Termination for BFS

- Suppose i_0 wants to know when the BFS tree is completed.
- Assume each search message receives a response, parent or non-parent.
  - Easy if edges are bidirectional, harder if unidirectional.

T-> O(diam) C->O(E)

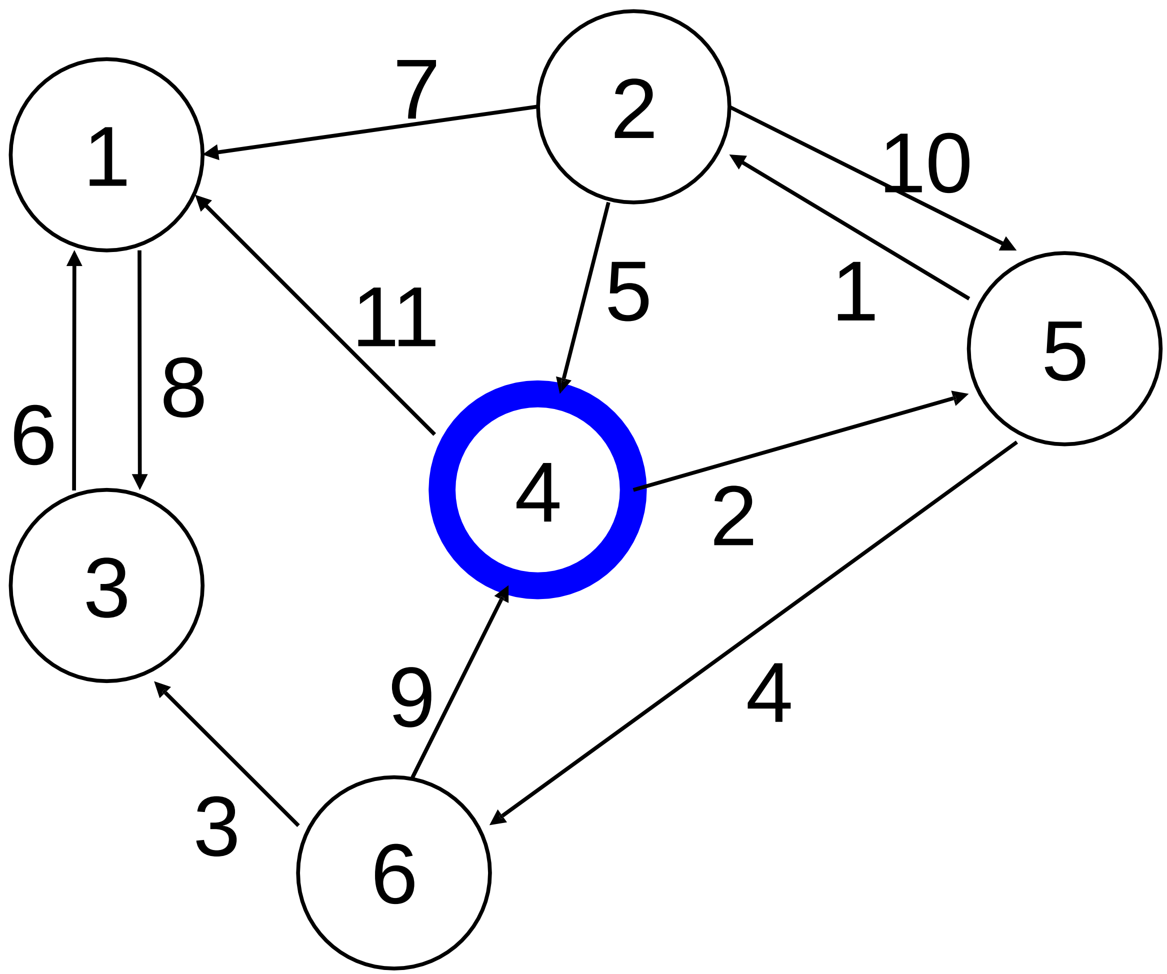
T->O(diam) C->O(diam * E)

- After a node has received responses to all its search messages, it knows who its children are, and knows they are all marked.
- Leaves of the tree discover who they are (receive all non-parent responses).
- Starting from the leaves, fan in complete messages to i_0.
- Node can send complete message after:
  - It has receives responses to all its search messages (so it knows who its children are), and
  - It has received complete messages from all its children.

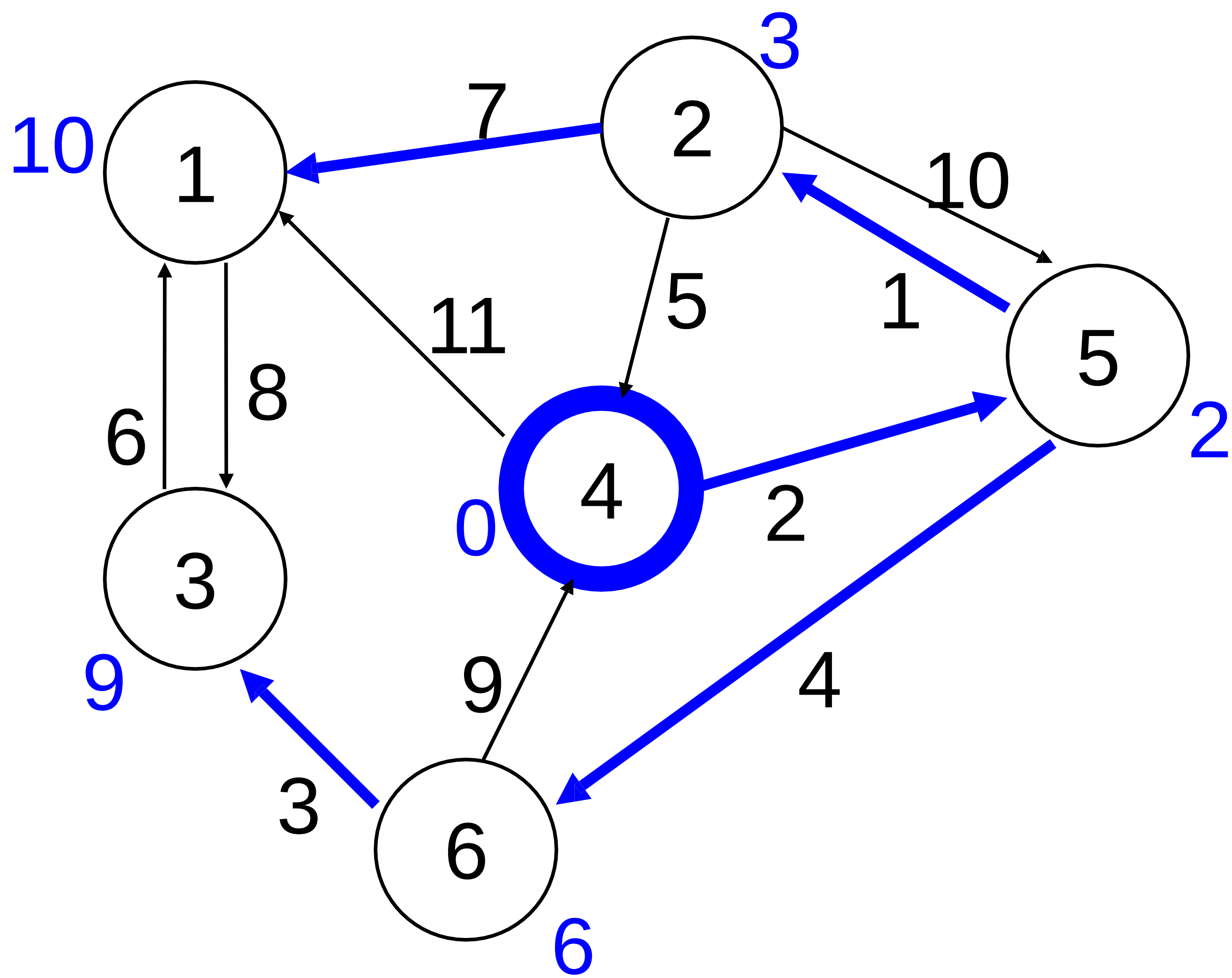
Shortest paths

- Motivation: Establish structure for efficient communication.
  - Generalization of Breadth-First Search.
  - Now edges have associated costs (weights).
- Assume:
  - Strongly connected digraph, root i_0.
  - Weights (nonnegative reals) on edges.
    - Weights represent some communication cost, e.g. latency.
  - UIDs.
  - Nodes know weights of incident edges.
  - Nodes know n (need for termination).
- Required:
  - Shortest-paths tree, giving shortest paths from i_0 to every other node.
  - Shortest path = path with minimum total weight.
  - Each node should output parent, “distance” from root (by weight).

Shortest paths



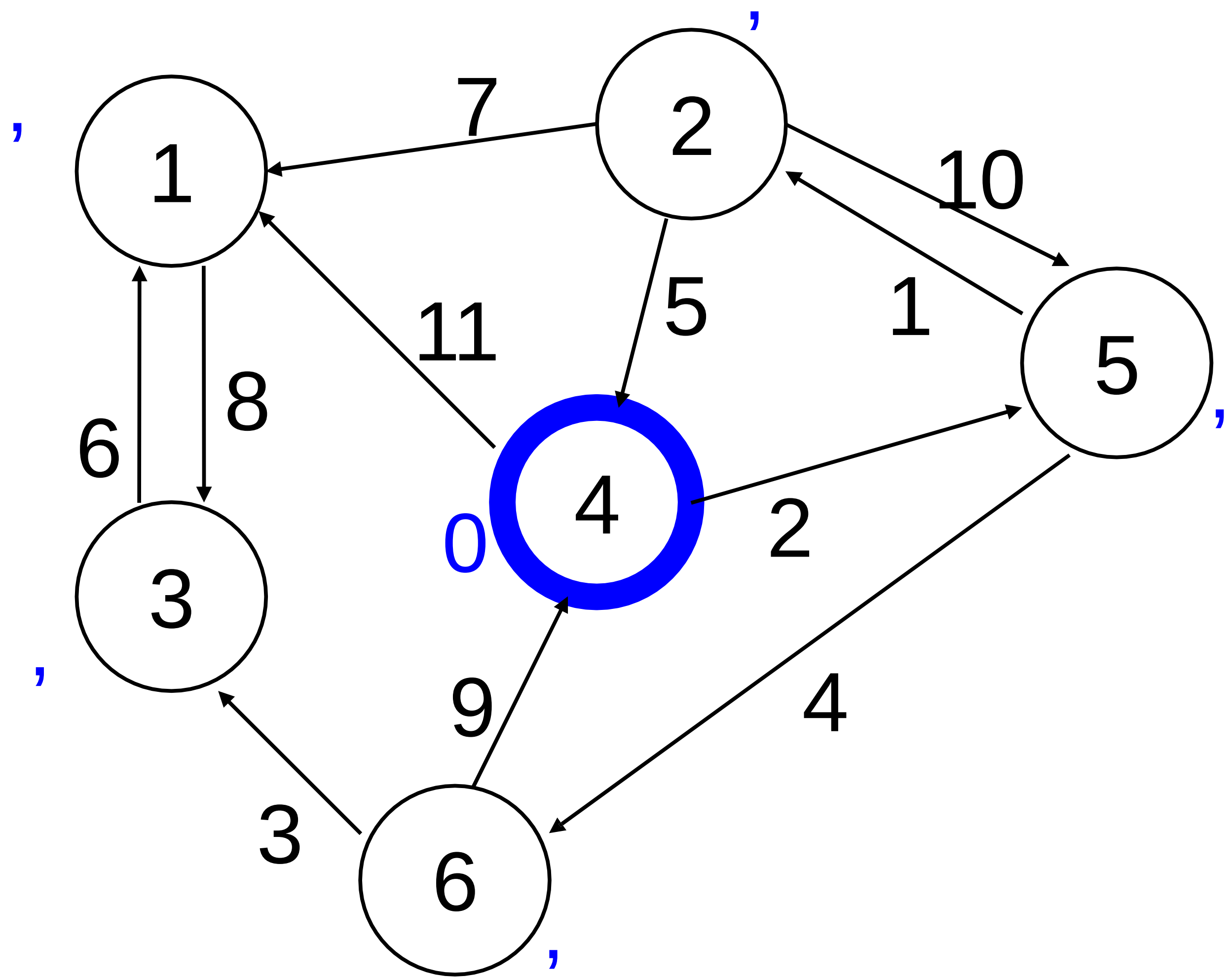
Shortest paths



Shortest paths algorithm

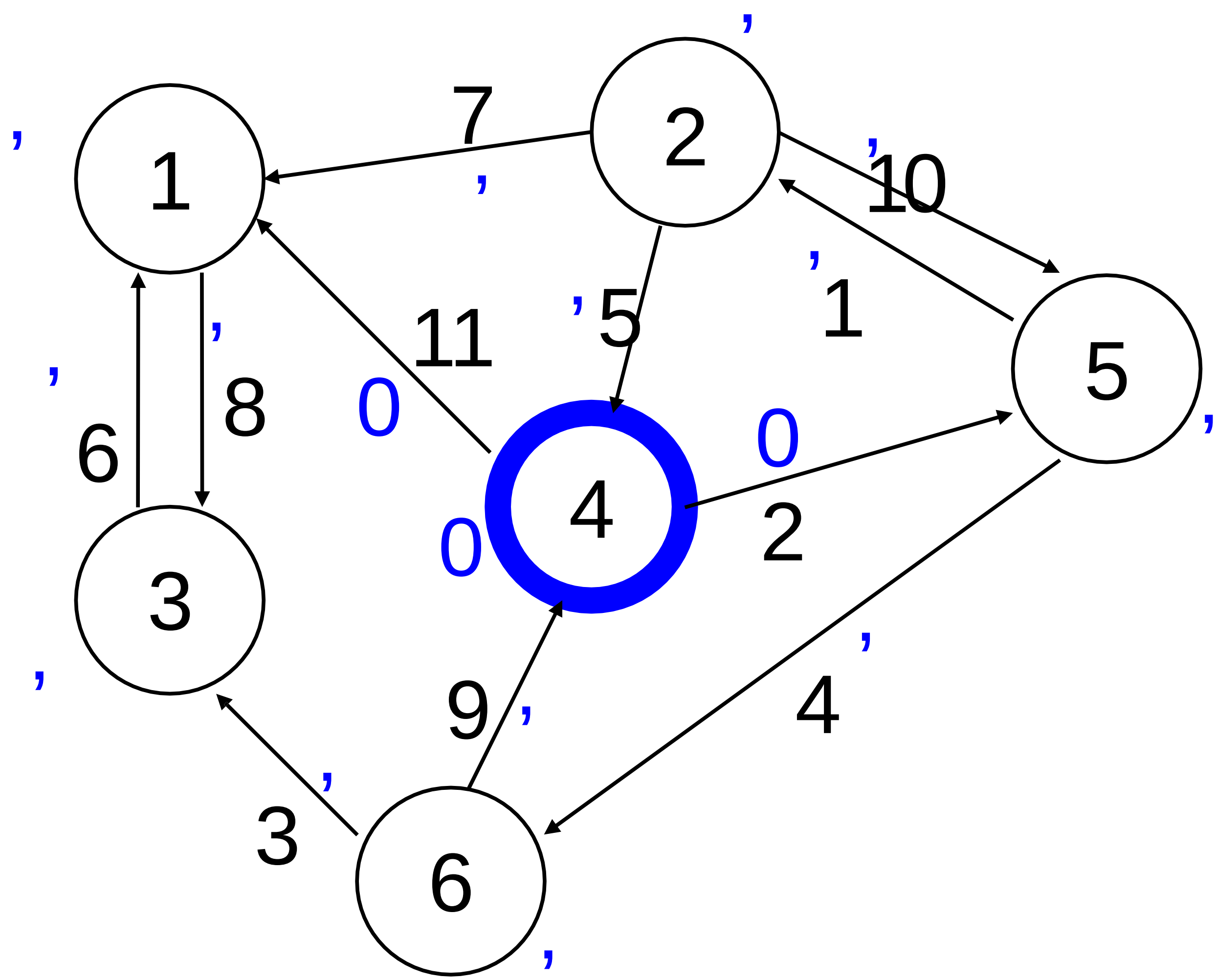
- Bellman-Ford (adapted from sequential algorithm)
- "Relaxation algorithm"
- Each node maintains:
  - dist, shortest distance it knows about so far, from i_0
  - parent, its parent in some path with total weight = dist
  - round number
- Initially i_0 has dist 0, all others ;parents all null
- At each round, each node:
  - Send dist to all out-nbrs
  - Relaxation step:
    - Compute new dist = min(dist, $\min_j(d_j + w_{ji})$).
    - Update parent if dist changes.
- Stop after n-1 rounds
- Then (claim) dist contains shortest distance, parent contains parent in a shortest-paths tree.

Shortest paths



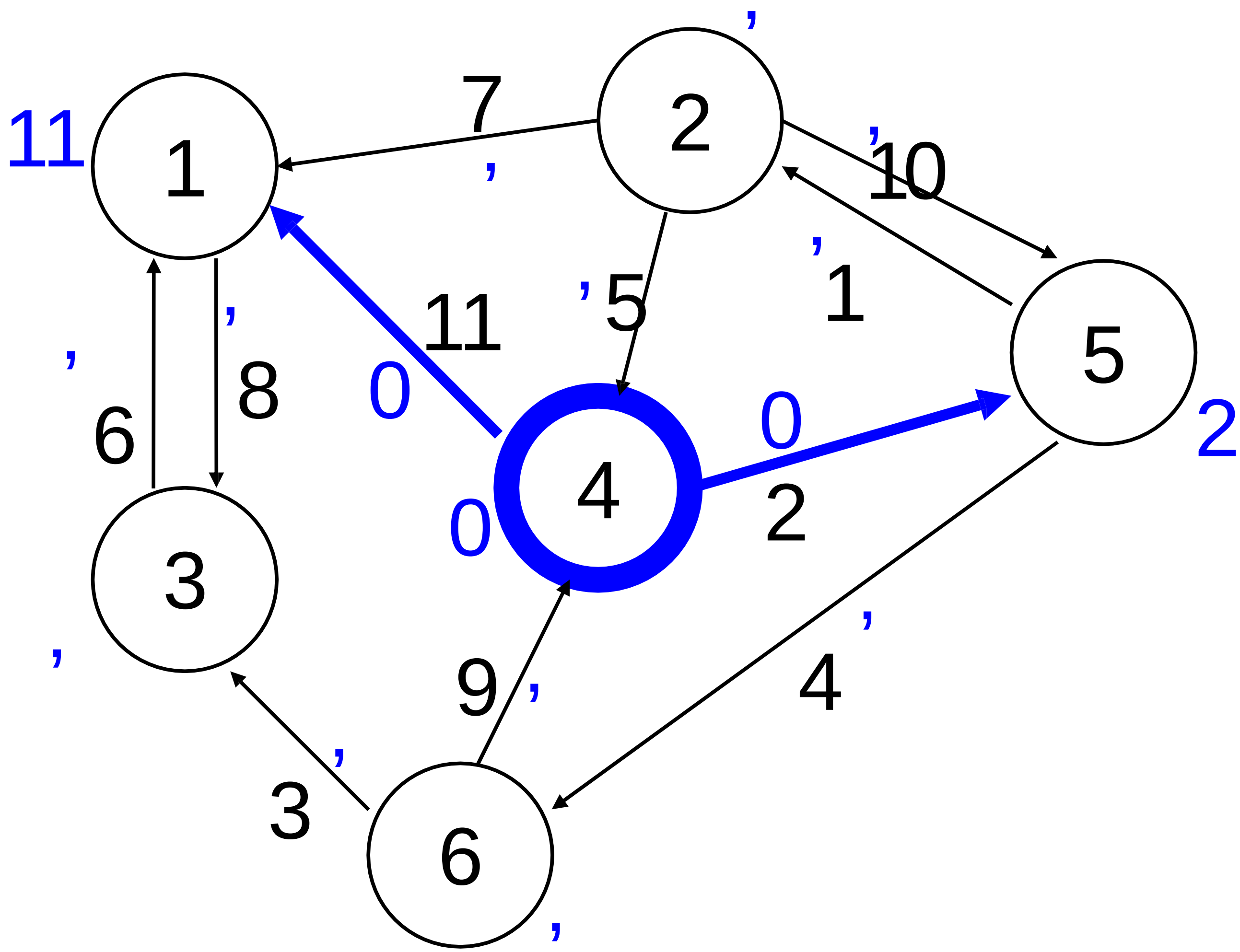
Round 1 (start)

Shortest paths



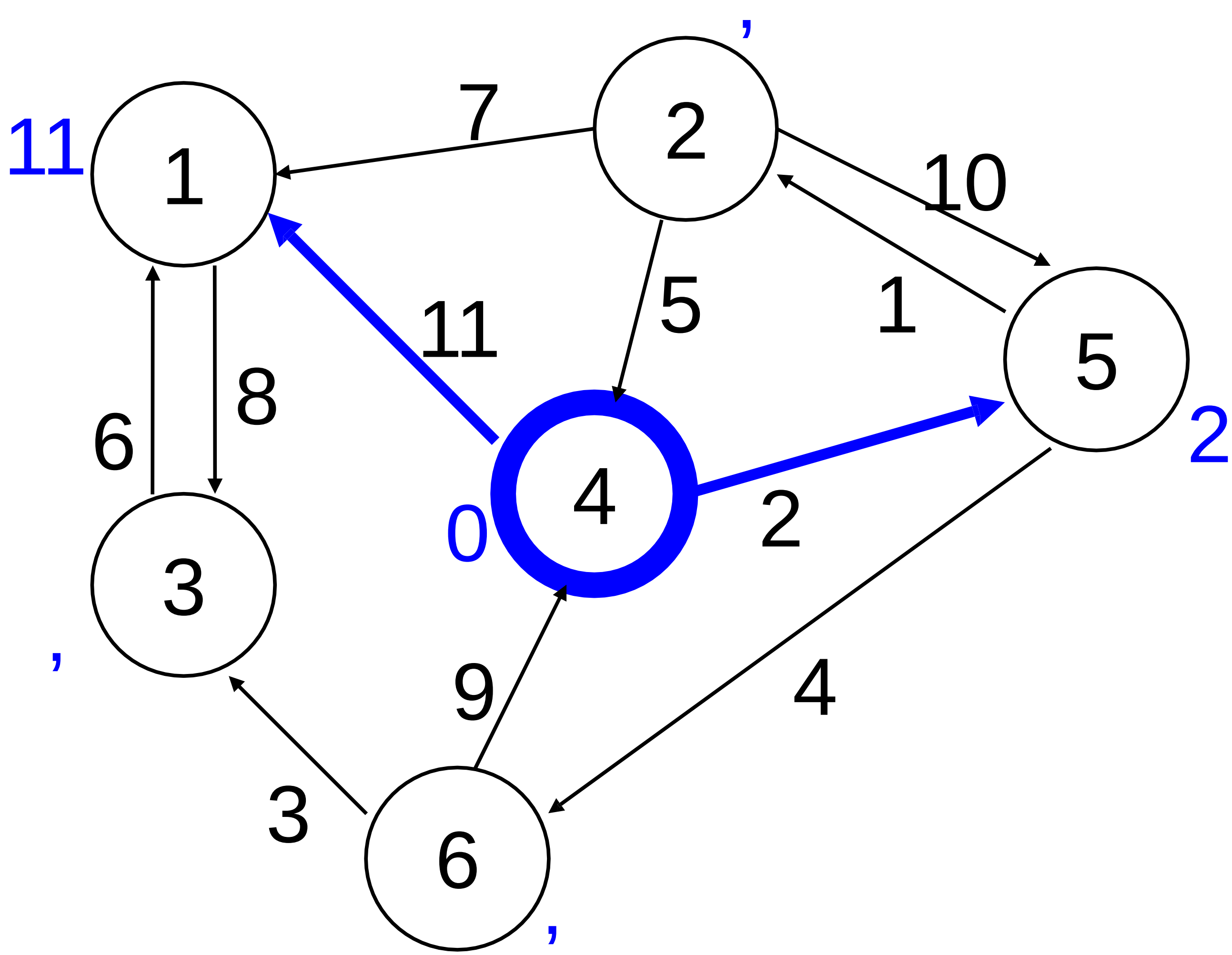
Round 1 (msgs)

Shortest paths



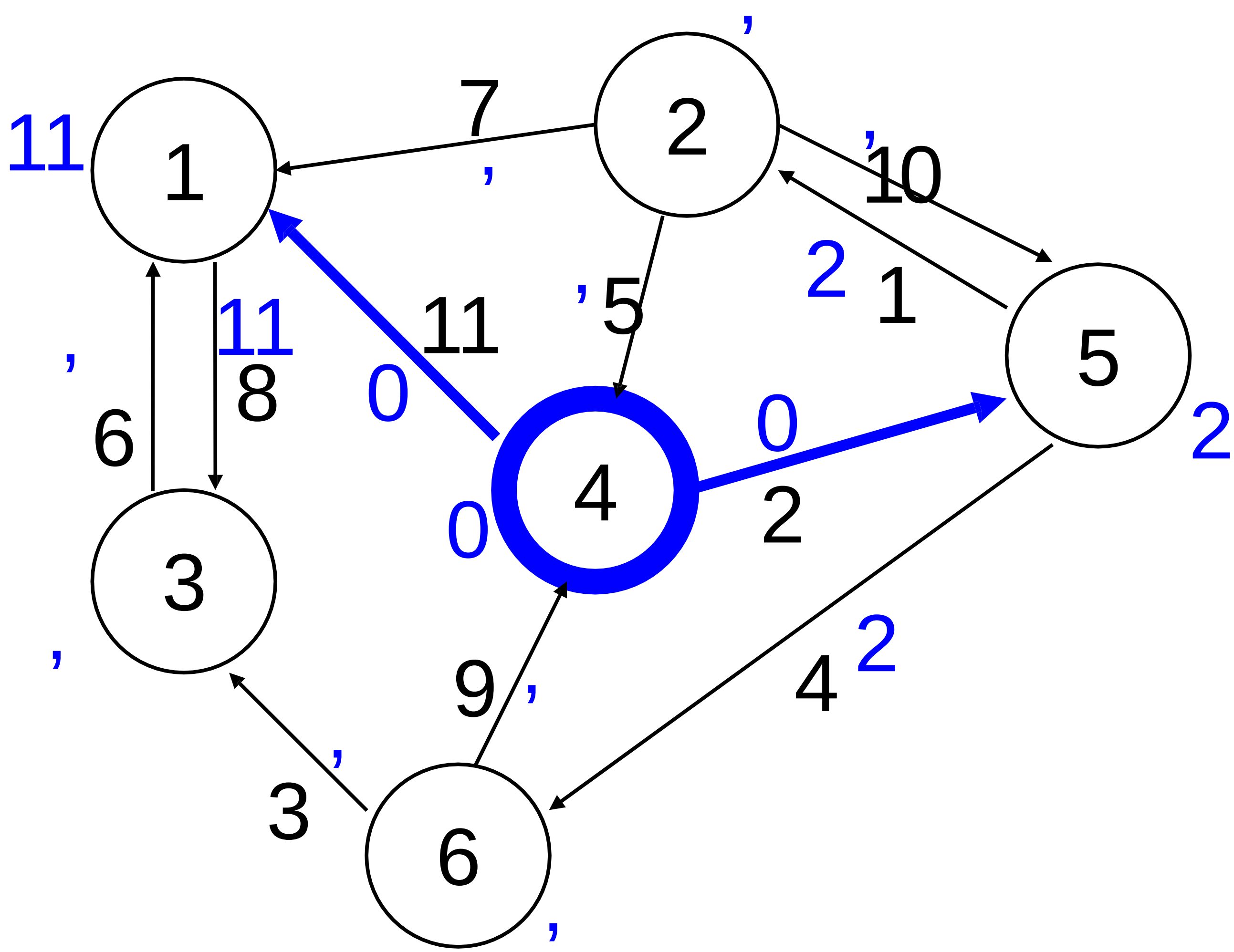
Round 1 (trans)

Shortest paths



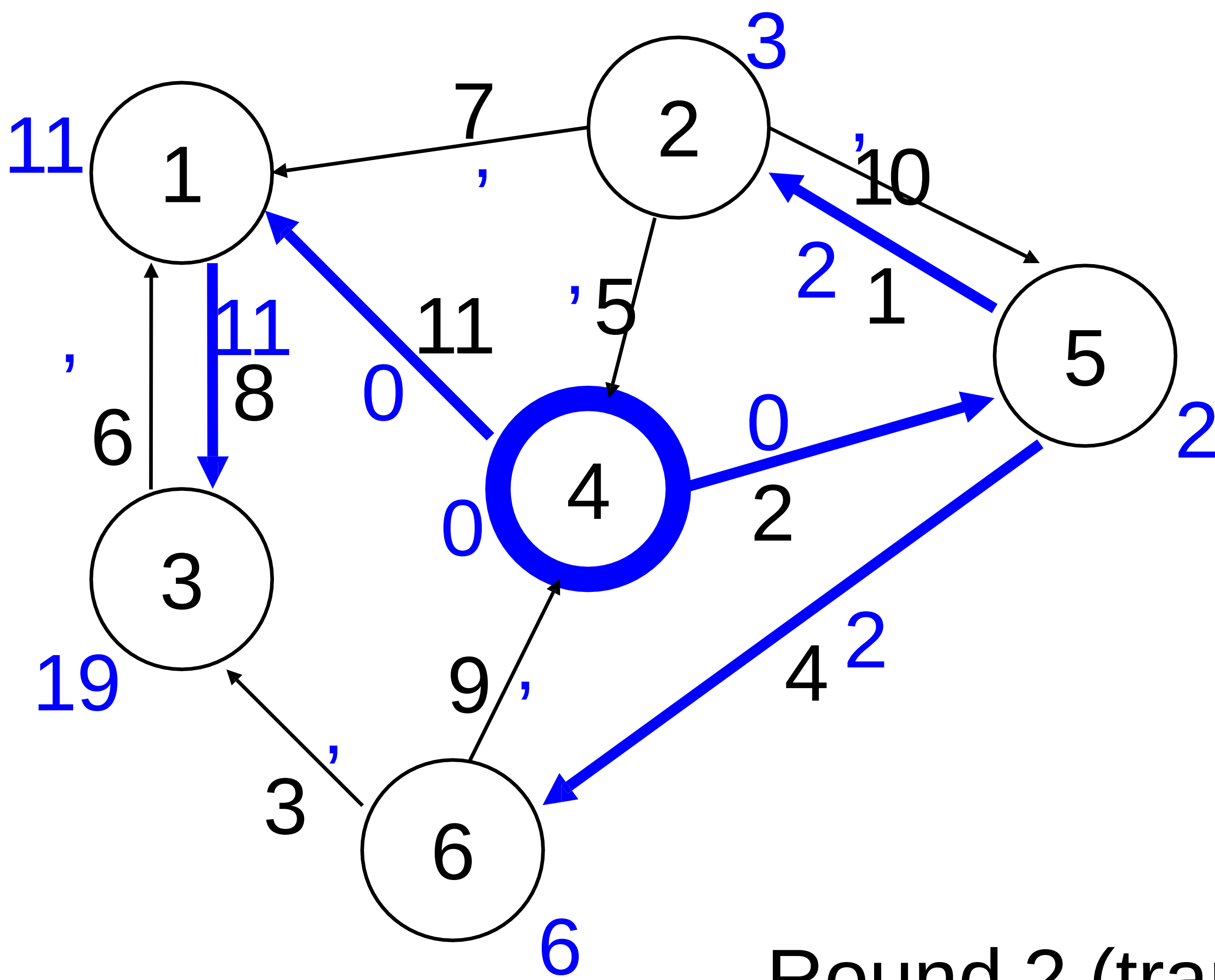
Round 2 (start)

Shortest paths



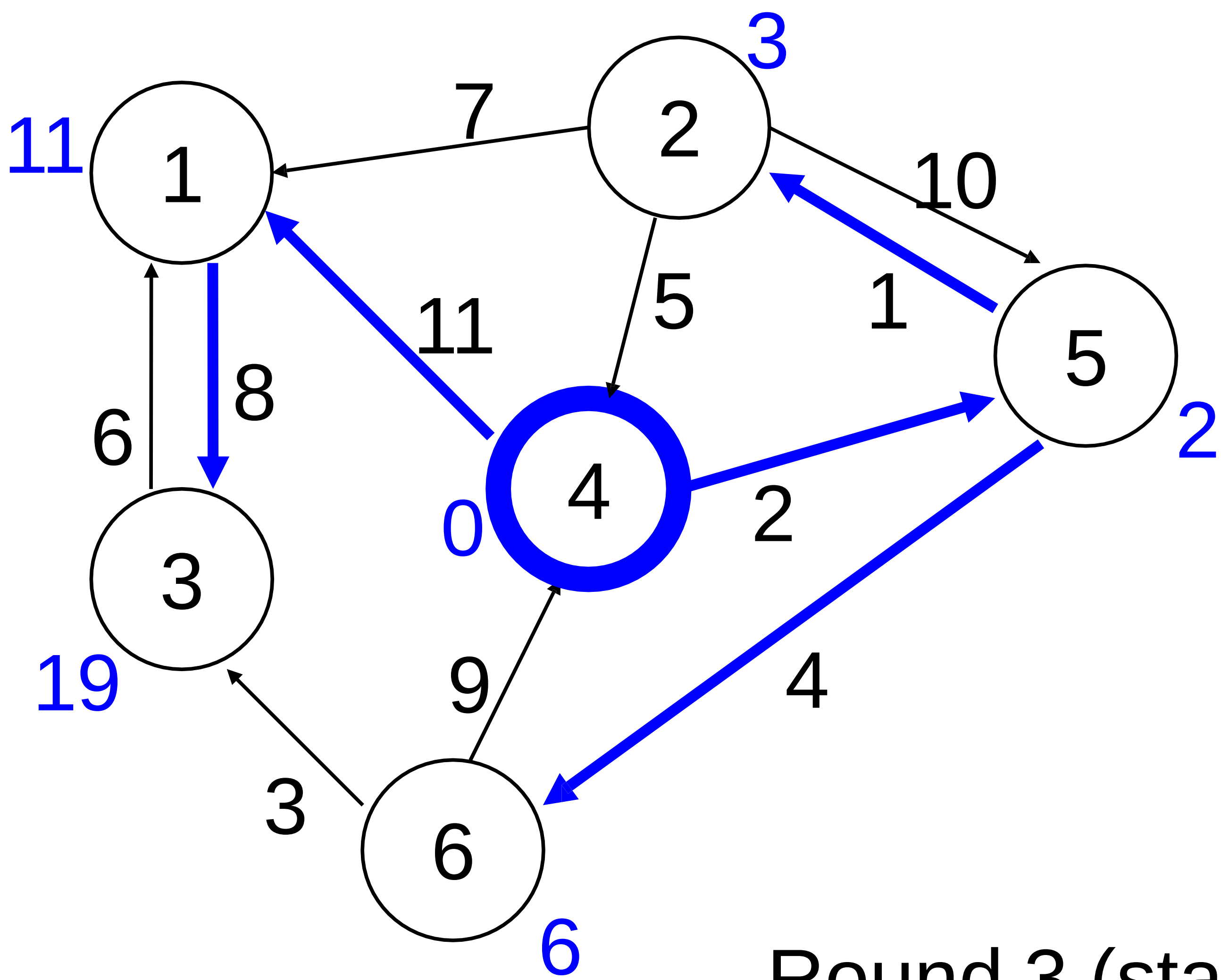
Round 2 (msgs)

Shortest paths



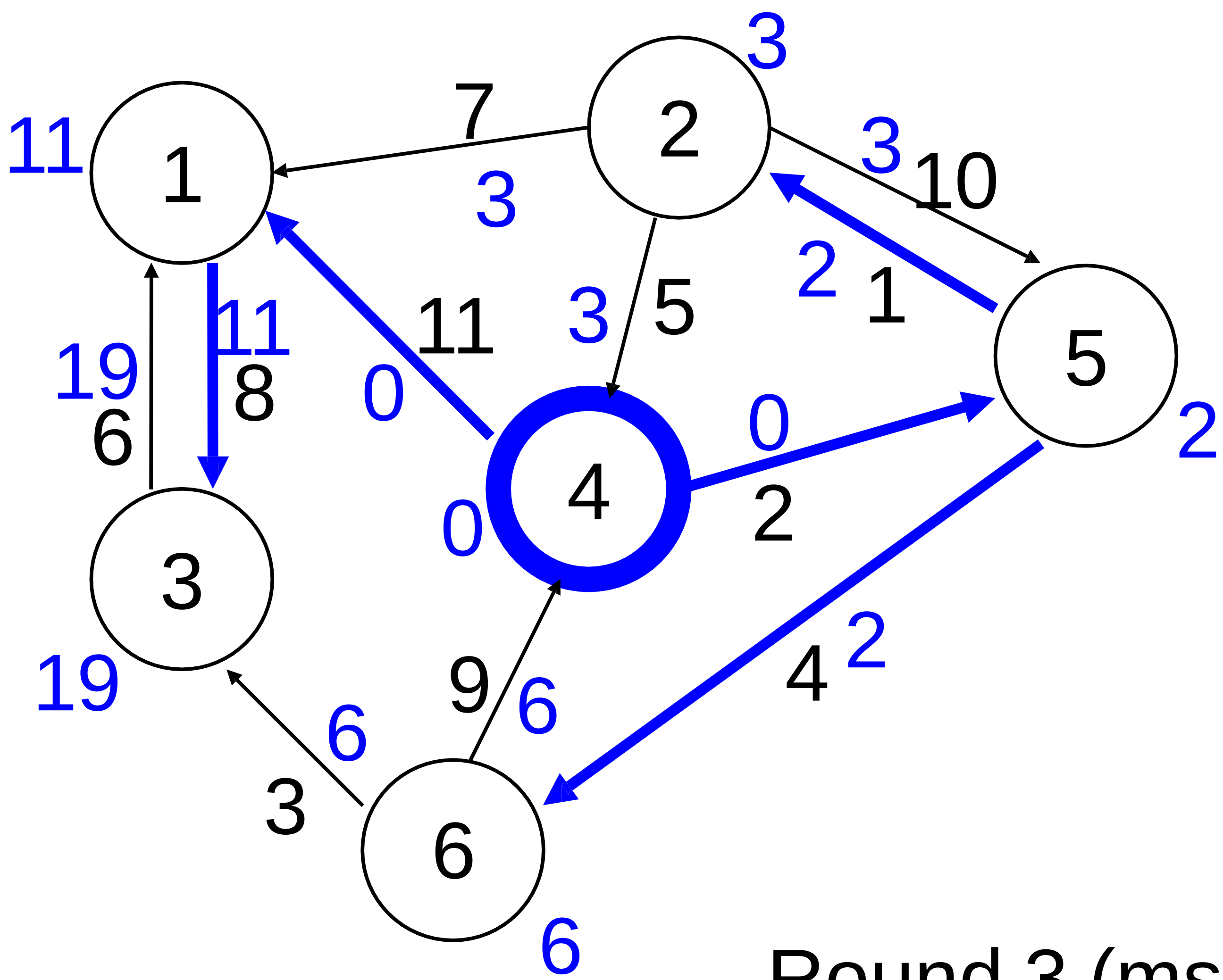
Round 2 (trans)

Shortest paths



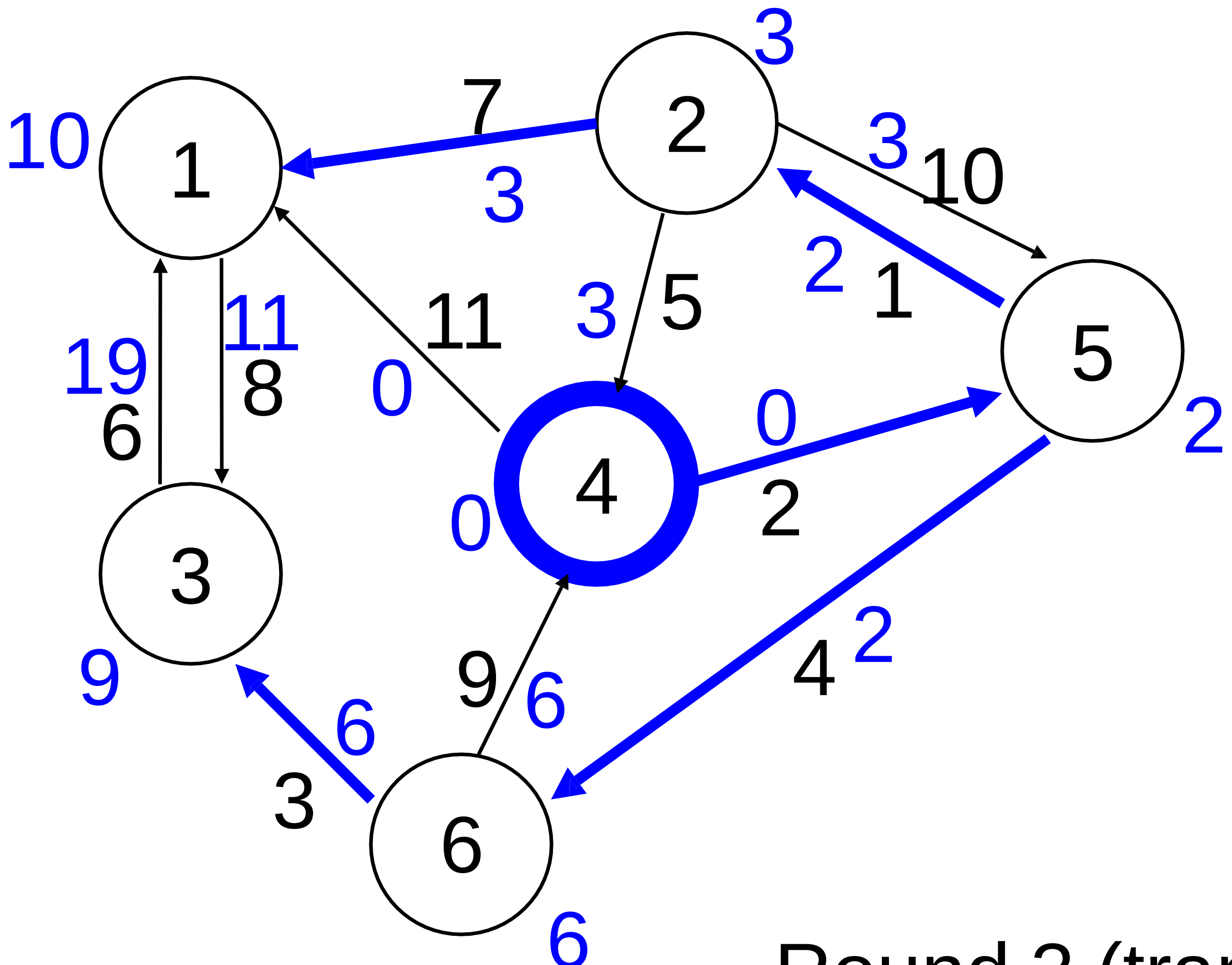
Round 3 (start)

Shortest paths



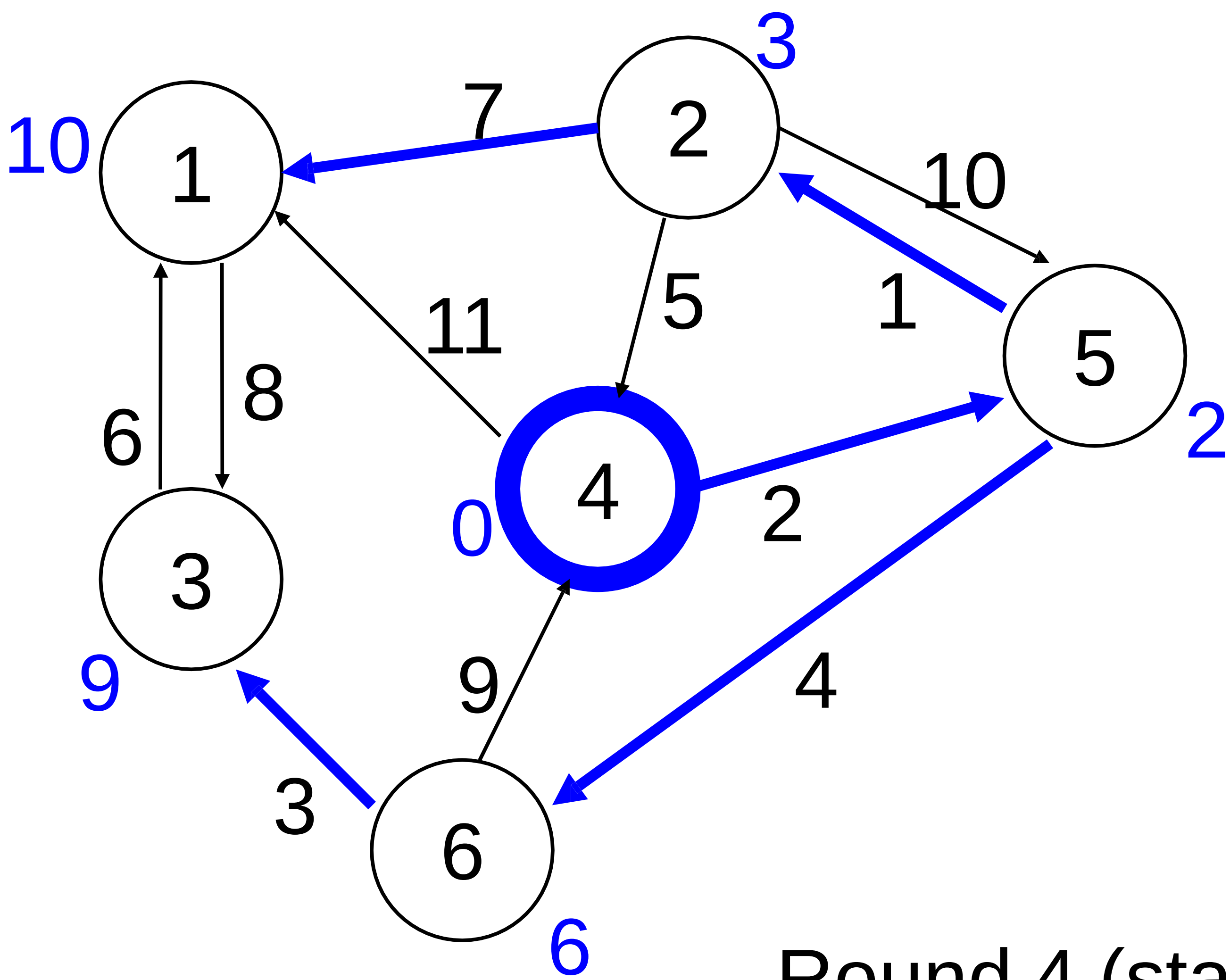
Round 3 (msgs)

Shortest paths



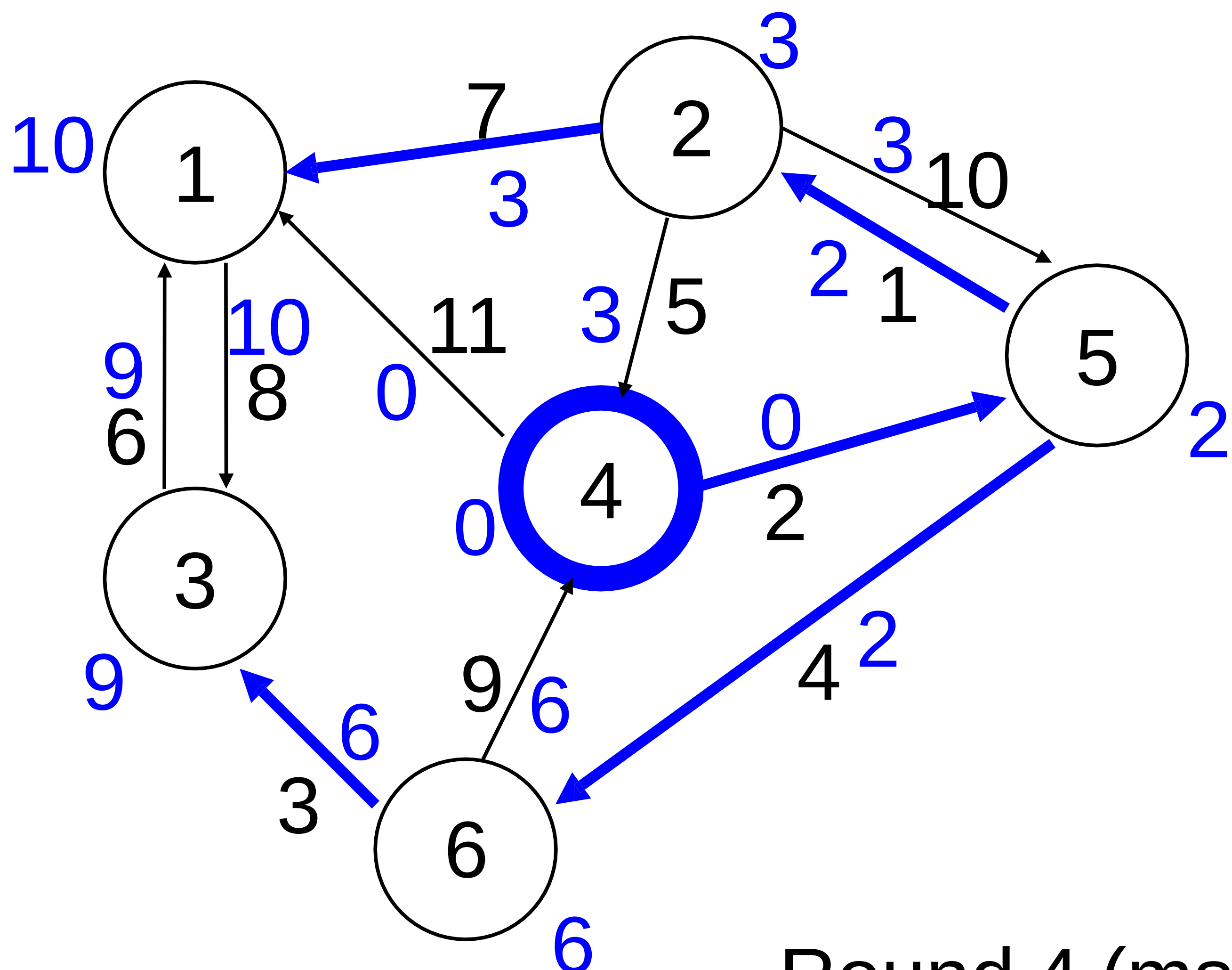
Round 3 (trans)

Shortest paths



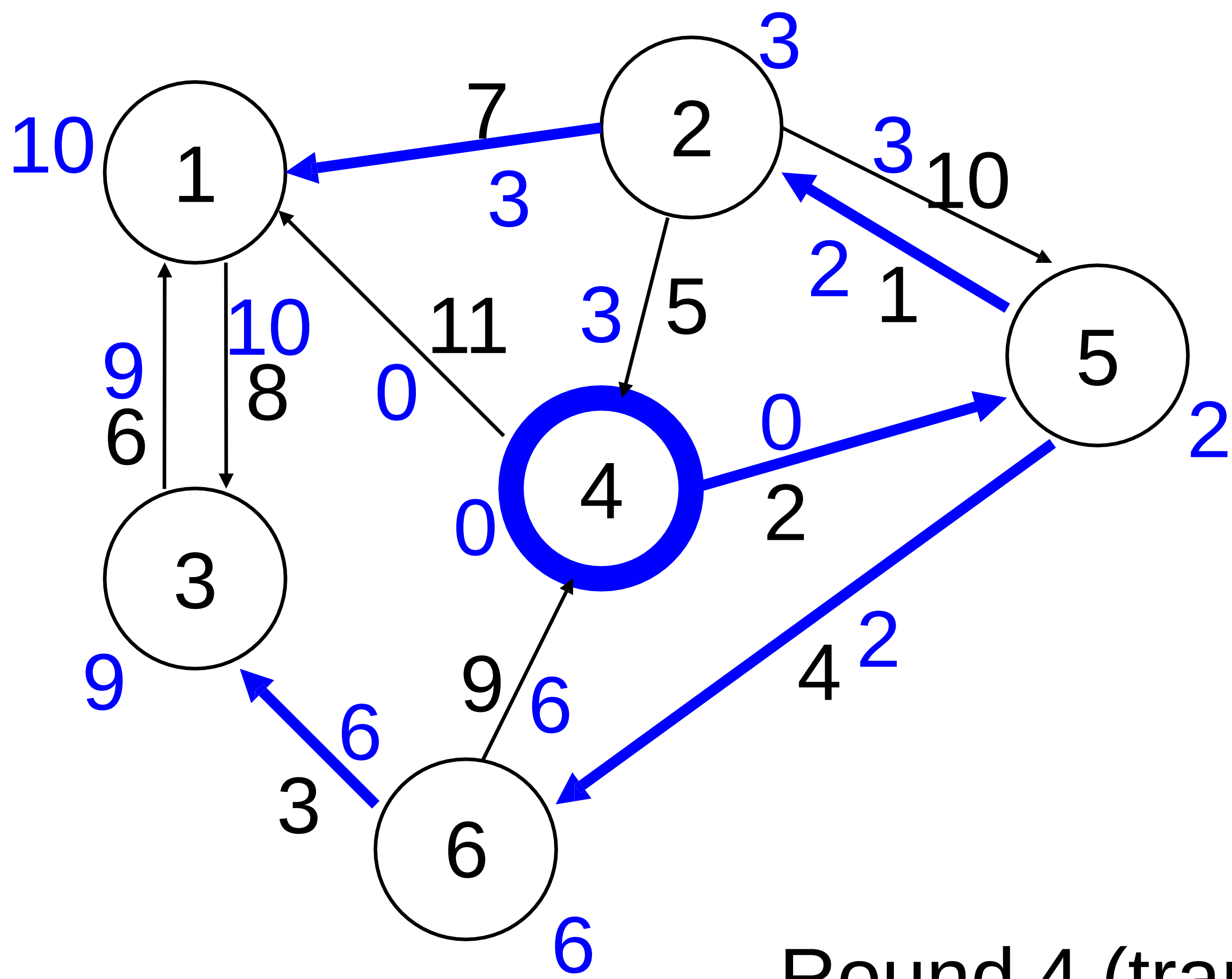
Round 4 (start)

Shortest paths



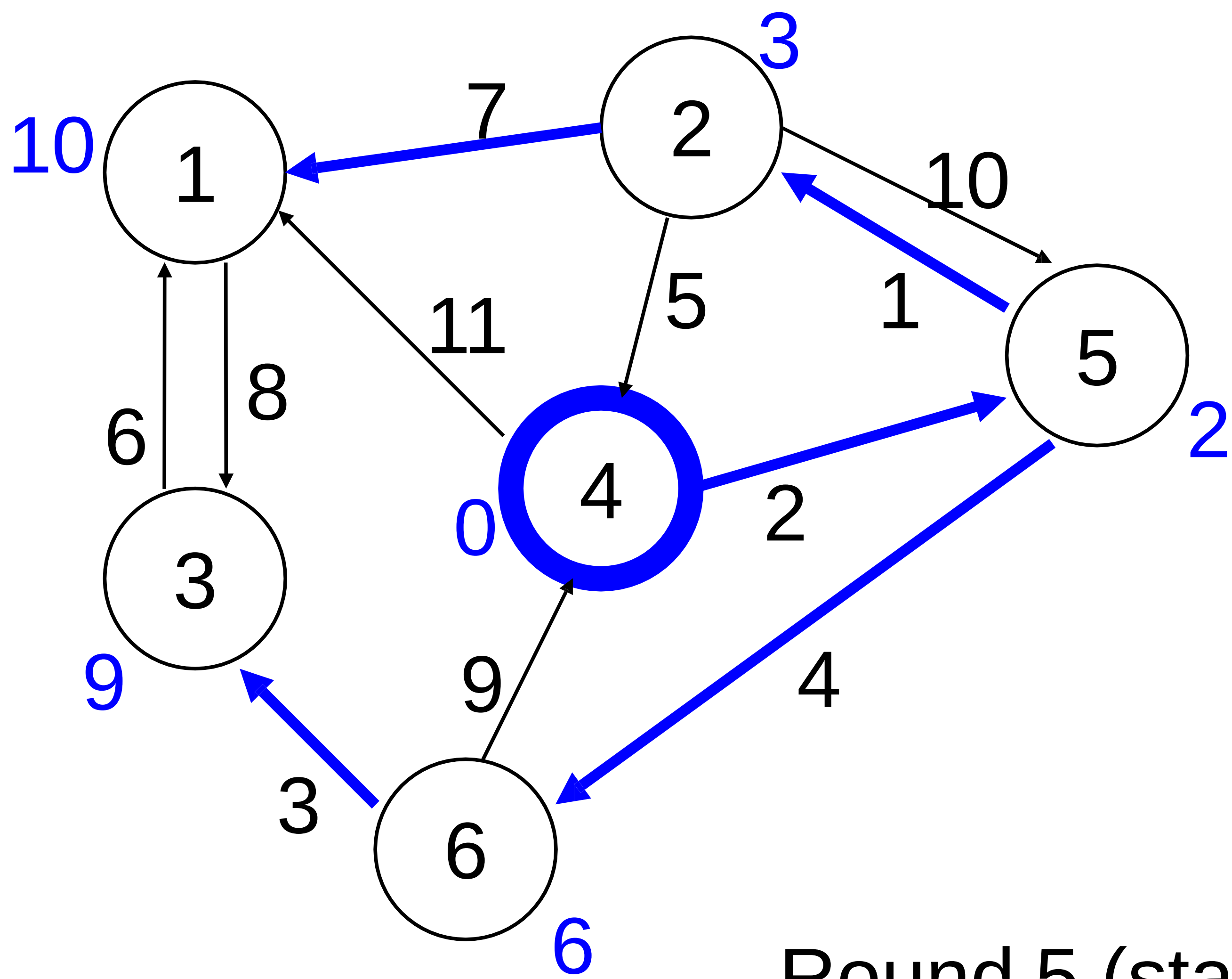
Round 4 (msgs)

Shortest paths



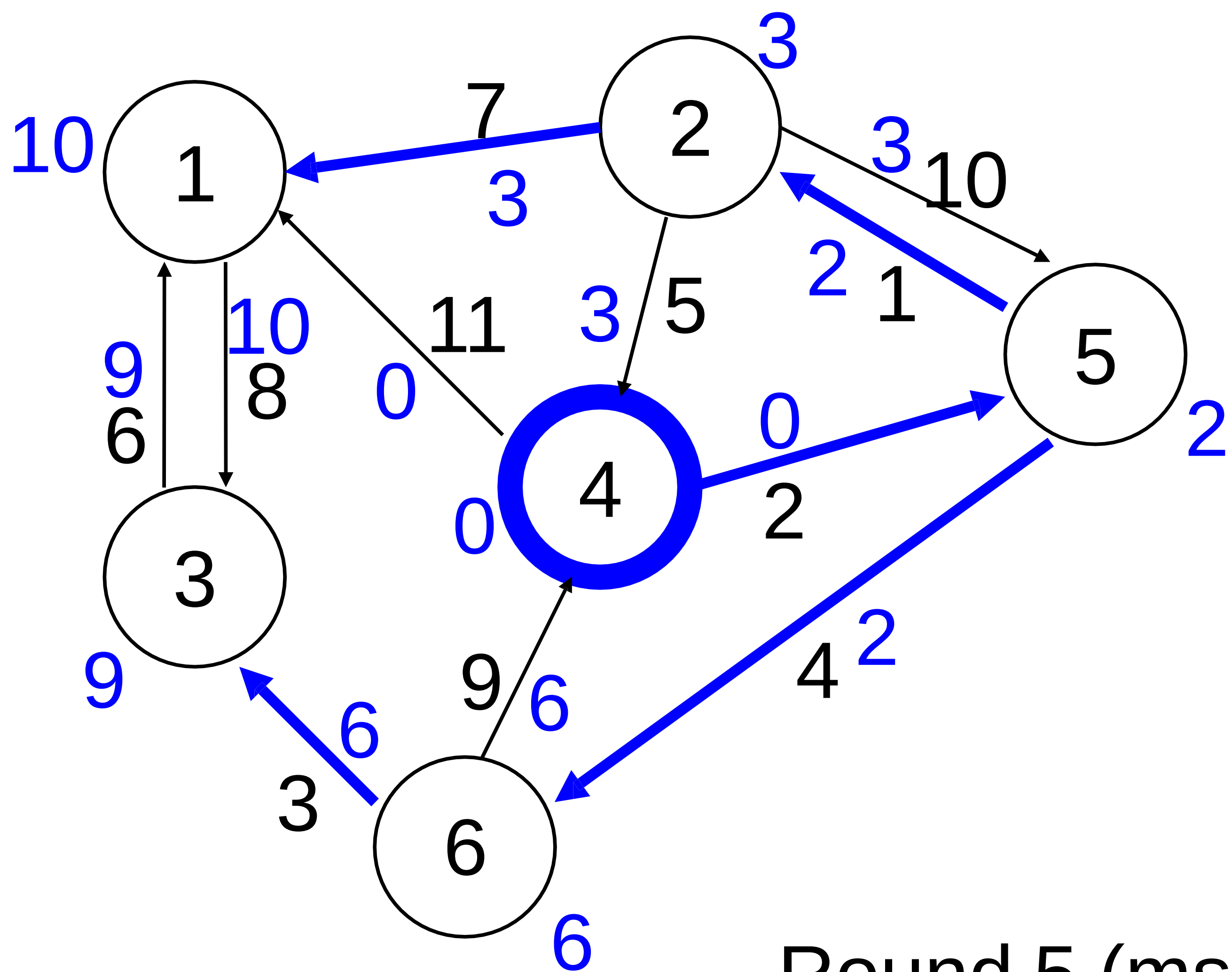
Round 4 (trans)

Shortest paths



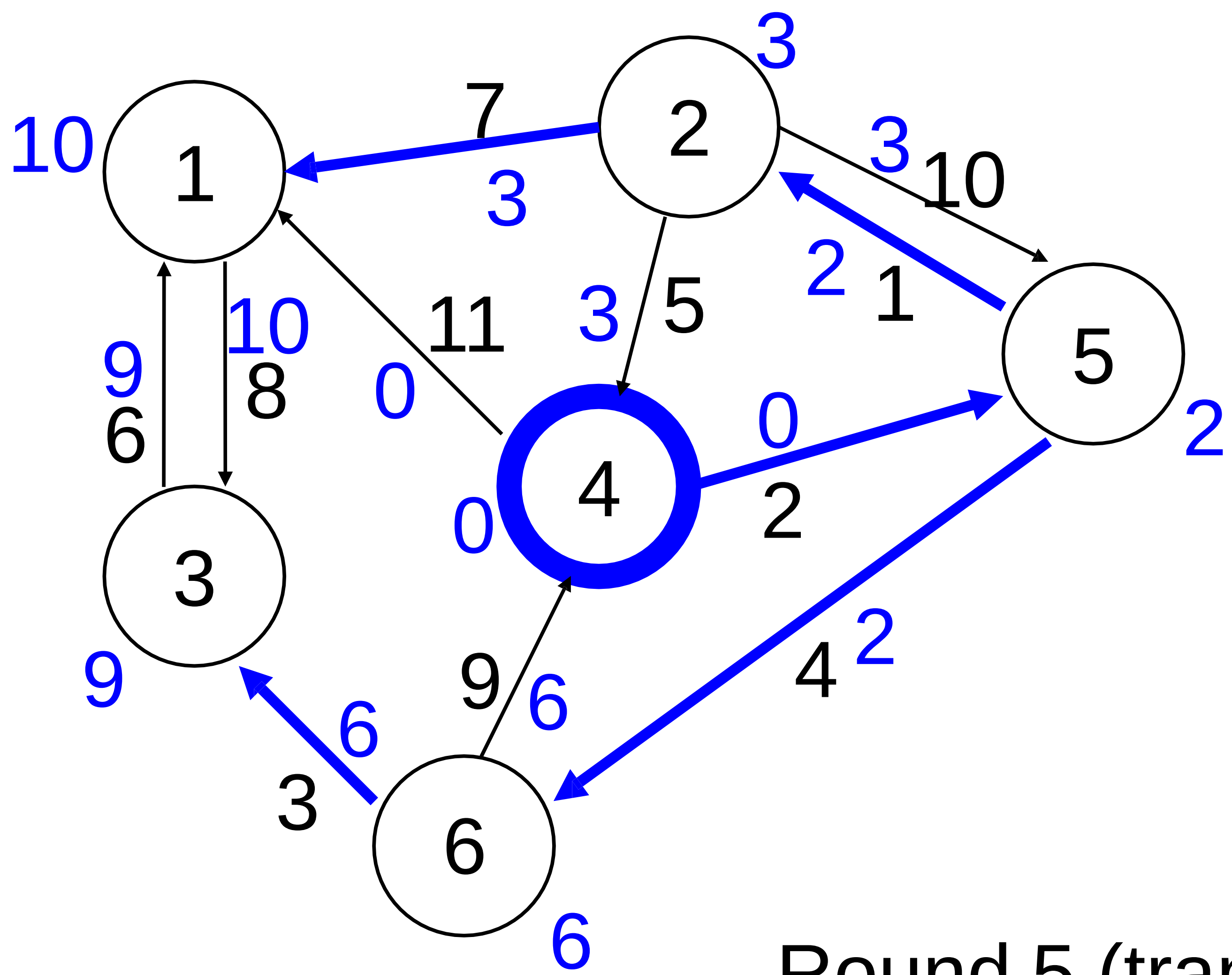
Round 5 (start)

Shortest paths



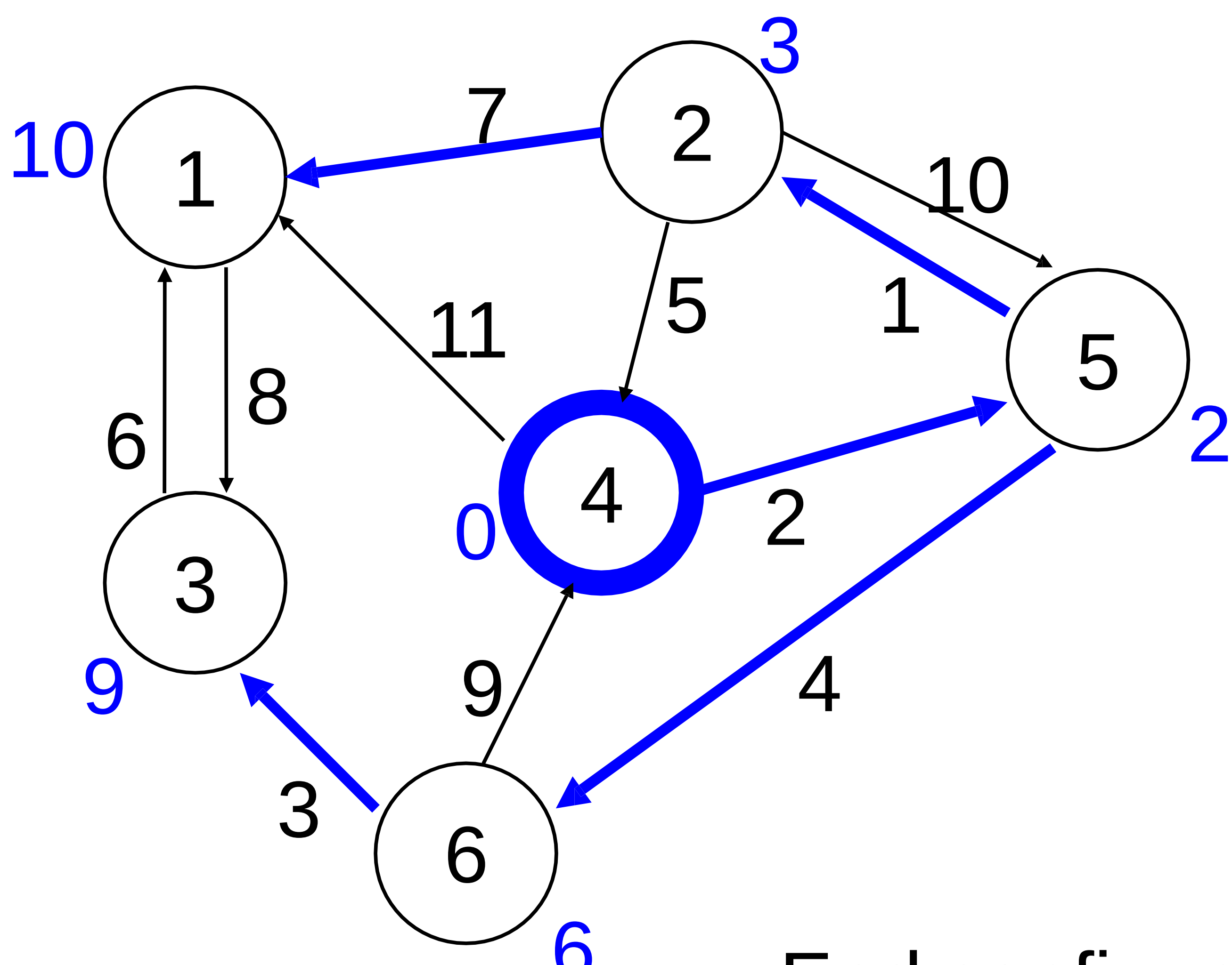
Round 5 (msgs)

Shortest paths



Round 5 (trans)

Shortest paths



End configuration

Correctness

- Need to show that, after round n-1, for each process i:
  - $dist_i$ = shortest distance from i_0
  - $parent_i$ = predecessor on shortest path from i_0
- Proof:
  - Induction on the number r of rounds.
  - But, what statement should we prove about the situation after r rounds?

Correctness

- Key invariant: After r rounds:
  - Every process i has its dist and parent corresponding to a shortest path from i_0 to i among those paths that consist of at most r hops (edges).
  - If there is no such path, then dist = ’and parent = null.

- Proof (sketch):
  - By induction on the number r of rounds.
  - Base: r = 0: Immediate from initializations.
  - Inductive step: Assume for r-1, show for r.
    - Fix i; must show that, after round r, $dist_i$ and $parent_i$ correspond to a shortest at-most-r-hop path.
    - First, show that, if $dist_i$ is finite, then it really is the distance on some at-most-r-hop path to i, and parent is its parent on such a path.
    - LTTR---easy use of inductive hypothesis.
    - But we must still argue that $dist_i$ and $parent_i$ correspond to a shortest at-most-r-hop path.

Correctness

- Key invariant: After r rounds:
  - Every process i has its dist and parent corresponding to a shortest path from i_0 to i among those paths that consist of at most r hops (edges).
  - If there is no such path, then dist = ’and parent = null.

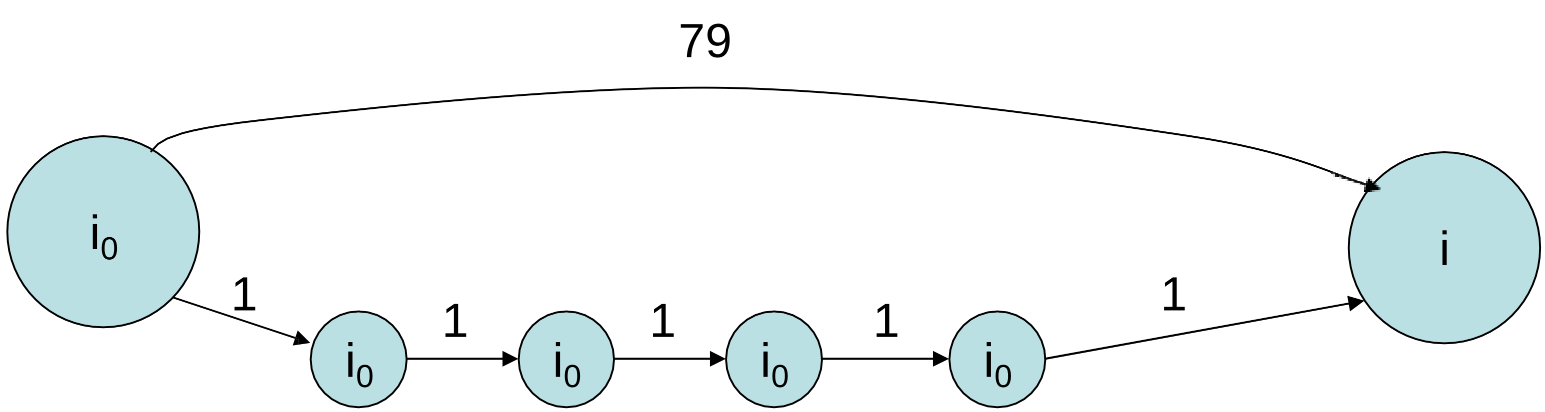
- Proof, inductive step:
  - Assume for r-1, show for r.
  - Fix i; must show that, after round r, dist_i and parent_i correspond to a shortest at-most-r-hop path.
  - If dist_i is finite, then it really is the distance on some at-most-r-hop path to i, and parent is its parent on such a path.
  - Claim that dist_i and parent_i correspond to a shortest at-most-r-hop path.
  - Any shortest at-most-r-hop path from i_0 to i, when cut off at i’s predecessor j on the path, yields a shortest (r-1)-hop path from i_0 to j.
  - By inductive hypothesis, after round r-1, for every such j, dist_j and parent_j correspond to a shortest at-most-(r-1)-hop path from i_0 to j.
  - At round r, all such j send i their info about their shortest at-most-(r-1)-hop paths, and process i takes this into account in calculating dist_i.
  - So after round r, dist_i and parent_i correspond to a shortest at-most-r-hop path.

Complexity

- Complexity:
  - Time: n-1 rounds
  - Messages: (n-1) |E|

- Worse that BFS, which has:
  - Time: diam rounds
  - Messages: |E|

- Q: Does the time bound really depend on n, or is it O(diam)?
- A: It's really n, since "shortest path" can be over a path with more links.
- Example:



Bellman-Ford Shortest-Paths Algorithm

- Will revisit Bellman-Ford shortly in asynchronous networks.
- Gets even more expensive there.
- Similar to old Arpanet routing algorithm.

Minimum spanning tree

- Another classical problem.
- Many sequential algorithms.
- Construct a spanning tree, minimizing the total weight of all edges in the tree.
- Assume:
  - Weighted undirected graph (bidirectional communication).
    - Weights are nonnegative reals.
    - Each node knows weights of incident edges.
  - Processes have UIDs.
  - Nodes know (a good upper bound on) n.
- Required:
  - Each process should decide which of its incident edges are in MST and which are not.

Minimum spanning tree theory

- Graph theory definitions (for undirected graphs)
  - Tree: Connected acyclic graph
  - Forest: An acyclic graph (not necessarily connected)
  - Spanning subgraph of a graph G: Subgraph that includes all nodes of G.
    - Spanning tree, spanning forest.
  - Component of a graph: A maximal connected subgraph.

- Common strategy for computing MST:
  - Start with trivial spanning forest, n isolated nodes.
  - Repeat (n-1 times):
    - Merge two components along an edge that connects them.
    - Specifically, add the minimum-weight outgoing edge (MWOE) of some component to the edge set of the current forest.

Why this works:

- Similar argument to sequential case.
- Lemma 1: Let $\{ T_i : 1 \leq i \leq k \}$ be a spanning forest of G. Fix any j, $1 \leq j \leq k$. Let e be a minimum weight outgoing edge of T_j.
  Then there is a spanning tree for G that includes all the T_is and e, and has minimum weight among all spanning trees for G that include all the T_is.
- Proof:
  - Suppose not---there's some spanning tree T for G that includes all the T_is and does not include e, and whose total weight is strictly less than that of any spanning tree that includes all the T_is and e.
  - Construct a new graph T' (not a tree) by adding e to T.
  - Contains a cycle, which must contain another outgoing edge, e', of T_j.
  - weight(e') $\geq$ weight(e), by choice of e (smallest weight).
  - Construct a new tree T'' by removing e' from T'.
  - Then T'' is a spanning tree, contains all the T_is and e.
  - weight(T'') $\leq$ weight(T).
  - Contradicts assumed properties of T.

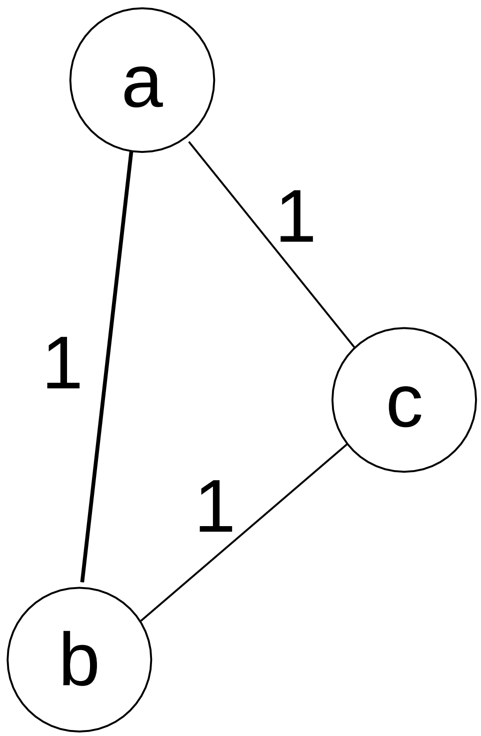
Minimum spanning tree algorithms

- General strategy:
  - Start with n isolated nodes.
  - Repeat (n-1 times):
    - Choose some component i.
    - Add the minimum-weight outgoing edge (MWOE) of component i.

- Sequential MST algorithms follow (special cases of) this strategy:
  - Dijkstra/Prim: Grows one big component by adding one more node at each step.
  - Kruskal: Always add min weight edge globally.

- Distributed?
  - All components can choose simultaneously.
  - But there is a problem…

Can get cycles:



Minimum spanning tree

- Avoid this problem by assuming that all weights are distinct.
- Not a serious restriction---could break ties with UIDs.
- Lemma 2: If all weights are distinct, then the MST is unique.
- Proof: Another cycle argument (LTTR).

- Justifies the following concurrent strategy:
  - At each stage, suppose (inductively) that the current forest contains only edges from the unique MST.
  - Now several components choose MWOEs concurrently.
  - Each of these edges is in the unique MST, by Lemma 1.
  - So OK to add them all (no cycles, since all are in the same MST).

- GHS (Gallager, Humblet, Spira) algorithm
  - Very influential (Dijkstra prize).
  - Designed for asynchronous setting, but simplified here.
  - We will revisit it in asynchronous networks.

GHS distributed MST algorithm

- Proceeds in phases (levels), each with O(n) rounds.
  - Length of phases is fixed, and known to everyone.
  - This is all that n is used for.
  - We'll remove use of n for asynchronous algorithm.

- For each $k \geq 0$, level k components form a spanning forest that is a subgraph of the unique MST.
- Each component is a tree rooted at a leader node.
  - Component identified by UID of leader.
  - Nodes in the component know which incident edges are in the tree.
- Each level k component has at least 2^k nodes.
- Every level k+1 component is constructed from two or more level k components.

- Level 0 components: Single nodes.
- Level $k \rightarrow$ level k+1:

Level k $\rightarrow$ Level k+1

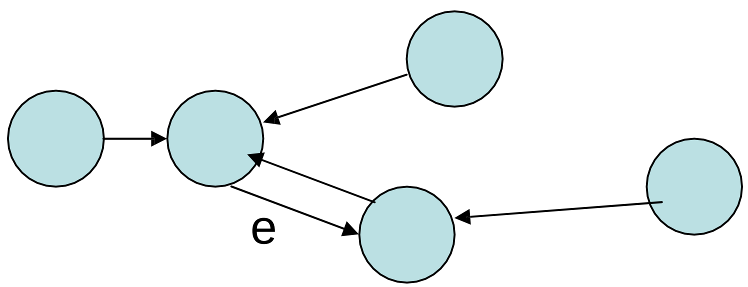
- Each level-k component leader finds MWOE of its component:
  - Broadcasts search (via tree edges).
  - Each process finds the mwoe among its own incident edges.
    - Sends test messages along non-tree edges, asking if node at the other end is in the same component (compare component ids).
  - Convergecast the min back to the leader (via tree edges).
  - Leader determines MWOE.

- Combine level-k components using MWOEs, to obtain level (k+1) components:
  - Wait long enough for all components to find MWOEs.
  - Leader of each level k component tells endpoint nodes of its MWOE to add the edge for level k+1.
  - Each new component has $\geq 2^{k+1}$ nodes, as claimed.

Level k $\rightarrow$ Level k+1, cont'd

- Each level-k component leader finds MWOE of its component.
- Combine level-k components using MWOEs, to obtain level-(k+1) components.

- Choose new leaders:
  - For each new, level k+1 component, there is a unique edge e that is the MWOE of two level k sub-components:



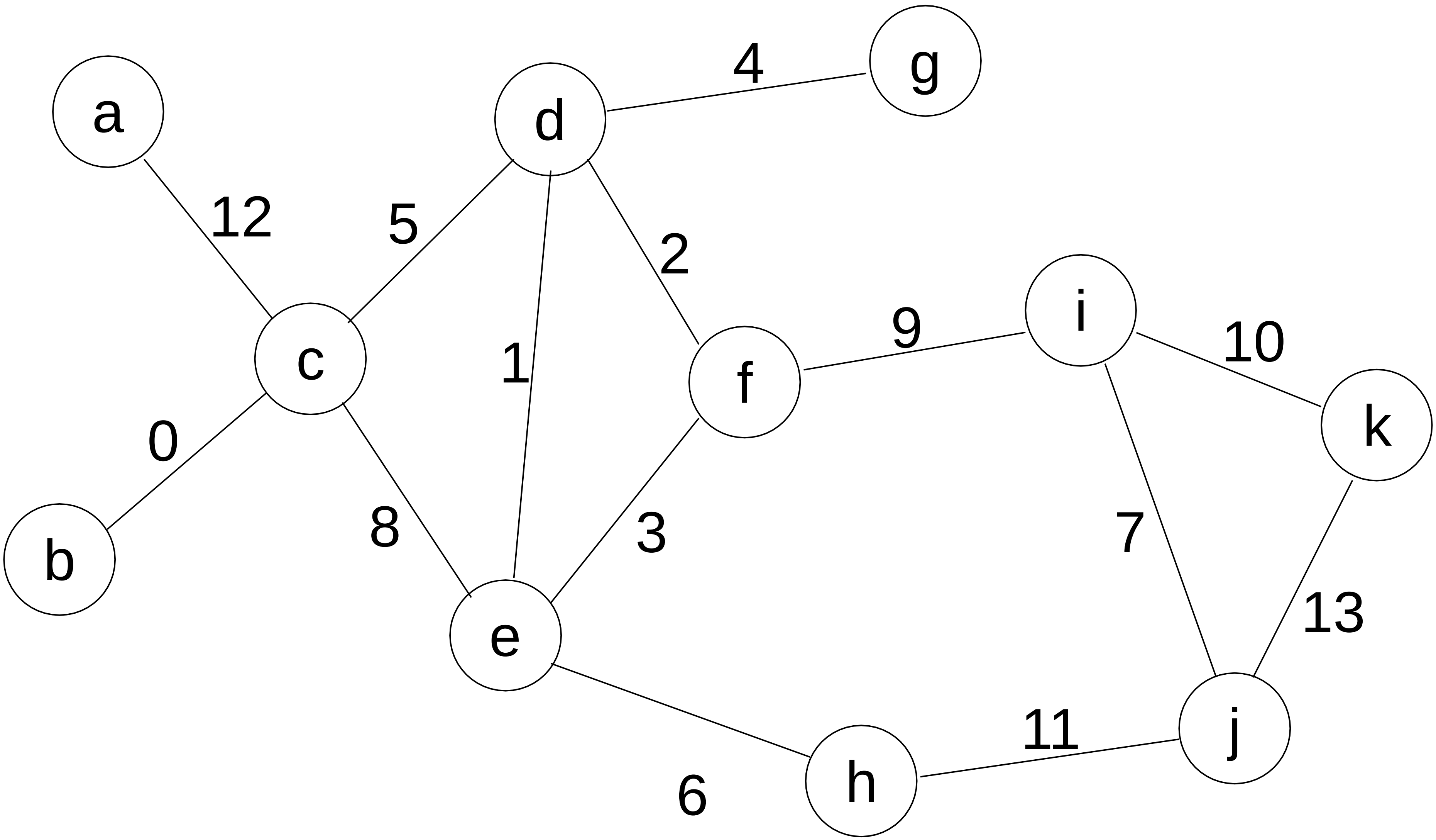
n edges, must have a cycle.
Cycle can't have length > 2, because weights of different edges on the cycle must decrease around the cycle.

  - Choose new leader to be the endpoint of e with the larger UID.
  - Broadcast leader UID to new (merged) component.
- GHS terminates when there are no more outgoing edges.

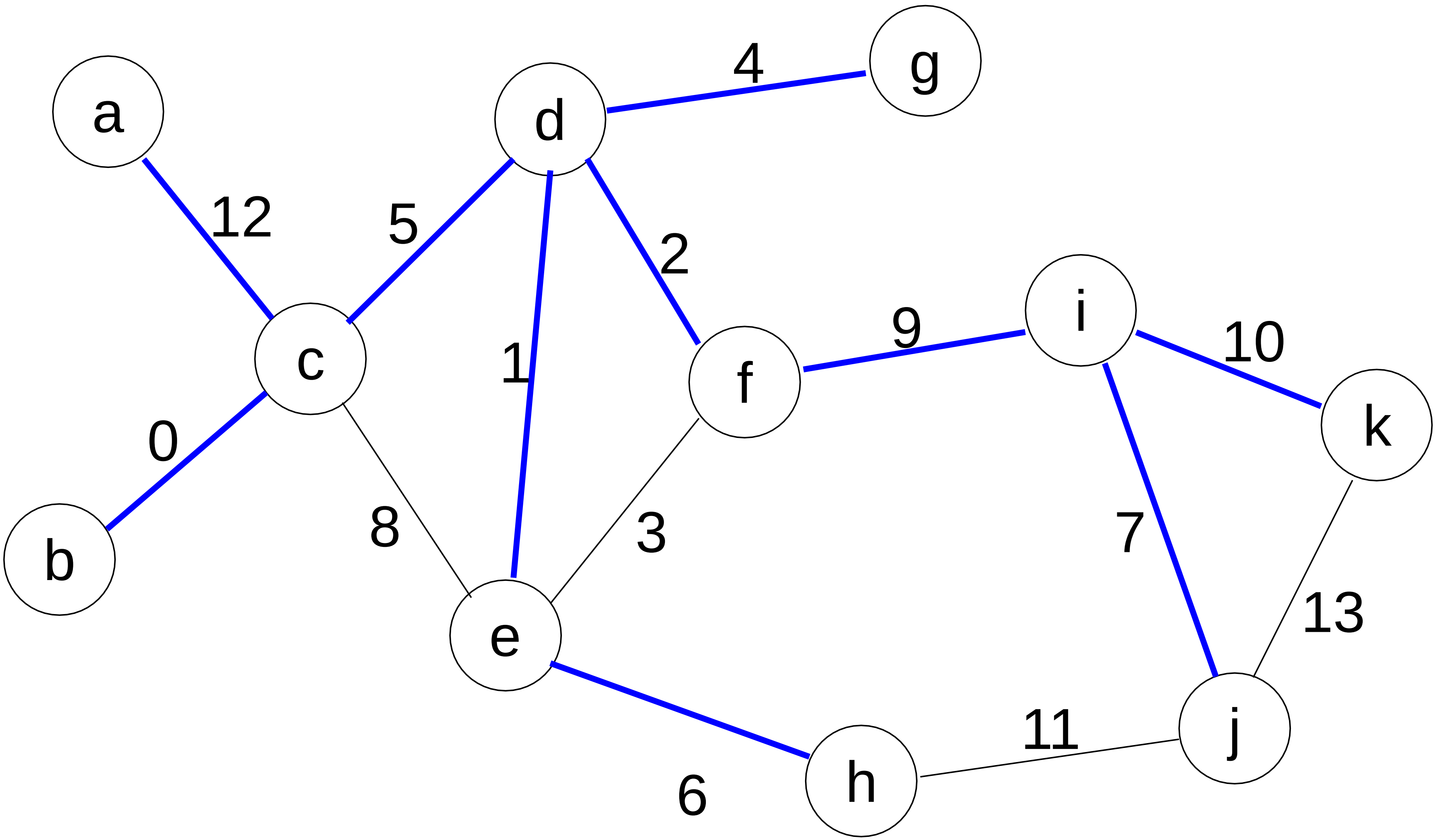
Note on synchronization

- This simplified version of GHS is designed to work with component levels synchronized.
- Difficulties can arise when they get out of synch (as we'll see).
- In particular, test messages are supposed to compare leader UIDs to determine whether endpoints are in the same component.
- Requires that the node being queried has up-to-date UID information.

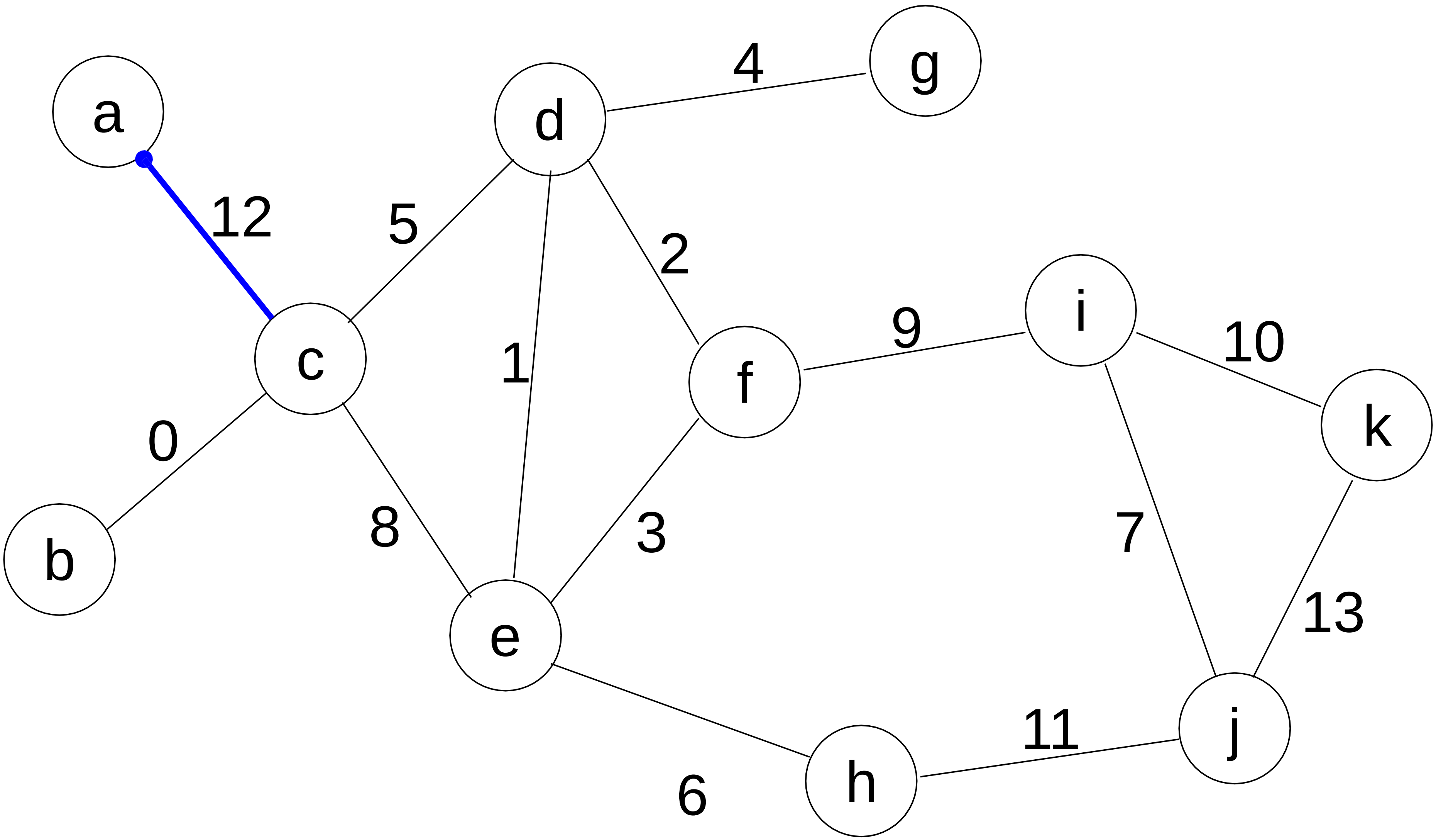
Minimum spanning tree



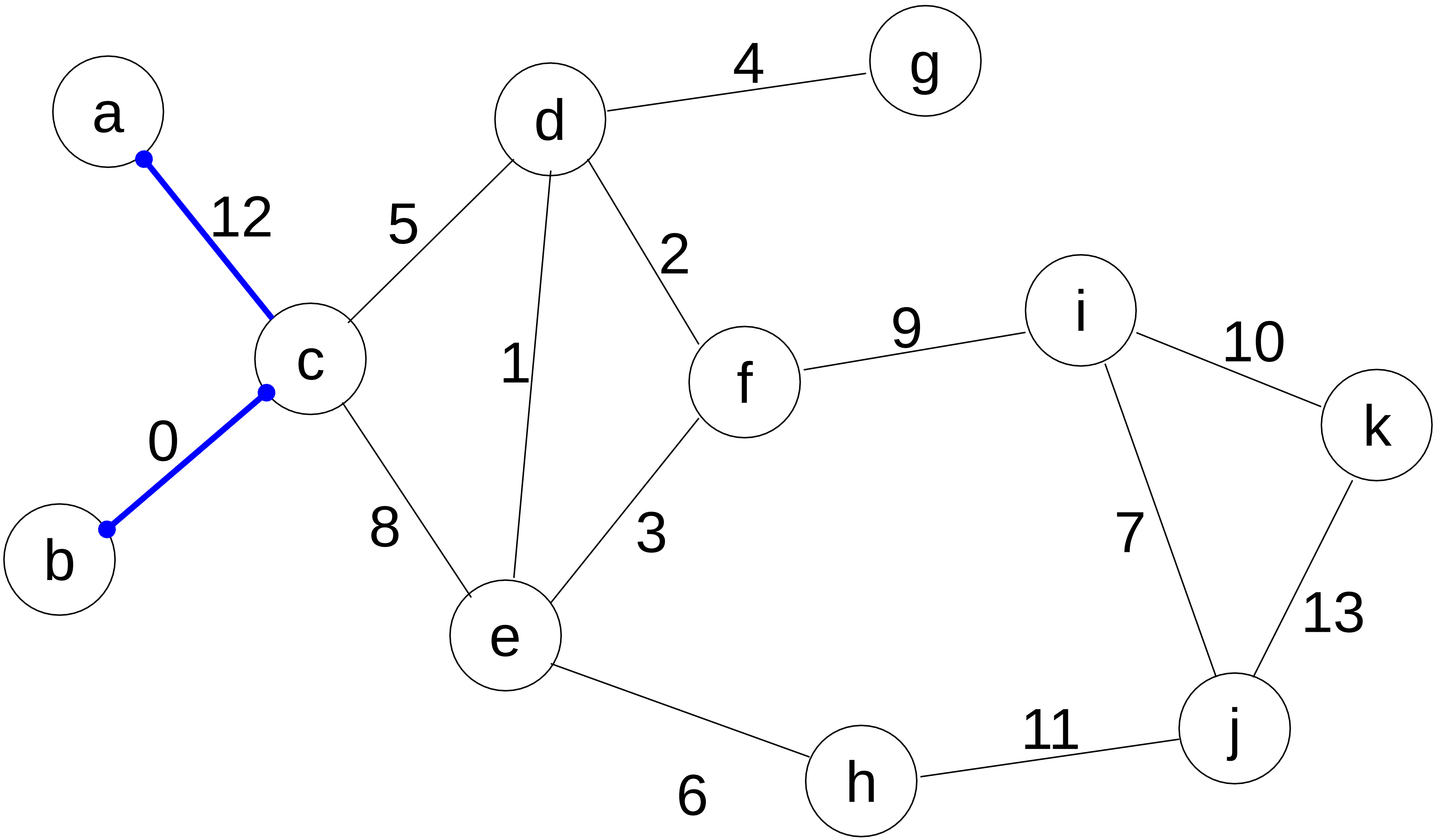
Minimum spanning tree



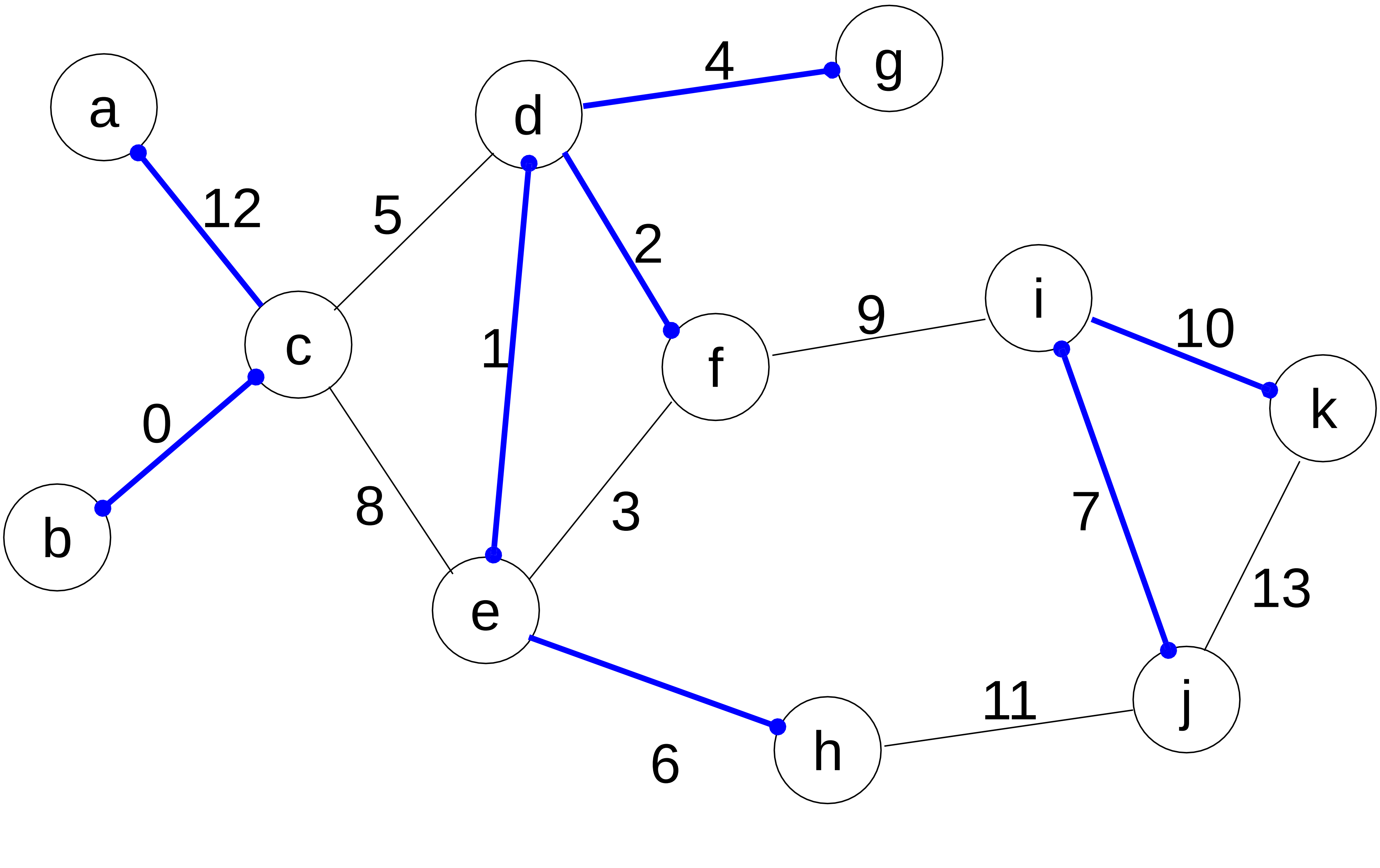
Minimum spanning tree



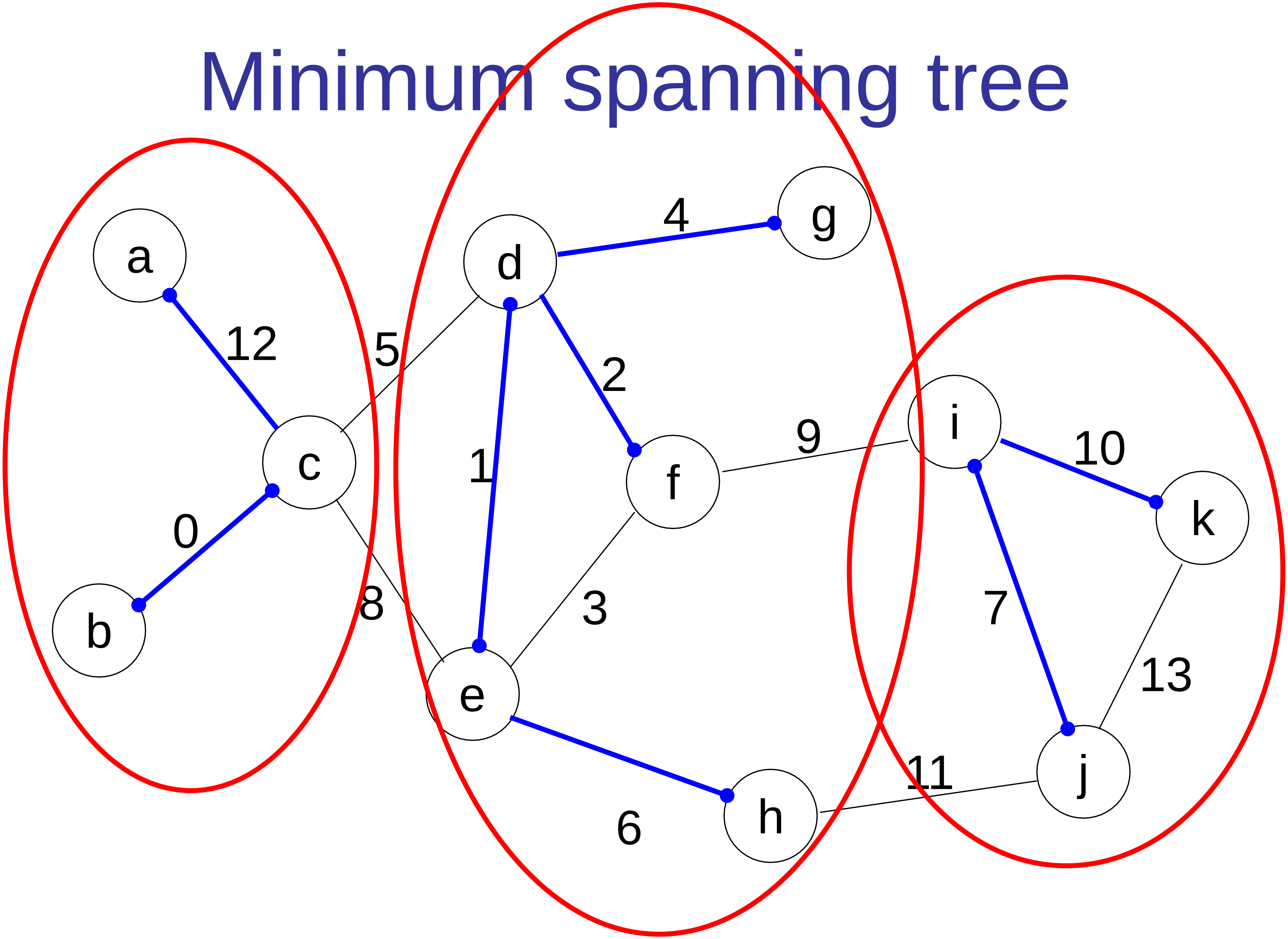
Minimum spanning tree



Minimum spanning tree



Minimum spanning tree



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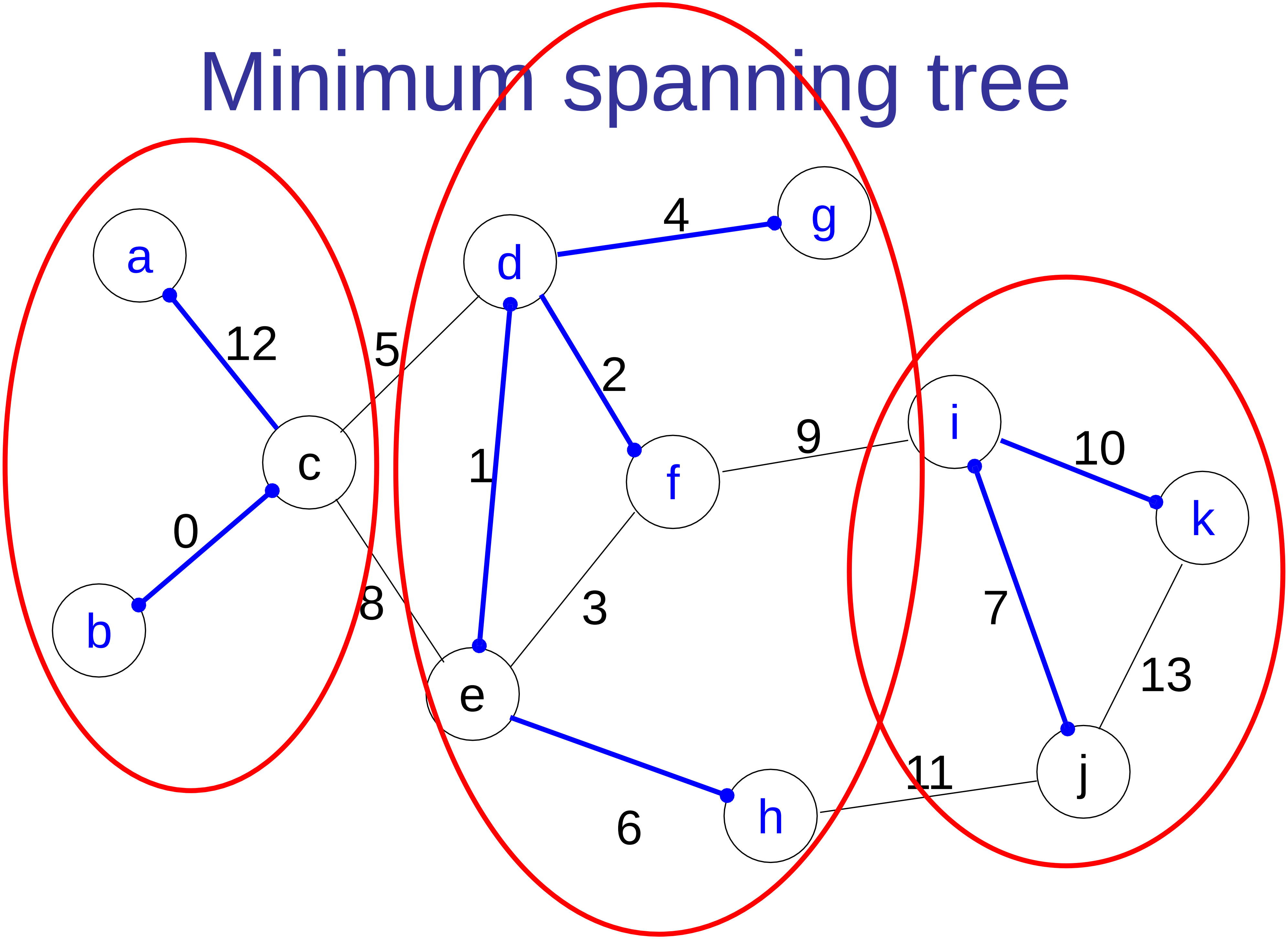
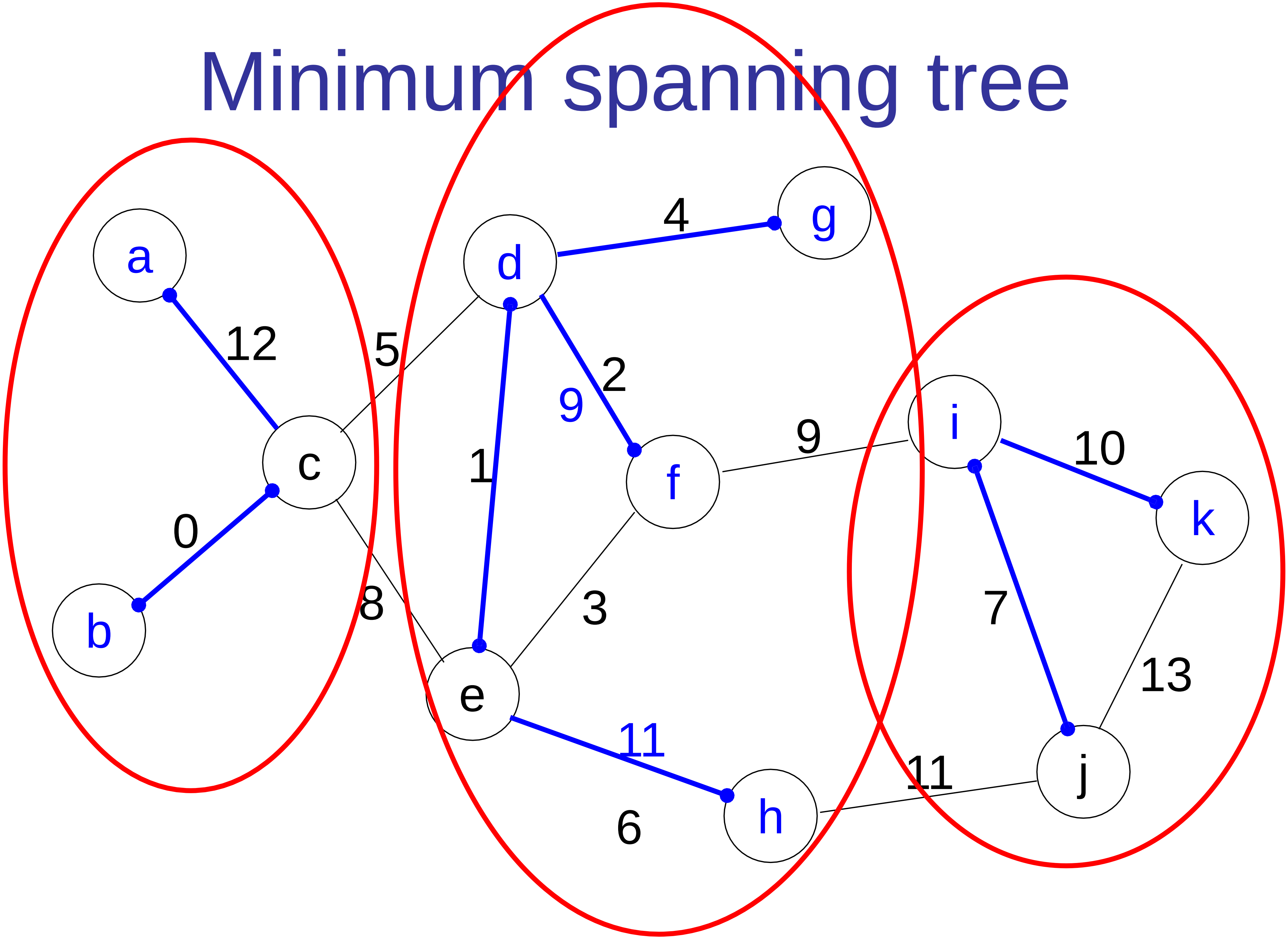
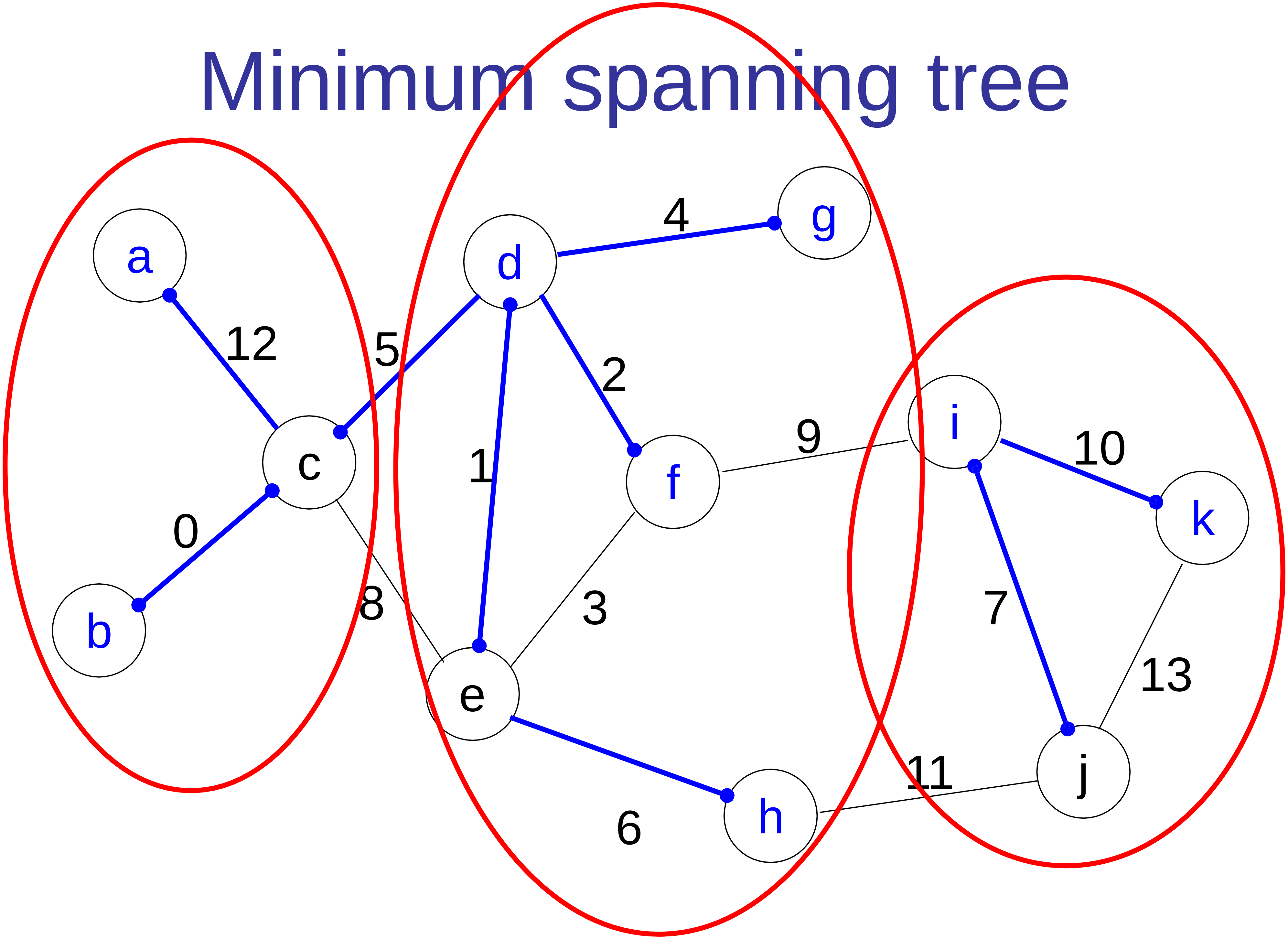


Minimum spanning tree



Minimum spanning tree



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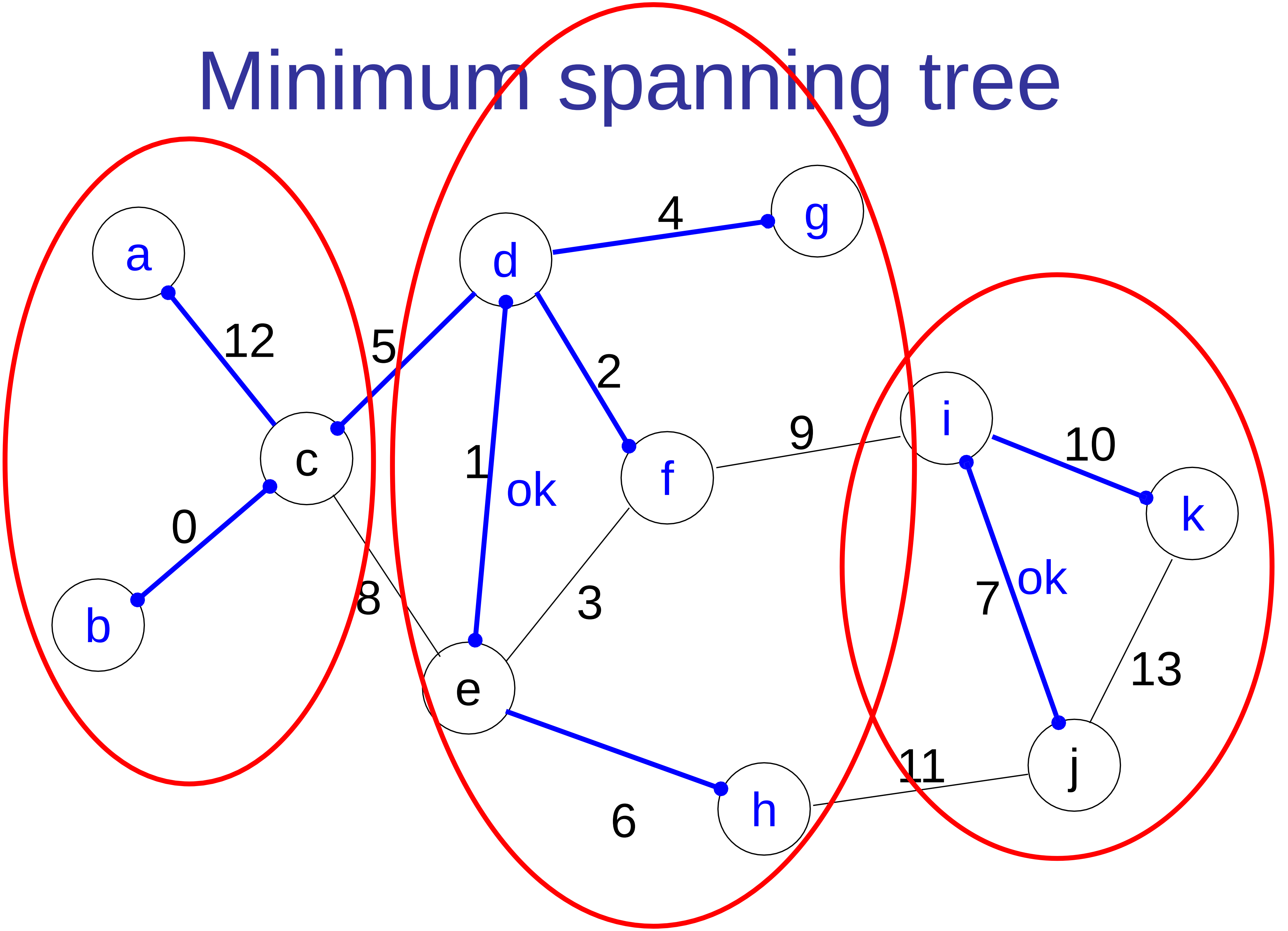
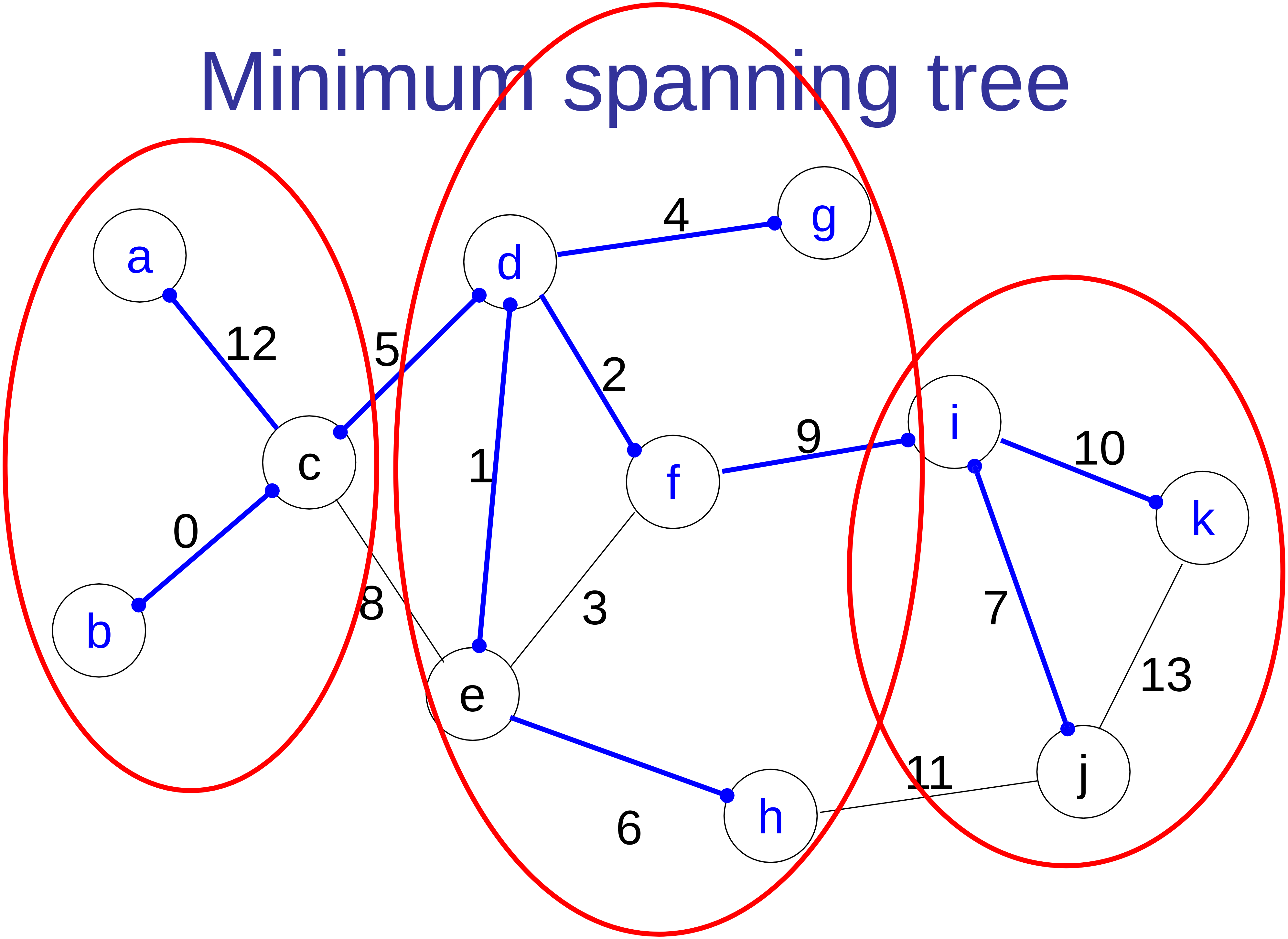
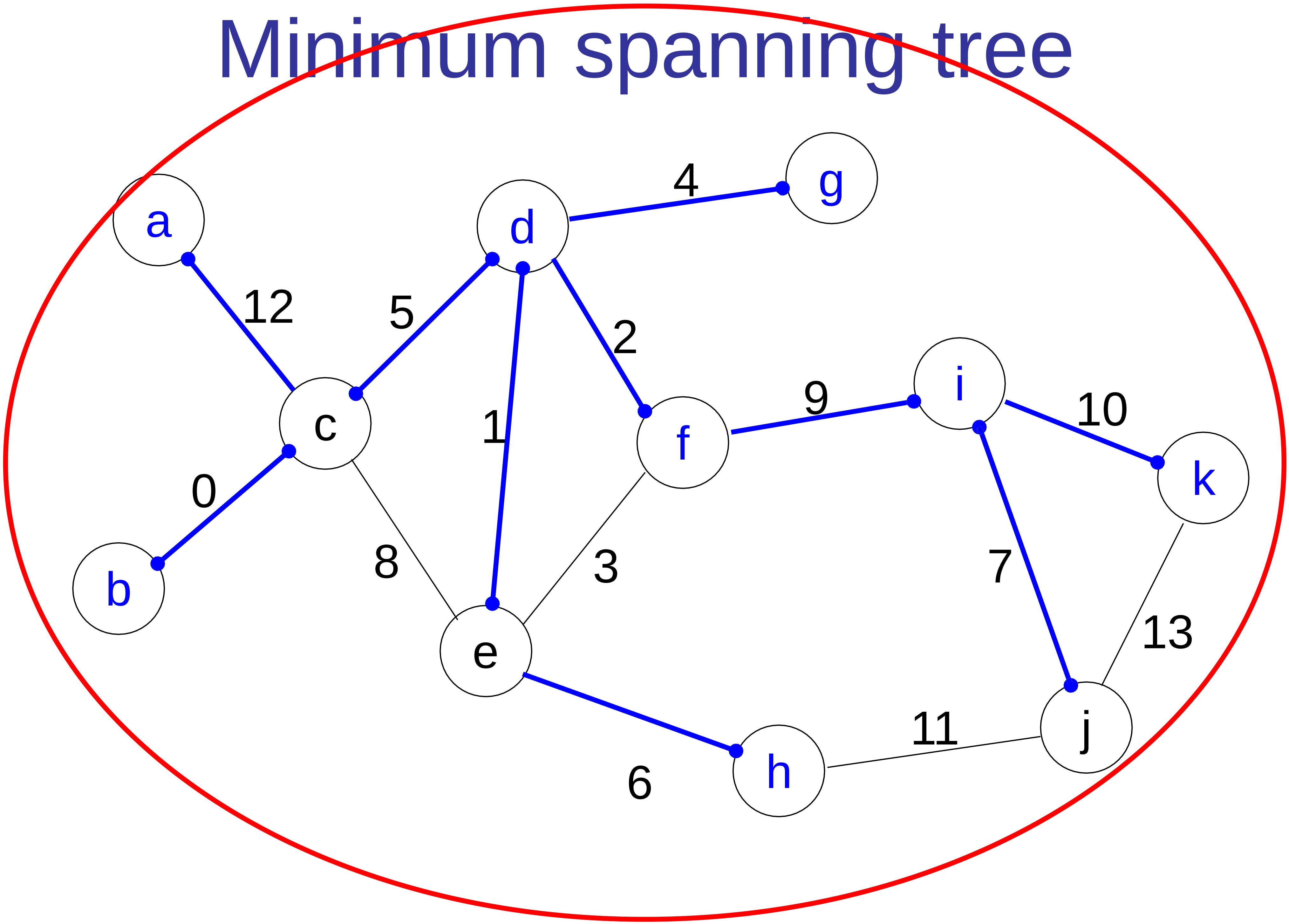


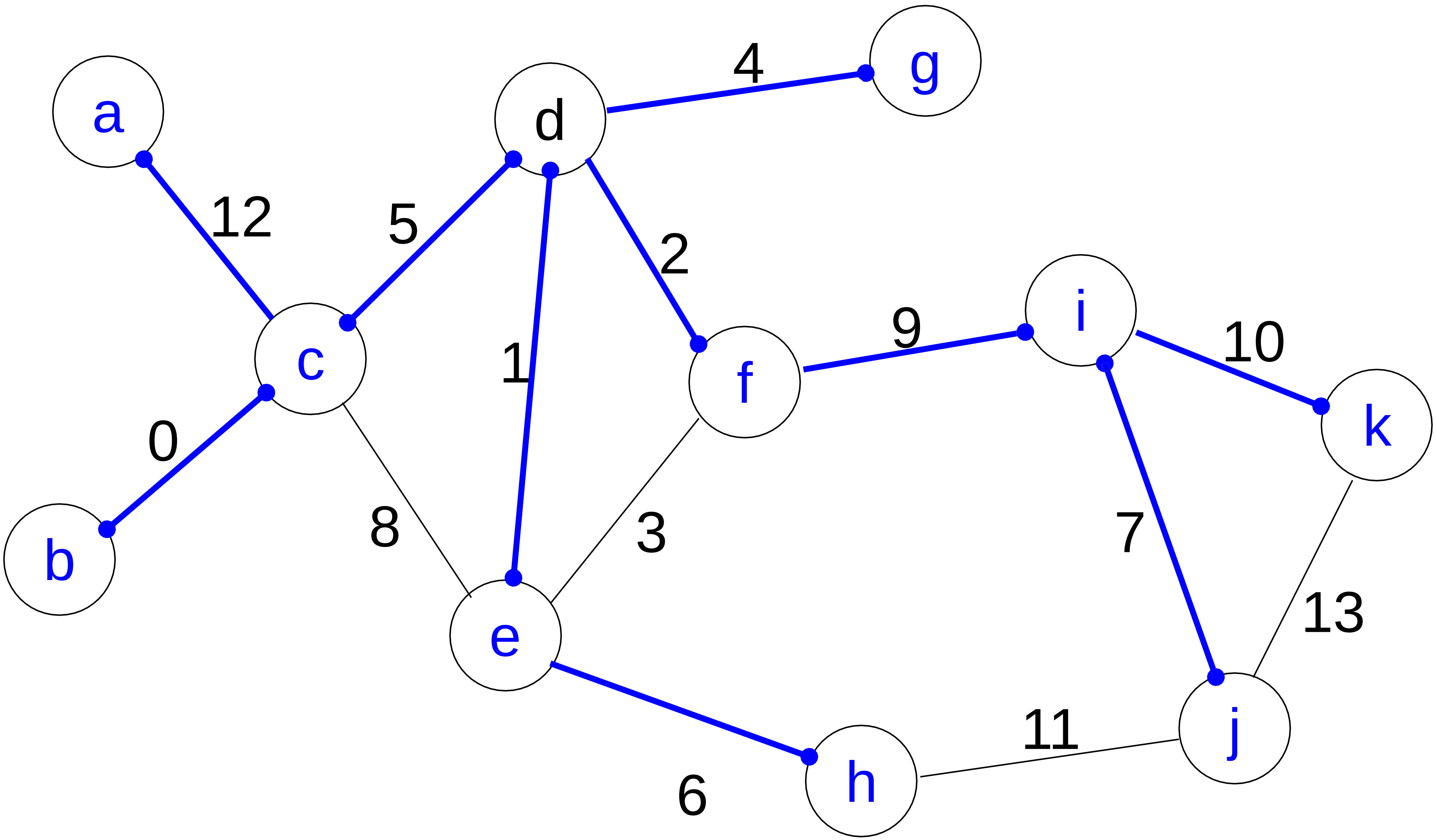
Minimum spanning tree



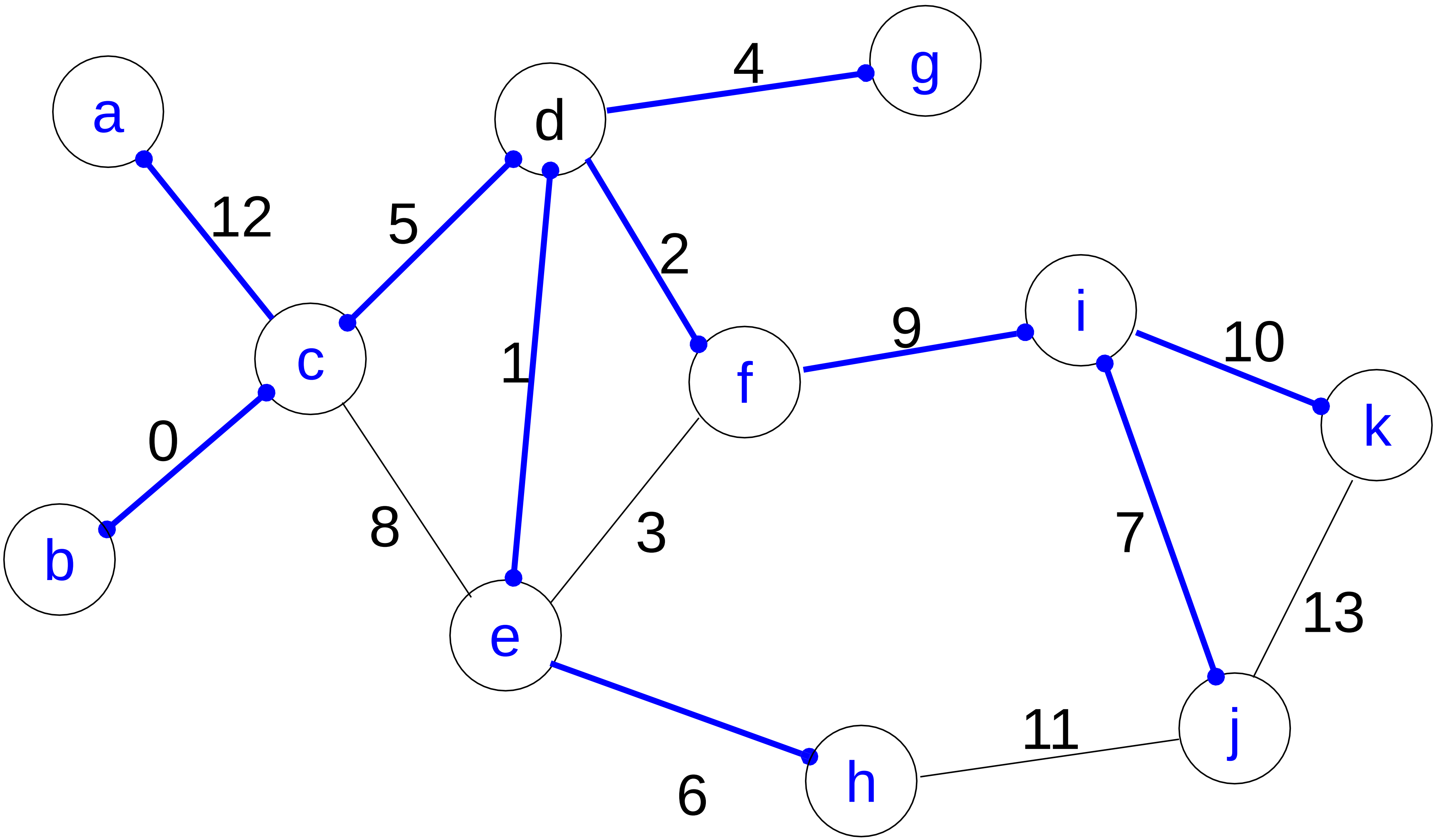
Minimum spanning tree



Minimum spanning tree



Minimum spanning tree



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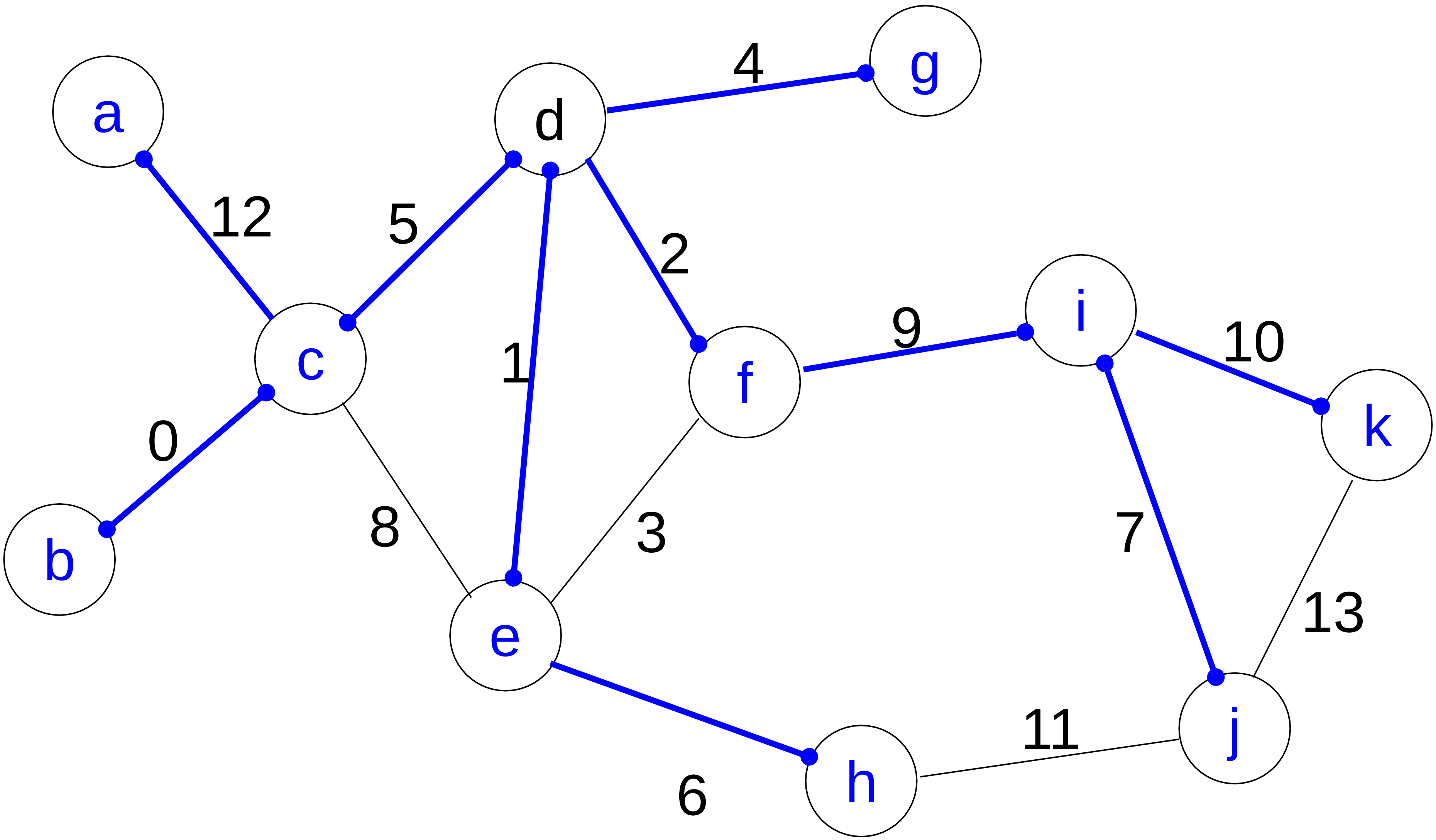

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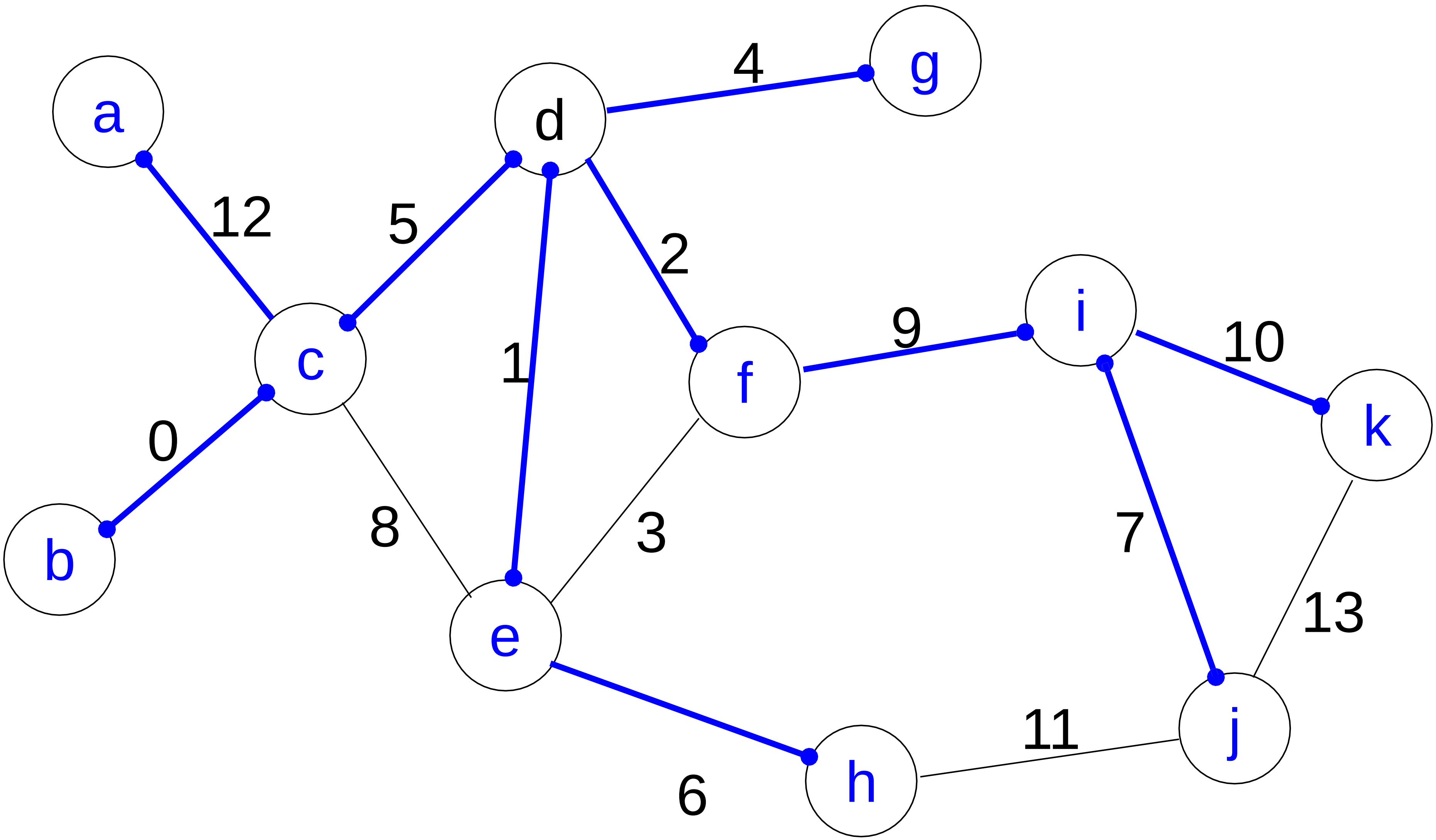

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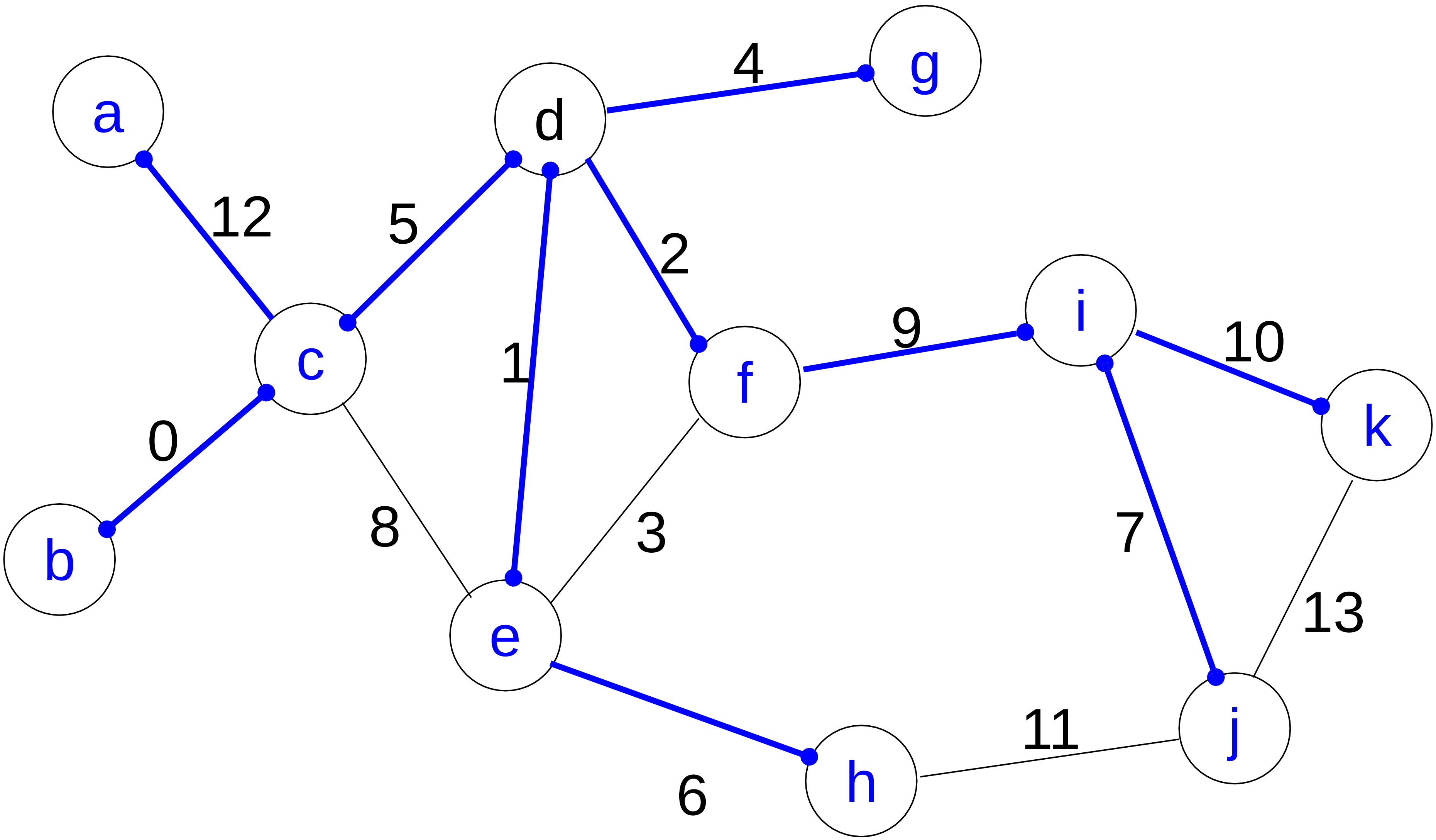

Simplified GHS MST Algorithm

- Proof?
- Use invariants; but this is complicated because the algorithm is complicated.
- Complexity:
  - Time: O(n log n)
    - n rounds for each level
    - log n levels, because there are $\geq 2^k$ nodes in each level k component.
  - Messages: O((n + |E|) log n)
    - Naïve analysis.
    - At each level, O(n) messages sent on tree edges, O(|E|) messages overall for all the test messages and their responses.
  - Messages: O(n log n + |E|)
    - A surprising, significant reduction.
    - Trick also works in asynchronous setting.
    - Has implications for other problems, such as leader election.

O(n log n + |E|) message complexity

- Each process marks its incident edges as rejected when they are discovered to lead to the same component; no need to retest them.
- At each level, tests candidate edges one a a time, in order of increasing weight, until the first one is found that leads outside (or exhaust candidates)
- Rejects all edges that are found to lead to same component.
- At next level, resumes where it left off.

- O(n log n + |E|) bound:
  - O(n) for messages on tree edges at each phase, O(n log n) total.
  - Test, accept (different component), reject (same component):
    - Amortized analysis.
    - Test-reject: Each (directed) edge has at most one test-reject, for O(|E|) total.
    - Test-accept: Can accept the same directed edge several times; but at most one test-accept per node per level, O(n log n) total.

Where/how did we use synchrony?

- Leader election
- Breadth-first search
- Shortest paths
- Minimum spanning tree

We will see these algorithms again
in the asynchronous setting.

Spanning tree $\rightarrow$ Leader

- Given any spanning tree of an undirected graph, elect a leader:
  - Convergecast from the leaves, until messages meet at a node (which can become the leader) or cross on an edge (choose endpoint with the larger UID).
  - Complexity: Time $O(n)$; Messages $O(n)$

- Given any weighted connected undirected graph, with known n, but no leader, elect a leader:
  - First use GHS MST to get a spanning tree, then use the spanning tree to elect a leader.
  - Complexity: Time $O(n \log n)$; Messages $O(n \log n + |E|)$.
  - Example: In a ring, $O(n \log n)$ time and messages.

Other graph problems…

- We can define a distributed version of practically any graph problem: maximal independent set (MIS), dominating set, graph coloring,…
- Most of these have been well studied.
- For example…

Maximal Independent Set

- Subset I of vertices V of undirected graph G = (V,E) is independent if no two G-neighbors are in V.
- Independent set I is maximal if no strict superset of I is independent.
- Distributed MIS problem:
  - Assume: No UIDs, nodes know (good upper bound on) n.
  - Required:
    - Compute an MIS I of the network graph.
    - Each process in I should output winner, others output loser.
- Application: Wireless network transmission
  - A transmitted message reaches neighbors in the graph; they receive the message if they are in "receive mode".
  - Let nodes in the MIS transmit messages simultaneously, others receive.
  - Independence guarantees that all transmitted messages are received by all neighbors (since neighbors don't transmit at the same time).
  - Neglecting collisions here---some strategy (backoff and retransmission, or coding) is needed for this.
- Unsolvable by deterministic algorithm, in some graphs.
- Randomized algorithm [Luby]:

Luby's MIS Algorithm (sketch)

- Each process chooses a random val in $\{1,2,\ldots,n^4\}$.
  - Large enough set so it's very likely that all numbers are distinct.
- Neighbors exchange vals.
- If node i's val > all neighbors' vals, then process i declares itself a winner and notifies its neighbors.
- Any neighbor of a winner declares itself a loser, notifies its neighbors.
- Processes reconstruct the remaining graph, eliminating winners, losers, and edges incident on winners and lowers.
- Repeat on the remaining graph, until no nodes are left.

- Theorem: If LubyMIS ever terminates, it produces an MIS.
- Theorem: With probability 1, it eventually terminates; the expected number of rounds until termination is O(log n).
- Proof: LTTR.

Termination theorem for Luby MIS

- Theorem: With probability 1, Luby MIS eventually terminates; the expected number of rounds until termination is O(log n).
- Proof: Key ideas
  - Define sum(i) = $\Sigma_{j \in nbrs(i)}$ 1/degree(j).
    - Sum of the inverses of the neighbors' degrees.
  - Lemma 1: In one stage of Luby MIS, for each in the graph, the probability that i is a loser (neighbor of a winner) is $\geq$ 1/8 sum(i).
  - Lemma 2: The expected number of edges removed from G in one stage is $\geq$ |E| / 8.
  - Lemma 3: With probability at least 1/16, the number of edges removed from G at a single stage is $\geq$ |E| / 16.