Tutorial Sheet – 1

(Fourier Series and Fourier Integral)

1. Derive the Fourier series of the following functions:

(a) $f(x)=\pi-|x|\,; \quad -\pi<x<\pi$ (b) $f(x)=2x|x|\,; \quad -1<x<1$

(c) $f(x)=\begin{cases} x, \text{ if } -\frac{\pi}{2}<x<\frac{\pi}{2} \\ \pi-x, \text{ if } \frac{\pi}{2}<x<\frac{3\pi}{2} \end{cases}$ (d) $f(x)=\begin{cases} 1-\frac{1}{2}|x|, \text{ if } -2<x<2 \\ 0, \quad \text{if } 2<x<6 \end{cases}$

(e) $f(x)=x+x^2; \quad -\pi<x<\pi$. Hence deduce that $\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\cdots$

(f) $f(x)=\begin{cases} 0, \text{ if } -\pi<x<0 \\ \sin x, \text{ if } 0<x<\pi \end{cases}$. Hence prove that $\frac{1}{4}(\pi-2)=\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots$

2. Derive the Fourier Cosine and Sine series of the following functions:

(a) $f(x)=2-x; \quad 0<x<2$ (b) $f(x)=x^2; \quad 0<x<L$

(c) $f(x)=\begin{cases} 1, \text{ if } 0<x<1 \\ 2, \text{ if } 1<x<2 \end{cases}$ (d) $f(x)=\begin{cases} x, \text{ if } 0<x<\frac{\pi}{2} \\ \frac{\pi}{2}, \text{ if } \frac{\pi}{2}<x<\pi \end{cases}$

(e) $f(x)=x(\pi-x); \quad 0<x<\pi$

Hence deduce that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^3}=\frac{\pi^3}{32}$.

3. Derive the Fourier Integrals of the following functions:

(a) $f(x)=\begin{cases} a, \text{ if } |x|<c \\ 0, \text{ if } |x|>c \end{cases}$. Hence evaluate $\int_0^{\infty}\frac{\sin c\omega\cos\omega x}{\omega}\,d\omega$.

(b) $f(x)=\begin{cases} 0, \quad \text{if } x<0 \\ \pi e^{-x}, \text{ if } x>0 \end{cases}$. Hence evaluate $\int_0^{\infty}\frac{\cos\omega x+\omega\sin\omega x}{1+\omega^2}\,d\omega$.

4. Derive the Fourier Cosine and Sine Integrals of the following functions:

(a) $f(x)=\begin{cases} x, \text{ if } 0<x<1 \\ 0, \text{ if } x>1 \end{cases}$ (b) $f(x)=\begin{cases} e^{-x}, \text{ if } 0<x<a \\ 0, \text{ if } x>a \end{cases}$

(c) $f(x)=\begin{cases} \pi-x, \text{ if } 0<x<\pi \\ 0, \quad \text{if } x>\pi \end{cases}$ (d) $f(x)=\begin{cases} 1-x^2, \text{ if } 0<x<1 \\ 0, \text{ if } x>1 \end{cases}$

5. Using Fourier Sine integral show that $\int_0^{\infty}\frac{1-\cos\omega\pi}{\omega}\sin\omega x\,d\omega=\begin{cases} \frac{\pi}{2}, \text{ if } 0<x<\pi \\ 0, \text{ if } x>\pi \end{cases}$.