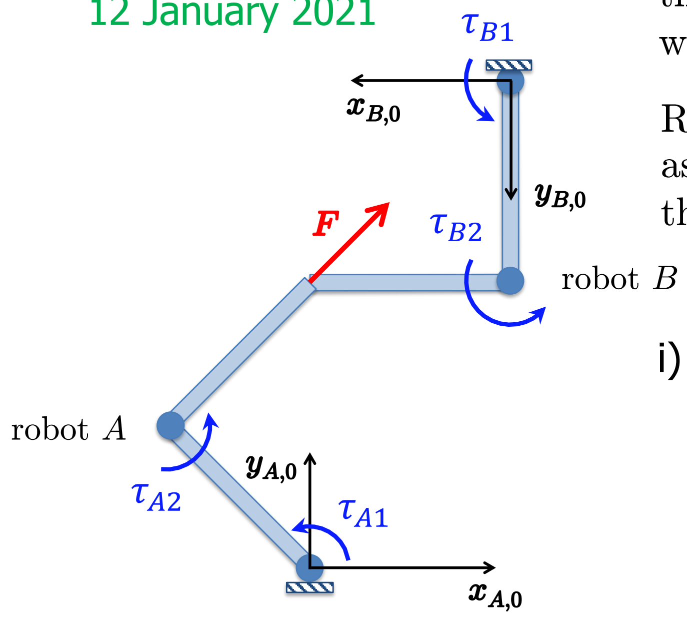
$$\tau^T dq - F^T \begin{pmatrix} dp \\ \omega dt \end{pmatrix} = \tau^T dq - F^T J dq = 0 \qquad \boxed{\forall dq}$$

$$\Rightarrow \quad \boxed{\tau = J^T(q) F}$$

Exercise on static balance

whiteboard …

Q#7 in Robotics 1 exam, 12 January 2021



Two 2R planar robots, A and B, having unitary link lengths are in their D-H configurations $\boldsymbol{q}_A = (3\pi/4, -\pi/2)$, $\boldsymbol{q}_B = (\pi/2, -\pi/2)$ [rad] w.r.t. their base frames, as in figure (no gravity!).

Robot A pushes against robot B with a force $\boldsymbol{F} \in \mathbb{R}^2$ of norm $\|\boldsymbol{F}\| = 10$ [N], as in figure. Compute the joint torques $\boldsymbol{\tau}_A \in \mathbb{R}^2$ and $\boldsymbol{\tau}_B \in \mathbb{R}^2$ (both in [Nm]) that keep the two robots in equilibrium.

solution

i) evaluate the task Jacobians of the two robots ($\dot{q}_A \to v_A$ and $\dot{q}_B \to v_B$)

$$\boldsymbol{J}_A(\boldsymbol{q}_A) = \begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) & -\sin(q_1+q_2) \\ \cos q_1 + \cos(q_1+q_2) & \cos(q_1+q_2) \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}_A} = \begin{pmatrix} -\sqrt{2} & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$\boldsymbol{J}_B(\boldsymbol{q}_B) = \begin{pmatrix} -\sin q_1 - \sin(q_1+q_2) & -\sin(q_1+q_2) \\ \cos q_1 + \cos(q_1+q_2) & \cos(q_1+q_2) \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}_B} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$$

ii) express the exchanged force in the proper frame(s) …

$${}^A\boldsymbol{F}_A = \|\boldsymbol{F}\| \cdot \begin{pmatrix} \cos(q_1+q_2) \\ \sin(q_1+q_2) \end{pmatrix}\Bigg|_{\boldsymbol{q}=\boldsymbol{q}_A} = 10 \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \text{ [N]}$$

$${}^B\boldsymbol{F}_B = {}^B\boldsymbol{R}_A \, {}^A\boldsymbol{F}_B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \left(-\,{}^A\boldsymbol{F}_A\right) = {}^A\boldsymbol{F}_A = 10 \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} \text{ [N]}$$

planar rotation matrix $\in SO(2)$

iii) … and compute the torque for each robot by the virtual work principle

$$\boldsymbol{\tau}_A = \boldsymbol{J}_A^T(\boldsymbol{q}_A)\, {}^A\boldsymbol{F}_A = \begin{pmatrix} -10 \\ 0 \end{pmatrix} \text{ [Nm]}$$

$$\boldsymbol{\tau}_B = \boldsymbol{J}_B^T(\boldsymbol{q}_B)\, {}^B\boldsymbol{F}_B = \begin{pmatrix} 0 \\ 5\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 7.0711 \end{pmatrix} \text{ [Nm].}$$

Duality between velocity and force

$$\dot{q} \xrightarrow{J(q)} v$$

velocity $\dot{q}$
(or displacement dq)
in the joint space

generalized velocity v
(or e-e displacement $\begin{pmatrix} dp \\ \omega\, dt \end{pmatrix}$)
in the Cartesian space

forces/torques τ
at the joints

generalized forces F
at the Cartesian e-e

$$\tau \xleftarrow{J^T(q)} F$$

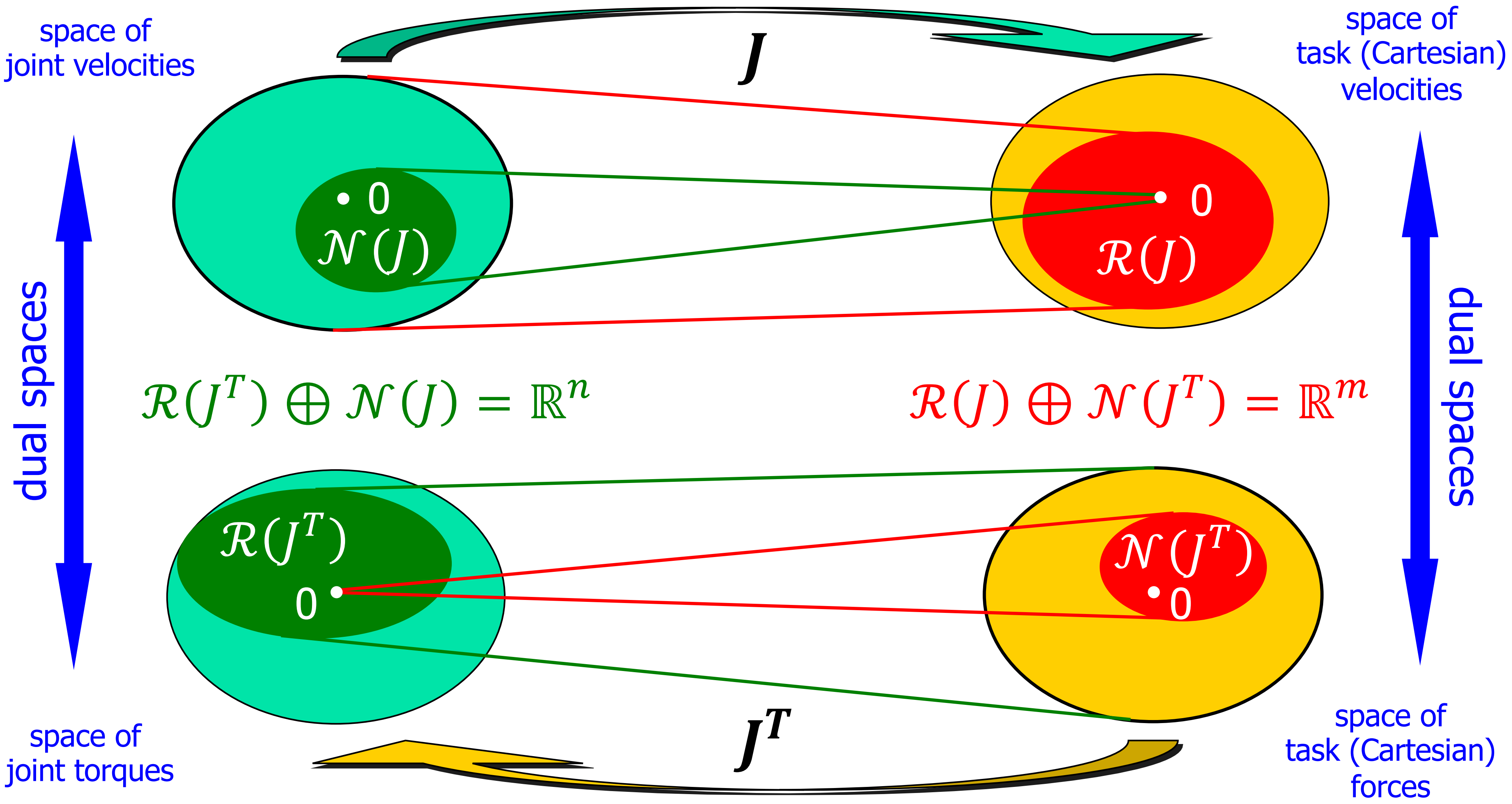
the singular configurations
for the velocity map are the same
as those for the force map

$$\boxed{\rho(J) = \rho(J^T)}$$

duplicate slide from block 11

Robot Jacobian

decomposition in linear subspaces and duality



(at a given configuration q)

Dual subspaces of velocity and force

summary of definitions

$$\mathcal{R}(J) = \{v \in I\!R^m : \exists \dot{q} \in I\!R^n, J\dot{q} = v\}$$

$$\mathcal{N}(J^T) = \{F \in I\!R^m : J^T F = 0\}$$

$$\mathcal{R}(J) \bigoplus \mathcal{N}(J^T) = I\!R^m$$

$$\mathcal{R}(J^T) = \{\tau \in I\!R^n : \exists F \in I\!R^m, J^T F = \tau\}$$

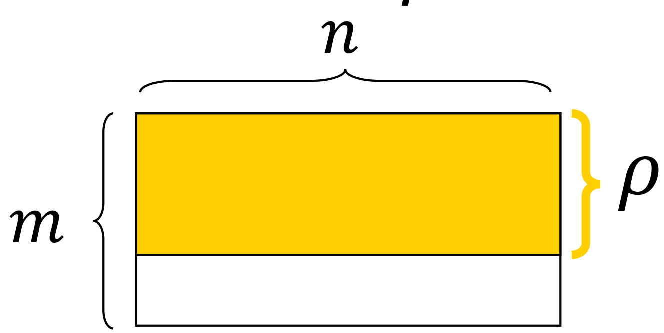
$$\mathcal{N}(J) = \{\dot{q} \in I\!R^n : J\dot{q} = 0\}$$

$$\mathcal{R}(J^T) \bigoplus \mathcal{N}(J) = I\!R^n$$

Velocity and force singularities

list of possible cases

$$\rho = \mathrm{rank}(J) = \mathrm{rank}(J^T) \leq \min(m, n)$$



1. $\rho = m$

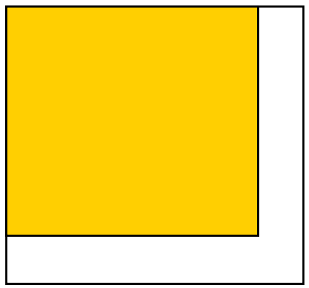
$$\exists \dot{q} \neq 0 : \; J\dot{q} = 0$$

$$\mathcal{N}(J^T) = \{0\}$$

2. $\rho < m$

$$\exists \dot{q} \neq 0 : \; J\dot{q} = 0$$

$$\exists F \neq 0 : \; J^T F = 0$$



1. $\det J \neq 0$

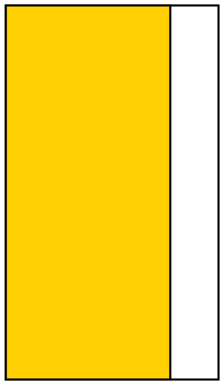
$$\mathcal{N}(J) = \{0\}$$

$$\mathcal{N}(J^T) = \{0\}$$

2. $\det J = 0$

$$\exists \dot{q} \neq 0 : \; J\dot{q} = 0$$

$$\exists F \neq 0 : \; J^T F = 0$$



1. $\rho = n$

$$\mathcal{N}(J) = \{0\}$$

$$\exists F \neq 0 : \; J^T F = 0$$

2. $\rho < n$

$$\exists \dot{q} \neq 0 : \; J\dot{q} = 0$$

$$\exists F \neq 0 : \; J^T F = 0$$

Singularity analysis

planar 2R arm with link lengths l_1 and l_2

$$J(q) = \begin{pmatrix} -(l_1 s_1 + l_2 s_{12}) & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \qquad \det J(q) = l_1 l_2 s_2$$

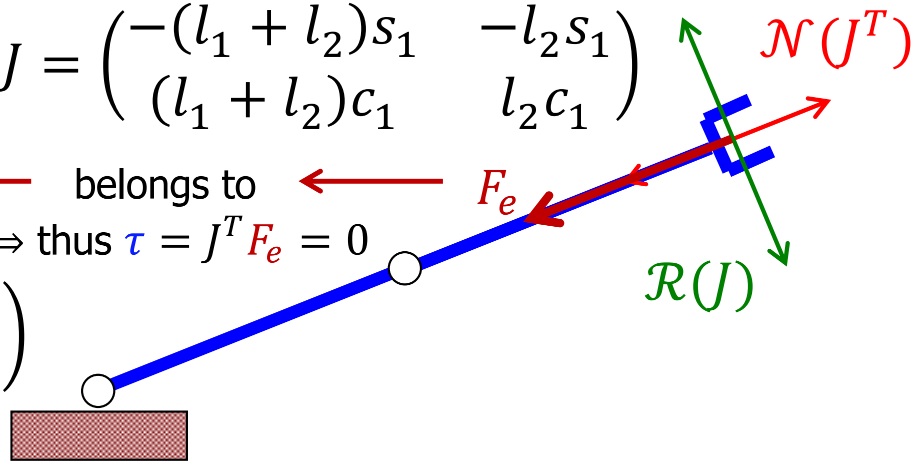
singularity at $q_2 = 0$ (arm stretched) ➡ $J = \begin{pmatrix} -(l_1 + l_2) s_1 & -l_2 s_1 \\ (l_1 + l_2) c_1 & l_2 c_1 \end{pmatrix}$

$$\mathcal{R}(J) = \alpha \begin{pmatrix} -s_1 \\ c_1 \end{pmatrix} \qquad \mathcal{N}(J^T) = \beta \begin{pmatrix} c_1 \\ s_1 \end{pmatrix}$$

$\xleftarrow{\text{belongs to}}$ F_e

$\Rightarrow$ thus $\tau = J^T F_e = 0$

$$\mathcal{R}(J^T) = \gamma \begin{pmatrix} l_1 + l_2 \\ l_2 \end{pmatrix} \qquad \mathcal{N}(J) = \delta \begin{pmatrix} l_2 \\ -(l_1 + l_2) \end{pmatrix}$$



singularity at $q_2 = \pi$ (arm folded) ➡ $J = \begin{pmatrix} (l_2 - l_1) s_1 & l_2 s_1 \\ -(l_2 - l_1) c_1 & -l_2 c_1 \end{pmatrix}$

$\mathcal{R}(J)$ and $\mathcal{N}(J^T)$ as above

$$\mathcal{R}(J^T) = \gamma \begin{pmatrix} l_2 - l_1 \\ l_2 \end{pmatrix} \text{ (for } l_1 = l_2\text{: } \gamma \begin{pmatrix} 0 \\ 1 \end{pmatrix}\text{)} \qquad \mathcal{N}(J) = \delta \begin{pmatrix} l_2 \\ -(l_2 - l_1) \end{pmatrix} \text{ (for } l_1 = l_2\text{: } \delta \begin{pmatrix} 1 \\ 0 \end{pmatrix}\text{)}$$

Cartesian forces F_e applied at the end-effector of the 2R arm along the stretched or folded direction need NO joint torques τ for balance

Force manipulability

- in a given configuration, evaluate how effective is the transformation between joint torques and end-effector forces
  - "how easily" can the end-effector apply generalized forces (or balance applied ones) in the various directions of the task space
  - in singular configurations, there are directions in the task space where external forces are balanced without the need of any joint torque
- we consider all end-effector forces that can be applied (or balanced) by choosing joint torque vectors of unit norm

$$\tau^T \tau = 1 \quad \Rightarrow \quad \boxed{F^T J J^T F = 1}$$

same directions of the principal axes of the velocity ellipsoid, but with semi-axes of inverse lengths

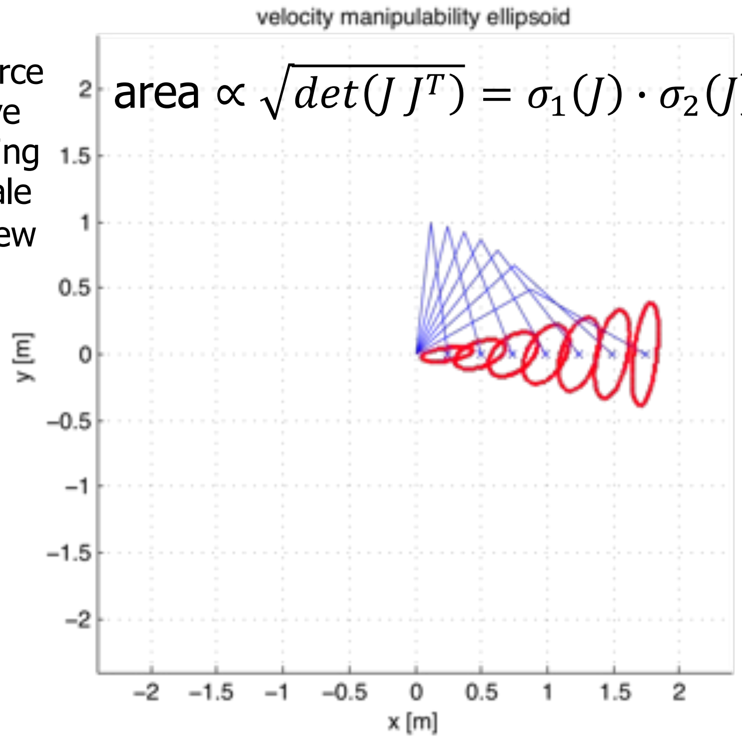
task force manipulability ellipsoid

Velocity and force manipulability

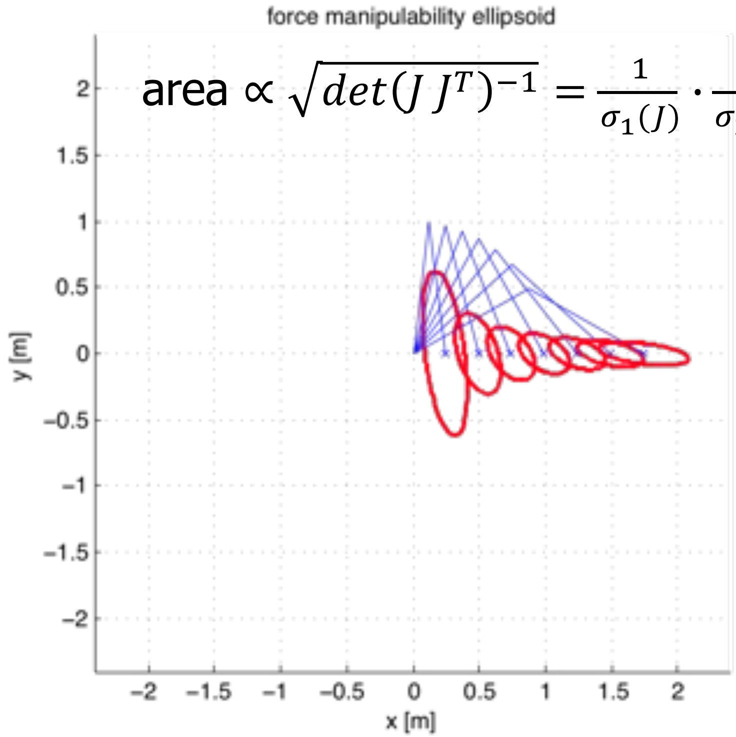
dual comparison of actuation vs. control

planar 2R arm with unitary links

note:
velocity and force ellipsoids have been drawn using a different scale for a better view



$$\text{area} \propto \sqrt{det(J\,J^T)} = \sigma_1(J) \cdot \sigma_2(J)$$



$$\text{area} \propto \sqrt{det(J\,J^T)^{-1}} = \frac{1}{\sigma_1(J)} \cdot \frac{1}{\sigma_2(J)}$$

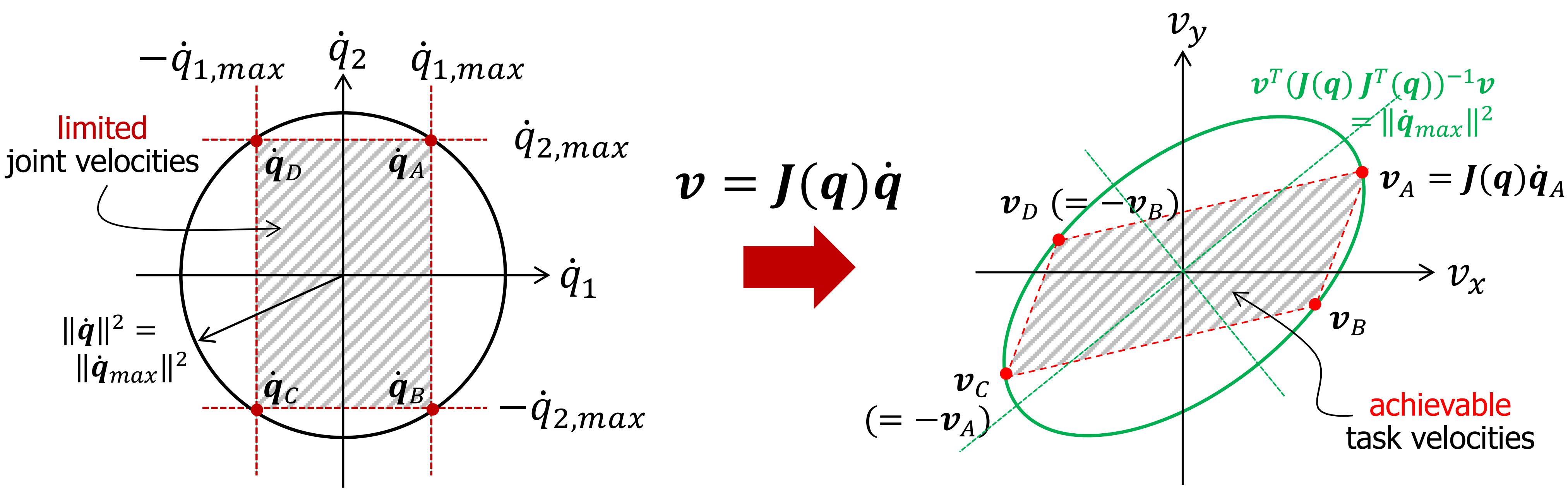
Cartesian **actuation** task (joint-to-task high transformation ratio):
preferred velocity (or force) directions are those where the ellipsoid stretches

⬍

Cartesian **control** task (low transformation ratio = high resolution):
preferred velocity (or force) directions are those where the ellipsoid shrinks

Ellipsoids and polytopes
manipulability versus task limits due to bounds

- manipulability: instantaneous capability of moving the end-effector (or of resisting to task forces) in different directions
- task limits: maximum velocity (or static balanced force) achievable in different task directions in the presence of joint velocity bounds



$$-\dot{q}_{i,max} \le \dot{q}_i \le \dot{q}_{i,max}$$

velocity ellipsoid and polytope at $\boldsymbol{q}$
for a 2R robot with joint velocity bounds

- a polytope is the convex hull of a set of p points in an Euclidean space
- linear maps transform polytopes into polytopes

Velocity and force transformations

- same reasoning made for relating end-effector to joint forces/torques (virtual work principle + static equilibrium) used also for transforming forces and torques applied at different places of a rigid body and/or expressed in different reference frames

transformation among generalized velocities

$$\begin{bmatrix} {}^A v_A \\ {}^A \omega \end{bmatrix} = \begin{bmatrix} {}^A R_B & -{}^A R_B S({}^B r_{BA}) \\ 0 & {}^A R_B \end{bmatrix} \begin{bmatrix} {}^B v_B \\ {}^B \omega \end{bmatrix} = J_{BA} \begin{bmatrix} {}^B v_B \\ {}^B \omega \end{bmatrix}$$

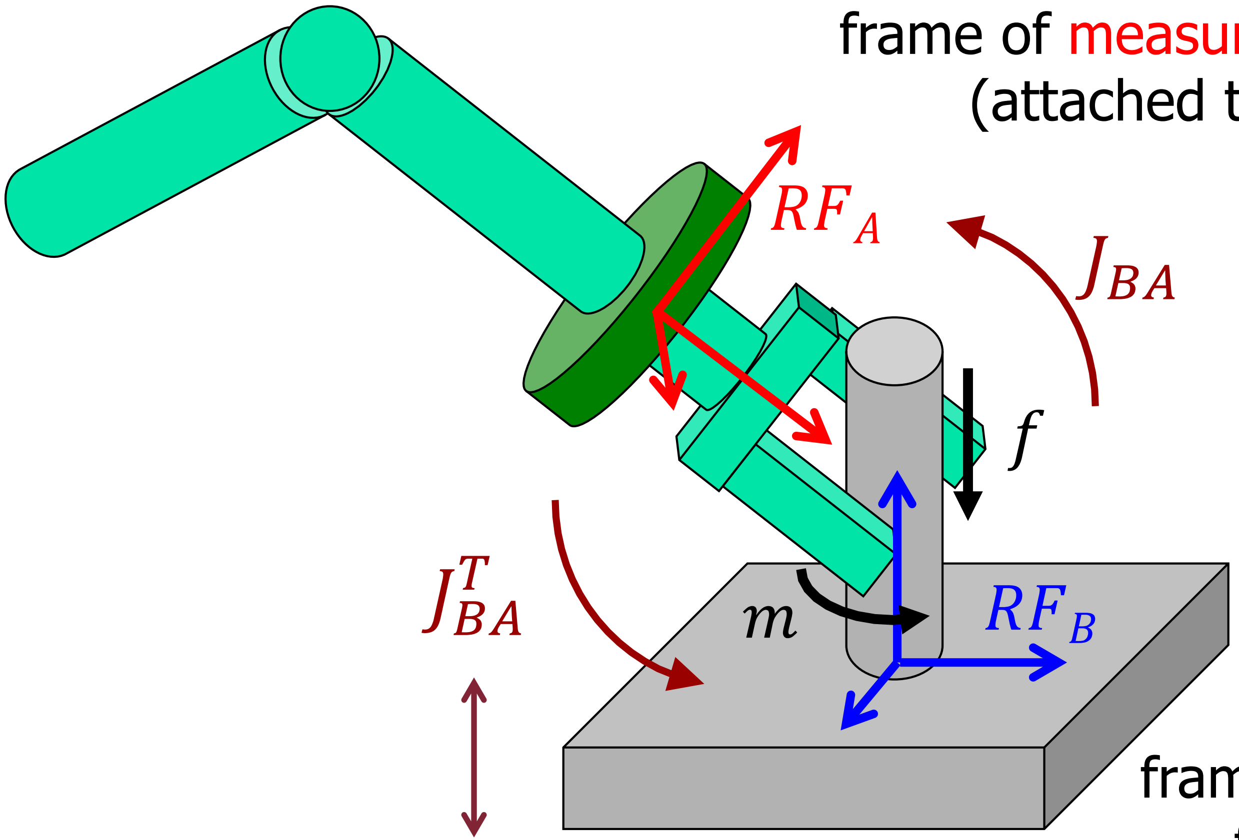
$\Updownarrow$

$$\begin{bmatrix} {}^B f_B \\ {}^B m \end{bmatrix} = J_{BA}^T \begin{bmatrix} {}^A f_A \\ {}^A m \end{bmatrix} = \begin{bmatrix} {}^B R_A & 0 \\ S({}^B r_{BA})\,{}^B R_A & {}^B R_A \end{bmatrix} \begin{bmatrix} {}^A f_A \\ {}^A m \end{bmatrix}$$

transformation among generalized forces

for skew-symmetric matrices, it is: $-S^T(r) = S(r)$

Example: 6D force/torque sensor

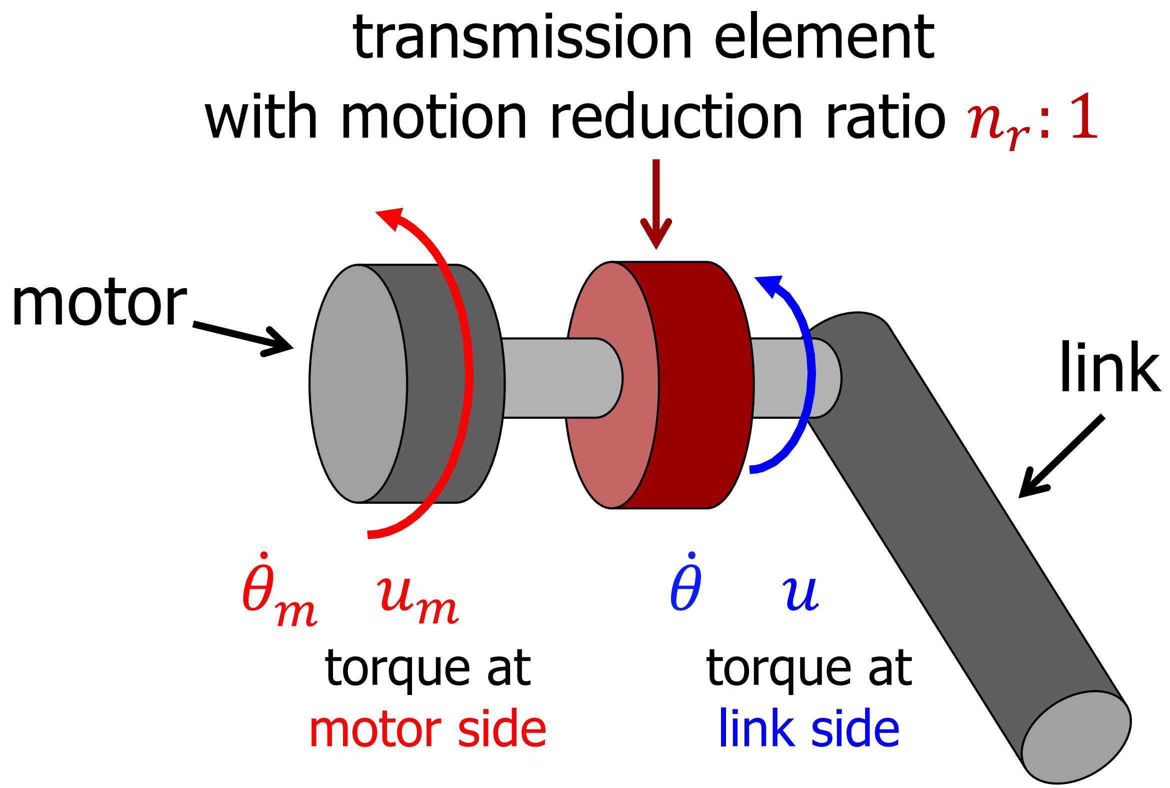


frame of measure for the forces/torques (attached to the wrist sensor)

frame of interest for evaluating forces/torques in a task with environment contact

$$\begin{bmatrix} {}^B f_B \\ {}^B m \end{bmatrix} = \begin{bmatrix} {}^B R_A & 0 \\ S({}^B r_{BA})\,{}^B R_A & {}^B R_A \end{bmatrix} \begin{bmatrix} {}^A f_A \\ {}^A m \end{bmatrix}$$

Example: Gear reduction at joints



one of the simplest applications of the principle of virtual work:

$$P_m = u_m \dot{\theta}_m = u\,\dot{\theta} = P$$

$$\boxed{\begin{aligned} \dot{\theta}_m &= n_r \dot{\theta} \\ u &= n_r u_m \end{aligned}}$$

here, $J = J^T = n_r$ (a scalar!)