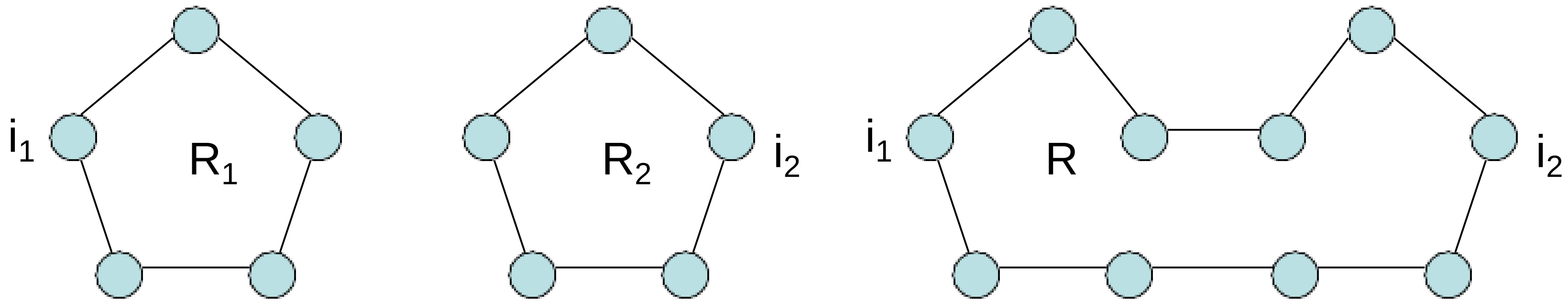
Lower bounds for leader election

- Q: Can we get lower time complexity?
- Easy n/2 lower bound (informal):
  - Suppose an algorithm always elects a leader in time < n/2.
  - Consider two separate rings of size n (n odd), R_1 and R_2.
  - Algorithm elects processes i_1 and i_2, each in time < n/2.



  - Now cut R_1 and R_2 at points furthest from the leaders, paste them together to form a new ring R of size 2n.
  - Then in R, both i_1 and i_2 get elected, because the time it takes for them to get elected is insufficient for information about the pasting to propagate from the pasting points to i_1 and i_2.

Lower bounds for leader election

- Q: Can we get lower message complexity?
- More difficult $\Omega(n \log n)$ lower bound.
- Assumptions
  - Comparison-based algorithm
  - Unique start state (except for UID), deterministic.

Comparison-based algorithms

- All decisions determined only by relative order of UIDs:
  - Identical start states, except for UID.
  - Manipulate UIDs only by copying, sending, receiving, and comparing them (<, =, >).
  - Can use results of comparisons to decide what to do:
    - State transition
    - What (if anything) to send to neighbors
    - Whether to elect self leader

Lower bound proof: Overview

- For any n, there is a ring R_n of size n such that in R_n, any leader election algorithm has:
  - $\Omega(n)$ "active" rounds (in which messages are sent).
  - $\Omega(n/i)$ msgs sent in active round i (for $i > \sqrt{n}$).
  - Thus, $\Omega(n \log n)$ msgs total.

- Choose ring R_n with a great deal of symmetry in ordering pattern of UIDs.
  - For n = power of 2: Bit-reversal rings.
  - For general n: c-symmetric rings.

- Key lemma: Processes whose neighborhoods "look the same" act the same, until information from outside their neighborhoods reaches them.
  - Need many active rounds to break symmetry.

Lower bound proof: Definitions

- A round is active if some (non-null) message is sent in the round.
- k-neighborhood of a process: The 2k+1 processes within distance k.

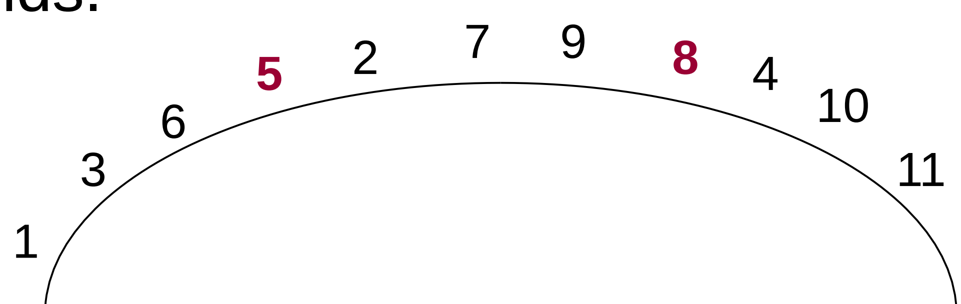
- $(u_1, u_2,..., u_k)$ & $(v_1, v_2,..., v_k)$ are order-equivalent provided that $u_i \leq u_j$ iff $v_i \leq v_j$ for all i,j.
  - Implies same (<, =, >) relationships for all corresponding pairs.
  - Example: (1 3 6 5 2 7 9) vs. (2 7 9 8 4 10 11)

- Two process states s and t correspond with respect to $(u_1, u_2,..., u_k)$ & $(v_1, v_2,..., v_k)$ if they are identical except that occurrences of u_i in s are replaced by v_i in t for all i.
  - Analogous definition for corresponding messages.

Lower bound proof: Key Lemma

- Lemma: Suppose A is a comparison-based algorithm on a synchronous ring network. Suppose i and j are processes whose sequences of UIDs in their k-neighborhoods are order-equivalent.

  Then at any point after ≤ k active rounds, the states of i and j correspond with respect to their k-neighborhoods' UID sequences.
- That is, processes with order-equivalent k-neighborhoods are indistinguishable until after "enough" active rounds.
- Enough: Information has had a chance to reach the processes from outside the k-neighborhoods.
- Example: 5 and 8 have order-equivalent 3-neighborhoods, so must remain in corresponding states through 3 active rounds.



Lower bound proof: Key lemma

- Lemma: Suppose A is a comparison-based algorithm on a synchronous ring network. Suppose i and j are processes whose sequences of UIDs in their k-neighborhoods are order-equivalent.

  Then at any point after $\leq$ k active rounds, the states of i and j correspond with respect to their k-neighborhoods' UID sequences.
- Proof:
  - Induction on r = number of completed rounds.
  - Base: r = 0.
    - Start states of i and j are identical except for UIDs.
    - Correspond with respect to k-neighborhoods for every k $\geq$ 0.
  - Inductive step: Assume for r-1, show for r.

Key lemma

- Lemma: Suppose i and j have order-equivalent k-neighborhoods. Then at any point after $\leq$ k active rounds, i and j are in corresponding states, with respect to their k-neighborhoods.
- Proof, inductive step:
  - Assume true after round r-1, for all i,j,k.
  - Prove true after round r, for all i,j,k.
  - Fix i,j,k, where i and j have order-equivalent k-neighborhoods.
  - Assume i $\neq$ j (trivial otherwise).
  - Assume at most k of first r rounds are active.
  - We must show that, after r rounds, i and j are in corresponding states with respect to their k-neighborhoods.
  - By inductive hypothesis, after r-1 rounds, i and j are in corresponding states with respect to their k-neighborhoods.
  - If neither i nor j receives a non-null message at round r, they make corresponding transitions, to corresponding states (with respect to their k-neighborhoods).
  - So assume at least one of i,j receives a message at round r.

Key lemma

- Lemma: Suppose i and j have order-equivalent k-neighborhoods. Then at any point after ≤ k active rounds, i and j are in corresponding states, with respect to their k-neighborhoods.
- Inductive step, cont'd:
  - So assume at least one of i,j receives a message at round r.
  - Then round r is active, and the first r-1 rounds include at most k-1 active rounds.
  - (k-1)-nbhds of i-1 and j-1 are order-equivalent, since they are included within the k-neighborhoods of i and j.
  - By inductive hypothesis, after r-1 rounds:
    - i-1 and j-1 are in corresponding states wrt their (k-1)-neighborhoods, and thus wrt the k-neighborhoods of i and j.
    - Similarly for i+1 and j+1.
  - Thus, messages from i-1 to i and from j-1 to j correspond.
  - Similarly for msgs from i+1 to i and from j+1 to j.
  - So i and j are in corresponding states and receive corresponding messages, so make corresponding transitions and end up in corresponding states.

Lower bound proof

- So, we have shown that many active rounds are needed to break symmetry, if there are large order-equivalent neighborhoods.
- It remains to show:
  - There exist rings with many, and large, order-equivalent neighborhoods.
  - This causes large communication complexity.
- First, see how order-equivalent neighborhoods cause large communication complexity…

Lower bound proof

- Corollary 1: Suppose A is a comparison-based leader-election algorithm on a synchronous ring network, and k is an integer such that for any process i, there is a distinct process j such that i and j have order-equivalent k-neighborhoods. Then A has more than k active rounds.

- Proof: By contradiction.
  - Suppose A elects i in at most k active rounds.
  - By assumption, there is a distinct process j with an order-equivalent k-neighborhood.
  - By Key Lemma, i and j are in corresponding states, so j is also elected—a contradiction.

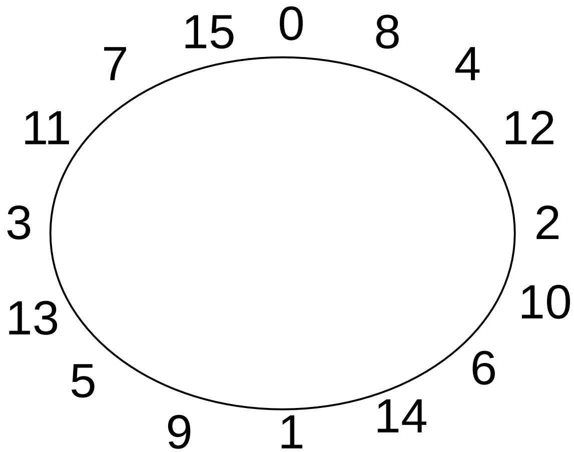
Lower bound proof

- Corollary 2: Suppose A is a comparison-based algorithm on a synchronous ring network, and k and m are integers such that the k-neighborhood of any process is order-equivalent to that of at least m-1 other processes. Then at least m messages are sent in A's k^{th} active round.

- Proof:
  - By definition, some process sends a message in the k^{th} active round.
  - By assumption, at least m-1 other processes have order-equivalent k-neighborhoods.
  - By the Key Lemma, immediately before this round, all these processes are in corresponding states. Thus, they all send messages in this round, so at least m messages are sent.

Highly symmetric rings

- That's how order-equivalent neighborhoods yield high communication complexity.
- Now, show existence of rings with many, large order-equivalent neighborhoods.
- For powers of 2: Bit-reversal rings
  - UID is bit-reversed process number.
  - Example:



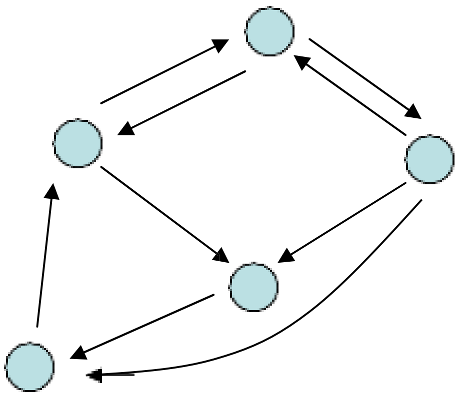
  - For every segment of length $n/2^b$, there are (at least) 2^b order-equivalent segments (including original segment).
  - So for every process i, there are at least n/4k processes (including i) with order-equivalent k-neighborhoods, for $k < n/4$.
  - More than n/8 active rounds.
  - Number of messages $\geq n/4 + n/8 + n/12 + ... + 2 = \Omega(n \log n)$

C-symmetric rings

- c-symmetric ring: For every l such that $\sqrt{n} < l < n$, and every sequence S of length l in the ring, there are at least $\lfloor cn/l \rfloor$ order-equivalent occurrences.
- [Frederickson-Lynch] There exists c such that for every positive integer n, there is a c-symmetric ring of size n.
- Given c-symmetric ring, argue similarly to before.

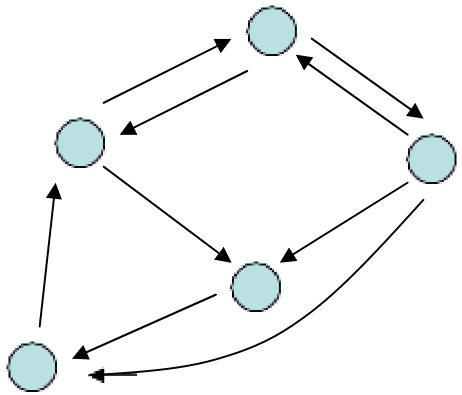
General Synchronous Networks

General synchronous networks



- Not just rings, but arbitrary digraphs.
- Basic tasks, such as broadcasting messages, collecting responses, setting up communication structures.
- Basic algorithms.
- No lower bounds.
- Algorithms are simplified versions of algorithms that work in asynchronous networks. We'll revisit them in asynchronous setting.

General synchronous network assumptions



- Digraph G = (V,E):
  - V = set of processes
  - E = set of communication channels
  - distance(i,j) = shortest distance from i to j
  - diam = max distance(i,j) for all i,j
  - Assume: Strongly connected (diam is finite), UIDs
- Set M of messages
- Each process has states, start, msgs, trans, as before.
- Processes communicate only over digraph edges.
- Generally don't know the entire network, just local neighborhood.
- Local names for neighbors.
  - No particular order for neighbors, in general.
  - But (technicality) if incoming and outgoing edges connect to same neighbor, the names are the same (so the node "knows" this).

Leader election in general synchronous networks

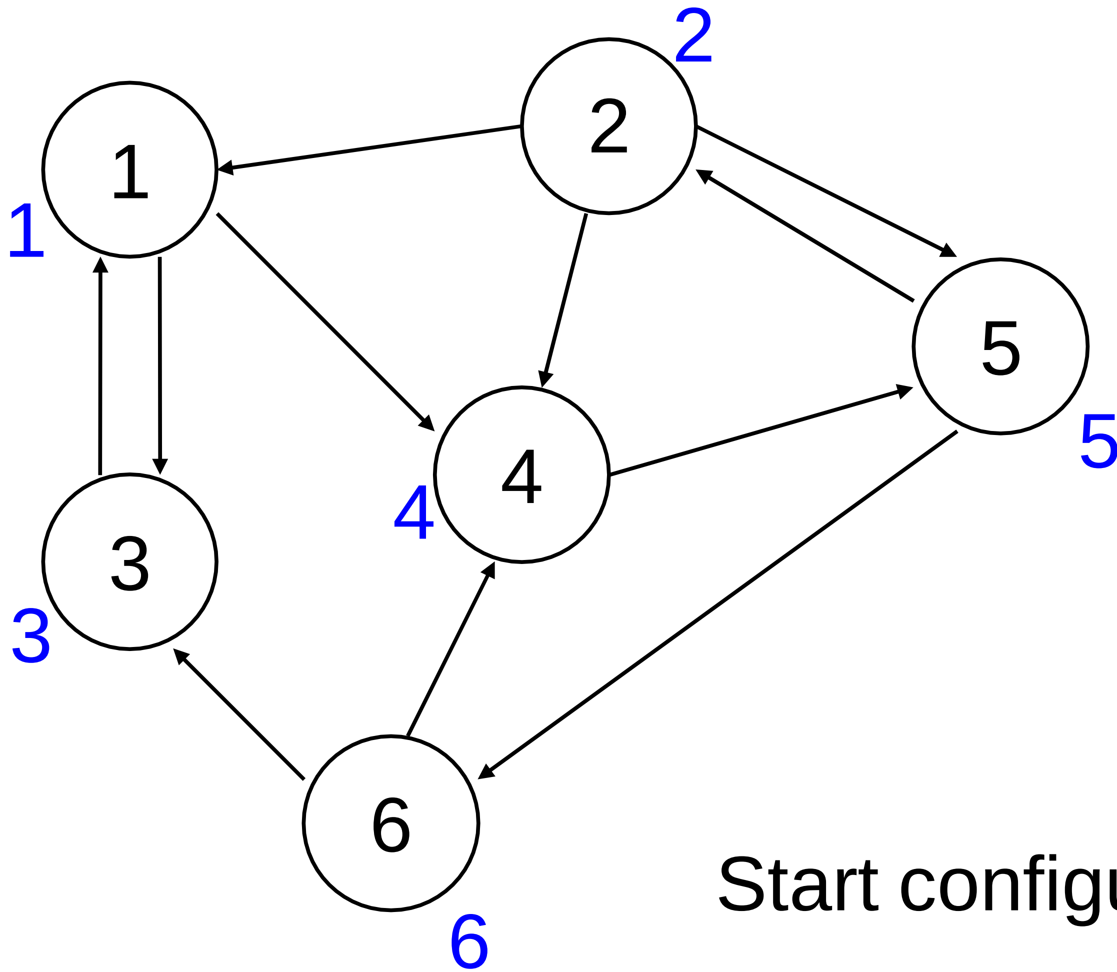
- Assume:
  - Use UIDs with comparisons only.
  - No constraints on which UIDs appear, or where they appear in the graph.
  - Processes know (upper bound on) graph diameter.
- Required: Everyone should eventually set status $\in$ {leader, non-leader}, exactly one leader.

- Show basic flooding algorithm, sketch proof using invariants, show optimized version, sketch proof by relating it to the basic algorithm.
- Basic flooding algorithm:
  - Every round: Send max UID seen to all neighbors.
  - Stop after diam rounds.
  - Elect self iff own UID is max seen.

Leader election in general synchronous networks

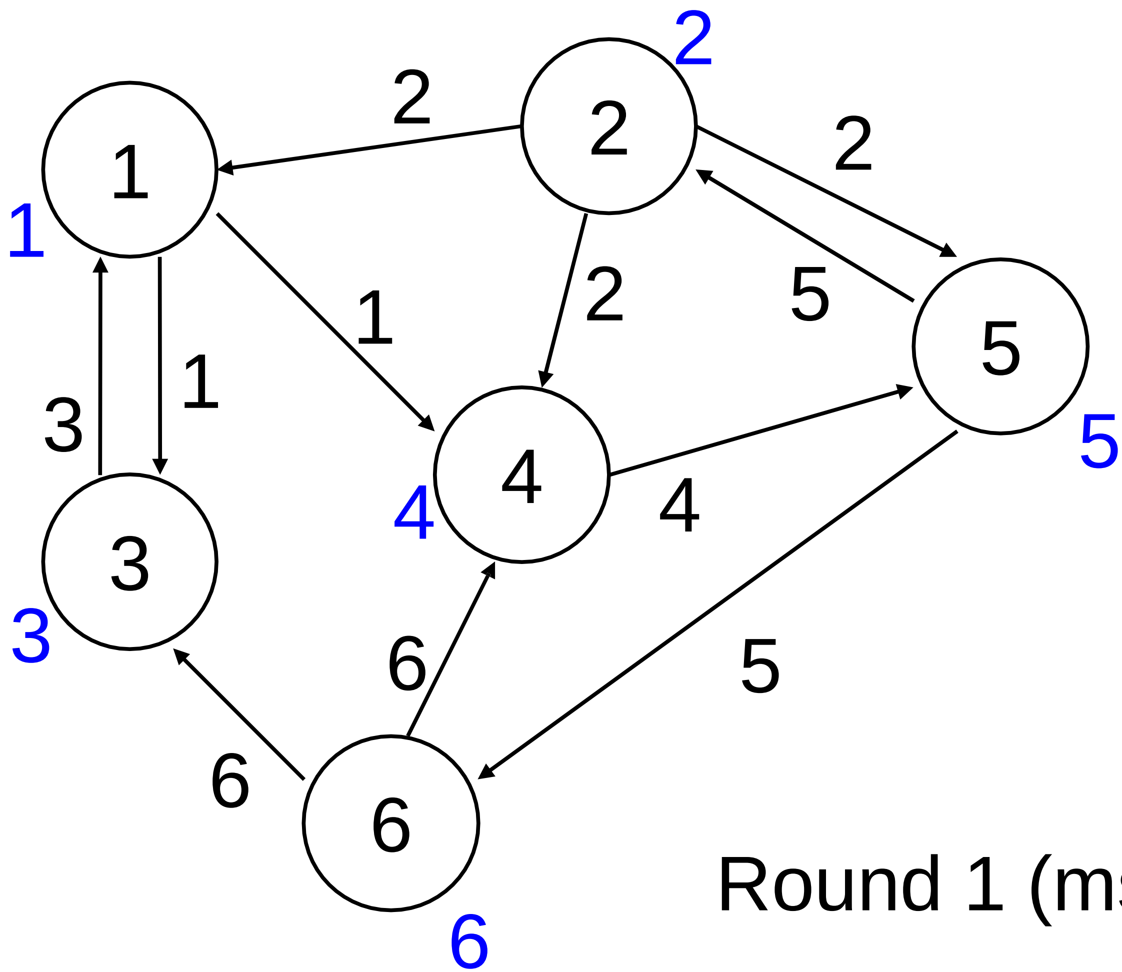
- states
  - u, initially UID
  - max-uid, initially UID
  - status ∈ {unknown, leader, not-leader}, initially unknown
  - rounds, initially 0
- msgs
  - if rounds < diam send max-uid to all out-nbrs
- trans
  - increment round
  - max-uid := max (max-uid, UIDs received)
  - if round = diam then
    - status := leader if max-uid = u, not-leader otherwise

Leader election in general network



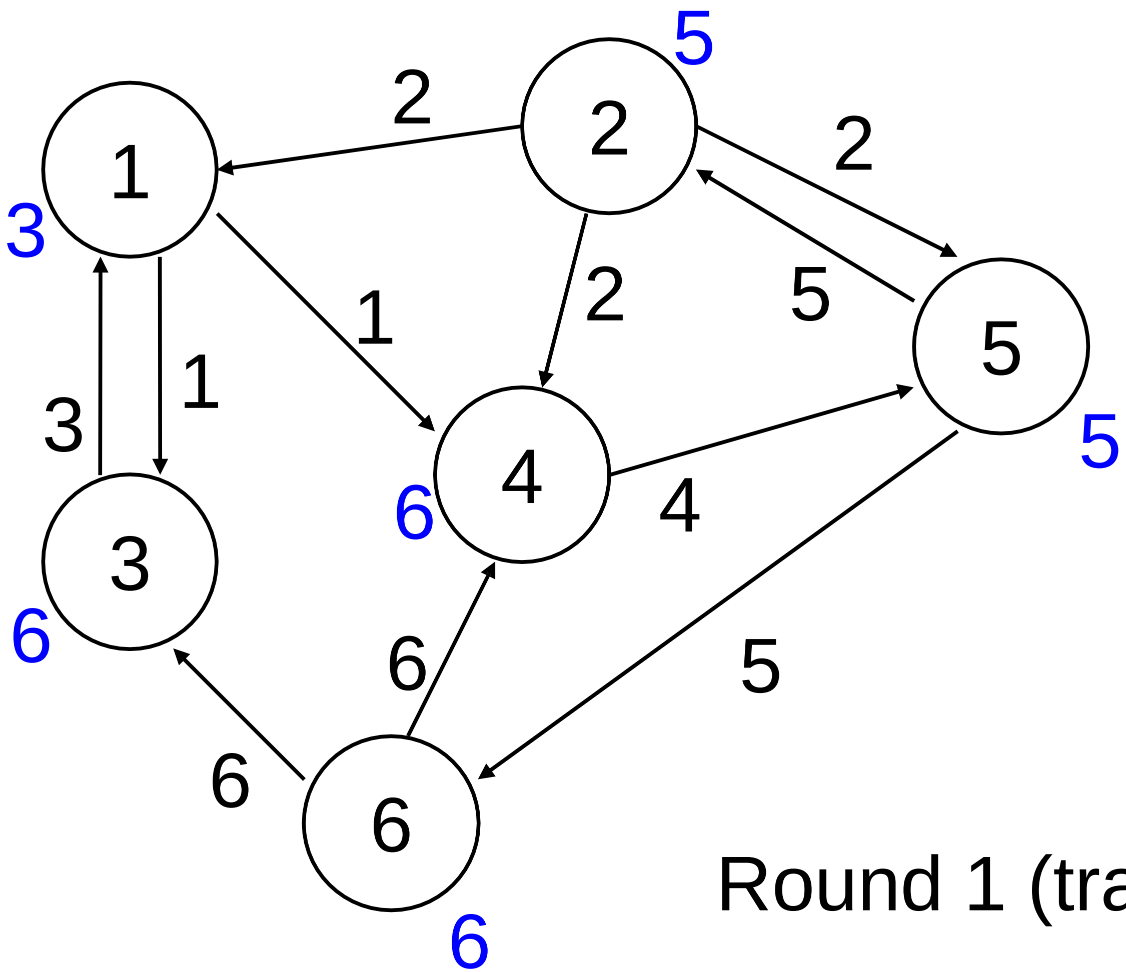
Start configuration

Leader election in general network



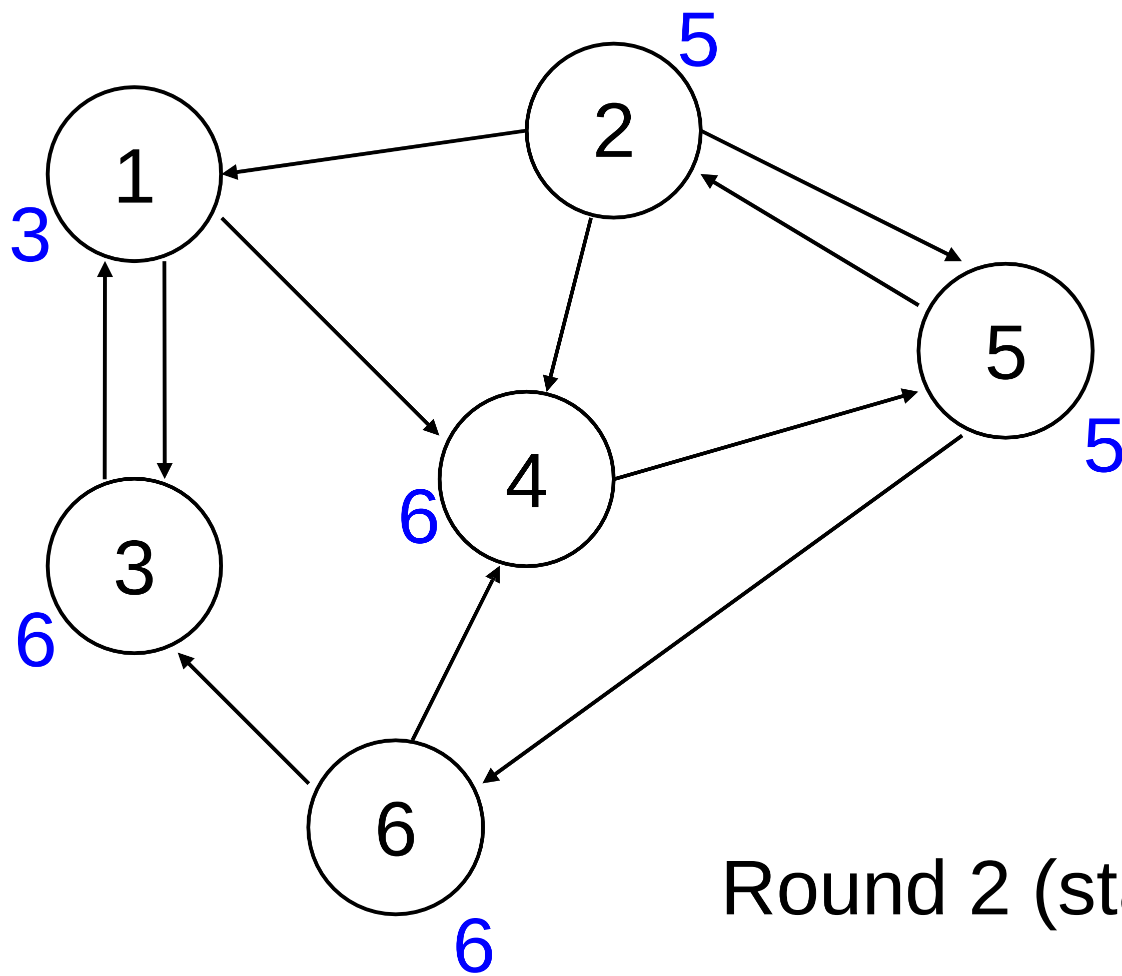
Round 1 (msgs)

Leader election in general network



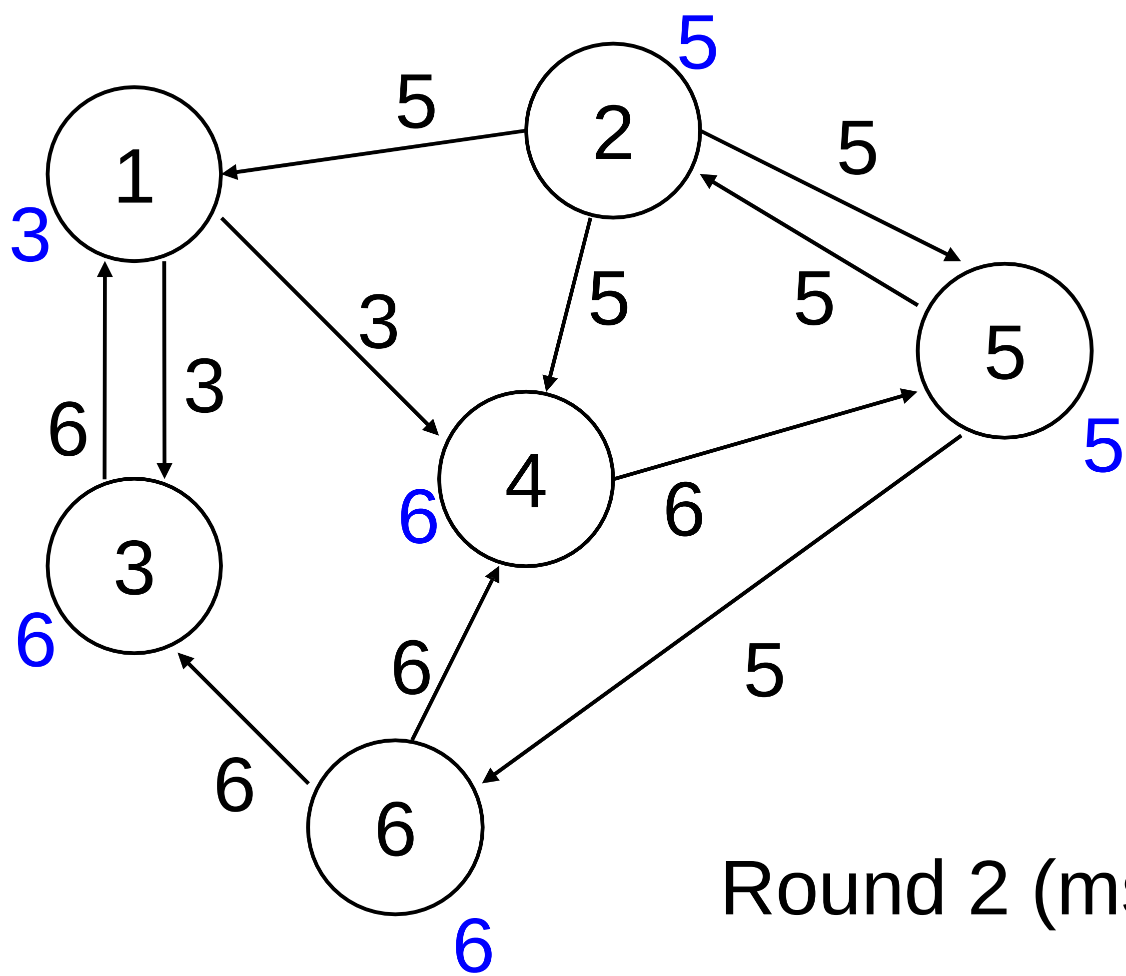
Round 1 (trans)

Leader election in general network



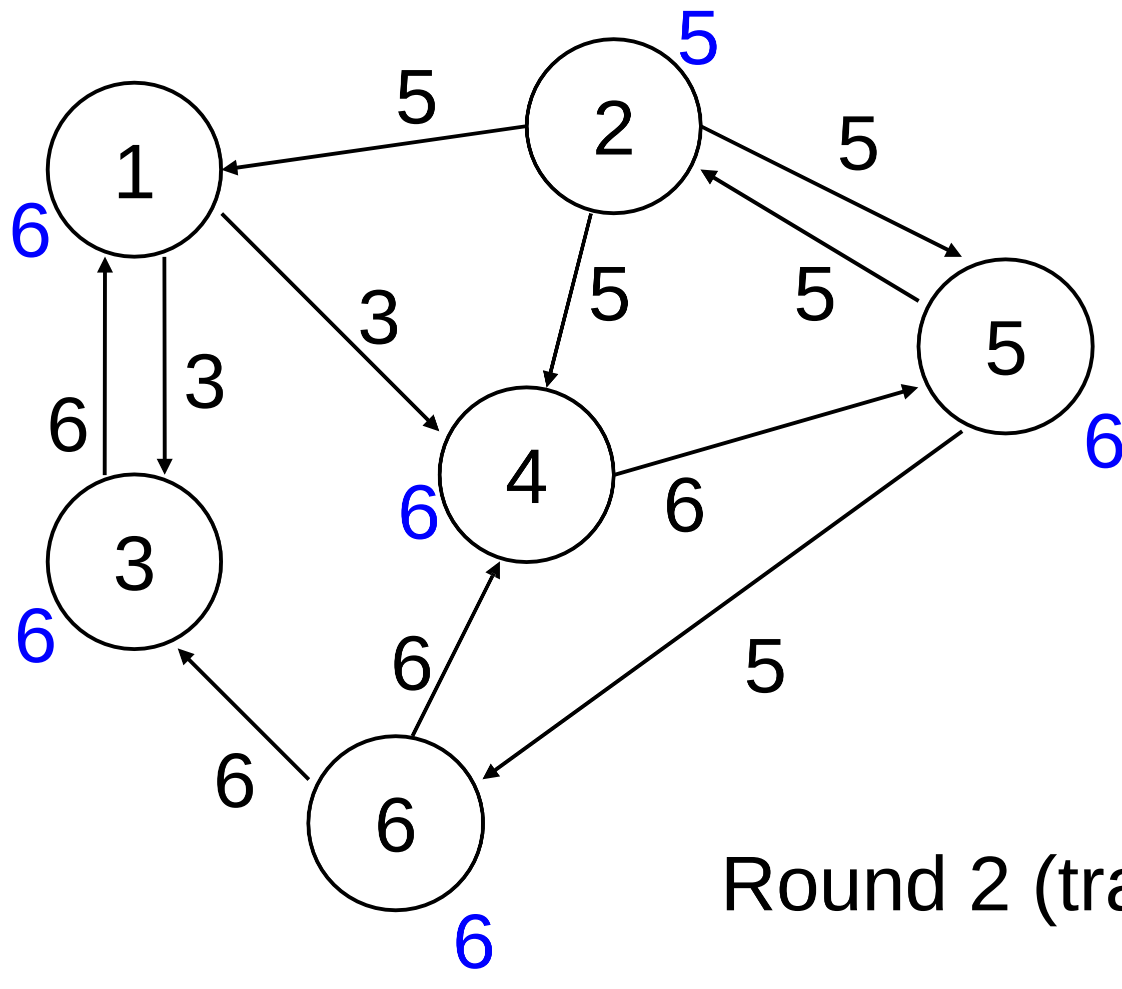
Round 2 (start)

Leader election in general network



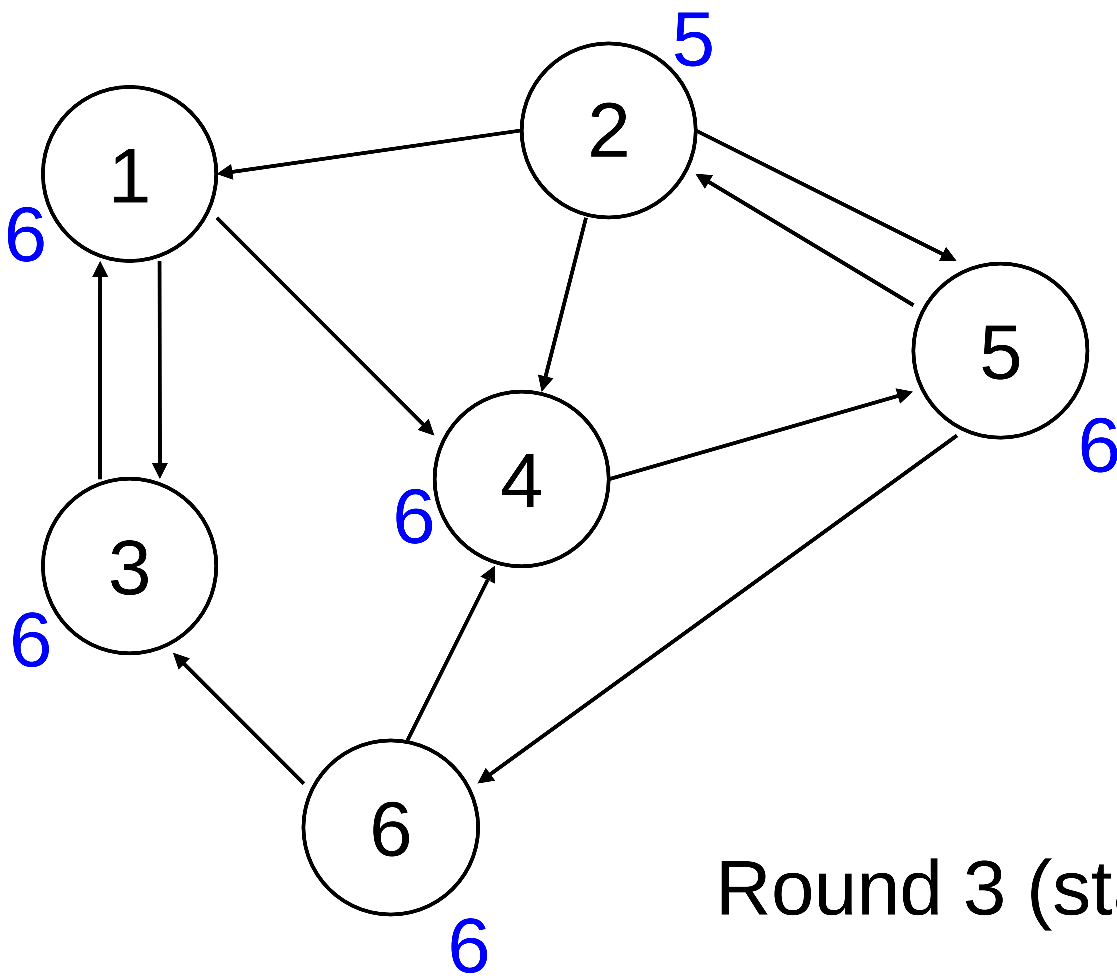
Round 2 (msgs)

Leader election in general network



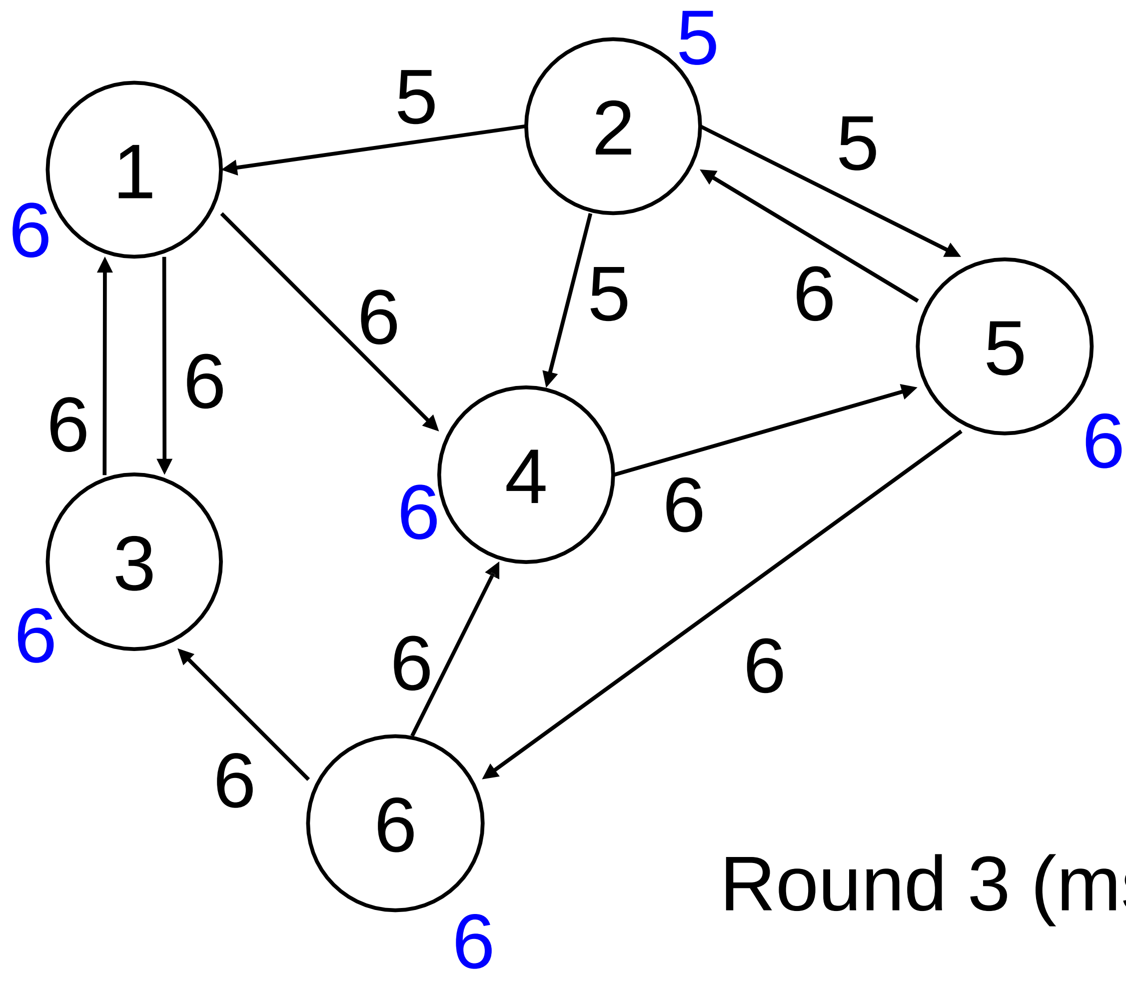
Round 2 (trans)

Leader election in general network



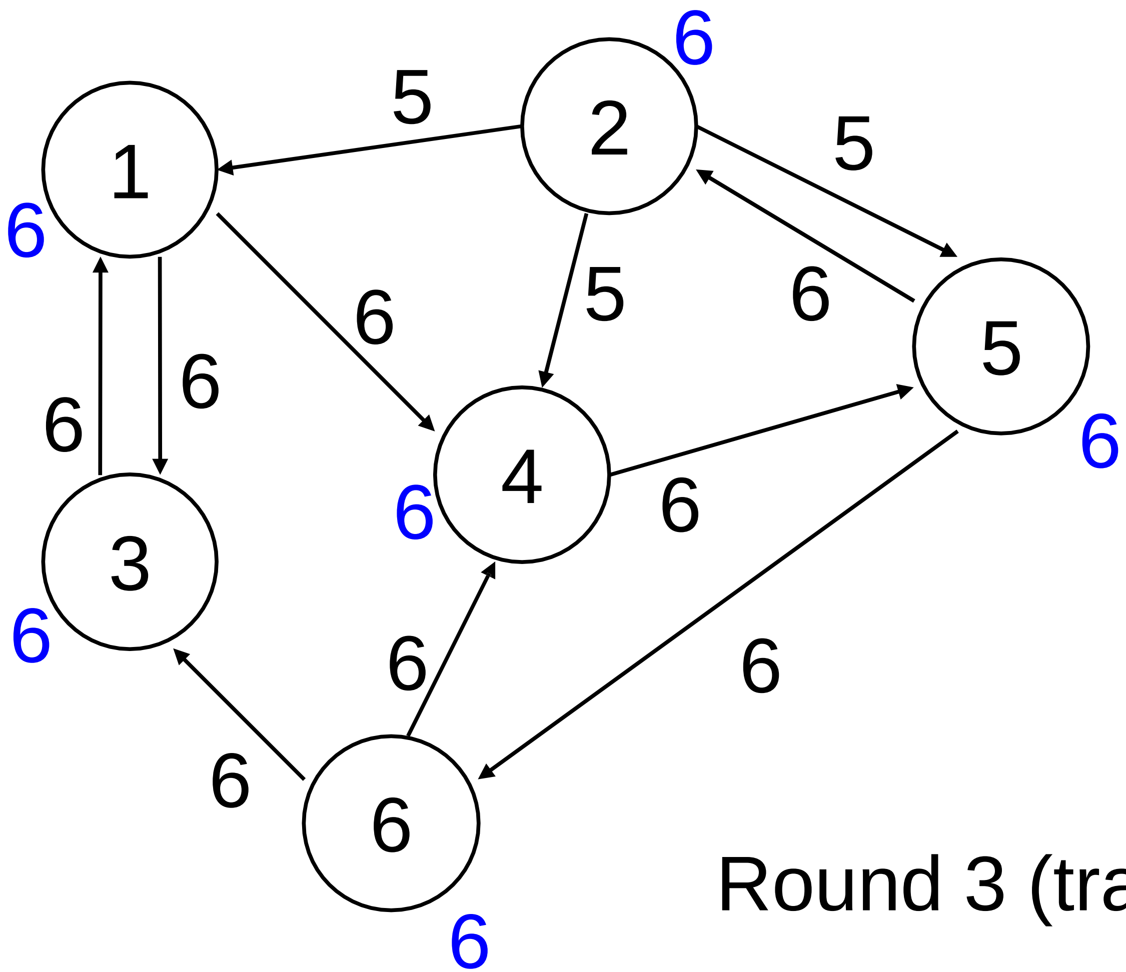
Round 3 (start)

Leader election in general network



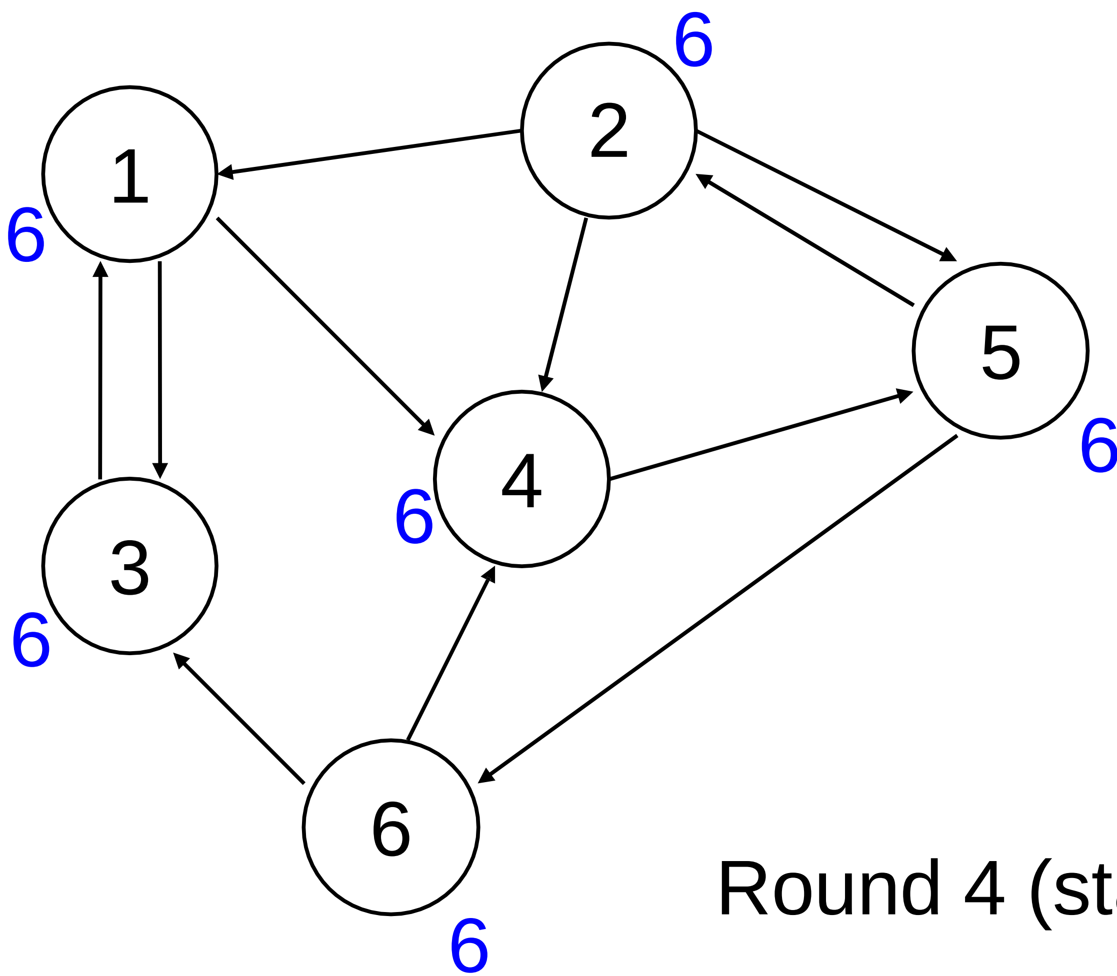
Round 3 (msgs)

Leader election in general network



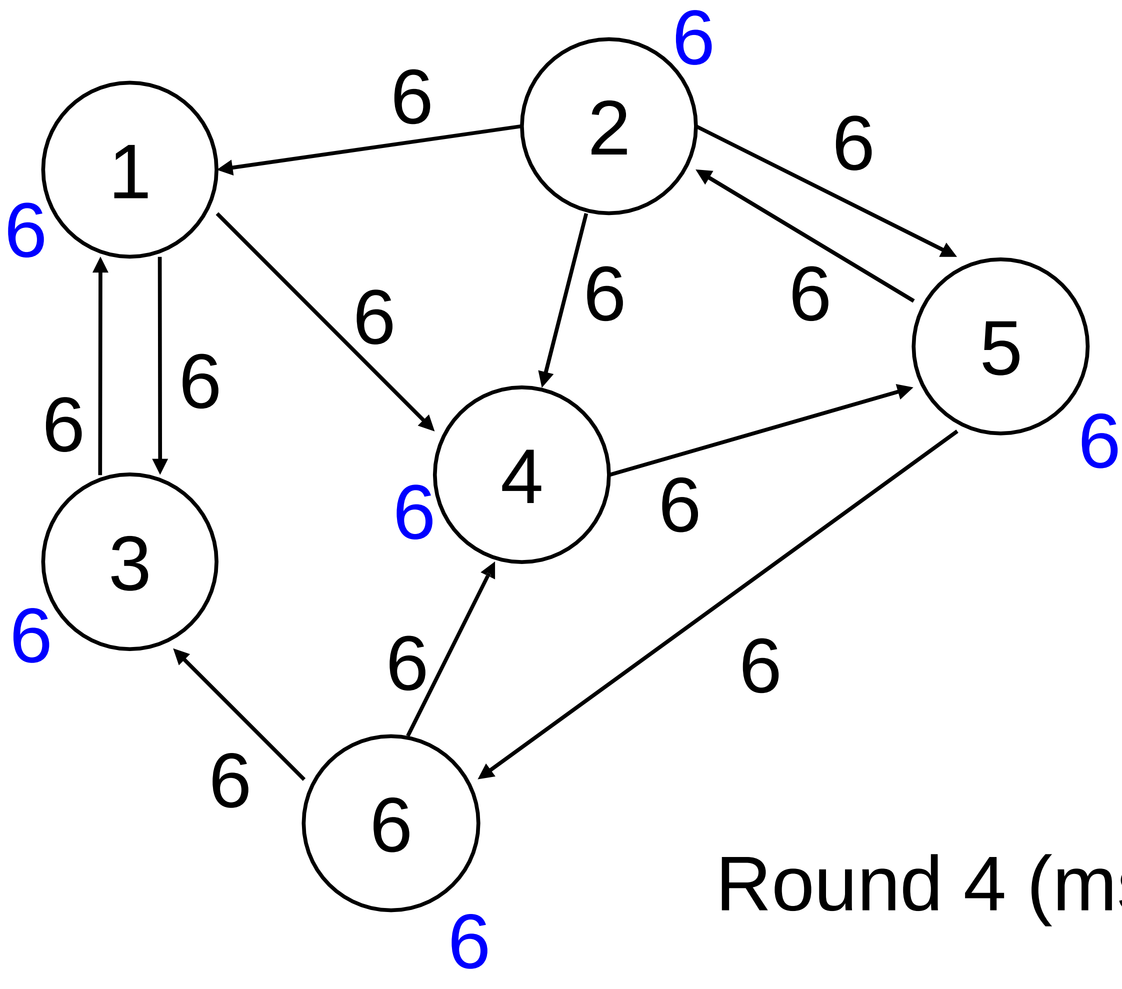
Round 3 (trans)

Leader election in general network



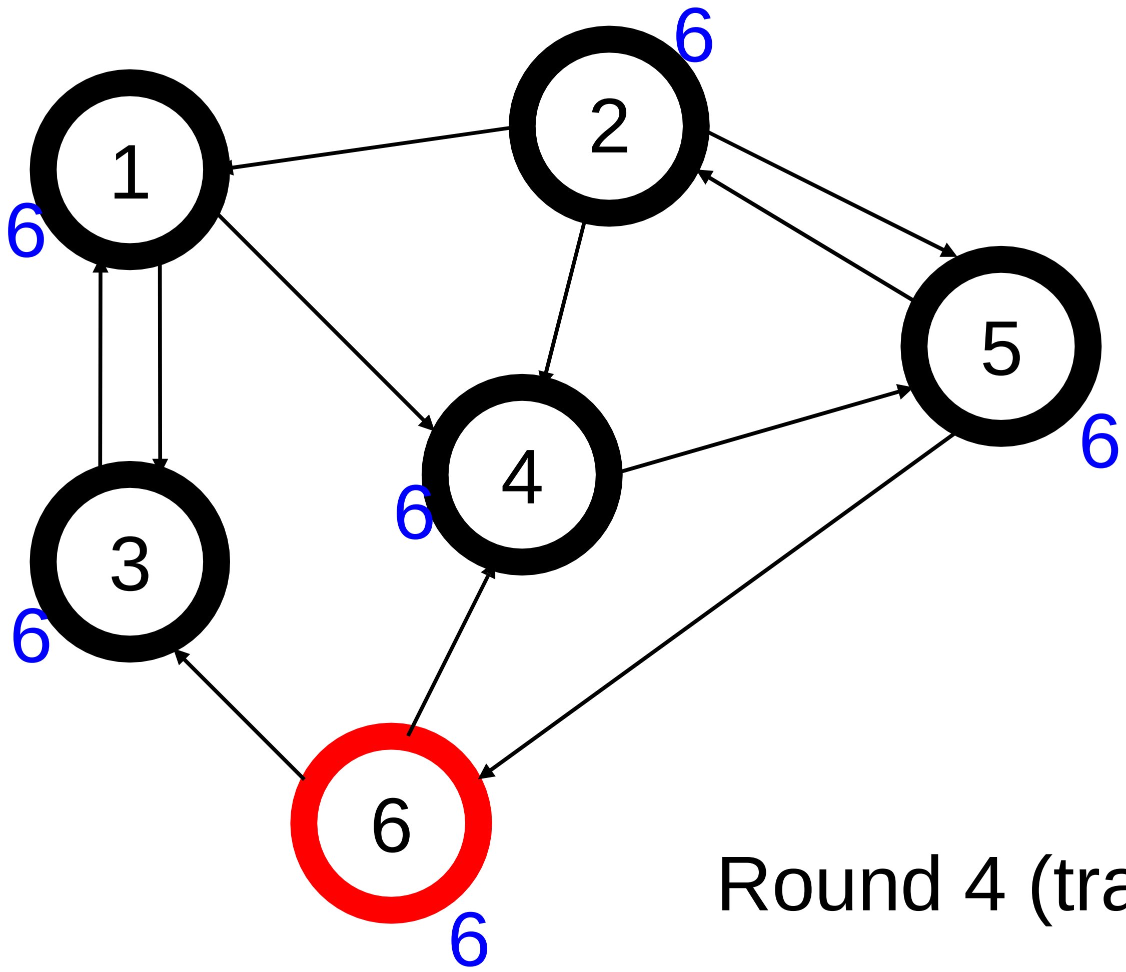
Round 4 (start)

Leader election in general network



Round 4 (msgs)

Leader election in general network



Round 4 (trans)

Leader election in general network

- Basic flooding algorithm (summary):
  - Assume diameter is known (diam).
  - Every round: Send max UID seen to all neighbors.
  - Stop after diam rounds.
  - Elect self iff own UID is max seen.

- Complexity:
  - Time complexity (rounds): diam
  - Message complexity: diam |E|
- Correctness proof?

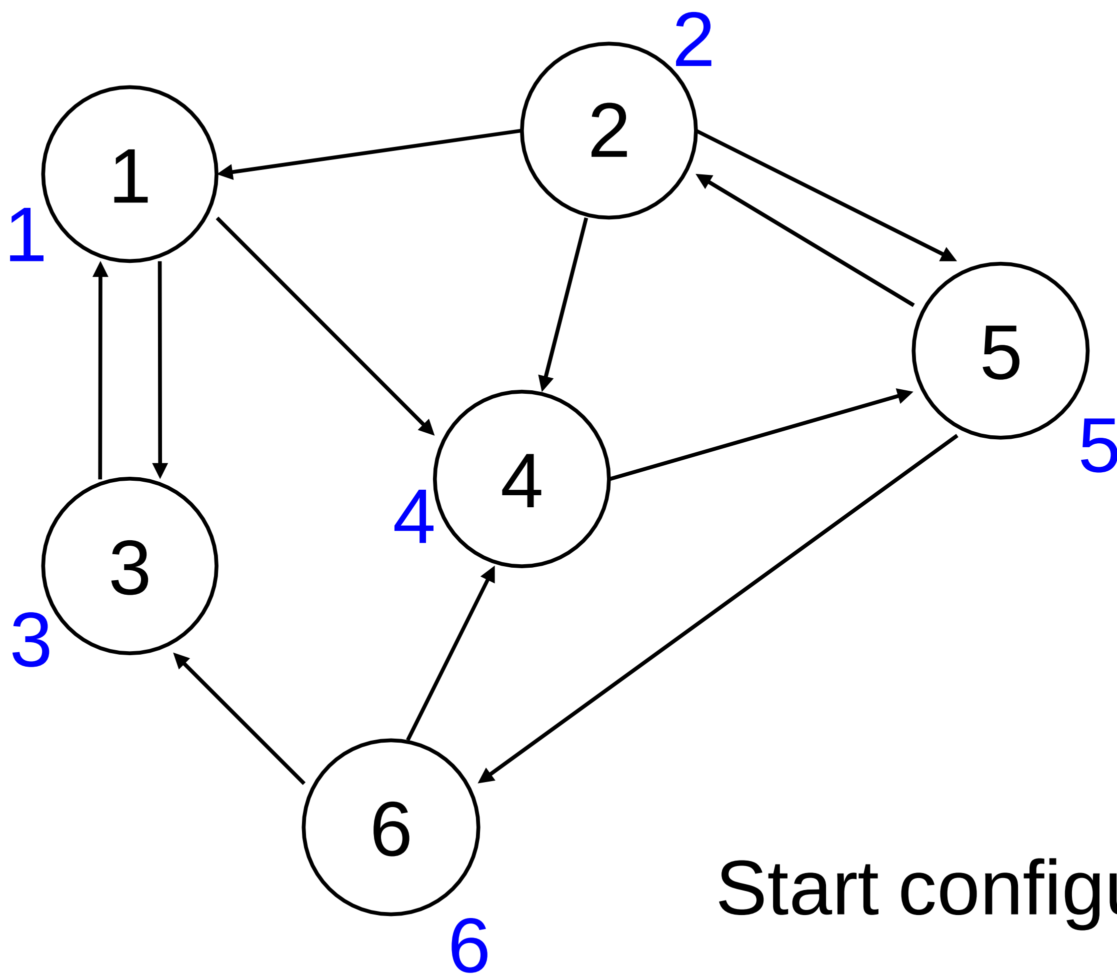
Key invariant

- Invariant: After round r, if distance(i,j) ≤ r then max-uid$_j$ ≥ UID$_i$.
- Proof:
  - Induction on r.
  - Base: r = 0
    - distance(i,j) = 0 implies i = j, and max-uid$_i$ = UID$_i$.
  - Inductive step: Assume for r-1, prove for r.
    - If distance(i,j) ≤ r then there is a node k in in-nbrs$_j$ such that distance(i,k) ≤ r -1.
    - By inductive hypotheses, after round r-1, max-uid$_k$ ≥ UID$_i$.
    - Since k sends its max to j at round r, max-uid$_j$ ≥ UID$_i$ after round r.

Reducing the message complexity

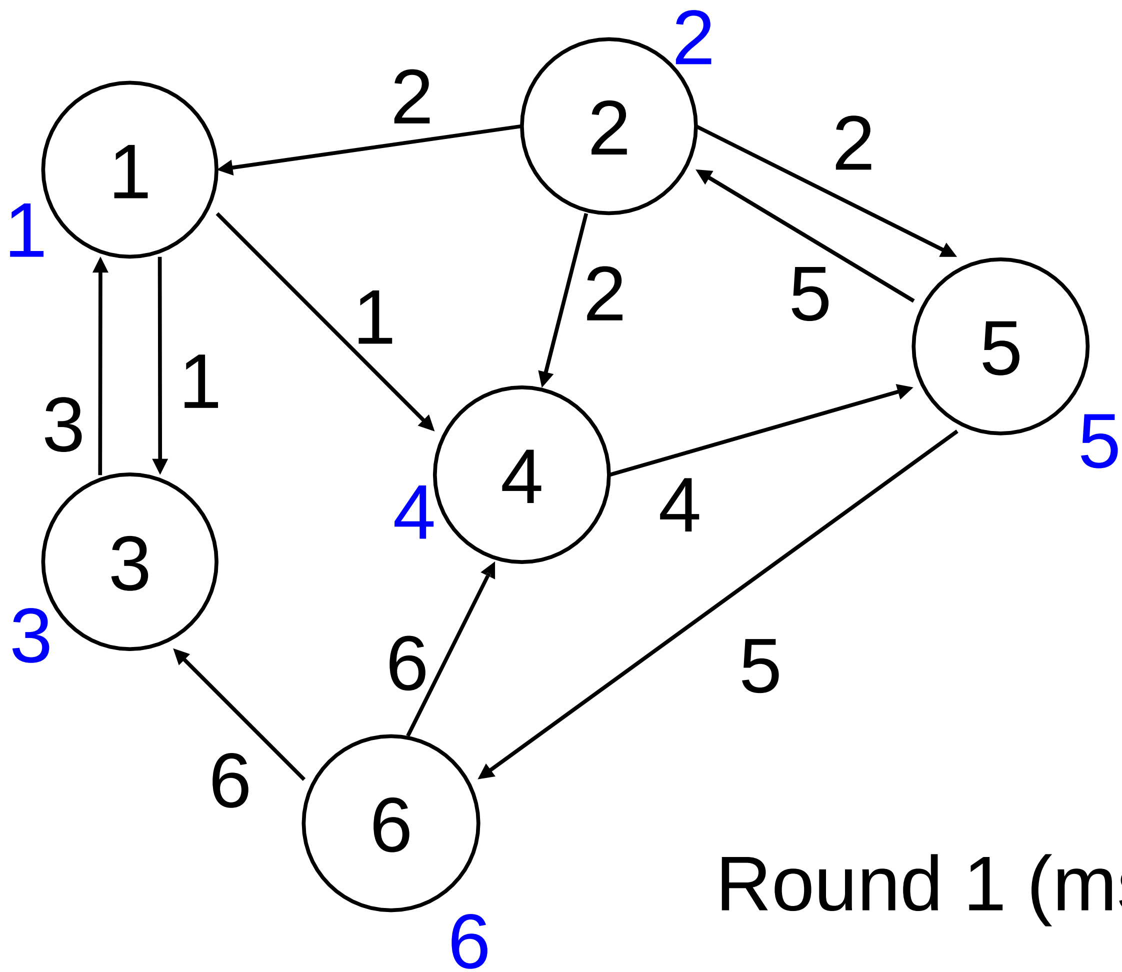
- Slightly optimized algorithm:
  - Don't send same UID twice.
  - New state var: new-info: Boolean, initially true
  - Send max-uid only if new-info = true
  - new-info := true iff max UID received > max-uid

Leader election in general network



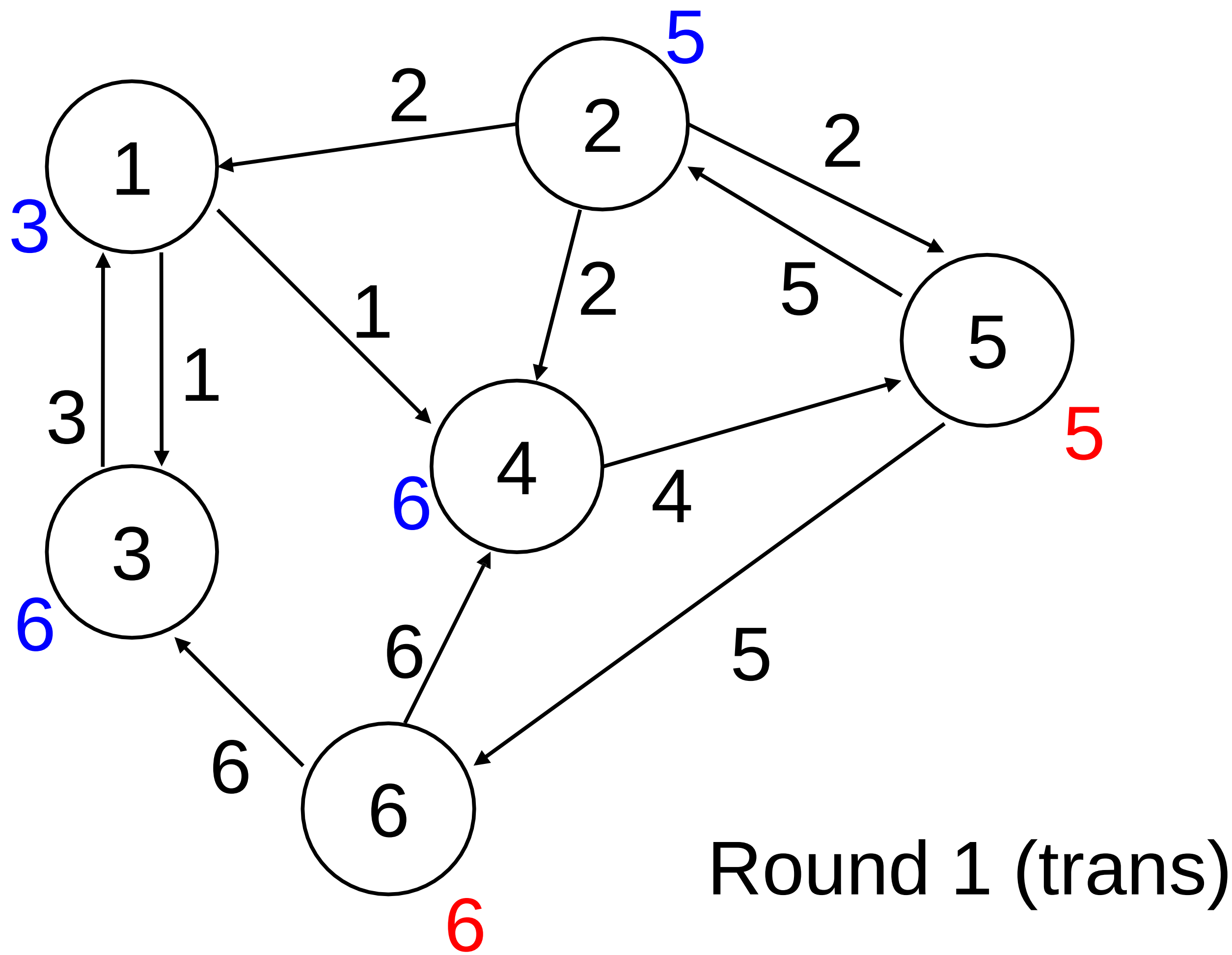
Start configuration

Leader election in general network

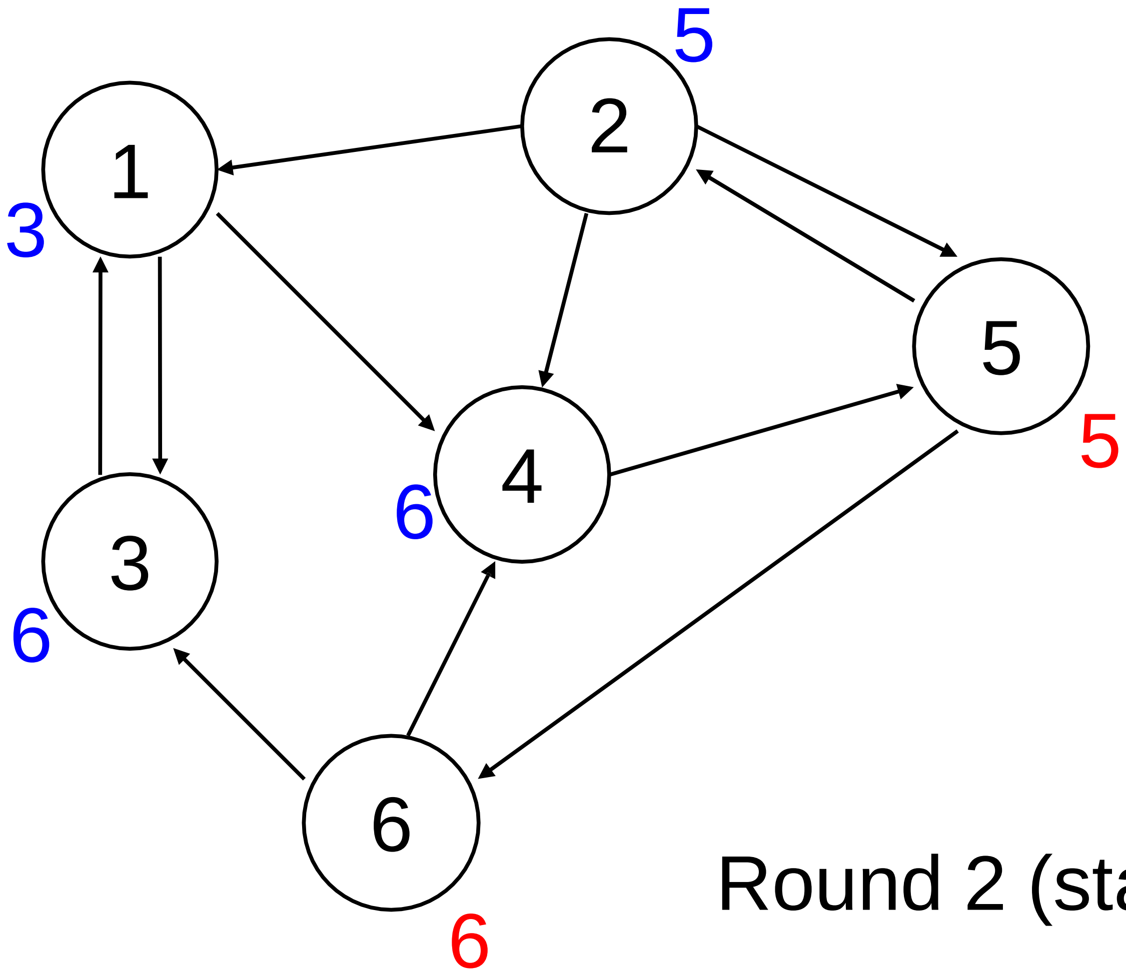


Round 1 (msgs)

Leader election in general network

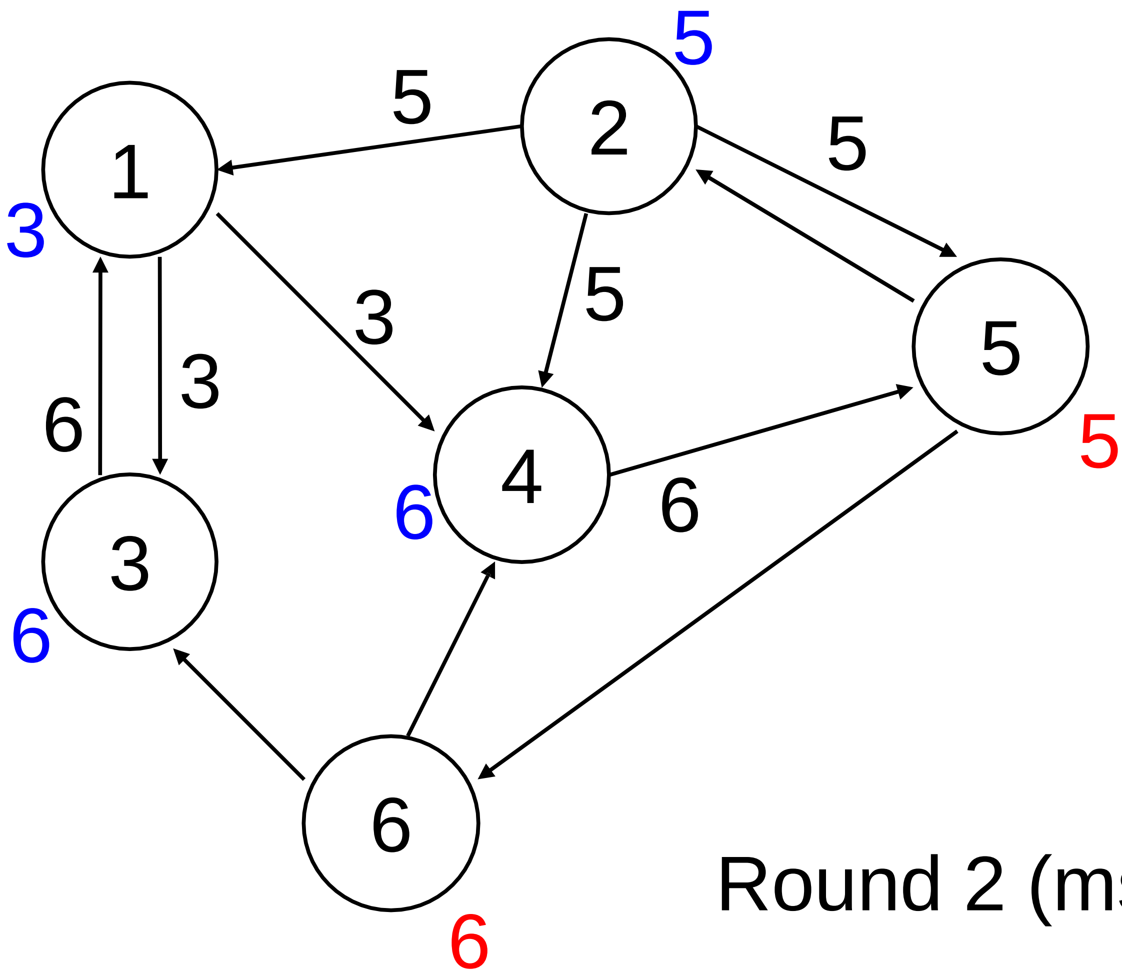


Leader election in general network



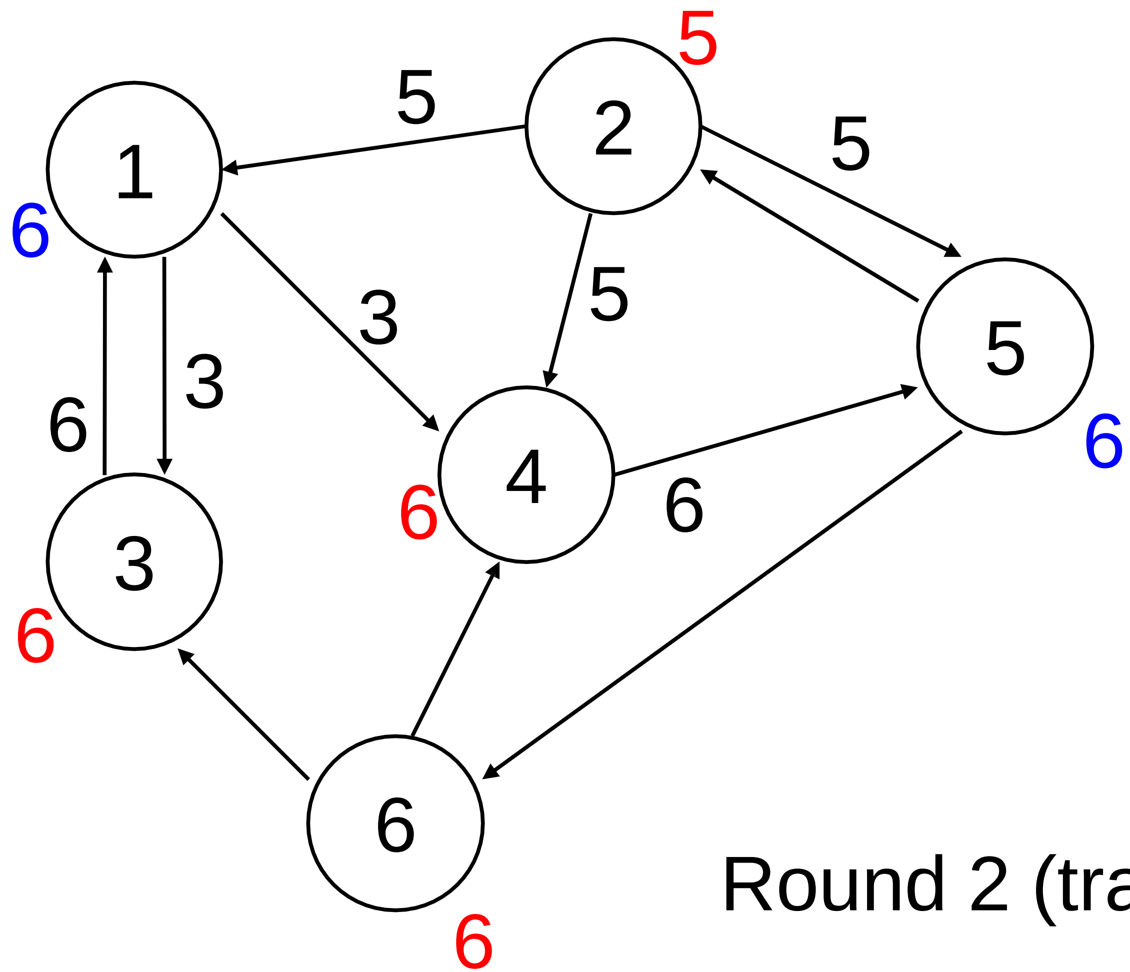
Round 2 (start)

Leader election in general network



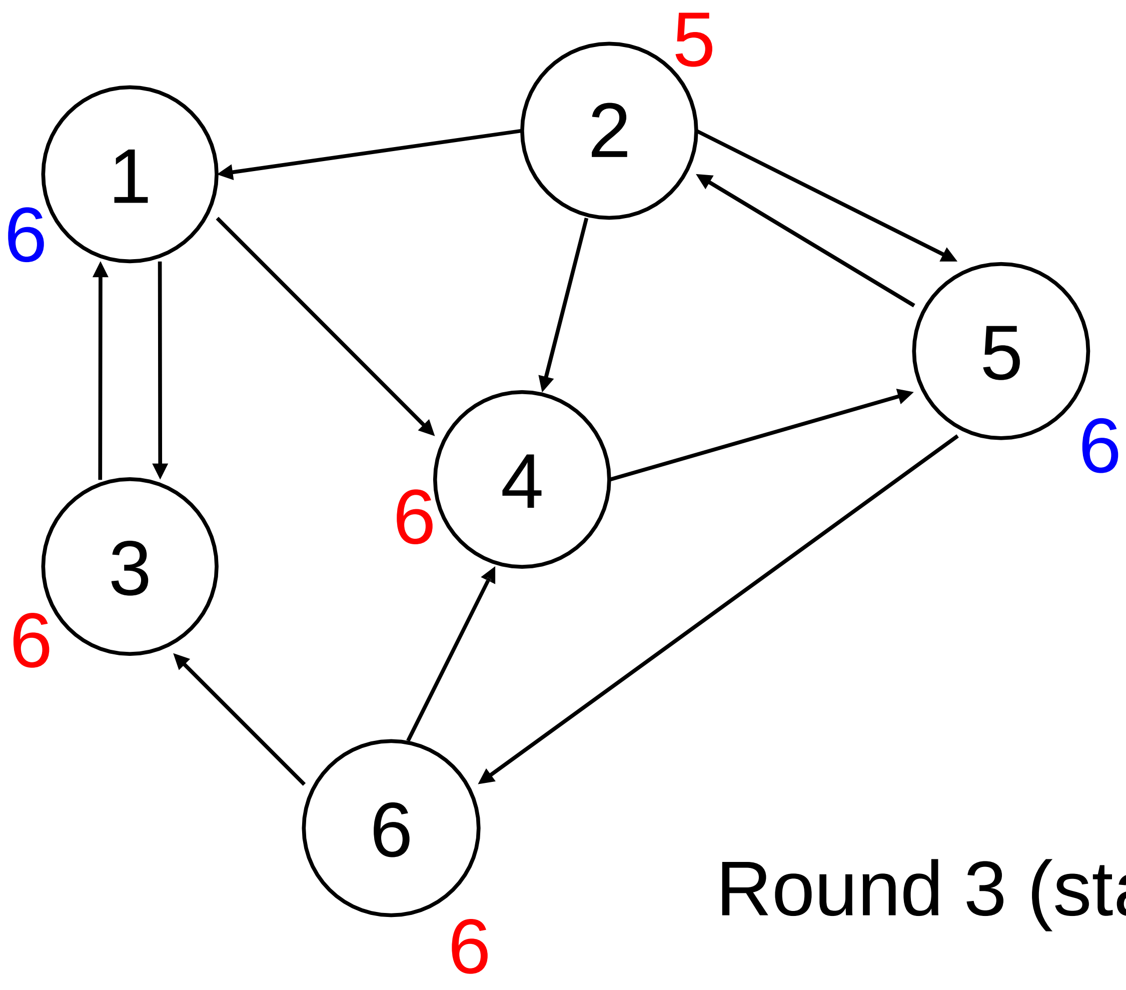
Round 2 (msgs)

Leader election in general network



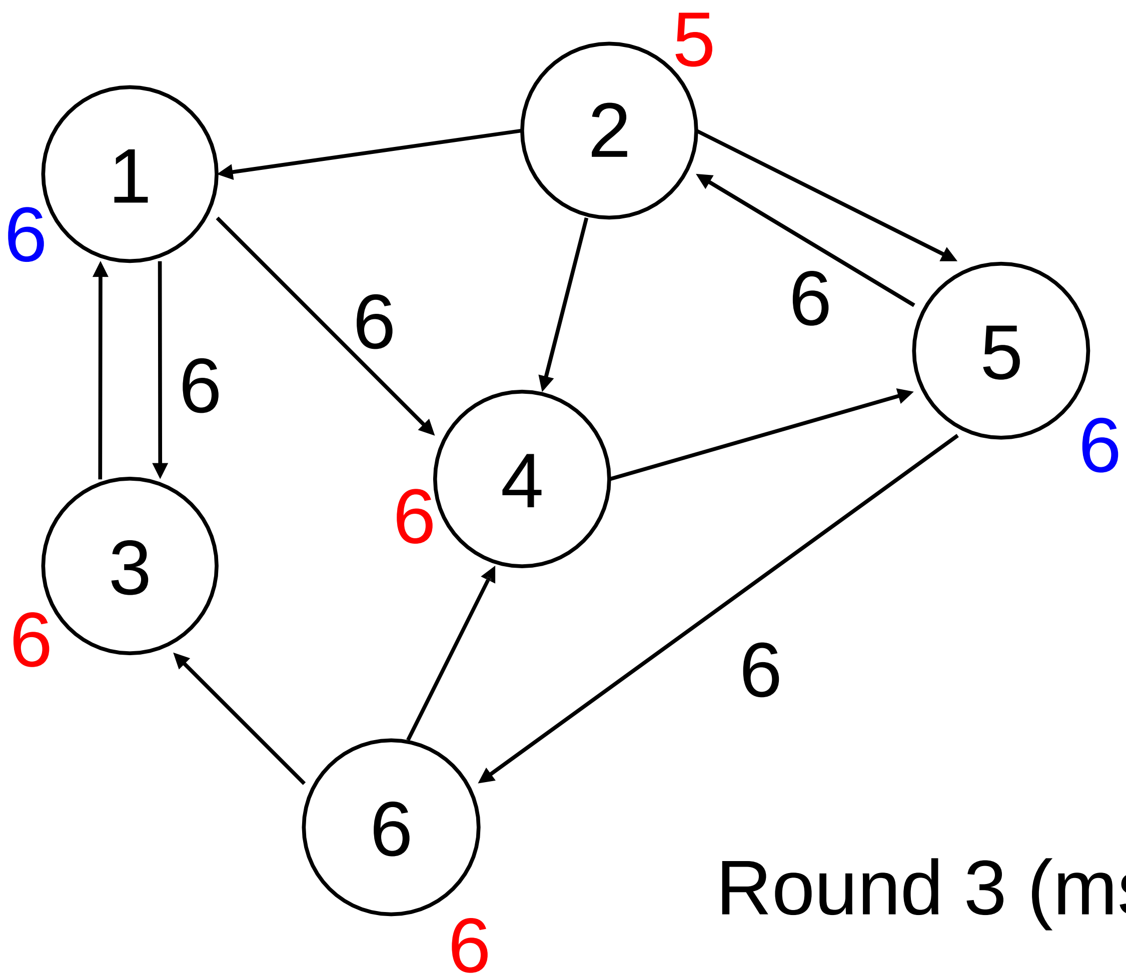
Round 2 (trans)

Leader election in general network



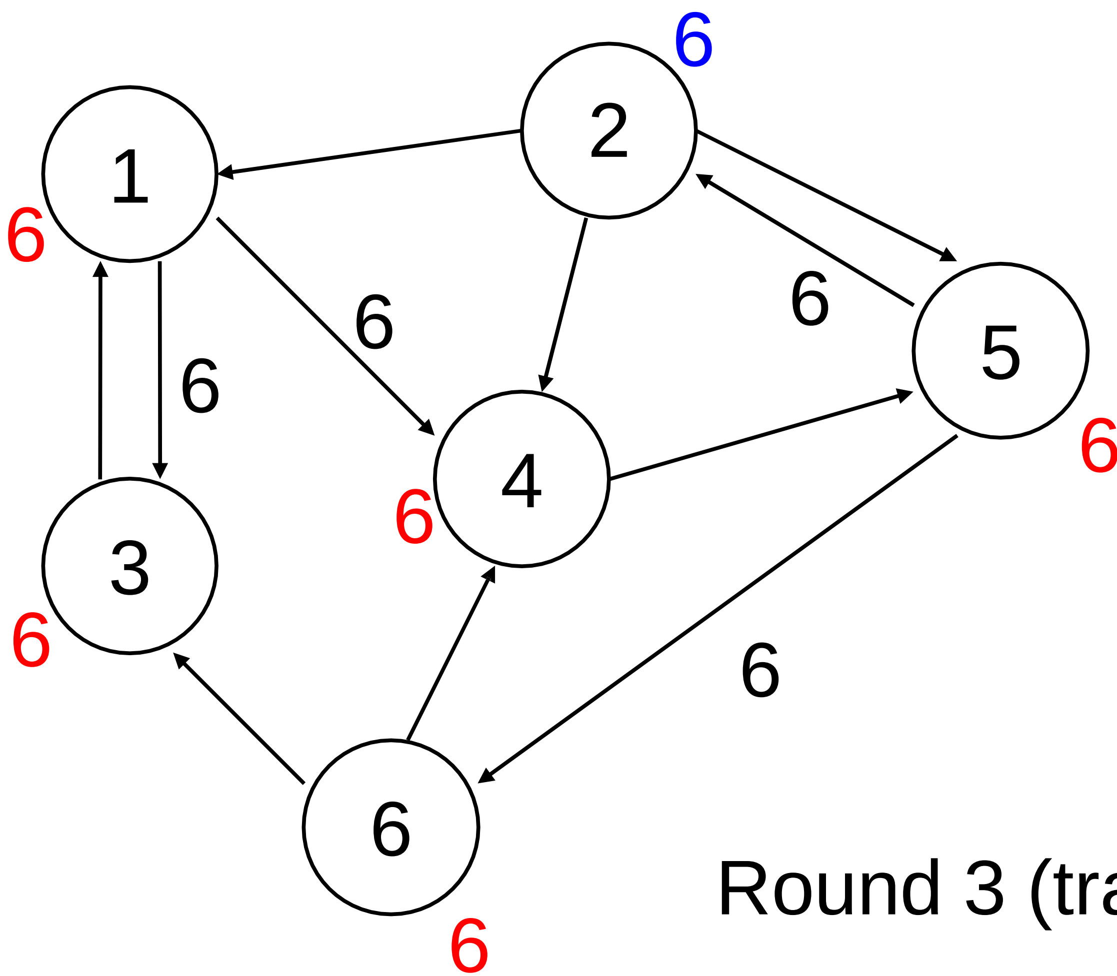
Round 3 (start)

Leader election in general network



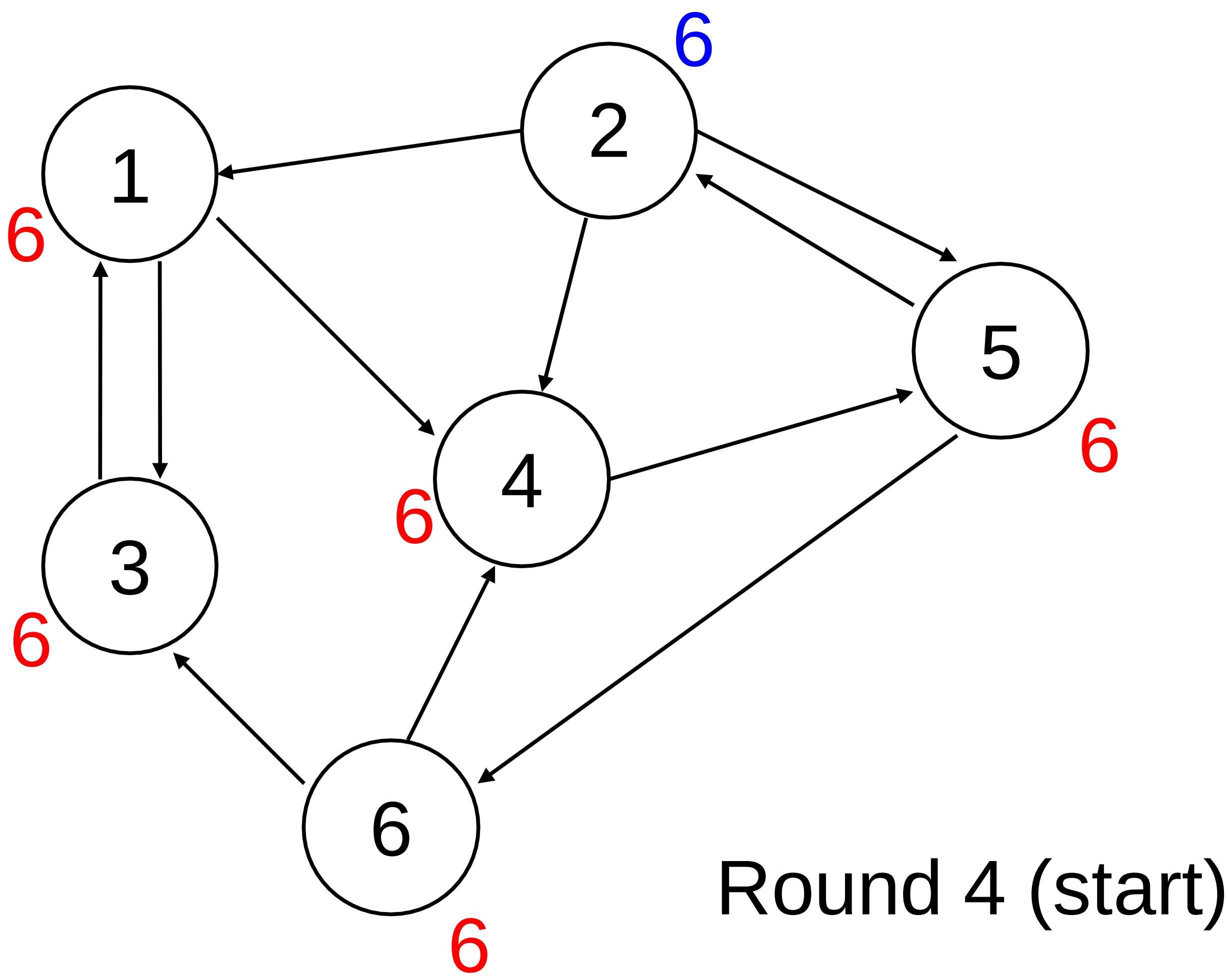
Round 3 (msgs)

Leader election in general network

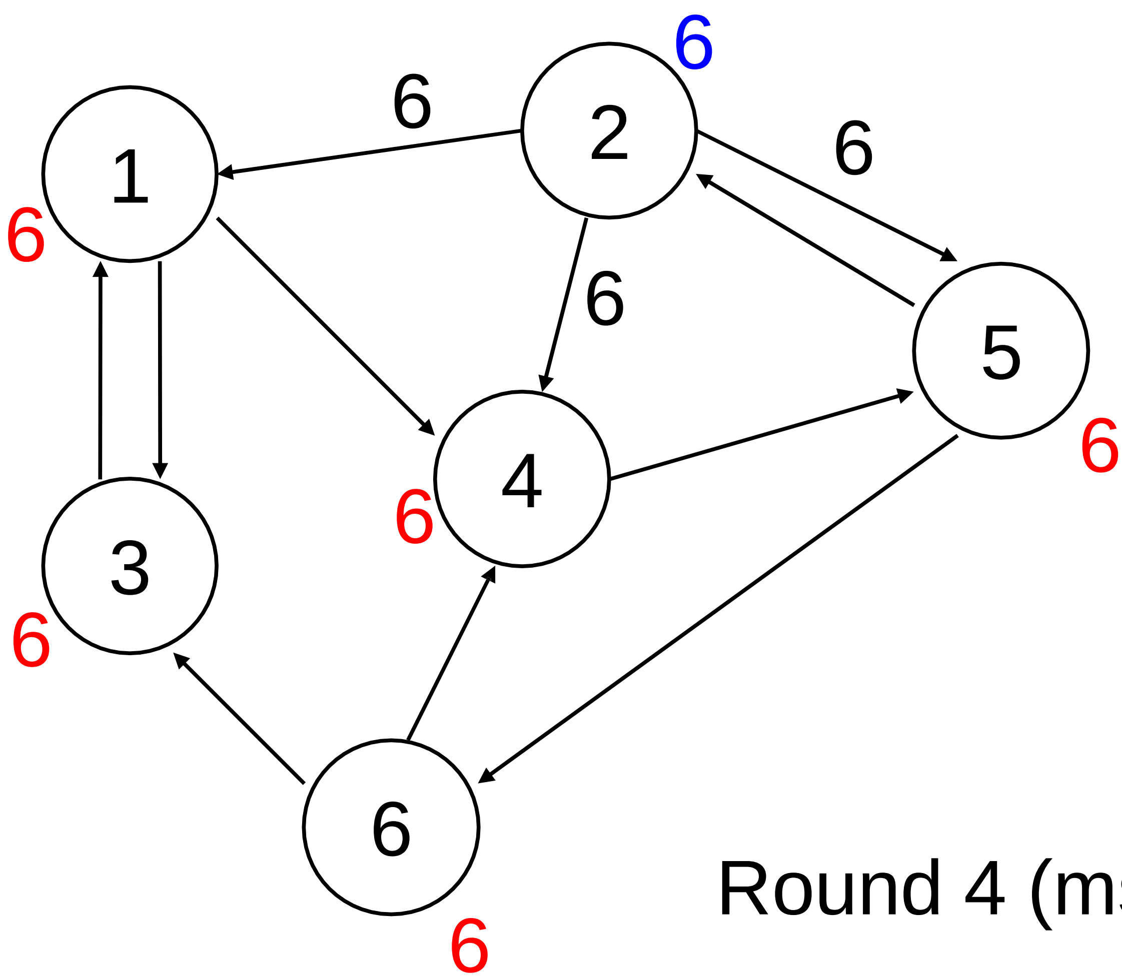


Round 3 (trans)

Leader election in general network

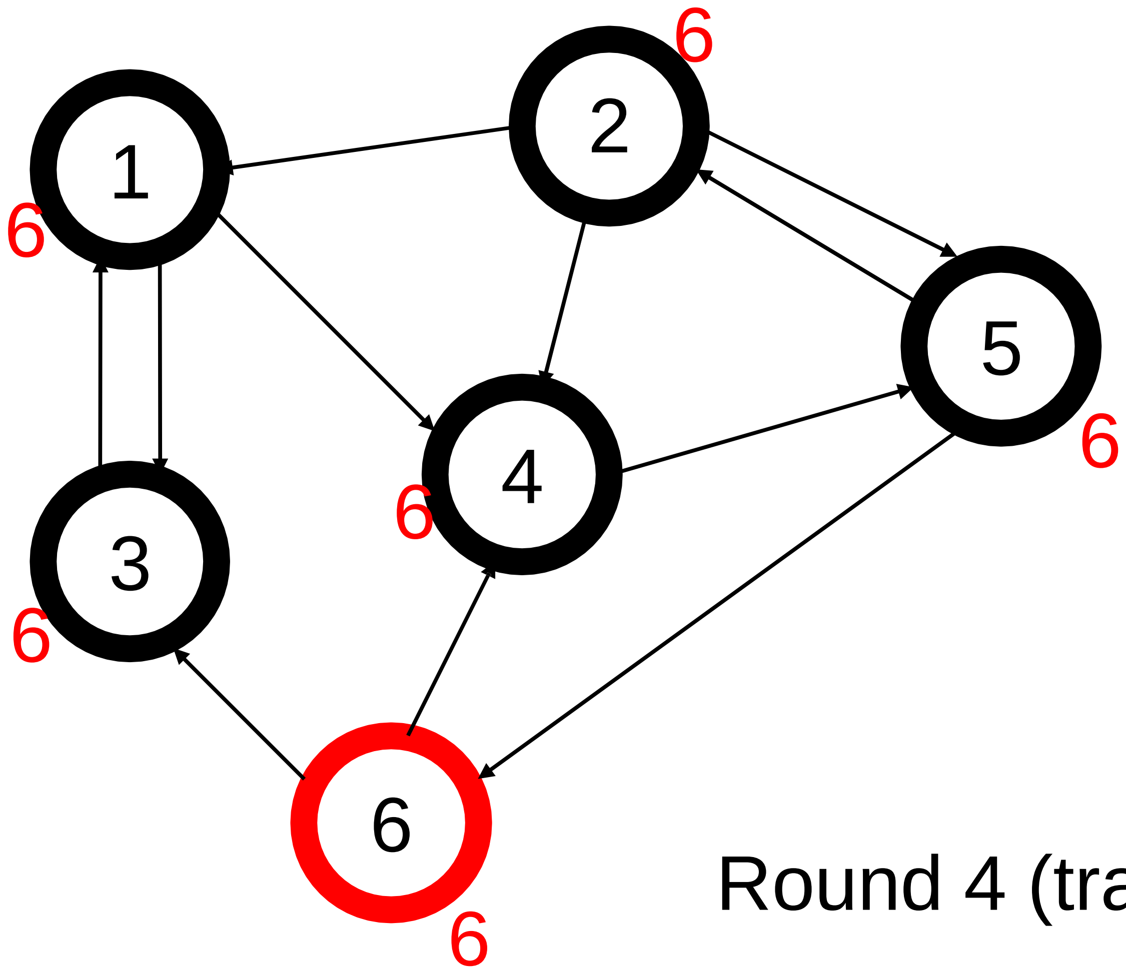


Leader election in general network



Round 4 (msgs)

Leader election in general network



Round 4 (trans)

Leader election in general network

- Slightly optimized algorithm (summary):
  - Don't send same UID twice
  - New state variable: new-info: Boolean, initially true
  - Send max-uid just when new-info = true
  - new-info := true iff max UID received > max-uid
  - Can improve communication cost drastically, though not the worst-case bound, diam |E|.
- Correctness Proof?
  - As before, or:
  - Can use another important proof method for distributed algorithms: simulation relations.

Simulation relation

- Relates new algorithm formally to an original one that has already been proved correct.
- Correctness then carries over to new algorithm.
- Often used to show correctness of optimized algorithms.
- Can repeat in several stages, adding more optimizations.

- "Run the two algorithms side by side."
- Define simulation relation between states of the two algorithms:
  - Satisfied by start states.
  - Preserved by every transition.
  - Outputs should be the same in related states.

Simulation relation for the optimized algorithm

- Key invariant of the optimized algorithm:
  - If $i \in$ in-nbrs$_j$ and max-uid$_i$ > max-uid$_j$ then new-info$_i$ = true.
  - That is, if i has better information than j has, then i is planning to send it to j on the next round.
  - Prove by induction.
- Simulation relation: All state variables of the basic algorithm (all but new-info) have the same values in both algorithms.
- Start condition: By definition.
- Preserved by every transition:
  - Key property: max-uids are always the same in the two algorithms.
  - Consider $i \in$ in-nbrs$_j$.
  - If new-info$_i$ = true before the step, then the two algorithms behave the same with respect to (i,j).
  - Otherwise, only the basic algorithm sends a message. However, by the invariant, max-uid$_i \leq$ max-uid$_j$ before the step, and the message has no effect.

Why all these proofs?

- Distributed algorithms can be quite complicated, subtle.
- Easy to make mistakes.
- So careful reasoning about algorithm steps is generally more important than for sequential algorithms.

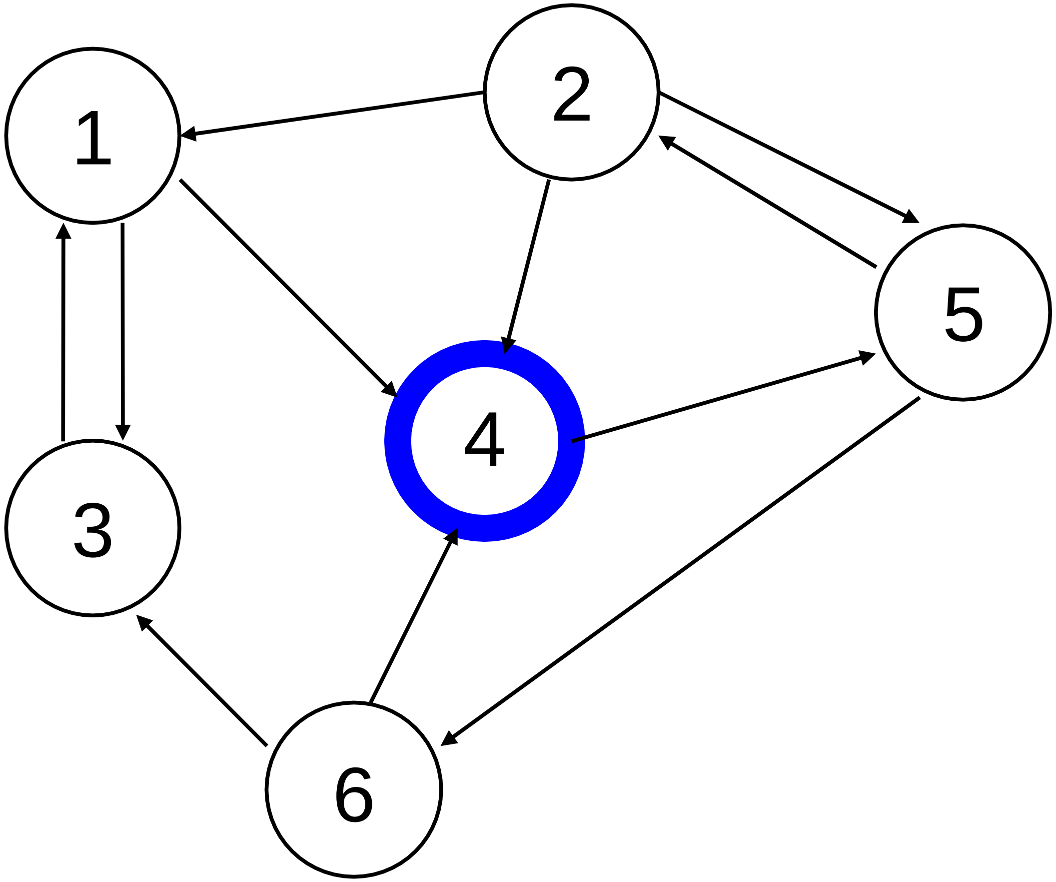
Other problems besides leader election…

- Breadth-first search
- Breadth-first spanning trees, shortest-paths spanning trees,…
- Minimum spanning trees
- Maximal independent sets

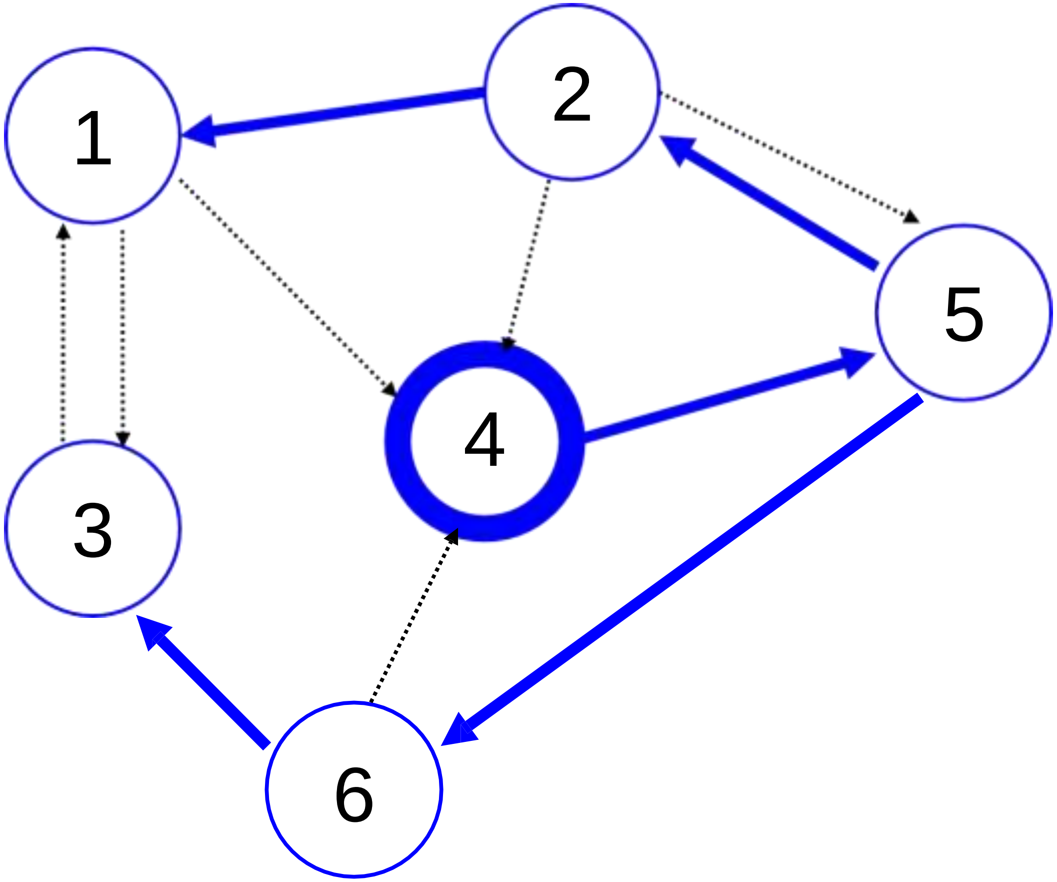
Breadth-first search

- Assume:
  - Strongly connected digraph, UIDs.
  - No knowledge of size, diameter of network.
  - Distinguished source node i_0.
- Required: Breadth-first spanning tree, rooted at source node i_0.
  - Branches are directed paths in the given digraph.
  - Spanning: Includes every node.
  - Breadth-first: Node at distance d from i_0 appears at depth d in tree.
  - Output: Each node (except i_0) sets a parent variable to indicate its parent in the tree.

Breadth-first search



Breadth-first search

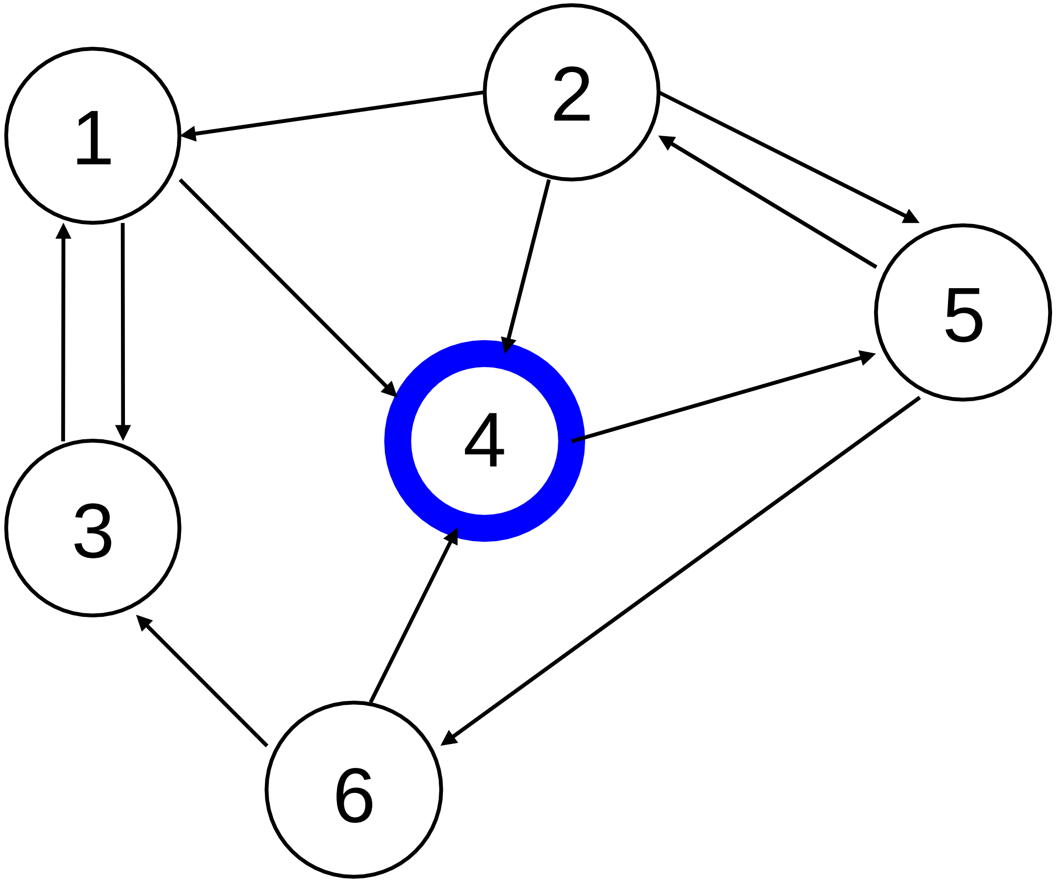


Breadth-first search algorithm

- Mark nodes as they get incorporated into the tree.
- Initially, only i_0 is marked.
- Round 1: i_0 sends search message to out-nbrs.
- At every round: An unmarked node that receives a search message:
  - Marks itself.
  - Designates one process from which it received search as its parent.
  - Sends search to out-nbrs at the next round.

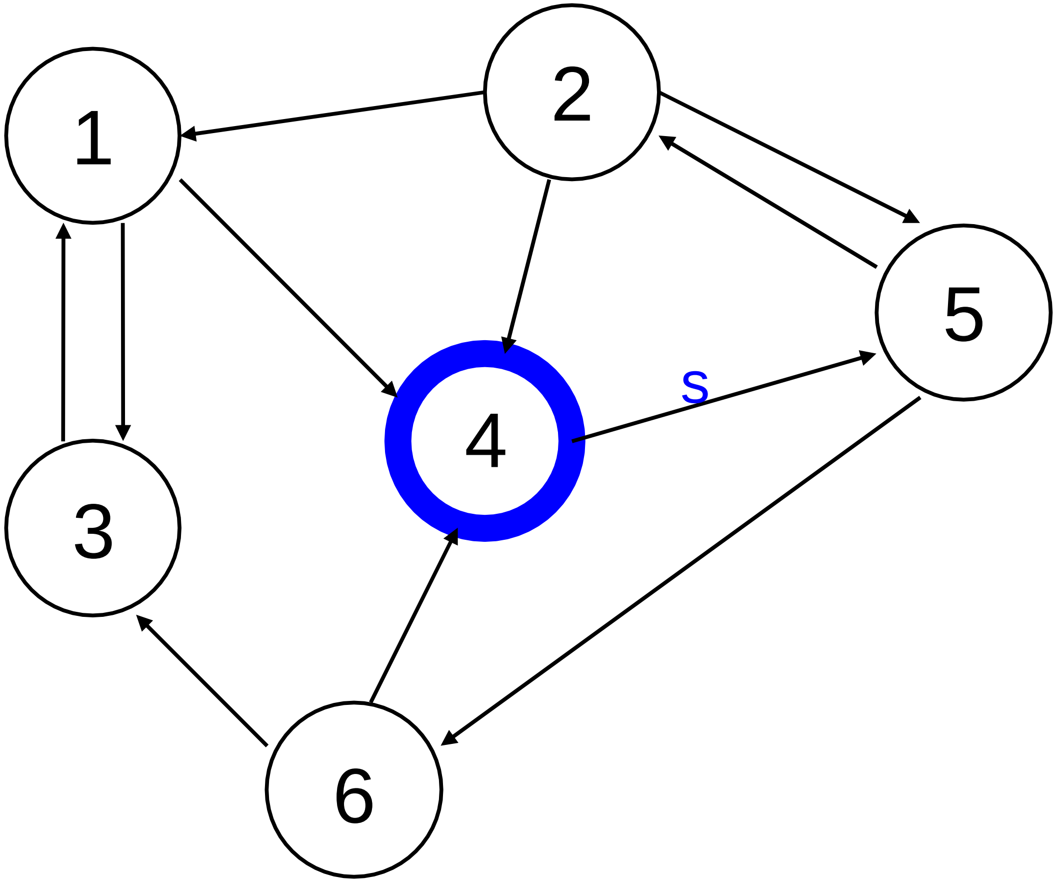
- Q: What state variables do we need?
- Q: Why does this yield a BFS tree?

Breadth-first search



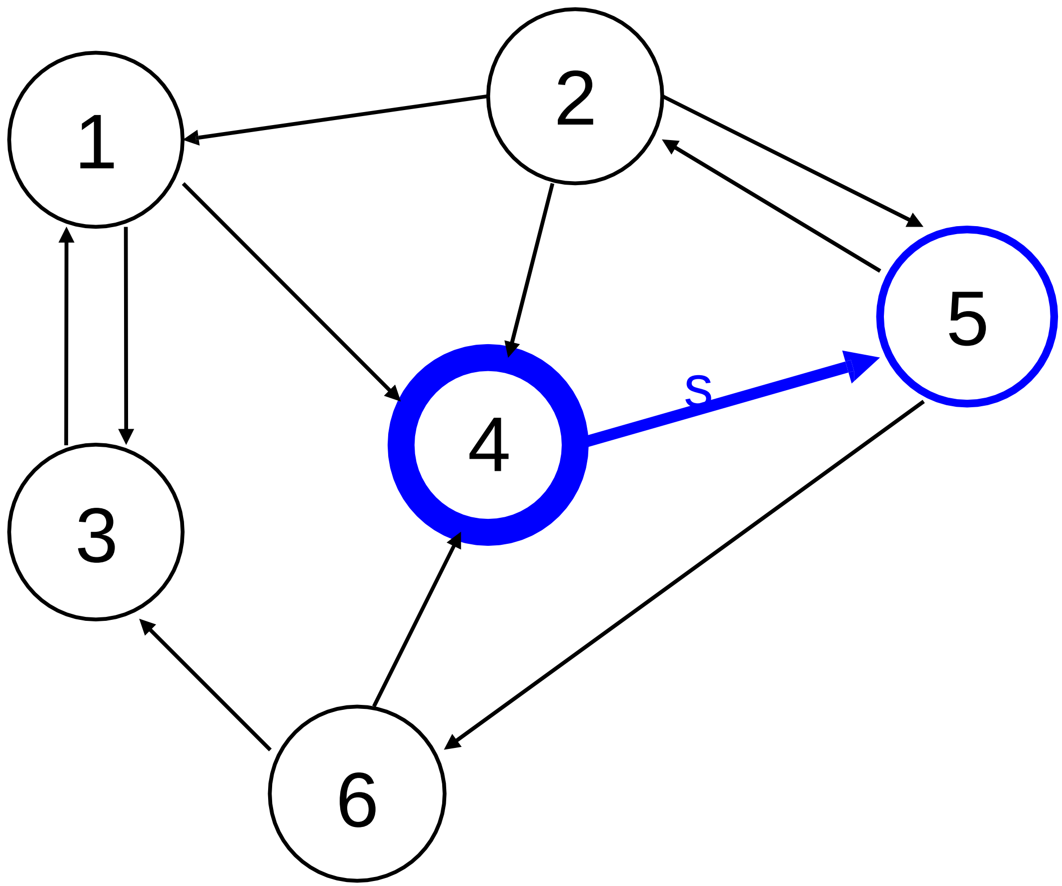
Round 1 (start)

Breadth-first search



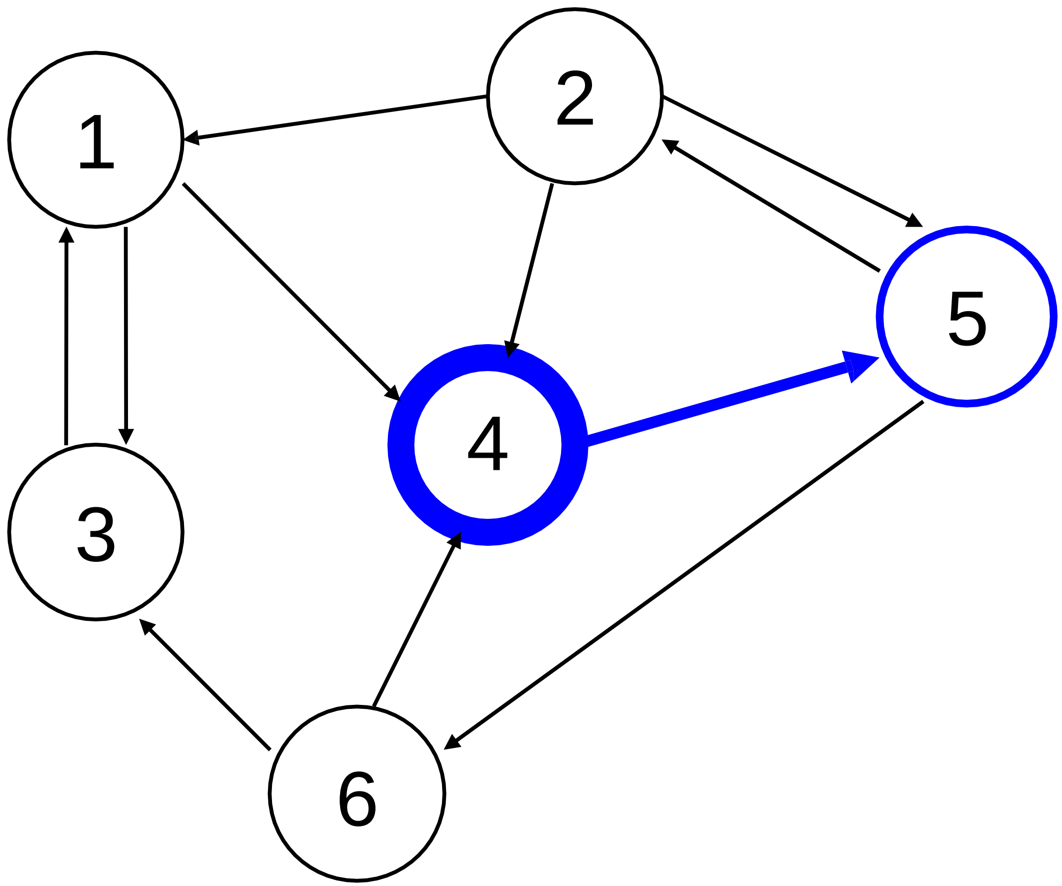
Round 1 (msgs)

Breadth-first search



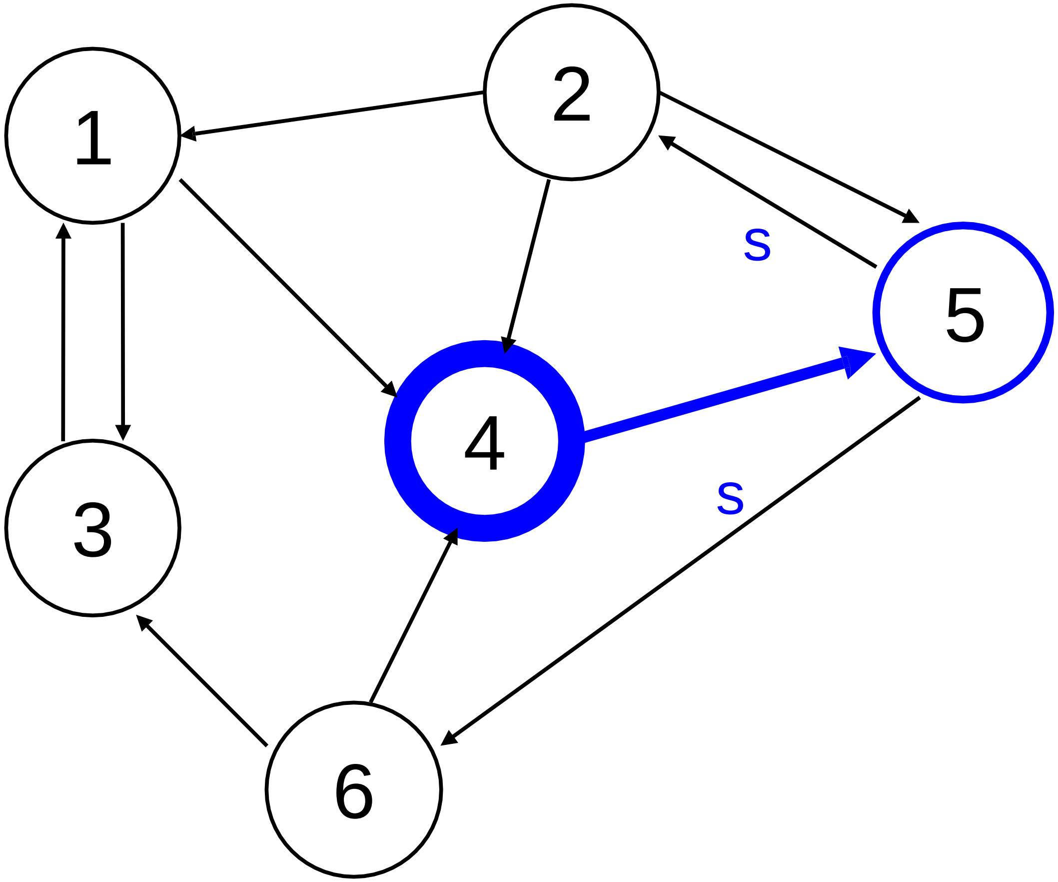
Round 1 (trans)

Breadth-first search



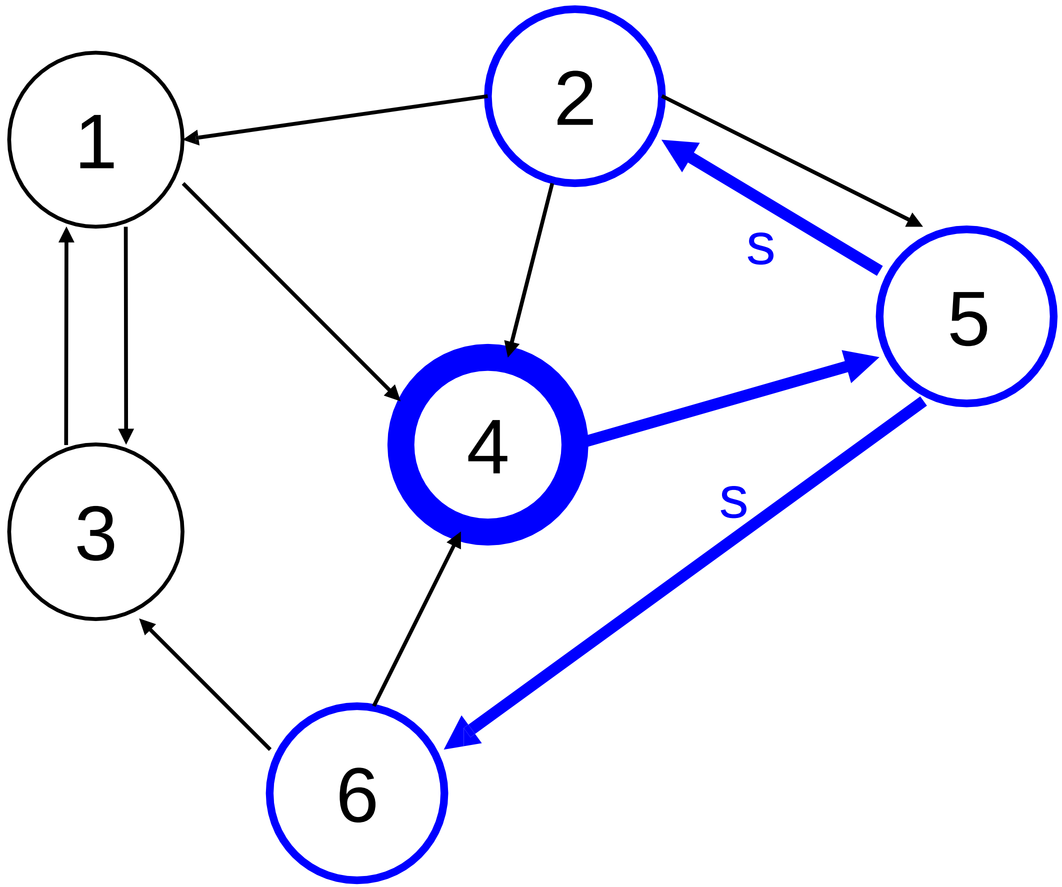
Round 2 (start)

Breadth-first search



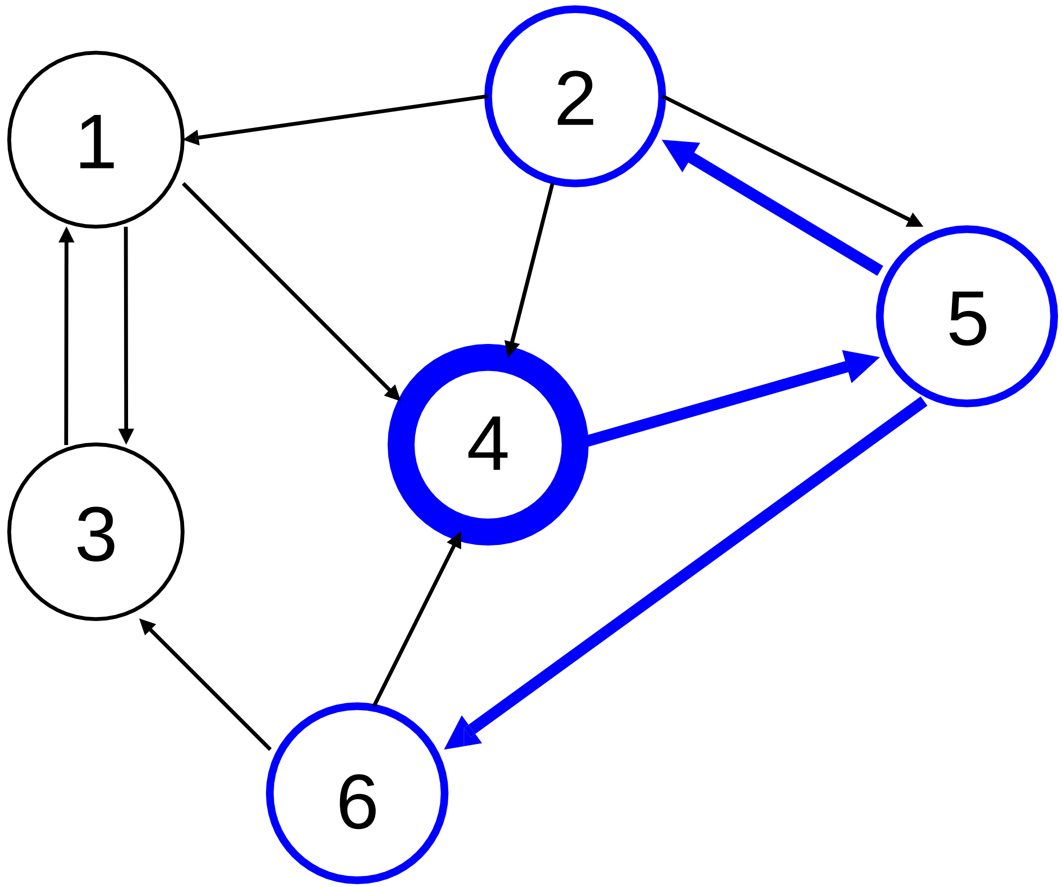
Round 2 (msgs)

Breadth-first search



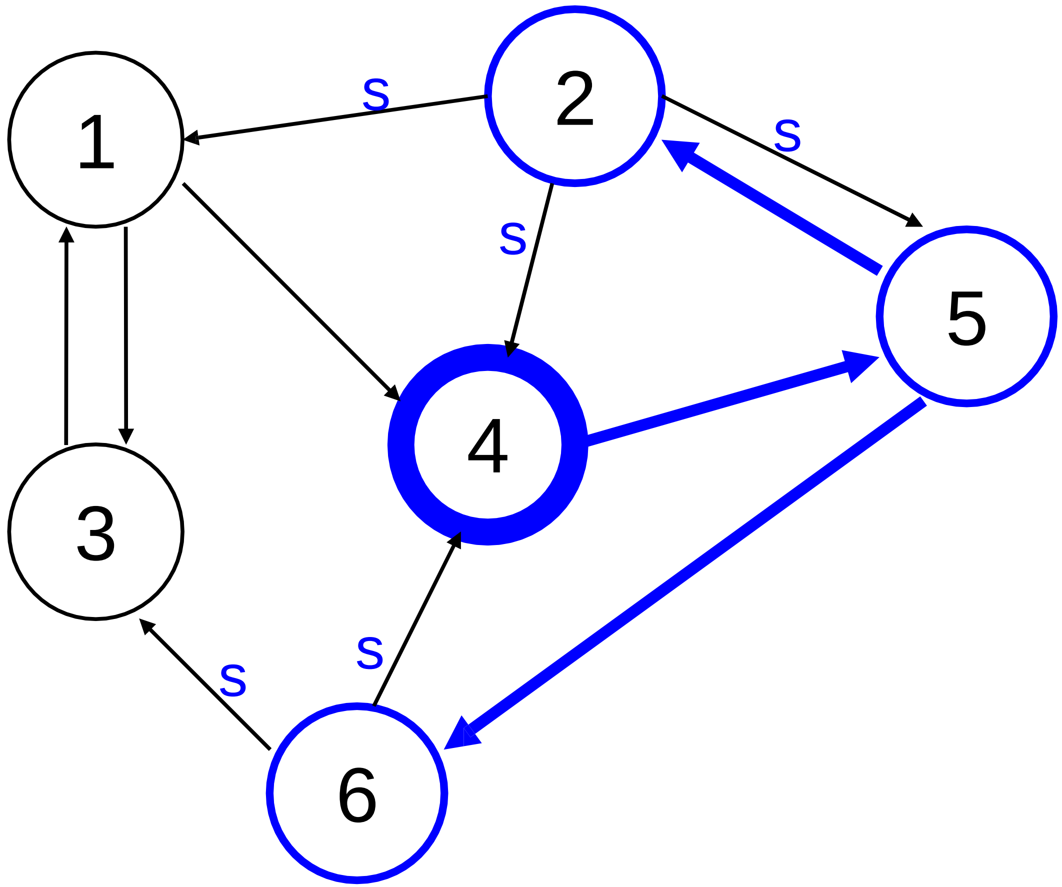
Round 2 (trans)

Breadth-first search



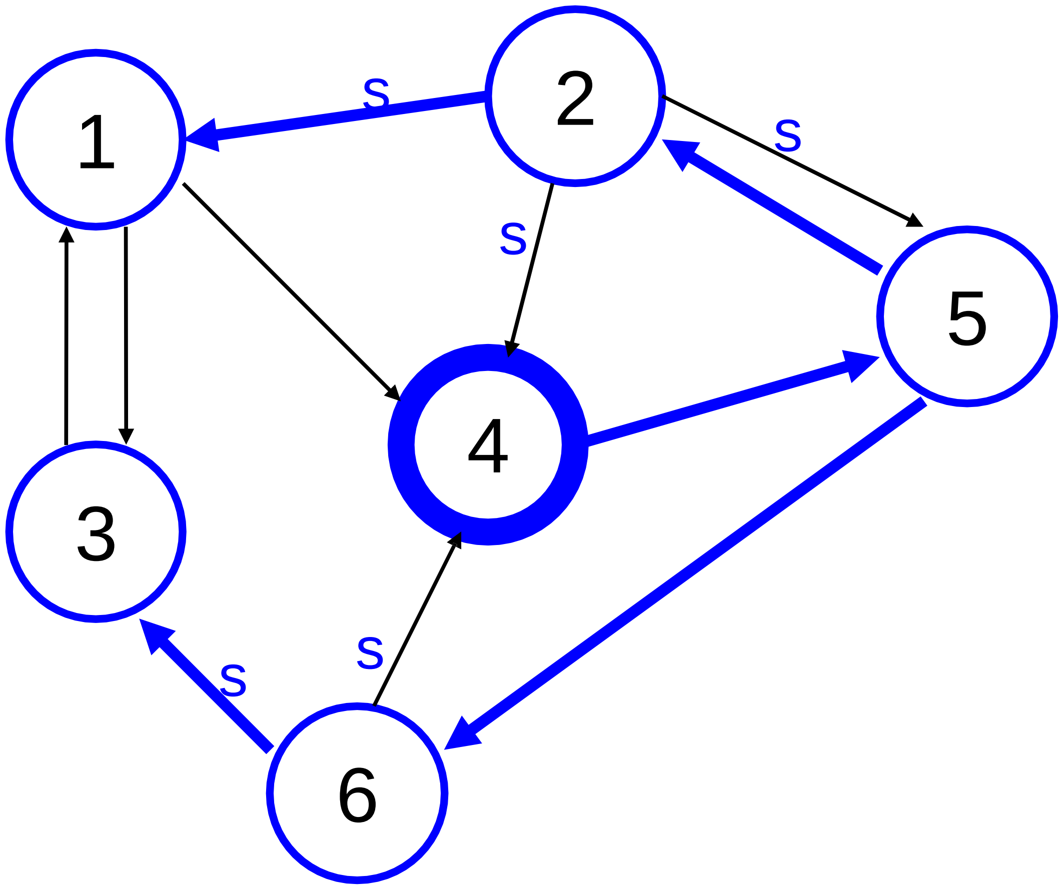
Round 3 (start)

Breadth-first search



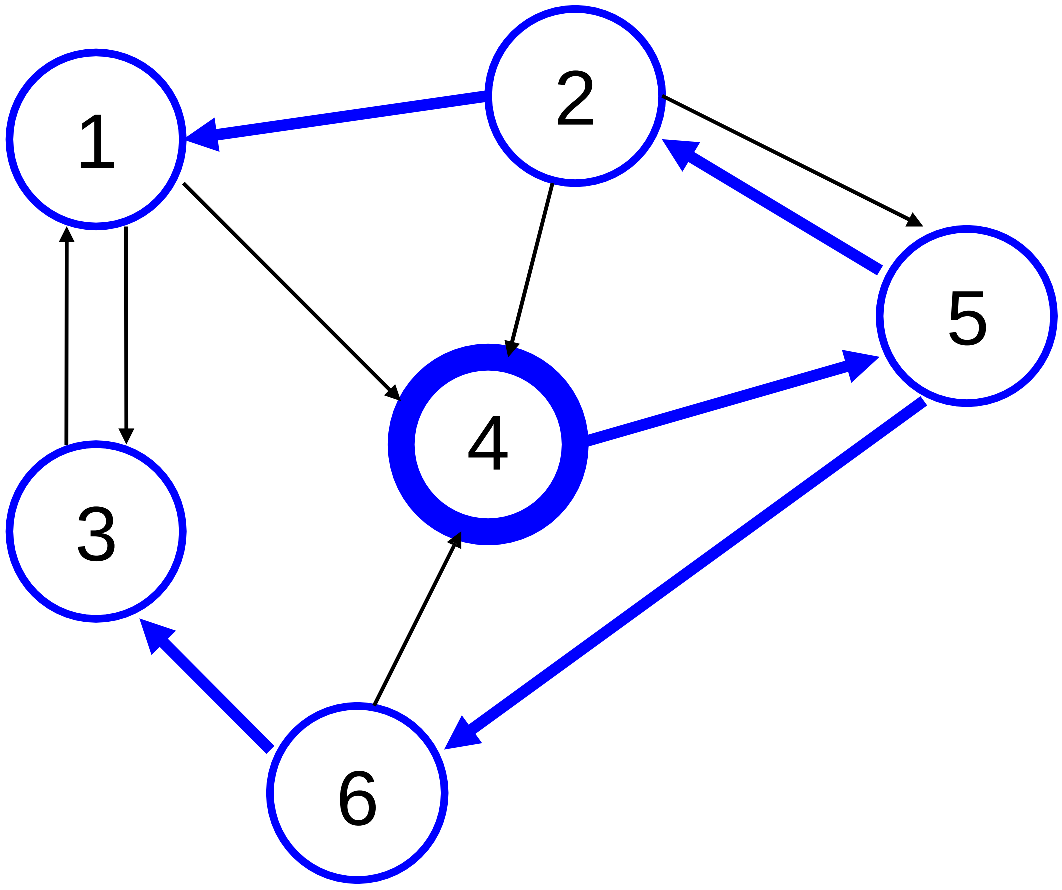
Round 3 (msgs)

Breadth-first search



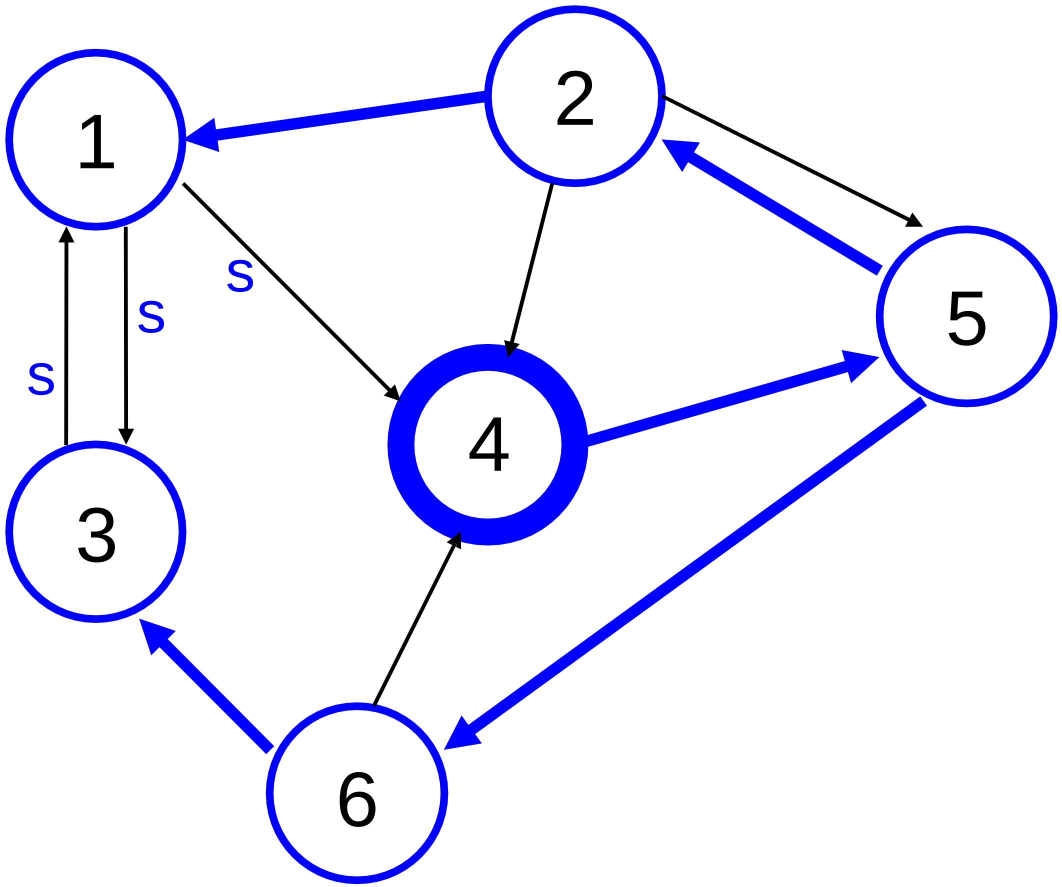
Round 3 (trans)

Breadth-first search



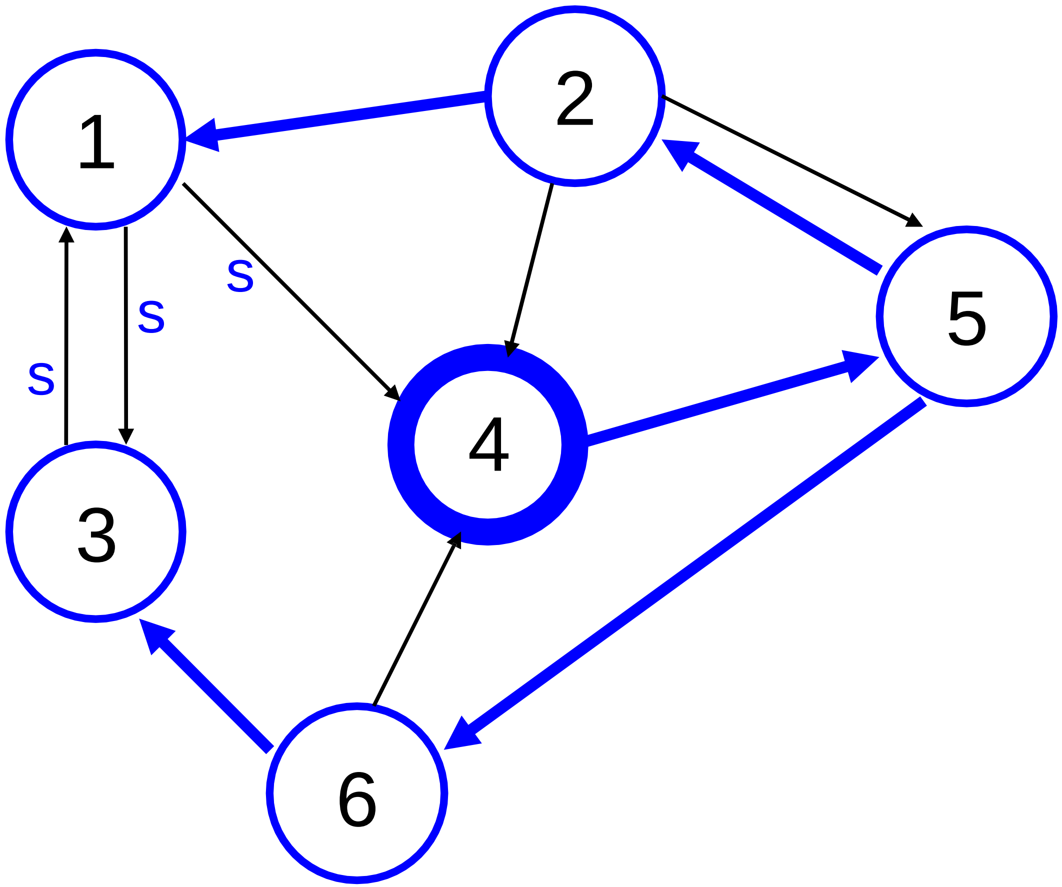
Round 4 (start)

Breadth-first search



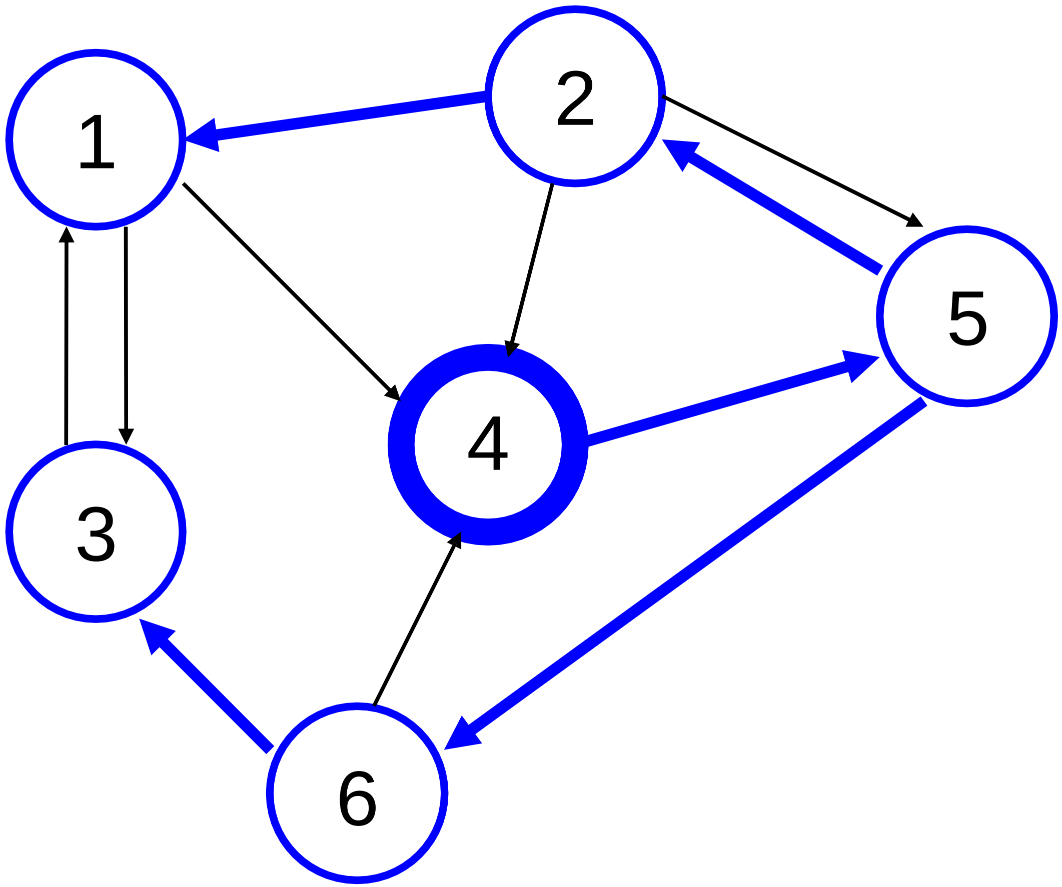
Round 4 (msgs)

Breadth-first search



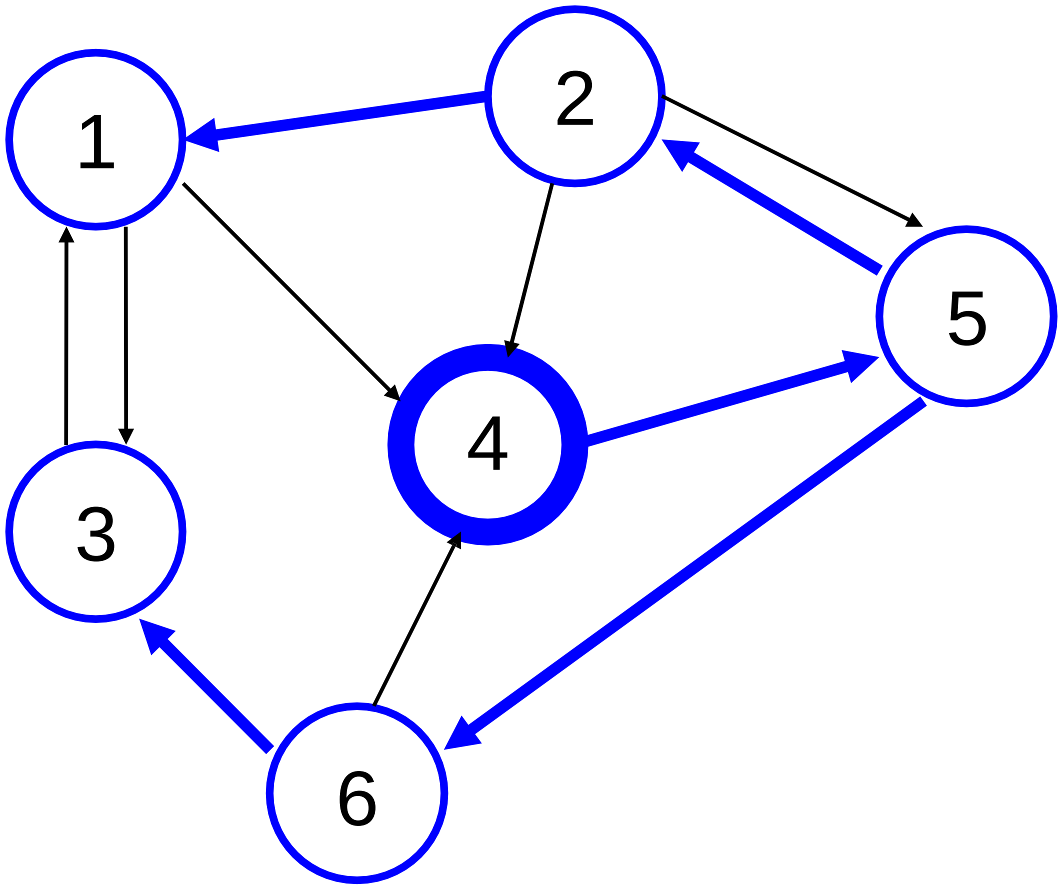
Round 4 (trans)

Breadth-first search



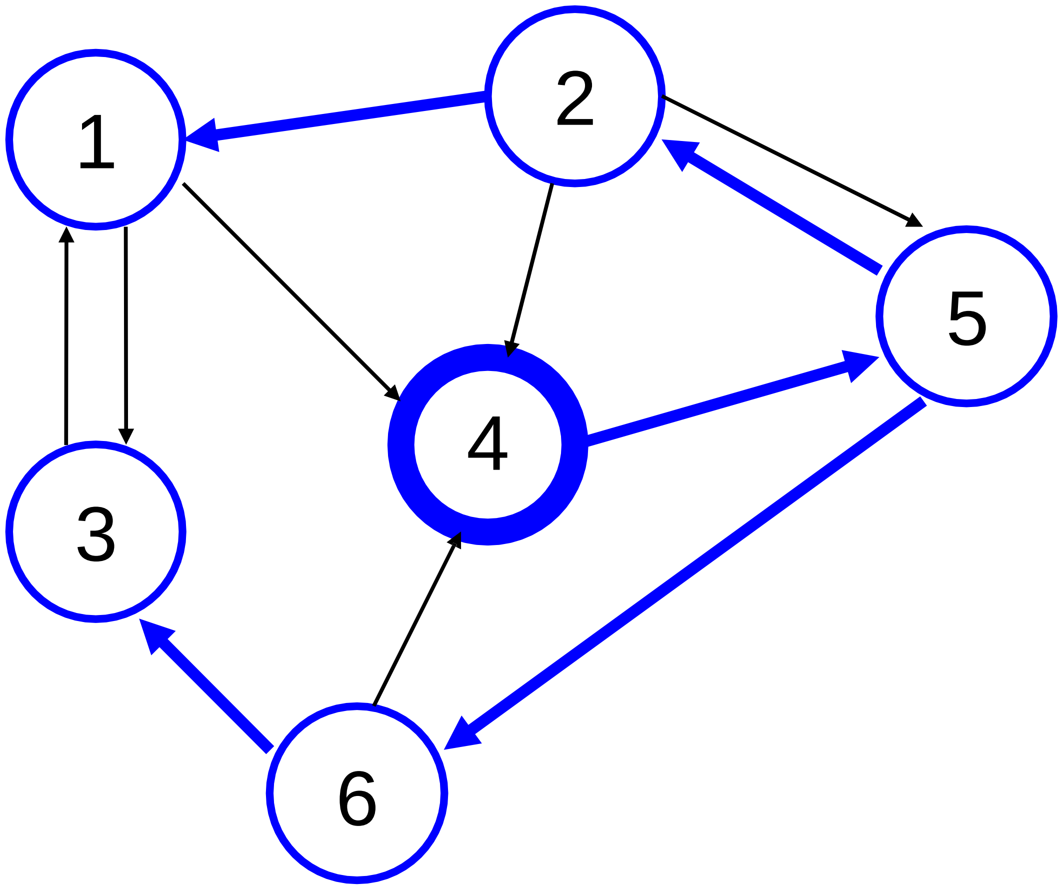
Round 5 (start)

Breadth-first search



Round 5 (msgs)

Breadth-first search



Round 5 (trans)

Breadth-first search algorithm

- Mark nodes as they get incorporated into the tree.
- Initially, only i_0 is marked.
- Round 1: i_0 sends search message to out-nbrs.
- At every round: An unmarked node that receives a search message:
  - Marks itself.
  - Designates one process from which it received search as its parent.
  - Sends search to out-nbrs at the next round.
- Yields a BFS tree because all the branches are created synchronously.
- Complexity: Time = diam + 1; Messages = |E|

BFS, bells and whistles

- Child pointers?
  - Easy with bidirectional communication.
  - What if not?
    - Could use BFS to search for parents.
    - High message bit complexity.
- Termination?
  - With bidirectional communication?
    - "Convergecast"
  - With unidirectional communication?

Applications of BFS

- Message broadcast:
  - Can broadcast a message while setting up the BFS tree ("piggyback" the message).
  - Or, first establish a BFS tree, with child pointers, then use it for broadcasting.
    - Can reuse the tree for many broadcasts
    - Each takes time only O(diameter), messages O(n).

- For the remaining applications, assume bidirectional edges (undirected graph).

Applications of BFS

- Global computation:
  - Sum, max, or any kind of data aggregation: Convergecast on BFS tree.
  - Complexity: Time O(diameter); Messages O(n)/
- Leader election (without knowing diameter):
  - Everyone starts BFS, determines max UID.
  - Complexity: Time O(diam); Messages O(n |E|) (actually, O(diam |E|)).
- Compute diameter:
  - All do BFS.
  - Convergecast to find height of each BFS tree.
  - Convergecast again to find max of all heights.