The Mechanics of Dynamic Shear Crack Propagation

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A recent trend in seismology has been to model the earthquake source as a dynamically extending shear crack, and several basic concepts which seem to be important in this modeling process are examined. First, the universal spatial dependence of the plane elastodynamic stress and velocity fields near a sharp propagating crack tip is demonstrated for both subsonic and transsonic crack speeds, and the corresponding energy release rates are considered. Next, a class of steady state shear crack propagation problems is analyzed, based on both a direct stress analysis approach and an energy integral approach which obviates the need for a complete stress analysis in some cases. Several distinct stress differences which correspond to some of the common definitions of stress drop are involved in the analysis of these simple problems. Finally, some analytical considerations are presented which are relevant to rupture velocity determination for transient shear crack growth according to various fracture criteria, such as the critical stress intensity factor criterion, the critical energy release rate criterion, and the critical stress level criterion. Possible effects of spatially nonuniform stress drop and frictional resistance on rupture propagation are also discussed.

INTRODUCTION

The mechanical modeling of an earthquake source has been a research objective in seismology for many years, and a recent trend in this area has been to view the source as a dynamically extending shear crack. In precracked laboratory specimens, shear cracks often tend to extend in a direction oblique to the initial crack direction under stress. On the other hand, it is almost universally assumed that shear cracks serving as earthquake source models extend as planar cracks, and this assumption appears to be consistent with observation. Possible reasons for the planar growth are that a preexisting fault provides a weakened path which is preferential for crack extension and that the confining pressure reduces the effect of tensile stresses near the crack tip which might otherwise lead to oblique crack growth.

The emphasis here is on basic concepts which seem to be important in dynamic shear crack analysis, and methods of analysis for specific problems are not considered in detail. A number of significant contributions which have concentrated on the dynamic shear fracture process have appeared in the literature in recent years, including those by *Burridge* [1973], *Burridge and Halliday* [1971], *Fossum and Freund* [1975], *Husseini et al.* [1975], *Kostrov* [1966, 1975], *Kostrov and Nikitin* [1970], *Richards* [1973], and *Weertman* [1975] on analytical solutions to particular problems and those by *Andrews* [1976*a*, *b*], *Das and Aki* [1977], and *Ida* [1973] on numerical solutions to particular problems. The present discussion is based primarily on these contributions, as well as on the results of research on dynamic tensile fracture which were recently reviewed by *Freund* [1976]. It should also be noted that a number of significant contributions which have concentrated on the stress wave radiation from a propagating shear crack have also appeared, including those by *Brune* [1973], *Husseini and Randall* [1976], *Madariaga* [1976], and *Richards* [1976].

SOME GENERAL RESULTS FOR PROPAGATING SHEAR CRACKS

Two general concepts which have played a major and fundamental role in the study of dynamic crack propagation are those of dynamic stress intensity factor and dynamic energy release rate. In this section these concepts are given mathematical definitions in terms of the elastodynamic stress and deformation fields which prevail in a body of isotropic elastic material during crack propagation.

Stress and Particle Velocity Near a Crack Tip

A number of analytical solutions for the propagation of a sharp crack through a plane elastic solid are available. It has been observed for some time that the dependence of the stress field on spatial coordinates local to the crack tip is common to all solutions. For running cracks the spatial distribution is dependent on the speed of crack propagation, and it reduces to the appropriate expression for the stationary crack when the crack speed is set equal to zero. It can be demonstrated that this common spatial dependence of the near-tip elastic field for running crack solutions is a general result, independent of the configuration of the body and the details of the loading system. The only quantity which varies from one specific problem to another is a time-dependent scalar multiple of the universal spatial dependence. The corresponding result for elastostatic fracture mechanics was first presented by *Irwin* [1957] and *Williams* [1957]. The general result for rapid propagation of a mode II shear crack may be derived by following the asymptotic analysis of *Freund and Clifton* [1974], who presented the corresponding results for dynamic propagation of a mode I tensile crack at subsonic speeds.

A Cartesian coordinate system is oriented in the body in such a way that the particle displacement is in the x, y plane (see Figure 1). The region of the plane occupied by the body is denoted by D, and the outer boundary by S. The inner boundary of D consists of the crack faces, except near the crack ends where the inner boundary is augmented by small loops surrounding the crack tips. The loops have arbitrary shape, but they are fixed with respect to the moving crack tips. The outer boundary S is subjected to traction boundary conditions, displacement boundary conditions, or some suitable combination of both. Tractions may be acting on the crack faces, but they are assumed to be bounded in magnitude near the crack tips, and no body forces are acting. For simplicity, it is assumed that conditions are such that crack extension occurs in mode II, the plane strain shearing mode.

Attention is directed to the crack tip surrounded by the loop L, and a local Cartesian coordinate system (ξ, η) and a local polar coordinate system (r, θ), both of which move with the

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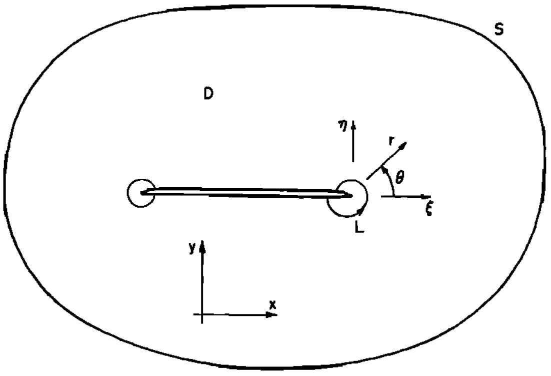

Fig. 1. Configuration of a body containing a crack at a fixed instant of time.

crack tip, are introduced as shown in Figure 1. The crack is assumed to be extending in its own plane, and the instantaneous rate of extension v is any continuously varying function of time. With the requirement that the local energy density must be integrable, the following general results may be established by application of the asymptotic method described by *Freund and Clifton* [1974]: For all plane elastodynamic solutions for running mode II cracks the shear stress component σ_{xy} and the particle velocity component $\dot{u}_x$ are given by

$$\sigma_{xy} = \frac{K_2(t)}{(2\pi)^{1/2}R(v)}\left\{4\alpha_l\alpha_s\frac{\cos(\theta_l/2)}{r_l^{1/2}} - (1+\alpha_s^2)\frac{\cos(\theta_s/2)}{r_s^{1/2}}\right\} \tag{1}$$

$$\dot{u}_x = \frac{v\alpha_s K_2(t)}{\mu R(v)(2\pi)^{1/2}}\left\{\frac{\sin(\theta_l/2)}{r_l^{1/2}} - (1+\alpha_s^2)\frac{\sin(\theta_s/2)}{r_s^{1/2}}\right\} \tag{2}$$

for $0 < v < v_s$ and by

$$\sigma_{xy} = \frac{K_2^*(t)}{(2\pi)^{1/2}}\left\{\frac{\cos(m\theta_l)}{r_l^m} + \frac{H(-\xi - \beta_s|\eta|)\sin(m\pi)}{(-\xi - \beta_s|\eta|)^m \tan m}\right\} \tag{3}$$

$$\dot{u}_x = \frac{vK_2^*(t)}{2\mu\alpha_l(2\pi)^{1/2}}\left\{\frac{\sin(m\theta_l)}{r_l^m} - 2\alpha_l\beta_s\frac{H(-\xi - \beta_s|\eta|)\sin(\pi m)}{(-\xi - \beta_s|\eta|)^m \tan m}\right\} \tag{4}$$

for $v_s < v < v_l$, as $r \to 0$. The subscripts l and s refer to the longitudinal and shear wave speeds v_l and v_s, μ is the elastic shear modulus, and

$$\alpha_l = (1 - v^2/v_l^2)^{1/2} \qquad r_l e^{i\theta_l} = \xi + i\alpha_l\eta \tag{5}$$

$$\alpha_s = (1 - v^2/v_s^2)^{1/2} \qquad r_s e^{i\theta_s} = \xi + i\alpha_s\eta \tag{6}$$

$$\beta_s = (v^2/v_s^2 - 1)^{1/2} \qquad \pi m = \tan^{-1}[4\alpha_l\beta_s/(1+\alpha_s^2)^2] \tag{7}$$

$$R(v) = 4\alpha_l\alpha_s - (1+\alpha_s^2)^2 \tag{8}$$

$$H(t) = 1 \quad t > 0 \qquad H(t) = 0 \quad t < 0 \tag{9}$$

Similar expressions may be written for the local crack tip variation of other components of stress and particle velocity for both crack velocity ranges. However, the above expressions are typical and perhaps are the most important for discussing mode II shear fractures, and the expressions for other components are not included here.

Expression (1) has been normalized with respect to the relation

$$K_2(t) = \lim_{r\to 0}(2\pi r)^{1/2}\sigma_{xy}(r, 0, t) \tag{10}$$

The time-dependent scalar $K_2(t)$ is then the mode II stress intensity factor of elastic fracture mechanics. For crack tip speeds in the range $0 < v < v_s$ the local stress and velocity vary as the inverse square root of distance from the crack tip. The algebraic sign of the coefficient of the square root singular factor depends on whether the crack speed is less than or greater than the Rayleigh surface wave speed v_r. The function $R(v)$, which appears in (1) and (2) and which is defined in (8), is the so-called Rayleigh wave function. This function has the properties that $R(\pm v_r) = 0$, $R(v) > 0$ for $0 < v < v_r$, and $R(v) < 0$ for $v_r < v < v_s$. Examination of (1) and (2) leads to the conclusion that shear traction on the prospective fracture plane $\theta = 0^\pm$ and particle velocity on the fracture surface $\theta = \pm\pi$ have opposite algebraic sign if $0 < v < v_r$ and the same sign if $v_r < v < v_s$. This observation has important implications in considering energy fluxes associated with crack growth. It should also be noted that expression (1) represents only the dominant singular term in the expansion of the elastic field about the crack tip, and the expansion could be continued. For example, if the crack faces are subjected to tractions such that $\sigma_{xy} \to \tau_0$ as $\xi \to 0^-$ and $\eta = \pm 0$, then the 'order one' term in the expansion is simply τ_0. The next term in the expansion is proportional to $(r)^{1/2}$, and so on. Of course, if $K_2(t) \equiv 0$, then the higher-order terms in the expansion take on added importance.

Expression (3) has been normalized with respect to the relation

$$K_2^*(t) = \lim_{r\to 0}(2\pi)^{1/2}r^m\sigma_{xy}(r, 0, t) \tag{11}$$

so that the time-dependent scalar $K_2^*(t)$ is also a mode II stress intensity factor, although the singularity in stress for $v_s < v < v_l$ is in general weaker than the inverse square root singularity which arises for $0 < v < v_s$. As can be seen from (7), the exponent m which appears in (3) and (4) varies continuously from $m = 0$ at $v = v_s$, up to a maximum value $m = \frac{1}{2}$ at $v = v_s(2)^{1/2}$, and back to $m = 0$ at $v = v_l$. It should be noted that the shear wave contribution to expressions (3) and (4) is very different from that in (1) and (2). Because the crack tip speed is greater than the shear wave speed, no shear wave radiated from the crack can propagate ahead of the running crack tip. Instead, shear wave motion can exist only behind discrete wave fronts which appear in the form of Mach waves trailing from the extending crack. In the local (ξ, η) coordinate system these Mach wave fronts coincide with the lines $\xi + \beta_s|\eta| = 0$. In the solution of a particular mode II crack propagation problem, *Burridge* [1973] noted that the stress singularity for crack speeds in the range $v_s < v < v_l$ has the form shown in (3). It is concluded from an application of the asymptotic method of *Freund and Clifton* [1974] that this is a general result, common to all mode II crack propagation problems.

Finally, the local crack tip field for dynamic propagation of a crack in the antiplane shear mode, or mode III, is included. Referring once again to Figure 1, the only nonzero component of displacement is in the direction normal to the x, y plane, which is, say, the z direction. The shear stress component σ_{yz} and the particle velocity $\dot{u}_z$ are

$$\sigma_{yz} = \frac{K_3(t)}{(2\pi r_s)^{1/2}}\cos(\theta_s/2) \tag{12}$$

$$\dot{u}_z = \frac{vK_3(t)}{\mu\alpha_s(2\pi r_s)^{1/2}}\sin(\theta_s/2) \tag{13}$$

as $r \to 0$ for any crack speed in the range $0 < v < v_s$. The notation is the same as that for mode II, except for the appearance of the mode III stress intensity factor

$$K_3(t) = \lim_{r\to 0} (2\pi r)^{1/2}\sigma_{yz}(r, 0, t) \quad (14)$$

As can be seen from (12) and (13), the local stress and velocity fields have the characteristic square root singular dependence on distance from the moving crack tip.

The Dynamic Energy Release Rate

Consider once again the two-dimensional body of linear elastic material containing a crack shown in Figure 1. For present purposes it is not necessary that the crack be mathematically sharp, but it is assumed that the loop L originates on one crack surface, that it terminates on the opposite crack surface, and that it completely surrounds the crack tip zone, whether this zone is a point or a diffuse region. As before, the loop is considered to be fixed with respect to the crack tip, which is moving at an arbitrary rate. The figure represents the body at a fixed instant of time. As was shown by *Freund* [1976], a general expression for energy absorption rate may be derived in terms of the local crack tip stress and velocity fields without specification of a particular mode of crack propagation.

As the crack tip surrounded by L moves under the action of the applied loading, mechanical energy flows through L at a rate which will be denoted by F. This energy flux F may be computed in terms of the elastic field of the body by application of an overall energy rate balance. Let P, T, and U denote the rate of work of the applied tractions, the total kinetic energy of the material in D, and the total strain energy of the material in D, respectively. Then the flux of energy through L is equal to the difference between the rate of work of the applied loads and the rate of increase of internal energy in D, that is,

$$F = P - (\dot{T} + \dot{U}) \quad (15)$$

If the terms on the right side of (15) are expressed in terms of the instantaneous stress and velocity fields in the body according to standard definitions and the divergence theorem is applied, then the following main result is obtained

$$F = \int_L [\sigma_{ij}n_j\dot{u}_i + \tfrac{1}{2}(\sigma_{ij}u_{i,j} + \rho\dot{u}_i\dot{u}_i)vn_x]\, dL \quad (16)$$

where σ_{ij} and $\dot{u}_i$ are the stress and velocity components, n_i is the normal to L pointing away from the crack tip, and ρ is the material mass density. The interpretation of the terms in (16) is straightforward. The first term is the rate of work of the material outside of L on the material inside L, that is, it is the sum along L of the inner product of traction and particle velocity, and if L were fixed with respect to the material, this would be the only contribution to the energy flux. Because L is moving through the material, however, material particles cross L. Associated with each material particle is an energy density, and the second term in (16) represents the contribution to the energy flux due to the flux of energy-bearing material particles.

Several remarks should perhaps be added here concerning previous work on the energy flux (16) or quantities related to it. First of all, in the original derivation of (16) by *Freund* [1972], only the limiting case in which L is shrunk onto the crack tip was considered. It seems that expression (16) provides a basis for unification of several apparently diverse results on dynamic energy release rate, however, and consideration of limiting cases of L is postponed. Second, F does reduce to the appropriate form of the path-independent J integral [*Rice,* 1968] for the special case of quasi-static deformation. In general, however, the value of F will indeed depend on the path employed to evaluate it (see, for example, *Atkinson and Eshelby* [1968] or *Freund* [1972]). The fact that F is path independent under the special condition that the complete elastic field is constant with respect to an observer moving with the crack tip has been observed by *Sih* [1970]. Finally, it is noted that result (16) is also valid for three-dimensional crack growth provided that L is understood to be a tube of fixed cross section surrounding the crack edge and moving with it and vn_x is the normal speed of the tube at each point on its surface.

Sharp crack tip. For extension of a sharp crack under plane conditions the rate of energy being supplied to the growing crack is given by F in (16) in the limit as L is shrunk onto the moving crack tip. To perform this calculation, F may be evaluated for a path within the region where the elastic field is adequately described by results (1)–(4) and (12)–(13) prior to taking the limit. Because the near-tip fields have universal spatial dependence, the integral defining F may be evaluated, and an expression for F in terms of the dynamic stress intensity factors may be obtained. Furthermore, the near-tip fields are steady with respect to an observer moving with the crack tip, so that the value of F which is calculated is independent of the shape of L within the crack tip region [*Freund,* 1976]. Some of the details of calculating F in terms of the dynamic stress intensity factor for extension of a mode I crack are included in [*Freund,* 1972]. For combined mode II and mode III crack growth at a speed in the range $0 < v < v_s$ the energy flux into the crack tip is given by

$$F = \frac{(1-\nu^2)}{E}\frac{v^3\alpha_s}{v_s^2R}K_2^2 + \frac{v}{2\mu\alpha_s}K_3^2 \quad (17)$$

where ν and E are Poisson's ratio and Young's modulus, respectively, of the material. The energy flux F is related to the energy released per unit crack advance, the dynamic energy release rate G, simply by $F = vG$. Equation (17) thus represents a relationship between the dynamic stress intensity factors and the dynamic energy release rate which is valid for all loading conditions and all geometrical configurations. In this sense it is the analog for dynamic shear cracks of Irwin's well-known relationship between energy release rate and the mode I stress intensity factor. It is noted from (17) that if $K_2 \neq 0$, then energy is absorbed by a mode II crack tip for speeds in the range $0 < v < v_r$, and energy is radiated from the tip for speeds in the range $v_r < v < v_s$. Although (17) remains valid for speeds in the range $v_r < v < v_s$, the possibility that a moving crack tip acts as a source of mechanical energy is normally rejected on physical grounds and the requirement that $K_2 = 0$ for $v_r < v < v_s$ is enforced in the solution of particular problems. As will be noted in the subsequent section on transient crack propagation, plane strain solutions for this speed range are not unique, and for the case to be considered, a nontrivial solution with $K_2 = 0$ can be found. For a mode III crack with $K_3 \neq 0$, energy is absorbed into the tip for all speeds in the range $0 < v < v_s$.

For propagation of a mode II crack with speed in the range $v_s < v < v_l$ the crack tip singularity in stress and velocity is weaker than the inverse square root singularity, unless $v = v_s(2)^{1/2}$, and therefore $F = 0$ for all speeds in this range except for $v = v_s(2)^{1/2}$. For this particular speed the shear waves trailing the crack tip are absent, and the energy flux into the tip

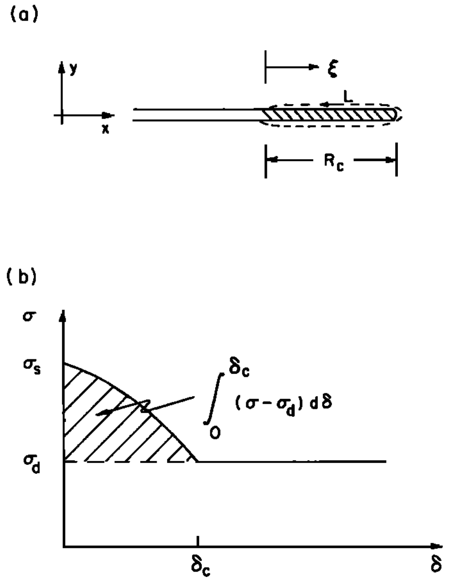

Fig. 2. The cohesive zone crack tip model. The amount of slip between the crack faces is denoted by δ, and the shear traction resisting slip within the cohesive zone is $\sigma(\delta)$. (*a*) The contour L is employed in the derivation of (19). (*b*) The cross-hatched area is the cohesive energy density of the interface.

due to longitudinal wave motion is

$$F = v(K_2^*)^2/4\mu\alpha_l \tag{18}$$

It must be noted that a stress distribution with an infinite singularity is clearly a mathematical idealization, in that no real material can actually support such a stress. The usual rationalization for admitting the singular stress distribution, the strength of which is measured by the stress intensity factor, is based on the concept of small-scale yielding [*Rice*, 1968]. It is thus assumed that in the immediate vicinity of the crack tip the potentially large stresses are relieved by some nonlinear process in a region whose dimensions are small compared to crack length and body dimensions. It is assumed further that the stress distribution in the elastic material adjacent to the small zone is adequately described by the dominant singular term in the elasticity solution. Under small-scale yielding conditions the stress intensity factor may be considered to be a one-parameter measure of the amplitude of the stress which is being applied to the material in the crack tip region. The stress intensity factor approach circumvents consideration of how the material in the crack tip region actually responds to the applied stress.

Cohesive zone crack tip model. In order to avoid infinitely large stresses on the fracture plane a number of models involving a one-dimensional cohesive zone extending ahead of the physical crack tip in the fracture plane have been proposed. In each case the model is analyzed by making the crack longer by an amount R_c, the cohesive zone size (see Figure 2*a*), with the cohesive stresses in this zone acting so as to restrain crack opening (in mode I) or crack face sliding (in modes II or III). The size of the cohesive zone is chosen so that the net stress intensity factor due to both the applied loads and the cohesive stress is zero. Cohesive zone models have been proposed for a variety of physical processes, for example, for pure cleavage tensile fracture by *Barenblatt et al.* [1962] and for fully developed plane stress plastic yielding at a tensile crack tip by *Dugdale* [1960]. A cohesive zone model which seems to be quite realistic for a number of geophysical applications involving frictional sliding is the so-called slip weakening model, according to which it is assumed that slip will commence at a point on a slip plane when the local shear stress on the slip plane is elevated to a certain level σ_s, that the shear stress required to sustain slip is reduced as the amount of slip δ is increased to some critical amount δ_c, and that the shear stress required to sustain slipping beyond the critical amount of slip is σ_d. The shear stress magnitudes σ_s and σ_d are usually associated with the static and dynamic frictional resistance of an interface to relative slip. The slip weakening model has been discussed by *Palmer and Rice* [1973], *Ida* [1972], and *Andrews* [1976*a*], among others.

In the slip weakening model for a mode II crack the local cohesive stress σ depends on the local slip $\delta(x, t) = u_x(x, 0^+, t) - u_x(x, 0^-, t)$ as shown in Figure 2*b*. The appropriate path L for computing energy flux into the cohesive zone during crack propagation is shown in Figure 2*a*. Because $n_x = 0$ for all points on L, the expression for F in terms of the cohesive stress and the slip reduces to

$$F = \int_{\substack{\text{cohesive}\\ \text{zone}}} \sigma(\delta) \frac{\partial \delta}{\partial t}\, dx \tag{19}$$

The slip may be viewed as a function of the crack tip coordinate ξ and time, in which case $\partial\delta(x, t)/\partial t$ may be replaced by $-v\partial\delta(\xi, t)/\partial\xi + \partial\delta(\xi, t)/\partial t$. Energy flux (13) then becomes the sum of two terms

$$F = v \int_0^{\delta_c} \sigma(\delta)\, d\delta + \int_0^{R_c} \sigma(\delta) \frac{\partial \delta}{\partial t}\, d\xi \tag{20}$$

where $\delta_c = \delta(0, t)$ is the crack tip opening displacement. Expression (20) is for the total energy flux into cohesive zone during crack propagation. The first term represents the rate at which energy must be supplied per unit crack surface area to overcome completely the cohesive stress, i.e., to produce a total slip of an amount δ_c. The second term represents the rate of work which must be supplied to change the amount of slip within the cohesive zone and to change zone size during nonsteady crack propagation. If the elastic field of the propagating crack is constant as seen by an observer moving with the crack tip, then δ depends on t only through ξ, that is, $\partial\delta(\xi, t)/\partial t = 0$, and the second term in (20) is zero.

Assuming such steady state crack propagation and assuming a linear relationship between cohesive stress and slip, $\sigma(\delta) = \sigma_s - (\sigma_s - \sigma_d)\delta/\delta_c$, and the energy flow into the cohesive zone per unit crack advance is $G = (\sigma_s + \sigma_d)\delta_c/2$. It should be noted that most authors view the excess of σ above σ_d as the cohesive stress, and in this case the energy absorbed per unit crack advance in overcoming cohesion is $G = (\sigma_s - \sigma_d)\delta_c/2$. The latter definition of fracture energy is particularly relevant to seismic rupture propagation because the characteristics of the process are generally viewed as being related to stress differences, rather than absolute stress levels. The energy release rate expressions for the slip weakening model applied to mode III crack extension are identical to those for mode II. It is noted that if conditions for small-scale yielding are met, that is, if the cohesive zone size R_c is much less than all other physical dimensions, then a relationship between the stress intensity factor and the physical parameters of the cohesive zone is obtained by equating the energy flow into the singular crack tip to the energy absorbed in the cohesive zone.

The results reported in this section provide a number of general relationships among parameters which characterize the mechanical conditions which prevail near the tip of a shear crack during rapid crack propagation. The actual dependence of any of these parameters on applied loading and geometrical configuration in any specific model of a dynamic shear rupture process can only be established by analyzing the corresponding boundary value problem. Although the collection of crack propagation problems which have been studied is rich in variety from the analytical, computational, and physical points of view, only a few analytical models are considered in the following sections. These particular models have been chosen because they address questions which appear to be important at the present time in the modeling of the dynamic crustal faulting process, they can be analyzed without undue complexity, and the results seem to provide some insight into the process of dynamic shear crack propagation.

Steady State Crack Propagation

The simplest dynamic crack propagation problems which can be analyzed are those of the steady state type, that is, those in which the crack propagation speed is assumed to be constant and the complete stress and deformation fields are taken to be fixed as seen by an observer moving with the crack tip. This idealization is clearly inadequate when considering the abrupt initiation or arrest of a dynamic fracture, but it may be quite acceptable for describing processes for which the duration of the acceleration and deceleration phases is short compared to the total duration of the process. Furthermore, in the present context the steady state assumption makes it possible to demonstrate in a simple way some of the main features of dynamic shear crack propagation which may be relevant in earthquake source modeling. Attention is directed toward several specific interrelated features in the steady state problems to be discussed below. Among these features is the phenomenon of fault plane healing. That is, analysis of a simple steady state model allows consideration not only of the onset of slipping between the fault faces at the leading edge of the dynamically extending fault but also of the cessation of slipping between the faces, or fault healing, at some distance behind the leading edge. A second specific aspect concerns the concept of the stress drop associated with fault extension. The inherent inhomogeneity of the stress distribution in the vicinity of a crack leads to the identification of several significant stress magnitudes in the description of the dynamic crack growth process. Thus several stress differences can be identified as stress drops, or, in other words, the definition of stress drop appears to be ambiguous. A third feature of the dynamic faulting process which can be considered by means of the analysis of steady state problems is the influence on the process of a change in remote loading conditions. An almost universal assumption in dynamic crack propagation analysis is that the process is driven by a quasi-static, spatially uniform stress which is applied far from the process region. As will be shown below, some interesting results may be deduced quite simply from a steady state crack model involving remote displacement conditions. In contrast to the remotely applied stress condition, which represents deadweight or perfectly soft loading, the remotely applied displacement condition represents the opposite extreme of perfectly stiff loading. These features will be discussed in greater detail in the remainder of this section on the basis of the analysis of a few particular problems. An approach whereby complete stress and deformation fields are established is briefly described first, and then an indirect approach based on an energy integral is considered. Before proceeding, it is noted again that the steady state assumption places severe limitations on the range of admissible responses of the system. Furthermore, for the steady state problems to be discussed below, each corresponding transient problem is too complex to permit a study of the tendency of the transient process to approach a steady state or of the stability of the steady state solution obtained.

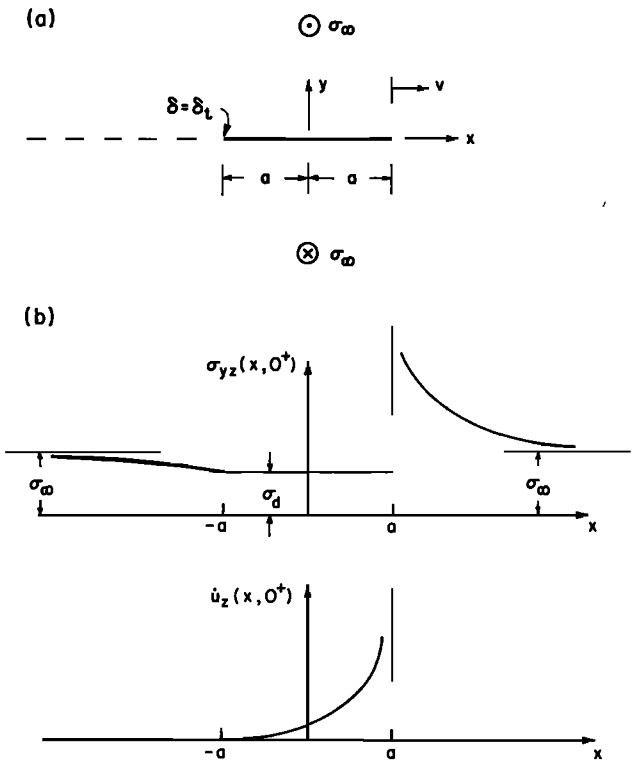

Fig. 3. (*a*) A typical mode III steady state crack propagation problem, where slip is resisted over a slipping region of length $2a$ by a shear traction σ_d. (*b*) The variation of shear stress and particle velocity along the slip plane.

A Stress Analysis Approach

Typical of the problems of the steady state type which can be analyzed by standard methods is the antiplane shear mode III problem represented in Figure 3*a*. Under the action of a uniform remote shear stress $\sigma_{yz} = \sigma_\infty$, the mode III crack grows in the x direction at speed v. The x, y coordinate system is fixed with respect to the moving crack. At the leading edge of the crack $x = a$, a singularity in $\sigma_{yz}(x, 0)$ will be admitted, and it is assumed that the strength of this singularity is governed by the cohesive strength of the fault plane $y = 0$. Slipping begins at $x = a - 0$, $y = 0$, and it continues throughout the interval $-a < x < a$, $y = 0$. Relative slipping of the crack faces is resisted by a uniform frictional stress $\sigma_{yz}(x, 0) = \sigma_d$, which is less than σ_∞ and which might be due to Coulomb friction arising from a uniform compressive stress in the y direction. At $x = -a$, $y = 0$ the relative particle velocity across the fault reduces to zero, and for $x < -a$, $y = 0$ no further slipping occurs. The total amount of slip between the fault surfaces will be denoted by δ_t, and the slipping process is assumed to terminate smoothly with bounded stresses at the healing point $x = -a$, $y = 0$.

An analytical technique of general applicability on problems of this type is that based on the theory of analytic functions of a complex variable. A general discussion of this method for steady state plane strain elastodynamic problems has been given by *Radok* [1956], and several specific applications to mode I fracture propagation are cited by *Freund* [1976]. An

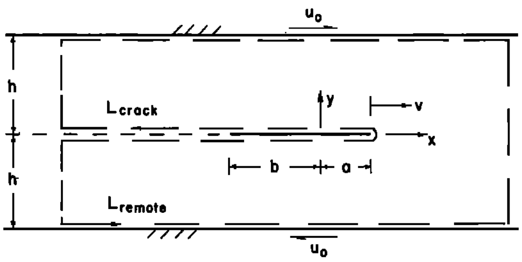

Fig. 4. A typical mode II steady state crack propagation problem which can be analyzed by means of the energy integral (25).

equivalent formulation for antiplane shear problems is straightforward. It is easily verified that the relevant field equations are satisfied if stress and particle velocity have the representations

$$\sigma_{xz} - i\sigma_{yz}/\alpha_s = g(\zeta) \qquad \mu\dot{u}_z = -v \; Re \; g(\zeta) \tag{21}$$

where $g(\zeta)$ is a function of the complex variable $\zeta = x + i\alpha_s y$ which is analytic in the complex ζ plane cut along $-a < x < a$, $y = 0$. The determination of stress and particle velocity then reduces to the determination of $g(\zeta)$ according to the powerful methods of analytic function theory [cf. *Rice*, 1968]. For the problem at hand the result is

$$g(\zeta) = i(\sigma_\infty - \sigma_d)\alpha_s^{-1}[1 - \zeta/(\zeta^2 - a^2)^{1/2}] - i\mu\delta_t/2\pi(\zeta^2 - a^2)^{1/2} - i\sigma_\infty\alpha_s^{-1} \tag{22}$$

from which the stress components and particle velocity may be extracted according to (21). Graphs of $\sigma_{yz}(x, 0)$ and $\dot{u}_z(x, 0)$ are shown in Figure 3*b*. The length of the slipping region is determined from the condition that stresses be bounded at $x = -a$, $y = 0$ as $a = \mu\alpha_s\delta_t/2\pi(\sigma_\infty - \sigma_d)$, and the stress intensity factor for the leading edge of the slipping zone is $K_3 = 2(\sigma_\infty - \sigma_d)(\pi a)^{1/2}$. From (12) the energy release rate at the leading edge is then

$$G = (\sigma_\infty - \sigma_d)\delta_t \tag{23}$$

A stress difference which appears naturally in this analysis is $(\sigma_\infty - \sigma_d)$, and (23) gives a value for this stress difference as the cohesive energy density of the fault surface divided by the total slip displacement across the fault. This cohesive energy is a measure of the static frictional strength of the fault, and it is usually assumed to be several orders of magnitude larger than the true surface energy of the material which is typically about 1 J/m^2.

If small-scale yielding conditions prevail and if the interface failure at the crack tip occurs in the slip weakening mode represented in Figure 2*b*, then G in (23) may be replaced by $\delta_c(\sigma_s - \sigma_d)/2$, where $\delta_c < \delta_t$. In this context the stress σ_s is viewed as a measure of fault resistance to onset of slipping, the resistance to slipping decreases from σ_s as the amount of slip increases to δ_c, and slipping is resisted by the stress σ_d for greater amounts of slip. In this case the relationship (23) may be rewritten as

$$2\delta_t/\delta_c = (\sigma_s - \sigma_d)/(\sigma_\infty - \sigma_d) \tag{24}$$

It is clear from (24) that a second stress difference $(\sigma_s - \sigma_d)$ has been introduced. Whereas the previously introduced stress difference $(\sigma_\infty - \sigma_d)$ represents the stress drop from the uniform remote tectonic stress σ_∞ to the frictional stress on the slipping part of the fault, the difference $(\sigma_s - \sigma_d)$ represents the stress drop from a point just ahead of the propagating edge of the slipping region of the fault to a point just behind the edge within the slipping region. It can be seen from (24) that the magnitudes of these two stress drops can be very different. For example, if δ_t is about 1 m and δ_c about 1 mm, then the stress drop $(\sigma_s - \sigma_d)$ is 3 orders of magnitude larger than $(\sigma_\infty - \sigma_d)$.

In the case of mode II crack propagation the same general analytical procedure may be applied, although the detailed formulation is somewhat more complicated. However, it turns out that certain key results for problems of this type, such as those in (23) and (24), may be extracted without actually solving the boundary value problems, that is, without determining the complete stress and deformation fields, as is discussed in the following subsection.

An Energy Integral Approach

The analysis is based on the energy flux integral (16) for the special case of steady state crack propagation in the x direction at speed v, for which $\partial/\partial t = -v\partial/\partial x$. If the ratio F/v for a given contour L surrounding the crack tip is denoted by $E(v; L)$, then it is a simple matter to show that

$$E = \int_L [\tfrac{1}{2}(\sigma_{ij}u_{i,j} + \rho v^2 u_{i,x}u_{i,x})n_x - \sigma_{ij}n_j u_{i,x}]\, dL \tag{25}$$

If L were a closed contour containing no singularities of the elastic field, then $E = 0$. This may be demonstrated by application of the divergence theorem to the integral in (25). A direct consequence of this result is that E is a path-independent integral. That is, the value of E is the same for all simple paths which originate at a particular point on one face of the crack, completely surround the crack tip region, and terminate at a particular point on the opposite face of the crack. If the crack faces are traction-free, then the integrand of (25) vanishes on the crack faces, and the value of E is independent of path even if the comparison contours do not originate and terminate at the same points on opposite faces of the crack [cf. *Rice*, 1968].

As a first example of the application of the path-independent integral (25), consider the antiplane shear problem which was discussed in the previous subsection and which is represented in Figure 3*a*. Suppose that a uniform stress $\sigma_{yz} = \sigma_\infty$ is subtracted from the stress state shown. The stresses and displacement gradients then decay as $(x^2 + y^2)^{-1/2}$ at large distances from the crack tip, which implies that the value of E along any remote contour will be zero ($E_{remote} = 0$). If a second path L is considered which originates on the lower crack surface at $x \to -\infty$, $y = -0$, runs along the lower crack face to $x = a$, surrounds the crack tip at $x = a$, and then runs along the upper crack face to $x \to -\infty$, $y = +0$, then the value of E for this path is $G + (\sigma_d - \sigma_\infty)\delta_t = E_{crack}$. Because the integral is path independent, $E_{remote} = E_{crack} = 0$, and result (23) is reproduced. It is noteworthy that a cohesive zone representing a slip weakening region could be included explicitly with this approach and result (24) could be derived directly without recourse to the small-scale yielding assumption. Similar results for a variety of other problems concerned with steady state crack propagation in unbounded bodies, such as that considered by ***Weertman*** [1975], can be directly extracted in much the same manner.

The energy integral approach becomes particularly useful in certain situations where the complete stress and deformation fields cannot be determined by any of the standard analytical methods but where enough information is available to compute the value of E. As typical of such a situation, consider the steady state propagation of a mode II shear crack in a strip of width $2h$ as shown in Figure 4. Under the action of applied displacements $u_x(x, \pm h) = \pm u_0$, $u_y(x, \pm h) = 0$, the mode II crack grows in the x direction at speed v. The x, y coordinate system is fixed with respect to the moving crack. Although a singularity in the stress component σ_{xy} at the leading edge of the slipping region could be admitted just as it was in the above mode III problem, a linear slip-weakening cohesive zone is introduced at the outset instead. Thus $\sigma_{xy}(a, 0) = \sigma_s$ with $\delta(a) = 0$, $\sigma_{xy}(0, 0) = \sigma_d$ with $\delta(0) = \delta_c$, and $\sigma_{xy}(-b, 0) = \sigma_d$ with $\delta(-b) = \delta_t > \delta_c$. Just as before, slipping begins on the fault plane at $x = a$, continues throughout the interval $-b < x < a$, and terminates with net displacement offset of δ_t at $x = -b$. Relative slipping is resisted by the cohesive/frictional stress $\sigma_{xy}(x, 0) = \sigma_s - (\sigma_s - \sigma_d)\delta/\delta_c$ in the interval $0 < x < a$ and by the uniform frictional stress $\sigma_{xy}(x, 0) = \sigma_d$ in the interval $-b < x < 0$. The frictional stress might be due to Coulomb friction arising from a superimposed uniform compressive stress in the y direction. If such a stress were actually introduced by specifying the alternate boundary condition $u_y(x, \pm h) = \pm\hat{u}_y$, however, it would have no influence on the main results to be obtained in the subsequent discussion.

The unique feature of this particular problem is that it represents crack propagation under displacement control or very stiff loading conditions, in contrast to the mode III problem of crack propagation under stress control or very soft loading conditions which was considered previously. Displacement control conditions make possible the consideration of stress relief by crack extension, that is, the shear stress on the fault plane at some point far behind the slipping region, say, $\sigma_- = \mu(u_0 - \delta_t/2)/h$, will be less than the shear stress on the fault plane at some distance ahead of the slipping region, say, $\sigma_+ = \mu u_0/h$. The relationship among the various stress magnitudes and slip magnitudes is obtained by direction application of the energy integral (25).

The two choices of contour L for evaluation of E are shown in Figure 4, and the value of E along the outer contour, say, E_{remote}, will be considered first. For points far ahead of or behind the crack tip the only nonzero stress component is $\sigma_{xy} = \sigma_\pm$, and the only nonzero displacement gradient component is $u_{x,y} = \sigma_\pm/\mu$, and these components are essentially uniform across the strip. For the portion of the remote contour along $y = \pm h$ the displacement components are uniform, and $n_x = 0$ so that the integrand of (25) vanishes identically and there is no contribution to the value of E. It is thus clear that

$$E_{\text{remote}} = (\sigma_+^2 - \sigma_-^2)h/\mu = (\sigma_+ + \sigma_-)\delta_t/2 \tag{26}$$

As shown in Figure 4, the other choice of contour L embraces the crack and $n_x = 0$ at all points of the inner contour. Then, according to (25), the value of E for the inner path is

$$E_{\text{crack}} = -\int_{-b}^{0} \sigma_d \frac{\partial \delta}{\partial x}\, dx - \int_0^a \sigma(\delta) \frac{\partial \delta}{\partial x}\, dx = (\sigma_s - \sigma_d)\delta_c/2 + \sigma_d \delta_t \tag{27}$$

In view of the path independence of E, $E_{\text{remote}} = E_{\text{crack}}$ or

$$\frac{\delta_t}{\delta_c} = \frac{(\sigma_s - \sigma_d)}{(\sigma_+ + \sigma_- - 2\sigma_d)} \tag{28}$$

This analysis introduces two additional stress differences which might be identified as stress drops, that is, $(\sigma_+ - \sigma_d)$ and $(\sigma_+ - \sigma_-)$. The second of these is particularly interesting because it satisfies the relationship

$$\sigma_+ - \sigma_- = \mu\delta_t/2h \tag{29}$$

If h can be identified in some way with the total distance of travel of the fault edge, which is not an entirely unreasonable identification, then the right side of (29) coincides with one of the standard definitions of stress drop [*Nur*, 1974]. It is noteworthy that σ_+ and σ_- do not represent frictional properties of the fault. The magnitudes of σ_+ and σ_- are arbitrary except that $\sigma_+ + \sigma_- > 2\sigma_d$ and $\sigma_+ > \sigma_-$ for the process to occur at all. Unfortunately, the energy integral approach provides no apparent means for computing the physical dimensions a and b in Figure 4.

The steady state crack propagation models considered in this section are representative of problems in this class. There are, of course, many other steady state crack propagation models involving other fault plane strength characterizations or other friction laws which might be profitably studied. The mode III problem represented in Figure 3*a* was chosen here because it is typical of the simplest steady state problems which can be analyzed, and the mode II strip problem represented in Figure 4 was chosen because it is typical of problems for which useful information can be extracted by means of the energy integral approach, even though the stress and deformation fields cannot be determined by known analytical methods. If a crack tip is moving more or less steadily, then a steady state analysis yields relationships among the various parameters used to characterize the process without specific reference to a crack propagation criterion. A study of how a crack tip or fault edge actually moves according to a particular crack propagation condition must be based on an analysis of transient crack propagation, and certain results of such analysis are summarized in the next section.

TRANSIENT CRACK PROPAGATION

The discussion of transient shear crack propagation is based on the model of a half-plane crack extending in mode II in an otherwise unbounded body. This model exhibits most of the conceptual features which have been considered in dynamic shear crack propagation analysis. In the geophysics literature, more emphasis has been placed on the analysis of the symmetrical expansion of a mode II shear crack of finite length and the three-dimensional expansion of an elliptical combined mode II and III shear crack than on the semi- infinite mode II or mode III cracks. The reason for this emphasis seems to be that actual faults have finite dimensions and if primary interest is on the details of seismic radiation due to fault motion, for example, then these fault dimensions must be included. If primary interest is on the fracture process, on the other hand, then the actual fault dimensions are of lesser importance, and the semi-infinite crack models appear to be suitable. In general, for a specific characterization of fracture resistance the main influence on crack tip motion is the increased area over which the stress drop acts as the crack increases in size, and this effect can be included in the semi-infinite plane crack models. The influence on crack tip motion at some point on a fault edge due to stress waves generated at some other point on the fault edge seems to be minor by comparison, and this effect is automatically precluded in the semi-infinite plane crack models. In his study of the symmetrical extension of a mode I tensile crack of finite length, *Rose* [1976] showed that the wave disturbance generated by either moving crack tip had a negligible effect on the motion of the other. Concerning three-dimensional shear cracks, the edge deformation can be resolved into a combination of mode II and mode III deformations, and with this point of view the results obtained from analysis of plane models have bearing on three-dimensional plane crack propagation as well.

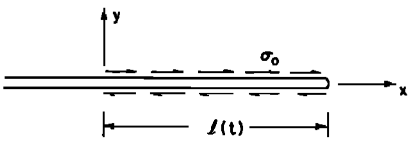

Fig. 5. Transient extension of a mode II semi-infinite shear crack due to a uniform stress drop σ_0 acting over the interval $0 < x < l(t)$.

The specific mode II shear crack model on which this discussion is based is represented in Figure 5. At the initial time $t = 0$ the half-plane crack begins to grow in the plane $y = 0$, and at time t the crack edge has advanced a distance $l(t)$. Equal and opposite shear tractions of magnitude σ_0 act on the crack faces in the interval $0 < x < l(t)$ as shown. The stress σ_0 may be viewed as the difference between a remotely applied stress of magnitude σ_∞ and a frictional stress which resists sliding σ_d. Thus σ_0 is a stress drop, and it acts over a region of increasing size as the crack grows. There are no other crack face tractions or remote loads acting on the body. Posed in this way, the problem is not suitable for consideration of fracture initiation. Initial loading of the body is not included explicitly, and the role of the preexisting crack as a stress concentrator in fracture initiation is not considered. These exclusions were made in order to keep the analysis as simple as possible, and loading conditions for this problem which might be more suitable for the study of fracture initiation are considered by *Fossum and Freund* [1975].

The solution of this problem for arbitrary nonuniform motion of the crack tip with speed $\dot{l}(t)$ in the range $0 < \dot{l} < v_r$ has been given by *Fossum and Freund* [1975]. A solution of this problem for crack speeds greater than the Rayleigh wave speed v_r has not appeared in the literature, although solutions for crack growth at constant speed $\dot{l} = v > v_r$ can be obtained by following the analytical procedure of *Fossum and Freund* [1975]. In terms of the notation established in (1)–(4) the shear stress on the prospective fracture surface $y = 0$ directly ahead of the crack tip at $x = l(t)$ is

$$\sigma_{xy} = \frac{K_2(t)}{[2\pi(x - l)]^{1/2}} - \sigma_0 \quad K_2(t) = 2\sigma_0[2l/\pi]^{1/2}k(\dot{l}) \tag{30}$$

for arbitrary $l(t)$ within the range $0 < \dot{l} < v_r$,

$$\sigma_{xy} = -\sigma_0 \quad K_2(t) = 0 \tag{31}$$

for constant speed v in the range $v_r < v < v_s$, and

$$\sigma_{xy} = \frac{K_2^*(t)}{(2\pi)^{1/2}(x - vt)^m} - \sigma_0 \quad K_2^*(t) = k^*(v)\sigma_0(vt)^m \tag{32}$$

for constant speed v in the range $v_s < v < v_l$. The function k in (30) is defined precisely in equation (12) of *Fossum and Freund* [1975], and a rough approximation of this function is $k(\dot{l}) = 1 - \dot{l}/v_r$; in particular, $k(0) = 1$, and $k(v_r) = 0$. For the crack tip speed range $v_r < v < v_s$ a solution with $K_2 \neq 0$ can be found. However, as can be seen from (17), a mode II crack tip propagating at a speed within this range with $K_2 \neq 0$ acts as a point source of energy. Such a result must be ruled out on physical grounds, so that $K_2 = 0$, and the next term in the expansion of stress with distance from the crack tip is that shown in (31). The function $k^*(v)$ has not yet been evaluated in detail, but it seems to be bounded, nonzero, and of order unity for speeds in the range $v_s < v < v_l$. When considered in conjunction with a fracture propagation criterion, results (30)–(32) imply crack tip motions having certain characteristics. These will be briefly discussed for several different fracture criteria.

Critical Stress Intensity Factor Criterion

Suppose that the shear crack is required to propagate in such a way that the stress intensity factor always has a fixed critical value. This is a generalization to the case of extending cracks of Irwin's well-known fracture initiation criterion. After the initiation phase has passed, the expression for K_2 in (30) suggests that the fracture condition can be satisfied as l increases indefinitely by having the crack tip gradually accelerate toward the Rayleigh wave speed so that the product $lk(\dot{l})$ is held constant. In this sense the Rayleigh wave speed represents the terminal velocity for crack propagation. Furthermore, if the traditional definition or interpretation of the stress intensity factor is understood, then crack speeds greater than the Rayleigh wave speed are not considered with this criterion because the coefficient of the square root singular contribution to the local stress field is zero (except for the special case $v = v_s(2)^{1/2}$).

Critical Energy Release Rate Criterion

Suppose that the shear crack is required to propagate in such a way that the energy release rate G always has a fixed critical value. This is a generalization of the classical Griffith criterion. The resulting crack motion will be almost identical to that for the critical stress intensity factor criterion, differing only in minor details. The reason for this similarity is that the energy release rate is proportional to the square of the stress intensity factor, and the energy release rate is nonzero only if the local crack tip stress field is square root singular. Stress fields with weaker singularities and nonsingular stress fields result in zero energy release rate for a sharp crack tip. Crack motion according to this criterion is discussed in detail by *Fossum and Freund* [1975]. The main qualitative result is that an unstable sharp-tipped mode II crack propagating with a constant energy release rate will accelerate toward the Rayleigh wave speed, but its speed will always be less than the Rayleigh wave speed. The energy release rate is also nonzero for the special crack speed $v_s(2)^{1/2}$, but according to (32), any specific value of energy release rate can be achieved at this speed only for a particular fixed crack length. Thus the fracture condition cannot be satisfied for a crack extending at the speed $v_s(2)^{1/2}$.

Critical Stress Level Criterion

Suppose that large, and possibly singular, shear stresses on the prospective fracture plane ahead of the crack tip are admitted, but that the interface can support such stresses without fracture only if the interval over which some critical stress level, say, σ_s, is exceeded in less than some critical length, say, λ. Further, suppose that fracture will begin if the fault plane shear stress is greater than σ_s over an interval of length λ and that the fracture will proceed with this condition satisfied. The shear stress can never exceed σ_s over a length interval greater than λ. This notion of a critical stress acting over a certain distance was used by *McClintock and Irwin* [1965] in their early studies of crack tip plasticity. In this case the critical stress was proportional to the plastic flow stress of the material, and the critical distance was the radius of the plastically deforming region around the crack tip. A similar viewpoint was adopted by *Congleton and Petch* [1967] in modeling the initiation of microcrack growth ahead of the tip of a much larger crack. In this model, growth of the large crack was achieved through coalescence of numerous microcracks. In this case the critical stress was the Griffith stress necessary to extend preexisting microcracks, and the corresponding critical distance was the mean spacing of these microcracks. A version of the same criterion was recently employed by *Das and Aki* [1977] in a form particularly well suited for numerical computation by a finite difference scheme. However, it seems that they used the crack tip stress distribution (30) for the entire velocity range $0 < v < v_l$.

After some fracture initiation phase has passed, the critical stress level fracture condition implies that the crack will accelerate according to

$$\frac{K_2(t)}{(2\pi\lambda)^{1/2}} - (\sigma_\infty - \sigma_d) = (\sigma_s - \sigma_\infty) \tag{33}$$

The term $-\sigma_\infty$ has been inserted on the right side of (33) to account for the fact that the remote loading contribution to the prospective fracture plane stress was not included in the formulation of the boundary value problem. With the explicit expression for $K_2(t)$ given in (30), result (33) becomes (cf. equation (27) of [*Das and Aki*, 1977])

$$\frac{2k(l)}{\pi}\left(\frac{l}{\pi}\right)^{1/2} = \frac{(\sigma_s - \sigma_d)}{(\sigma_\infty - \sigma_d)} \tag{34}$$

After any appreciable amount of crack growth, $l \gg \lambda$. Therefore k must approach zero, or the wave speed must approach the Rayleigh wave speed.

During the acceleration phase the shear stress on the fracture plane is usually small compared to σ_s at distances ahead of the crack tip which are large compared to λ. However, as the crack propagates steadily at speeds near the Rayleigh wave speed, the shear stress distribution on the fracture plane develops a sharp peak at a point traveling with the shear wave speed. This behavior was also observed for the symmetrically growing shear crack by *Burridge* [1973]. The shear stress magnitude at this peak is $n(\sigma_\infty - \sigma_d)$. An estimate of the integral defining n yielded a value of about 3, which is not too different from the numerically computed value $n = 1.63$ reported by *Burridge* [1973]. If the interface is relatively weak, i.e., if σ_s is not much larger than σ_∞, then it is quite possible that $n(\sigma_\infty - \sigma_d) > (\sigma_s - \sigma_\infty)$. If this inequality is satisfied at the shear stress peak, then it will simply be a matter of time before it is satisfied over an interval of length λ. Thus if the interface is relatively weak, then a secondary fracture will initiate at some distance ahead of the main fracture at some time.

A question then arises as to how this secondary fracture will grow. A detailed analysis of the growth process would be prohibitively complicated. There are not many possibilities, however, and the growth of the secondary fracture may be discussed in qualitative terms. The end of the secondary fracture nearest the main crack tip and the main crack tip will likely coalesce shortly after formation of the secondary fracture, so that the other tip of the secondary fracture becomes the main crack tip. One possibility is that this tip will move at a speed just below the Rayleigh wave speed, another secondary fracture will be initiated some time later, and this process will be repeated over and over. A second possibility is that this tip will move at a speed greater than the Rayleigh wave speed. If $\sigma_s > \sigma_\infty$, then it can be seen from (31) that speeds in the range $v_r < v < v_s$ are ruled out and possible speeds in the range $v_s < v < v_l$ must be sought. For this purpose, use is made of (32) with the interpretation of vt in (32) as the amount of crack growth since the formation of the secondary fracture, say, Δl. The crack growth criterion then takes the form

$$k^*(v)\left(\frac{\Delta l}{\lambda}\right)^m = \frac{(\sigma_s - \sigma_d)}{(\sigma_\infty - \sigma_d)} \tag{35}$$

If k^* is insensitive to variations in speed v, then m must decrease as Δl increases for fixed λ. As can be seen from its definition (7), the value of m decreases from $m = \frac{1}{2}$ at $v = v_s(2)^{1/2}$ to $m = 0$ at $v = v_l$. Thus the secondary crack would begin to grow at a speed equal to or greater than $v_s(2)^{1/2}$, and it would accelerate toward the speed v_l. The second of these two possibilities is qualitatively consistent with the numerical calculations of *Andres* [1976*b*] and of *Das and Aki* [1977].

In summary, it seems that for a relatively strong interface with $(\sigma_s - \sigma_\infty)/(\sigma_\infty - \sigma_d) > n$ the crack will accelerate rapidly to speeds approaching the Rayleigh wave speed but will continue to propagate indefinitely at speeds below the Rayleigh wave speed. For a relatively weak interface with $(\sigma_s - \sigma_\infty)/(\sigma_\infty - \sigma_d) < n$ the crack tip will quickly accelerate to speeds approaching the Rayleigh wave speed, and it will propagate for some short time at sub-Rayleigh wave speeds. Then due to secondary fracturing, the crack tip speed will abruptly increase to $v_s(2)^{1/2}$ or beyond, and it will continue to accelerate toward the longitudinal wave speed v_l. The qualitative picture of crack propagation based on the critical stress level fracture criterion and the simple mode II crack problem represented in Figure 5 is generally consistent with the numerical results of *Andrews* [1976*b*] and *Das and Aki* [1977]. One fundamental difficulty with the critical stress level criterion is that the existence of a critical length λ seems to have no particular physical basis. However, as discussed by *Das and Aki* [1977], the criterion appears to be well suited for numerical computation by a finite difference approach where λ can be identified with the difference mesh spacing on the fracture plane. Furthermore, in such a numerical scheme the shear stress can be reduced from σ_s to σ_d according to some cohesive zone model to simulate energy uptake in overcoming an intrinsic material cohesion in the crack tip region. Unless σ_s is larger than σ_∞ and σ_d by 2 or 3 orders of magnitude, however, the rate of work against cohesion is negligible by comparison to the rate at which energy is absorbed through frictional sliding of the crack faces. A direct comparison of the results of the numerical calculations of *Andrews* [1976*b*] and *Das and Aki* [1977] suggests that the crack motion is quite insensitive to whether or not a finite nonzero crack tip fracture energy exists. *Andrews* [1976*b*] uses a cohesive zone crack tip model to simulate nonzero energy uptake at the crack tip at all crack speeds. On the other hand, *Das and Aki* [1977] simulate crack growth in their finite difference scheme by abruptly releasing nodes in the difference mesh whenever the nodal stress reached a critical value. As was observed by *Rice* [1978], crack growth consisting of abrupt release of stress over a finite interval (the mesh spacing) occurs without energy absorption at the crack tip. The fact that the crack motion predictions in these two studies are qualitatively identical suggests that the influence on crack motion of crack tip energy absorption rates is small, probably because such rates are usually small compared to other energy rates involved as noted above.

Nonuniform Stress Drop and Fracture Resistance

This section is concluded with a brief discussion of crack arrest. As was noted by *Husseini et al.* [1975], a running fracture can be arrested by either of two mechanisms, either a reduction of the crack driving force to a subcritical level or an increase in the resistance of the material to a supercritical level. A simple illustration of these effects can be given in terms of the present mode II crack propagation model and the critical stress intensity fracture criterion, for example. The dynamic stress intensity factor for the shear crack shown in Figure 5 for

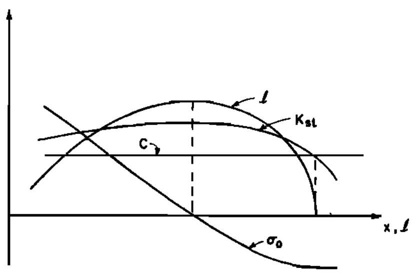

Fig. 6. A sketch of the variation with position along the slip plane of the physical quantities appearing in the crack tip equation of motion (36) for the special case when the critical stress intensity factor C is a constant but the stress drop decreases along the slip plane.

crack face traction representing a stress drop $\sigma_0(x)$ which varies in an arbitrary manner along the slip plane is given in equation (18) of *Fossum and Freund* [1975] for sub-Rayleigh wave speeds. In terms of this arbitrary stress drop, the critical stress intensity factor fracture condition requires that the crack move in such a way that the dynamic stress intensity factor is always equal to some critical value, say, $C(l)$, which may also vary in an arbitrary manner along the slip plane. The equation of motion of the crack tip, i.e., the equation governing $l(t)$, is then

$$k(\dot{l})K_{st}(l) = C(l) \tag{36}$$

where $K_{st}(l)$ is the equivalent static crack stress intensity factor

$$K_{st}(l) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^l \frac{\sigma_0(x)\,dx}{(l-x)^{1/2}} \tag{37}$$

Some specific implications of (36) are shown schematically in Figures 6 and 7. The case considered in Figure 6 is that with constant critical stress intensity factor C and with continuously decreasing stress drop σ_0 as shown. Equation (37) implies that the equivalent static stress intensity factor will first increase after initiation and then decrease, with a maximum at the point where $\sigma_0 = 0$. The equation of motion (36) implies that the crack tip velocity will also increase and then decrease, with a maximum at the same point. The crack will arrest when K_{st} has been reduced to C. With reference to the sketch, the speed $\dot{l} = 0$ when $K_{st} = C$.

A similar example in which the stress drop σ_0 is constant but the fracture resistance C increases along the slip plane is shown schematically in Figure 7. Equation (37) implies that K_{st} will continuously increase with crack length, and according to (36) the crack will continue to accelerate as long as the slope of K_{st} versus l is greater than the slope of C versus l. The crack tip reaches a maximum velocity when these slopes coincide, and the crack tip will come to rest if C reaches the value K_{st} at some point. With reference to the sketch, $\dot{l} = 0$ when $K_{st} = C$.

Finally, it is noted that the functions $\sigma_0(x)$ and $C(l)$ need not be monotonic, but they could vary in an arbitrary way along the slip plane in order to simulate variations in driving stress, frictional resistance, and fracture resistance in the faulting process. It is clear from (36) that if these functions varied in an oscillatory manner, then crack motions $l(t)$ which solve (36) or other equations of motion similar to it could be very complicated indeed. The effects of nonuniform driving stress and fracture resistance are likely to be important in future development of dynamic shear crack models.

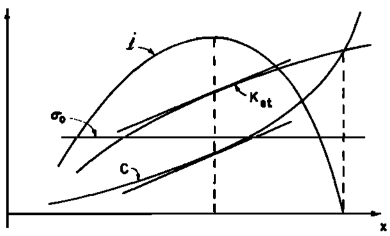

Fig. 7. Same as Figure 6 for the special case when the stress drop σ_0 is uniform but the critical stress intensity factor increases along the slip plane.

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