Name : Subhojit Ghimire

Scholar Id. : 1912160

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Q.1.

Soln :- Given,
$$f(x) = \begin{cases} -\pi & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$$

Here, limit is $[-\pi, \pi]$

So, $b-a = \pi - (-\pi) = 2\pi$

Now, fourier series is defined as,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{b-a}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{b-a}\right)$$

Substituting $b-a = 2\pi$,
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Calculating separately for a_0, a_n, b_n by Euler's formula,
$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$
$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\,dx + \int_0^{\pi} x\,dx\right]$$
$$= \frac{1}{\pi}\left[-\pi\,|x|_{-\pi}^{0} + \left|\frac{x^2}{2}\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(0-(-\pi)\right) + \frac{1}{2}\left((\pi)^2 - 0^2\right)\right]$$

$$= \frac{1}{\pi}\left[-\pi^2 + \frac{1}{2}\pi^2\right]$$

$$= \frac{1}{\pi} \times \left(\frac{-\pi^2}{2}\right)$$

$$\therefore a_0 = -\frac{1}{2}\pi$$

Solving for a_n.

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{2n\pi x}{b-a}\right)dx$$

$$= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi \cos nx \, dx + \int_0^{\pi} x \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[-\pi \left| \frac{\sin nx}{n} \right|_{-\pi}^{0} + \left| x\frac{\sin x}{n} + \frac{\cos nx}{n^2} \right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi \frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]$$

We know, $\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$\therefore a_n = \frac{(-1)^n - 1}{\pi n^2}$$

Checking for odd and even possibilities,

$$a_n = \frac{-2}{\pi n^2}, \text{ if } n \text{ is odd}$$

and $a_n = 0$, if n is even.

i.e., only odd case exists.

$$\therefore a_n = \frac{-2}{\pi n^2}$$

Solving for b_n,

$$b_n = \frac{2}{b-a}\int_a^b f(x).\sin\left(\frac{2n\pi x}{b-a}\right)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x).\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,\sin nx\, dx + \int_{\pi}^{0} x\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[(-\pi)\left|\frac{\cos nx}{n}\right|_{-\pi}^{0} + \left| x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-(-1)^n) - \pi\frac{(-1)^n}{n}\right]$$

$$\therefore b_n = \frac{1}{n}[1-2(-1)^n]$$

Substituting the calculated values of a_0, a_n and b_n in the main fourier series equation,

$$f(x) = \frac{\left(-\frac{1}{2}\right)\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{-2}{\pi n^2}\right)\cos nx + \sum_{n=1}^{\infty}\left(\frac{1-2(-1)^n}{n}\right)\sin nx$$

$$\therefore f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \cdots\right) + \left(3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} + \cdots\right)$$

This is the required solution.

Q.2.

Sol^n:- Given, $f(x) = \begin{cases} 1-x^2, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$

Applying fourier cosine transform,

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x \, dx$$

$$= \int_0^1 (1-x^2)\cos\omega x \, dx$$

$$= \left[\frac{(1-x^2)\sin\omega x}{\omega}\right]_0^1 - \int_0^1 \frac{(-2x)\sin\omega x}{\omega} dx$$

$$= 2\left[\frac{x(-\cos\omega x)}{\omega^2}\right]_0^1 + \int_0^1 \frac{\cos\omega x}{\omega^2} dx$$

$$= 2\left[\frac{-\cos\omega}{\omega^2} + \frac{\sin\omega x}{\omega^2}\right]_0^1$$

$$\therefore F_c(\omega) = 2\left[\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right]$$

Applying inverse fourier cosine transform,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos\omega x \, d\omega$$

$$= \frac{2}{\pi}\int_0^\infty 2\left[\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right]\cos\omega x \, d\omega$$

$$= \frac{4}{\pi}\int_0^\infty \left(\frac{\sin\omega - \omega\cos\omega}{\omega^3}\right)\cos\omega x \, d\omega$$

$$= f(x)$$

$$= \begin{cases} 1-x^2 ; & 0 \le x \le 1 \\ 0 ; & x > 1 \end{cases}$$

Hence, Proved.

Q.3.

Sol^n:-

Given, $f(x) = e^{-ax}$, $a > 0$.

We know,

fourier sine transform,

$$F_s(s) = \int_0^{\infty} f(x) \sin sx \, dx$$

where, $f(x)$ in $0 < x < \infty$

So,

$$F_s(s) = \int_0^{\infty} e^{-ax} \sin sx \, dx$$

$$= \left| \frac{e^{-ax}}{(-a)^2 + (s)^2} \left((-a)\sin sx - s\cos sx \right) \right.$$

$$= \left| \frac{e^{-ax}}{a^2 + s^2} \left(-a\sin sx - s\cos sx \right) \right|_0^{\infty}$$

$$\therefore F_s(s) = \frac{s}{a^2 + s^2}$$

This is the required solution.

Q.04.

Soln:- Given, $\frac{\delta^2 x}{\delta x^3} - \frac{\delta^3 z}{\delta x \delta y^2} - \frac{\delta^2 z}{\delta x^2} + \frac{\delta^2 z}{\delta x \delta y} = \frac{x+2}{x^3}$

Or, $(D^3 - DD'^2 - D^2 + DD')z = \frac{x+2}{x^3}$

finding C.F.,

The auxiliary eqn is,

$$m^3 - m - m^2 + m = 0$$

$$\text{or, } m^3 - m^2 = 0$$

$$\text{or, } m^2(m-1) = 0$$

$$\therefore m = 0, 0, 1$$

$\therefore$ C.F. $= f_1(y) + f_2(y+x) + e^x f_3(y-x)$

finding P.I.,

$$P.I. = \frac{1}{(D+D'-1)(D-D')} \int \left(\frac{1}{x^2} + \frac{2}{x^3}\right) dx$$

$$= \frac{1}{(D+D'-1)(D-D')} \left(\frac{-1}{x} + 2\left(\frac{x^{-2}}{-2}\right)\right)$$

$$= \frac{1}{(D+D'-1)(D-D')} \left(\frac{-1}{x^2} - \frac{1}{x}\right)$$

$$= \frac{1}{(D+D'-1)} \int \left(\frac{-1}{x^2} - \frac{1}{x}\right) dx$$

$$= \frac{1}{(D+D'-1)} \left(\frac{1}{x} - \log x\right)$$

$$= e^x \int e^{-x} \left(\frac{1}{x} - \log x\right) dx$$

$\therefore PI = \log x.$

So, the PDE solution is,

$$z = CF + PI$$

$$\therefore z = f_1(y) + f_2(y+x) + e^x f_3(y-x) + \log x.$$

Q. 5.

Sol^n:- Given, $\frac{\partial^2 u}{\partial t^2} = \frac{c^2 \partial^2 u}{\partial x^2}$, c is constant

Let the solution of wave equation be,

$$u(x,t) = T(t).X(x)$$

Now,

$$\frac{\partial u}{\partial x} = TX' \quad ; \quad \frac{\partial^2 u}{\partial x^2} = TX''$$

and,

$$\frac{\partial u}{\partial t} = T'X \quad ; \quad \frac{\partial^2 u}{\partial t^2} = T''X$$

So, the given equation can be written as,

$$TX'' = c^2 T''X$$

or,

$$\frac{T''}{c^2 T} = \frac{X''}{X} = K \quad (\text{say})$$

i.e, $\frac{T''}{c^2 T} = K$ and $\frac{X''}{X} = K$.

or, $T'' - Kc^2 T = 0$ — (i)

and $X'' - KX = 0$ — (ii)

Case I : When $K = 0$,

$$T'' = 0 \Rightarrow T' = C_1 \Rightarrow T = C_1 t + C_2$$

Similarly, $X'' = 0 \Rightarrow X' = C_3 \Rightarrow X = C_3 x + C_4$

So, the solution for this is given by,

$$u(x,t) = (C_1 t + C_2)(C_3 x + C_4)$$

Case II : When $K > 0$

Let $K = n^2$

So,

$T'' - n^2c^2T = 0$ and $X'' - n^2X = 0$

or, $(D^2 - n^2c^2)T = 0$ $\left(D = \frac{d}{dt} ; D' = \frac{d}{dx}\right)$

and $(D'^2 - n^2)X = 0$.

finding auxiliary equation, replacing D with m and D' with 1,

for the function of T,

$m^2 - n^2c^2 = 0$

$\therefore m = \pm nc$

$\therefore T = c_1e^{nct} + c_2e^{-nct}$

for the function of X,

$m^2 - n^2 = 0$

$\therefore m = \pm n$

$\therefore X = c_3e^{nx} + c_4e^{-nx}$

So, the solution for this is given by,

$u(x,t) = [c_1e^{nct} + c_2e^{-nct}][c_3e^{nx} + c_4e^{-nx}]$

Case III : when $K < n$

Let, $k = -n^2$

So,

$T'' + n^2c^2T = 0$ and $X'' + n^2X = 0$

or, $(D^2 + n^2c^2)T = 0$

and $(D'^2 + n^2)X = 0$

finding auxiliary equation,

for function of T,

$m^2 + n^2c^2 = 0$

$m = \pm inc$

$\therefore T = c_1\cos nct + c_2\sin nct$

for function of X,

$$m^2 + n^2 = 0$$

$$\therefore m = \pm in$$

$$\therefore X = C_3 \cos nx + C_4 \sin nx$$

$\therefore$ the solution for this is given by,

$$u(x,t) = (C_1 \cos nct + C_2 \sin nct)(C_3 \cos nx + C_4 \sin nx)$$