The Generalized PID Controller and its Application to Control of Ultrasonic and Electric Motors

Ilan Rusnak
Nanomotion Ltd., P.O.Box 223, Yokneam 20692, Israel

Abstract: In this paper by the use of optimal stochastic control theory we formulate a tracking problem and show those cases when their solution gives the PI, PD and PID controllers. Thus, we are avoiding heuristics and giving a systematic approach to explanation for their excellent performance. It is shown that the PI controller is optimal for a first order system, the PID controller is optimal for a 2nd order system with no zero. The same approach is applied to a general linear strictly proper system and a generalized PID controller is derived. Such controller is called here PID^{n-1} controller. As an example, generalized PID controllers for an ultrasonic motor and an electric DC motor are presented. *Copyright ©2000 IFAC*

Keywords: generalized PID, PID controller architecture, PID controller structure, optimal PID, PID^{n-1} controller.

1. Introduction

The PD, PI and PID controllers are successfully applied controllers to many applications, almost from the beginning of controls applications (D'Azzo and Houpis,1988; Franklin, *et al.*, 1994).

The facts of their successful application, good performance, easiness of tuning are speaking for themselves and are sufficient rational for their use, although their structure is justified by heuristics: "These ... controls - called proportional-integral-derivative (PID) control - constitute the heuristic approach to controller design that has found wide acceptance in the process industries." (Franklin, *et al.*, 1994, pp. 168).

In (Morari and Zafiriou, 1989, pp. 114) it is shown that Internal Model Control - "IMC leads to PID controllers for virtually all models common in industrial practice."

Structure of PID controllers is discussed in (Gawthorp, 1996; He, Q. and Garvey, 1996) with respect to self tuning algorithms and automatic selection of structures.

In (Rusnak, 1998) the linear quadratic regulator (LQR) theory has been used to formulate tracking problems and to show those cases when their solution gives the PID controllers. Namely, a problem has been stated whose solution leads to the PD, PI and PID controllers. This enabled avoiding heuristics and is giving a systematic approach to explanation for the good performance of the PID controllers. The main contribution of the results in (Rusnak, 1998; Rusnak, 1999) is that it shows for what problems the PID controllers are the optimal controllers and for which they are not. In (Rusnak, 1999) the generalized PID structure had been introduced and applied up to a fifth order system.

The importance of the results in (Rusnak, 1998; Rusnak, 1999) and the further generalization presented in this paper is:
1) From systems theoretical point of view it is important to know that a widely used control architecture can be derived from an optimal control problem.
2) The solution shows for what kind of systems the PID controllers are optimal and will show for which systems it is not, thus enabling to show why a PID controller does not perform well for all systems. This will enable to forecast what control designs not to apply a PID controller.
3) For those systems that the PID is not the optimal controller architecture the optimal control approach shows what is the optimal controller architecture, thus achieving generalization.
4) The present approach advises how to design PID controller on finite time interval, i.e., when the gains are time varying.

5) This generalization can be used in deriving the PID^{n-1} (Rusnak, 1999) controllers for SISO high order system, for MIMO systems, for time-varying and nonlinear systems; thus enabling a systematic generalization of the PID controller architecture-structure-topology.
6) The current design procedures of PID controllers are assuming noise free environment and are considering it as disturbances. The present approach advises how to utilize the presented approach to design PID structure controllers in presence of noises and stochastic disturbance by the use of LQG theory.

It is shown in (Rusnak, 1998) that the PI controller is optimal for a first order system. the PD controller is optimal for a 2nd order systems with no zero when constant disturbances rejection is not required. The PID controller is optimal for a 2nd order systems with no zero with requirement on rejection of constant disturbances.

The reference trajectory is generated by a system identical to the plant. The differences are the initial conditions and the input to the reference trajectory generator. The tracking error is the position error, and zero steady state (constant disturbances rejection) is imposed by integral action on the tracking error. This is the reason that the PID controllers are so well performing in servo applications and chemical processes, as these are of this type.

In this paper we apply the approach that has been developed in (Rusnak, 1998) to a more general case. Namely, in (Rusnak, 1998; Rusnak, 1999) it was assumed that the reference trajectory is generated by a deterministic linear system evolving from its deterministic initial conditions. Here we assume that the reference trajectory is generated by a linear system linear system driven by a stochastic process and its initial conditions are random.

The problem is stated and those cases for which the PI, PD and PID controllers are the optimal controller structures are presented. These are the controllers for first and second order system without zeros, respectively. The approach is applied to a general linear strictly proper system and the generalized PID controller is derived, the PID^{n-1} controller. As an example, a generalized PID controller for an ultrasonic and electric motors are presented.

2. Optimal Tracking Problem

We assume the n-th order multi-input multi-output system

$$\begin{aligned} &\dot{x} = Ax + Bu, \quad x(t_o) = x_o, \\ &y = Cx \end{aligned} \tag{1}$$

where $x \in R^n$ is the state; $u \in R^m$ is the input and $y \in R^p$ is the measured output; $A \in R^{nxn}$, $B \in R^{nxm}$ and $C \in R^{pxn}$. The initial condition x_o is a zero mean random vector. The reference trajectory generator is

$$\begin{aligned} &\dot{x}_r = A_r x_r + w_r, \quad x_r(t_o) = x_{ro}, \\ &y_r = C_r x_r \end{aligned} \tag{2}$$

where $x_r \in R^{\nu}$ is the state; $w_r \in R^{\mu}$ is the reference trajectory generator input and $y_r \in R^p$ is the reference output; $A_r \in R^{\nu x \nu}$, and $C_r \in R^{px\nu}$. The reference trajectory generator input, w_r, is a zero mean white stochastic process. The initial condition x_{ro} is a zero mean random vector.

The integral action is introduced into the control in order to "force" zero tracking errors for polynomial inputs, and to attenuate constant disturbances. This is done by introducing the auxiliary variable (the integral of the tracking error)

$$\dot{\eta} = e = y - y_r, \quad \eta(t_o) = \eta_o, \tag{3}$$

where η_o is a zero mean random vector. The control objective is

$$\begin{aligned} J = \tfrac{1}{2}E\{&(y(t_f) - y_r(t_f))^T G_1 (y(t_f) - y_r(t_f)) \\ &+ \eta(t_f)^T G_2 \eta(t_f) \\ &+ \int_{t_o}^{t_f} [y(t) - y_r(t))^T Q_1 (y(t) - y_r(t)) \\ &+ \eta(t)^T Q_2 \eta(t) + u(t)^T R u(t)]dt\} \end{aligned} \tag{4}$$

The expectation is taken with respect to the reference trajectory generator input and the initial conditions. The optimal tracking problem (Kwakernaak and Sivan, 1972). is to find an admissible input u(t) such that the tracking objective (4) is minimized subject to the dynamic constraints (1, 2, 4). All vectors and matrices are of the proper dimensions.

3. Solution of the Tracking Problem

In order to solve the Stochastic Optimal Tracking Problem we augment the state variables to the form (Kwakernaak and Sivan, 1972).

$$\overline{x}=\begin{bmatrix}x\\ \eta\\ x_r\end{bmatrix},\overline{x}_o=\begin{bmatrix}x_o\\ \eta_o\\ x_{ro}\end{bmatrix},\overline{A}=\begin{bmatrix}A&0&0\\ C&0&-C_r\\ 0&0&A_r\end{bmatrix},$$
$$\overline{B}=\begin{bmatrix}B\\ 0\\ 0\end{bmatrix},\ \overline{C}=\begin{bmatrix}C&0&-C_r\end{bmatrix},\overline{w}_r=\begin{bmatrix}0\\ 0\\ w_r\end{bmatrix}, \tag{5}$$

then the problem is minimization of (4) subject to (1, 2) is the problem of minimization of

$$J=\tfrac{1}{2}E\{x(t_f)^T Gx(t_f) + \int_{t_o}^{t_1}[x(t)^T Qx(t)+u(t)^T Ru(t)]dt\} \tag{6}$$

subject to

$$\dot{\overline{x}}=\overline{A}\overline{x}+\overline{B}u+\overline{w}_r,\quad \overline{x}(t_o)=\overline{x}_o \tag{7}$$

where

$$Q=\overline{C}^T Q_1\overline{C}+\begin{bmatrix}0\\1\\0\end{bmatrix}Q_2\begin{bmatrix}0&1&0\end{bmatrix}$$
$$G=\overline{C}^T G_1\overline{C}+\begin{bmatrix}0\\1\\0\end{bmatrix}G_2\begin{bmatrix}0&1&0\end{bmatrix}.$$

The problem above is the stochastic linear optimal regulator problem defined and solved in (Bryson and Ho, 1964; Kwakernaak and Sivan, 1972). The solution is

$$u=-R^{-1}\overline{B}P\overline{x}$$
$$-\dot{P}=P\overline{A}+\overline{A}^T P+Q-P\overline{B}R^{-1}\overline{B}P,\quad P(t_f)=G, \tag{8}$$

Notice that the solution is not affected by w_r. This is due to the assumption that it is a zero mean stochastic process. If we write $P=\{[P_{ij}];\ i,j=1,2,3\}$, then

$$u=-R^{-1}\ [\,B^T P_{11}\ \ B^T P_{12}\ \ B^T P_{13}\,],\begin{bmatrix}x\\ \eta\\ x_r\end{bmatrix} \tag{9}$$

4. General solution - The Generalized PID Controller

For the general case $A = A_r$ and $C = C_r = I$, i.e. the reference trajectory generator is identical to the plant, it can be shown that $P_{13} = -P_{11}$. Then

$$u=-R^{-1}\begin{bmatrix}B^T P_{11}&B^T P_{12}&B^T P_{13}\end{bmatrix}\begin{bmatrix}x\\ \eta\\ x_r\end{bmatrix} \tag{10}$$
$$=K_1 e+K_2\int e\,dt$$

where

$$K_1=-R^{-1}B^T P_{11},\ K_2=R^{-1}B^T P_{12}. \tag{11}$$

This is the generalized PID controller. It is shown in (Rusnak, 1998; Rusnak, 1999) that this approach leads for a first order system to the P and PI controllers.

5. Ultrasonic Motor

As an example we consider the control of an ultrasonic motor. A low order model of the ultrasonic motor is a second order system with no zero (Uchino, 1997; NanoMotion, 2000). That is we assume that the plant and the trajectory generator are

$$m\ddot{x}=k_F V-k_v\dot{x} \tag{12}$$

where
- m - is the mass
- x - is the position
- k_F - is the force constant
- V - is the voltage of the motor
- k_v - is the velocity damping coefficient.

$$A=A_r=\begin{bmatrix}0&1\\0&-a_1\end{bmatrix},B=\begin{bmatrix}0\\ b_2\end{bmatrix},$$
$$C=C_r=\begin{bmatrix}1&0\end{bmatrix}, \tag{13}$$

and we have

i.e.. $H(s)=H_r(s)=\dfrac{b_2}{s(s+a_1)}$

The state of the plant and trajectory generator are denoted $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix}$, respectively. From equation (10), one can see that our method applies a PI controller on each state. A PI controller on a single state is denoted as

$$C = k + \frac{k_I}{s} \tag{14}$$

We consider the following feedback structures:

a) full state feedback structure - parallel PI architecture;
Then

$$u = C_1(x - x_r) + C_2(\dot{x} - \dot{x}_r) \tag{15}$$

This structure is presented in figure 1.

b) full state feedback structure - cascade PI architecture;
the control engineers prefer the architecture, that is presented in figure 2

$$\begin{aligned} u &= C_2\left[(\dot{x} - \dot{x}_r) + \frac{C_1}{C_2}(x - x_r)\right] \\ &= C_v\left[(\dot{x} - \dot{x}_r) + C_p(x - x_r)\right] \end{aligned} \tag{16}$$

and we have:
the controller of the velocity loop

$$C_v = C_2 = k_2 + \frac{k_{I2}}{s},$$

and the controller of the position loop

$$C_p = \frac{C_1}{C_2} = \frac{k_1 s + k_{I1}}{k_2 s + k_{I2}},$$

There are four free gains and not six as one might think. This structure is presented in figure 2.

c) output feedback structure - PID architecture.

c.1) We want to force zero steady state tracking error on the output. Here $y = x_1, y_r = x_{1r}$ and $\dot{y} = x_2, \dot{y}_r = x_{2r}$. Then, since the tracking error is $e = y - y_r$, we have

$$\dot{\eta} = e = y - y_r \tag{17}$$

$$\begin{aligned} u &= \begin{bmatrix} k_1 & k_2 & k_3 & -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \eta \\ y_r \\ \dot{y}_r \end{bmatrix} \\ &= k_1 e + k_2 \dot{e} + k_3 \int e dt \end{aligned} \tag{18}$$

That is, we get PID controller. Without requirement on the integral action, $k_3=0$, and we get the PD controller.

c.2) Here we want to force zero steady state tracking error on the rate of the output as well,

$$\eta = \begin{bmatrix} y - y_r \\ \dot{y} - \dot{y}_r \end{bmatrix} = \begin{bmatrix} e \\ \dot{e} \end{bmatrix}, \quad \eta = \begin{bmatrix} \int e dt \\ e \end{bmatrix}, \tag{19}$$

$$\begin{aligned} u &= \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & -k_1 & -k_2 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \eta \\ \dot{\eta} \\ y_r \\ \dot{y}_r \end{bmatrix} \\ &= (k_1 + k_4)e + k_2 \dot{e} + k_3 \int e dt \end{aligned} \tag{20}$$

These are PID controllers. This structure is presented in figure 3. For other second order plants, or when the trajectory generator is not identical to the plant we will not get generally such structure. For example for a second order system with one zero see results in (Rusnak, 1998; Rusnak, 1999). The output feedback PID controller induces zero steady state errors on position and velocity simultaneously.

6. An electric DC motor

As an example of a third order system we consider a DC motor. The differential equation that are describing a linear DC motor are

$$\begin{aligned} m\ddot{x} &= k_F I - D\dot{x} \\ v - k_E \dot{x} &= L\frac{dI}{dt} + RI \end{aligned} \tag{21}$$

where

m	- is the mass
x	- is the position
k_F	- is the force constant
I	- is the current of the motor
D	- is the friction coefficient

v	- is the applied voltage
k_E	- is the back emf coefficient
L	- is the inductance of the motor
R	- is the resistance of the motor.

an we have

$$\begin{bmatrix} ms^2+Ds & -k_F \\ k_E s & Ls+R \end{bmatrix} \begin{bmatrix} x \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix} \tag{22}$$

$$\begin{bmatrix} x \\ I \end{bmatrix} = \frac{1}{(ms^2+Ds)(Ls+R)+k_F k_E s} \begin{bmatrix} Ls+R & k_F \\ -k_E & ms^2+Ds \end{bmatrix} \begin{bmatrix} 0 \\ v \end{bmatrix} \tag{23}$$

$$\frac{x}{v} = \frac{k_F}{(ms^2+Ds)(Ls+R)+k_F k_E s}$$

$$\frac{I}{v} = \frac{ms^2+Ds}{(ms^2+Ds)(Ls+R)+k_F k_E s} \tag{24}$$

$$\frac{x}{I} = \frac{k_F}{ms^2+Ds}$$

a) full state feedback structure - parallel PI architecture;

$$u = C_1(x - x_r) + C_2(\dot{x} - \dot{x}_r) + C_3(I - I_r) \tag{25}$$

This structure is presented in figure 4.

b) full state feedback structure - cascade PI architecture;

Usually the control engineers prefer the architecture, that is presented in figure 2, the cascade architecture. Then the controller is

$$\begin{aligned} u &= C_3\left\{(I - I_r) + \frac{C_2}{C_3}\left[(\dot{x} - \dot{x}_r) + \frac{C_1}{C_2}(x - x_r)\right]\right\} \\ &= C_I\left\{(I - I_r) + C_v\left[(\dot{x} - \dot{x}_r) + C_P(x - x_r)\right]\right\} \end{aligned} \tag{26}$$

and we have the controller of the current loop

$$C_I = C_3 = k_3 + \frac{k_{13}}{s};$$

the controller of the velocity loop;

$$C_v = \frac{C_2}{C_3} = \frac{k_2 s + k_{12}}{k_3 s + k_{13}},$$

the controller of the position loop;

$$C_p = \frac{C_1}{C_2} = \frac{k_1 s + k_{11}}{k_2 s + k_{12}},$$

i.e. PI controller in the current loop and lead-lag/lag-lead controllers in the velocity and position loops. Notice that there are only six free parameters and not ten as one might guess.

c) output feedback structure - the generalized PID architecture.

In the previous cases we assumed measurement of all state variables. If we assume that only the error is measured then the generalized PID controller is

$$\begin{aligned} \frac{u}{e} &= [C_1 + C_2 s + C_3 \frac{I}{x}] = k_1 + \frac{k_{11}}{s} + [k_2 + \frac{k_{12}}{s}]s + [k_3 + \frac{k_{13}}{s}]\frac{ms^2+Ds}{k_F} \\ &= \frac{\frac{m}{k_F}k_3 s^3 + (k_2 + k_{13}\frac{m}{k_F} + \frac{D}{k_F}k_3)s^2 + (k_1 + k_{12} + \frac{D}{k_F}k_{13})s + k_{11}}{s} \\ &= \frac{C_a s^3 + C_v s^2 + C_p s + C_I}{s} \end{aligned} \tag{28}$$

This is a generalized PID controller denoted here PID^2 controller. This controller is not proper and it might create problems in realization.

7. Discussion

Similarly, for higher order systems generalized PID controller can be derived. These controllers have the structure of proper and not proper PID^{n-1}, where n is the order of the plant. The controller's poles cancel out the plant's zeros, see (Rusnak, 1998; Rusnak, 1999) for details.

Although from input-output transfer function, there is no difference between the full state feedback PI controller and the generalized PID^{n-1} controller and other architecture-structure topology, there are differences with respect to the response to initial conditions, effects of saturation etc..

8. Conclusions

By the use of The stochastic Linear Quadratic Tracking theory we formulated a control-tracking problem and showed those cases when their solution gives the PID family of controllers. This way we avoided heuristics and gave a systematic approach to explanation for the good performance of the PID controllers. The PID controller architecture is optimal for a 2nd order systems with no zero. The reference trajectory is generated by a system identical to the plant.

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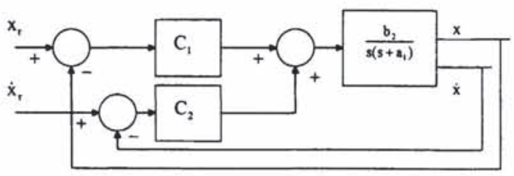

Figure 1: The parallel PI architecture for second order system.

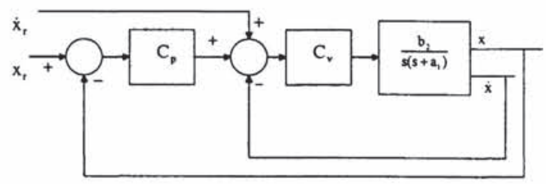

Figure 2: The cascade PI architecture for second order system

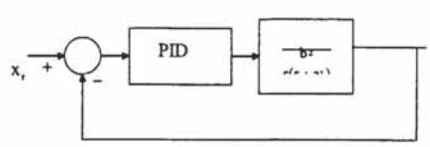

Figure 3: The PID architecture for second order system.

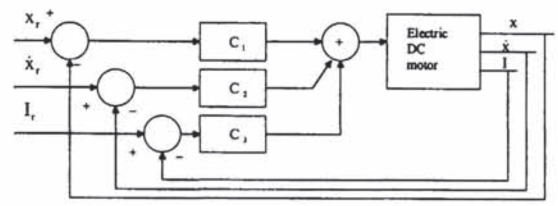

Figure 4: The parallel PI architecture for an electric DC motor.

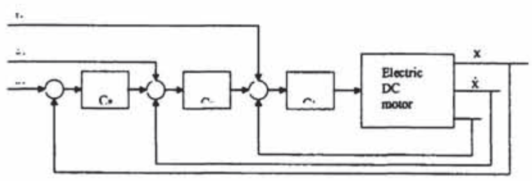

Figure 5: The cascade PI architecture for an electric DC motor.

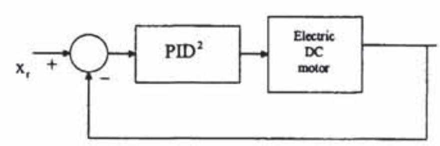

Figure 6: The PID^2 architecture for third order system.