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Q.10

Solution: Given, $X(t) = Y \sin \omega t$

Y is uniformly distributed random variable in $(-1, 1)$.

Let, $Y = A$.

$F_y = \int_{-\infty}^{\infty} A \, dt = 1$

$\therefore A[1-(-1)] = 1$ as it is defined in $(-1, 1)$

$\therefore A = 1/2 = Y$.

To check for wide sense stationary,

$E(Y) = \frac{1}{2}(b+a) = \frac{1}{2}(1+(-1)) = 0$

$\therefore \mu = 0$

Also, $Var(Y) = E(Y^2) - (E(Y))^2$

$= \frac{1}{12}(b-a)^2$

$= \frac{1}{12}(1-(-1))^2$

$= \frac{1}{3}$

Since, $X(t) = Y \sin \omega t$

$E(X(t)) = E(Y \sin \omega t) = E(Y) \sin \omega t = 0$

$\therefore$ mean, $\mu = 0$

As, $f_x(t) = \frac{1}{2} \sin \omega t = K$, $X(t)$ is wide sense stationary process

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Q.3.

Solution: Let, $\{X(t): -\infty < t < \infty\}$ be a zero-mean stationary.

Autocorrelation function: $R_{xx}(\tau) = \begin{cases} 1 - \frac{|\tau|}{T} & -T \leq \tau \leq T \\ 0 & , \text{otherwise.} \end{cases}$

Let, $\{X(t_i): i = 1, 2, \dots n\}$ be sequence of n samples.

$$t_i = \frac{iT}{2}, \quad i = 1, 2, \dots n$$

To find mean and variance,

Given, Sample mean $\mu_n = \frac{1}{n} \sum_{i=1}^{n} X(t_i)$

Since, X(t) is zero-mean stationary.

$$E[X(t_i)] = 0$$

Also, $R_x(t_i, t_k) = E[X(t_i) X(t_k)]$

$\therefore R_x(t_k - t_i) = R_x\left[(k-i)\frac{T}{2}\right]$

To find $E(\mu_n)$,

$$E\left[\frac{1}{n} \sum_{i=1}^{n} X(t_i)\right] = \frac{1}{n} \sum_{i=1}^{n} E[X(t_i)] = 0$$

To find $\text{var}(\mu_n)$,

$$\text{var}(\mu_n) = E\{[\mu_n - E(\mu_n)]^2\} = E(\mu_n^2)$$

And,

$$E(\mu_n^2) = E\left[\frac{1}{n} \sum_{i=1}^{n} X(t_i)\right]\left[\frac{1}{n} \sum_{i=1}^{n} X(t_i)\right]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} E[X(t_i) X(t_k)]$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} R_x\left[(k-i)\frac{T}{2}\right]$$

Replacing these obtained values in $R_x(\tau)$,

$$R_x\left[(k-i)\frac{T}{2}\right] = \begin{cases} 1 & , k = i \\ 1/2 & , |k-i| = 1 \\ 0 & , |k-i| > 2 \end{cases}$$

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$\therefore$ Variance,

$$var(\mu_n) = \frac{1}{n^2}\left[n(1) + 2(n-1)\left(\frac{1}{2}\right) + 0\right]$$

$$= \frac{1}{n^2}(2n-1)$$

Q. 40

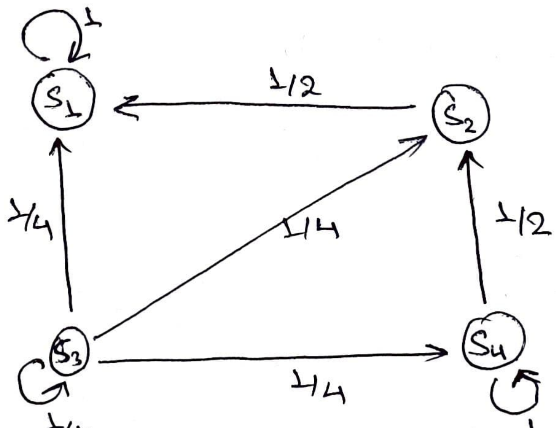
Solution: An absorbing Markov chain is a state that

$$P(X_{t+1} = i \mid X_t = i) = 1$$

An absorbing Markov chain is a matrix chain that becomes impossible to leave some states, and any state with $P_{ij} > 0$ can reach such a state.

Given,
$$\begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \end{array}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, S_1 and S_4 are the absorbing states.



Since we can get from S_2 and S_3 to both absorbing states, S_1 and S_4 don't reach any further.

$\therefore$ The given matrix is a transition matrix.

$\therefore$ The given matrix is an absorbing Markov chain.

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Q.5.

Solution: Simple random walk $X(n)$ with absorbing barriers at $\underbrace{\text{state } 0}_{P_{00}=1}$ and $\underbrace{\text{state } 3}_{P_{33}=1}$.

To find transition probability matrix P,

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

To find probability of absorption into states 0 and 3.

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 3 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 3 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & p \\ 0 & p & q & 0 \end{bmatrix} \end{array}$$

$$R = \begin{bmatrix} P_{10} & P_{13} \\ P_{20} & P_{23} \end{bmatrix} = \begin{bmatrix} q & 0 \\ 0 & p \end{bmatrix}$$

$$Q = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0 & p \\ q & 0 \end{bmatrix}$$

Probability of absorption in state 0,

$$\mu_{10} = \frac{q}{1-pq} \quad ; \quad \mu_{20} = \frac{q^2}{1-pq}$$

Probability of absorption in state 3

$$\mu_{13} = \frac{p^2}{1-pq} \quad ; \quad \mu_{23} = \frac{p}{1-pq}$$

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Q. 6.

Solution:

Given, $P = \begin{bmatrix} 0.40 & 0.30 & 0.30 \\ 0.20 & 0.50 & 0.30 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$

And, $V_i = (0.3,\ 0.6,\ 0.1)$

To find probabilities with his fourth purchase,

$$V_i^4 = V_i P^3$$

$$P^3 = P^2 \times P = P \times P \times P$$

$$= \begin{bmatrix} 0.40 & 0.30 & 0.30 \\ 0.20 & 0.50 & 0.30 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}^2 \times \begin{bmatrix} 0.40 & 0.30 & 0.30 \\ 0.20 & 0.50 & 0.30 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.295 & 0.345 & 0.36 \\ 0.255 & 0.385 & 0.36 \\ 0.275 & 0.325 & 0.4 \end{bmatrix} \begin{bmatrix} 0.40 & 0.30 & 0.30 \\ 0.20 & 0.50 & 0.30 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

$$= \begin{bmatrix} 0.277 & 0.351 & 0.372 \\ 0.269 & 0.359 & 0.372 \\ 0.275 & 0.345 & 0.38 \end{bmatrix}$$

$$\therefore V_i P^3 = \begin{bmatrix} 0.3 & 0.6 & 0.1 \end{bmatrix} \begin{bmatrix} 0.277 & 0.351 & 0.372 \\ 0.269 & 0.359 & 0.372 \\ 0.275 & 0.345 & 0.38 \end{bmatrix}$$

$$= \begin{bmatrix} 0.272 & 0.3552 & 0.3728 \end{bmatrix}$$

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Q. 7

Solution: Markov chain, in mathematics, is a stochastic model describing a sequence of possible events where the probability of each event depends upon the outcome of the previous event.

A fair dice is tossed repeatedly.

Let, X_n be max. number of occurings in first n toss

To find. transition probability matrix P of the Markov Chain $\{X_n\}$.

We have 6 sample space : $\{1,2,3,4,5,6\}$

Let $X_n = 3$

Then, $X_{n+1} = 3$, if $(n+1)^{th}$ trial results in 1, 2 or 3.

$X_{n+1} = 4$, if $(n+1)^{th}$ trial results in 4

$X_{n+1} = 5$, if $(n+1)^{th}$ trial results in 5

$X_{n+1} = 6$, if $(n+1)^{th}$ trial results in 6.

$\therefore P[X_{n+1}=3 \mid X_n = 3] = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6}$

$P[X_{i+1} = i \mid X_n = 3] = \frac{1}{6}$ for $i = 4, 5, 6$

$$\therefore P = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{bmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 2/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 3/6 & 1/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 4/6 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 5/6 & 1/6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

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To find P^2;

$$P^2 = \frac{1}{36}\begin{bmatrix} 1 & 3 & 5 & 7 & 9 & 11 \\ 0 & 4 & 5 & 7 & 9 & 11 \\ 0 & 0 & 9 & 7 & 9 & 11 \\ 0 & 0 & 0 & 16 & 9 & 11 \\ 0 & 0 & 0 & 0 & 25 & 11 \\ 0 & 0 & 0 & 0 & 0 & 36 \end{bmatrix}$$

To find $P(x=6)$

$$P[X=6] = \sum_{i=1}^{6} P[X=6 \mid X_0=i] \times P[X_0=i]$$

$$= \frac{1}{6}\sum_{i=1}^{6} P^2_{i=6}$$

$$= \frac{1}{6} \times \frac{1}{36} \times (11+11+11+11+11+36)$$

$$= \frac{91}{216}$$