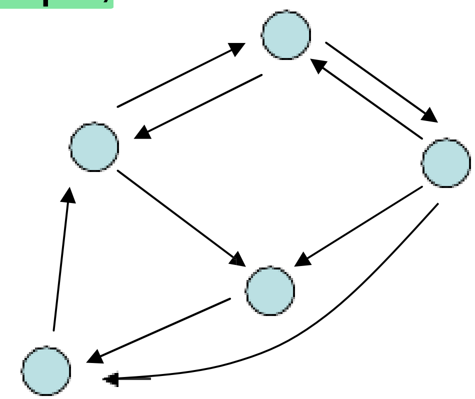
6.852: Distributed Algorithms
Fall, 2009

Distributed algorithms

- Algorithms that are supposed to work in distributed networks, or on multiprocessors.
- Accomplish tasks like:
  - Communication
  - Data management
  - Resource management
  - Synchronization
  - Consensus
- Must work in difficult settings:
  - Concurrent activity at many locations
  - Uncertainty of timing, order of events, inputs
  - Failure and recovery of machines/processors, of communication channels.
- So the algorithms can be complicated:
  - Hard to design
  - Hard to prove correct, analyze.

Synchronous network model



- Processes (or processors) at nodes of a network digraph, communicate using messages.
- Digraph: G = (V,E), n = |V|
  - $out\text{-}nbrs_i$, $in\text{-}nbrs_i$
  - distance(i,j), number of hops on shortest path
  - diam = $\max_{i,j}$ distance(i,j)
- M: Message alphabet, plus $\perp$ placeholder
- For each i in V, a process consisting of :
  - $states_i$, a nonempty, not necessarily finite, set of states
  - $start_i$, a nonempty subset of $states_i$
  - $msgs_i$: $states_i \times out\text{-}nbrs_i \rightarrow M \cup \{\perp\}$
  - $trans_i$: $states_i \times$ vectors (indexed by $in\text{-}nbrs_i$) of $M \cup \{\perp\} \rightarrow states_i$

V V I !

- Executes in rounds:
  1. Apply $msgs_i$ to determine messages to send,
  2. Send and collect messages,
  3. Apply $trans_i$ to determine new state.

V.V I !

Remarks

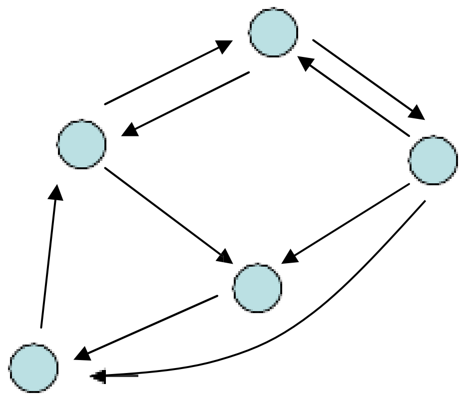
- No restriction on amount of local computation.
- Deterministic (a simplification).
- Can define "halting states", but not used as accepting states as in traditional automata theory.
- Later, we will consider some complications:
  - Variable start times
  - Failures
  - Random choices
- Inputs and outputs: Can encode in the states, e.g., in special input and output variables.

Executions

- An execution is a mathematical object used to describe how an algorithm operates.
- Definition (p. 20):
  - State assignment: Mapping from process indices to states.
  - Message assignment: Mapping from ordered pairs of process indices to $M \cup \{\perp\}$.
  - Execution: $C_0, M_1, N_1, C_1, M_2, N_2, C_2, \ldots$
    - C's are state assignments.
    - M's are messages sent.
    - N's are messages received.
    - Infinite sequence (but could consider finite prefixes)

$\left.\begin{array}{l}\text{M's}\\ \text{N's}\end{array}\right\}$ Message assignments

Leader election

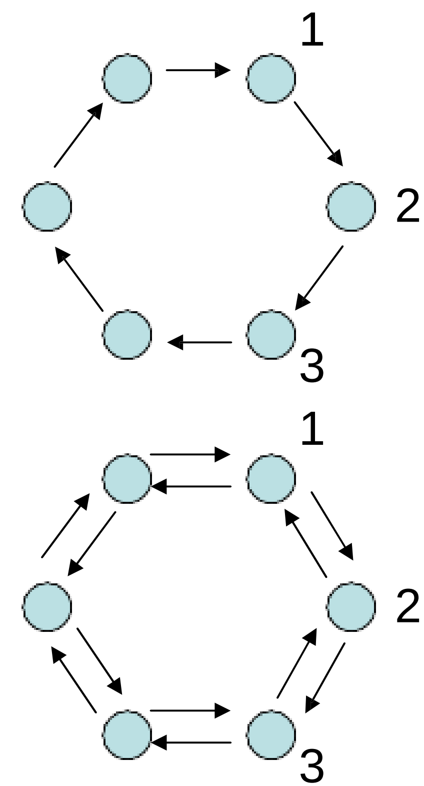


- Network of processes.
- Want to distinguish exactly one, as the “leader”.
- Eventually, exactly one process should output “leader” (set special status variable to “leader”).
- Motivation: Leader can take charge of:
  - Communication
  - Coordinating data processing (e.g., in commit protocols)
  - Allocating resources
  - Scheduling tasks
  - Coordinating consensus protocols
  - …

Simple case: Ring network

- Variations:
  - Unidirectional or bidirectional
  - Ring size n known or unknown
- Numbered clockwise
- Processes don't know the numbers; know neighbors as "clockwise" and "counterclockwise".

- Theorem 1: Suppose all processes are identical (same sets of states, transition functions, etc.).
  Then it's impossible to elect a leader, even under the best assumptions (bidirectional communication, ring size n known to all).



//A// Leader

Proof of Theorem 1

- By contradiction. Assume an algorithm that solves the problem.
- Assume WLOG that each process has exactly one start state (could choose same one for all).
- Then there is exactly one execution

  $C_0, M_1, N_1, C_1, M_2, N_2, C_2, \ldots$
- Prove by induction on the number r of completed rounds that all processes are in identical states after r rounds.
  - Generate same messages, to corresponding neighbors.
  - Receive same messages.
  - Make the same state changes.
- Since the algorithm solves the leader election problem, someone eventually gets elected.
- Then everyone gets elected, contradiction.

So we need something more…

- To solve the problem at all, we need something more---some way of distinguishing the processes.
- E.g., assume processes have unique identifiers (UIDs), which they “know”.
  - Formally, each process starts with its own UID in a special state variable.
- UIDs are elements of some data type, with specified operations, e.g.:
  - Arbitrary totally-ordered set with just (<, =, >) comparisons.
  - Integers with full arithmetic.
- Different UIDs can appear anywhere in the ring, but each can appear only once.

A basic algorithm [LeLann] [Chang, Roberts]

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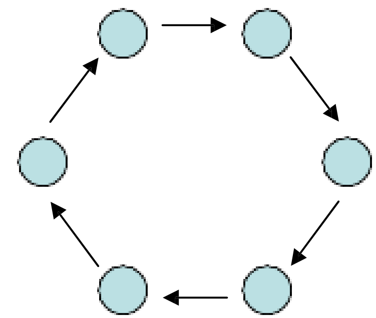


- Assumes:
  - Unidirectional communication (clockwise)
  - Processes don’t know n
  - UIDs, comparisons only
- Idea:
  - Each process sends its UID in a msg, to be relayed step-by-step around the ring.
  - When process receives a UID, compares with its own.
  - If incoming is:
    - Bigger, pass it on. pass the bigger UID for next process comparison
    - Smaller, discard. and returns null
    - Equal, process declares itself the leader. and halt all other process (algo ends)
  - Elects process with the largest UID.

~~In~~ terms of our formal model:

- M, the message alphabet: Set of UIDs
- $states_i$: Consists of values for state variables:
  - u, holds its own UID
  - send, a UID or ⊥, initially its own UID
  - status, one of {?, leader}, initially ?
- $start_i$: Defined by the initializations.
- $msgs_i$: Send contents of send variable, to clockwise nbr.
- $trans_i$:
  - Defined by pseudocode (p. 28):

```
if incoming = v, a UID, then
    case
        v > u:  send := v
        v = u: status := leader
        v < u: no-op
    endcase
```

  - Entire block of code is treated as atomic.

Correctness proof

- Prove exactly one process ever gets elected leader.
- More strongly:
  - Let i_{max} be the process with the max UID, u_{max}.
  - Prove:
    - i_{max} outputs “leader” by end of round n.
    - No other process ever outputs “leader”.

i_{max} outputs "leader" after n rounds

- Prove by induction on number of rounds.
- But need to strengthen, to say what happens after r rounds, $0 \leq r \leq n$.

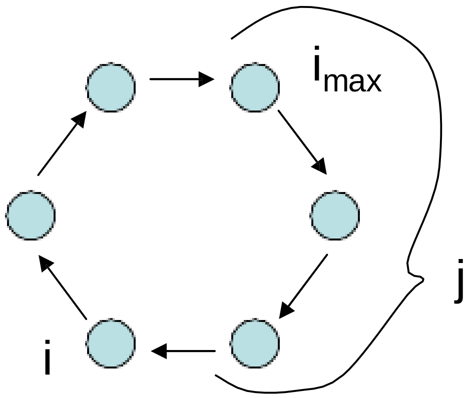
- Lemma 2: For $0 \leq r \leq n-1$, after r rounds, send at process $(i_{max} + r) \bmod n$ contains u_{max}.
- That is, u_{max} is making its way around the ring.

- Proof of Lemma 2:
  - Induction on r.
  - Base: Check the initialization.
  - Inductive step: Key fact: Everyone else lets u_{max} pass through.

- Use Lemma 2 for r = n-1, and a little argument about the n^{th} round to show the correct output happens.
- Key fact: i_{max} uses arrival of u_{max} as signal to set its status to leader.

Uniqueness

- No one except i_{max} ever outputs "leader".
- Again, strengthen claim:
- Lemma 3: For any $r \geq 0$, after r rounds, if $i \neq i_{max}$ and j is any process in the interval $[i_{max}, i)$, then j's send doesn't contain u_i.
- Thus, u_i doesn't get past i_{max} when moving around the ring.
- Proof:
  - Induction on r.
  - Key fact: i_{max} discards u_i (if no one has already).



- Use Lemma 3 to show that no one except i_{max} ever receives its own UID, so never elects itself.

Invariant proofs

- Lemmas 2 and 3 are examples of invariants---properties that are true in all reachable states.
- Another invariant: If $r = n$ then the status variable of i_{max} = leader.
- Invariants are usually proved by induction on the number of steps in an execution.
  - May need to strengthen, to prove by induction.
  - Inductive step requires case analysis.
- In this class:
  - We'll outline key steps of invariant proofs, not present all details.
  - We'll assume you could fill in the details if you had to.
  - Try some examples in detail.
- Invariant proofs may seem like overkill here, but:
  - Similar proofs work for much harder synchronous algorithms.
  - Also for asynchronous algorithms, and partially synchronous algorithms.
  - The properties, and proofs, are more subtle in those settings.
- Invariants provide the main method for proving properties of distributed algorithms.

Complexity bounds

- What to measure?
  - Time = number of rounds until "leader": n
  - Communication = number of single-hop messages: $\leq n^2$
- Variations:
  - Non-leaders announce "non-leader":
    - Any process announces "non-leader" as soon as it sees a UID higher than its own.
    - No extra costs.
  - Everyone announces who the leader is:
    - At end of n rounds, everyone knows the max.
      - No extra costs.
      - Relies on synchrony and knowledge of n.
    - Or, leader sends a special "report" message around the ring.
      - Time: $\leq 2n$
      - Communication: $\leq n^2 + n$
      - Doesn't rely on synchrony or knowledge of n.

1)

imp

2) O(1)

store max UID each node

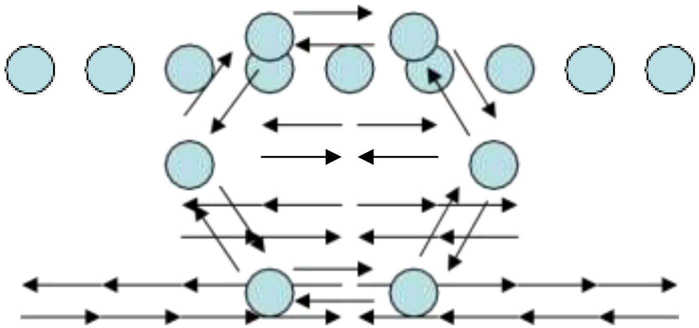
O(n)

Halting

- Add halt states, special "looping" states from which all transitions leave the state unchanged, and that generate no messages.
- For all problem variations:
  - Can halt after n rounds.
    - Depends on synchrony and knowledge of n.
  - Or, halt after receiving leader's "report" message.
    - Does not depend on synchrony or knowledge of n
- Q: Can a process just halt (in basic problem) after it sees and relays some UID larger than its own?
- A: No---it still has to stay alive to relay messages.

Reducing the communication complexity [Hirschberg, Sinclair]

- $O(n \log n)$, rather than $O(n^2)$
- Assumptions:
  - Bidirectional communication
  - Ring size not known.
  - UIDs with comparisons only



- Idea:
  - Successive doubling strategy
    - Used in many distributed algorithms where network size is unknown.
  - Each process sends a UID token in both directions, to successively greater distances (double each time).
  - Going outbound: Token is discarded if it reaches a node whose UID is bigger.
  - Going inbound: Everyone passes the token back.
  - Process begins next phase only if/when it gets both its tokens back.
  - Process that gets its own token in outbound direction, elects itself the leader.

V. Imp

In terms of formal model:

- Needs local process description.
- Involves bookkeeping, with hop counts.
- LTTR (p. 33)

Complexity bounds

- ~~Time:~~
  - ~~Worse than [LCR] but still O(n).~~
  - Time for each phase is twice the previous, so total time is dominated by last complete phase (geometric series).
  - Last phase is O(n), so total is also.

Communication bound: O(n log n)

$k \leftrightarrow \lambda$

- $1 + \lceil \log n \rceil$ phases: 0,1,2,…
- Phase 0: All send messages both ways, $\leq 4n$ messages.
- Phase $k > 0$:
  - Within any block of $2^{k-1} + 1$ consecutive processes, at most one is still alive at the start of phase k.
    - Others' tokens are discarded in earlier phases, stop participating.
  - So at most $\lfloor n / (2^{k-1} + 1) \rfloor$ start phase k.
  - Total number of messages at phase k is at most $4\,(2^k \lfloor n / (2^{k-1} + 1) \rfloor) \leq 8n$

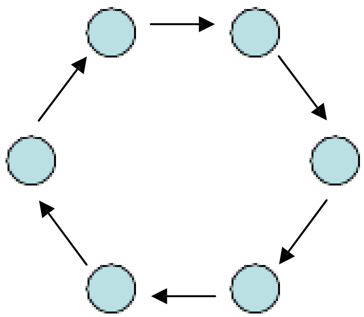
Out and back, both directions

New distance

- So total communication is as most $8\,n\,(1 + \lceil \log n \rceil) = O(n \log n)$

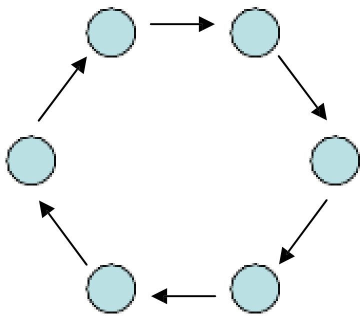
Non-comparison-based algorithms

- Q: Can we improve on worst-case O(n log n) messages to elect a leader in a ring, if UIDs can be manipulated using arithmetic?
- A: Yes, easily!
- Consider case where:
  - n is known
  - Ring is unidirectional
  - UIDs are positive integers, allowing arithmetic.



- Algorithm:
  - Phases 1,2,3,…each consisting of n rounds
  - Phase k
    - Devoted to UID k.
    - If process has UID k, circulates it at beginning of phase k.
    - Others who receive it pass it on, then become passive (or halt).
  - Elects min

Complexity bounds



- Communication:
  - Just n (one-hop) messages
- Time:
  - $u_{min}\, n$
  - Not practical, unless the UIDs are small integers.

- Q: What if n is unknown?
- A: Can still get O(n) messages, though now the time is even worse: $O(2^{u_{min}}\, n)$.
  - VariableSpeeds algorithm, Section 3.5.2
  - Different UIDs travel around the ring at different speeds, smaller UIDs traveling faster
  - UID u moves 1 hop every 2^u rounds.
  - Smallest UID gets all the way around before next smallest has gone half-way, etc.