تمرین سری سوم بهینه‌سازی   محمدرضا دلد   ۴۰۰۱۳۱۰۱۸

سوال ①

$$\begin{aligned} & x+y+z=1 \\ + \; & -x-y+z=0 \\ \hline & 0+0+2z=1 \rightarrow z=\tfrac{1}{2} \end{aligned}$$

z مشخص شد.

$$y=t \rightarrow -x-t+\frac{1}{2}=0 \rightarrow \boxed{x=-t+\frac{1}{2}}$$

معادله نقطه‌ای برداری خط تلاقی دو صفحه:

$$P(t) = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} + t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \begin{array}{c} x \\ y \\ z \end{array}$$

کمترین فاصله تا مبدا $\longleftarrow$ $\min \|P\|^2$

چون محدب است مشتق می‌گیریم و برابر صفر می‌گذاریم:

$$\|P\|^2 = \left(\tfrac{1}{2}-t\right)^2 + t^2 + \left(\tfrac{1}{2}\right)^2$$

$$-2\left(\tfrac{1}{2}-t\right) + 2t = 0 \rightarrow t=\frac{1}{4}$$

نقطه مورد نظر: $P\left(\frac{1}{4}\right) = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{pmatrix}$

سوال 2

$$\begin{pmatrix} 1 & 0 \\ 1 & 5 \\ 1 & 10 \\ 1 & 15 \end{pmatrix} \begin{pmatrix} \ln a \\ b \end{pmatrix} = \begin{pmatrix} \ln(227) \\ \ln(237) \\ \ln(249) \\ \ln(262) \end{pmatrix}$$

$$x^* = (A^TA)^{-1} A^T b \rightarrow A^TA = \begin{pmatrix} 4 & 30 \\ 30 & 350 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 10 & 15 \end{pmatrix} \begin{pmatrix} 5.42 \\ 5.46 \\ 5.51 \\ 5.56 \end{pmatrix} = \begin{pmatrix} 21.97 \\ 166.03 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 30 \\ 30 & 350 \end{pmatrix} \begin{pmatrix} \ln a \\ b \end{pmatrix} = \begin{pmatrix} 21.97 \\ 166.03 \end{pmatrix} \rightarrow \left(\begin{array}{cc|c} 1 & 0 & 5.42 \\ 0 & 1 & 0.0095 \end{array}\right)$$

$5.42 \rightarrow \ln a$

$0.0095 \rightarrow b$

$$\ln a = 5.42 \rightarrow a = 226.5$$

$$P(t) = 226.5\, e^{0.00927t}$$

$$P(30) = 226.5\, e^{0.0095 \times 30} = \boxed{301.19}$$

سوال ③

$$f(x,\theta) = (1-\theta)^{x-1}\,\theta$$

$$\text{likelihood}: \; L(\theta) = (1-\theta)^{x_1-1}\theta\,(1-\theta)^{x_2-1}\theta \cdots (1-\theta)^{x_n-1}\theta = \theta^n (1-\theta)^{\sum_1^n x_i - n}$$

$$\text{log likelihood}: \; \ln L(\theta) = n\ln\theta + \sum_{i=1}^{n} x_i - n \,.\, \ln(1-\theta)$$

$$\frac{d}{d\theta}\ln L(\theta) = \frac{n}{\theta} - \frac{\sum_1^n x_i - n}{1-\theta} = 0 \rightarrow \theta = \frac{n}{\sum_{i=1}^{n} x_i} = \boxed{\frac{1}{\bar{X}}}$$

سوال ④

$$\begin{cases} P(0): \; P(y=0) = 1 - Q(a^T u + b) = P(-a^T u - b) \\ P(1): \; P(y=1) = Q(a^T u + b) \end{cases}$$

$$Q(z) = \frac{1}{2\pi}\int_{z}^{\infty} e^{t^2/2}\, dt$$

$$\text{log likelihood}: \sum_{y_i=1} \log Q(a^T u_i + b) + \sum_{y_i=0} \log Q(-a^T u_i - b)$$

می‌دانیم یک تابع مقعر است.

سوال ⑤

$$\min \sum_{1}^{n} x_i \log(x_i)$$
$$\text{s.t.} \quad Ax \leq b$$
$$[1]^T x = 1$$

Linear program: $\sum x_i \log(x_i) + \lambda^T(Ax-b) + v(1^T x - 1)$

$$g(\lambda, v) = \inf_x L_p = -\lambda^T b - v + \inf \big(\sum x_i \log x_i$$
$$+ (\lambda^T A + v 1^T) x_i$$
$$= -\sup (-A^T\lambda - v_1)^T x - \sum_{1}^{n} x_i \log x_i$$
$$\rightarrow y = -A^T\lambda - v_1$$
$$f^{*}(-A^T\lambda - v_1) = f^{*}(y) = -\sup (y^T A - f(x))$$
$$f^{*}(y) = \sum_{i=1}^{n} e^{y_i - 1} \rightarrow y = -A^T\lambda - v_1 \rightarrow f^{*}(-A^T\lambda - v_1)$$
$$\sum_{i=1}^{n} e^{-a_i^T\lambda - v - 1}$$

dual $\quad g(\lambda, v) = -\lambda^T b - v - \sum e^{-a_i^T\lambda - v - 1}$

$\max \ g(\lambda, v) \ \& \ \lambda \geq 0 \rightarrow \sup(y^T x - \sum x_i \log x_i) \leftarrow$ کب

$$\begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} - \begin{pmatrix} \log x_1 + 1 \\ \vdots \\ \log x_n + 1 \end{pmatrix} = 0 \rightarrow y_i = \log x_i + 1$$
$$x_i = e^{y_i - 1}$$

ادامه سوال ⑤

$$f(y) = \sum_{i=1}^{n} y_i e^{y_i - 1} - \sum e^{y_i-1}(y_i - 1)$$

$$= \sum e^{y_i-1}(y_i - y_i + 1) = \sum_{i=1}^{n} e^{y_i - 1}$$

سوال ⑥

$$\min_x f(x) = \|x\|^2$$

$$\text{s.t. } c(x) = a^T x + \alpha \geq 0 \qquad L(x,y) = f(x) - \lambda c(x)$$

$$\|x\|^2 - \lambda(a^T x + \alpha)$$

مشتق: $\nabla_x L(x,\lambda) = 2x - \lambda a$, $\nabla_{xx} L(x,\lambda) = 2I$; $2I > 0$

KKT : $\begin{cases} \nabla_x L(x^*, \lambda^*) = 0 \\ \lambda^* c(x^*) = 0 \\ \lambda^* \geq 0 \end{cases} \Rightarrow x^* = \frac{\lambda^*}{2} a$

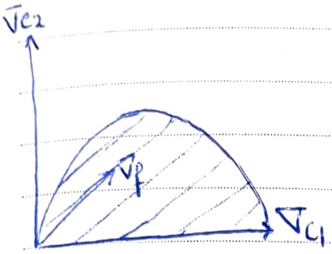
$$\lambda^* = 0 \perp a^T x^* + \alpha = \frac{\lambda^* \|a\|^2}{2} + \alpha = 0$$

if $\alpha \geq 0 \rightarrow \lambda^* = 0 \rightarrow (x^*, \lambda^*) = (0, 0)$

if $\alpha < 0 \rightarrow (x^*, \lambda^*) = -\left(\frac{\alpha}{\|a\|^2} a, \frac{2}{\|a\|^2}\right)$

سوال (7)

$$\min \quad x_1 + x_2$$
$$\text{s.t.} \quad x_1^2 + x_2^2 \leq 2$$
$$x_2 \geq 0$$



ناحیه feasible :

$$\mathcal{L}(x,\lambda) = f(x) - \lambda_1 c_1(x) - \lambda_2 c_2(x)$$

$$\nabla_x L(x^*, \lambda^*) = 0, \quad \lambda^* \geq 0$$

$$\lambda_1^* c_1(x^*) = 0, \quad \lambda_2^* c_2(x^*) = 0$$

$$x^* = \begin{pmatrix} -\sqrt{2} \\ 0 \end{pmatrix} \longrightarrow \nabla f(x^*) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\nabla c_1(x^*) = \begin{pmatrix} 2\sqrt{2} \\ 0 \end{pmatrix} \quad \nabla c_2(x^*) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \boxed{\lambda^* = \begin{pmatrix} \frac{1}{2\sqrt{2}} \\ 1 \end{pmatrix}}$$