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Branch - CSE

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Q.1.

Soln:- Given, $f_{xy}(x,y) = \begin{cases} Kx^2(4-y) & , x<y<2x,\ 0<x<2 \\ 0 & , \text{otherwise.} \end{cases}$

a) Ans) To find value of constant K, we take

$$\int_0^2\int_x^{2x} Kx^2(4-y)\,dy\,dx = 1$$

Or, $$K\int_0^2 \left[4x^2(2x) - \frac{x^2(4x^2)}{2} - 4x^3 + \frac{x^4}{2}\right]dx = 1$$

Or, $$K\left[2^5 - \frac{2^6}{5} - 2^4 + \frac{2^5}{10}\right] = 1$$

Or, $$K\left[\frac{80-64+16}{5}\right] = 1$$

Or, $$\frac{32K}{5} = 1$$

$$\therefore K = \frac{5}{32}$$

b) Ans) Marginal density functions,

$$f_x = \int_x^{2x} \frac{5}{32}x^2(4-y)\,dy$$

$$= \frac{5}{32}x^2\left(4x - \frac{3x^2}{2}\right)$$

$$= \frac{5}{32}x^3\left(4 - \frac{3x}{2}\right)$$

$$f_y = \int_0^2 \frac{5}{32} x^2 (4-y)\, dx$$

$$= \frac{5}{32} (4-y) \left(\frac{8}{3}\right)$$

$$= \frac{5}{3} \left(1 - \frac{y}{4}\right)$$

c) Ans) Conditional probability density functions,

$$f(x|y) = \frac{f(xy)}{f(y)} = \frac{\frac{5}{32} x^2 (4-y)}{\frac{5}{12} (4-y)} = \frac{3}{8} x^2$$

$$f(y|x) = \frac{f(xy)}{f(x)} = \frac{\frac{5}{32} x^2 (4-y)}{\frac{5}{32}\left(4x^3 - \frac{3x^4}{2}\right)} = \frac{4-y}{x\left(4 - \frac{3}{2}x\right)}$$

f) Ans) To find if X and Y are independent,

$$f_{XY} = \frac{5}{32} x^2 (4-y)$$

$$f_X = \frac{5}{32} x^3 \left(4 - \frac{3x}{2}\right)$$

$$f_y = \frac{5}{3}\left(1 - \frac{y}{4}\right)$$

$$f_X \cdot f_Y = \frac{5}{32} x^3 \left(4 - \frac{3x}{2}\right) \cdot \frac{5}{3}\left(1 - \frac{y}{4}\right)$$

$$= \frac{25}{32} x^3 \left(4 - \frac{3}{2}x\right)(4-y)$$

$$\neq f_{xy}$$

$\therefore$ X and Y are not independent.

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2) soln)

Given, X and Y are independent random variables,

$$Z = X + 2Y$$
$$W = X - Y$$

(a) To find: joint pdf of Z and W.

from the given equations, we can obtain

$$z - w = 3y$$
$$\therefore y = \frac{1}{3}(z-w)$$

and, $x = w+y = w + \frac{1}{3}(z-w)$

$$\therefore x = \frac{1}{3}(z+2w)$$

$$J(z,w) = \begin{vmatrix} \frac{\delta x}{\delta z} & \frac{\delta y}{\delta z} \\ \frac{\delta x}{\delta w} & \frac{\delta y}{\delta y} \end{vmatrix} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{vmatrix}$$

Choosing corresponding values, we get,

$$\frac{\delta x}{\delta z} = \frac{1}{3}$$

$$\frac{\delta y}{\delta z} = \frac{1}{3}$$

$$\frac{\delta x}{\delta w} = \frac{2}{3}$$

$$\frac{\delta y}{\delta w} = -\frac{1}{3}$$

$$\therefore J(z,w) = \frac{1}{3}$$

$$\therefore f_{ZW}(z,w) = \begin{cases} \frac{1}{3}, & 0 < \frac{1}{3}(z+2w) < 1;\ 0 < \frac{1}{3}(z-w) < 1. \\ 0, & \text{otherwise} \end{cases}$$

~~(a) To find joint pdf of X and Y~~

(b) To find: joint moment generating function of X and Y.

$$f_X(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

Combining,

$$\therefore f_{XY}(x,y) = \begin{cases} 1, & 0 < x < 1;\ 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

3) Soln) Given,

Markov Chain with state $\{0,1,2\}$

Transition probability matrix, $P = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix}$

$$P^2 = \begin{pmatrix} 0.75 & 0 & 0.25 \\ 0.5 & 0.25 & 0.25 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 0.25 & 0.375 & 0.375 \\ 0.375 & 0.25 & 0.375 \\ 0.75 & 0 & 0.25 \end{pmatrix}$$

$$P^4 = \begin{pmatrix} 0.5625 & 0.125 & 0.3125 \\ 0.5 & 0.1875 & 0.3125 \\ 0.25 & 0.375 & 0.375 \end{pmatrix}$$

To find periodicity,

In the matrix, the states are defined as,

$\longrightarrow$ States of x_n

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 1 & 0 & 0 \end{pmatrix} \end{array}$$

$\downarrow$ States of x_{n-1}

Similar for other powers of P.

Here,

Period of state 0,

$$d_0 = GCD\{2,3,4,\ldots\} = 1$$

Hence, the state is aperiodic.

4) Soln) Given,

Two-state Markov chain with transition probability matrix,

$$P = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}$$

To find: n-step transition probability matrix P^n.

$$P = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 1 & 0 \\ 3/4 & 1/4 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 1 & 0 \\ 7/8 & 1/8 \end{pmatrix}$$

for n-step, it is clearly,

$$P^n = \begin{pmatrix} 1 & 0 \\ 1-\frac{1}{2^n} & 1/2^n \end{pmatrix}$$

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(d) Conditional means and Conditional variances,

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f_x(x)\,dx \quad , 0 < x < 2 \quad = \frac{3}{2}$$

$$\mu_Y = E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy \quad , 0 < y < 2x \quad = \frac{40}{9}$$

$$= 0 < y < 4$$

(e) $\text{cov}(X,Y) = E(XY) - E(X)E(Y)$

$$E(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{xy}(x,y)\,dx\,dy$$

$$= \int_0^2 \int_{y-x}^{2x} xy \, ($$