FOURIER SERIES AND FOURIER TRANSFORM

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Q.1. Obtain a half range cosine series for,

$$f(x) = \begin{cases} Kx, & 0 \le x \le l/2 \\ K(l-x), & l/2 \le x \le l \end{cases}$$

Hence deduce, $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty = \frac{\pi^2}{8}$

Sol^n:- Half range cosine series for f(x) is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right)$$, interval given by $(0, c)$

for the given question, interval is $(0, l)$.

Now

$$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$$

$$= \frac{2}{l} \left\{ \int_0^{l/2} Kx\,dx + \int_{l/2}^{l} K(l-x)\,dx \right\}$$

$$= \frac{2}{l} \left\{ \frac{K}{2} \left[x^2\right]_0^{l/2} + Kl\left[x\right]_{l/2}^{l} - \frac{K}{2}\left[x^2\right]_{l/2}^{l} \right\}$$

$$= \frac{2}{l} \left\{ \frac{Kl^2}{8} + \frac{Kl^2}{2} - \frac{3Kl^2}{8} \right\}$$

$$= \frac{Kl^2 + 4Kl^2 - 3Kl^2}{4l}$$

$$= \frac{Kl}{2} \qquad \text{— (i)}$$

Again,

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \left\{ \left[\frac{x \sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} + \frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right]_0^{l/2} \right.$$

$$\left. + \left[(l-x)\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} + (-1)\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2} \right]_{l/2}^{l} \right\}$$

$$= \frac{2k}{l} \cdot \frac{l^2}{n^2\pi^2} \left\{ \frac{1}{2} \sin\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right) - 1 - \frac{1}{2} \sin\left(\frac{n\pi}{2}\right) - \left((-1)^n - \cos\left(\frac{n\pi}{2}\right)\right) \right\}$$

$$= \frac{2kl}{n^2\pi^2} \left[2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1 \right] \quad \text{— (ii)}$$

So, from (i) and (ii), we get,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{kl}{4} + \frac{2kl}{\pi^2} \sum_{n=1}^{\infty} \left\{ \left[\frac{2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1}{n^2} \right] \right\}$$

$$= \frac{kl}{4} + \frac{2kl}{\pi^2} \left(0 - 1 + 0 - 0 + 0 - \frac{4}{36} + 0 - 0 + 0 - \frac{4}{100} \cdots \right)$$

$$= \frac{kl}{4} - \frac{2kl}{\pi^2} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty \right)$$

Taking $x = l$,

$$f(x) = 0$$

Or, $$\frac{2kl}{\pi^2} \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty \right) = \frac{kl}{4}$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \infty = \frac{\pi^2}{8}$$

Proved

Q.2. Express $f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{for } |x| > 1 \end{cases}$

as a fourier sine integral. Hence evaluate $\int_0^\infty \frac{\sin\lambda x \cos\lambda x}{\lambda} d\lambda$

Soln: fourier sine integral of $f(x)$,

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin\lambda x \int_0^x f(t) \sin\lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \int_0^x 1 \cdot \sin\lambda t \, dt \, d\lambda$$

$$= \frac{2}{\pi} \int_0^\infty \sin\lambda x \frac{[\cos\lambda t]_0^x}{\lambda} d\lambda$$

Or, $$f(x) = \frac{2}{\pi} \int_0^\infty \sin\lambda x \frac{(1 - \cos\lambda x)}{\lambda} d\lambda \quad \text{— (i)}$$

$$\therefore \frac{\pi}{2} f(x) = \int_0^\infty \frac{\sin\lambda x}{\lambda} - \int_0^\infty \frac{\sin\lambda x \cos\lambda x}{\lambda} d\lambda$$

$$= \int_0^\infty \frac{\sin \lambda u}{\lambda u} - \frac{\pi}{2} f(x)$$

$$\therefore \int_0^\infty \frac{\sin \lambda x \cos \lambda x}{\lambda} d\lambda = \frac{\pi}{2} - \frac{\pi}{2} f(x)$$

Now, $$\int_0^\infty \frac{\sin \lambda x \cos \lambda x}{\lambda} d\lambda = \begin{cases} 0, & |x| \le 1 \\ \pi/2, & |x| > 1 \end{cases}$$

At $x=1$, which is the point of discontinuity of $f(x)$, the value of the above integral,

i.e., $$\int_0^x \frac{\sin \lambda x \cos \lambda x}{\lambda} dx \Big|_{at\, x=1} = \frac{\pi}{2} - \frac{\pi}{2}\left[\frac{f(1^-)+f(1^+)}{2}\right]$$

$$= \frac{\pi}{2} - \frac{\pi}{2}\left[\frac{0+\pi/2}{2}\right]$$

$$= \frac{\pi}{2} - \frac{\pi^2}{8}$$

$$= \frac{\pi(4-\pi)}{8}$$

Q.3. Find fourier transform of $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$

Sol^n:- F.T. of $f(x)$ is,

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx} dx$$

$$= \int_{-\infty}^{-a} 0 . e^{isx} dx + \int_{-a}^{a} x^2 . e^{isx} dx + \int_a^\infty 0 . e^{isx} dx$$

$$= 0 + \int_{-a}^{a} x^2 . e^{isx} dx + 0$$

$$= \left[x^2 \frac{e^{isx}}{is} - (2x)\frac{e^{isx}}{(is)^2} + 2\frac{e^{isx}}{(is)^3}\right]_{-a}^{a}$$

$$= a^2\left[\frac{e^{isa}}{is} - \frac{e^{-isa}}{is}\right] - 2\left[\frac{a e^{isa}}{-s^2} - \frac{(-a)e^{-isa}}{-s^2}\right]$$
$$+ 2\left[\frac{e^{isa}}{-is^3} - \frac{e^{-isa}}{-is^3}\right]$$

$$= a^2 . \frac{2\sin(as)}{s} + \frac{2a}{s^2} . 2\cos(as) - \frac{2}{s^3} 2\sin(as)$$

$$= \frac{2a^2 \sin(as)}{s} + \frac{4a}{s^2}\cos(as) - \frac{4}{s^3}\sin(as)$$

$\therefore F\{f(x)\} = 2\left\{\frac{a^2\sin(as)}{s} + \frac{2a}{s^2}\cos(as) - \frac{2}{s^3}\sin(as)\right\}$

Q.4o Find FCT of e^{-x^2} and hence evalaate FST of xe^{-x^2}

Soln! FCT of e^{-x^2} is,

$$f(x) = e^{-x^2}$$

$$F_c\{f(x)\} = \int_0^\infty f(x)\cos(sx)\,dx$$

$$= \int_0^\infty e^{-x^2}\cos(sx)\,dx = I, \text{ say}$$

$$\therefore \frac{dI}{ds} = \int_0^\infty e^{-x^2}(-\sin sx)\,x\,dx \quad -(i)$$

$$= \frac{1}{2}\int_0^\infty e^{-x^2}\{(-2x)\sin(sx)\}\,dx$$

$$= \frac{1}{2}\left[(\sin sx\, e^{-x^2})_0^\infty - \int_0^\infty s\cos sx\, e^{-x^2}\,dx\right]$$

$$= \frac{-s}{2} I$$

$$\therefore \int \frac{dI}{I} = -\int \frac{s}{2}\,ds$$

$$\Rightarrow \log I = \frac{-s^2}{4} + \log c$$

$$\therefore I = ce^{-s^2/4} \quad -(ii)$$

for $s = 0$,

$$c = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$\therefore I = \frac{\sqrt{\pi}}{2}e^{-\frac{s^2}{4}} \quad -(\text{from (ii)})$$

from (i),

$$\text{FST of } xe^{-x^2} = \frac{-dI}{ds}$$

$$= \frac{sI}{2}$$

$$= \frac{s\sqrt{\pi}}{4}e^{-\frac{s^2}{4}}$$