MATHEMATICS ASSIGNMENT

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1. Evaluate: $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}$

Soln:- Let $z = e^{i\theta}$

or, $dz = ie^{i\theta}\, d\theta = iz\, d\theta$

$\therefore d\theta = \dfrac{dz}{iz}$

So,

$$\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2} = \frac{1}{iz}\oint_c \frac{1}{\left(2+\frac{z+\frac{1}{z}}{2}\right)^2}\, dz \quad , \text{ where, } c \text{ is } |z|=1$$

$$= -4i \oint_c \frac{z}{(z^2+4z+1)^2}\, dz$$

$$= -4i \oint_c \frac{z}{(z+2-\sqrt{3})^2(z+2+\sqrt{3})^2}\, dz$$

$$= -4i \oint_c f(z)\, dz \quad , \text{ where, } f(z) = \frac{z}{(z+2-\sqrt{3})^2(z+2+\sqrt{3})^2}$$

Poles of $f(z)$ are given by,

$$(z+2-\sqrt{3})^2(z+2+\sqrt{3})^2 = 0$$

$\Rightarrow z = -2+\sqrt{3}$, which is pole of order 2

$\Rightarrow z = -2-\sqrt{3}$, which is pole of order 2.

Pole, $z = -2-\sqrt{3}$ lies outside the contour circle.

Residue at $z = -2+\sqrt{3}$ (R_1)

$$= \frac{1}{1!} \frac{d}{dz} \frac{z(z+2-\sqrt{3})^2}{(z+2+\sqrt{3})^2(z+2-\sqrt{3})^2}$$

$$= \frac{d}{dz} \frac{(z)}{(z+2+\sqrt{3})^2}$$

$$= \frac{(z+2+\sqrt{3})^2 - z \cdot 2(z+2+\sqrt{3})}{(z+2+\sqrt{3})^4}$$

$(\because z = -2+\sqrt{3})$

$$\therefore R_1 = \frac{(2\sqrt{3})^2 - 2(-2\sqrt{3})(2\sqrt{3})}{(2\sqrt{3})^4}$$

$$= \frac{\sqrt{3}}{18}$$

By Residue Theorem,

$$-4i \oint_c f(z)\,dz = -4i \times 2\pi i \times R_1$$

$$= 2\pi \times 4 \times \frac{\sqrt{3}}{18}$$

$$= \frac{4\pi\sqrt{3}}{9}$$

$$\therefore -4i \oint_c f(z)\,dz = \frac{4\pi\sqrt{3}}{9}$$

2. Use Newton-Raphson method to find all the roots of the equation $2-x^2 = \sin x$ correct to six places after decimal.

Soln:-

$$2 - x^2 = \sin x$$

$$\therefore f(x) = x^2 - 2 + \sin x = 0$$

$$\therefore f'(x) = 2x + \cos x$$

Here,

$$f(1) = -0.158 < 0$$

$$f(2) = 2.909 > 0$$

Hence, there lies a root between $x=1$ and $x=2$. Let, $x_0 = 1.1$ be our initial approximation.

By Newton Rapson's formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.061800$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.061549$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.061549$$

$\therefore x_2 - x_3 \Rightarrow x = 1.061549$ is a root of $f(x)$

Also,

$$f(-1) = -1.841$$

$$f(-2) = 1.090$$

There lies a root between $x=-2$ and $x=-1$

Let, $x_0 = -1.7$ be our initial approximation.

By Newton Rapson's formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1.728809$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.728466$$

$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = -1.728466$$

$$\therefore x_2 = x_3$$

$\therefore x = -1.728466$ is a root of $f(x)$

3. The matrix $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ is transformed to the diagonal form $D = T^{-1}AT$ where, $T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. Find the value of θ which gives diagonal transformation.

Soln:

$$A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

$$T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$|T| = \cos\theta + \sin^2\theta = 1$$

$$\therefore T^{-1} = \frac{1}{|T|} adj.[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We have,

$$D = T^{-1}AT$$

or, $$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

or, $$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} a\cos\theta + h\sin\theta & h\cos\theta + b\sin\theta \\ -a\sin\theta + h\cos\theta & -h\sin\theta + b\cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

or, $$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} a\cos^2\theta + h\sin\theta\cos\theta + h\cos\theta\sin\theta + b\sin^2\theta & -a\sin\theta\cos\theta - h\sin^2\theta + h\cos^2\theta + b\sin\theta\cos\theta \\ -a\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta + b\sin\theta\cos\theta & a\sin^2\theta - h\cos\theta\sin\theta - h\cos\theta\sin\theta + b\cos^2\theta \end{bmatrix}$$

Equating corresponding elements,
Let's take elements from first column, second row:

$$-a\sin\theta\cos\theta + h\cos^2\theta - h\sin^2\theta + b\sin\theta\cos\theta = 0$$

Or, $\sin\theta\cos\theta(b-a) + h\cos 2\theta = 0$

Or, $h\cos 2\theta = \frac{(a-b)}{2}\sin 2\theta$

Or, $\frac{2h}{a-b} = \tan 2\theta$

Or, $2\theta = \tan^{-1}\left(\frac{2h}{a-b}\right)$

$\therefore \theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$

This is the required value of θ.