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A constitutive model for anisotropic damage in fiber-composites

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Abstract

A constitutive model for anisotropic damage is developed to describe the elastic–brittle behavior of fiber-reinforced composites. The main objective of the paper focuses on the relationship between damage of the material and the effective elastic properties for the purpose of stress analysis of structures. A homogenized continuum is adopted for the constitutive theory of anisotropic damage and elasticity. Internal variables are introduced to describe the evolution of the damage state under loading and as a subsequence the degradation of the material stiffness. The corresponding rate-equations are subjected to the laws of thermomechanics. Emphasis is placed on a suitable coupling among the equations for the rates of the damage variables with respect to different damage modes. Evolution equations for the progression of the passive damage variables complete the kinetic equations. Most material parameters are obtained from uniaxial and simple shear tests as demonstrated by the example.

Keywords: Fiber-reinforced composites; Anisotropic damage, Rate equations; Internal variables; Passive damage; Damage mechanics; Failure mechanics; Dissipation potential

1. Introduction

Damage plays an important role in many fibrous composite materials with non-ductile matrices. Their elastic–brittle behavior is characterized by the formation and evolution of microcracks (surface discontinuities) and cavities (volume discontinuities). Pronounced irreversibility of these defects is a consequence. These defects cause primarily stiffness degradation and only small permanent deformations remain in the stress-free body after unloading as long as the material is not close to complete deterioration. The main objective of this paper is the construction of a simple damage model for stress analysis of fiber-composite structures. The organization of the article is as follows.

Before the constitutive equations are developed, the mechanical behavior of a broad range of laminated fiber-reinforced composites is outlined briefly. Glass- or carbon-fiber reinforced vinylester or epoxy resins typically fall into this class of materials. Their mechanical response to deformations will be cast into a mathematical model. Special emphasis is given to the interaction between fiber damage due to fiber stress and matrix damage due to transverse and shear stress on the elastic response and the ability to transmit various states of stresses.

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As an outgrowth of various failure mechanisms – also denoted as failure modes – in fiber-composites the loading surfaces in strain space for unidirectional (UD-) laminae are obtained by identifying them with the corresponding failure surfaces in stress space. A set of "internal (hidden) variables", denoted as damage variables, is introduced to indicate the state of anisotropic damage within the limits of the theory for "homogenized continua". These unknowns are treated as phenomenological internal variables, since they have no direct relation to the micromechanics of crack and void growth.

A simple algebraic structure of the constitutive tensor for the damaged lamina is proposed next. Its dependence on the damage variables gives rise to the application of the theory of internal variables for irreversible mechanical processes. It provides the necessary kinetic equations for the hidden variables of the damaged material. For this purpose a potential function – denoted as the damage potential or dissipation potential – is introduced mainly for convenience.

The Kelvin inequality, resulting from thermodynamics, poses well-known restrictions on possible candidates for the dissipation potential. The second law of thermodynamics is assumed to hold for the proposed set of internal variables, despite the fact that the list might be incomplete. Nonnegative dissipation, however, must be a necessary ingredient for any constitutive model if the growth of the total energy shall be excluded. A family of growth laws is finally suggested to complete the "rate-independent" kinetic equations.

The notion of "passive damage" is addressed afterwards. Evolution equations for the non-active damage parameters are introduced to advance a second state of damage, which is crucial when the influence of a given state of defects (microcracks and voids) on the effective elastic properties is different for the material behavior in tension and compression. This phenomena is typical for many brittle materials and frequently denoted as "one-sidedness". Finally, the proposed damage model is subjected to uniaxial loading along the privileged axis. These examples, together with simple shear tests, provide all the data for the material parameters of the model in its simplest version.

2. Mechanical behavior of composites

2.1. *Lamina behavior under loading*

2.1.1. *In fiber direction*

Stresses in fiber direction are predominantly transmitted through the fibers because of their high stiffness and strength in comparison to the properties of the matrix material. The transmission of tensile stresses in the fibers is hardly impaired by the state of damage in the matrix, since fibers straighten under high tension. The straightening of the fibers may contribute to matrix damage in the absence of fiber damage. The load carrying capacity of fibers in compression, however, is severely affected by the effective stiffness and strength of the surrounding matrix phase. The matrix acts like an "elastic foundation" for the fibers, treated in mechanical models as beams under compression (Rosen and Dow, 1972; Evans and Adler, 1978; Steif, 1990). Both fiber rupture due to tension, and buckling or kinking of fibers due to compression cause damage evolution in resin matrices. As a consequence, all stiffness components of the constitutive tensor for the damaged unidirectional laminae are degraded.

2.1.2. *In transverse direction*

Normal stresses, acting transverse to the fibers and shear stresses are transmitted through both matrix and fibers. However, their damaging effect mainly takes place in the matrix or in the fiber–matrix interface, leading to debonding. Usually the bond strength of the interface zone between fibers and matrix is the lowest in comparison to the data for the strength of the single constituents. Advancing cracks in the matrix soon pass into the fiber–matrix interface and propagate along the fibers without crossing into the fiber material. Progressive opening of existing cracks is characteristic for tensile loading in transverse direction, whereas "crushing" in the sense of "fragmentation" of brittle matrix materials is very typical for compression in transverse direction. Under impact loading brittle materials completely crumble or pulverize in extreme cases.

A common fractographic observation of graphite–epoxy composites are "hackles" in the damaged resin due to shear loading. During the process of microcracking under shear stresses lacerations and scallops in a saw tooth or wave-like pattern are formed in the resin between the fibers – see Hibbs and Bradley (1987) or Smith and Grove (1987). In view of these physical observations it is reasonable to state the important assumption for the mechanical model: transverse and shear stresses have no influence on the damage in fiber direction as far as the tensile load-carrying capacity of the fibers is concerned. Although the elastic resistance of the damaged resin against straightening of the fibers is reduced if waviness and misalignment of fibers are present, damage of the matrix has a secondary effect on the actual elastic properties of tension in fiber direction. Therefore, the contribution of transverse and shear stresses to tensile fiber damage will be neglected.

2.2. *Passive damage*

A given set of microcracks and cavities has a different influence on the effective elastic properties of the material under tension or compression. Since damage is understood as a phenomenological concept, different states of damage need to be introduced for each sign of the "stress", i.e. individual stress component, spherical or deviatoric part of stress tensor, invariants of the stress tensor or others. The growth of defects under loading not only changes the "active damage state" (Krajcinovic and Fonseka, 1981), compatible with the sign of the corresponding stress state, but also the "passive damage state" related to the opposite sign of the current stresses.

For cyclic loading passive damage evolution becomes a major concern. Fiber kinking and buckling cause breakage of brittle fibers and complete loss of the tensile strength. Debonding of fibers takes place at the ends close to the ruptured cross-section due to tensile failure of the fibers. These ruptured fibers do not transmit compressive stresses after load reversal. In a cycle of loading from tension to compression in transverse direction the cracks in the resin – typically oriented in fiber direction – close again. Transverse compression in the lamina is possible in the form of contact stresses across the crack surfaces at a later loading stage.

The converse in the case of a reversed loading pattern is not true. If the transverse stresses in compression initiate microcracks, leading to crushing of the resin, the change of loading from compression to tension has the effect of crack opening. The solid is stress free across the crack surface in terms of transverse tension and shear.

The "hackles" in the resin, caused by shear loading, are inclined under approximately 45° towards the fiber direction and coincide with the principle stress direction – see Fig. 1. Under increasing shear strains the propagation of the crack tips in the matrix is stopped at the fibers and the new direction of crack propagation is turned parallel to the axis of reinforcement. Coalescence of neighboring microcracks takes place in the fiber–matrix interface.

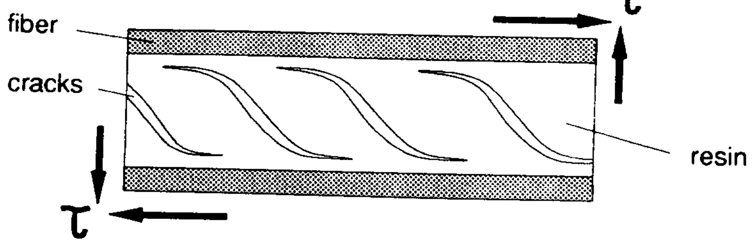

Fig. 1. Microcrack pattern due to shear loading.

If the shear stresses are reversed, the principal stresses just interchange their directions. Although the closing microcracks between the fibers allow compression in the principle direction, the damage due to the microcracks in the fiber–matrix interface still remains active. This justifies the approximation that the influence of damage due to shearing on the effective shear stiffness is the same after load reversal.

2.3. Unloading and reloading

Rough crack surfaces frequently do not close elastically without additional damage after tensile stresses have been removed. Small particles of the damaged material dislocate when the crack surfaces are opening. They partially block or hamper the closing of cracks and might be the cause of permanent deformations. A very localized, self-equilibrated state of residual stresses can develop and further advance the crack tips. However, elastic–brittle materials are assumed to behave elastically during unloading under ideal conditions. This mechanical model will be based on the assumption that the damage state does not change during unloading and no permanent deformations are present in the stress-free body. During reloading, damage growth is typically very small in comparison to first loading as long as the previous strains are not exceeded. These additional damage contributions along the reloading path will also be neglected in the model, since it is not intended to account for fatigue. It is important to keep in mind that the effective elastic properties on the unloading and reloading path only depend on the current state of damage.

2.4. Constitutive assumptions

A summary of the mechanical idealizations for the constitutive model of a lamina comprises the following assumptions in addition to the one previously discussed:
- A homogenized continuum provides the theoretical basis for the constitutive model of each UD-lamina. Plane stress conditions are assumed adequate to model the constitutive behavior of UD-laminae.
- The rules of mixtures lead to the elasticity moduli and strength parameters of the undamaged UD-lamina, calculated from the properties of fiber and resin data together with the volume ratios of fibers and matrix.
- The stress–strain response, obtained from test data of unidirectional laminae, is known to be very nonlinear especially for shearing (Hahn and Tsai, 1973). However, linear elasticity is assumed to hold if the damage state (state of defects) does not change. This implies linear elastic unloading and reloading in stress–strain space. All nonlinear effects of the constitutive behavior are attributed to damage. In particular no plastic (irreversible) deformations are supposed to occur.
- The orthotropic nature of the lamina as a homogenized continuum is maintained throughout the damaging process. This means: the defects in the composite material are treated in the mathematical model as having the equivalent effect on the elastic properties as disk-like cracks would exert, if they are only oriented either tangential or normal to the fiber direction. Therefore, the symmetry class of the UD-lamina remains the same for all states of damage.

3. Failure mechanisms of fiber composites

3.1. Damage modes and specification of failure modes

Although the evolution of damage in the model will continuously depend on the strains in all states of loading, this section concentrates on failure as a single event, which is independent of the past deformation history. Later, this point of view enables the extension of the concept of failure modes to the one on damage modes. It allows the separate modelling of damage evolution in the individual damage modes. Each associated failure mode is viewed as a distinct state in the history of its damage mode, where a specified combination of stress components reaches a maximum. In other words: failure in a particular mode is defined when a scalar valued function – called a "criterion" – of the stress components, attains its maximum. The criteria are obtained by the equations for the failure surfaces in stress space. They will serve together with damage thresholds for the derivation of loading conditions in strain space.

The different failure mechanisms due to tension or compression are specified as: those caused by normal stress σ_{11} in fiber direction and those caused by normal stress σ_{22} transverse to the fibers as well as by shear stresses τ. Subsequently, failure criteria are formulated. The stress components are referred to the afore-mentioned lamina coordinates – see Fig. 3. Each failure mode indicates the limit of the load bearing capacity of the lamina under tension, compression and shear. The major failure mechanisms of laminae, which are accounted for in this context are summarized and related to the strength parameters for simple shear and uniaxial loading in the direction of the privileged axis:

- *Mode I: Fiber rupture*

  Fiber rupture is primarily caused by tensile stresses σ_{11} in fiber direction. Graphite- or glass-epoxy tension specimens show a linear load–displacement diagram up to rupture at critical load – see Carlsson and Pipes (1987). Failure is initiated by a number of fiber breaks in the vicinity of one another. The ruptured fibers debond from the matrix material and cavities are formed between the broken fiber ends. In a homogenized continuum model the strains between the debonded ends of the fiber rapidly grow, whereas outside of this region the strains decrease and unloading takes place. This behavior of the continuum model is typical for strain localization. After the rupture strain is exceeded, the particles of the continuum in the localization (damage) zone pass through a sequence of damage states under increasing strains. Thus, damage mechanics is only used to describe the nonlinear stress–strain relationship in the postfailure regime, whereas hardly any damage is observed in the prefailure branch of the load–displacement diagram.

  The strength parameter X_t, associated with critical loading, is governed by the tensile strength of the fibers and the fiber–matrix volume ratio. Its value is assumed to be given for the lamina. The matrix strength contributes little to X_t, because the failure strains of resin matrices are usually higher than the ones of the fibers. However, uniaxial off-axis tests show a pronounced sensitivity of the failure load with respect to the shear strength of the resin or the fiber–matrix bond strength, if the loading direction slightly deviates from the fiber direction.

- *Mode II: Fiber buckling and kinking*

  Buckling and kinking of fibers, accompanied by matrix fragmentation, is observed in uniaxial compression tests with loading in the fiber direction. In Hahn and Williams (1984) longitudinal splitting is distinguished as another mode of compressive failure. It results from transverse tensile stresses due to Poisson's ratio differences between the matrix and the fibers and the weakness of the fiber–matrix interface. The compressive strength of the lamina X_c is not only influenced by the compressive strength of its constituents, but also by the elastic stiffness and by the shear strength of the matrix. The latter becomes especially important for high volume ratios of fibers in the lamina – see Rosen and Dow (1972) or Steif (1990). It is concluded in Hahn and Williams (1984) that the microbuckling of fibers start with the buckling of a single fiber and progressively involves additional fibers as the damage propagates in the kink-band. The buckled fibers reduce the load carrying part of the cross-section.

  Stress–strain curves from compression tests of composites with a stiff matrix show little nonlinearity prior to failure. However, a nonlinear prefailure response is typical for composites with a soft matrix reinforced by fibers, which fail at high strain – see Hahn and Williams (1984). The nonlinearity is due to the lateral deflection of imperfect or misaligned fibers.

- *Mode III: Matrix cracking under transverse tension and shearing*

  Microcrack growth in a graphite- or glass-epoxy lamina under transverse tension is unstable under load control and sudden failure is initiated by a few macrocracks along the fiber–matrix interface. If, however,

the lamina is stabilized as the 90°-ply in a multidirectional laminate, failure of the lamina under transverse tension is no longer abrupt and the transverse stiffness gradually decreases under strain control. The matrix cracks along the fibers are modelled as continuously distributed defects in a narrow zone (damage region). For actual computations the width of the damage region may be given by the size of the finite elements or by a localization limiter and it is resorted to a nonlocal concept, e.g. Bažant (1986) or Bažant and Pijaudier-Cabot (1988). Failure in the sense of matrix cracking crucially depends on the tensile strength Y_t and shear strength S_c of the lamina. Both strength parameters are roughly proportional to the tensile and shear strength of the resin or to the fiber–matrix bond strength, if that is lower.
The nonlinearity of the stress–strain diagram is usually small and damage mechanics comes into play for modelling the postfailure range of the stress–strain response similar to mode I.

- *Mode IV: Matrix crushing under transverse compression and shearing*
  Microcracking of the resin is observed in graphite- or glass-epoxy laminae under inplane shear and transverse compression. Local tensile stresses in the resin are due to the differences of Poisson's ratio between the fibers and the matrix and the splitting effect of a fiber pressed against the neighboring fibers close to the lamina surface. With increasing shear strains, the direction of crack growth is turned parallel to the fibers at the fiber–matrix interface – see Fig. 1. Fragmentation of the resin is the result of both effects, whereas the fibers suffer little damage. The effective crack orientation for stiffness reduction in the fragmented matrix is primarily parallel to the fibers. It is not unreasonable to assume negligible coupling between shear and normal response for the privileged coordinate axes and to maintain the orthotropic symmetry for the damaged lamina. Failure is related to the compressive strength Y_c and shear strength S_c of the composite material.
  The stress–strain diagram for the normal components is almost linearly elastic in contrast to the shear stress–strain response. As will be shown later, the shear response is linearly approximated in the proposed model.

Delamination of individual layers or separation of stacks of layers is typically encountered in composite structures. It is a failure form of the laminate and not of the lamina. Debonding of fibers from the surrounding matrix or fiber pullout are consequences of interfacial failure. Their mathematical treatment and modelling is a micromechanical task beyond the scope of this topic.

3.2. Failure criteria

The methodology developed by Hashin (1980) is best suited for the purpose of finding failure criteria in the above outlined failure modes of the lamina. A brief discussion on the relation between the concept of failure and damage modes is given next.

Starting from the three-dimensional transversely isotropic continuum, the general form of a failure criterion is approximated by a complete quadratic polynomial in terms of the integrity basis in stress space. Failure planes are introduced to split the general failure criterion into four separate criteria for the previously defined modes I through IV.

The failure planes of the matrix modes III and IV are normal to the fibers, whereas the fiber modes I and II are associated with the planes perpendicular to the fibers. Failure is assumed to be caused by normal and shear stresses, acting on the failure plane – see Fig. 2. The normal and shear stress components, σ_{nn}, σ_{1n}, σ_{tn}, on the planes tangential to the fibers are functions of σ_{22}, σ_{33}, σ_{12}, σ_{13} and σ_{23} but not of σ_{11}. Only the σ_{11}-, σ_{12}- and σ_{13}-stress components contribute to the modes for fiber failure. The general failure criterion is applied to both matrix and fiber modes, providing two different algebraic expressions. In each criterion for matrix and fibers, only those invariants appear which depend on the stress components from one of the afore-mentioned sets, defined by means of the failure planes.

Each of the failure criteria for matrix and fibers is subdivided into tensile and compressive modes next. The coefficients of the quadratic polynomials are related to the four strength parameters X_t, X_c, Y_t and Y_c, available

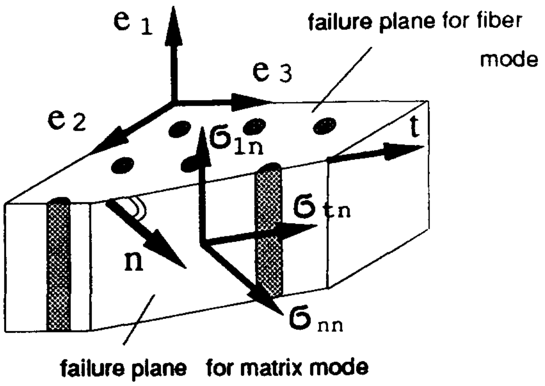

Fig. 2. Failure planes of the lamina.

from simple uniaxial tension and compression tests. In many practical applications the transverse and axial shear strength parameters are considered as equal and will be denoted as S_c in this context. Based on physical reasoning and due to limited material data, the contribution of distinct invariants to the various failure criteria is considered as insignificant (Hashin, 1980; Matzenmiller and Schweizerhof, 1991). After the plane stress assumptions are applied to the four failure criteria, they are reduced to the following simple forms:

Tensile fiber mode I: $\sigma_{11} \geq 0$:

$$e_m^2 = \left(\frac{\sigma_{11}}{X_t}\right)^2 - 1 \begin{cases} \geq 0 \text{ failed} \\ < 0 \text{ elastic} \end{cases} \tag{1a}$$

Compressive fiber mode II: $\sigma_{11} < 0$:

$$e_c^2 = \left(\frac{\sigma_{11}}{X_c}\right)^2 - 1 \begin{cases} \geq 0 \text{ failed} \\ < 0 \text{ elastic} \end{cases} \tag{1b}$$

Tensile matrix mode III: $\sigma_{22} \geq 0$:

$$e_m^2 = \left(\frac{\sigma_{22}}{Y_t}\right)^2 + \left(\frac{\tau}{S_c}\right)^2 - 1 \begin{cases} \geq 0 \text{ failed} \\ < 0 \text{ elastic} \end{cases} \tag{1c}$$

Compressive matrix mode IV: $\sigma_{22} < 0$:

$$e_d^2 = \left(\frac{\sigma_{22}}{Y_c}\right)^2 + \left(\frac{\tau}{S_c}\right)^2 - 1 \begin{cases} \geq 0 \text{ failed} \\ < 0 \text{ elastic} \end{cases} \tag{1d}$$

The failure criteria may be interpreted as loading criteria, a terminology encountered in strain space plasticity. The role played by the yield stress in plasticity will be taken by the threshold variables r in damage mechanics. In classical continuum damage mechanics as proposed by Kachanov only the undamaged (whole) part of the cross-section A (net-area) for the uniaxial case is supposed to carry loading, i.e. transmit stresses. Consequently, the stresses σ_{ij} in the failure criteria should be interpreted as effective stresses $\hat{\sigma}_{ij}$, referred to the net area. This means that the failure criteria are assumed to hold in terms of the effective stresses rather than the nominal stresses.

4. Damage variables, effective stresses and constitutive tensor

4.1. Damage variables and the concept of effective stresses

As already indicated at the outset, the orthotropic nature of the mechanical response is maintained at all states of damage for treating the lamina as a homogenized continuum in plane stress. This assumption permits the modeling of damage by two arrays of parallel cracks, coinciding with the failure planes in Fig. 2. The orientation of each crack-array is given by the unit normal vectors $\boldsymbol{n}_1$ and $\boldsymbol{n}_2$. They are equal to the base vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ of the lamina coordinate system, defined in Fig. 3. The two non-negative damage parameters ω_{11} and ω_{22} are introduced to quantify the relative size of the disk-like cracks, also denoted as the loss area, A_{loss}, in reference to the uniaxial case.

Without conceptual difficulties the orientation of the damage normal $\boldsymbol{n}_i$ could be chosen as an additional unknown, leading to the method of rotating cracks. Crack orientations, deviating from the privileged axis are investigated by Buczek and Herakovich (1983). However, the acceptance of the physical arguments of rotating crack models is still unsettled.

Obviously the effective normal stresses $\hat{\sigma}_{11}$ and $\hat{\sigma}_{22}$ are related to ω_{11} and ω_{22} in the classical sense of continuum damage mechanics. Similarly, shear stresses can only be transmitted across the intact part of the cross-section in a representative volume element. However, rough crack surfaces are assumed and the disk-like cracks may also be penetrated by fibers. Usually the calculated loss areas for the normal and shear stresses differ. Therefore, it is reasonable to deal with the damage parameter ω_{12} for shear as an independent unknown. A simple relationship between effective stresses ($\hat{\boldsymbol{\sigma}}$) and nominal (true) stresses $\boldsymbol{\sigma}$ holds:

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{M}\boldsymbol{\sigma} \tag{2a}$$

and in Voigt notation:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau \end{bmatrix}, \quad \hat{\boldsymbol{\sigma}} = \begin{bmatrix} \hat{\sigma}_{11} \\ \hat{\sigma}_{22} \\ \hat{\tau} \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_{11} \\ \omega_{22} \\ \omega_{12} \end{bmatrix}, \quad \boldsymbol{M} = \begin{bmatrix} \dfrac{1}{1-\omega_{11}} & 0 & 0 \\ 0 & \dfrac{1}{1-\omega_{22}} & 0 \\ 0 & 0 & \dfrac{1}{1-\omega_{12}} \end{bmatrix}, \tag{2b–e}$$

where $\boldsymbol{M}$ represents the rank-four damage operator. Instead of postulating the damage operator, it may be determined by means of the "hypothesis of strain equivalence" and by making use of the constitutive tensors for the virgin and damaged material – see Chaboche (1982). The shortcoming of this approach is a complete interaction between the normal components of the effective and nominal stresses, which unnecessarily complicates the loading criteria and growth functions. In addition, the Poisson effect obstructs the physical significance of the damage variables ω_{11} and ω_{22} as measures of the loss area for stress transmittance.

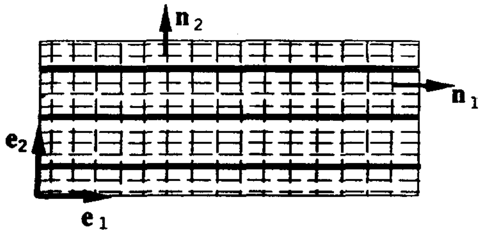

Fig. 3. Damaged lamina with failure planes.

The two damage parameters ω_{11} and ω_{22} assume different values for tension (ω_{11t} and ω_{22t}) and compression (ω_{11c} and ω_{22c}) in order to account for the phenomenon of one-sidedness. In contrast to ω_{11} and ω_{22} the damage parameter for shear ω_{12} is independent of the sign of the shear stress τ.

4.2. Constitutive tensor of damaged UD-laminae

It is assumed that the constitutive law is linear between the stresses and strains and the influence of damage on the elastic response may be described by the damage variables $\boldsymbol{\omega}$. Instead of taking the elasticity constants themselves as unknowns (Ortiz, 1985; Yazdani and Schreyer, 1988; Ju, 1989) the components of the constitutive tensor are represented as functions of $\boldsymbol{\omega}$ and the material parameters of the undamaged lamina. In accordance with the assumption of orthogonal crack arrays, the constitutive tensor $\boldsymbol{C}(\boldsymbol{\omega})$ is derived by physical arguments and information of the dependencies between effective elastic properties and individual damage variables. The tensor representation theory (Talreja, 1985a; Weitsman, 1987) and other postulates such as energy equivalence (Cordebois and Sidoroff, 1982a; Chow and Wang, 1987) or strain equivalence (Lemaitre and Chaboche, 1978; Cordebois and Sidoroff, 1982b) offer alternatives for finding the constitutive tensor of the damaged material.

Generally for a given arbitrary damage operator, the postulate of strain equivalence yields an unsymmetrical constitutive tensor, which should be rejected as a model for elastic behavior. This hypothesis serves here as a first guidance together with physical arguments to set up the constitutive tensor $\boldsymbol{C}(\boldsymbol{\omega})$ for the damaged laminae. For the purpose of building the dependence on damage into the constitutive law, the compliance relationship is more easily accessible in order to relate the material parameters to the mechanical response in the privileged axes. The compliance relationship for orthotropic elasticity in plane stress is recalled in terms of the effective stresses $\hat{\boldsymbol{\sigma}}$:

$$\boldsymbol{\epsilon} = \boldsymbol{H}_0\,\hat{\boldsymbol{\sigma}}, \qquad \boldsymbol{H}_0 = \begin{bmatrix} \dfrac{1}{E_\|} & -\dfrac{\nu_{21}}{E_\|} & 0 \\ -\dfrac{\nu_{12}}{E_\perp} & \dfrac{1}{E_\perp} & 0 \\ 0 & 0 & \dfrac{1}{G} \end{bmatrix}, \qquad \boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix}, \tag{3a–c}$$

with $E_\|$, $E_\perp$, G, ν_{21} and ν_{12} as the elasticity parameters of the undamaged lamina. Positive definiteness of $\boldsymbol{H}_0$ requires: $\nu_{12}\nu_{21} < 1$.

Eqs. (2a) and (3a) result in $\boldsymbol{\epsilon} = \boldsymbol{H}_0\,\boldsymbol{M}\,\boldsymbol{\sigma}$. The final form of the compliance tensor for the damaged lamina, $\boldsymbol{H}(\boldsymbol{\omega})$, is obtained after Poisson's ratios $\nu_{12}(\boldsymbol{\omega})$ and $\nu_{21}(\boldsymbol{\omega})$ are adjusted. On this account the two functions $h_{12}(\boldsymbol{\omega})$ and $h_{21}(\boldsymbol{\omega})$ of the damage variables $\boldsymbol{\omega}$ are brought into the $(2,1)$- and $(1,2)$-entries of $\boldsymbol{H}(\boldsymbol{\omega})$,

$$\boldsymbol{H}(\boldsymbol{\omega}) = \begin{bmatrix} \dfrac{1}{(1-\omega_{11})E_\|} & \dfrac{-h_{21}(\boldsymbol{\omega})\,\nu_{21}}{(1-\omega_{22})E_\|} & 0 \\ \dfrac{-h_{12}(\boldsymbol{\omega})\,\nu_{12}}{(1-\omega_{11})E_\perp} & \dfrac{1}{(1-\omega_{22})E_\perp} & 0 \\ 0 & 0 & \dfrac{1}{(1-\omega_{12})G} \end{bmatrix} \tag{4}$$

4.2.1. Remarks on the physical significance of the compliance tensor $\boldsymbol{H}(\boldsymbol{\omega})$:

- The elastic stiffness parameter $E_{11} = (1-\omega_{11})\,E_\|$ in fiber direction only depends on the variable for fiber damage, described by ω_{11} but is independent of matrix damage.

  The transverse elastic stiffness $E_{22} = (1-\omega_{22})\,E_\perp$ is influenced by the ω_{22}, accounting for the density of cracks parallel to the fibers and not by the number of ruptured fibers, measured by ω_{11}. If fiber damage causes matrix cracks, their density will be monitored by ω_{22}.

- The correct dependence of the shear stiffness on the parameter for shear damage ω_{12} is an immediate consequence of the definition for ω_{12} in Eq. (2a).
- As ω_{22} approaches unity, the transverse stress σ_{22} approaches zero and the Poisson effect from the (1,2)-matrix entry must vanish. Then the mechanical response of the lamina resembles the behavior of a "loose bundle of fibers" as the matrix damage parameter ω_{22} indicates complete damage due to parallel cracks in the resin. However, the fibers may even be undamaged. A similar observation is made for the (2,1)-matrix entry.
- Quantitative assessments for the degradation of Poisson's ratio were made by Highsmith and Reifsnider (1982), Laws et al. (1983), Talreja (1985b), Laws and Dvorak (1987), Herakovich et al. (1988a,b) and Gottesman et al. (1980) among others. As it may be expected – and is concluded by Laws et al. (1983) for polymer and metal-matrix composites – E_{11} turns out to be virtually independent of the crack density parameter for cracks parallel to the fibers. The assessment of damage in modes III and IV is recorded by ω_{22}. Consequently, the algebraic expression $(-h_{21}(\boldsymbol{\omega})\,\nu_{21})\,/\,(1-\omega_{22})$, as part of the (2,1)-matrix entry, is attributed to the Poisson effect $\hat{\nu}_{21}(\boldsymbol{\omega})$ of the damaged lamina. The minor Poisson effect does not significantly change with varying crack spacings as reported in Laws et al. (1983). Thus, $\hat{\nu}_{21}(\boldsymbol{\omega})$ is assumed to be constant with respect to ω_{22}. It gives rise to the conclusion that $h_{21}(\boldsymbol{\omega})$ should be of the form:

$$h_{21}(\boldsymbol{\omega}) = (1-\omega_{22})\,h(\omega_{11})\ , \tag{5a}$$

where $h(\omega_{11})$ is assumed to depend only on the variable ω_{11}, responsible for fiber failure. In addition, the same decoupling of damage effects on the components of the constitutive tensor is supposed to hold as it does for the elastic constitutive response between normal and shear components in case of orthotropic symmetry. Thus, $h_{21}(\boldsymbol{\omega})$ is chosen as a function of ω_{11} and ω_{22} only, but not of ω_{12}.

- A thermodynamic potential is assumed to exist, if the deformation is elastic without changing the damage state under some loading conditions. Consequently, the equality $\nu_{12}/E_{\perp} = \nu_{21}/E_{\parallel}$ must remain valid not only for the virgin material – and in fact it does for most applications, but also for the elasticity moduli of the damaged material. The necessary major symmetry of the constitutive tensor is retained, if

$$h_{12}(\boldsymbol{\omega}) = (1-\omega_{11})\,h(\omega_{11}). \tag{5b}$$

- Strong dependence on the spacing of parallel cracks is reported by Laws et al. (1983) for the effective Young's modulus $\hat{E}_{\perp}(\boldsymbol{\omega})$ in the transverse direction and the major Poisson's ratio $\hat{\nu}_{12}(\boldsymbol{\omega})$. If $\hat{E}_{\perp}(\boldsymbol{\omega})$ diminishes with $(1-\omega_{22})$, then the effective major ratio becomes:

$$\hat{\nu}_{12}(\boldsymbol{\omega}) = h_{12}(\boldsymbol{\omega})\frac{1-\omega_{22}}{1-\omega_{11}}\nu_{12} = (1-\omega_{22})\nu_{12}\,h(\omega_{11}). \tag{5c}$$

The degradation of the effective Poisson's ratio $\hat{\nu}_{12}$ with ω_{22} is at least in qualitative agreement with the result of the cited authors, since no explicit relation between the geometry of the crack arrays and the phenomenological damage variable ω_{22} has been established.

- The analogous assumption that cracks orthogonal to the fibers do not affect the major Poisson's ratio $\hat{\nu}_{12}(\boldsymbol{\omega})$ leads to $h(\omega_{11}) = 1$, which agrees with the result in Nuismer (1980) and Carlsson and Pipes (1987).
- The elastic response between shear stresses and shear strains is assumed to be linear. If, however, the elastic nonlinearity in the resin shear behavior is pronounced, the artificial inflation of the ω_{12}-variable due to nonlinear elasticity should be kept low by carefully selecting the value for the shear modulus G.

Under the aforementioned presuppositions the compliance tensor finally becomes:

$$H(\omega) = \begin{bmatrix} \dfrac{1}{(1-\omega_{11})E_{\parallel}} & -\dfrac{\nu_{21}}{E_{\parallel}} & 0 \\ -\dfrac{\nu_{12}}{E_{\perp}} & \dfrac{1}{(1-\omega_{22})E_{\perp}} & 0 \\ 0 & 0 & \dfrac{1}{(1-\omega_{12})G} \end{bmatrix}. \tag{6}$$

Its inverse always exists as long as the damage variables are less than one ($\omega_{ij} < 1$),

$$C(\omega) = \frac{1}{D}\begin{bmatrix} (1-\omega_{11})E_{\parallel} & (1-\omega_{11})(1-\omega_{22})\nu_{21}E_{\perp} & 0 \\ (1-\omega_{11})(1-\omega_{22})\nu_{12}E_{\parallel} & (1-\omega_{22})E_{\perp} & 0 \\ 0 & 0 & D(1-\omega_{12})G \end{bmatrix} \tag{7}$$

with

$$D = 1 - (1-\omega_{11})(1-\omega_{22})\nu_{12}\nu_{21} > 0$$

and H_0 is positive definite. It is noted that the normal stress contributions, induced by the Poisson effect, vanish as either of the damage variables ω_{11} or ω_{22} approaches unity. For non-zero Poisson's ratios the effective elastic stiffness in fiber direction slightly decreases for practical composites as matrix damage continues to develop.

5. Loading surfaces and loading conditions

5.1. Loading criteria

The state of damage is unchanged along a path of strain in the interior of a well-defined region, called the elastic range $\mathcal{E}$. Further it is supposed that any state of stress or strain lies either inside $\mathcal{E}$ or on the boundary $\partial\mathcal{E}$ of the elastic range, defined by the loading criterion in stress space $f(\sigma, \omega, r)$ or strain space $g(\epsilon, \omega, r)$ in terms of either σ or ϵ and the damage variables ω as well as the threshold r. If the elastic range is bounded by several surfaces f_i or g_i, each surface can have its own threshold r_i. The damage thresholds r measure the size of the elastic region. As it is assumed here, their evolution is governed by the "persistence" condition $\dot{g} = 0$, which relates the evolution of the thresholds r to the one for the damage variables ω. Therefore, the thresholds r are no longer independent internal variables.

The shape of the loading surfaces in the space of effective stresses $\hat{\sigma}$ may be obtained from the failure criteria in Eqs. (1a)–(1d). After the stress components σ_{ij} in the failure criteria of Eqs. (1a)–(1d) are formally replaced by the effective stresses $\hat{\sigma}_{ij}$, use is made of Eq. (2a) in order to derive the loading surfaces $f_{\parallel}$ for the fiber modes I, II and $f_{\perp}$ for the matrix modes III, IV:

$$f_{\parallel} = \frac{\sigma_{11}^2}{(1-\omega_{11\,c,t})^2 X_{c,t}^2} - r_{\parallel\,c,t} = 0,$$

$$f_{\perp} = \frac{\sigma_{22}^2}{(1-\omega_{22\,c,t})^2 Y_{c,t}^2} + \frac{\tau^2}{(1-\omega_{12})^2 S_c^2} - r_{\perp} = 0.$$

The shorthand notation $(\ )_{c,t}$ is used for quantities such as $X_{c,t}$, $\omega_{11c,t}$ and $r_{\parallel c,t}$. It means for example:

$$X_{c,t} = \begin{cases} X_t & \text{if} \quad \sigma_{11} \geq 0 \\ X_c & \text{if} \quad \sigma_{11} < 0 \end{cases}$$

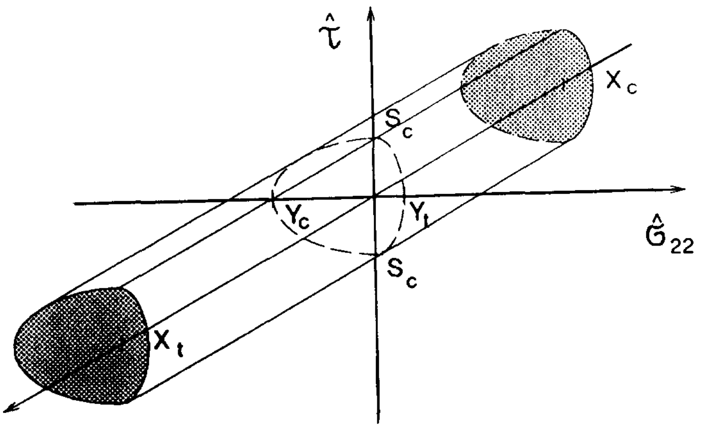

Fig. 4. Multisurface loading criteria, formed by $f_\perp$ and $f_\parallel$ in the space of effective stresses.

Likewise for $Y_{c,t}$ and $\omega_{22c,t}$ with σ_{22} as the indicator.

The requirement of a continuous boundary of the elastic range $\partial\mathcal{E}$ at $\sigma_{22} = 0$ implies that the threshold $r_\perp$ for damage in the matrix modes III and IV assumes the same value for transverse tension and compression. The surfaces $f_\parallel$ and $f_\perp$, described by quadratic forms, specify the boundary of a connected, convex set in stress space – see Fig. 4,

$$f_\parallel = \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{F}_\parallel \boldsymbol{\sigma} - r_\parallel = 0, \tag{8a}$$

$$f_\perp = \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{F}_\perp \boldsymbol{\sigma} - r_\perp = 0, \tag{8b}$$

where

$$\boldsymbol{F}_\parallel = \begin{bmatrix} \left[(1-\omega_{11\,c,t})X_{c,t}\right]^{-2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{F}_\perp = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \left[(1-\omega_{22\,c,t})Y_{c,t}\right]^{-2} & 0 \\ 0 & 0 & \left[(1-\omega_{12})S_c\right]^{-2} \end{bmatrix}.$$

It is well-known that the stress–strain relationship in descending branches ("softening") of its graph is stable under strain control only. In plasticity theory either the loading conditions are postulated as functions of strains and strain rates (Casey and Naghdi, 1981), or they come out as solvability requirements (Lubliner, 1990) of the consistency condition in terms of the strain rates. For the transformation of f into g the constitutive tensor is employed to change the defining spaces

$$g_\parallel = \boldsymbol{\epsilon}^{\mathrm{T}} \boldsymbol{G}_\parallel \boldsymbol{\epsilon} - r_\parallel = 0, \tag{9a}$$

with

$$\boldsymbol{G}_\parallel = \boldsymbol{C}\,\boldsymbol{F}_\parallel\,\boldsymbol{C} = \frac{1}{D^2}\frac{E_\parallel^2}{X_{c,t}^2} \begin{bmatrix} 1 & (1-\omega_{22\,c,t})\nu_{12} & 0 \\ (1-\omega_{22\,c,t})\nu_{12} & (1-\omega_{22\,c,t})^2\nu_{12}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and gradient:

$$\nabla_\epsilon g_\| = \frac{\partial g_\|}{\partial \boldsymbol{\epsilon}} = 2\frac{\epsilon_{11} + (1-\omega_{22\,c,t})\nu_{12}\epsilon_{22}}{D^2}\frac{E_\|^2}{X_{c,t}^2}\begin{bmatrix} 1 \\ (1-\omega_{22\,c,t})\nu_{12} \\ 0 \end{bmatrix}$$

Likewise:

$$g_\perp = \boldsymbol{\epsilon}^{\mathrm{T}}\boldsymbol{G}_\perp\boldsymbol{\epsilon} - r_\perp = 0, \tag{9b}$$

$$\boldsymbol{G}_\perp = \boldsymbol{C}\boldsymbol{F}_\perp\boldsymbol{C} = \frac{1}{D^2}\begin{bmatrix} \dfrac{(1-\omega_{11\,c,t})^2\nu_{21}^2E_\perp^2}{Y_{c,t}^2} & \dfrac{(1-\omega_{11\,c,t})\nu_{21}E_\perp^2}{Y_{c,t}^2} & 0 \\ \dfrac{(1-\omega_{11\,c,t})\nu_{21}E_\perp^2}{Y_{c,t}^2} & \dfrac{E_\perp^2}{Y_{c,t}^2} & 0 \\ 0 & 0 & \left(D\dfrac{G}{S_c}\right)^2 \end{bmatrix},$$

$$\nabla_\epsilon g_\perp = \frac{\partial g_\perp}{\partial \boldsymbol{\sigma}} = \frac{2}{D^2}\begin{bmatrix} (1-\omega_{11\,c,t})\,\nu_{21}\,\dfrac{E_\perp^2}{Y_{c,t}^2}\left[(1-\omega_{11\,c,t})\nu_{21}\,\epsilon_{11} + \epsilon_{22}\right] \\ \dfrac{E_\perp^2}{Y_{c,t}^2}\left[(1-\omega_{11\,c,t})\nu_{21}\,\epsilon_{11} + \epsilon_{22}\right] \\ 2\left(D\dfrac{G}{S_c}\right)^2\epsilon_{12} \end{bmatrix}.$$

5.2. Loading conditions

The boundary $\partial\mathcal{E}$ in strain space moves (at least locally) "outwards" with increasing strains, if the strain rate forms an acute angle with the gradient $\nabla_\epsilon g$ at the given state of strain on the loading surface. The damage state changes under this condition, called "loading":

$$\begin{aligned} &g = 0 \quad \text{and} \quad \frac{\partial g}{\partial \boldsymbol{\epsilon}}\cdot\dot{\boldsymbol{\epsilon}} > 0 \quad \text{loading} \\ &g = 0 \quad \text{and} \quad \frac{\partial g}{\partial \boldsymbol{\epsilon}}\cdot\dot{\boldsymbol{\epsilon}} = 0 \quad \text{neutral loading} \\ &g = 0 \quad \text{and} \quad \frac{\partial g}{\partial \boldsymbol{\epsilon}}\cdot\dot{\boldsymbol{\epsilon}} < 0 \quad \text{unloading} \end{aligned} \tag{10}$$

Besides the damage variables $\boldsymbol{\omega}$ and the damage thresholds r no other internal variables have been introduced to describe the location of the loading surface in strain space. The origin is contained in the elastic range $\mathcal{E}$ at all states of damage. Therefore, $\mathcal{E}$ has to expand at least locally when its boundary $\partial\mathcal{E}$ moves "outwards" during loading, and the time derivative of the damage threshold $\dot{r}$ must be non-negative $\dot{r} \geq 0$. The persistence condition in strain space has to provide a monotonically increasing threshold r for a meaningful loading surface $g = 0$ in strain space

$$\dot{g} = \frac{\partial g}{\partial \boldsymbol{\epsilon}}\dot{\boldsymbol{\epsilon}} + \frac{\partial g}{\partial \boldsymbol{\omega}}\dot{\boldsymbol{\omega}} - \dot{r} = 0. \tag{11}$$

For a given loading surface g, Eq. (11) restricts the amount or direction of damage growth $\dot{\boldsymbol{\omega}}$:

$$-\frac{\partial g}{\partial \boldsymbol{\omega}}\dot{\boldsymbol{\omega}} \leq \frac{\partial g}{\partial \boldsymbol{\epsilon}}\dot{\boldsymbol{\epsilon}} \tag{12}$$

This restriction is comparable to critical softening in elastoplasticity (Lubliner, 1990). It states that a possible shrinkage of the elastic range $\mathcal{E}$ due to growing damage is less than the expansion of the loading surface, resulting from strain increase. If Poisson's ratio is zero, $\nu_{12} = \nu_{21} = 0$, this condition is trivially satisfied for the proposed loading surfaces. In the case that inequality (12) does not hold for some $\dot{\boldsymbol{\omega}}$, the loading criterion g may be multiplied by an appropriately chosen function of $\boldsymbol{\omega}$ in order to redefine the damage threshold r. Thus, r becomes a monotonically increasing function.

The loading conditions in terms of the loading surfaces $f_\perp$ and $f_\parallel$ in stress space follow from the transformation of the gradient $\nabla_\epsilon g$ into $\partial f/\partial \boldsymbol{\sigma}$ by means of the persistence conditions $\dot{g} = \dot{f} = 0$ and the rate-equations for $\boldsymbol{\omega}$. In combination with the constitutive law in rate-form

$$\dot{\boldsymbol{\epsilon}} = \boldsymbol{H}\,\dot{\boldsymbol{\sigma}} + \frac{\partial \boldsymbol{\epsilon}}{\partial \boldsymbol{\omega}}\dot{\boldsymbol{\omega}}, \tag{13}$$

the stress space loading criterion yields:

$$\dot{f} = \frac{\partial f}{\partial \boldsymbol{\sigma}}\boldsymbol{C}\,\dot{\boldsymbol{\epsilon}} + \left(-\frac{\partial f}{\partial \boldsymbol{\sigma}}\boldsymbol{C}\frac{\partial \boldsymbol{\epsilon}}{\partial \boldsymbol{\omega}} + \frac{\partial f}{\partial \boldsymbol{\omega}}\right)\dot{\boldsymbol{\omega}} - \dot{r} = 0\,. \tag{14}$$

6. Kinetic law

6.1. General form of damage rule

Microcracks and voids grow under the change in the external controls of applied forces and prescribed boundary conditions. The rate of evolution of the damage variables is assumed to be locally governed by the local state variables $\boldsymbol{\sigma}, \boldsymbol{\omega}$. In the absence of "healing" the damage variables must be monotonically increasing ($\dot{\boldsymbol{\omega}} \geq 0$) when damage takes place. The rate-equations for the damage variables $\boldsymbol{\omega}$ – denoted as the "damage rule" – have to allow independent growth of the damage parameters or appropriate coupling among them, i.e. damage growth in the resin might not effect fiber damage on the one hand, whereas fiber rupture causes defect growth in the resin, on the other hand.

In the presence of strain softening, the rate of evolution for $\dot{\boldsymbol{\omega}}(\boldsymbol{\sigma}, \boldsymbol{\omega}, \dot{\boldsymbol{\epsilon}})$ is supposed to be locally controllable under the strain rate $\dot{\boldsymbol{\epsilon}}$. Under these conditions the rate-equations must be of the form:

$$\dot{\boldsymbol{\omega}} = \sum_i \phi_i\, \boldsymbol{q}_i. \tag{15}$$

The scalar functions $\phi_i(\boldsymbol{\sigma}, \boldsymbol{\omega}, \dot{\boldsymbol{\epsilon}})$ control the amount of growth and the vector-valued functions $\boldsymbol{q}_i(\boldsymbol{\sigma}, \boldsymbol{\omega})$ accommodate the coupling of growth for the individual damage variables in the various damage modes. It was tacitly assumed that the coupling among the damage variables through $\boldsymbol{q}_i$ is independent of the strain-rate itself.

The next objective is to construct expressions for the scalar functions ϕ_i by means of an appropriately chosen norm of the strain-rate tensor $\dot{\boldsymbol{\epsilon}}$.

6.2. Growth functions

Among the various definitions of a continuous metric, the inner product norm between the strain rate and the normal to the level curves of the loading surface $\nabla_\epsilon g$ has geometrical significance. The strain increment $\dot{\boldsymbol{\epsilon}}dt$ indicates the direction of the loading path $\boldsymbol{\epsilon}$ as it crosses the loading surface $g = 0$ in strain space. The scalar

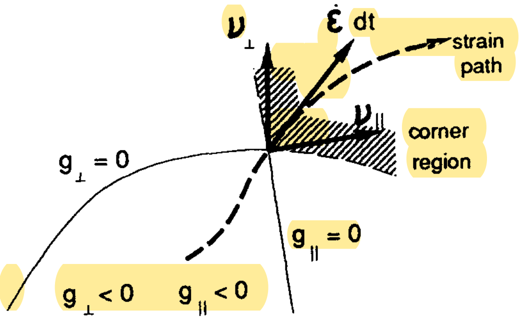

Fig. 5. Corner region (shaded) for non-smooth loading surfaces in strain space.

$\phi_i \geq 0$ has to be associated with each loading direction $\dot{\epsilon}$ in relation to the orientation of the normal $\nabla_\epsilon g$ and must be in accordance with the loading conditions:

(1) $\phi_i = 0$ *when* $g_i = f_i \leq 0$
The strain path is in the interior of the elastic range $\mathcal{E}$ or comes into contact with the boundary $\partial\mathcal{E}$. In the latter case the tangent vectors $\dot{\epsilon}$ to the strain path and to the loading surface $\nabla_\epsilon g_i$ might even coincide.

(2) $\phi_i > 0$ *when* $g_i = f_i = 0+$
The strain path crosses the loading surface $f_i = g_i = 0$, indicated by the "+" sign at the end of the criterion. The strain increment has a non-zero component pointing in the direction of the gradient $\nabla_\epsilon g_i$.

Provided the constitutive model is considered as rate-independent (either the material response is rate-insensitive or sufficiently small strain rates are applied), then ϕ_i has to be linear in the strain rates $\dot{\epsilon}$. Without loss of generality the inner product, written as $(\cdot,\cdot)$, of $\dot{\epsilon}$ with a unit vector $\boldsymbol{\nu}_i(\boldsymbol{\sigma},\boldsymbol{\omega})$ represents a linear mapping of $\dot{\epsilon}$ into the one-dimensional space of each ϕ_i. Subsequently, it is combined with a growth function $\gamma_i(\boldsymbol{\sigma},\boldsymbol{\omega})$ to form

$$\phi_i = \gamma_i \langle \boldsymbol{\nu}_i, \dot{\boldsymbol{\epsilon}} \rangle, \tag{16}$$

where for notational convenience the following definition is used:

$$\langle x, y \rangle = \begin{cases} (x, y) & \text{if } (x, y) \geq 0 \\ 0 & \text{if } (x, y) < 0. \end{cases}$$

The inner product between $\boldsymbol{\nu}_i$ and $\dot{\boldsymbol{\epsilon}}$ satisfies the above stated requirements for the functions ϕ_i, if $\boldsymbol{\nu}_i$ is identified with the gradient $\partial g_i/\partial\boldsymbol{\epsilon}$ to the loading surface g_i up to a scalar multiplier, since any proportionality factor may be merged with the growth function $\gamma_i(\boldsymbol{\sigma},\boldsymbol{\omega})$ – see also Lubliner (1991)

$$\boldsymbol{\nu}_i \propto \frac{\partial g_i}{\partial \boldsymbol{\epsilon}}. \tag{17}$$

At most two loading surfaces $g_\perp$, $g_\parallel$ (resp. $f_\perp$, $f_\parallel$) for either tension or compression are "active", if the loading path crosses into the "corner region", indicated in Fig. 5.

Similar to the classical Bailey–Norton model of nonlinear creep, the growth functions $\gamma_i(\boldsymbol{\sigma},\boldsymbol{\omega})$ may be constructed as power functions in terms of the state variables $\boldsymbol{\sigma},\boldsymbol{\omega}$. Crack and void growth is driven by the peak stresses around the crack tips. This suggests that the effective stresses rather than the nominal stresses control the growth of damage. Therefore, it is reasonable to have the growth functions $\gamma_i\left(\hat{\boldsymbol{\sigma}}\left(\boldsymbol{\sigma},\boldsymbol{\omega}\right),\boldsymbol{\omega}\right)$ depend on the stresses $\boldsymbol{\sigma}$ through the effective stresses $\hat{\boldsymbol{\sigma}}$. In the sense of the theory for this damage model, the growth function for each damage mode should only be governed by the stress components acting on the corresponding failure planes. The growth functions γ_1 and γ_2 for fiber and matrix modes of damage are adjustable to material data by means of the coefficients α_{ij} and exponents n_{ij}, which may be different for tension and compression:

$$\gamma_1 = \alpha_{11\,c,t}\left(\frac{\hat{\sigma}_{11}}{X_{c,t}}\right)^{n_{11\,c,t}}, \qquad \gamma_2 = \alpha_{22\,c,t}\left(\frac{\hat{\sigma}_{22}}{Y_{c,t}}\right)^{n_{22\,c,t}} + \alpha_{12}\left(\frac{\hat{\tau}}{S_c}\right)^{n_{12}} \tag{18a–b}$$

Comparable attempts with power laws have their origin in the context of rate-dependent evolution equations for the purpose of estimating the necessary time for brittle fracture.

6.3. Damage potential

The vector-valued functions $\boldsymbol{q}_i(\boldsymbol{\sigma}, \boldsymbol{\omega})$ indicate the relative growth among the damage variables in the ith damage mode at the current local state. Each vector $\boldsymbol{q}_i$ may be thought of as the gradient to the surface $Q_i(\boldsymbol{\sigma}, \boldsymbol{\omega}) = 0$, represented by a potential function. Potential functions are introduced just for convenience in order to determine the direction of growth in the multidimensional space of internal variables. Before the notion of generalized potentials is introduced, a remark on the "positive work assumption" follows.

- Like the flow rule in plasticity theory, the damage rule may be obtained as a necessary condition of the work assumption. A work assumption similar to Drucker's postulate of plasticity theory is not expected to hold for softening. Il'iushin's postulate – see Il'iushin (1961) or Naghdi and Trapp (1975) – assumes that in any closed cycle of strain the inequality

$$\oint \sigma_{ij}\, d\epsilon_{ij} \geq 0$$

holds. This integration has the form of specific work along the strain path. Sufficient conditions on the damage rule, emerging as consequences of the above inequality, are derived by Carroll (1987) for the case where the constitutive tensor $\boldsymbol{C}$ depends on a second-order tensor $\boldsymbol{\omega}$. Besides the existence of an elastic potential and the convexity of the loading surface, the work inequality also enforces the direction of damage evolution normal to the level curves of the function g in strain space:

$$\frac{\partial(\boldsymbol{C}\boldsymbol{\epsilon})}{\partial\boldsymbol{\omega}}\,\dot{\boldsymbol{\omega}} = -\dot{\lambda}\,\frac{\partial g}{\partial\boldsymbol{\epsilon}}$$

For the current study it is convenient to investigate this condition in stress space:

$$\frac{\partial(\boldsymbol{H}\boldsymbol{\sigma})}{\partial\boldsymbol{\omega}}\,\dot{\boldsymbol{\omega}} = \dot{\lambda}\,\frac{\partial f}{\partial\boldsymbol{\sigma}}.$$

Substituting Eqs. (6), (8a) and (8b) into the above restriction yields the evolution equation of the multi-surface damage model:

$$\begin{bmatrix} \dfrac{\sigma_{11}}{(1-\omega_{11\,c,t})^2 E_{\parallel}}\,\dot{\omega}_{11} \\ \dfrac{\sigma_{22}}{(1-\omega_{22\,c,t})^2 E_{\perp}}\,\dot{\omega}_{22} \\ \dfrac{\tau}{(1-\omega_{12})^2 G}\,\dot{\omega}_{12} \end{bmatrix} = \phi_1 \begin{bmatrix} \dfrac{2\sigma_{11}}{(1-\omega_{11\,c,t})^2 X_{c,t}^2} \\ 0 \\ 0 \end{bmatrix} + \phi_2 \begin{vmatrix} 0 \\ \dfrac{2\sigma_{22}}{(1-\omega_{22\,c,t})^2 Y_{c,t}^2} \\ \dfrac{2\tau}{(1-\omega_{12})^2 S_c^2} \end{vmatrix}.$$

The system of ordinary differential equations for the damage rule is equivalent to:

$$\dot{\omega}_{11} = \phi_1 \frac{2E_{\parallel}}{X_{c,t}^2}, \qquad \dot{\omega}_{22} = \phi_2 \frac{2E_{\perp}}{Y_{c,t}^2}, \qquad \dot{\omega}_{12} = \phi_2 \frac{2G}{S_c^2}.$$

The decoupling of the rate-equations into the variables for fiber and matrix damage is not surprising in case of the above damage rule in stress space, where the loading criteria are postulated with respect to the separation of fiber and matrix modes. As the elasticity constants $\nu_{12} = \nu_{21} = 0$ vanish due to damage increase, the same decoupling pattern in strain space is discovered as it is in stress space.

The serious deficiency of this damage rule is manifested by the vector-valued function $\boldsymbol{q}_1$, which excludes growth of the variables ω_{22} and ω_{12} for matrix damage in case of fiber loading. However, both fiber rupture and fiber buckling usually cause matrix damage in test specimens made of brittle resin.

The contribution of $\boldsymbol{q}_2$ to the damage rule in Eq. (15) seems to be in better physical agreement, since fiber damage ω_{11} is not affected by increasing matrix damage, $\dot{\omega}_{22}$, $\dot{\omega}_{12} > 0$. The matrix damage parameters ω_{22} and ω_{12} increase in proportion to each other with a ratio of $E_{\perp}/Y_{c,t}^2 : G/S_c^2$, which is a measure of the "degree of anisotropy" for the laminae. Practical composites, having ratios of 5 : 1 to 10 : 1 for transverse tension, undergo – according to this rule – complete damage in terms of the variable ω_{22} for the transverse stress σ_{22}, but predict very little damage ω_{12} in the shear response at the same time. In the compression case the situation is reversed, where the ratio between the rates of the two damage variables $\dot{\omega}_{22} : \dot{\omega}_{12}$ is of the order of 2 : 3 or 1 : 2.

It is concluded that the work assumption restricts the validity of the resulting evolution laws for the damage variables to a very special class of materials. The same restrictions on the damage rule follow, if instead of the work assumptions the principle of maximum damage dissipation is postulated, which also holds for softening.

6.3.1. Generalized potentials

Since the previous damage rule does not lead to plausible results, a more general but concise thermodynamical approach is undertaken. Damage as a local process is treated by the assumption that the global Clausius–Duhem inequality also holds for any "small" region of the lamina during elastic and inelastic deformation. As a consequence the Kelvin inequality (Coleman and Gurtin, 1967; Lubliner, 1972) – also denoted as the dissipation inequality – arises

$$\mathcal{D} = (\boldsymbol{Y}, \dot{\boldsymbol{\omega}}) \geq 0 \tag{19}$$

This inequality may be viewed as an inner product between the rates of damage $\dot{\boldsymbol{\omega}}$ and the "energy release rates" $\boldsymbol{Y}$, a terminology (Lemaitre, 1985) borrowed from fracture mechanics. In continuum damage mechanics the variables $\boldsymbol{Y}$ have the meaning of energy, released per volume, due to the advancement of damage. They are the "thermodynamic forces", conjugate to the damage variables $\boldsymbol{\omega}$.

$$\boldsymbol{Y} = -\frac{\partial W}{\partial \boldsymbol{\omega}}$$

The forces $\boldsymbol{Y}$ are obtained from the strain energy function $W(\boldsymbol{\epsilon}, \boldsymbol{\omega})$, which is assumed to exist. The requirement of non-negative internal dissipation $\mathcal{D}$ gives motivation to introduce a potential function $Q(\boldsymbol{Y}, \boldsymbol{\omega})$ – called "generalized potential" (Germain et al. 1983), "dissipation potential" or "damage potential"- in the space of the thermodynamic forces $\boldsymbol{Y}$. If Q is convex and the origin of the coordinate system in the space $\boldsymbol{Y}$ is an interior point of the set, bounded by Q (or Q is at least star convex with respect to the origin), then the inner product of the gradient $\frac{\partial Q}{\partial \boldsymbol{Y}}$ with its position vector $\boldsymbol{Y}$ is non-negative. The gradient of Q defines the vector-valued function $\boldsymbol{q}_i$ for the damage rule. For each damage mode a generalized potential Q_i may be specified, which leads to the notion of a multi-surface potential,

$$\dot{\boldsymbol{\omega}} = \sum_i \phi_i \frac{\partial Q_i}{\partial \boldsymbol{Y}} \tag{20}$$

The direction of damage growth, governed by generalized potentials, no longer depends on the stress components σ_{ij} directly, but rather through the thermodynamic conjugate forces $\boldsymbol{Y}$. Since the stresses are considered as state variables, it is straight forward to explicitly derive the forces $\boldsymbol{Y}$ from the Gibbs potential energy $U(\boldsymbol{\sigma}, \boldsymbol{\omega})$, obtained by the Legendre–Fenchel transformation with respect to strain. The inverse $\boldsymbol{H}$ of the constitutive tensor $\boldsymbol{C}$ always exists for the proposed model as long as complete damage is not approached in the limit:

$$U = (\boldsymbol{\epsilon}, \boldsymbol{\sigma}) - W \quad \text{and} \quad \boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\epsilon}}$$

An additive split of the complementary energy into an elastic (U^{e}) and an inelastic (U^{i}) part is discussed by Lubliner (1972), Nguyen (1985) and Ortiz (1985) in the context of elasto-plasticity. In the sense of continuum damage mechanics the Gibbs energy is written as:

$$U = U^{\mathrm{e}}(\boldsymbol{\sigma}, \boldsymbol{\omega}) + U^{\mathrm{i}}(\boldsymbol{\omega}).$$

The inelastic part $U^{\mathrm{i}}(\boldsymbol{\omega})$ is independent of the stress state and is of importance, if damage evolves in the stress free body. Stiffness degradation due to environmental causes or the evolution of the passive damage state are examples for the application of such a form. Another example is kinematic hardening in the theory of elastoplasticity (Nguyen, 1985). The significance of the inelastic contribution $U^{\mathrm{i}}(\boldsymbol{\omega})$ becomes apparent in the algebraic expression of the “driving forces” $\boldsymbol{Y}$ for damage growth in the stress free body,

$$\boldsymbol{Y}\big|_{\boldsymbol{\sigma}=0} = \frac{\partial U^{\mathrm{i}}(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} \tag{21}$$

In addition, the evolution of the passive damage state may be modelled by such inelastic contributions to the forces $\boldsymbol{Y}$.

If the elasticity part of the model is taken as linear, the complementary energy reads:

$$U = U^{\mathrm{e}} = \tfrac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{H}(\boldsymbol{\omega})\,\boldsymbol{\sigma}.$$

The forces $\boldsymbol{Y}$ become:

$$\boldsymbol{Y} = \frac{\partial U^{\mathrm{e}}}{\partial \boldsymbol{\omega}} = \frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{\omega}}\boldsymbol{\sigma} \quad \text{with components:} \quad \boldsymbol{Y} = \begin{bmatrix} Y_{11} \\ Y_{22} \\ Y_{12} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \dfrac{\sigma_{11}^2}{(1-\omega_{11\,c,t})^2 E_{\parallel}} \\ \dfrac{\sigma_{22}^2}{(1-\omega_{22\,c,t})^2 E_{\perp}} \\ \dfrac{\tau^2}{(1-\omega_{12})^2 G} \end{bmatrix}. \tag{22}$$

In special cases the existence of a generalized potential function Q has been proven, e.g. by Kestin and Bataille (1978) for the case of linear transformations between “fluxes” and forces and in Rice (1971), where the rate-equation combines each component of the flux vector only with its conjugate force component.

6.3.2. Remarks on energy release rates

- All components of the energy release rates $\boldsymbol{Y}$ are positive. Different states of forces $\boldsymbol{Y}$ are associated with both states of damage for tension and compression.
- As a result of the proposed constitutive tensor $\boldsymbol{C}$ for damage, each stress component in Eq. (22b) corresponds to just one component of the thermodynamical forces $\boldsymbol{Y}$. Thus, the Jacobian of the map $\boldsymbol{\sigma} \mapsto \boldsymbol{Y}$ is diagonal:

$$\frac{\partial \boldsymbol{Y}}{\partial \boldsymbol{\sigma}} = \frac{\partial^2 U}{\partial \boldsymbol{\sigma}\,\partial \boldsymbol{\omega}} = \mathrm{diag}\left[\frac{\sigma_{11}}{(1-\omega_{11\,c,t})^2 E_{\parallel}} \quad \frac{\sigma_{22}}{(1-\omega_{22\,c,t})^2 E_{\perp}} \quad \frac{\tau}{(1-\omega_{12})^2 G}\right].$$

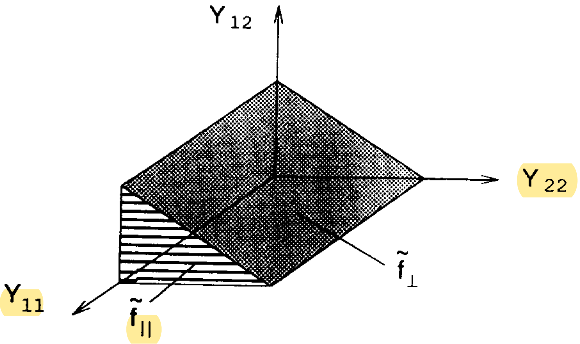

Fig. 6. Loading surface in force space $\boldsymbol{Y}$.

– The two potentials $U(\boldsymbol{\sigma}, \boldsymbol{\omega})$ and $Q(\boldsymbol{Y}, \boldsymbol{\omega})$ govern the inelastic behavior of the solid (Germain et al., 1983) completely.

The loading criteria $\tilde{f}(\boldsymbol{Y}, \boldsymbol{\omega})$ in Eqs. (8a), (8b) may be represented in the force space $\boldsymbol{Y}$, where they are piecewise linear – see Fig. 6:

$$\tilde{f}_{\parallel} = 2\frac{E_{\parallel}}{X_{c,t}^2} Y_{11} - r_{\parallel\, c,t} = 0, \qquad \tilde{f}_{\perp} = 2\frac{E_{\perp}}{Y_{c,t}^2} Y_{22} + 2\frac{G}{S_c^2} Y_{12} - r_{\perp} = 0.$$

In force space they may be interpreted as predicting damage growth when the forces $\boldsymbol{Y}$ attain a maximal force threshold. It is formally possible to state the loading criteria in the space of the conjugate forces in order to control the onset of damage evolution with conditions on the forces and their rates rather than the stresses or strains. The necessary transformation of $\langle \boldsymbol{\nu}_i, \dot{\boldsymbol{\epsilon}} \rangle$ into $(\partial \tilde{f}/\partial \boldsymbol{Y}, \dot{\boldsymbol{Y}})$ follows from the persistence condition $\dot{\tilde{f}} = \dot{g} = 0$. However, neither the forces $\boldsymbol{Y}$ nor their rates are directly controllable in a loading process. Therefore, it is reasonable to keep the loading criteria and the evolution laws as functions of strains and strain rates $\dot{\boldsymbol{\epsilon}}$.

Together with a suitable redefinition of the growth functions γ_i of Eqs. (18a–b) in terms of the thermodynamic conjugate forces $\boldsymbol{Y}$

$$\bar{\gamma}_1(\boldsymbol{Y}, \boldsymbol{\omega}) = \bar{\alpha}_{11\,c,t} Y_{11}^{\bar{n}_{11\,c,t}}, \qquad \bar{\gamma}_2(\boldsymbol{Y}, \boldsymbol{\omega}) = \bar{\alpha}_{22\,c,t} Y_{22}^{\bar{n}_{22\,c,t}} + \bar{\alpha}_{12} Y_{12}^{\bar{n}_{12}} \tag{23a–b}$$

the rate-equations during loading finally become:

$$\dot{\boldsymbol{\omega}} = \sum_i \bar{\gamma}_i \left(\frac{\partial Q_i}{\partial \boldsymbol{Y}} \otimes \frac{\partial g_i}{\partial \boldsymbol{\epsilon}} \right) \dot{\boldsymbol{\epsilon}}. \tag{24}$$

It can be shown that the growth functions $\bar{\gamma}_i(\boldsymbol{Y}, \boldsymbol{\omega})$ are strictly monotonously increasing in $\boldsymbol{Y}$ when loading takes place.

6.4. Potential surfaces in thermodynamic force space

It suffices to define the potential functions $Q_i(\boldsymbol{Y}, \boldsymbol{\omega})$ in the thermodynamic force space $\boldsymbol{Y}$ where all components are positive. The evolution of damage in each individual mode is subordinated to dissipation potentials, represented as surfaces in force space $\boldsymbol{Y}$. Although smooth surfaces are preferable for computational reasons, it is not attempted to steer damage growth in all modes by a single, continuous differentiable function. Of particular interest are piecewise linear or quadratic surfaces, resulting in constant and linear laws for the relative damage growth. They may be viewed as truncated Taylor series to approximate a stronger nonlinear coupling.

6.4.1. Piecewise linear and quadratic potentials

Different surfaces $Q_{\|}$ and $Q_{\perp}$ determine the rate-equations in the damage modes for the fibers and the matrix material. The coefficients $l_{\|ij}(\boldsymbol{\omega})$ and $l_{\perp ij}(\boldsymbol{\omega})$ of the piecewise linear potentials $Q_{\|}$ and $Q_{\perp}$ for fiber and matrix damage are functions of the state variable $\boldsymbol{\omega}$,

$$Q_{\|} = l_{\|ij} Y_{ij} + c_1, \qquad Q_{\perp} = l_{\perp ij} Y_{ij} + c_2, \tag{25}$$

with constants c_1, c_2 and summation over indices i, j. As a result of non-negative internal dissipation

$$(\boldsymbol{Y}, \dot{\boldsymbol{\omega}}) = \phi_1\,(\boldsymbol{Y}, \boldsymbol{l}_{\|}) + \phi_2\,(\boldsymbol{Y}, \boldsymbol{l}_{\perp}) \geq 0$$

the normal vectors $\boldsymbol{l}_{\|}$ and $\boldsymbol{l}_{\perp}$ on the planes $Q_{\|}$ and $Q_{\perp}$ must point to the first octant of the Cartesian coordinate system in $\boldsymbol{Y}$-space at zero damage and $c_1 = c_2 = 0$. The implication of non-negative components $l_{\|ij}$, $l_{\perp ij} \geq 0$ is in agreement with the physical behavior, which entails monotonic damage growth.

As already mentioned, the dissipation function is non-negative, if the set $\mathcal{Q}$, bounded by the coordinate planes and the surfaces $Q_i \leq 0$ is star convex with respect to the origin, i.e. every line segment joining the origin and a point $\boldsymbol{Y} \in \mathcal{Q}$ is in $\mathcal{Q}$.

The coupling of the rate-equations for the fluxes is controlled by $\boldsymbol{l}_{\|}$ and $\boldsymbol{l}_{\perp}$, which may be different for tension and compression. In view of the previous discussion on the interaction of damage evolution the following proposal satisfies the physical aspects:

$$\boldsymbol{q}_1 = \boldsymbol{l}_{\|} = \begin{bmatrix} 1 \\ l_{\|\sigma} \\ l_{\|\tau} \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_2 = \boldsymbol{l}_{\perp} = \begin{bmatrix} 0 \\ l_{\perp\sigma} \\ l_{\perp\tau} \end{bmatrix} \tag{26}$$

The components $l_{\|\sigma}$ and $l_{\|\tau}$ indicate the relative growth of the variables ω_{22} and ω_{12} for matrix damage with respect to fiber damage ω_{11}, if the fiber mode is active. For a given growth function γ_2 only one of the parameters $l_{\perp\sigma}$ and $l_{\perp\tau}$ is independent, since the ratio between them is significant and specifies damage growth for ω_{12} with respect to ω_{22} in the matrix modes.

Any singular point $\boldsymbol{\epsilon}_0$ on the loading surface $\boldsymbol{\epsilon}_0 \in \left\{\boldsymbol{\epsilon} \mid g_{\perp} = g_{\|} = 0\right\}$ corresponds to a singular point $\boldsymbol{Y}_0$ on the intersection of the planes $\tilde{Q}_{\|}(\boldsymbol{\epsilon}, \boldsymbol{\omega})$ and $\tilde{Q}_{\perp}(\boldsymbol{\epsilon}, \boldsymbol{\omega})$, described by the potential functions of Eqs. (25) in strain space. This determines the constants c_1 and c_2. Eq. (15) can be viewed as a generalization of "Koiter's rule for normality on non-smooth yield surfaces" (Koiter, 1953).

$$\dot{\boldsymbol{\omega}} = \gamma_1\, \boldsymbol{l}_{\|} \langle \boldsymbol{\nu}_1, \dot{\boldsymbol{\epsilon}} \rangle + \gamma_2\, \boldsymbol{l}_{\perp} \langle \boldsymbol{\nu}_2, \dot{\boldsymbol{\epsilon}} \rangle$$

In component form:

$$\begin{bmatrix} \dot{\omega}_{11\,\mathrm{c,t}} \\ \dot{\omega}_{22\,\mathrm{c,t}} \\ \dot{\omega}_{12} \end{bmatrix} = \left\{ \bar{\gamma}_1\, \bar{\epsilon}_{11} \begin{bmatrix} 1 & (1-\omega_{22\,\mathrm{c,t}})\nu_{12} & 0 \\ l_{\|\sigma} & l_{\|\sigma}(1-\omega_{22\,\mathrm{c,t}})\nu_{12} & 0 \\ l_{\|\tau} & l_{\|\tau}(1-\omega_{22\,\mathrm{c,t}})\nu_{12} & 0 \end{bmatrix} \right. \tag{27}$$

$$\left. + \bar{\gamma}_2 \begin{bmatrix} 0 & 0 & 0 \\ l_{\perp\sigma}(1-\omega_{11\,\mathrm{c,t}})\nu_{21}\bar{\epsilon}_{22} & l_{\perp\sigma}\bar{\epsilon}_{22} & 2l_{\perp\sigma}\bar{\epsilon}_{12} \\ l_{\perp\tau}(1-\omega_{11\,\mathrm{c,t}})\nu_{21}\bar{\epsilon}_{22} & l_{\perp\tau}\bar{\epsilon}_{22} & 2l_{\perp\tau}\bar{\epsilon}_{12} \end{bmatrix} \right\} \begin{bmatrix} \dot{\epsilon}_{11\,\mathrm{c,t}} \\ \dot{\epsilon}_{22\,\mathrm{c,t}} \\ 2\dot{\epsilon}_{12} \end{bmatrix},$$

with

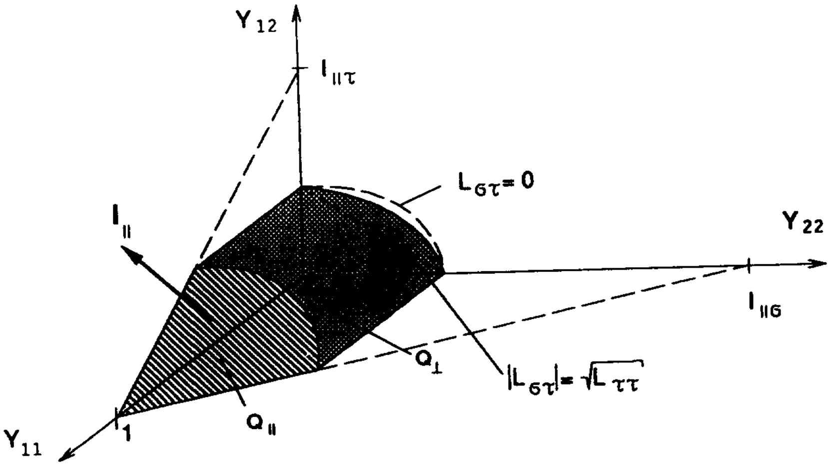

Fig. 7. Piecewise linear and quadratic damage potential.

$$\tilde{\epsilon}_{11} = 2\left(\frac{E_{\|}}{DX_{c,t}}\right)^2 \left[\epsilon_{11} + (1-\omega_{22\,c,t})\nu_{12}\epsilon_{22}\right], \qquad \tilde{\epsilon}_{22} = 2\left(\frac{E_{\perp}}{DY_{c,t}}\right)^2 \left[(1-\omega_{11\,c,t})\nu_{21}\epsilon_{11} + \epsilon_{22}\right],$$

$$\tilde{\epsilon}_{12} = 2\left(\frac{G}{S_c}\right)^2 \epsilon_{12}.$$

It is only the Poisson effect, which introduces additional non-zero entries into the second column of the fiber mode and into the first column of the matrix mode in Eq. (27).

Quadratic potentials for Q may be written as $Q = \boldsymbol{Y}^T \boldsymbol{L}\boldsymbol{Y} + c$, where the coefficient matrix $\boldsymbol{L}$ in the quadratic form must be positive semi-definite in order to satisfy the Kelvin inequality. By Onsager's principle $\boldsymbol{L}$ is symmetric, so a potential exists at least if the growth functions are constant. The surface Q is convex as long as $\boldsymbol{L}$ is non-negative. Q simplifies for the matrix modes to:

$$Q_{\perp} = \boldsymbol{Y}^T\boldsymbol{L}_{\perp}\boldsymbol{Y} + c, \qquad \boldsymbol{L}_{\perp} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & L_{\sigma\tau} \\ 0 & L_{\sigma\tau} & L_{\tau\tau} \end{bmatrix} \quad \text{and} \quad L_{\tau\tau} \geq L_{\tau\sigma}^2. \tag{28}$$

The dissipation function is non-smooth at the "corners" of the potential – see Fig. 7. Again, the surface Q only needs to be star-shaped with respect to the origin in order to satisfy positive dissipation.

7. Passive damage evolution

As indicated, the influence of the defects on the elastic behavior is treated phenomenologically. In order to model passive damage it becomes necessary to introduce separate states of damage for tension and compression. In general, the variables for both damage states evolve, if the state of defects changes. It is worth noting: damage models for a realistic material behavior will lie somewhere in between the two limiting cases:

(i) The damage states for tension $\boldsymbol{\omega}_t$ and compression $\boldsymbol{\omega}_c$ are the same (orientationally invariant damage), i.e. $\boldsymbol{\omega}_t = \boldsymbol{\omega}_c$.

(ii) The damage evolution for tension has no influence on the one for compression (unilateral damage).

A reasonable way to define the evolution of the passive damage state $\boldsymbol{\varpi}$ is simply by scaling the contributions in the rate-equation (15) for the variables $\boldsymbol{\omega}$ of the active damage modes. Since the scale factors may also depend

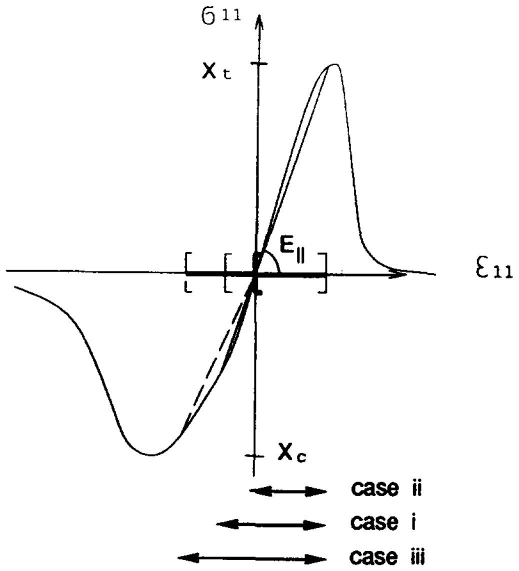

Fig. 8. Stress–strain-diagram with elastic range for cases i–iii.

on the state variables $\boldsymbol{\sigma}$ and $\boldsymbol{\omega}$, growth functions $\Gamma_i(\boldsymbol{\sigma}, \boldsymbol{\omega})$ can be introduced into the evolution equations for the passive damage state $\boldsymbol{\varpi}$,

$$\dot{\boldsymbol{\varpi}} = \sum_i \Gamma_i \langle \boldsymbol{\nu}_i, \dot{\boldsymbol{\epsilon}} \rangle \boldsymbol{q}_i.$$

The functions Γ_i play a role similar to the functions $\gamma_i(\boldsymbol{\sigma}, \boldsymbol{\omega})$ for the active damage variables. Formally the evolution of the passive damage is strain controlled by the same norm $\langle \boldsymbol{\nu}_i, \dot{\boldsymbol{\epsilon}} \rangle$ as it is the active one. Fig. 8 illustrates the extension of the elastic range $\mathcal{E}$ – indicated by [ ] – in tension and compression due to strain control of active and passive damage for a specimen under uniaxial tension.

The enforcement of continuity along the boundary of the elastic range – see section 5.1 – may serve as a guideline for a third choice of passive damage evolution.

(iii) If the loading surface evolves the same in tension as in compression, then the function γ_i may be used for the passive growth Γ_i.

These three possibilities are generalized by introducing a variable $\boldsymbol{\beta}$, which may be viewed as an "equivalent strain tensor" for the passive damage state. It is obtained from the linear map by means of the fourth-order tensor $\boldsymbol{\Pi}$:

$$\boldsymbol{\beta} = \boldsymbol{\Pi}\, \boldsymbol{\epsilon}.$$

If $\boldsymbol{\Pi}$ is assumed constant, the replacement of the inner product norm $\langle \boldsymbol{\nu}_i, \dot{\boldsymbol{\epsilon}} \rangle$ by $\langle \boldsymbol{\nu}_i, \dot{\boldsymbol{\beta}} \rangle$ yields:

$$\dot{\boldsymbol{\varpi}} = \sum_i \Gamma_i \langle \boldsymbol{\nu}_i, \boldsymbol{\Pi}\, \dot{\boldsymbol{\epsilon}} \rangle \boldsymbol{q}_i.$$

The previously introduced types i–iii of passive damage are recovered if:

(i) $\dot{\boldsymbol{\varpi}} = \dot{\boldsymbol{\omega}}$.
(ii) $\boldsymbol{\Pi} = 0$, $\quad \Gamma_i = 0$.
(iii) $\boldsymbol{\Pi} = \boldsymbol{I}$, $\quad \Gamma_i = \gamma_i$.

Alternatively, the evolution of the passive damage state may be modelled by utilizing the inelastic contribution $U^{i}(\boldsymbol{\omega})$ of the complementary energy function to the thermodynamic forces $\boldsymbol{Y}$ in Eq. (21). An evolution equation for the center of the elastic range needs to be introduced similar to the concept of kinematic hardening in elastoplasticity. The resulting motion of the boundary $\partial\mathcal{E}$ of the elastic range in tension and compression leads to the evolution equations for both damage states. It is a generalization of type (iii) with types (i) and (ii) as special cases.

8. Application to uniaxial tests

Most of the previously introduced parameters in the kinetic equations can be extracted from experimental data of uniaxial tension, compression and simple shear tests. The stress-strain diagram of damage models typically shows the degression of the tangent stiffness during loading before the ultimate strength is reached. After the peak loading is passed, the load–displacement diagram under displacement control shows a steep descending branch. The validity of the corresponding stress-strain diagram might be restricted in face of localization for softening materials. Practical computations, however, crucially rely on representative diagrams for the local response as schematically sketched in Fig. 8.

Analytical solutions of the model are possible under some uniaxial loading conditions. The case of uniaxial fiber tension is studied as an example. For uniaxial fiber compression a coupled set of rate equations is obtained, since the the compressive strength X_c of the composite (Rosen and Dow, 1972; Steif, 1990) depends on the elasticity moduli G_m or E_m of the matrix material and thus on the damage state. For most practical reinforcement ratios v the so-called "shear-mode" of compressive failure determines the compressive strength. An estimate for the fiber buckling stress $\sigma_{f,crit}$ is given by

$$\sigma_{f,crit} = \frac{G_m}{v(1-v)}$$

for reinforcement ratios v between 0.18 and 0.6 in the case of fiber-glass-epoxy or v between 0.08 and 0.5 for boron–epoxy laminae - see Rosen and Dow (1972), Jones (1975). The lamina strength for compression is:

$$X_c = v\,\sigma_{f,crit} = \frac{G_m}{1-v}.$$

In the presence of shear damage the compressive strength decreases linearly with ω_{12}, since $G_m(\boldsymbol{\omega}) = G_{m0}(1-\omega_{12})$, where G_{m0} represents the shear modulus of the undamaged lamina. This yields:

$$X_c = (1-\omega_{12})\,\frac{G_{m0}}{1-v} = (1-\omega_{12})\,X_{c0},$$

with X_{c0} as the compressive strength of the undamaged composite. However, experiments rather predict:

$$X_c = (1-\omega_{12})^{\alpha}\,X_{c0},$$

where the parameter α is of the order of 0.1.

In the case of a brittle matrix material the evolution equations for ω_{11} and ω_{12} are coupled through the strength X_c, because every buckled fiber completely damages the surrounding resin of the matrix material or at least the interface zone between the fiber surface and the matrix. This means: $\dot{\omega}_{22} > 0$ and $\dot{\omega}_{12} > 0$ when $\dot{\omega}_{11} > 0$. Analytical solutions for the resulting system of coupled nonlinear ordinary differential equations are difficult to obtain.

In the case of reinforced rubber the buckled fibers hardly damage the neighboring matrix material, i.e. $\omega_{22} = \omega_{12} = 0$, when fiber damage evolves, $\dot{\omega}_{11} > 0$. The solution of the governing equations is similar to the example for uniaxial tension.

8.1. Example: uniaxial tension in fiber direction

The lamina is loaded in fiber direction only: $\sigma_{22} = \tau = 0$. The non-zero strain components and energy release rates are

$$\epsilon_{11} = \frac{\sigma_{11}}{(1-\omega_{11t})E_\parallel} \quad \text{and} \quad \epsilon_{22} = -\frac{\nu_{12}}{E_\perp}\sigma_{11} \quad \text{resp.} \quad Y_{11} = \frac{\sigma_{11}^2}{2(1-\omega_{11t})^2 E_\parallel} = \tfrac{1}{2}E_\parallel\, \epsilon_{11}^2,$$

The transverse strain ϵ_{22} is controlled by the fiber strain ϵ_{11} and damage component ω_{11}:

$$\epsilon_{22} = -(1-\omega_{11t})\, \nu_{21}\, \epsilon_{11}.$$

Loading is predicted by the fiber mode criterion $g_\parallel = 0$. By noting that the transverse strain rate $\dot{\epsilon}_{22}$ depends on $\dot{\epsilon}_{11}$ and $\dot{\omega}_{11t}$, the time derivative becomes:

$$\dot{\epsilon}_{22} = -(1-\omega_{11t})\nu_{21}\, \dot{\epsilon}_{11} + \dot{\omega}_{11t}\, \nu_{21}\, \epsilon_{11}.$$

The evolution equations for the damage variables take the form:

$$\begin{bmatrix} \dot{\omega}_{11t} \\ \dot{\omega}_{22c,t} \\ \dot{\omega}_{12} \end{bmatrix} = \bar{\gamma}_1\, \tilde{\epsilon}_{11} \begin{bmatrix} 1 \\ l_{\parallel\sigma} \\ l_{\parallel\tau} \end{bmatrix} \left\{ \left[1-(1-\omega_{11t})^2 \nu_{12}\nu_{21}\right] \dot{\epsilon}_{11} + (1-\omega_{11t})\nu_{12}\nu_{21}\, \epsilon_{11}\, \dot{\omega}_{11t} \right\},$$

where the definition

$$\tilde{\epsilon}_{11} = 2\left(\frac{E_\parallel}{DX_t}\right)^2 \left[1-(1-\omega_{11t})^2\nu_{12}\nu_{21}\right] \epsilon_{11}.$$

has been used.

The Poisson effect formally modifies the kinetic equation for ω_{11} in comparison to its one-dimensional counterpart of the damage model. Complete agreement with regard to the ω_{11}-component is achieved by an appropriate choice of $\bar{\alpha}_{11}$ in the growth function $\bar{\gamma}_1$,

$$\bar{\gamma}_1 = \bar{\alpha}_{11}\left(\frac{\hat{\sigma}_{11}}{X_t}\right)^{n_{11t}} = \bar{\alpha}_{11}\left(\frac{\epsilon_{11}}{X_t\,/\,E_\parallel}\right)^{n_{11t}}.$$

The evolution equation for ω_{11} can be brought into the simple form:

$$\dot{\omega} = \alpha\left(\frac{\epsilon}{X_t\,/\,E}\right)^{m-1} \frac{\dot{\epsilon}}{X_t\,/\,E}.$$

with σ_{11}, ϵ_{11}, ω_{11}, α_{11}, $n_{11}+1$ and $E_\parallel$ replaced by: σ, ϵ, ω, α, $m-1$ and E.

The quotient X_t/E is equal to the "nominal" failure strain ϵ_f for a linear elastic material. The criteria for loading in fiber direction,

$$g_\parallel = \left(\frac{\epsilon_{11}}{X_t/E_\parallel}\right)^2 - r_{\parallel t} = 0,$$

obviously satisfies inequality (12). Two special cases for α are investigated:

(i) $\alpha = \frac{1}{e}(1-\omega)$ implies:

$$\dot{\omega} = \frac{1}{e}(1-\omega)\left(\frac{\epsilon}{\epsilon_f}\right)^{m-1} \frac{\dot{\epsilon}}{\epsilon_f}.$$

The differential equation can be integrated in closed form:

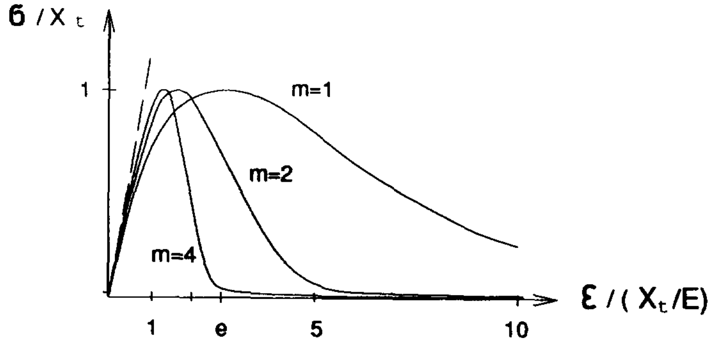

Fig. 9. Uniaxial stress-strain diagrams.

$$\omega = 1 - \exp\left[-\frac{1}{me}\left(\frac{\epsilon}{\epsilon_f}\right)^m\right]$$

with the constant of integration set to zero ($\epsilon = 0$ if $\omega = 0$). The same result may be derived from statistical mechanics by means of the probability theory for the failure of a bundle of fibers with initial defects, subjected to the stress σ. The Weibull distribution function of the defects in the fibers yields the previous result – see Weibull (1939), Krajcinovic (1989). When all fibers have the same failure strain, then the perfectly elastic–brittle failure model is recovered.

The stress σ and strain ϵ are normalized with respect to the strength X_t and the nominal failure strain ϵ_f in the constitutive law for the case of loading.

$$\frac{\sigma}{X_t} = \exp\left[-\frac{1}{me}\left(\frac{\epsilon}{\epsilon_f}\right)^m\right]\frac{\epsilon}{\epsilon_f}.$$

For $m = 1$, 2 and 4 the graphical representation of the stress–strain response in case of uniaxial loading is given in Fig. 9.

(ii) $\alpha = \alpha_0$ is constant and the exponent is set to $m = 1$ – see also Janson and Hult (1977) and Krajcinovic (1989):

$$\dot{\omega} = \alpha_0 \frac{\dot{\epsilon}}{\epsilon_0}$$

The constant ω_0 can be interpreted as the initial damage in the resulting "state equation", obtained by integration:

$$\omega = \max\left(\alpha_0 \frac{\epsilon}{\epsilon_0} - \omega_0\,,\, 0\right).$$

The solution is linear in the strain and guarantees non-negative damage ω. The evolution of damage is initiated when the strain ϵ exceeds the value $\epsilon_0 = (\epsilon_f)\,/\,(\alpha_0)\,\omega_0 \leq \epsilon_f$ for the first time with loading indicated by $(\partial g_{\|})\,/\,(\partial \epsilon)\,\dot{\epsilon} > 0$. The initial damage threshold, $r = r_0$, in the criterion for loading in fiber direction is determined by the initial damage value ω_0,

$$r_0 = \left(\frac{\epsilon_0}{\epsilon_f}\right)^2 = \left(\frac{\omega_0}{\alpha_0}\right)^2.$$

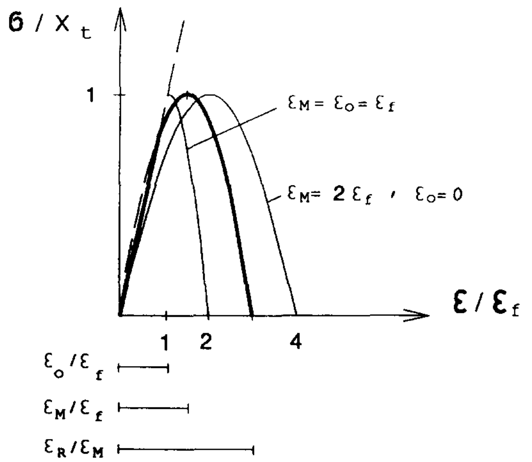

Fig. 10. Dimensionless stress–strain diagram with limiting cases.

In general, the material parameter α_0 is not directly accessible from test data and can be expressed in terms of the strain ϵ_M at peak stress, $\sigma|_{\epsilon_M} = X_t$. During loading it follows from the condition $d\sigma/d\epsilon = 0$:

$$\alpha_0 = \frac{1+\omega_0}{2\epsilon_M}\epsilon_f \quad \text{with } \epsilon_M \text{ bounded by:} \quad \epsilon_f \leq \epsilon_M \leq 2\epsilon_f$$

and

$$\sigma = (1+\omega_0)\; E \left(1 - \frac{\epsilon}{2\epsilon_M}\right) \epsilon \,.$$

The second root of the stress–strain function is the strain ϵ_R at complete rupture, where $\sigma = 0$. It is equal to twice the strain ϵ_M at maximum stress, i.e. $\epsilon_R = 2\epsilon_M$.

For the approximation of stress–strain diagrams it is more appropriate to replace the initial damage ω_0 by the threshold strain ϵ_0 at the onset of damage evolution, where $\omega_0 = (\alpha_0) \,/\, (\epsilon_f)\, \epsilon_0$. The stress–strain response for loading results in

$$\sigma = \frac{2\epsilon_M - \epsilon}{2\epsilon_M - \epsilon_0} E\epsilon.$$

Its graph is shown in Fig. 10 for an ϵ_M between ϵ_f and $2\,\epsilon_f$ together with both limit curves. The area under the σ–ϵ curve is a measure for the strain energy at rupture (complete damage) of the lamina. By appropriately choosing the exponent m either the rupture strain or the "fracture" energy can be accommodated.

Frequently, the Young's modulus E, the strength X_t and the associated strain ϵ_M at peak stress are available, which give the threshold strain as

$$\epsilon_0 = \epsilon_M \left(2 - \frac{\epsilon_M}{X_t/E}\right) \quad \text{and stress as} \quad \sigma = \frac{2\epsilon_M - \epsilon}{\epsilon_M{}^2/\epsilon_f} E\epsilon.$$

9. Concluding remarks and summary

A simple model for the nonlinear analysis of composite material has been presented within the framework of continuum damage mechanics. The complicated process of defect development is reduced to a few essential features, which include the specification of an elastic and an inelastic range of deformation, failure planes and damage modes. Damage variables are introduced for the phenomenological treatment of the state of defects and its implications on the degradation of the stiffness properties. The loading criteria and kinetic laws are condensed to the dominating physical quantities. The significant material parameters are obtained from data readily available. The parameters for the growth functions and relative damage growth may be estimated on the basis of uniaxial tests. The model allows considering more refined loading criteria or evolution laws, especially with respect to interaction terms, e.g. products between different components of stresses or conjugate forces. More elaborate dependencies between the components of the constitutive tensor and the governing internal variables may be taken into consideration without conceptual difficulties. Since those interdependencies are often difficult to obtain they are kept to a minimum number.

The model is well suited for the application and implementation in standard finite element codes, since it falls into the class of strain-controlled continuum models. The history date for the defect evolution is kept low. The algebraic structure of the constitutive equations allows to employ well-established numerical algorithms of elastoplasticity with minor modifications. The issues of stability and localization due to strain softening for the analysis of boundary value problems have yet to be addressed.

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