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Comparative Analysis of Traditional, Hypercomplex, and Pre-trained Deep Neural Networks for Audio Emotion Recognition

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Audio Emotion Recognition

AER: computational task of identifying and classifying the emotional content embedded within audio signals.

discrete emotional model: sadness, happiness, fear, anger, disgust, and neutral.

Three main approaches:

1. 1D and 2D Hypercomplex NN on log-Mel / MFCCs

2. 1D and 2D Hypercomplex NN on log-Mel / MFCCs + 1st and 2nd derivatives

3. Hypercomplex projection-head with pre-trained Wav2Vec backbone.

With:
- Hyperparameter tuning (Algebra signature, conv. kernel size, conv. stride)
- Comparison with 1D and 2D traditional NN analogues of 1. 2. And 3.

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Three main approaches:

1. 1D and 2D Hypercomplex NN on log-Mel / MFCCs → PureCliffSER, CliffSER

2. ~~1D and 2D Hypercomplex NN on log-Mel / MFCCs + 1st and 2nd derivatives~~

   1D and 2D hypercomplex NN on log-Mel / MFCCs of the EMD decomposed signal

3. Hypercomplex projection-head with pre-trained Wav2Vec backbone.

With:
- Hyperparameter tuning (Algebra signature, conv. kernel size, conv. stride)
- Comparison with 1D and 2D traditional NN analogues of 1. 2. And 3.

Features: Audio Speech Processing (1)

Voiced Speech signal: produced when the vocal cords vibrate.

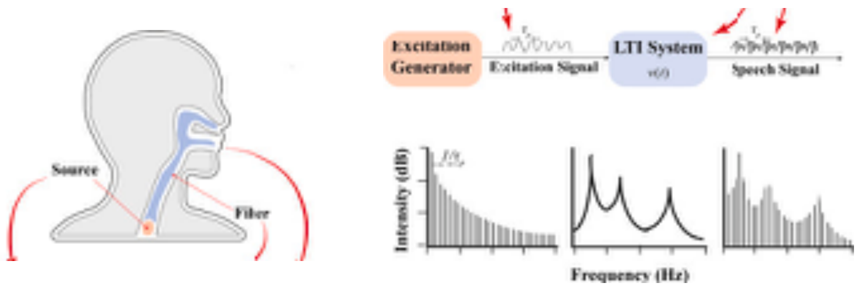
$$y[t] = \delta_P[t] * h_V[t] \qquad h_V[t] = A_V\,(g[t] * v[t] * r[t])$$

$g[t]$: glottal pulse signal.

$v[t]$: vocal tract impulse response.

$r[t]$: radiation load response at the lips.

A_V : voice gain.



Unvoiced Speech signal: produced without vibration (turbulent airflow through constrictions in the vocal tract).

$$y[t] = \sigma_P[t] * h_U[t] \qquad h_U[t] = A_U\,(v[t] * r[t])$$

In both cases: lots of convolutions!

Features (2) : Audio Speech Processing

Cepstral analysis : log-magnitude spectrum emphasizes the 'periodicity' of the harmonics. It is treat as a periodic signal subjected to further Fourier analysis.

Time	Frequency	Quefrency
Convolution	Multiplication	Sum

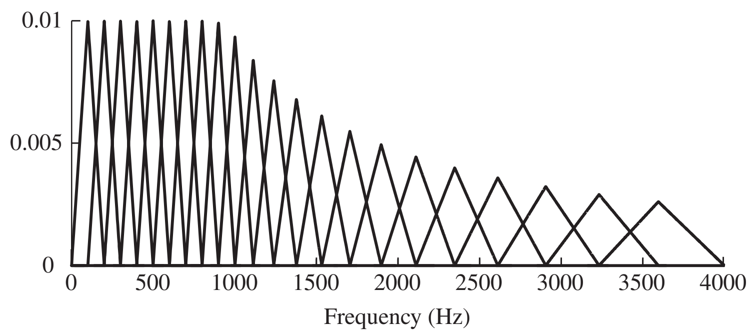
$$\Xi(\eta) = \int_0^1 \log(|X(\phi)|)\, e^{j2\pi\phi n}\, d\phi$$

$$\Xi\big[g(t) * v(t) * r(t)\big] = \Xi\big[g(t)\big] + \Xi\big[v(t)\big] + \Xi\big[r(t)\big]$$

Mel-frequency cepstral coefficients (MFCCs) : first M points of the cepstral transform (using DCT) of R logMel-spectrogram sub-bands.

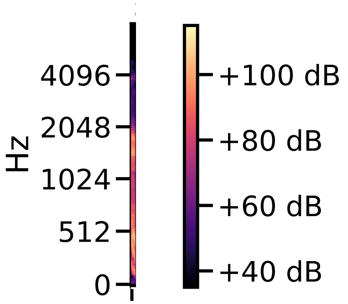
$$\text{MFFC}_\tau(m) = \frac{1}{R}\sum_{r=1}^{R} \log\big(\tilde{X}_\tau(r)\big)\cos\Big(\frac{2\pi}{R}\big(r + \frac{1}{2}\big)m\Big)$$

$$\tilde{X}_\tau[r] = \frac{1}{\sum_{k=0}^{N-1} |\text{Tri}_r[k]|^2} \sum_{k=0}^{N-1} \big|\text{Tri}_r[k] X_\tau[k]\big|^2$$

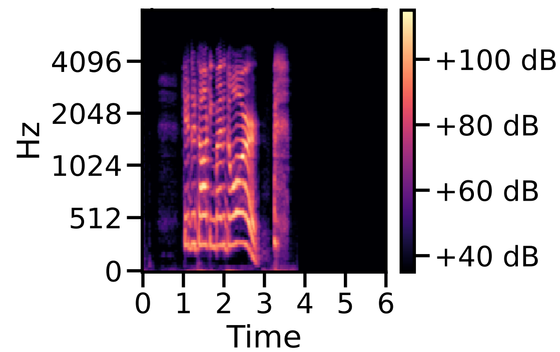


Features (3) : Selection and clustering

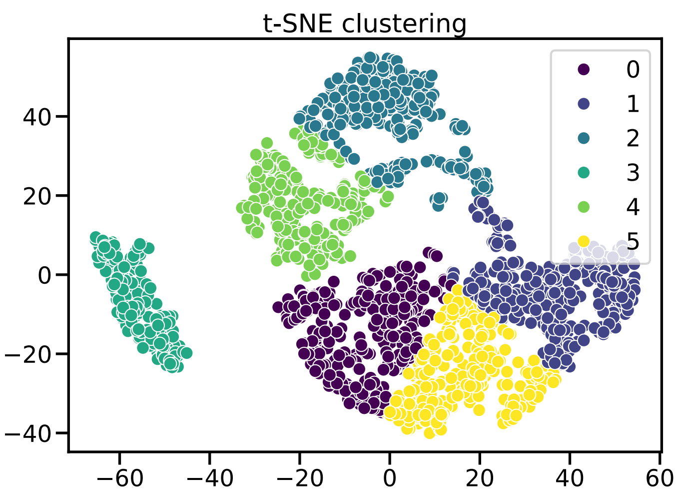
1D: 128 frequncy / quefrency points

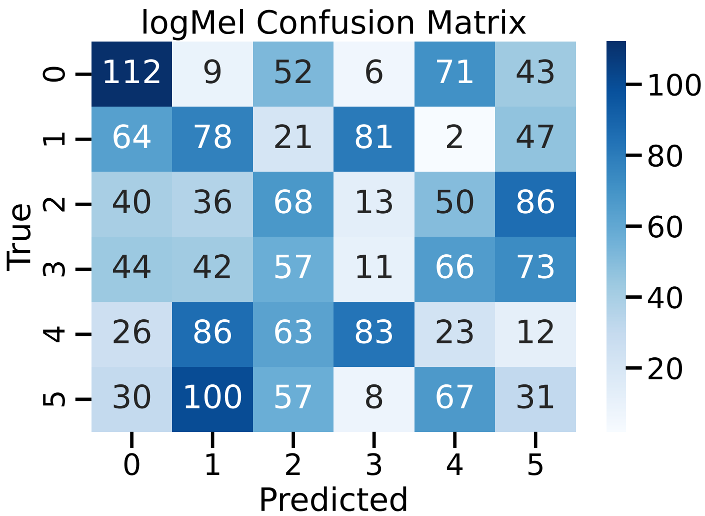


2D: 128 frequncy / quefrency points, 188 temporal points



Feature space clustering:





Features (4) : EMD

1) local maximum and local minimum of x[t]

2) cubic spline interpolation: $e_1^{max}[t]$, $e_1^{min}[t]$

3) mean envelope: $e_1^{mean}[t] = \dfrac{e_1^{max}[t] + e_1^{min}[t]}{2}$

4) IMF extracted by removing the mean envelope from the data: $h_1[t] = x[t] - e_1^{mean}[t]$

5) IMF condition on $h_1[t]$:

A. Not satisfied

replace $x[t]$ with $h_1[t]$ in the previous 1) to 4) steps:

$$e_{11}^{mean}[t] = \frac{e_{11}^{max}[t] + e_{11}^{min}[t]}{2}$$

and:

$$h_{11}[t] = h_1[t] - e_{11}^{mean}[t]$$

If also $h_{11}[t]$ doesn't satisfy the IMF conditions, repeat k times the previous steps, obtaining:

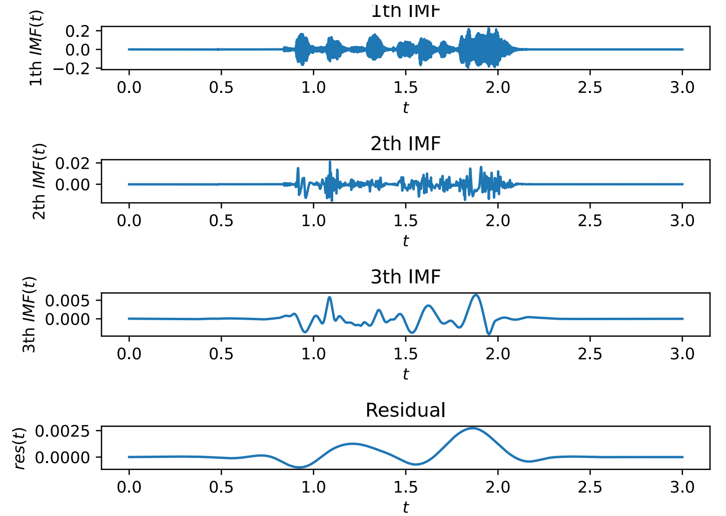
$$h_{1k}[t] = h_{1(k-1)}[t] - e_{1k}^{mean}[t]$$

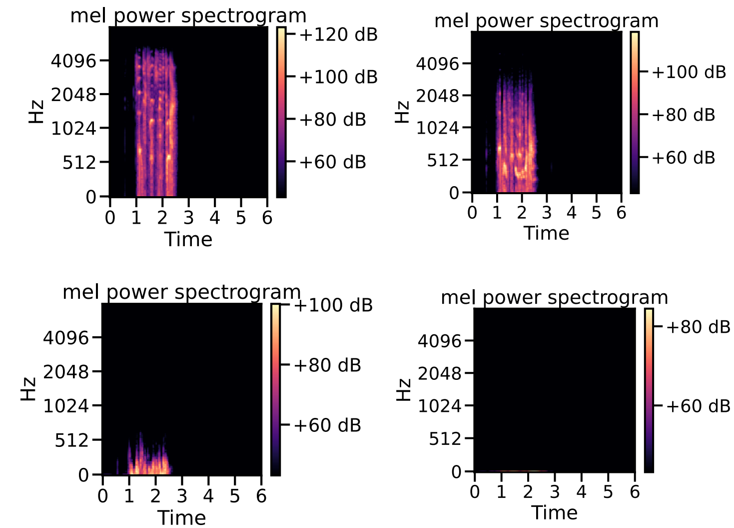
B. Satisfied

Separate the current IMF from the data and use the residual component to repeat steps 1) to 5):

$$r_i[t] = x[t] - h_1[t]$$

IMF condition: $$\sum_{t=1}^{T} \frac{(h_{1(k-1)}[t] - h_{1k}[t])^2}{h_{1(k-1)}[t]^2} \leq 0.1$$





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Models (1): Clifford Algebra

Real Clifford Algebra: The Clifford Algebra over the real field $\mathbb{R}$ is an Algebra that enriches $\mathbb{R}^n$ with a specific bilinear operator known as the Clifford product.

Consider the vector space $(\mathbb{R}^n, +_{\mathbb{R}^n}, \cdot_{\mathbb{R}\mathbb{R}^n})$ over the field $(\mathbb{R}, +_{\mathbb{R}}, \cdot_{\mathbb{R}})$, equipped with a non-degenerate quadratic form $q_A(\cdot): \mathbb{R}^n \to \mathbb{R}$ s.t. the corresponding diagonalized and normalized form $q_G(\cdot)$ satisfies:

$$q_G(\mathrm{e}_i) = \begin{cases} 1 & 1 \le i \le p \\ -1 & p < i \le p+q \end{cases}$$

with: $\{\mathrm{e}_1, \dots, \mathrm{e}_n\} \in \mathbb{R}^n$ canonical basis of $\mathbb{R}^n$

The Clifford algebra $(Cl_{p,q}, +_{Clp,q}, \cdot_{Clp,q})$ is the real associative algebra constructed on $\mathbb{R}^n$ by requiring that the Clifford product $\cdot_{Cl_{p,q}}$ between elements of the algebra satisfies the following conditions:

$$e_i^2 = q_A(\mathrm{e}_i), \quad \forall i = 1, \dots, p+q$$
$$e_i e_j = -e_j e_i \quad \forall i < j$$

e.g. from $\mathbb{R}^2$ we can generate $Cl_{2,0}$ and $Cl_{0,2}$ as follows:

$$\begin{aligned} q_G(\mathrm{a}) &= \mathrm{a}^T G \mathrm{a} \\ &= (\mathrm{a}_1, \mathrm{a}_2)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \mathrm{a}_1 \\ \mathrm{a}_2 \end{pmatrix} \\ &= \mathrm{a}_1^2 + \mathrm{a}_2^2 \\ &= ||\mathrm{a}||^2_{\mathbb{R}^2} \end{aligned}$$

$$\begin{aligned} q_G(\mathrm{a}) &= \mathrm{a}^T G \mathrm{a} \\ &= (\mathrm{a}_1, \mathrm{a}_2)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \mathrm{a}_1 \\ \mathrm{a}_2 \end{pmatrix} \\ &= -\mathrm{a}_1^2 - \mathrm{a}_2^2 \\ &= -||\mathrm{a}||^2_{\mathbb{R}^2} \end{aligned}$$

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Models (2): Clifford Product

Consider the Clifford $Cl_{2,0}$, it has basis $\{1, e_1, e_2, e_{12}\}$ with multiplication table:

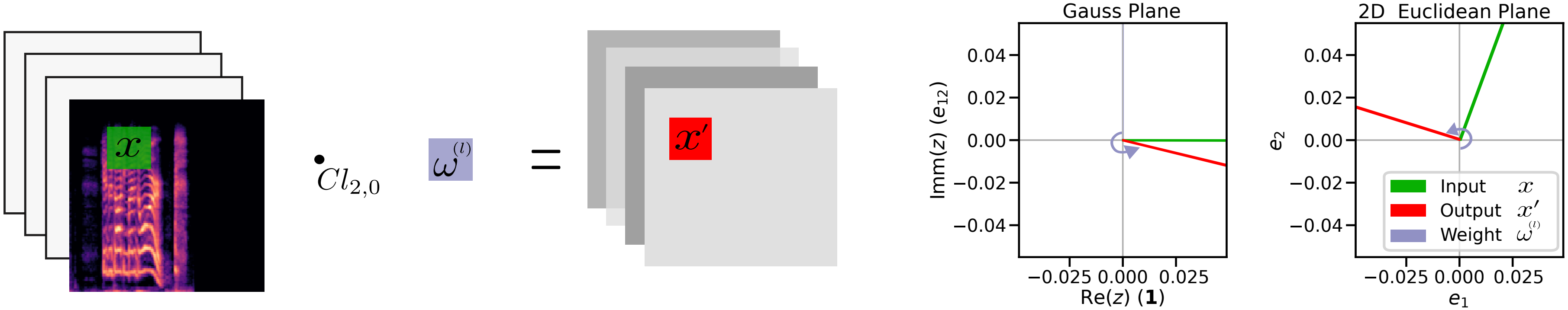
	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

$$\begin{array}{rl} \text{even part} & \mathcal{C}\ell_2^+ = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2 \simeq \mathbb{C} \\ \text{odd part} & \mathcal{C}\ell_2^- = \mathbb{R}^2. \end{array}$$

Product between original vectors:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (x_1 e_1 + y_1 e_2)(x_2 e_1 + y_2 e_2) \\ &= x_1 x_2 e_1^2 + y_1 y_2 e_2^2 + x_1 y_2 e_{12} + y_1 x_2 e_{21} \\ &= (x_1 x_2 + y_1 y_2) 1 + (x_1 y_2 - y_1 x_2) e_{12} \end{aligned}$$

Geometric interpretation of the Clifford product between Algebra elements (no more vectors, but multi-vectors / hypercomplex numbers):



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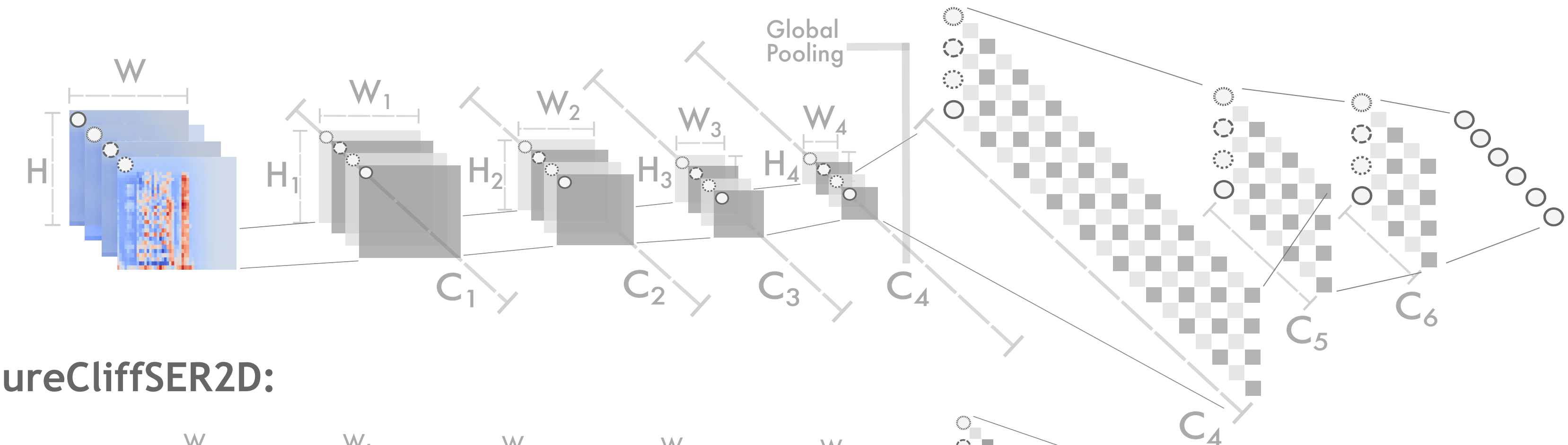
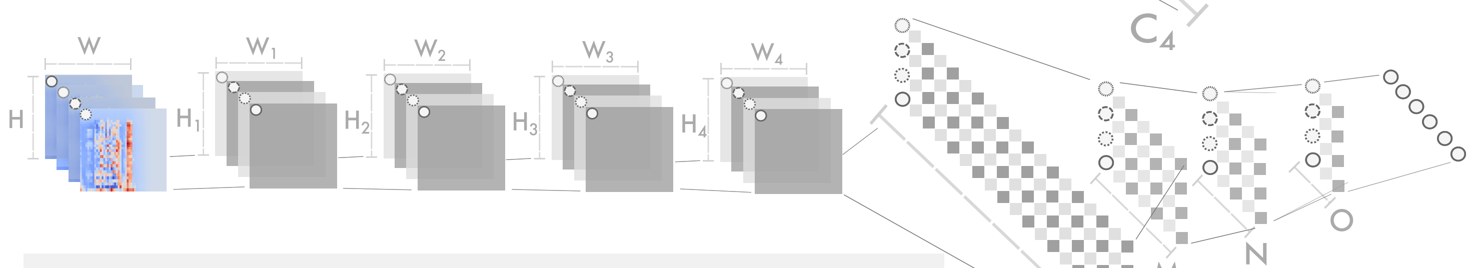
Models (3): Architectures

CliffSER2D: Signatures: [(2,0), (0,2), (3,0), (0,3)] Feature expansion: [32, 64, 128, 256, 128, 64]



PureCliffSER2D:



$$\sum_{v,u=-\lfloor\frac{K}{2}\rfloor}^{\lfloor\frac{K}{2}\rfloor} \begin{pmatrix} w_0 & \eta_1 w_1 & \eta_2 w_2 & -\eta_1\eta_2 w_{12} \\ w_1 & w_0 & \eta_2 w_{12} & \eta_2 w_2 \\ w_2 & \eta_1 w_{12} & w_0 & -\eta_1 w_1 \\ w_{12} & w_2 & -w_1 & w_0 \end{pmatrix}_{(v,u)} \cdot \begin{pmatrix} \circ \\ \circ \\ \circ \\ \circ \end{pmatrix}_{(i-v,\ j-u)}$$

CliffW2V: Projection-head with 2 Clifford linear layers + 1 traditional output layer

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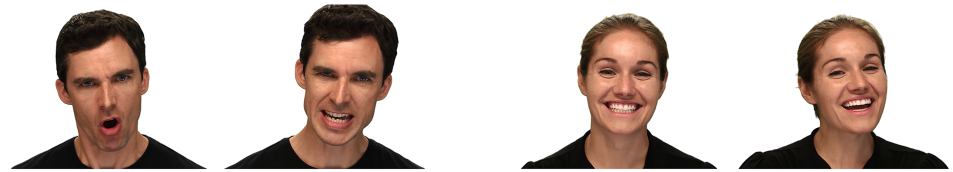
Experimental set-up

Dataset: Ryerson Audio-Visual Database of Emotional Speech and Song (RAVDESS)

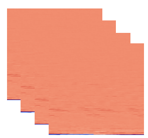
- 7356 recordings featuring acted emotional expressions
- 24 professional actors (12 female, 12 male)
- two vocal channels: speech and song.



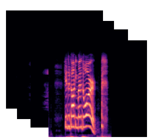
Angry Speech Angry Song Happy Speech Happy Song

Pipeline: Feature extraction + Clustering + data augmentation + data standardization, then:

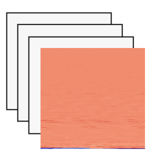
EMD MFCCs



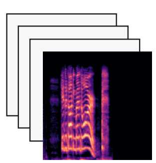
EMD logMel



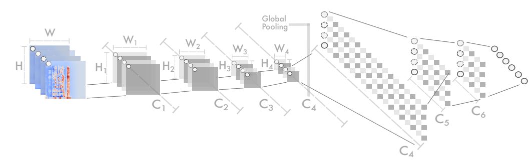
NO EMD MFCCs



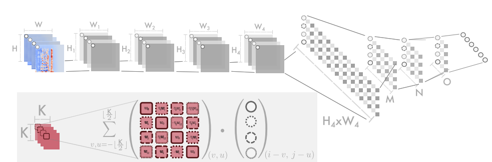
NO EMD logMel



CliffSER 1D / 2D



PureCliffSER 1D / 2D



Training: CrossEntropy loss

Evaluation: 5-fold CV

Metrics:

- avg. accuracy
- avg. F1 score
- avg. precision
- avg. Recall

Raw Data → **CliffW2V** →

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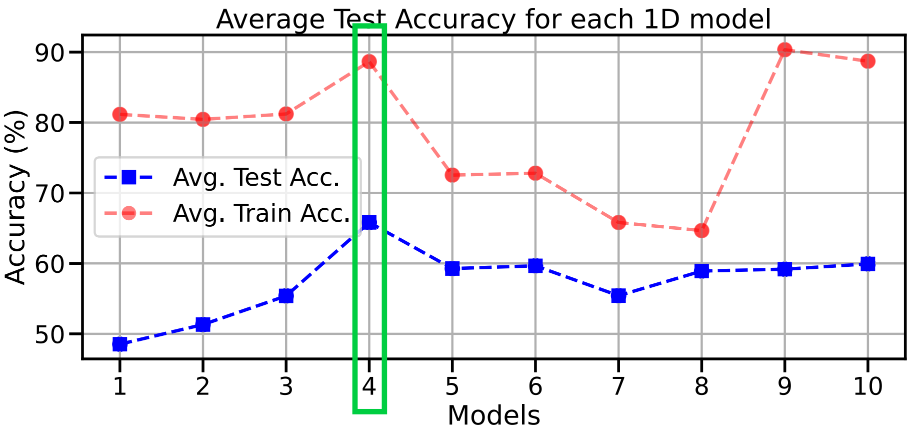
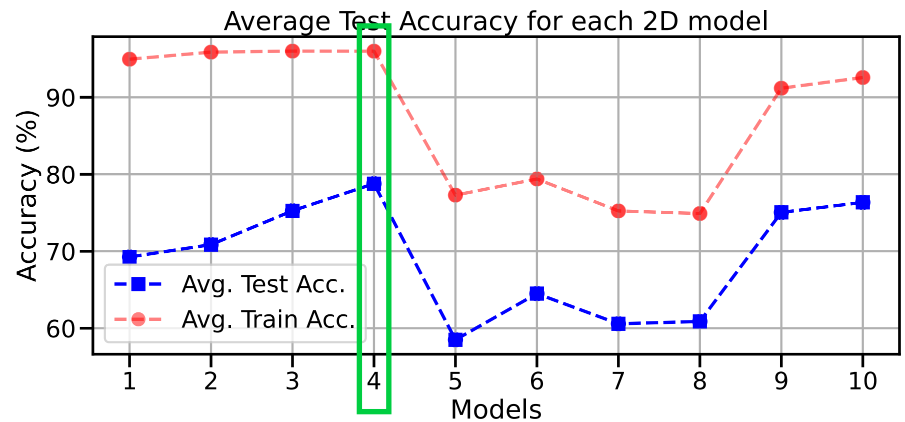
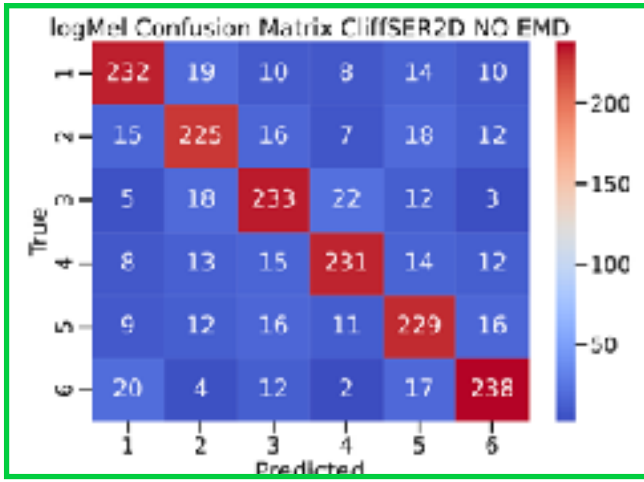
Results:





		ALGEBRA		KERNEL		STRIDE	
N°	NAME	1D	2D	1D	2D	1D	2D
1	EMD MFCCs CliffSER	3,0	2,0	15	5	1	2
2	EMD logMEL CliffSER	0,3		15	5	1	2
3	NO EMD MFCCs CliffSER	3,0	0,3	15	5	1	2
4	NO EMD logMEL CliffSER	0,3		15	5	1	2
5	EMD MFCCs PureCliffSER	2,0	0,2	15	5	1	2
6	EMD logMEL PureCliffSER	2,0	0,2	15	5	1	2
7	NO EMD MFCCs PureCliffSER	2,0		15	5	1	2
8	NO EMD logMEL PureCliffSER	0,2	0,3	15	5	1	2
9	CNN MFCCs	—		15	5	1	2
10	CNN logMEL	—		15	5	1	2



Facts:

- CliffSER1D, CliffSER2D> CNN1D and CNN2D
- CliffSER > PureCliffSER
- NO EMD > EMD
- (0,3) signature > all the others
- Wav2Vec > CliffW2V

- Wav2Vec: 90.65%
- CliffW2V: 84.13

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Thank you!

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Appendix: Algebra and geometry

Algebra (0): recap

Group: $(G, @_G)$, s.t. :

$@_G : G \times G \rightarrow G$ is:

- Associative
- Has inverse
- Has identity
- (If commutative -> abelian group)

Field: $(K, +_K, \bullet_K)$, s.t. :

$+_K : K \times K \rightarrow K$ is:

- Associative
- Has inverse
- Has identity
- Commutative

i.e. $(G, +_G)$ is an abelian group

$\bullet_K : G \backslash \{0_k\} \times G \backslash \{0_k\} \rightarrow G \backslash \{0_k\}$ is:

- Associative
- Has inverse
- Has identity
- Commutative

i.e. $(G \backslash \{0_k\}, \bullet_K)$ is an abelian group

Vector/linear space: $(V, +_V, \cdot_{KV})$ on $(K, +_K, \bullet_K)$ s.t. $(V, +_V)$ is an abelian group and:

- $\cdot_{KV}$ distributive w.r.t. $+_V$ and $\cdot_{KV}$
- $\cdot_{KV}$ has identity, with $1_{KV} = 1_K$
- $\forall \underline{v} \in V \ , \ \forall h, k \in K, \ \ h \cdot_{KV} (k \cdot_{KV} \underline{v}) = (h \bullet_K k) \cdot_{KV} \underline{v}$

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Algebra (1): linear groups

Linear groups : subgroups of the general linear group which satisfy conditions about preserving properties that come from linear algebra.

Group	Composition Law	Identity	Inverse
$GL_n(\mathbb{R})$[a]	matrix multiplication	I_n	$A \mapsto A^{-1}$

$$
\begin{tikzcd}[column sep=small]
& GL_n(\mathbb{R}) \arrow[dl] \arrow[dr] & \\
SL_n(\mathbb{R}) \left\{ \begin{matrix} A \in GL_n(\mathbb{R}) \\ \det A = 1 \end{matrix} \right\} \arrow[dr] & & O_n(\mathbb{R}) \left\{ \begin{matrix} A \in GL_n(\mathbb{R}) \\ A^t A = I \end{matrix} \right\} \arrow[dl] \\
& SO_n(\mathbb{R}) \left\{ \begin{matrix} A \in GL_n(\mathbb{R}) \\ A^t A = I, \det A = 1 \end{matrix} \right\} &
\end{tikzcd}
$$

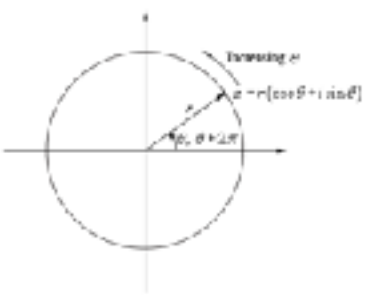
Rotations in $\mathbb{R}^n$!!!

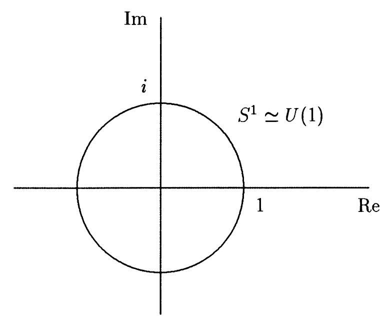
$$
\begin{tikzcd}[column sep=small]
& GL_n(\mathbb{C}) \arrow[dl] \arrow[dr] & \\
SL_n(\mathbb{C}) \left\{ \begin{matrix} A \in GL_n(\mathbb{C}) \\ \det A = 1 \end{matrix} \right\} \arrow[dr] & & U_n \left\{ \begin{matrix} A \in GL_n(\mathbb{C}) \\ A^* A = I \end{matrix} \right\} \arrow[dl] \\
& SU_n \left\{ \begin{matrix} A \in GL_n(\mathbb{C}) \\ A^* A = I, \det A = 1 \end{matrix} \right\} &
\end{tikzcd}
$$

Notice that: $S^1 = \{(x_1, x_2) \,|\, x_1^2 + x_2^2 = 1\}$, endowing S^1 with $\bullet_{\mathbb{C}}$ leads to:

$$
\begin{aligned}
(S^1, \bullet_{\mathbb{C}}) &\simeq \{z = a + ib \in \mathbb{C} \,|\, |z| \\
&= z\bar{z} = \sqrt{a^2 + b^2} = 1\} \\
&= U(1)
\end{aligned}
$$

Since: $|z| = z\bar{z} = \sqrt{a^2 + b^2} = 1$





Algebra (2): algebra over a field

Algebra V over a Field K: vector space $(V, +_V, \cdot_{KV})$, implicitly defined on the field $(K, +_K, \bullet_K)$ of scalars, together with a bilinear law of composition of elements of the vector space, $\bullet_V : V \times V \to V$, that is 'inherited' from a field structure $(\mathbb{F}, +_\mathbb{F}, \bullet_\mathbb{F})$, i.e. $\bullet_V$ mimics $\bullet_\mathbb{F}$.

Endow $(\mathbb{F}, +_\mathbb{F}, \bullet_\mathbb{F})$ with a linear structure $\cdot_{KV}$ (which is not typical of a field structure)

Endow $(V, +_V, \cdot_{KV})$ with a field structure $\bullet_\mathbb{F}$ (which is not typical for a vector space)

Endow $(\mathbb{C}, +_\mathbb{C}, \bullet_\mathbb{C})$ with a linear structure $\cdot_{\mathbb{R}\mathbb{R}^2}$

Algebra over a field

Endow $(\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{R}\mathbb{R}^2})$ with a field structure $\bullet_{\mathbb{R}^2}$ (which mimics $\bullet_\mathbb{C}$)

Identify a specific way to fix $(K, +_K, \bullet_K)$ as a distinguished sub-structure of $(\mathbb{F}, +_\mathbb{F}, \bullet_\mathbb{F})$

Notice that $\mathbb{C} \simeq \mathbb{R}^2$ by fixing a specific representations of $z \in \mathbb{C}$ as an element in $\mathbb{R}^2$ (i.e. fix a way to represent $\mathbb{R}$ as a sub-field of $\mathbb{C}$)

Exercise: build the **real algebra** $\mathbb{C}$:

- Field: $(\mathbb{C}, +_\mathbb{C}, \bullet_\mathbb{C})$ replaces $(\mathbb{F}, +_\mathbb{F}, \bullet_\mathbb{F})$
- Vector space: $(\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{R}\mathbb{R}^2})$ or $(\mathrm{Mat}_2(\mathbb{R}), +_{\mathrm{Mat}_2(\mathbb{R})}, \cdot_{\mathbb{R}\mathrm{Mat}_2(\mathbb{R})})$ replace $(V, +_V, \cdot_{KV})$

  $(\mathbb{R}, +_\mathbb{R}, \bullet_\mathbb{R})$ replaces $(K, +_K, \bullet_K)$

$\to$ build $\mathbb{C}$ as an algebra over $\mathbb{R}$

Algebra (3): algebra $\mathbb{C}$ on $\mathbb{R}$

Exercise: build the **real algebra** $\mathbb{C}$ (i.e. endow $(\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{R}\mathbb{R}^2})$ with a bilinear operator $\bullet_{\mathbb{R}^2}$)
(i.e. endow $(\mathbb{C}, +_{\mathbb{C}}, \bullet_{\mathbb{C}})$ with a 'scalar-vector' product $\cdot_{\mathbb{R}\mathbb{R}^2}$)

Consider $\mathbb{C}$ as a field:

$$(\mathbb{C}, +_{\mathbb{C}}, \bullet_{\mathbb{C}}) \;:\; z \in \mathbb{C} \;:\; z = x_1 + iy_1$$

$$+_{\mathbb{C}} : (x_1 + iy_1) + (x_2 + iy_2) \;=\; (x_1 + x_2) + i(y_1 + y_2) \qquad \bullet_{\mathbb{C}} : (x_1 + iy_1)(x_2 + iy_2) \;=\; (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

We can write $\mathbb{C}$ as the set of ordered pairs of real numbers, i.e. emphasize the isomorphism:

$$\mathbb{C} \simeq \mathbb{R}^2 : (\mathbb{C}, +_{\mathbb{C}}, \bullet_{\mathbb{C}}) \to (\mathbb{R}^2, +_{\mathbb{R}^2}, \bullet_{\mathbb{R}^2})$$

$$\forall\, z = x + iy \;\in \mathbb{C}, \quad z \mapsto (x, y) \in \mathbb{R}^2$$

Denoting $1 = (1,0)$, $i = (0,1)$

We can't introduce $\bullet_{\mathbb{R}^2}$ in $(\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{R}\mathbb{R}^2})$ s.t.:

$$\bullet_{\mathbb{R}^2} : (x_1, y_1)(x_2, y_2) \;=\; (x_1x_2 - y_1y_2,\;\; x_1y_2 + y_1x_2) \quad \text{(i.e. } \bullet_{\mathbb{C}}\text{)}$$

without distinguish $\mathbb{R}$ as represented by $\mathbb{R} \simeq \{(x,0)\,|\,x \in \mathbb{R}\} \subseteq \mathbb{R}^2 \simeq \mathbb{C}$, and letting $\bullet_{\mathbb{R}^2}$ reduce to $\cdot_{\mathbb{R}\mathbb{R}^2}$ for $(x_1, x_2) = (\lambda,0)$:

$$(x_1, y_1) \bullet_{\mathbb{R}^2} (x_2, y_2) \;=\; (\lambda,0) \bullet_{\mathbb{R}^2} (x_2, y_2) \;=\; (\lambda x_2 - 0y_2,\;\; \lambda y_2 + 0x_2) \;=\; (\lambda x_2, \lambda y_2) = \;\lambda \bullet_{\mathbb{R}\mathbb{R}^2} (x_2, y_2)$$

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Algebra (4): algebra $\mathbb{C}$ on $\mathbb{R}$

This is not the unique way to construct the real algebra $\mathbb{C}$, e.g. we can write $\mathbb{C}$ exploiting the isomorphism $\mathbb{C} \to$ Mat$_2(\mathbb{R})$: $\forall\, z = x + iy \,\in \mathbb{C}, z \mapsto \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$

$$\begin{array}{ccccc}
\begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} & \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} & = & \begin{pmatrix} x_1x_2 - y_1y_2 & -x_1y_2 - y_1x_2 \\ y_1x_2 + x_1y_2 & -y_1y_2 + x_1x_2 \end{pmatrix} \\
\text{Mat}_2(\mathbb{R}) \simeq \mathbb{C} \ \big\updownarrow & \big\updownarrow\ \text{Mat}_2(\mathbb{R}) \simeq \mathbb{C} & & \big\updownarrow\ \text{Mat}_2(\mathbb{R}) \simeq \mathbb{C} \\
x_1 + iy_1 & x_2 + iy_2 & = & (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)
\end{array}$$

By doing so, the unit norm condition in $\mathbb{C}$, i.e.:

$$z \in U(1) \;\Leftrightarrow\; |z| = z\bar{z} = \sqrt{x^2 + y^2} = 1, \quad U(1) \subset \mathbb{C}$$

has its counterpart in Mat$_2(\mathbb{R})$ with the following condition:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2) \Leftrightarrow \text{det} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = 1, \quad SO(2) \subset \text{Mat}_2(R)$$

i.e., exploiting Mat$_2(\mathbb{R}) \simeq \mathbb{C}$, $z \in \mathbb{C} : z\bar{z} = 1$ represents a rotation in $\mathbb{R}^2$

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Clifford algebra (0): Clifford algebra $Cl_{2,0}$ on the vector plane $\mathbb{R}^2$

$Cl_{2,0}$ is build in a similar way we built the real algebra $\mathbb{C}$, i.e. endow $(\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{R}\mathbb{R}^2})$ with a bilinear operator. Instead of considering $\bullet_{\mathbb{R}^2}$ as a way to mimic $\bullet_{\mathbb{C}}$, we would like to equipe $\mathbb{R}^2$ with a product which satisfies the following conditions:

- It is associative, distributive and (possibly) commutative (commutativity can't be ensured for $\mathbb{R}^n$ with $n \geq 3$).
- It preserves the norm $|\underline{r}\underline{r}'| = |\underline{r}||\underline{r}'|$, i.e the product equals the square of the length: $\underline{r}\,\underline{r} = \underline{r}^2 \Rightarrow \underline{r}^2 = |\underline{r}|^2$

Consider the vector space $(\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{R}\mathbb{R}^2})$, it is spanned by $\{\underline{e}_1, \underline{e}_2\}$, i.e.:

$$\forall \underline{r} \in \mathbb{R}^2, \quad \underline{r} = x\underline{e}_1 + y\underline{e}_2$$

Endow it with the above-mentioned product (considering $\mathbb{R}^2$ as a field now):

$$a, b \in \mathbb{R}^2 \mapsto ab \in \mathbb{R}^2 \quad s.t.\ a \in \mathbb{R}^2 \mapsto a^2 = |a|^2$$

Using distributivity, without assuming commutativity:

$$a^2 = |a|^2$$

$$(xe_1 + ye_2)^2 = x^2 + y^2$$

$$x^2e_1^2 + y^2e_2^2 + xy(e_1e_2 + e_2e_1) = x^2 + y^2$$

The only way to get satisfied the equation is to set:

$$\boxed{\begin{array}{l} e_1^2 = e_2^2 = 1 \\ e_1e_2 + e_2e_1 = 0 \;\Rightarrow\; e_1e_2 = -e_2e_1, \quad \text{with } e_1e_2 = e_{12} \end{array}}$$

Therefore for two elements of $\mathbb{R}^2$, $r = x_1e_1 + y_1e_2$ and $r' = x_2e_1 + y_2e_2$:

$$\begin{aligned}(x_1e_1 + y_1e_2)(x_2e_1 + y_2e_2) &= x_1x_2e_1^2 + y_1y_2e_2^2 + x_1y_2e_{12} + y_1x_2e_{21} \\ &= (x_1x_2 + y_1y_2) + (x_1y_2 - y_1x_2)e_{12} \\ &= \langle \underline{a}, \underline{b} \rangle + \underline{a} \wedge \underline{b}\end{aligned}$$

i.e. the elements that result from the product are linear combinations of $\{1, e_{12}\}$, which do not belong to $\mathbb{R}^2$. In particular e_{12} squares to -1 and therefore it is neither a scalar nor a vector, it is a **bivector** (i.e. an **oriented plane area**).

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Clifford algebra (1): Clifford algebra $Cl_{2,0}$ on the vector plane $\mathbb{R}^2$

Exactly as in the case of the real algebra $\mathbb{C}$, **we're fixing a specific and unique way to represent** $\mathbb{R}$ **(and, in this case, also** $\mathbb{R}^2$ **and** $\mathbb{C}$**) as subfields of the field (not explicitly stated) that we're trying to turn into an algebra (i.e into the** $Cl_{2,0}$ **real algebra):**

$$\mathbb{R} \simeq \{(x,0,0,0) \,|\, x \in \mathbb{R}\} \simeq \{x1 + 0e_1 + 0e_2 + 0e_{12} \,|\, x \in \mathbb{R}\} \subseteq CL_{2,0}$$

$$\mathbb{R}^2 \simeq \{(0,x,y,0) \,|\, (x,y) \in \mathbb{R}^2\} \simeq \{01 + xe_1 + ye_2 + 0e_{12} \,|\, (x,y) \in \mathbb{R}^2\} \subseteq CL_{2,0}$$

$$\mathbb{C} \simeq \{x1 + 0e_1 + 0e_2 + ye_{12} \,|\, x + iy \in \mathbb{C}\} \subseteq CL_{2,0}$$

$$\left(\bigwedge^2 \mathbb{R}^2 \simeq \{(0,0,0,x) \,|\, x \in \mathbb{R}\} \simeq \{01 + 0e_1 + 0e_2 + xe_{12} \,|\, x \in \mathbb{R}\} \subseteq CL_{2,0}\right)$$

The Clifford product $\bullet_{Cl_{2,0}}$ should:

- equal the product ab (introduced in the previous slide) for $a = x_1e_1 + y_1e_2\,,\, b = x_2e_1 + y_2e_2 \;\in \mathbb{R}^2$
- extend the product ab also for elements $a = x_1 1 + y_1e_{12}\,,\, b = xe_{12}\,, \ldots \;\in \mathbb{C}, \mathbb{R}, \bigwedge^2 \mathbb{R}^2$

Exercise: build the **real algebra** $Cl_{2,0}$. Notice that we could have followed step by step the construction used for the real algebra $\mathbb{C}$, to construct $Cl_{2,0}$. However, doing so would have sacrificed some of the intuitive meaning behind the attempt to introduce the Clifford product directly into $\mathbb{R}^2$:

- Field: $(Cl_{2,0}, +_{Cl_{2,0}}, \bullet_{Cl_{2,0}})$ replaces $(\mathbb{F}, +_{\mathbb{F}}, \bullet_{\mathbb{F}})$
- Vector space: $(\mathbb{R}^4, +_{\mathbb{R}^4}, \cdot_{\mathbb{R}\mathbb{R}^4})$ replace $(V, +_V, \cdot_{KV})$
  $(\mathbb{R}, +_{\mathbb{R}}, \bullet_{\mathbb{R}})$ replaces $(K, +_K, \bullet_K)$

$\rightarrow$ build $Cl_{2,0}$ as an algebra over $\mathbb{R}$

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Clifford algebra (2): Clifford algebra $Cl_{2,0}$ on the vector plane $\mathbb{R}^2$

The Clifford $Cl_{2,0}$ is a 4-dimensional real algebra with a basis $\{1, e_1, e_2, e_{12}\}$ following the multiplication table:

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	1	$\mathbf{e}_{12}$	$\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	1	$-\mathbf{e}_1$
$\mathbf{e}_{12}$	$-\mathbf{e}_2$	$\mathbf{e}_1$	-1

1	$\mathbb{R}$	scalars
$\mathbf{e}_1, \mathbf{e}_2$	$\mathbb{R}^2$	vectors
$\mathbf{e}_{12}$	$\bigwedge^2 \mathbb{R}^2$	bivectors.

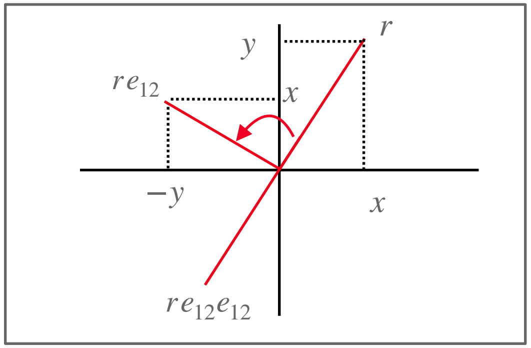
$Cl_{2,0}$ contains copies of $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{C}$ as it is a direct sum of its 0,1,2-graded subspaces:

$$\mathcal{C}\ell_2 = \mathbb{R} \oplus \mathbb{R}^2 \oplus \bigwedge^2 \mathbb{R}^2 \qquad \begin{array}{ll} \text{even part} & \mathcal{C}\ell_2^+ = \mathbb{R} \oplus \bigwedge^2 \mathbb{R}^2 \simeq \mathbb{C} \\ \text{odd part} & \mathcal{C}\ell_2^- = \mathbb{R}^2. \end{array}$$

Clifford product between vector and unit bivector in $Cl_{2,0}$:

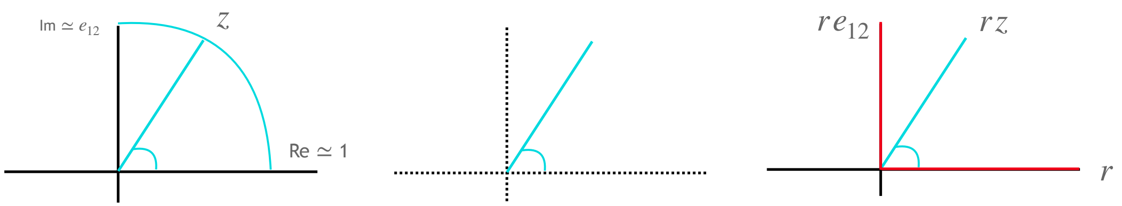
$$\begin{array}{r} e_{12} = 01 + 1e_{12} \\ r = xe_1 + ye_2 \end{array} \Rightarrow \begin{aligned} re_{12} &= xe_1e_1e_2 + ye_2e_1e_2 \\ &= xe_2 - ye_2e_2e_1 \\ &= xe_2 - ye_1 \end{aligned} \Rightarrow \text{Counter-clockwise rotation by } \frac{\pi}{2}$$

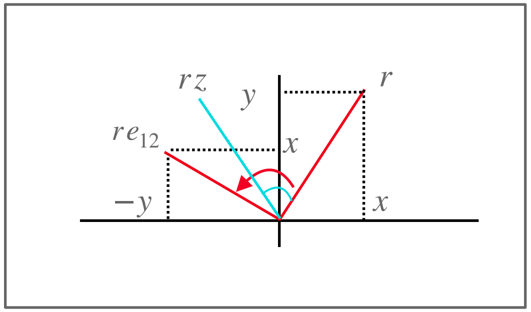
$$re_{12}e_{12} = r(-1) \quad \Rightarrow \text{rotation by } \pi$$



Clifford product between vector and unit complex number in $Cl_{2,0}$:

$$\begin{array}{l} z = \cos\theta + i\sin\theta \\ r = xe_1 + ye_2 \end{array} \Rightarrow \begin{aligned} rz &= r\cos\theta + ri\sin\theta \\ &= r\cos\theta + re_{12}\sin\theta \end{aligned} \Rightarrow \text{Counter-clockwise rotation by } \theta$$





while $z = \cos\theta + i\sin\theta$ had coordinates $(\cos\theta, \sin\theta)$ w.r.t. basis $\{1, e_{12}\}$,
$rz = r\cos\theta + re_{12}\sin\theta$ has the same coordinates $(\cos\theta, \sin\theta)$ but w.r.t. basis $\{r1, re_{12}\}$

Quaternions (0)

Quaternions $\mathbb{H}$ are generalized complex numbers (i.e. hypercomplex numbers) characterized by three imaginary units i, j, k. They form the anti-commutative field ($\mathbb{H}$, $+_{\mathbb{H}}$, $\bullet_{\mathbb{H}}$) in which:

$$\forall q \in \mathbb{H}, \quad q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad \Rightarrow q = (q_0, q_1, q_2, q_3)^T,$$ **exploiting:** $\mathbb{H} \simeq \mathbb{R}^4$, **as we did for** $\mathbb{C} \simeq \mathbb{R}^2$

$$+_{\mathbb{H}} : q + p \ = \ q = (q_0 + p_0) + (q_1 + p_1)\mathbf{i} + (q_2 + p_2)\mathbf{j} + (q_3 + p_3)\mathbf{k}$$

$$\begin{aligned}\bullet_{\mathbb{H}} : pq &= (p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) \\ &= p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3 + \\ &\quad +(p_1q_0 + p_0q_1 - p_3q_2 + p_2q_3)\mathbf{i} + \\ &\quad +(p_2q_0 + p_3q_1 + p_0q_2 - p_1q_3)\mathbf{j} + \\ &\quad +(p_3q_0 - p_2q_1 + p_1q_2 + p_0q_3)\mathbf{k}\end{aligned}$$

$$\Rightarrow pq = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

$$= p_0q_0 + \langle \underline{p}\,\underline{q} \rangle + p_0\underline{q} + \underline{p}q_0 + \underline{p} \times \underline{q}$$

Involutions:

$$\forall q \in \mathbb{H}, \quad q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \quad \Rightarrow \bar{q} = q_0 - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$$ Conjugate

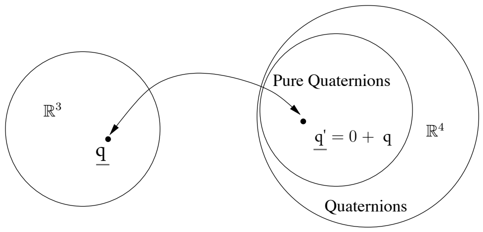
$$\Rightarrow \tilde{q} = q_0 + q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}$$ Reverse

$$\Rightarrow \hat{q} = q_0 - q_1\mathbf{i} + q_2\mathbf{j} - q_3\mathbf{k}$$ Grade involuted

Norm: $\forall q \in \mathbb{H}, \ |q| = \sqrt{q\bar{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$

Inverse: $\forall q \neq 0 \in \mathbb{H}, \ q^{-1} = \dfrac{\bar{q}}{|q|^2}$

Pure quaternions: quaternion whose real part is zero



$\forall q \in$ **Pu**$(\mathbb{H})$, $\ \bar{q} = -q$

Unit quaternions: quaternion whose norm is 1

$q \in$ **Unit**$(\mathbb{H})$, $\ \Leftrightarrow \ |q| = 1$

Quaternions (1): group $SU(2)$

Recall that $SU(2)$ is defined as :

$$SU(2) = \{A \in GL_2(C)\,|\,A^*A = 1_2,\ \det(A) = 1\}$$

$SU(2)$ stands for $(SU(2), \bullet_{Mat_2(\mathbb{C})})$

Let A be expressed as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}^T$,

with: $m_{ij} = (-1)^{i+j}\det(A_{ij})$

Condition $A^*A = 1_2$ implies $A^{-1} = A^*$, let's compute A^{-1} with Cramer's method

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$, since $A \in SU(2)$ then $\det(A) = 1 \quad \Rightarrow \quad A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

for $A^{-1} = A^*$, we must have $A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} = A^*$, i.e. it must be: $d = \bar{a},\quad b = -\bar{c}$

$$SU(2) = \left\{A \in GL_2(C)\,|\,A = \begin{pmatrix} a & b \\ -\bar{b} & -\bar{a} \end{pmatrix}\right\}$$

Let be $a = x_0 + ix_1$ and $b = x_1 + ix_2$, therefore any element in $SU(2)$ can be written as a linear combination of basis elements:

$$A = \underbrace{x_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{1}} + \underbrace{x_1\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}}_{\mathbf{i}} + \underbrace{x_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\mathbf{j}} + \underbrace{x_3\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}}_{\mathbf{k}}$$

Satisfying, w.r.t. $\bullet_{Mat_2(\mathbb{C})}$:

$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$

$$SU(2) \simeq \text{Unit}(\mathbb{H}) = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1\} \subset \mathbb{H}$$

Quaternions (2): unit and pure quaternions

Notice that $(x_0, x_1) : x_0^2 + x_1^2 = 1$ is an element of S^1 , this means that $(x_0, x_1, x_2) : x_0^2 + x_1^2 + x_2^2 = 1$ is an element of S^2 and a unit quaternion $(x_0, x_1, x_2, x_3) : x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1$, exploiting the isomorphism between $\mathbb{H}$ and $\mathbb{R}^4$, is an element of S^3, therefore:

$$\textbf{Unit}(\mathbb{H}) \simeq SU(2) \simeq S^3$$

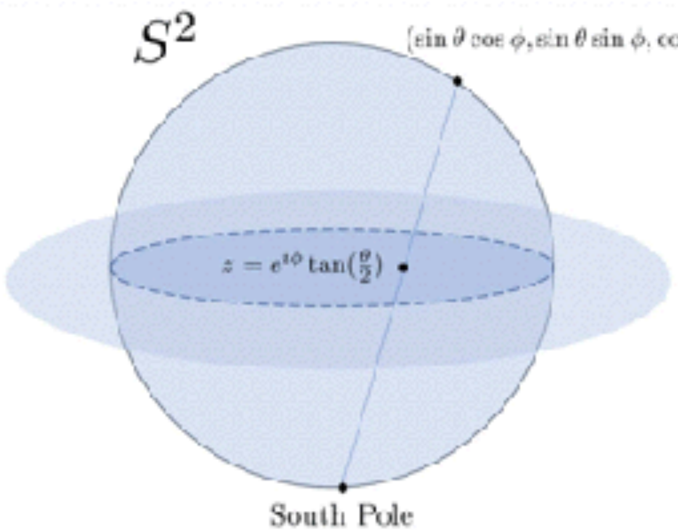
To understand the geometry of S^3, let's look more closely to S^2. A 'location' in S^2 can be represented by **latitudes** and **longitudes**.

- Latitudes: horizontal slices of the sphere (a point at the north, circles of increasing radius, equator, circle of decreasing radius and a point at the south pole):

$$S^2 = \bigcup_{c=-1}^{c=1} \{(x_0, x_1, x_2) \in S^2 \,|\, x_0 = c\}$$

  the equator is:

$$E^2 = \{(x_0, x_1, x_2) \in S^2 \,|\, x_0 = 0\}$$



In 4 dimensions (i.e considering S^3) latitudes are still horizontal slices, but they are 2D spheres of different radius:

$$S^3 = \bigcup_{c=-1}^{c=1} \{(x_0, x_1, x_2, x_3) \in S^3 \,|\, x_0 = c\}$$

the equator in S^3 is isomorphic to the pure quaternions : $E^3 = \{(x_0, x_1, x_2, x_3) \in S^3 \,|\, x_0 = 0\} \simeq \textbf{Pu}(\mathbb{H})$

Clifford algebra (3): Clifford algebra $Cl_{0,2}$ on the anti-euclidean space $\mathbb{R}^{0,2}$

The Clifford $Cl_{0,2}$ is a 4-dimensional real algebra with a basis $\{1, e_1, e_2, e_{12}\}$ following the multiplication table:

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_{12}$
$\mathbf{e}_1$	-1	$\mathbf{e}_{12}$	$-\mathbf{e}_2$
$\mathbf{e}_2$	$-\mathbf{e}_{12}$	-1	$\mathbf{e}_1$
$\mathbf{e}_{12}$	$\mathbf{e}_2$	$-\mathbf{e}_1$	-1

$CL_{0,2} \simeq \mathbb{H}$

$\mathbf{1}$	$\mathbf{1}$
e_1, e_2	$\mathbf{i}, \mathbf{j}$
e_{12}	$\mathbf{k}$

$Cl_{0,2}$ contains copies of $\mathbb{R}$ and $\mathbb{R}^3$ as it is a direct sum of its 0,1,2-graded subspaces:

$$Cl_{0,2} = \mathbb{R} \oplus \mathbb{R}^3 \simeq \mathbb{H} \simeq \mathbb{R}^4$$

$Cl_{0,2}$ is build endowing $(\mathbb{R}^2, +_{\mathbb{R}^2}, \cdot_{\mathbb{R}\mathbb{R}^2})$ with a bilinear operator which satisfies the following conditions:

- It is associative, distributive and (anti) commutative.
- It preserves the norm $|\underline{r}\,\underline{r}'| = |\underline{r}||\underline{r}'|$, i.e the product equals the square of the length: $\underline{r}\,\underline{r} = \underline{r}^2 \Rightarrow \underline{r}^2 = -|\underline{r}|^2$

Rotation of a vector in $\underline{r} \in \mathbb{R}^3$ along an arbitrary rotation axis $\underline{u} \in \mathbb{R}^3$:

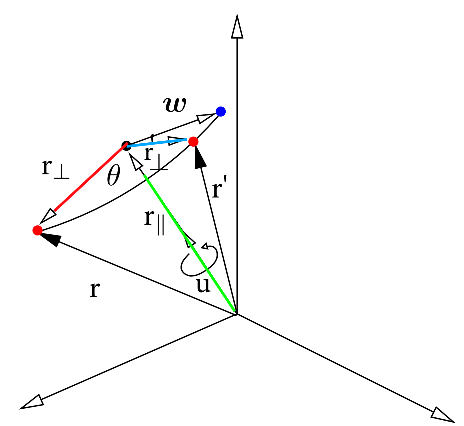
$$\underline{r} = (x, y, z) \qquad \underline{u} = (u_1, u_2, u_3) \Rightarrow \underline{r} = \underline{r}_{\|u} + \underline{r}_{\perp u} \Rightarrow$$

$$\underline{r}_{\|u} = \langle \underline{r}\,\underline{u} \rangle \underline{u}$$

Not affected by the rotation

$$\underline{r}_{\perp u} = \underline{r} - \underline{r}_{\|u} = \underline{r} - \langle \underline{r}\,\underline{u} \rangle \underline{u}$$

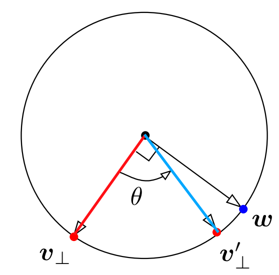
Affected by the rotation



Recall that a rotation on a plane is a 2D vector multiplied by a unit complex number, i.e, $\underline{r}'_{\perp u}$ has coordinates $(\cos\theta, \sin\theta)$ w.r.t basis $\{\underline{r}_{\perp u}, \underline{w}\}$:

$$\underline{r}'_{\perp u} = \underline{r}_{\perp u} \cos\theta + \underline{w} \sin\theta$$

where: $\underline{w} = \underline{u} \times \underline{r}_{\perp u} = \underline{u} \times \underline{r}$



Clifford algebra (3): Clifford algebra $Cl_{0,2}$ on the anti-euclidean space $\mathbb{R}^{0,2}$

$$
\begin{aligned}
\underline{r}_{\perp u} &= \underline{r}_{\parallel u} + \underline{r}'_{\perp u} \\
&= \langle \underline{r}, \underline{u} \rangle \underline{u} + \underline{r}_{\perp u} \cos\theta + \underline{w} \sin\theta \\
&= \langle \underline{r}, \underline{u} \rangle \underline{u} + \big(\underline{r} - \langle \underline{r}, \underline{u} \rangle \underline{u}\big) \cos\theta + (\underline{u} \times \underline{r}) \sin\theta \\
&= \langle \underline{r}, \underline{u} \rangle \underline{u} + \underline{r} \cos\theta - \langle \underline{r}, \underline{u} \rangle \underline{u} \cos\theta + (\underline{u} \times \underline{r}) \sin\theta \\
&= \underline{u} \langle \underline{r}, \underline{u} \rangle \big(1 - \cos\theta\big) + \underline{r} \cos\theta + (\underline{u} \times \underline{r}) \sin\theta
\end{aligned}
$$

$$\underline{r}_{\parallel u} = \langle \underline{r}\, \underline{u} \rangle \underline{u}$$

$$\underline{r}'_{\perp u} = \underline{r}_{\perp u} \cos\theta + \underline{w} \sin\theta$$

$$
\begin{aligned}
\underline{r}_{\perp u} &= \underline{r} - \underline{r}_{\parallel u} \\
&= \underline{r} - \langle \underline{r}\, \underline{u} \rangle \underline{u}
\end{aligned}
$$

$$\underline{w} = \underline{u} \times \underline{r}_{\perp u} = \underline{u} \times \underline{r}$$

Rodrigues' rotation formula, which can be also written as matrix multiplication using $A_{\underline{u}}(\theta) \in SO(3)$:

$$
A_{\underline{u}}(\theta) = \begin{pmatrix}
u_1^2(1-\cos\theta) + \cos\theta & u_1 u_2 (1-\cos\theta) - u_3 \sin\theta & u_1 u_3 (1-\cos\theta) + u_2 \sin\theta \\
u_1 u_2 (1-\cos\theta) + u_3 \sin\theta & u_2^2(1-\cos\theta) + \cos\theta & u_2 u_3 (1-\cos\theta) - u_1 \sin\theta \\
u_1 u_3 (1-\cos\theta) - u_2 \sin\theta & u_2 u_3 (1-\cos\theta) + u_1 \cos\theta & u_3^2(1-\cos\theta) + \cos\theta
\end{pmatrix}
$$

$$
\underline{r}\cos\theta = \begin{pmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 0 & \cos\theta \end{pmatrix} \underline{r}
\qquad
\underline{u} \times \underline{r} = \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix} \underline{r}
\qquad
\underline{u} \langle \underline{u}\, \underline{r} \rangle = \begin{pmatrix} u_1^2 & u_1 u_2 & u_1 u_3 \\ u_2 u_1 & u_2^2 & u_2 u_3 \\ u_3 u_1 & u_3 u_2 & u_3^2 \end{pmatrix} \underline{r}
$$

Clifford algebra (3): Clifford algebra $Cl_{0,2}$ on the anti-euclidean space $\mathbb{R}^{0,2}$

Clifford product between vector $\underline{r}$ (pure quaternion) and unit quaternions q in $Cl_{0,2} \simeq \mathbb{H}$:

TH: For any unit quaternion $q \in \text{Unit}(\mathbb{H})$ and for any vector (pure quaternion) $r = 0 + \underline{r} \in \text{Pu}(\mathbb{H})$, s.t. $\underline{r} \in \mathbb{R}^3$, the mapping:

$$\mathbb{R}^3 \to \mathbb{R}^3 \quad : \quad \underline{r} \mapsto q\underline{r}q^{-1} \quad = (-q)\underline{r}(-q)^{-1}$$

Is a rotation of $\underline{r}$ towards an axis $\underline{u}$.

Fact: any unit quaternion $q \in \text{Unit}(\mathbb{H})$ has the following trigonometric expression:

$$q = q_0 + \underline{q} = \cos\frac{\theta}{2} + \underline{u}\sin\frac{\theta}{2} = e^{\underline{u}\frac{\theta}{2}}, \quad s.t.\ \underline{u} = \frac{\underline{q}}{|\underline{q}|} \text{ implying } |\underline{q}| = \sin\frac{\theta}{2}$$

Proof:

$$q\underline{r}q^{-1} = (q_0 - |\underline{q}|^2)\underline{r} + 2(\langle \underline{q}, \underline{r}\rangle)\underline{q} + 2q_0(\underline{q} \times \underline{r})$$

$$= (\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2})\underline{r} + 2\langle \underline{u}\sin\frac{\theta}{2}, \underline{r}\rangle \underline{u}\sin\frac{\theta}{2} + 2\cos\frac{\theta}{2}(\underline{u}\sin\frac{\theta}{2} \times \underline{r})$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$ $\qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$ $\qquad \sin 2\theta = 2\cos\theta\sin\theta$

(Duplication and bisection formulas)

$$= \underline{r}\cos\theta + \underline{u}\langle \underline{u}, \underline{r}\rangle(1-\cos\theta) + (\underline{u} \times \underline{r})\sin\theta$$

Which is exactly the Rodrigues' rotation formula

But also $q\underline{r}q^{-1} = (-q)\underline{r}(-q)^{-1}$, therefore $SO(3) \simeq \text{Unit}(\mathbb{H})/\{\pm 1\}$ (Notice that we have also previously proved that $SU(2) \simeq$ **Unit($\mathbb{H}$)**)

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Clifford algebra (4): Clifford algebra $Cl_{0,2}$ on the anti-euclidean space $\mathbb{R}^{0,2}$

Clifford product between vector $\underline{r}$ in $\mathbb{R}^4$ (quaternion) and a pair of unit quaternions q, p in $Cl_{0,2} \simeq \mathbb{H}$:

TH: For any pair unit quaternions $q, p \in$ Unit($\mathbb{H}$) and for any vector (in $\mathbb{R}^4$) $r = r_0 + \underline{r} \in \mathbb{H}$, the mapping:

$$\mathbb{R}^4 \to \mathbb{R}^4 \quad : \quad r \mapsto qrp^{-1} \quad = (-q)\underline{r}(-p)^{-1}$$

is a rotation in 4 dimensions (i.e. $SO(4) \simeq$ Unit($\mathbb{H}$) $\times$ Unit($\mathbb{H}$)$/\{(1,1),(-1,-1)\}$)

5.6 Matrix representation of quaternion multiplication

The product of two quaternions $q = w + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$ and $u = u_0 + \mathbf{i}u_1 + \mathbf{j}u_2 + \mathbf{k}u_3$ can be represented by matrix multiplication:

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & -z & y \\ y & z & w & -x \\ z & -y & x & w \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

where $qu = v$. Swapping the multiplication to the right, that is, $uq = v'$, gives a partially transformed matrix:

$$\begin{pmatrix} w & -x & -y & -z \\ x & w & z & -y \\ y & -z & w & x \\ z & y & -x & w \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} v'_0 \\ v'_1 \\ v'_2 \\ v'_3 \end{pmatrix} .$$

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