MINI - PROJECT ON MATHEMATICS - II

Reg. No.: 1912160
Phone. No.: 9864391349
Name : Subhojit Ghimire

Subject Code: MA-102

TOPIC:

GAUSS DIVERGENCE THEOREM AND APPLICATIONS

* Overview:

Gauss divergence theorem states that, the sum of all sources of the field in a region gives the net flux out of the region. In vector calculus, Gauss divergence theorem is the result that relates to the flux of a vector field through a closed surface to the divergence of the field in the volume enclosed.

If V be a volume by bounded by a surface S with outward unit normal $\vec{n}$, and $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ is a continuously differentiable vector field in V, then,

$$\iiint_V (\nabla \cdot \vec{F})\, dV = \oiint_S (\vec{F} \cdot \vec{n})\, dS$$

This relation between surface and volume integrals can also be written as,

$$\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint \mathrm{div}\, \vec{F}\, dV$$

The divergence theorem is an important result for the mathematics of physics and engineering, particularly in electro statics and fluid dynamics.

* Proof :

Proof for Gauss divergence theorem -

Let, 'S' be a closed surface.

Let, any line drawn parallel to coordinate axes cut 'S' in almost two points.

Let, S_1 and S_2 be the surface at the top and bottom of S represented by $z = f(x,y)$ and $z = \phi(x,y)$ respectively.

Now,

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}, \text{ then,}$$

We have,

$$\iiint \frac{\delta F_3}{\delta z} dV = \iiint \frac{\delta F_3}{\delta z} dx dy dz$$

$$= \iint_R \left[\int_{z=\phi(x,y)}^{z=f(x,y)} \frac{\delta F_3}{\delta z} \right] dx dy$$

Also, $\iint_R [F_3(x,y,z)]_z = f(x,y) z = \phi(x,y) dx dy$

can be written as, $\iint_R [F_3(x,y,f) - F_3(x,y,\phi)] dx dy$ -(i)

So, for the upper surface S_2,

$$\frac{dy}{dx} = \cos\gamma_2 \, dS = k \cdot n_2 dS$$

Since the normal vector n_2 to S_2 makes an acute angle γ_2 with $\vec{K}$ vector,

$$dx dy = -\cos\gamma_2 \, dS_1 = -\vec{K} \cdot \vec{n} \cdot dS_1$$

And, the normal vector n_1 to S_1 makes an obtuse angle γ_1 with $\vec{K}$ vector,

Then,

$$\iint_R F_3(x,y,z) dx dy = \iint_{S_2} F_3 \vec{K} \cdot \vec{n_2} dS_2 \quad \text{-(ii)}$$

$$\iint_R F_3(x,y,\phi) dx dy = \iint_{S_1} F_3 \vec{K} \cdot \vec{n_1} dS_1 \quad \text{-(iii)}$$

Now,

The expression (i) can be written as,

$$\iint_R F_3(x,y,z) dx dy - \iint_R F_3(x,y,\phi) dx dy \quad \text{-(iv)}$$

Substituting values of (ii) and (iii) in (iv), we get,

$$\iint_{S_2} F_3 \vec{K} \cdot \vec{n_2}\, dS_2 - \iint_{S_1} F_3 \vec{K} \cdot \vec{n_1}\, dS_1$$

Thus, the above expression can be written as,

$$\iint_S F_3 \vec{K} \cdot \vec{n}\, dS$$

Similarly,
projecting the surface S on coordinate plane,
we get,

$$\iiint \frac{\delta F_3}{\delta z}\, dV = \iint F_3 \vec{K} \cdot \vec{n}\, dS$$

$$\iiint \frac{\delta F_2}{\delta y}\, dV = \iint F_2 \vec{j} \cdot \vec{n}\, dS$$

$$\iiint \frac{\delta F_3}{\delta x}\, dV = \iint F_1 \vec{i} \cdot \vec{n}\, dS$$

Adding the above three equations, we get,

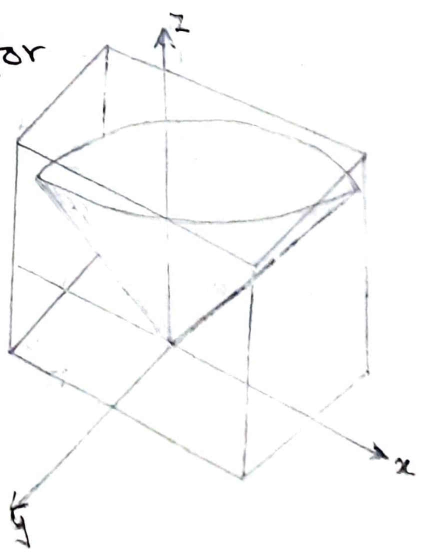
$$\iint_V \int \left[\frac{\delta F_1}{\delta z} + \frac{\delta F_2}{\delta y} + \frac{\delta F_3}{\delta z}\right] dV = \iint_S \left[F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{K}\right] \cdot \vec{n}\, dS$$

Therefore, the divergence theorem can be written as,

$$\iiint_V \nabla \vec{F} \cdot dV = \iint_S \vec{F} \cdot \vec{n} \cdot dS$$

* VERIFICATION :

Q. Verify the divergence theorem for vector field $\vec{F} = (x-y, x+z, z-y)$ and surface S that consists of cone $x^2+y^2=z^2$, $0 \le z \le 1$, and the circular top of the cone. Assume this surface is positively oriented.



Known factors:

field : $F = (x-y, x+z, z-y)$

Cone : $x^2+y^2 = z^2$

Solution:-

Let E be the solid cone enclosed by S.

To verify the theorem, we show that,

$$\iiint_E \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

by calculating each integral separately.

To Compute the triple integral, $\operatorname{div} \vec{F} = P_x + Q_y + R_z = 2$

Therefore, $\iiint_E \operatorname{div} \vec{F}\, dV = 2 \iiint_E dV = 2$ (vol. of E)

The volume of right circular cone is given by $\frac{\pi r^2 h}{3}$.

Here, $h = r = 1$.

So, $\iiint_E \operatorname{div} \vec{F}\, dV = 2$ (volume of E) $= \frac{2\pi}{3}$

To compute the flux integral:

S can be written as union of smooth surfaces.

So, flux integral can be broken into two pieces,

(i) one flux integral across circular top of cone

(ii) one flux integral across remaining portion of cone

Let, S_1 be the circular top

and, S_2 be the portion under the top.

For S_1,

$$\vec{r}(u,v) = \langle u\cos v, u\sin v, 1\rangle, \quad 0 \le u \le 1;\ 0 \le v \le 2\pi$$

Then, tangent vectors $\vec{t_u} = \langle \cos v, \sin v, 0\rangle$

and $\vec{t_v} = \langle -u\cos v, u\sin v, 0\rangle$

Therefore, flux across S_1,

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{2\pi} \vec{F}(\vec{r}(u,v)) \cdot (\vec{t_u} \times \vec{t_v})\, dA$$

$$= \int_0^1 \int_0^{2\pi} \langle u\cos v - u\sin v, u\cos v + 1, 1 - u\sin v\rangle \cdot \langle 0, 0, v\rangle\, dv\, du$$

$$= \int_0^1 \int_0^{2\pi} u - u^2 \sin v\, dv\, du$$

$$= \pi$$

For S_2,

$\vec{r}(u,v) = \langle u\cos v, u\sin v, u \rangle, \quad 0 \le u \le 1, \; 0 \le v \le 2\pi$

Then, tangent vectors $\vec{t_u} = \langle \cos v, \sin v, 1 \rangle$

and $\vec{t_v} = \langle -u\sin v, u\cos v, 0 \rangle$

$$\vec{t_u} \times \vec{t_v} = \langle -u\cos v, -u\sin v, u \rangle$$

Therefore, flux across S_2,

$$\iint_{S_2} \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^{2\pi} \vec{F}(\vec{r}(u,v)) \cdot (\vec{t_u} \times \vec{t_v})\, dA$$

$$= \int_0^1 \int_0^{2\pi} \langle u\cos v - u\sin v, u\cos v + u, u - \sin v \rangle \cdot \langle u\cos v, u\sin v, -u \rangle \, dv\, du$$

$$= \int_0^1 \int_0^{2\pi} u^2\cos^2 v + 2u^2 \sin v - u^2 \, dv\, du$$

$$= \frac{\pi}{3}$$

Therefore, the total flux across S is,

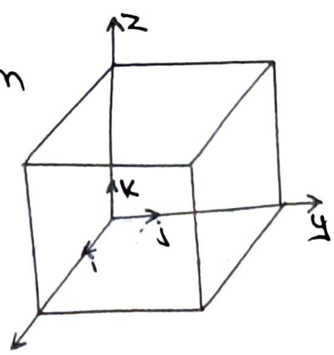
$$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S} = \frac{2\pi}{3} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$$\therefore \iiint_E \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

Hence, the divergence theoren has been verified.

* Solved Problems:

Q. Use Gauss Divergence Theorem for $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$, taken over the rectangular parallelopiped $0 \le x \le a, \; 0 \le y \le b, \; 0 \le z \le c$.



Solution :- We know, Gauss divergence theorem is given by, $\iint_S \vec{F} \cdot \hat{n}\, dS = \iiint_V \operatorname{div} \vec{F}\, dV$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \right) \cdot \left[(x^2 - yz)i + (y^2 - z)j + (z^2 - xy)k \right]$$

$$= 2(x + y + z)$$

Now, $\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$

$$= \int_0^a\int_0^b\int_0^c 2(x+y+z)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^b\left[\int_0^c 2(x+y+z)\,dz\right]dx\,dy$$

$$= \int_0^a\left[\int_0^b 2\left(xc+yc+\frac{c^2}{2}\right)\right]dy$$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = abc(a+b+c)$

Q. From Gauss Divergence Theorem, find $\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 4xi - 2y^2j + z^2k$ is taken in the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Soln: By Gauss divergence theorem, we know,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(4xi - 2y^2j + z^2k)$$

$$= 4 - 4y + 2z$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\left[\int_0^3 (4-4y+2z)\,dz\right]dx\,dy$$

$$= \int_{2}^{2}\left[\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12-12y+6)\,dy\right]dx$$

$$= \int_{-2}^{2}\left[21\left(\sqrt{4-x^2}+\sqrt{4-x^2}\right)-6\right]dx$$

$$= 42\left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right]$$

$$= 42\left[2\times\frac{\pi}{2} + 2\times\frac{\pi}{2}\right]$$

$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 84\pi$

* Applications :

The use of Gauss-divergence theorem can be seen in the fields of electrostatic and fluid dynamics.

* Application in Electrostatics:

Q. Derive Gauss' theorem of electrostatic using divergence theorem.

Solution:- Gauss divergence theorem states that

$$\iiint_V \nabla \cdot \vec{F} \, dV = \iint_S \vec{F} \hat{n} \, dS \quad -(i)$$

We have,

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

Integrating this on a closed volume V whose surface is S, it becomes,

$$\iiint_V \nabla \cdot E \, dv = \frac{Q}{\epsilon_0} \quad -(ii)$$

where, Q is the total charge in V.

, for any field F, we get,

$$\iiint_V \nabla \cdot E \, dV = \iint_S E \cdot dS = \Phi_S(E) \quad -(iii)$$

where, $\Phi_S(E)$ is the flux of E through S

Combining (ii) and (iii), we get,

$$\Phi_S(E) = \frac{Q}{\epsilon_0} \quad (\Phi_S \text{ is electric flux})$$

which is Gauss Law or Gauss' Flux theorem.

Q. Suppose we have four stationary point charges in space, all with a charge of 0.002 Coulombs (C). The charges are located at (0, 0, 1), (1, 1, 4), (-1, 0, 0) and (-2, -2, 2). Let $\vec{E}$ denote the electrostatic field generated by these point charges. If S is the sphere of radius 2 oriented outward and centered at the origin, then find: $\iint_S \vec{E} \cdot d\vec{S}$

Solution:-

According to Gauss' law, the flux of $\vec{E}$ across S is the total charge inside of S divided by the electric constant.

Since S has radius of 2 units, only two of the charges are inside of S, charges at (0, 1, 1) and (-1, 0, 0).

Therefore, the total encompassed charge by S is

$$Q = (0.002) \times 2 = 0.004.$$

Applying Gauss' law,

$$\iint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$$

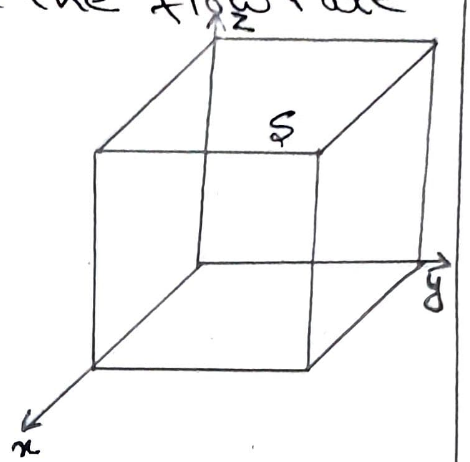
$$= \frac{0.004}{8.854 \times 10^{-12}}$$

$$\approx 4.418 \times 10^9 \text{ Vm}$$

$$\therefore \iint_S \vec{E} \cdot d\vec{S} = 4.42 \times 10^9 \text{ Vm}$$

* Application in Fluid Dynamics:

Q. Let $\vec{V} = \langle -\frac{y}{z}, \frac{x}{z}, 0 \rangle$ be the velocity field of a fluid. Let C be the solid cube given by $1 \le x \le 4$, $2 \le y \le 5$, $1 \le z \le 4$, and let S be the boundry of this cube. Find the flow rate of the fluid across S.



Solution:

Given, vector field: $\vec{V} = \langle -\frac{y}{z}, \frac{x}{z}, 0 \rangle$

The flow rate of the fluid across S is $\iint_S \vec{V} \cdot d\vec{S}$.

In the figure, we can see that the rate of fluid entering the cube is the same as the rate of the fluid exiting the cube. The field is rotational in nature and for a given circle parallel to xy-plane that has a center on the z-axis, the vectors along that circle are all the same magnitude. This means, the flow rate of the fluid across the cube should be zero.

Using divergence theorem to verify this assumption,

$$\iint_S \vec{V} \cdot d\vec{S} = \iiint_C \operatorname{div} \vec{V} \, dV$$
$$= \iiint_C 0 \, dV$$
$$= 0$$

Therefore, the flux is zero.
Hence, verified.