Assignment 3
Applications of a 200-Year-Old Technique in Image Processing

Homeworks Guidelines and Policies

- ***What you must hand in.*** It is expected that the students submit an assignment report (HW3_[student_id].pdf) as well as required source codes (.m or .py) into an archive file (HW3_[student_id].zip).
- ***Pay attention to problem types.*** Some problems are required to be solved *by hand* (shown by the icon), and some need to be implemented (shown by the icon).
  Please don't use implementation tools when it is asked to solve the problem by hand, otherwise you'll be penalized and lose some points.
- ***Don't bother typing!*** You are free to solve by-hand problems on a paper and include picture of them in your report. Here, cleanness and readability are of high importance. Images should also have appropriate quality.
- ***Reports are critical.*** Your work will be evaluated mostly by the quality of your report. Don't forget to explain what you have done, and provide enough discussions when it's needed.
- ***Appearance matters!*** In each homework, 5 points (out of a possible 100) belongs to compactness, expressiveness and neatness of your report and codes.
- ***Python is also allowable.*** By default, we assume you implement your codes in MATLAB. If you're using Python, you have to use equivalent functions when it is asked to use specific MATLAB functions.
- ***Be neat and tidy!*** Your codes must be separated for each question, and for each part. For example, you have to create a separate .m file for part b. of question 3. Please name it like p3b.m.
- ***Use bonus points to improve your score.*** Problems with bonus points are marked by the icon. These problems usually include uncovered related topics or those that are only mentioned briefly in the class.
- ***Moodle access is essential.*** Make sure you have access to Moodle because that's where all assignments as well as course announcements are posted on. Homework submissions are also done through Moodle.

- ***Assignment Deadline.*** Please submit your work **before the end of June 4th**.
- ***Delay policy.*** During the semester, students are given 7 free late days which they can use them in their own ways. Afterwards there will be a 25% penalty for every late day, and no more than three late days will be accepted.
- ***Collaboration policy.*** We encourage students to work together, share their findings and utilize all the resources available. However you are not allowed to share codes/answers or use works from the past semesters. Violators will receive a zero for that particular problem.
- ***Any questions?*** If there is any question, please don't hesitate to contact us through the following email addresses: **ali.the.special@gmail.com** and **fardin.aiar@gmail.com**.

1. Fundamentals of Fourier Transform (I): DCT, DFT, and Sampling Theorem (16 Pts.)

Keywords: *Frequency Domain, Fourier Transform, Inverse Fourier Transform, Discrete Fourier Transform, Discrete Cosine Transform, Sampling Theorem, Aliasing, Nyquist Sampling Rate*

The **Fourier Transform** is probably the most common transformation occurring in the nature. While its applications are mainly prominent in signal processing and differential equations, many other applications also make the Fourier transform and its variants universal elsewhere in almost all branches of science and engineering. No wonder it has such an enormous impact in the area of image processing and computer vision as well.

Trying to mathematically master **Fourier Transform** and related topics in the **Frequency Domain**, here we will start with three topics: **Discrete Cosine Transform**, **Discrete Fourier Transform** and **Sampling Theorem**.

I. Discrete Cosine Transform

The definition of DCT is

$$F(u,v) = \frac{2C(u)C(v)}{\sqrt{MN}} \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \cos\frac{(2i+1)u\pi}{2M} \cos\frac{(2j+1)v\pi}{2N} f(i,j)$$

where $i,u = 0,1,\ldots,M-1$, $j,v = 0,1,\ldots,N-1$, and constants $C(u)$ and $C(v)$ are defined by

$$C(\xi) = \begin{cases} \frac{\sqrt{2}}{2}, & \xi = 0 \\ 1, & \text{otherwise} \end{cases}$$

Assume the 8×8 image $f(x,y)$ given in Figure 1-a.

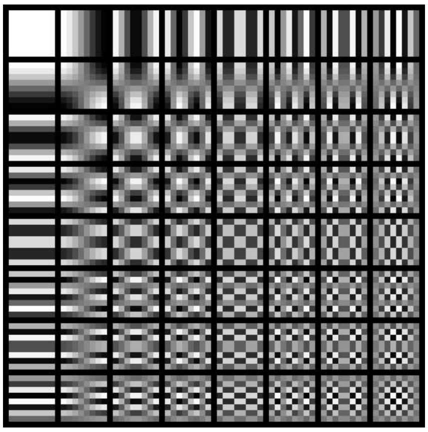
a. Determine the value of $F(0,0)$.

b. $F(0,0)$ is known as a DC coefficient. Using the computation experience from the previous part, explain what the meaning of DC coefficient is and why it is called a “DC value”.

c. Figure 1-b displays 8×8 2-D DCT basis functions. Using the definition of DCT transform, mathematically prove that $F(0,2)$ is actually related to the pattern in the position $(0,2)$.

10	90	160	80	10	90	160	80
20	100	150	70	20	100	150	70
30	110	140	60	30	110	140	60
40	120	130	50	40	120	130	50
50	130	120	40	50	130	120	40
60	140	110	30	60	140	110	30
70	150	100	20	70	150	100	20
80	160	90	10	80	160	90	10

(a)



(b)

Figure 1 (a) An 8x8 image given for the first part. (b) 8x8 2-D DCT basis functions.

II. Discrete Fourier Transform

If the Fourier transform of a function $f(x)$ is $F(u)$, determine the Fourier transform of the following functions:

d. $f(x-3)$
e. $f^2(x)$
f. $f(-x)$
g. $f(4x-3)$
h. $F(x)$

Next, consider the following 4×4 grayscale image.

33	23	13	03
32	22	12	02
31	21	11	01
30	20	10	00

i. Calculate the 2D DFT of the image.
j. Multiply the image by $(-1)^{x+y}$ and repeat the previous part.
k. How do the results relate to each other? Explain.

Now, Consider the given matrix below.

l. Compute its 2-D inverse DFT.

$$F(u,v)=\begin{bmatrix}-1 & 0 & 2\\ 2 & 0 & 1\\ 1 & 0 & -2\end{bmatrix}$$

III. Sampling Theorem

Assume $F(u,v)$ (shown in Figure 2) is the 2D Fourier transform of the function $f(x,y)$. $F(u,v)=1$ in the shaded region and zero outside.

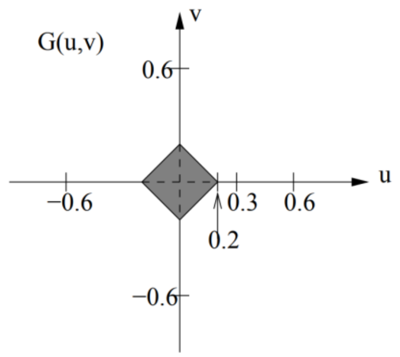


Figure 2 Sketch of the Fourier transform of the given function.

m. The function $f(x,y)$ is sampled by a 2D impulse train to

$$g(x,y)=f(x,y)\cdot\sum_n\delta\left(x-n/0.6\right)\cdot\sum_m\delta\left(y-m/0.6\right)$$

In other words, the sampling distance is $\Delta=1/0.6$ in both directions. Sketch $G(u,v)$ in the (u,v)-plane and grade the axes.

n. The same function is sampled by another 2D impulse train to

$$h(x,y)=f(x,y)\cdot\sum_n\delta\left(x-n/0.3\right)\cdot\sum_m\delta\left(y-m/0.3\right)$$

Sketch $H(u,v)$ in the (u,v)-plane and grade the axes.

o. Which one of the functions $g(x,y)$ and $h(x,y)$ show aliasing? Specify which sampling distance Δ will give aliasing for $F(u,v)$ and which not. Mark the aliasing in your sketch.

p. Now if we consider $V(u,v)=\mathcal{R}_{45^\circ}\left[F(u,v)\right]$ (i.e. we get $V(u,v)$ if we rotate $F(u,v)$ 45°). Find the 2D inverse Fourier transform of $V(u,v)$.

q. Specify which sampling distance Δ shows aliasing for $V(u,v)$ and which not.

Note: You may complete the sketch in Figure 2 which is available in the assignment folder.

2. Fundamentals of Fourier Transform (II): 2D Convolution (16 Pts.)

Keywords: *Frequency Domain, 2D Convolution, Image Filtering*

Following the previous problem, here we extend our notion of image processing in the frequency domain to an extremely beneficial process: **2D Convolution**. The whole idea of **Image Filtering** in the frequency domain is theoretically based upon convolution and its properties.

First, consider the following 3×3 filters in spatial domain:

$$h_1(x,y)=\begin{bmatrix}0 & -1 & 0\\ -1 & 5 & -1\\ 0 & -1 & 0\end{bmatrix},\quad h_2(x,y)=\begin{bmatrix}1 & 2 & 1\\ 2 & 4 & 2\\ 1 & 2 & 1\end{bmatrix}$$

a. What's the equivalent filters $H_1(u,v)$ and $H_2(u,v)$ in the frequency domain?

b. Are they low-pass or high-pass filters? Prove mathematically.

Next, assume the following filter transfer functions in the frequency domain:

$$H_1(u,v)=8-2\cos(2\pi u)-2\cos(2\pi v)-2\cos(2\pi(u+v))-2\cos(2\pi(u-v))$$

$$H_2(u,v)=\frac{j}{2}\left[\sin\left(2\pi u/M\right)-\sin\left(2\pi v/N\right)\right],\quad u=0,1,\dots,M-1,\quad v=0,1,\dots,N-1$$

c. Find the coefficients of the spatial filters and present them as 3×3 operators.

d. Are these filters low-pass, high-pass, band-pass or band-reject filters? Explain.

Now, assume an input image if filtered in the frequency domain using the following filters. Explain the effects of each filter on the output image.

e. $V_1(u,v)=-4\pi^2(u^2+v^2)$

f. $V_2(u,v)=\left[1+4\pi^2(u^2+v^2)\right]$

g. $V_3(u,v)=\exp\left(-\dfrac{u^2+v^2}{2D_0}\right),\quad D_0=1$

h. $V_4(u,v)=\exp\left(-\dfrac{u^2+v^2}{2D_0}\right),\quad D_0\to 0$

i. $V_5(u,v)=\exp\left(-\dfrac{u^2+v^2}{2D_0}\right),\quad D_0\to\infty$

Finally, the relationship between two images $f_1(x,y)$ and $f_2(x,y)$ in the frequency domain is:

$$F_1(u,v)=F_2(u,v)\left(2-\frac{\sin(3\pi u)}{3\pi u}\frac{\sin(3\pi v)}{3\pi v}\right)$$

j. Find the relationship between the images in spatial domain.

k. Which images will have sharper edges? Why?

l. What's the name of filtering which transforms $f_2(x,y)$ to $f_1(x,y)$?

m. Assuming the following new relationship between the two images in the frequency domain, specify which image will have sharper edges:

$$F_1(u,v)=F_2(u,v)\left(1-\frac{\sin(3\pi u)}{3\pi u}\frac{\sin(3\pi v)}{3\pi v}\right)$$

3. Image Analysis in Frequency Domain is Literally an Abstract Art! (15 Pts.)

Keywords: *Frequency Domain, Fourier Analysis, Magnitude, Phase*

The focus of this problem is mainly on what it means to represent images in the **Frequency Domain**, what can be inferred from **Fourier Representation** of an image and how image representation in spatial and frequency domain relate to each other. First, consider the images in Figure 3.

a. Specify which image(s) have Fourier transforms $F(u,v)$ with the following properties.

i. The real part of $F(u,v)$ is zero at all u,v. ii. The imaginary part of $F(u,v)$ is zero at all u,v.

iii. $F(u,v)$ is purely real and positive for all u,v. iv. $F(0,0)=0$ v. $F(u,v)$ has circular symmetry

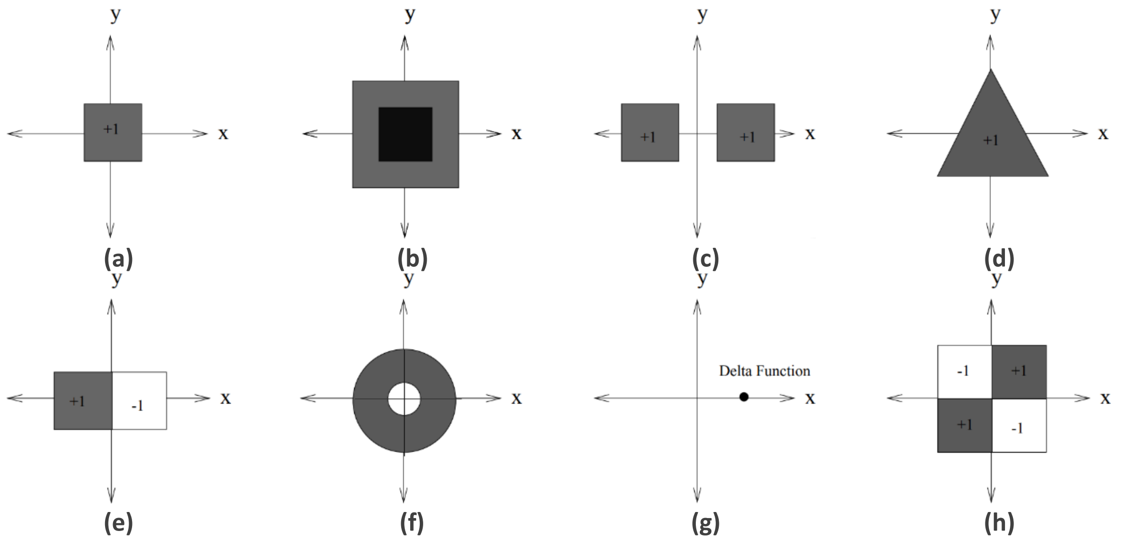


Figure 3 Images with different structures given for the first part.

In the next part, we will get a hands-on experience with analysis of image representation in frequency domain. Consider the paintings in Figure 4.

b. Compute and display the magnitude and phase of the images. Explain which specific structure in each image yields the results you obtained. Also specify the relations between the images in the spatial domain and the frequency domain.

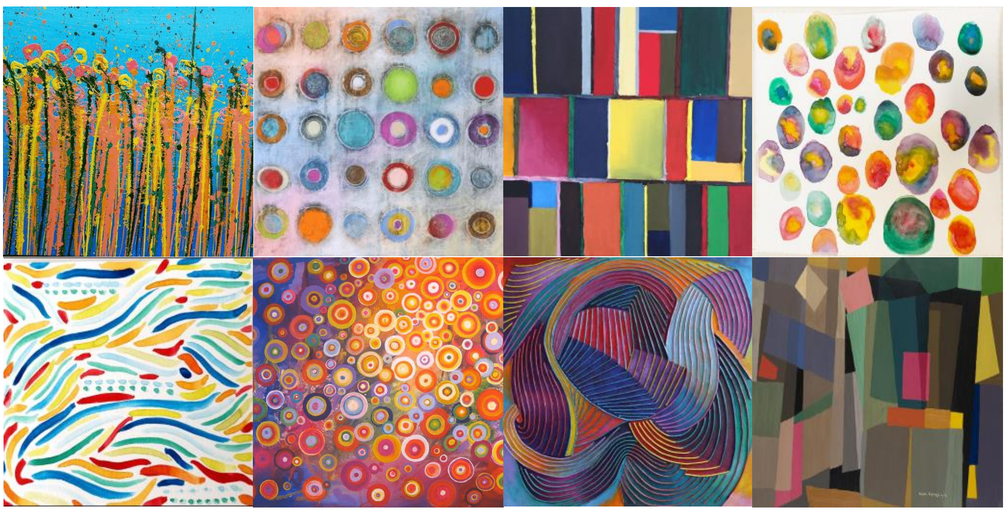


Figure 4 Some abstract paintings containing different patterns and structures.

4. Don't Always Trust What You See! (20 Pts.)

Keywords: *Image Filtering, Frequency Domain, Fourier Transform, Notch Filter, Color Spaces*

Figure 5 This image is actually a grayscale version of the famous "Girl with a Pearl Earring" painting. Only the grid lines are colored.

Take a look at the image in Figure 5. This is a famous oil painting by Dutch painter Johannes Vermeer, known as *Girl with a Pearl Earring*. You've probably seen it many times before. But is this image exactly the one painted by Vermeer back in 1665? Well, not quite.

The image you see is in fact a grayscale version of that painting. If you look more closely, you will notice tiny color grid lines which trick your brain to perceive and fill in the missing colors itself. This process is known as *Color Assimilation Grid Illusion*.

In order to prove that this effect is actually an illusion, we are going to use **Fourier Transform** to capture and then **Notch Filters** to remove these patterns. Note that, in contrary to the previous problem, you are now dealing with color images. Therefore you need to filter each channel separately, and then combine them again to obtain a filtered color result. On the other hand, based on the pattern characteristics, it might be better to first convert the images into other color spaces in order to better capture those grid lines patterns, and then convert the filtered result back to RGB space again to visualise it. Figure 6 compares amplitude of the Fourier transform in different channels and different color spaces.

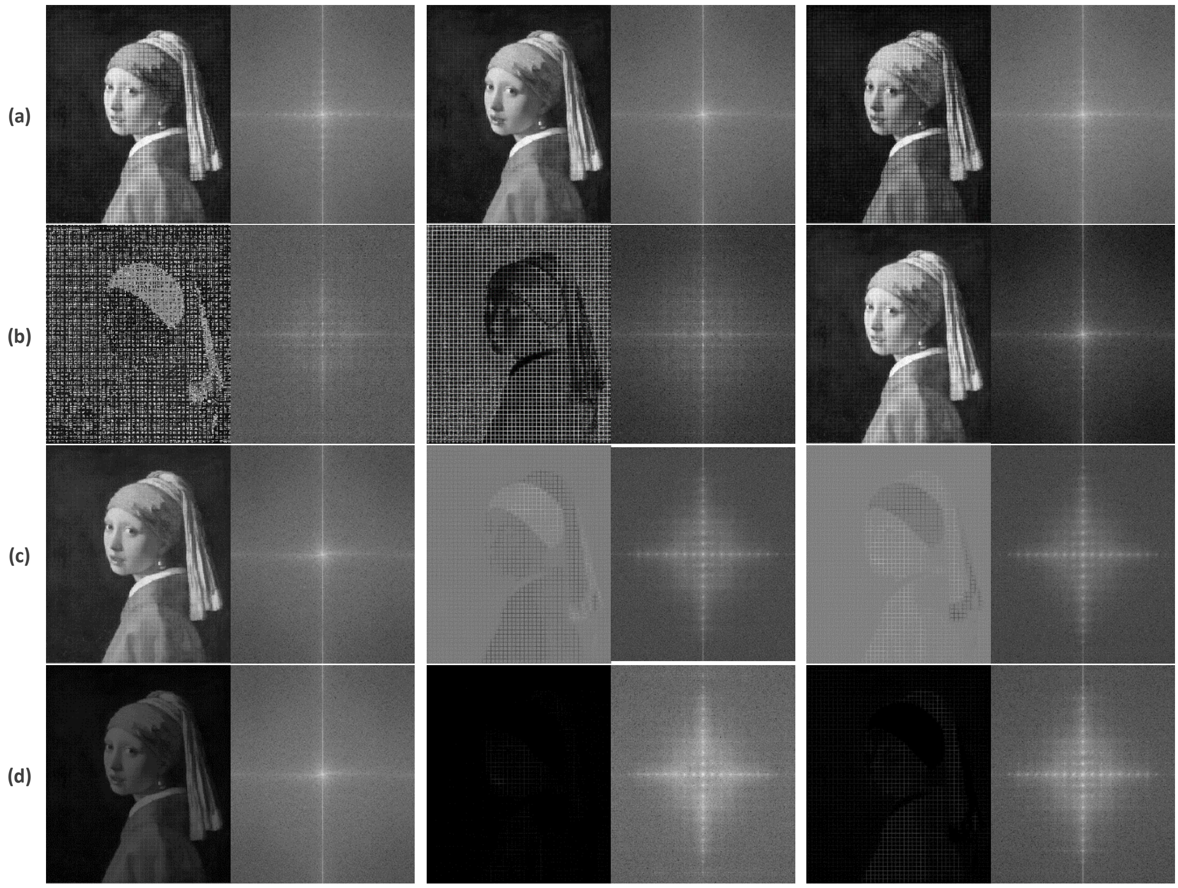
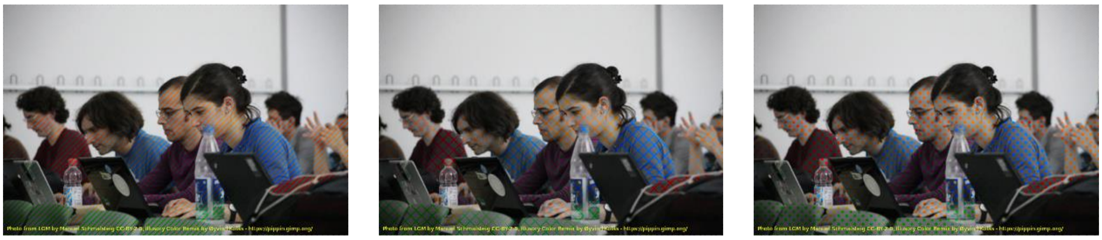


Figure 6 Comparison of Fourier transform of each channel in different color spaces (a) RGB (b) HSV (c) YCbCr (d) Lab

I. When Your Brain Paints!

First we want to prove that the color assimilation grid illusion is really an illusion.

a. For each one of the images in Figure 7, display the amplitude of the Fourier transform of each one of the channels in the color space which is best to highlight its color grid variations.



(a) **(b)** **(c)**

Figure 7 Different grids still put the same effect over the image. (a) Diagonal lines. (b) Diagonal grid. (c) Dot grid.

b. Repeat the previous part for the following images. This time, try to remove the patterns and obtain clear grayscale images by applying appropriate bandpass filters.

Figure 8 Removing the color patterns in these images would restore the original grayscale images

II. Impossible Munker's Illusions: Are They Actually The Same?

A similar technique can be applied to create images containing two or more shapes that appear to be in different colors, but are actually of the same color. These illusions is called *Munker's Illusions*.

c. Prove that the shapes inside the images in the following figure are of the same color.

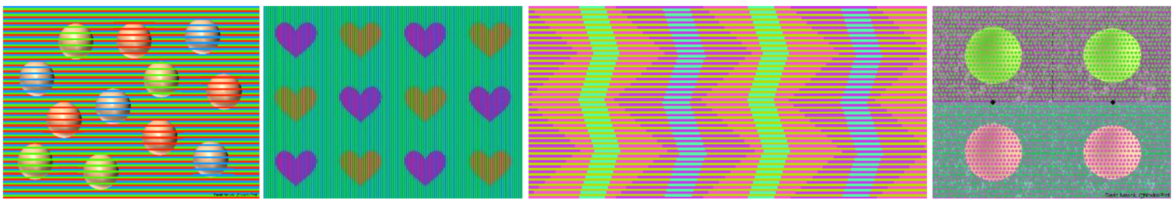


Figure 9 Shapes in these images (spheres, hearts and zigzag lines) are all identical

d. Repeat the previous part for the following images, considering the fact that the patterns are now much more complicated.

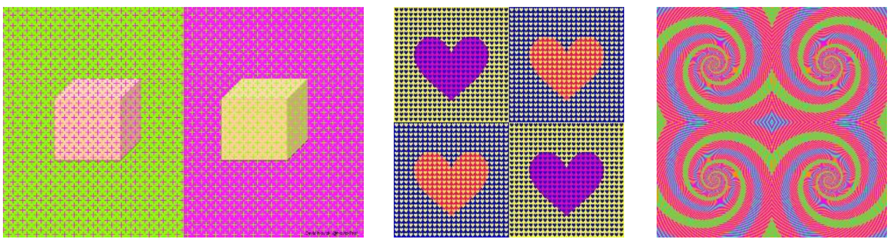


Figure 10 Color patterns in these Munker's Illusions examples are more complicated, but the shapes (cubes, hearts, blue and green spirals) are still identical

5. Albert Einstein or Marilyn Monroe? It Depends on Where You are Standing! (18 Pts.)

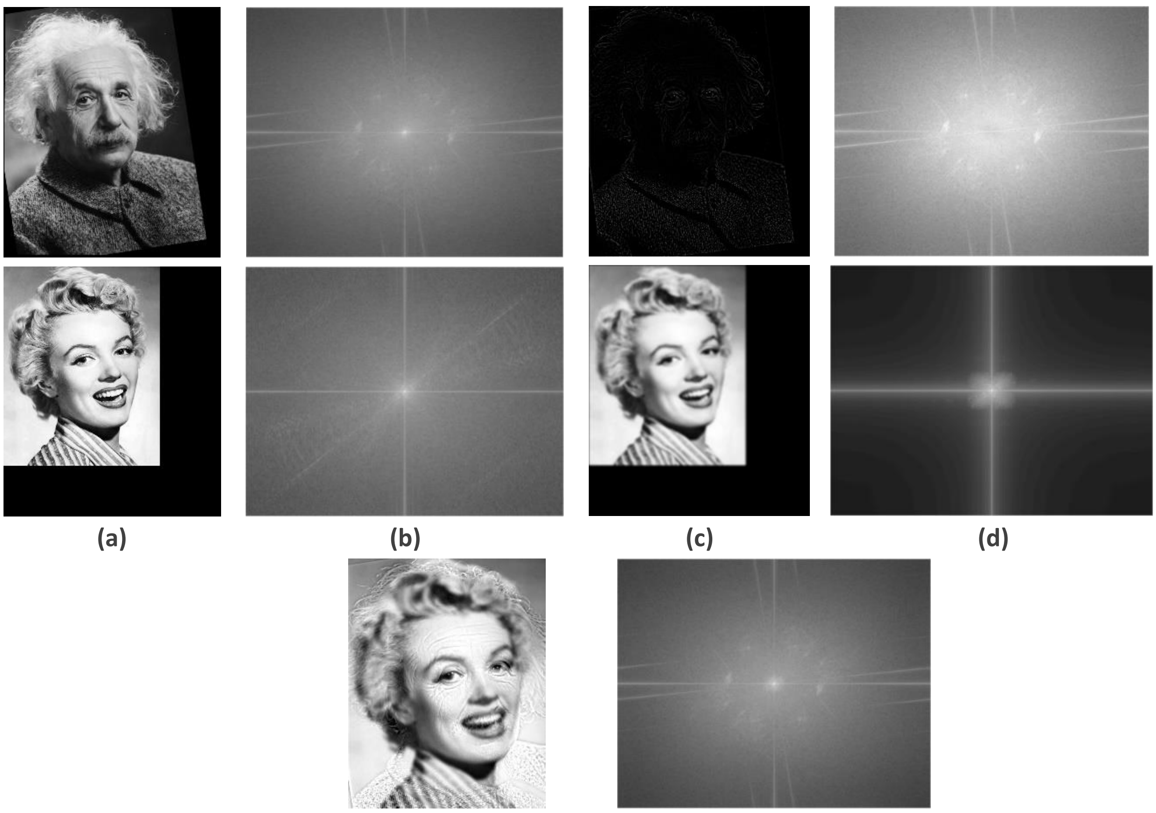
Keywords: *Image Filtering, Frequency Domain, Bandpass Filters, Hybrid Images, Image Alignment*

This problem aims to get you familiar with an amazing application of **Image Filtering** in **Frequency Domain**: *hybrid images*. Hybrid images are statistic images whose interpretation depends on the distance between the image and the observer. In an image, **High Frequency** details tend to be more noticeable from close range, while only the **Low Frequency** signal can be perceptible from a distance. The idea is to merge the high frequency content of one image with the low frequency content of the other, and obtain an image whose interpretation varies based on the viewing distance.

(a) (b) (c)

Figure 11 An example of hybrid images (a) Albert Einstein portrait (b) Marilyn Monroe portrait (c) Hybrid images containing both portraits

More precisely, given two images, the procedure starts with aligning their contents (here, faces) so that the final hybrid image makes more sense. Then, it continues with applying a low pass filter on one image and a high pass filter on the other. Finally, all that is left to do is to add the results of the filters, or calculate their averages. The cut-off frequency of each filter should be chosen empirically.



(a) (b) (c) (d)

(e) (f)

Figure 12 The process of creating a hybrid image (a) Aligned input images (b) Fourier spectrum of the aligned input images (c) The result of applying band-pass filters on the images (d) Fourier spectrum of the filtered images (e) Resultant hybrid image (f) Fourier spectrum of the hybrid image

So, read the provided input image pairs (Figure 13) and perform the following steps for each pair.

a. **Image alignment.** You are given a function (`align_imgs.m` for MATLAB and `align_imgs.py` for Python) which takes two images and two pairs of points, and align them so that the two pairs of points will have approximately equal coordinates. Use this function to align the input images. Display the resultant aligned images and their amplitudes of the Fourier transform.
Note: The Python function is not tested, hence it is not guaranteed to work flawlessly.
b. **Image filtering.** Apply low-pass filter on the first image using a standard Gaussian filter, and high-pass filter on the other by subtracting the image filtered with Gaussian filter from the original one. Choose proper values for cut-off frequencies. Display the results as well as their logarithmic amplitude of the Fourier transform.
c. **Merge Images.** Merge the images you obtained in the previous part, and display the final hybrid image and the corresponding amplitude of the Fourier transform.
d. **Visualisation.** Apply a Gaussian filter with five increasing cut-off values on the hybrid images in order to illustrate the process of transformation of one image into another.

Figure 13 Two set of input images. These images must be aligned before adapting the algorithm

6. Some Explanatory Questions (10 Pts.)

Please answer the following questions as clear as possible:

a. What would be the result of the convolution of an even function and an odd function? Prove your answer.
b. How does aliasing appear in an image? Assume an image containing diagonal black and white lines (sinusoidal image). Will the aliasing cause the distance between the lines become longer or shorter? Will it change the direction of the lines?
c. Consider two types of padding with equal total number of zeros; appending zeros to the ends of rows and columns of the image, and surround the image by a border of zeros. Do you think these two types would make any difference in image filtering in the frequency domain? Explain.
d. Suppose for a discrete Fourier transform of an image, we multiply -1 to the phase image and then recombine it with the magnitude and use inverse Fourier transform to obtain a new image. Does the new image bear any resemblance with the previous one? Explain mathematically.
e. Mathematically explain how rotation and scaling affects phase plot of an image in the Fourier domain.

Good Luck!
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