An electronic Duffing Oscillator

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I. ELECTRONIC DUFFING OSCILLATOR

Figure 1 presents the schematics of the electronic circuit used to implement a Duffing oscillator, which is a slightly modified version of the circuit used in [1]. It contains three operational amplifiers that make up two integrators and an inverting amplifier, as well as two analog multipliers.

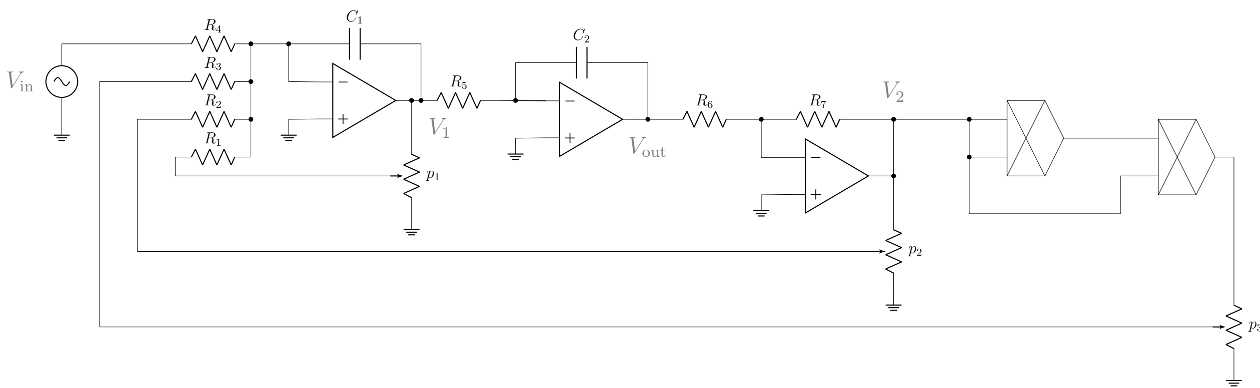

FIG. 1: Schematics of the electronic Duffing oscillator.

Since the inverting input of the leftmost operational amplifier is (virtually) grounded and the currents going into R_1 to R_4 all go into C_1:

$$\dot{V}_1 = -\frac{1}{C_1} I_1 = -\frac{1}{C_1}\left(\frac{p_1 V_1}{R_1} + \frac{p_2 V_2}{R_2} + \frac{p_3 V_3}{R_3} + \frac{V_{\text{in}}}{R_4}\right). \tag{1}$$

And since the inverting input of the middle operational amplifier is (virtually) grounded, and since the current going into R_5 also goes into C_2,

$$\dot{V}_{\text{out}} = -\frac{V_1}{C_2 R_5}. \tag{2}$$

Finally, since the inverting input of the rightmost amplifier is virtually grounded and the current going through R_6 also goes into R_7,

$$V_2 = -\frac{R_7 V_{\text{out}}}{R_6} \tag{3}$$

The output of the first multiplier is $g_m V_2^2$, and that of the third is

$$V_3 = g_m^2 V_2^3. \tag{4}$$

Assembling Equations (1)-(4), one obtains a second-order differential equation for V_2:

$$C_1 C_2 R_4 R_5 \ddot{V}_{\text{out}} + p_1 \frac{C_2 R_4 R_5}{R_1} \dot{V}_{\text{out}} + p_2 \frac{R_4 R_7}{R_2 R_6} V_{\text{out}} + p_3 \frac{g_m^2 R_4 R_7^3}{R_3 R_6^3} V_{\text{out}}^3 = V_{in}. \tag{5}$$

where p_1, p_2 and p_3 are the division ratios of the potentiometers, and g_{m} is the gain of the electronic multipliers.

Figure 2 shows a photograph of the realization of the circuit in Figure 1 built on a breadboard using Texas Instrument LM741CN operational amplifiers [2] and Analog Devices AD633ANZ multipliers [3].

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FIG. 2: Picture of the electronic Duffing oscillator.

A. Design choices

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) + k_3x^3(t) = f\cos(\omega t) \tag{6}$$

Assume the oscillator is at phase resonance and exhibits a one-harmonic response

$$x(t) = a\sin(\omega t). \tag{7}$$

Using the harmonic balance method, i.e., inserting this ansatz into Equation (6) and balancing fundamental sine and cosine coefficients, one finds

$$\begin{cases} -m\omega^2 a + ka + \dfrac{3k_3}{4}a^3 = 0 \\ c\omega a = f \end{cases} \tag{8}$$

Inserting the second equation into the first one,

$$m\omega^2 - k - \frac{3k_3}{4}\frac{f^2}{c^2\omega^2} = 0, \tag{9}$$

from which we deduce the phase resonance backbone expression

$$\omega = \sqrt{\frac{k + \sqrt{k^2 + \dfrac{3mk_3f^2}{c^2}}}{2m}}. \tag{10}$$

We can also normalize this frequency with the natural frequency of the underlying linear oscillator ($\omega_0^2 = k/m$)

$$\frac{\omega}{\omega_0} = \sqrt{\frac{1 + \sqrt{1 + \dfrac{3mk_3f^2}{k^2c^2}}}{2}}, \tag{11}$$

and the amplitude of the phase resonance backbone can be found with the second relation in Equation (8)

$$a = \frac{f}{c\omega}. \tag{12}$$

There are four free parameters, m, c, k and k_3. We prescribe the linear natural frequency ω_0, the damping ratio $\zeta_0 = 2c/(m\omega_0)$, as well as the frequency ratio ω/ω_0 and the forcing level f when the oscillator undergoes maximum-amplitude oscillations, when $a = a_{\max}$.

We also need to remember that, from Equation (2),

$$|V_1|_{\max} = C_2R_5\omega a_{\max} \tag{13}$$

and from Equation (3)

$$|V_2|_{\max} = \frac{R_7}{R_6}a_{\max}. \tag{14}$$

Neither of these voltages should exceed the maximum allowed by the power supply. As for V_3, since $g_m = 0.1$, the multipliers will saturate only if V_{out} saturates. Choosing $R_6 = R_7$ also allows us not to worry about V_2 as there will be no amplification.

Polypropylene (PP) film capacitors were selected for C_1 and C_2 due to their low dissipation.

Bypass capacitors of 0.1μF were used for all integrated circuits [TECHNICAL NOTE], and bulk capacitors of 10μF were used for the power supply.

Parameter	Value
R_1 (kΩ)	100
R_2 (kΩ)	10
R_3 (kΩ)	1
R_4 (kΩ)	100
R_5 (kΩ)	1
R_6 (kΩ)	10
R_7 (kΩ)	10
C_1 (μF)	1
C_2 (μF)	1
g_{m} (V^{-1})	0.1

TABLE I: Electrical parameters of the electronic Duffing oscillator.

B. Validation

Table I gathers the electrical parameters of the electronic oscillator. The potentiometers were adjusted to minimize the dissipation ($p_1 = 0$), maximize the nonlinearity ($p_3 = 1$) and set the linear resonance frequency of the oscillator to a relatively small value ($p_2 =$). From these values and those in I, using Equations (**??**)-(**??**), it was possible to deduce the theoretical characteristics of the oscillator given in Table II. These theoretical values were also validated with a least-squares fit of the parameters in Equation (**??**) against experimental data with the results of a swept-sine excitation at $f_1 = XX$V, yielding the identified values in Table II. The fitted conservative ($\beta x_1 + \gamma x_1^3$) and non-conservative (αx_2) terms are plotted against measurements in Figures 3(a) and 3(b), respectively.

Parameter	α	β	γ	δ
Theoretical value	0			
Identified value				

TABLE II: Parameters of the Duffing oscillator.

(a) (b)

FIG. 3: Conservative $\beta x_1 + \gamma x_1^3$ (a) and non-conservative αx_2 (b) terms: experimental measurements (•) and fitted law (—).

We observe small difference between theoretical and experimental values, as well as a normalized least-squares error of the surface fit of 5%. This indicates that there are significant non-idealities in the circuit that may come from non-ideal operational amplifiers and multipliers characteristics, and loading effects on the potentiometers. Nevertheless, a precise realization of a Duffing oscillator was not mandatory in this work and the qualitative agreement of the circuit with its theoretical counterpart was deemed good enough for this study.

[1] K. Srinivasan, K. Thamilmaran, and A. Venkatesan, Effect of nonsinusoidal periodic forces in Duffing oscillator: Numerical and analog simulation studies, Chaos, Solitons & Fractals **40**, 319 (2009).

[2] *LM741 Operational Amplifier*, Texas Instruments (2015).

[3] *Low Cost Analog Multiplier*, Analog Devices (2015), rev. K.