Robotics 1

Trajectory planning

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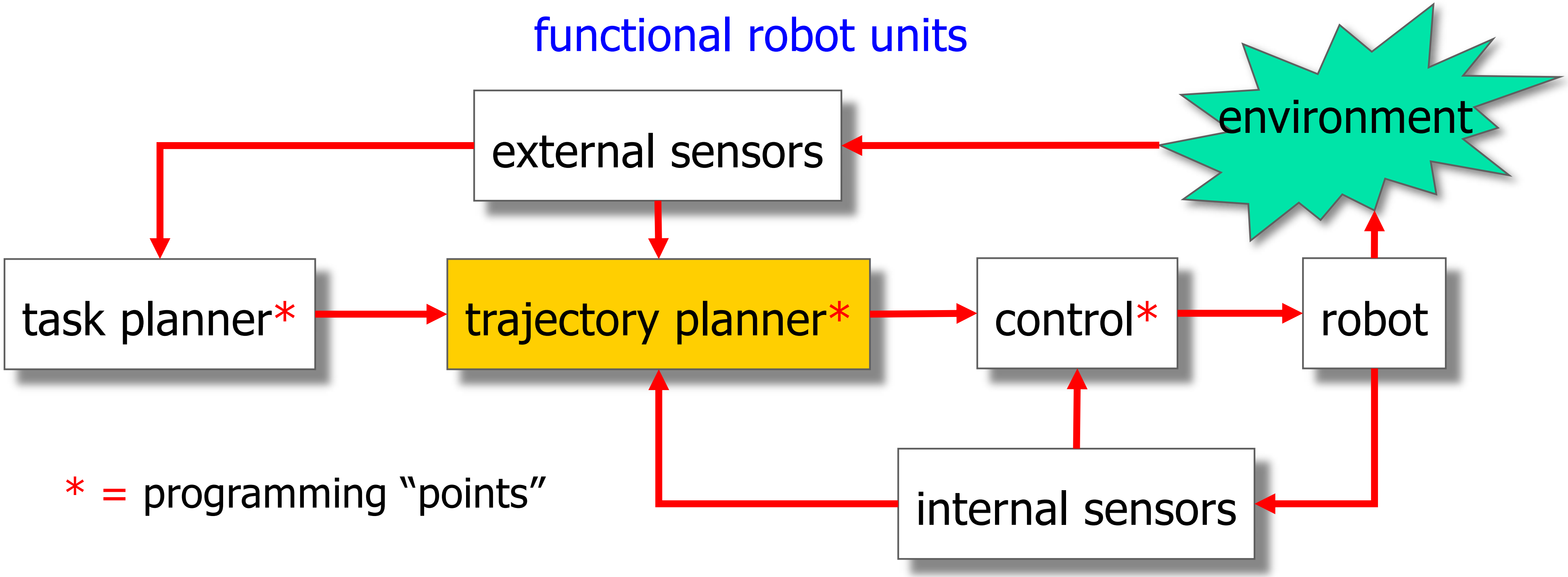
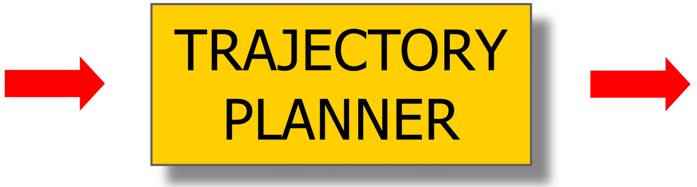
Trajectory planner interfaces

functional robot units



* = programming "points"



robot action described as a sequence of poses or configurations (with possible exchange of contact forces)

reference profile/values (continuous or discrete) for the robot controller

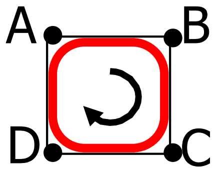
Trajectory definition

a standard procedure for industrial robots

1. define Cartesian pose points (position+orientation) using the teach-box
2. program an (average) velocity between these points, as a 0-100% of a maximum system value (different for Cartesian- and joint-space motion)
3. linear interpolation in the joint space between points sampled from the built trajectory

examples of additional features

a) over-fly



b) sensor-driven STOP

c) circular path through 3 points

main drawbacks

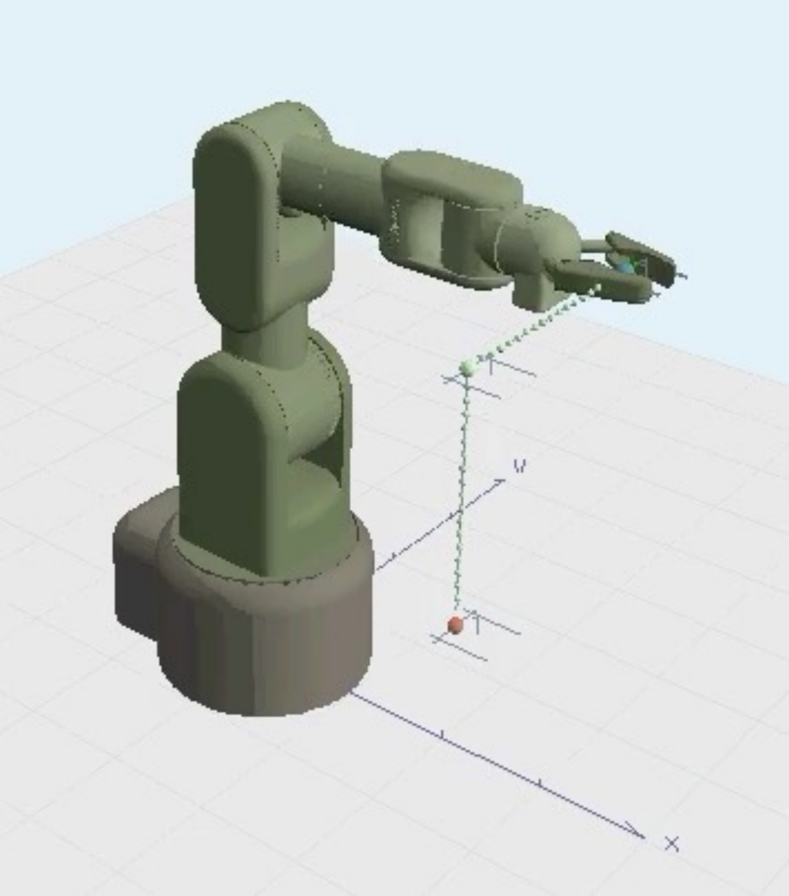
- semi-manual programming (as in "first generation" robot languages)
- limited visualization of motion

⇒ a mathematical formalization of trajectories is useful/needed

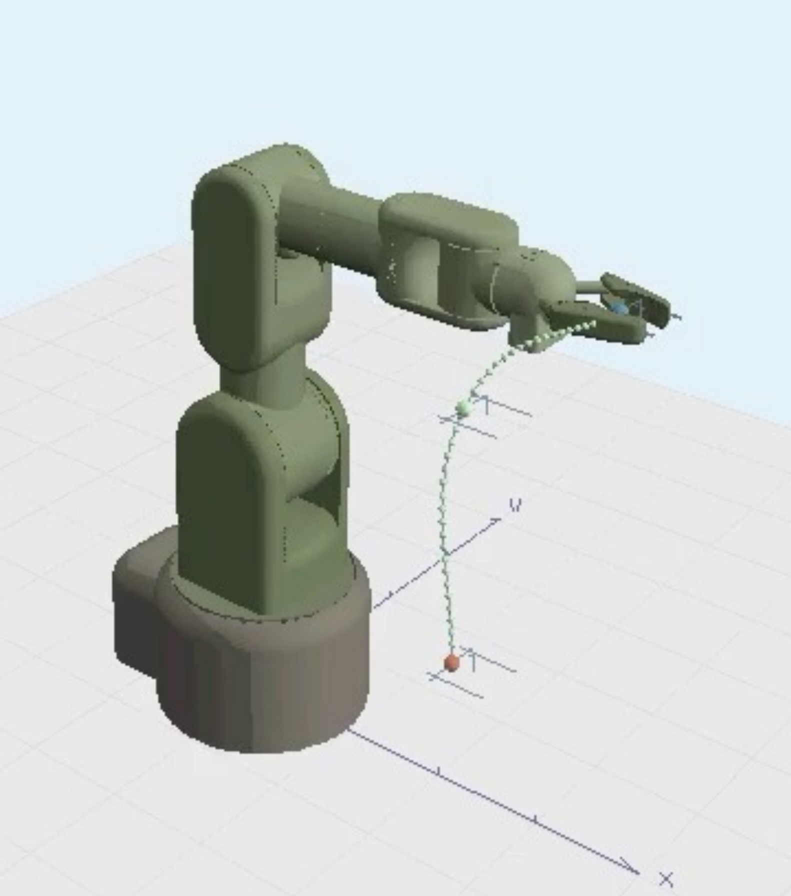
Some typical trajectories

- Point-to-point Cartesian motion with an intermediate point

video



Straight lines as Cartesian path



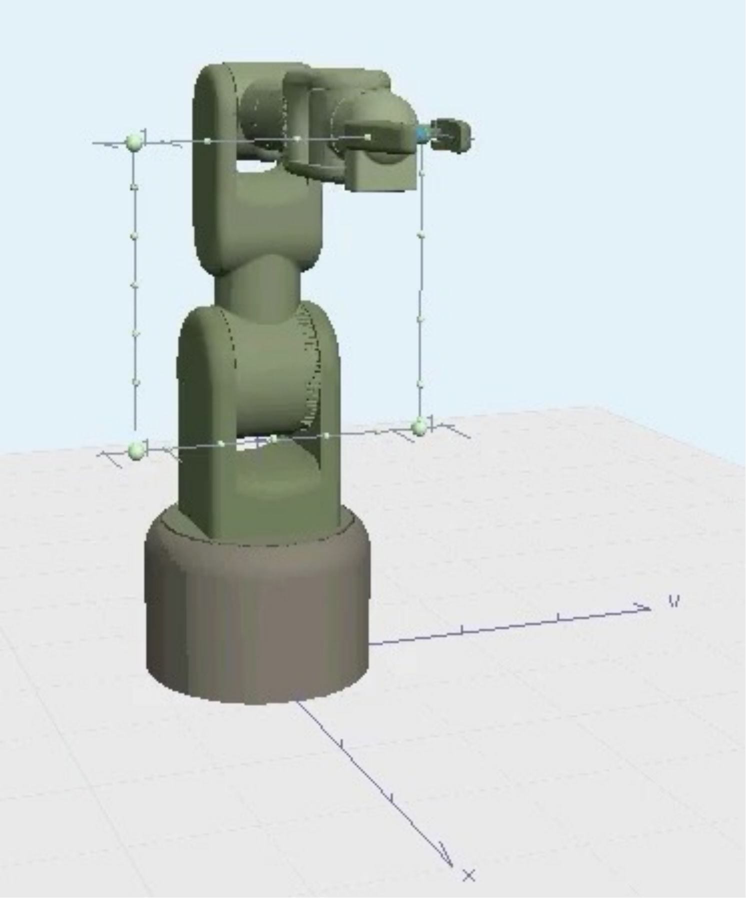
video

Interpolation with Bezier curves

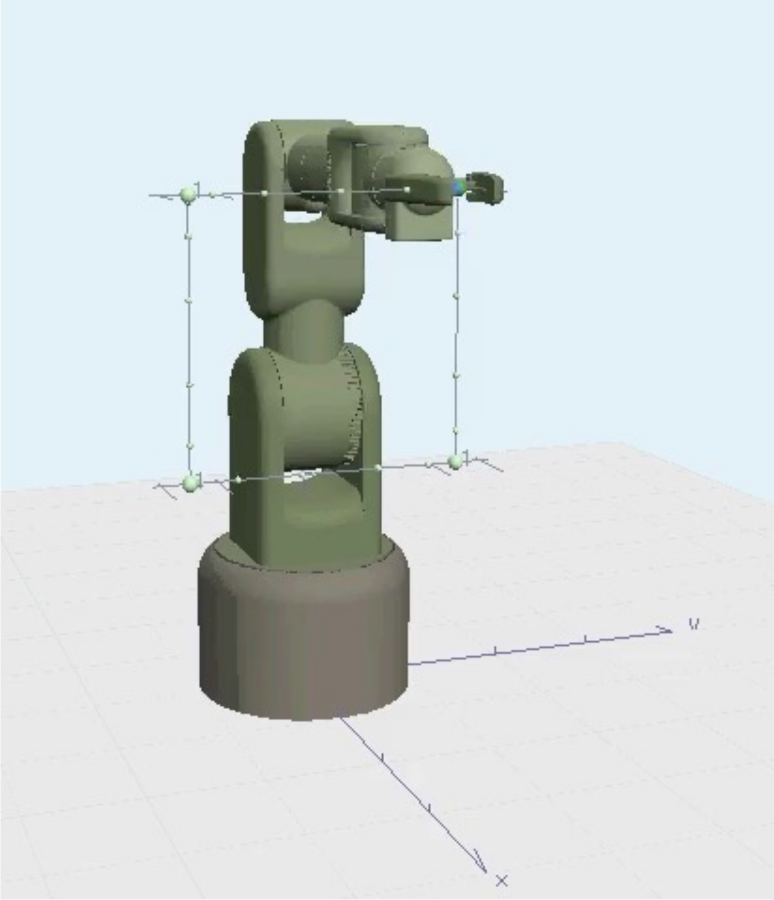
Some typical trajectories

- Timing laws: Cartesian path with (dis-)continuous tangent

video



Square path at constant speed

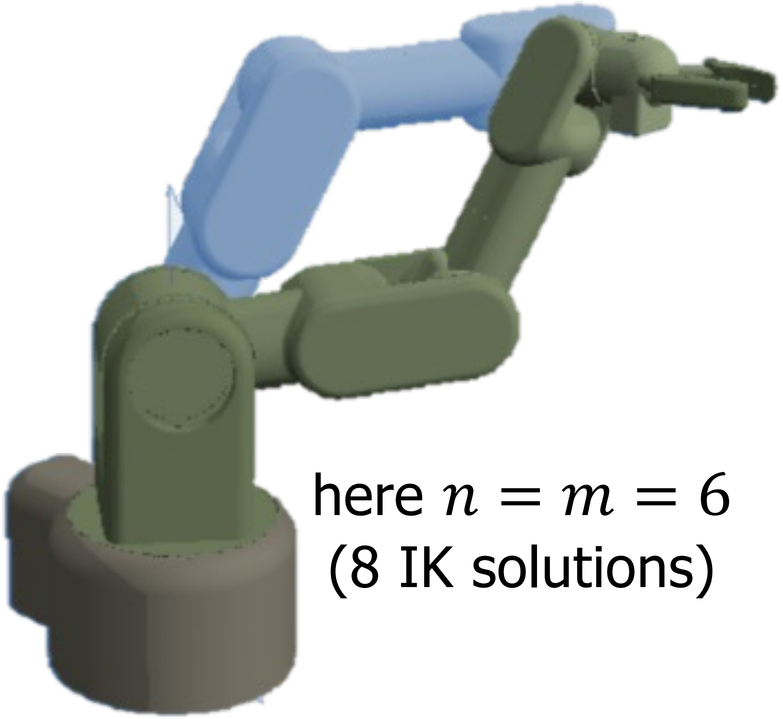


video

Square path with trapezoidal speed profile

Joint and Cartesian trajectories

- assigned task: arm reconfiguration between two inverse kinematic solutions associated to a given end-effector pose



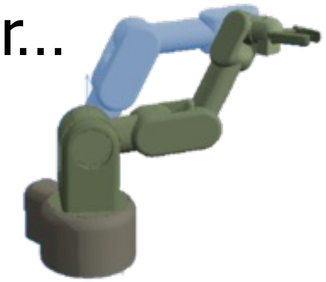
here $n = m = 6$
(8 IK solutions)

- **initial** and **final** configuration
- same Cartesian pose (no change!): the motion cannot be fully specified in the Cartesian space
- to perform this task, the robot should leave the given end-effector pose and then return to it
- a self-motion could be sufficient
  - if there is (task) redundancy ($m < n$)
  - if the robot starts in a singularity

for "simple" manipulators (e.g., all industrial robots) and $m = n$, the execution of these tasks will require the passage through a singular configuration

Joint and Cartesian trajectories

- a reconfiguration task (or… passing through singularity)



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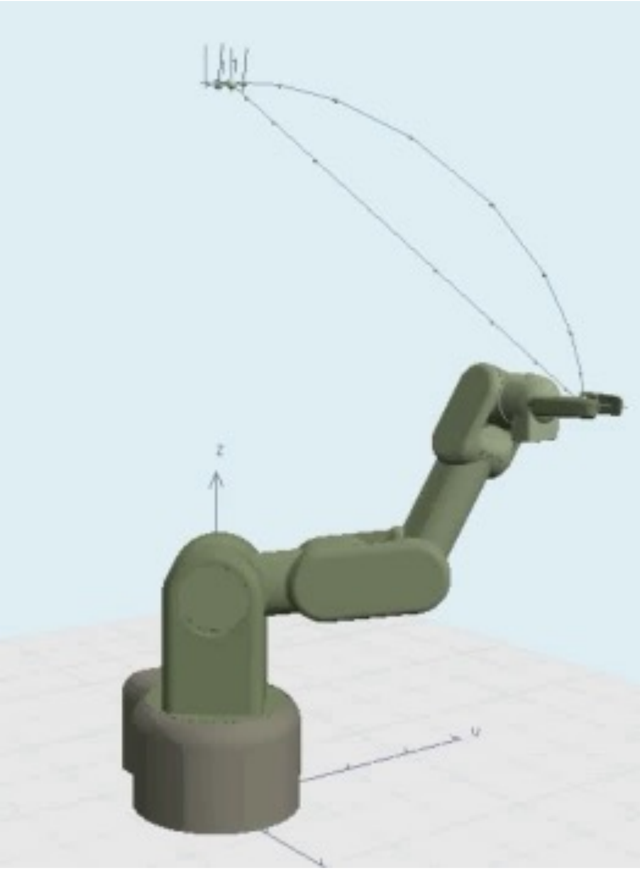
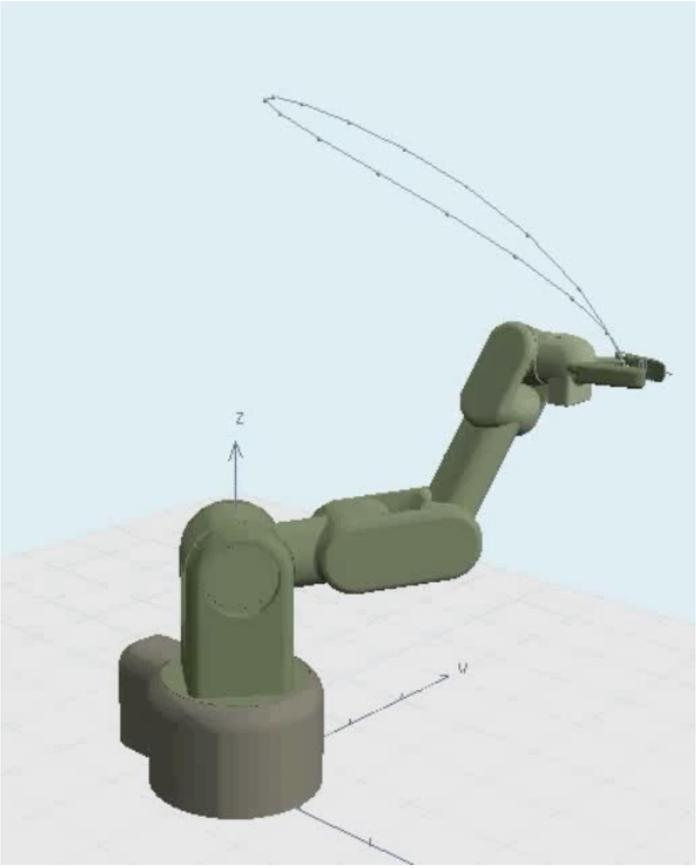


three-phase trajectory:
circular arc + self-motion + linear path



video

single-phase trajectory
in the joint space (no stops)

From task to trajectory

TRAJECTORY $\begin{cases} \text{of motion } p_d(t) \text{ (or } q_d(t)) \\ \text{of interaction } F_d(t) \end{cases}$

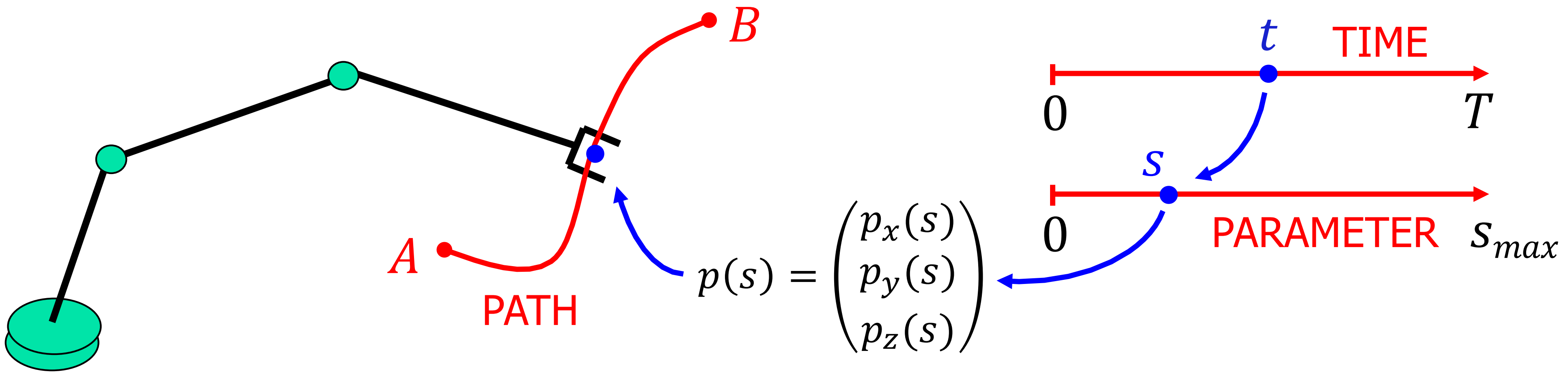
||

GEOMETRIC PATH parameterized by s: $p = p(s)$
(e.g., s is the arc length)

+

TIMING LAW describes the time evolution of $s = s(t)$

$\Big\}\; p(s(t))$

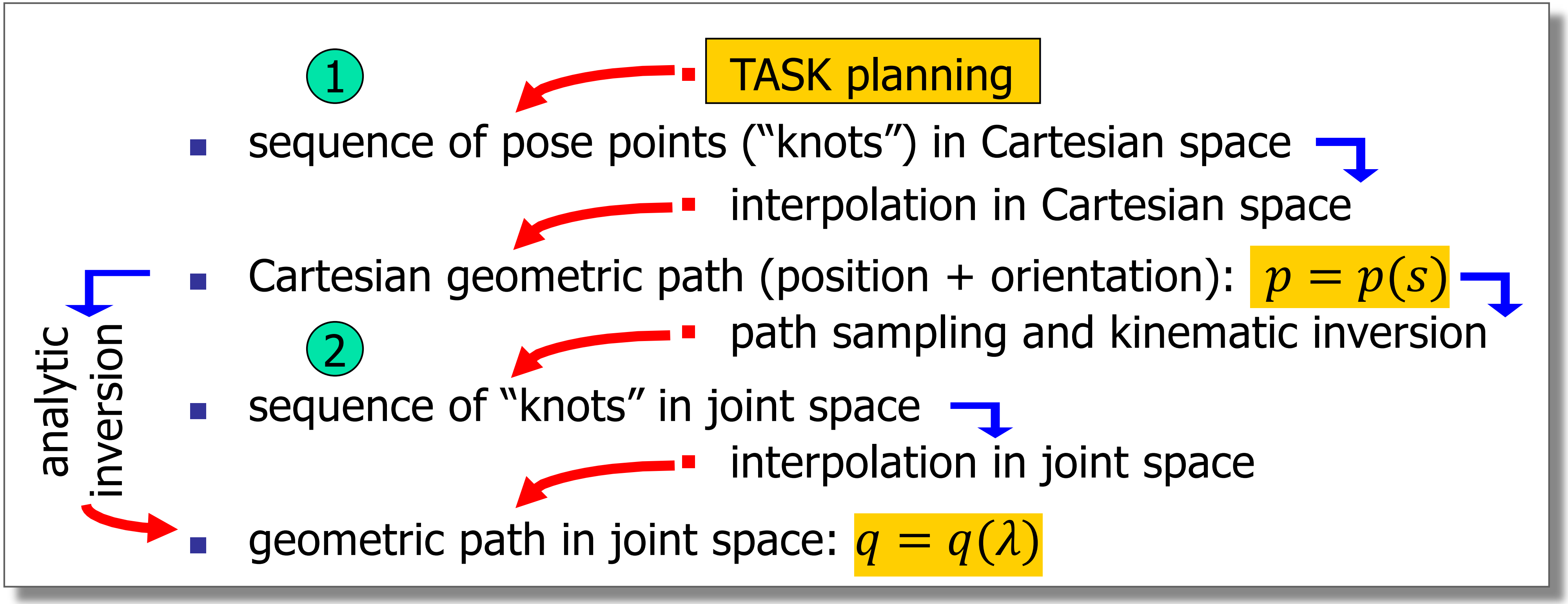


$$p(s) = \begin{pmatrix} p_x(s) \\ p_y(s) \\ p_z(s) \end{pmatrix}$$

example: TASK planner provides A, B
TRAJECTORY planner generates $p(t)$

Trajectory planning

operative sequence



1

TASK planning

- sequence of pose points (“knots”) in Cartesian space
  - interpolation in Cartesian space
- Cartesian geometric path (position + orientation): $p = p(s)$
  - path sampling and kinematic inversion

2

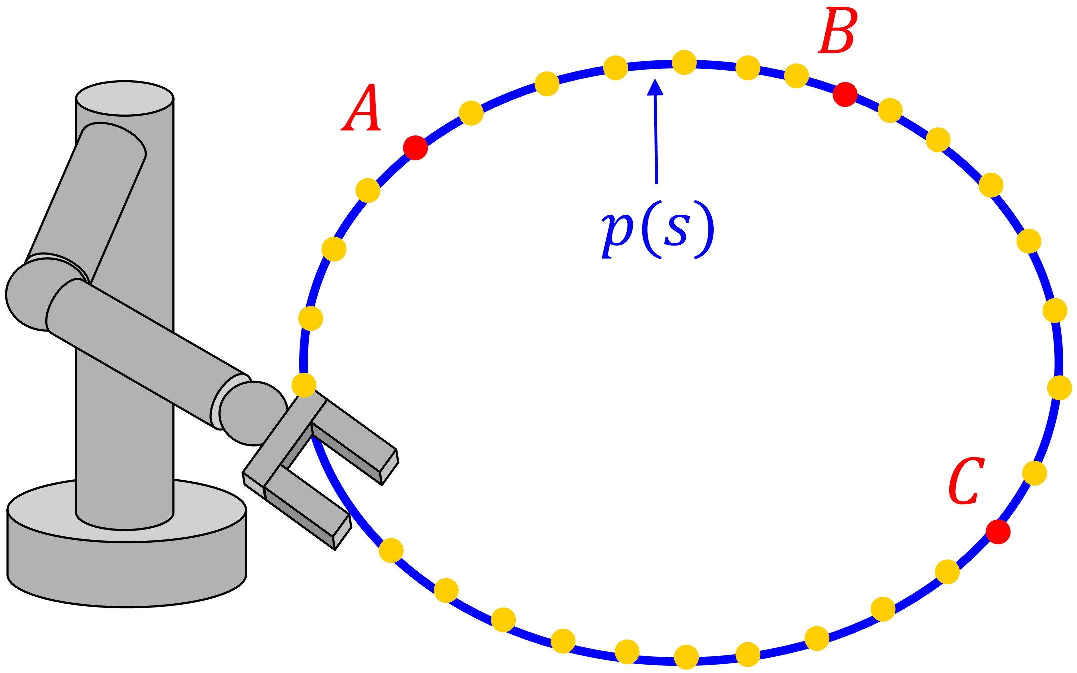
- sequence of “knots” in joint space
  - interpolation in joint space
- geometric path in joint space: $q = q(\lambda)$

analytic inversion

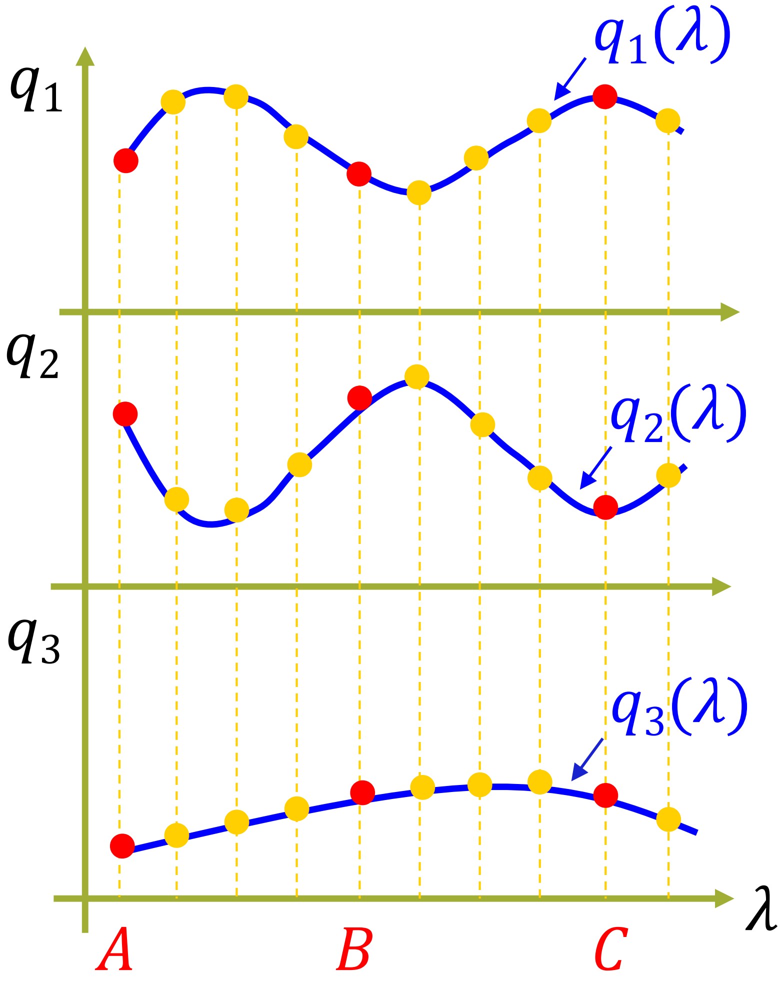
additional issues to be considered in the planning process

- obstacle avoidance
- on-line/off-line computational load
- sequence 2 is more “dense” than 1

Example



Cartesian space



joint space

Path and timing law

- after choosing a path, the trajectory definition is completed by the choice of a timing law

$$p = p(s) \quad \Rightarrow s = s(t) \qquad \text{(Cartesian space)}$$

$$q = q(\lambda) \quad \Rightarrow \lambda = \lambda(t) \qquad \text{(joint space)}$$

  - if s(t) = t, path parameterization is the natural one given by time
- the timing law
  - is chosen based on task specifications (stop in a point, move at constant velocity, and so on)
  - may consider optimality criteria (min transfer time, min energy,…)
  - constraints are imposed by actuator capabilities (max torque, max velocity,…) and/or by the task (e.g., max acceleration on payload)

note: on parameterized paths, a space-time decomposition takes place

e.g., in Cartesian space

$$\dot{p}(t) = \frac{dp}{ds}\dot{s} \qquad \ddot{p}(t) = \frac{dp}{ds}\ddot{s} + \frac{d^2p}{ds^2}\dot{s}^2$$

Trajectory classification

- space of definition
  - Cartesian, joint
- task type
  - point-to-point (PTP), multiple points (knots), continuous, concatenated
- path geometry
  - rectilinear, polynomial, exponential, cycloid, …
- timing law
  - bang-bang in acceleration, trapezoidal in velocity, polynomial, …
- coordinated or independent
  - motion of all joints (or of all Cartesian components) starts and ends at the same instants (say, $t = 0$ and $t = T$) = single timing law

  or
  - motions are timed independently (according to the requested displacement and robot capabilities) – mostly only in the joint space

Cartesian vs. joint trajectory planning

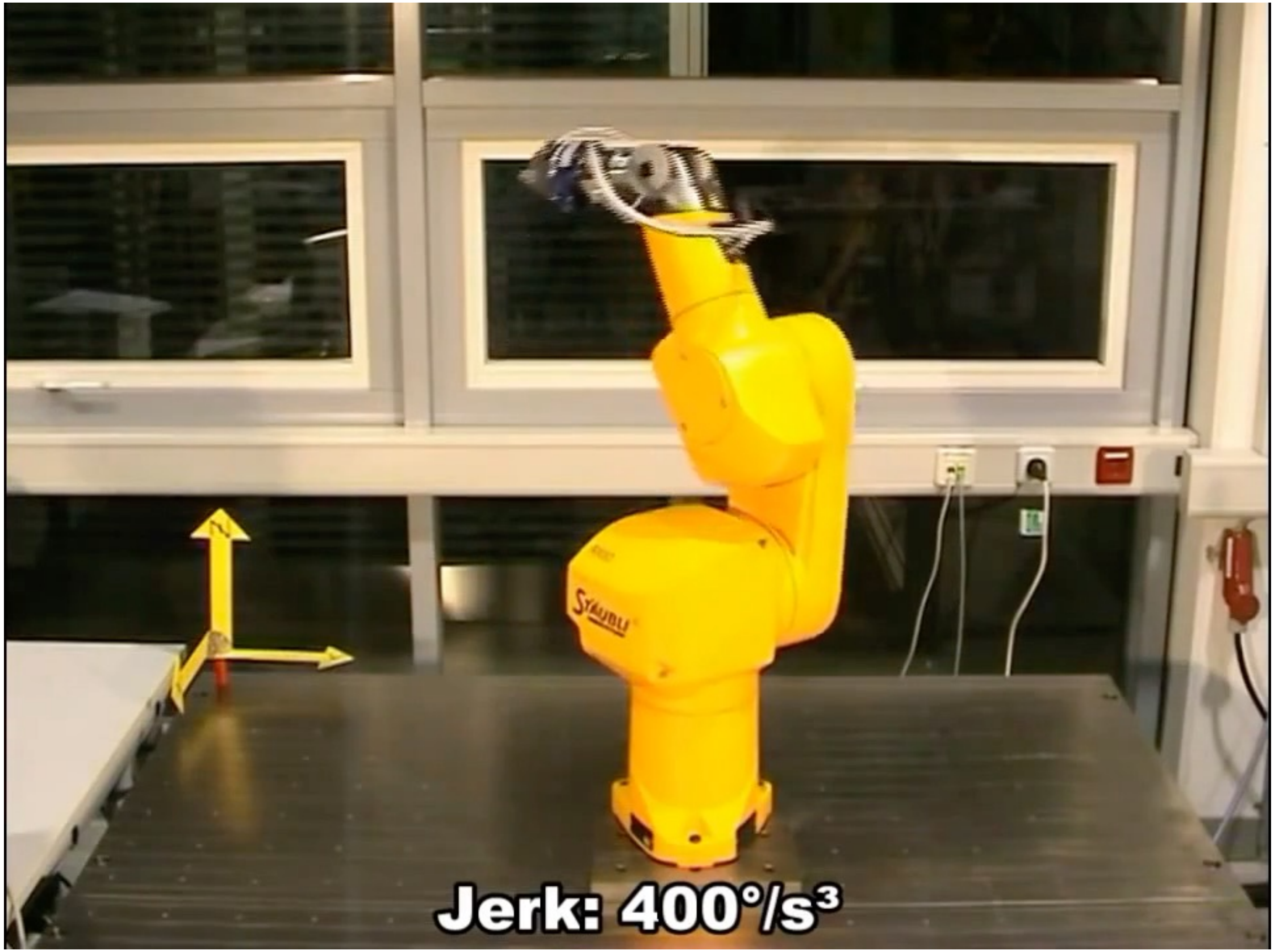
- planning in Cartesian space
  - allows a more direct visualization of the generated path
  - obstacle avoidance, lack of "wandering"
- planning in joint space
  - does not need on-line kinematic inversion
- issues in kinematic inversion
  - $\dot{q}$ and $\ddot{q}$ (or higher-order derivatives) may also be needed
    - Cartesian task specifications involve the geometric path, but also bounds on the associated timing law
  - for redundant robots, choice among ∞^{n-m} inverse solutions, based on optimality criteria or additional auxiliary tasks
  - off-line planning in advance is not always feasible
    - e.g., when environment interaction occurs or when sensor-based motion is needed

Relevant characteristics

- computational efficiency and memory space
  - e.g., store only the coefficients of a polynomial function
- predictability and accuracy
  - vs. “wandering” out of the knots
  - vs. “overshoot” on final position
- flexibility
  - allowing concatenation of primitive segments
  - over-fly
  - …
- continuity
  - in space and/or in time
  - at least C^1, but also up to jerk = third derivative in time

A robot trajectory with bounded jerk

video



Trajectory planning in joint space

- $q = q(t)$ in time or $q = q(\lambda)$ in space (then with $\lambda = \lambda(t)$)
- it is sufficient to work component-wise (q_i in vector q)
- an implicit definition of the trajectory, by solving a problem with specified boundary conditions in a given class of functions
- typical classes: polynomials (cubic, quintic,…), trigonometric (cosine, sines, combined, …), clothoids, …
- imposed conditions
  - passage through points = interpolation
  - initial, final, intermediate velocity (or geometric tangent for paths)
  - initial, final acceleration (or geometric curvature)
  - continuity of time-(or space-)derivative up to the k-th order: class C^k

many of the following methods and remarks can be directly applied also to Cartesian trajectory planning (and vice versa)!

Cubic polynomial in space

$\boxed{q(0) = q_0}$ $\boxed{q(1) = q_1}$ $\boxed{q'(0) = v_0}$ $\boxed{q'(1) = v_1}$ ← 4 conditions

$$q(\lambda) = q_0 + \Delta q\underbrace{(a\lambda^3 + b\lambda^2 + c\lambda + d)}$$

$\Delta q = q_1 - q_0$

$\lambda \in [0,1]$

4 coefficients → "doubly normalized" polynomial $q_N(\lambda)$

$q_N(0) = 0 \Leftrightarrow d = 0$ $\quad$ $q_N(1) = 1 \Leftrightarrow a + b + c = 1$

$q'_N(0) = dq_N/d\lambda|_{\lambda=0} = c = v_0/\Delta q$ $\quad$ $q'_N(1) = dq_N/d\lambda|_{\lambda=1} = 3a + 2b + c = v_1/\Delta q$

special case: $v_0 = v_1 = 0$ (zero tangent)

$$q'_N(0) = 0 \Leftrightarrow c = 0$$

$$\left.\begin{array}{l} q_N(1) = 1 \Leftrightarrow a + b = 1 \\ q'_N(1) = 0 \Leftrightarrow 3a + 2b = 0 \end{array}\right\} \Leftrightarrow \begin{array}{l} a = -2 \\ b = 3 \end{array}$$

Cubic polynomial in time

$\boxed{q(0) = q_{in}}$ $\boxed{q(T) = q_{fin}}$ $\boxed{\dot{q}(0) = v_{in}}$ $\boxed{\dot{q}(T) = v_{fin}}$ ← 4 conditions

$$\Delta q = q_{fin} - q_{in}$$

$$\tau = t/T \in [0,1]$$

$$q(\tau) = q_{in} + \Delta q \underbrace{(a\tau^3 + b\tau^2 + c\tau + d)}$$

4 coefficients → "doubly normalized" polynomial $q_N(\tau)$

$q_N(0) = 0 \Leftrightarrow d = 0$

$q_N(1) = 1 \Leftrightarrow a + b + c = 1$

$q'_N(0) = dq_N/d\tau|_{\tau=0} = c = \dfrac{v_{in}T}{\Delta q}$

$q'_N(1) = dq_N/d\tau|_{\tau=1} = 3a + 2b + c = \dfrac{v_{fin}T}{\Delta q}$

special case: $v_{in} = v_{fin} = 0$ (rest-to-rest)

$q'_N(0) = 0 \Leftrightarrow c = 0$

$$\left.\begin{array}{l} q_N(1) = 1 \Leftrightarrow a + b = 1 \\ q'_N(1) = 0 \Leftrightarrow 3a + 2b = 0 \end{array}\right\} \Leftrightarrow \begin{array}{l} a = -2 \\ b = 3 \end{array}$$

A trigonometric alternative

$$\boxed{q(0) = q_{in}} \quad \boxed{q(T) = q_{fin}} \quad \boxed{\dot{q}(0) = 0} \quad \boxed{\dot{q}(T) = 0} \longleftarrow$$

boundary conditions
(rest-to-rest)

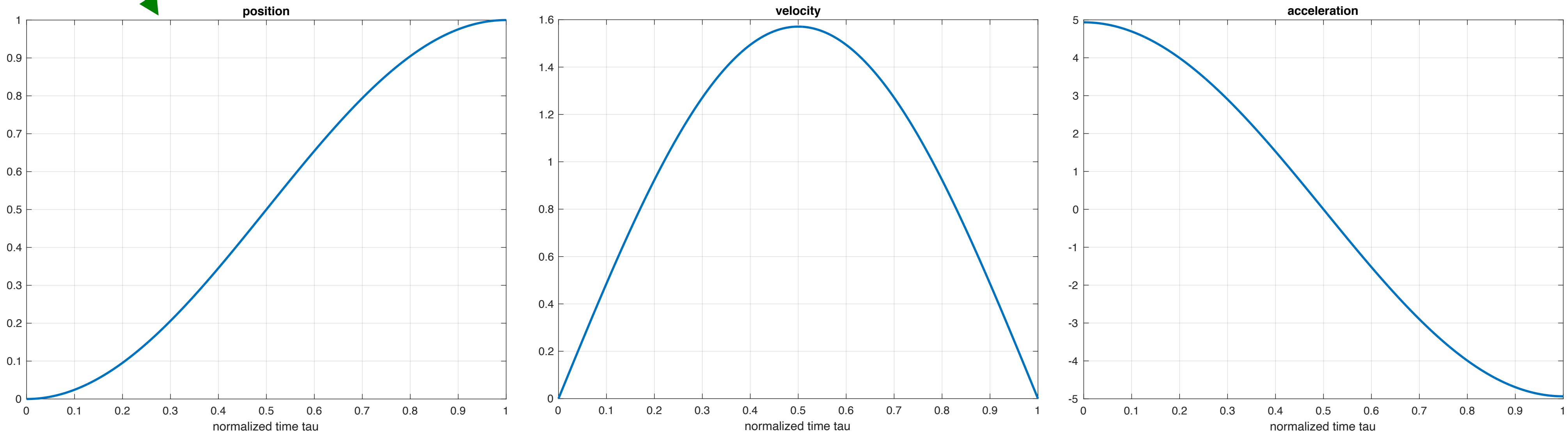
$$\boxed{q(\tau) = q_{in} + \Delta q \underbrace{\frac{1 - \cos \pi\tau}{2}}}$$

$$\Delta q = q_{fin} - q_{in}$$

$$\tau = t/T \in [0,1]$$

doubly normalized

$$\dot{q}(\tau) = \frac{\Delta q}{T} \frac{\pi}{2} \sin \pi\tau \qquad \ddot{q}(\tau) = \frac{\Delta q}{T^2} \frac{\pi^2}{2} \cos \pi\tau$$



$$\max \dot{q}(\tau) = \dot{q}(0.5) = \frac{\Delta q}{T} \frac{\pi}{2} \qquad \max|\ddot{q}(\tau)| = \ddot{q}(0) = -\ddot{q}(1) = \frac{\Delta q}{T^2} \frac{\pi^2}{2}$$

Quintic polynomial

$$q(\tau) = a\tau^5 + b\tau^4 + c\tau^3 + d\tau^2 + e\tau + f$$

6 coefficients

$$\tau \in [0,1]$$

allows to satisfy 6 conditions, for example (in normalized time $\tau = t/T$)

$\boxed{q(0) = q_0}$ $\boxed{q(1) = q_1}$ $\boxed{q'(0) = v_0 T}$ $\boxed{q'(1) = v_1 T}$ $\boxed{q''(0) = a_0 T^2}$ $\boxed{q''(1) = a_1 T^2}$

$$\begin{aligned} q(\tau) = {} & (1-\tau)^3 (q_0 + (3q_0 + v_0 T)\tau + (a_0 T^2 + 6v_0 T + 12 q_0)\tau^2/2) \\ & + \tau^3 (q_1 + (3q_1 - v_1 T)(1-\tau) + (a_1 T^2 - 6 v_1 T + 12 q_1)(1-\tau)^2/2) \end{aligned}$$

special case: $v_0 = v_1 = a_0 = a_1 = 0$

$$q(\tau) \;= q_0 + \Delta q (6\tau^5 - 15\tau^4 + 10\tau^3)$$

$$\Delta q = q_1 - q_0$$

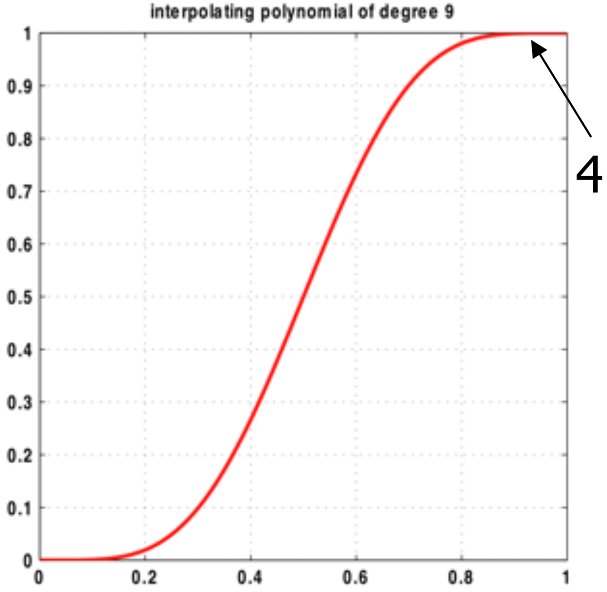
Higher-order polynomials

- a suitable solution class for satisfying symmetric boundary conditions (in a PTP motion) that impose zero values on higher-order derivatives
  - the interpolating polynomial is always of odd degree
  - the coefficients of such (doubly normalized) polynomial are always integers, alternate in sign, sum up to unity, and are zero for all terms up to the power = (degree-1)/2
- in all other cases (e.g., for interpolating a large number N of points), their use is not recommended
  - there is a unique polynomial of degree $N-1$ interpolating N points
  - k-th degree polynomials have $k-1$ maximum and minimum points
  - oscillations arise out of the interpolation points (wandering)

Interpolating $N = 2$ knots

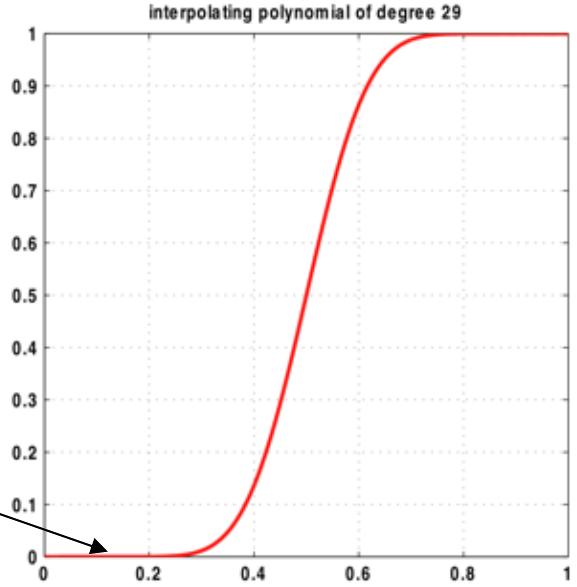
with high-order polynomials and zero boundary conditions

9th degree



4 derivatives are zero

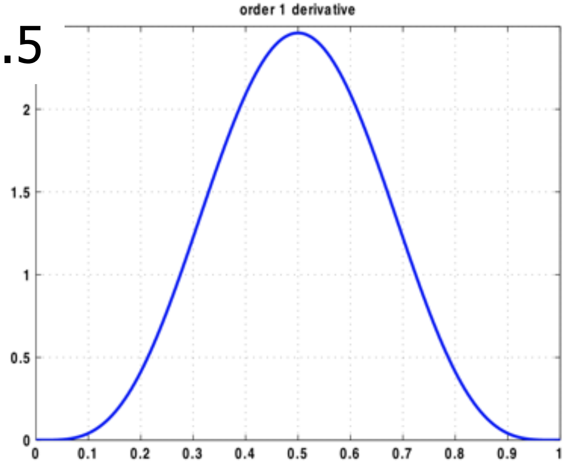
14 derivatives are zero!



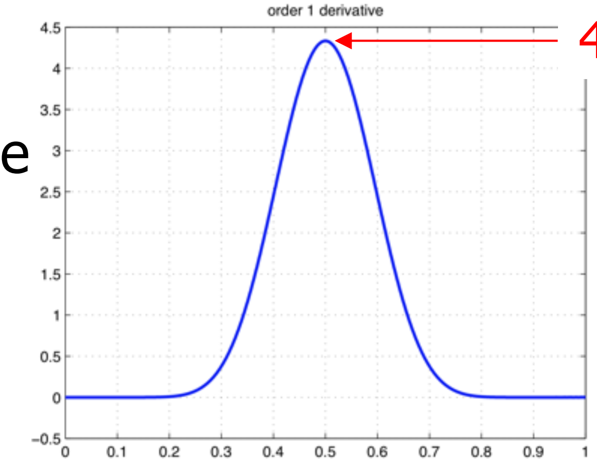
29th degree

no overshoot nor wandering

2.5



normalized first derivative (velocity in time)



4.5!!

peaking at midpoint

Interpolating N knots $q_1 \ldots q_N$

with a **unique** polynomial of degree $N-1$

$N = 2 \;\Rightarrow\;$ a line

$$q(\tau) = a_0 + a_1\tau$$
$$= q_1 + (q_2 - q_1)\tau$$

$N \;\Rightarrow\;$ a polynomial of degree $N-1$

$$q(\tau) = a_0 + a_1\tau + \cdots + a_{N-1}\tau^{N-1}$$

$$\boxed{\tau = \frac{t}{T} \in [0,1]}$$

$N = 3 \;\Rightarrow\;$ a quadric

$$q(\tau) = a_0 + a_1\tau + a_2\tau^2$$

$$a_0 = q_1$$

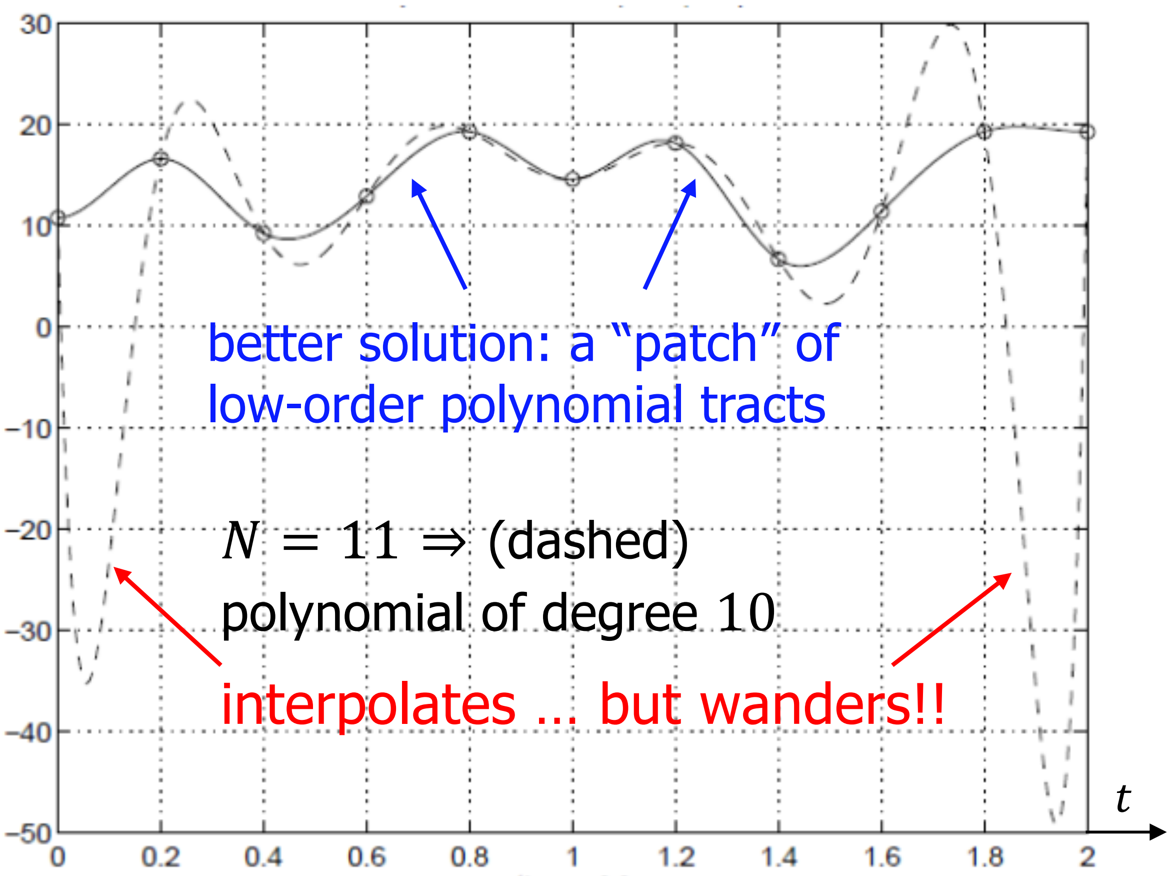
$$a_1 = \frac{(q_3 - q_1){\tau_m}^2 - (q_2 - q_1)}{\tau_m(\tau_m - 1)}$$

$$a_2 = \frac{(q_2 - q_1) - (q_3 - q_1)\tau_m}{\tau_m(\tau_m - 1)}$$

at $\tau_m \in (0,1)$, $q(\tau_m) = q_2$

$N = 4 \;\Rightarrow\;$ a cubic

$$q(\tau) = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3$$



better solution: a "patch" of low-order polynomial tracts

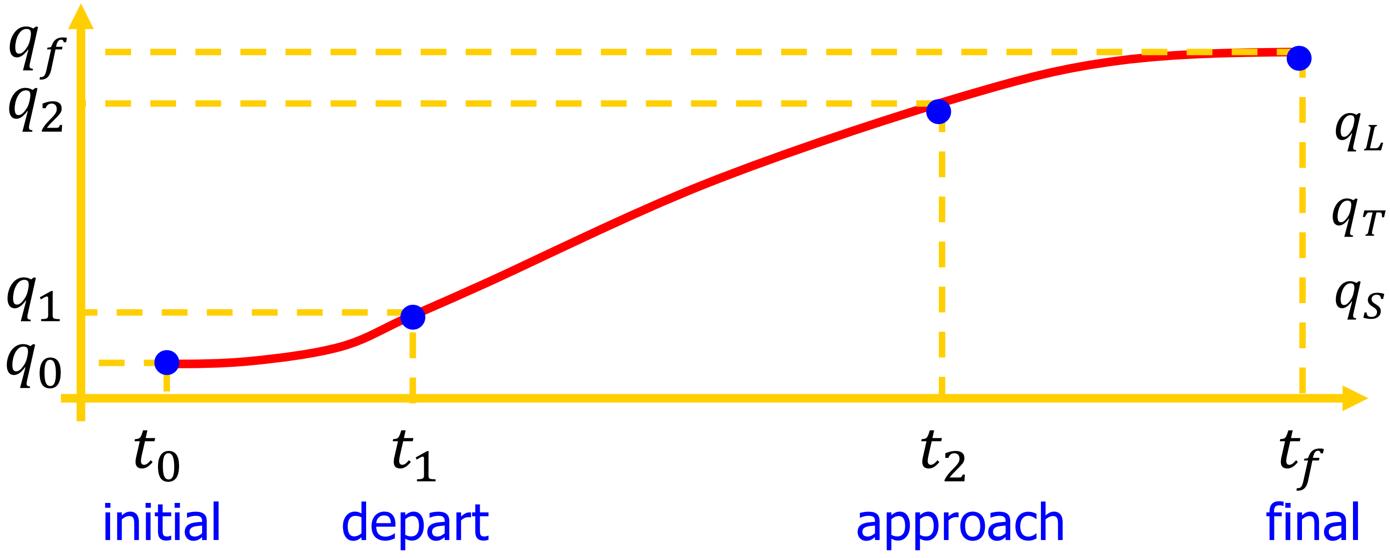
$N = 11 \Rightarrow$ (dashed) polynomial of degree 10

interpolates … but wanders!!

$T = 2$

4-3-4 polynomials

three phases (Lift off, Travel, Set down) in a pick-and-place operation in time



$$\left.\begin{aligned} q_L(t) &= \text{4th order polynomial} \\ q_T(t) &= \text{3rd order polynomial} \\ q_S(t) &= \text{4th order polynomial} \end{aligned}\right.$$

$$\underbrace{\phantom{q_S(t) = \text{4th order polynomial}}}_{\text{14 coefficients}}$$

boundary conditions

$$q(t_0) = q_0 \quad q(t_1^-) = q(t_1^+) = q_1 \quad q(t_2^-) = q(t_2^+) = q_2 \quad q(t_f) = q_f \;\Big\}\; \text{6 passages}$$

$$\dot{q}(t_0) = \dot{q}(t_f) = 0 \qquad \ddot{q}(t_0) = \ddot{q}(t_f) = 0 \;\Big\}\; \text{4 initial/final velocity/acceleration}$$

$$\dot{q}(t_i^-) = \dot{q}(t_i^+) \qquad \ddot{q}(t_i^-) = \ddot{q}(t_i^+) \qquad i = 1,2 \;\Big\}\; \text{4 continuity up to acceleration}$$

Interpolation using splines

- problem

  interpolate N knots, with continuity up to the second derivative

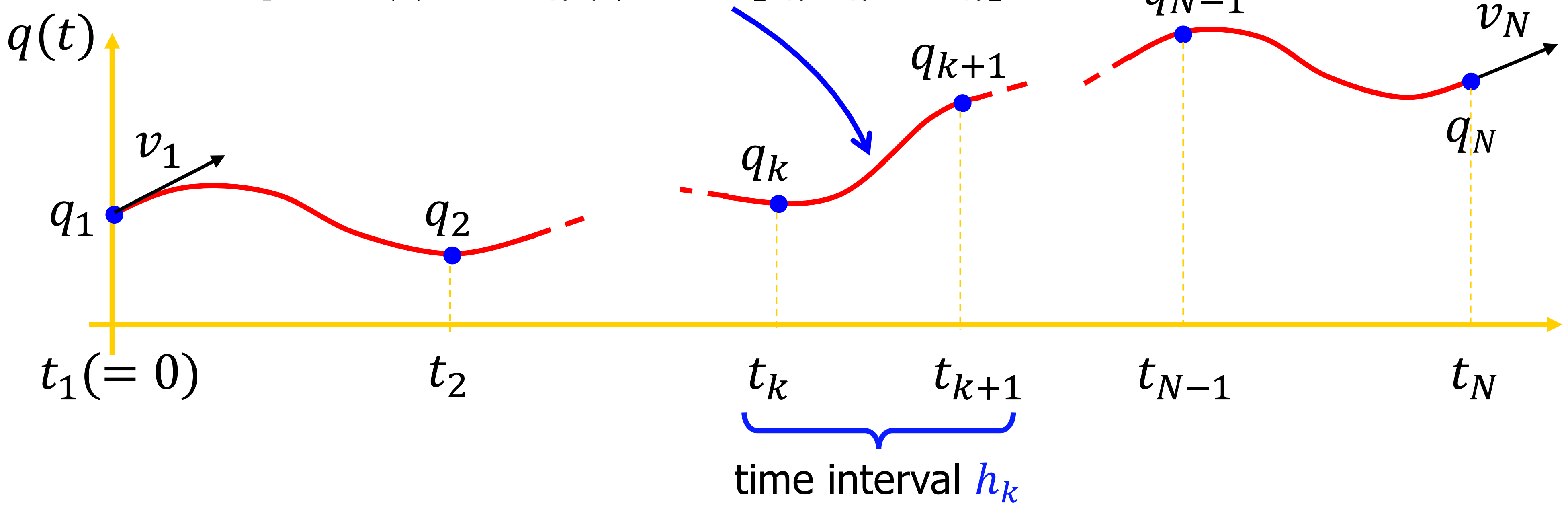
- solution $\longleftrightarrow$ de Casteljau@Citroën, Bézier@Renault, de Boor@General Motors (late 1950) …

  spline: $N-1$ cubic polynomials, concatenated so to pass through N knots, and continuous up to the second derivative at the $N-2$ internal knots

- $4(N-1)$ coefficients
- $4(N-1)-2$ conditions, or
  - $2(N-1)$ of passage (for each cubic, in the two knots at its ends)
  - $N-2$ of continuity for first derivative (at the internal knots)
  - $N-2$ of continuity for second derivative (at the internal knots)
- 2 free parameters are still left over
  - can be used, e.g., to assign initial and final derivatives, v_1 and v_N
- presented next in terms of time t, but similar in terms of space λ
  - here: first derivative = velocity, second derivative = acceleration

Building a cubic spline

$$q = \theta(t) = \{\theta_k(t), t \in [t_k, t_k + h_k]\}$$



$$\theta_k(\tau) = a_{k0} + a_{k1}\tau + a_{k2}\tau^2 + a_{k3}\tau^3$$

$$\tau = t - t_k \in [0, h_k]$$

$$(k = 1, \cdots, N-1)$$

continuity conditions
for velocity and acceleration →

$$\boxed{\begin{array}{ll} \dot{\theta}_k(h_k) = \dot{\theta}_{k+1}(0) & \\ & k = 1, \cdots, N-2 \\ \ddot{\theta}_k(h_k) = \ddot{\theta}_{k+1}(0) & \end{array}}$$

An efficient algorithm

1. if all velocities v_k at internal knots were known, then each cubic in the spline would be uniquely determined by

$$\begin{aligned}\theta_k(0) &= q_k = a_{k0}\\ \dot{\theta}_k(0) &= v_k = a_{k1}\end{aligned} \qquad \begin{pmatrix} h_k^2 & h_k^3 \\ 2h_k & 3h_k^2 \end{pmatrix}\begin{pmatrix} a_{k2} \\ a_{k3} \end{pmatrix} = \begin{pmatrix} q_{k+1} - q_k - v_k h_k \\ v_{k+1} - v_k \end{pmatrix} \quad ①$$

2. impose the continuity for accelerations ($N-2$ conditions)

$$\ddot{\theta}_k(h_k) = 2a_{k2} + 6a_{k3}h_k = 2a_{k+1,2} = \ddot{\theta}_{k+1}(0)$$

3. expressing the coefficients $a_{k2}, a_{k3}, a_{k+1,2}$ in terms of the still unknown knot velocities (see step 1.) yields a linear system of equations that is always solvable

$$\begin{pmatrix} \\ \boldsymbol{A}(h_1, \cdots, h_{N-1}) \\ \\ \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \\ \vdots \\ \vdots \\ v_{N-1} \end{pmatrix} = \begin{pmatrix} \vdots \\ \boldsymbol{b}(h_1, \cdots, h_{N-1}, q_1 \cdots, q_N, v_1, v_N) \\ \vdots \end{pmatrix}$$

tri-diagonal matrix always invertible

unknown

known vector

to be substituted then back in ①

Structure of $A(\boldsymbol{h})$

$$\begin{pmatrix}
2(h_1+h_2) & h_1 & & & & & \\
h_3 & 2(h_2+h_3) & h_2 & & & & \\
 & \cdots & & & & & \\
 & & & \cdots & & & \\
 & & & & & \cdots & \\
 & & & & h_{N-2} & 2(h_{N-3}+h_{N-2}) & h_{N-3} \\
 & & & & & h_{N-1} & 2(h_{N-2}+h_{N-1})
\end{pmatrix}$$

diagonally dominant matrix (for $h_k > 0$)

[the same tridiagonal matrix for all joints]

Structure of $b(\boldsymbol{h}, \boldsymbol{q}, v_1, v_N)$

$$\begin{pmatrix} \frac{3}{h_1 h_2}\left(h_1^2(q_3 - q_2) + h_2^2(q_2 - q_1)\right) - h_2 v_1 \\ \frac{3}{h_2 h_3}\left(h_2^2(q_4 - q_3) + h_3^2(q_3 - q_2)\right) \\ \vdots \\ \vdots \\ \frac{3}{h_{N-3} h_{N-2}}\left(h_{N-3}^2(q_{N-1} - q_{N-2}) + h_{N-2}^2(q_{N-2} - q_{N-3})\right) \\ \frac{3}{h_{N-2} h_{N-1}}\left(h_{N-2}^2(q_N - q_{N-1}) + h_{N-1}^2(q_{N-1} - q_{N-2})\right) - h_{N-2} v_N \end{pmatrix}$$

Properties of splines

- a spline (in space) is the solution with minimum curvature among all interpolating functions having continuous second derivative
- for cyclic tasks ($q_1 = q_N$), it is preferable to simply impose continuity of first and second derivatives (i.e., velocity and acceleration in time) at the first/last knot as "squaring" conditions
  - choosing $v_1 = v_N = v$ (for a given v) doesn't guarantee in general the continuity up to the second derivative (when in time, the acceleration)
  - in this way, the first = last knot will be handled as all other internal knots
- a spline is uniquely determined from the set of data $q_1, \cdots, q_N$, $h_1, \cdots, h_{N-1}, v_1, v_N$
- in time, the total motion occurs in $T = \sum_k h_k = t_N - t_1$
- the time intervals h_k can be chosen so as to minimize T (linear objective function) under (nonlinear) bounds on velocity and acceleration in $[0, T]$
- spline construction can be suitably modified when the second derivative (in time, the acceleration) is also assigned at the initial and final knots

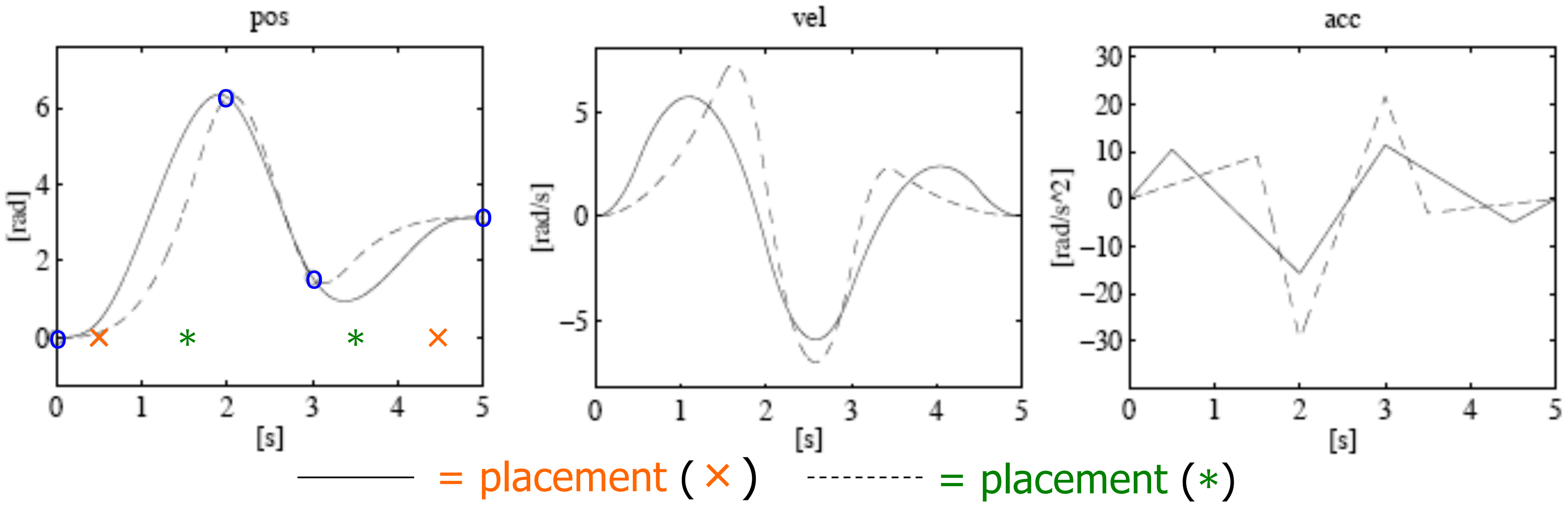
A modification
handling assigned initial and final accelerations

- two more parameters are needed in order to impose also the initial acceleration α_1 and final acceleration α_N
- two "fictitious knots" are inserted in the first and the last original intervals, increasing the number of cubic polynomials from $N-1$ to $N+1$
- in these two knots only continuity conditions on position, velocity and acceleration are imposed

  ⇒ two free parameters are left over (one in the first cubic and the other in the last cubic), which are used to satisfy the boundary conditions on acceleration
- depending on the (time) placement of the two additional knots, the resulting spline changes …

A numerical example

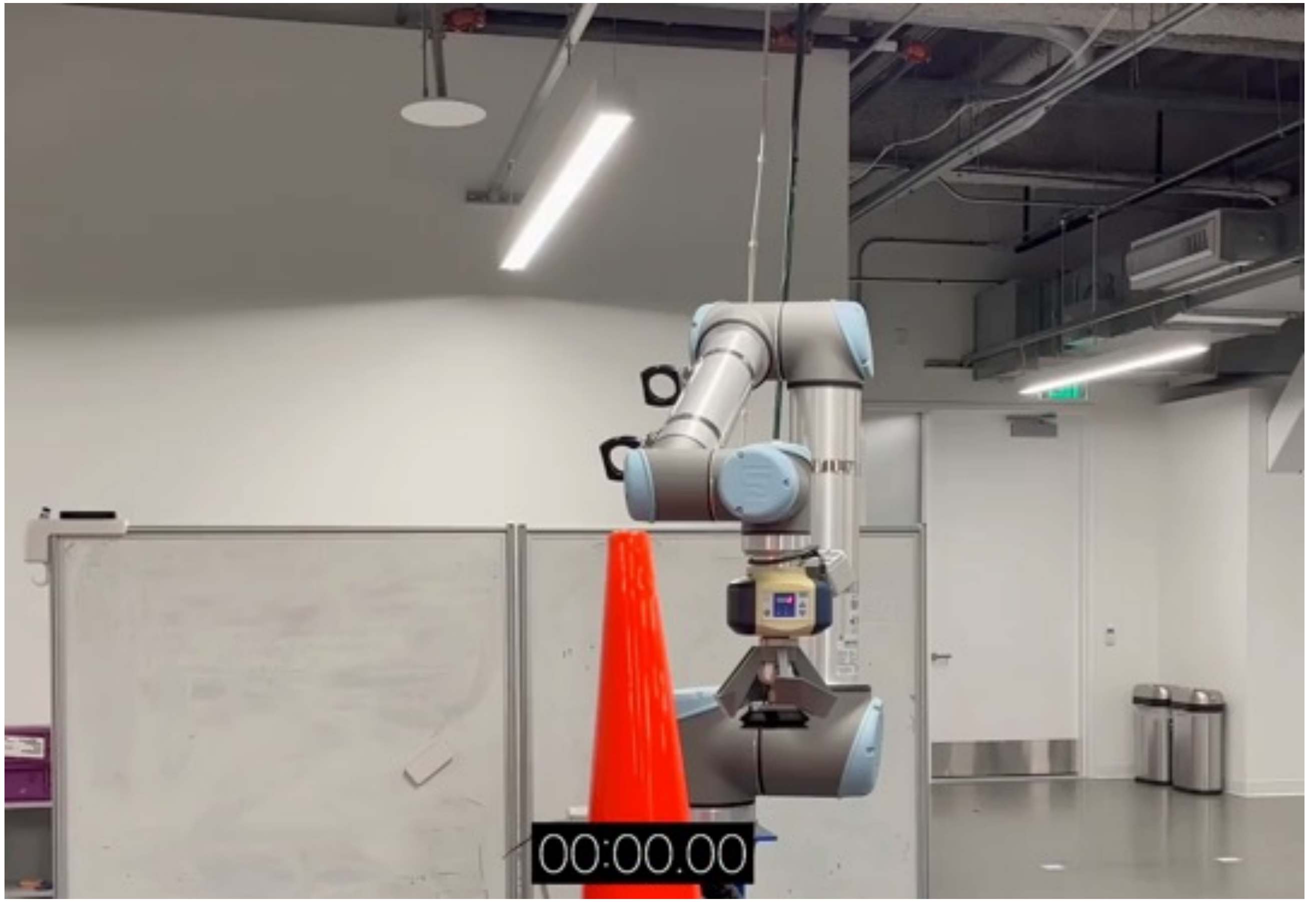
- $N = 4$ knots (o) ⇒ 3 cubic polynomials
  - joint values $q_1 = 0,\ q_2 = 2\pi,\ q_3 = \pi/2\,,\ q_4 = \pi$
  - at $t_1 = 0, t_2 = 2, t_3 = 3, t_4 = 5 \Rightarrow h_1 = 2, h_2 = 1, h_3 = 2$
  - boundary velocities $v_1 = v_4 = 0$
- 2 added knots to impose accelerations at both ends (5 cubic polynomials)
  - boundary accelerations $\alpha_1 = \alpha_4 = 0$
  - two placements: at $t_1' = 0.5$ and $t_3' = 4.5$ (×); or at $t_1'' = 1.5$ and $t_4'' = 3.5$ ($*$)

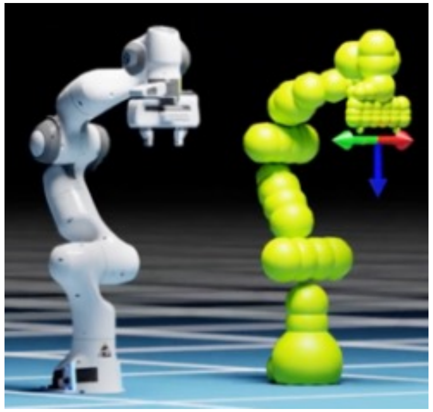


Point-to-point optimal motion

computation in real time

- point-to-point motion without prescribed interpolation path (infinite feasible trajectories …)
- in the presence of obstacles (robot modeled with bubbles)
- optimization algorithm penalizes jerk and acceleration, leading to smooth and short trajectories
- real-time computation (100 ms) with a CUDA (Compute Unified Device Architecture) library using NVIDIA parallel GPUs





video: see https://curobo.org/index.html#overview

library content:
forward kinematics (URDF), numerical inverse kinematics (L-BFGS), collision checking, motion generation, model predictive control