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MATHEMATICS ASSIGNMENT - II

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Q. 1. Find fourier transform of the following functions:

(a) $f(x) = \frac{1}{\sqrt{x}}$

Sol^n:-

$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

for $x > 0$,

$F\{f(x)\} = \int_0^{\infty} \frac{1}{\sqrt{x}}\, e^{isx}\, dx$

Let, $-t = isx$

$-dt = is\,dx$

$\therefore dx = \frac{dt}{-is}$

So, $F\{f(x)\} = \int_0^{\infty} \left(\frac{t}{is}\right)^{\frac{1}{2}-1} \cdot e^{-t} \frac{dt}{-is}$

$= \frac{1}{\sqrt{-is}} \sqrt{\frac{1}{2}}$

$= \sqrt{\frac{\pi}{-is}}$

$= \sqrt{\frac{\pi i}{s}}$

(b) $f(x) = \begin{cases} e^{-ikx}, & \text{if } a<x<b \\ 0, & \text{if } x<a \text{ and } x>b \end{cases}$

Sol^n:-

$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$= \int_a^b e^{-ikx}\, e^{isx}\, dx$

$= \int_a^b e^{i(s-k)x}\, dx$

$= \frac{1}{i(s-k)} \left[e^{i(s-k)x} \right]_a^b$

$= \frac{e^{i(s-k)b} - e^{i(s-k)a}}{i(s-k)}$

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(c) $f(x) = \begin{cases} a^2 - x^2 & ; \text{ if } |x| < a \\ 0 & ; \text{ if } |x| > a \end{cases}$. Deduce, $\int_0^\infty \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4}$

Solⁿ: $F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$

$= \int_{-a}^{a} (a^2 - x^2)\, e^{isx}\, dx$

$= \frac{a^2}{is}(e^{isa} - e^{-isa}) - \left[\frac{x^2 e^{isx}}{is} - \frac{2x e^{isx}}{(is)^2} + \frac{2e^{isx}}{(is)^3}\right]_{-a}^{a}$

$= \frac{a^2}{is}(e^{isa} - e^{-isa}) - \left[\frac{a^2 e^{isa}}{is} - \frac{2ae^{isa}}{(is)^2} + \frac{2e^{isa}}{(is)^3} - \left\{\frac{a^2 e^{-isa}}{is} + \frac{2ae^{-isa}}{(is)^2} + \frac{2e^{-isa}}{(is)^3}\right\}\right]$

$= \frac{2a}{(is)^2}(e^{isa} + e^{-is}) - \frac{2}{(is)^3}(e^{isa} - e^{-isa})$

$= \frac{4a}{i^2 s^2}\cos(sa) - \frac{4}{i^2 s^3}\sin(sa)$

$= \frac{4}{i^2 s^3}(a.s.\cos(sa) - \sin(sa))$

$= \frac{4}{s^3}[\sin(sa) - as(\cos(sa))]$

By inverse fourier transform,

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^2}(\sin sa - as\cos sa)\, e^{-isx}\, ds$

Putting $sa = t$,

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 4a^3 \frac{(\sin t - t\cos t)}{t^3} \cdot \frac{dt}{a} \cdot e^{\frac{-itx}{a}}$

$= \frac{2a^2}{\pi}\int_{-\infty}^{\infty} \frac{\sin t - t\cos t}{t^3} \cdot e^{\frac{-itx}{a}}\, dt$

Putting $x = 0$, $f(0) = a^2$,

$a^2 = \frac{2a^2}{\pi} \cdot 2\int_0^\infty \frac{\sin t - t\cos t}{t^3}\,dt \left(\because \frac{\sin t - t\cos t}{t^3} \text{ is even function}\right)$

or $\frac{\pi}{4} = \int_0^\infty \frac{\sin t - t\cos t}{t^3}\, dt$

$\therefore \int_0^\infty \frac{\sin x - x\cos x}{x^3}\, dx = \frac{\pi}{4}$. verified

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Q.2. Find the fourier Sine transform of the following functions.

(a) $f(x) = x^{-\frac{1}{2}}$

Sol^n:- $F_s\{f(x)\} = \int_0^\infty f(x)\sin sx \, dx$

$= \int_0^\infty x^{-\frac{1}{2}} \frac{(e^{isx} - e^{-isx})}{2i} dx$

$= \frac{1}{2i}\left[\int_0^\infty x^{-\frac{1}{2}} e^{isx} dx - \int_0^\infty x^{-\frac{1}{2}} e^{-isx} dx\right]$

Let, $I_1 = \int_0^\infty x^{-\frac{1}{2}} e^{isx} dx$

$= \frac{1}{\sqrt{-is}} \int_0^\infty (t)^{\frac{1}{2}-1} e^{(-t)} dt$

$= \frac{\sqrt{\frac{1}{2}}}{\sqrt{-is}} = \sqrt{\frac{\pi}{-is}} = \sqrt{\frac{\pi i}{s}}$

Let, $-t = isx$, $-dt = is\,dx$, $\therefore dx = \frac{-dt}{is}$

Let, $I_2 = \int_0^\infty x^{-\frac{1}{2}} e^{-isx} dx$

$= \frac{1}{is} \int_0^\infty t^{\frac{1}{2}-1} e^{-t} dt$

$= \frac{\sqrt{\frac{1}{2}}}{\sqrt{is}} = \frac{\sqrt{\pi}}{\sqrt{is}}$

Let, $t = isx$, $\therefore dx = \frac{dt}{is}$

$\therefore F\{f(x)\} = \frac{1}{2i}\left[\sqrt{\frac{\pi i}{s}} - \sqrt{\frac{\pi}{is}}\right]$

$= \sqrt{\frac{\pi}{s}} \frac{1}{2i}(\sqrt{i} - \sqrt{-i})$

$= \sqrt{\frac{\pi}{s}} \frac{1}{2\sqrt{i}}\left(\frac{\sqrt{i} - \sqrt{-i}}{\sqrt{i}}\right)$

$= \frac{1}{2}\sqrt{\frac{\pi}{is}}(1 - \sqrt{-1})$

$= \frac{1}{2}\sqrt{\frac{\pi}{is}}(1 - i)$

(b) $f(x) = \begin{cases} x & ; & 0<x<1 \\ 2-x & ; & 1<x<2 \\ 0 & ; & x>2 \end{cases}$

Soln:- $F_s\{f(x)\} = \int_0^1 x \sin sx\,dx + \int_1^2 (2-x)\sin sx\,dx$

$$= \left| x\left(\frac{-\cos x}{s}\right) - 1\left(\frac{-\sin sx}{s^2}\right)\right|_0^1 + \left|(2-x)\left(\frac{-\cos sx}{s}\right) - (-1)\left(\frac{-\sin sx}{s^2}\right)\right|_1^2$$

$$= \frac{-\cos s}{s} + \frac{\sin s}{s^2} - \frac{\sin 2s}{s^2} - \left\{\frac{-\cos s}{s} - \frac{\sin s}{s^2}\right\}$$

$$= \frac{2\sin s - \sin 2s}{s^2}$$

Q.3. Find fourier cosine transform of $f(x) = \begin{cases} x^2, & 0<x<1 \\ 0, & x>1 \end{cases}$

Soln: $F_c\{f(x)\} = \int_0^\infty f(x)\cos x\,dx$

$$= \int_0^\infty x^2 \cos x\,dx$$

$$= \left| \frac{x^2 \sin sx}{s} - 2x\left(\frac{-\cos x}{s^2}\right) + 2\left(\frac{-\sin sx}{s^3}\right)\right|_0^1$$

$$= \frac{\sin s}{s} + \frac{2\cos s}{s^2} - \frac{2\sin s}{s^3}$$

Q.4. Find fourier Cosine and sine transform of the function, $f(x)\, e^{-ax}$; $a>0$. Hence, find the values of integrals $\int_0^\infty \frac{\omega \sin \omega x}{a^2+\omega^2}\,d\omega$ and $\int_0^\infty \frac{\cos \omega x}{a^2+\omega^2}\,d\omega$

Soln:- $F_c\{f(x)\} = \int_0^\infty e^{-ax} \cos \omega x\,dx$

$$= \left| \frac{e^{-ax}}{a^2+\omega^2}\left(-a\cos\omega x + \omega \sin \omega x\right)\right|_0^\infty$$

$$= -\left\{\frac{1}{a^2+\omega^2}(-a)\right\}$$

$$= \frac{a}{a^2+\omega^2}$$

$$F_s\{f(x)\} = \int_0^\infty e^{-ax} \sin \omega x \, dx$$

$$= \left| \frac{e^{-ax}}{a^2+\omega^2} (-a \sin \omega x - \omega \cos \omega x) \right|_0^\infty$$

$$= \frac{\omega}{a^2+\omega^2}$$

By inverse of fourier cosine transform,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(\omega) \cos \omega x \, d\omega$$

or, $$\int_0^\infty \frac{a \cdot \cos \omega x}{a^2+\omega^2} d\omega = \frac{\pi}{2} e^{-ax}$$

$$\therefore \int_0^\infty \frac{\cos \omega x}{a^2+\omega^2} d\omega = \frac{\pi}{2a} e^{-ax}$$

By inverse of fourier sine transform,

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(\omega) \sin \omega x \, d\omega$$

or, $$\frac{2}{\pi} = \int_0^\infty \frac{\omega \sin \omega x}{a^2+\omega^2} d\omega$$

$$\therefore \int_0^\infty \frac{\omega \sin \omega x}{a^2+\omega^2} d\omega = \frac{\pi}{2} e^{-ax}$$