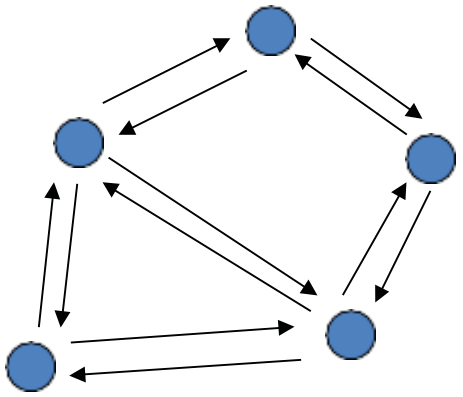
Asynchronous Distributed Algorithms

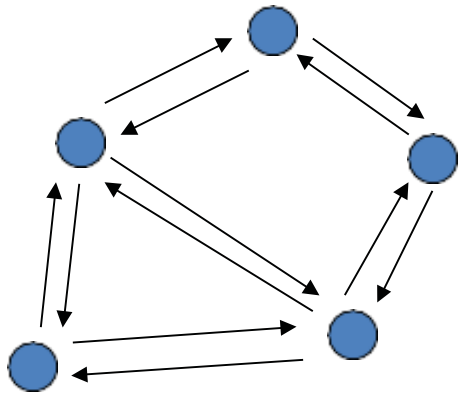


Asynchronous Network Model

- Complications so far:
  - Processes act concurrently
  - A little nondeterminism.
- Now things get much worse:
  - No rounds---process steps and message deliveries happen at arbitrary times, in arbitrary orders.
  - Processes get out of synch.
  - Much more nondeterminism.

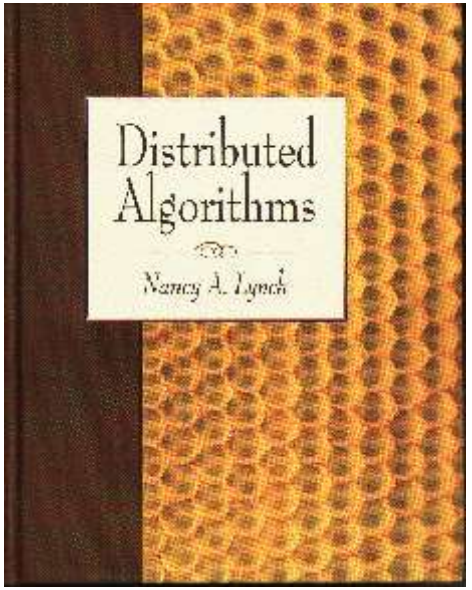
synchronous nw model

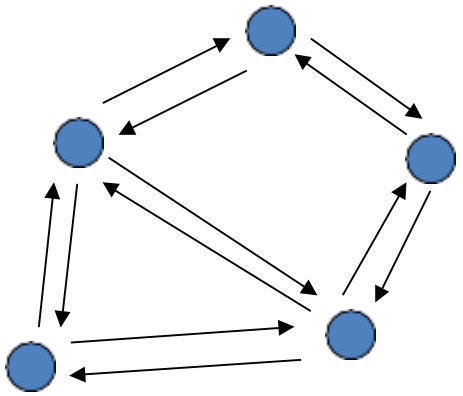
- Understanding asynchronous distributed algorithms is hard because we can't understand exactly how they execute.
- Instead, we must understand abstract properties of executions.



Aynchronous Network Model

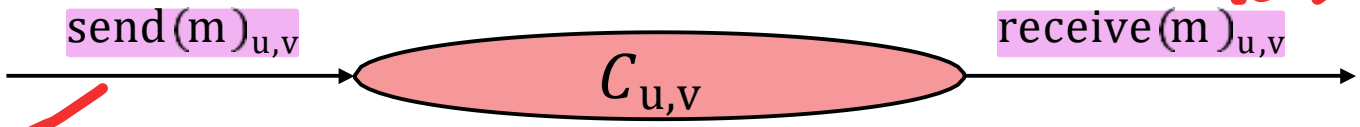
- Lynch, Distributed Algorithms, Chapter 8.
- Processes at nodes of an undirected graph $G = (V, E)$, communicate using messages.
- Communication channels associated with edges (one in each direction on each edge).
  - $C_{u,v}$, channel from vertex n to vertex v.
- Each process has output ports and input ports that connect it to its communication channels.
- Processes need not be distinguishable.





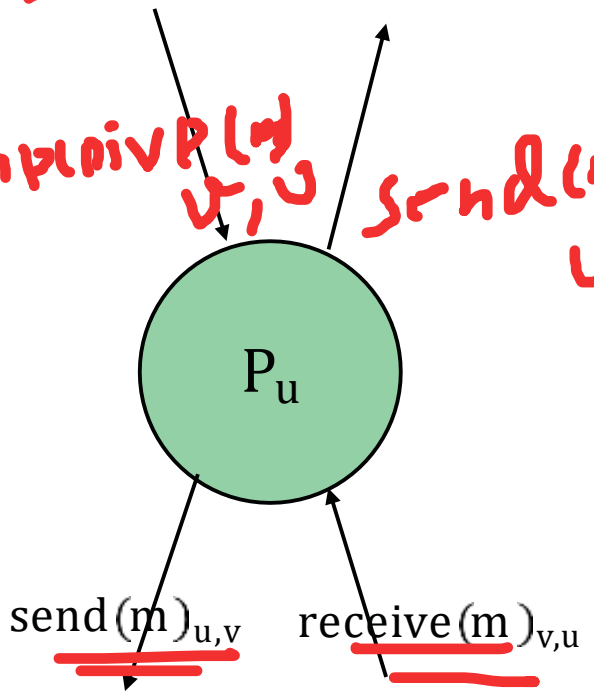
Channel Automaton $C_{u,v}$

- Formally, an input/output automaton.
- Input actions: $\text{send}(m)_{u,v}$
- Output actions: $\text{receive}(m)_{u,v}$
- State variable:
  - $mqnene$, a FIFO queue, initially empty.
- Transitions:
  - $\text{send}(m)_{u,v}$
    - Effect: add m to $mqnene$.
  - $\text{receive}(m)_{u,v}$
    - Precondition: m = head($mqnene$)
    - Effect: remove head of $mqnene$



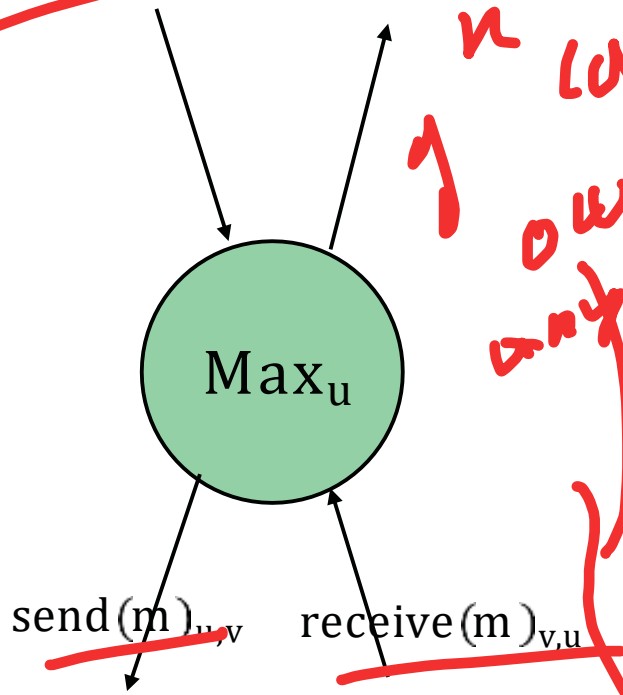
Process Automaton P_u

- Associate a process automaton with each vertex of G.
- To simplify notation, let P_u denote the process automaton at vertex n.
  - But the process does not "know" n.
- P_u has $send(m)_{u,v}$ outputs and $receive(m)_{v,u}$ inputs.
- May also have external inputs and outputs.
- Has state variables.
- Keeps taking steps (eventually).



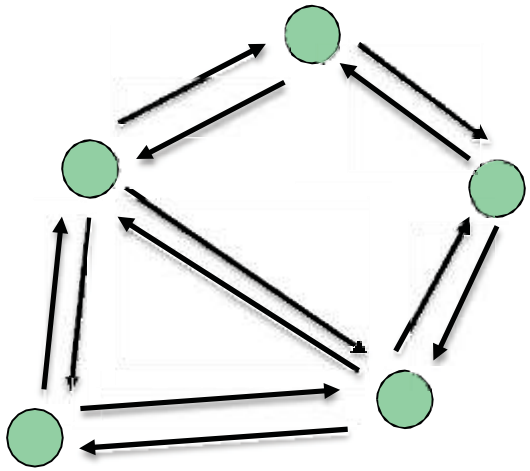
Example: Max_u Process Automaton

- Input actions: $\mathrm{receive}(m)_{v,u}$
- Output actions: $\mathrm{send}(m)_{u,v}$
- State variables:
  - max, a natural number, initially x_u
  - For each neighbor v:
    - $send(v)$, a Boolean, initially $true$
- Transitions:
  - $\mathrm{receive}(m)_{v,u}$
    - Effect: if $m > max$ then
      - $max := m$
      - for every w, $send(w) := true$
  - $\mathrm{send}(m)_{u,v}$
    - Precondition: $send(v) = true$ and $m = max$
    - Effect: $send(v) := false$



Combining Processes and Channels

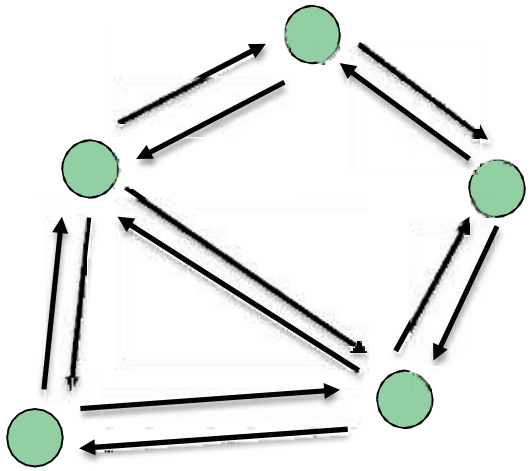
- Undirected graph $G = (V, E)$.
- Process P_u at each vertex n.
- Channels $C_{u,v}$ and $C_{v,u}$, associated with each edge $\{n, v\}$.
- $send(m)_{u,v}$ output of process P_u gets identified with $send(m)_{u,v}$ input of channel $C_{u,v}$.
- $receive(m)_{v,u}$ output of channel $C_{v,u}$ gets identified with $receive(m)_{v,u}$ input of process P_u.
- Steps involving such a shared action involve simultaneous state transitions for a process and a channel.



Execution

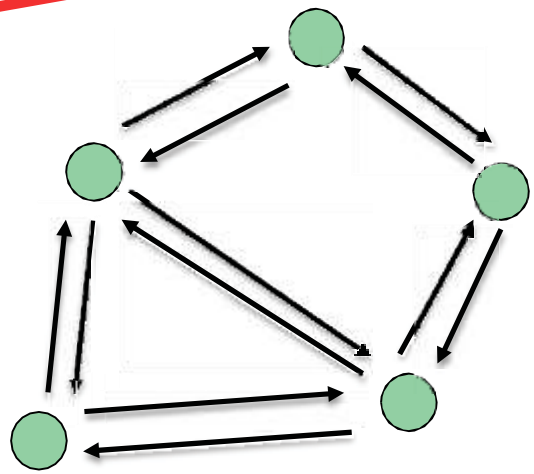
- No synchronous rounds anymore.
- The system executes by performing enabled steps, one at a time, in any order.
- Formally, an execution is modeled as a sequence of individual steps.
- Different from the synchronous model, in which all processes take steps concurrently at each round.

- Assume enabled steps eventually occur:
  - Each channel always eventually delivers the first message in its queue.
  - Each process always eventually performs some enabled step.

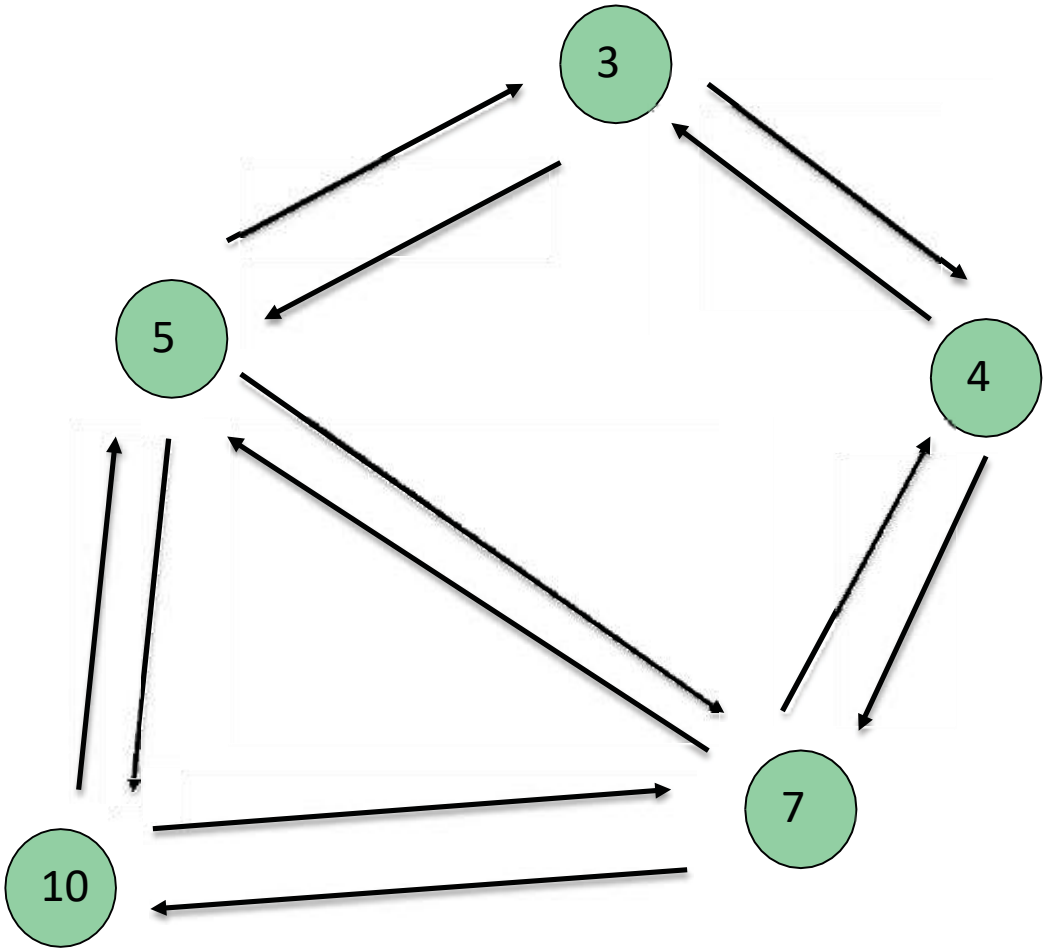


Combining Max Processes and Channels

- Each process Max_u starts with an initial value x_u.
- They all send out their initial values, and propagate their $\max$ values, until everyone has the globally-maximum value.
- Sending and receiving steps can happen in many different orders, but in all cases the global max will eventually arrive everywhere.



Max System



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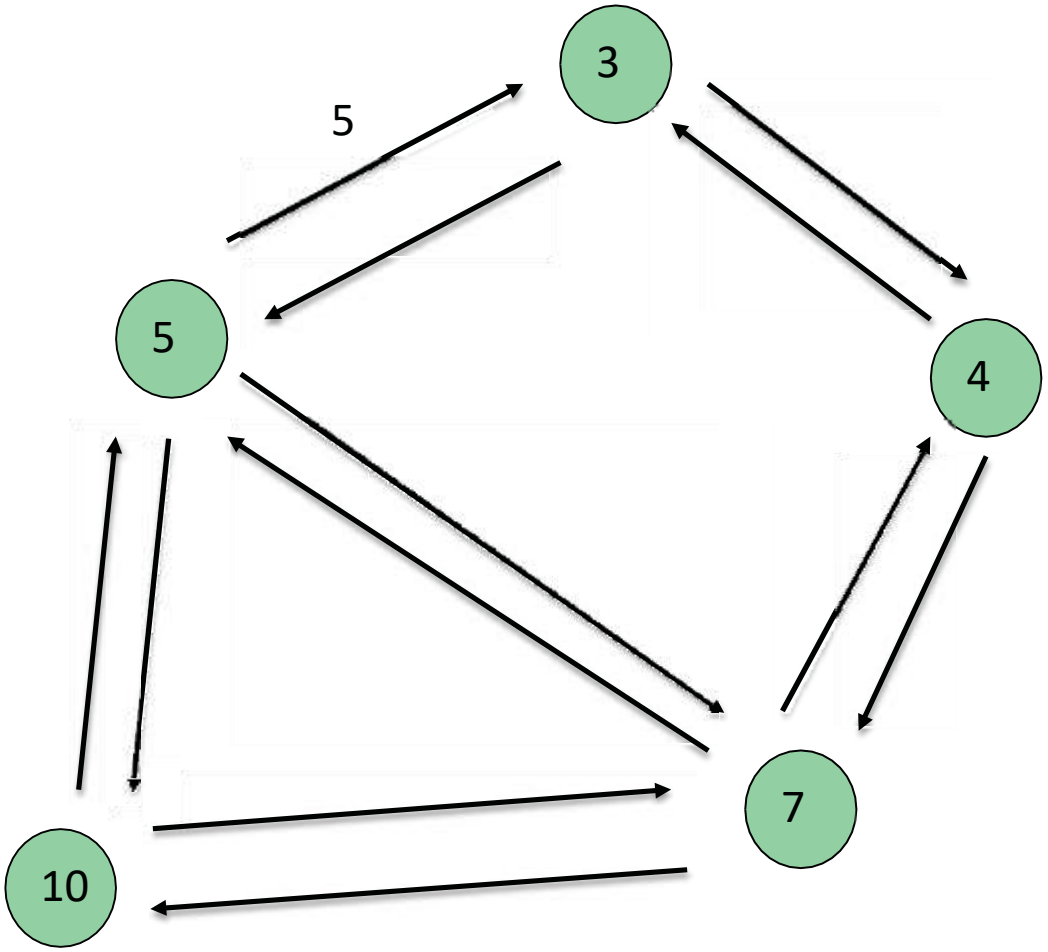

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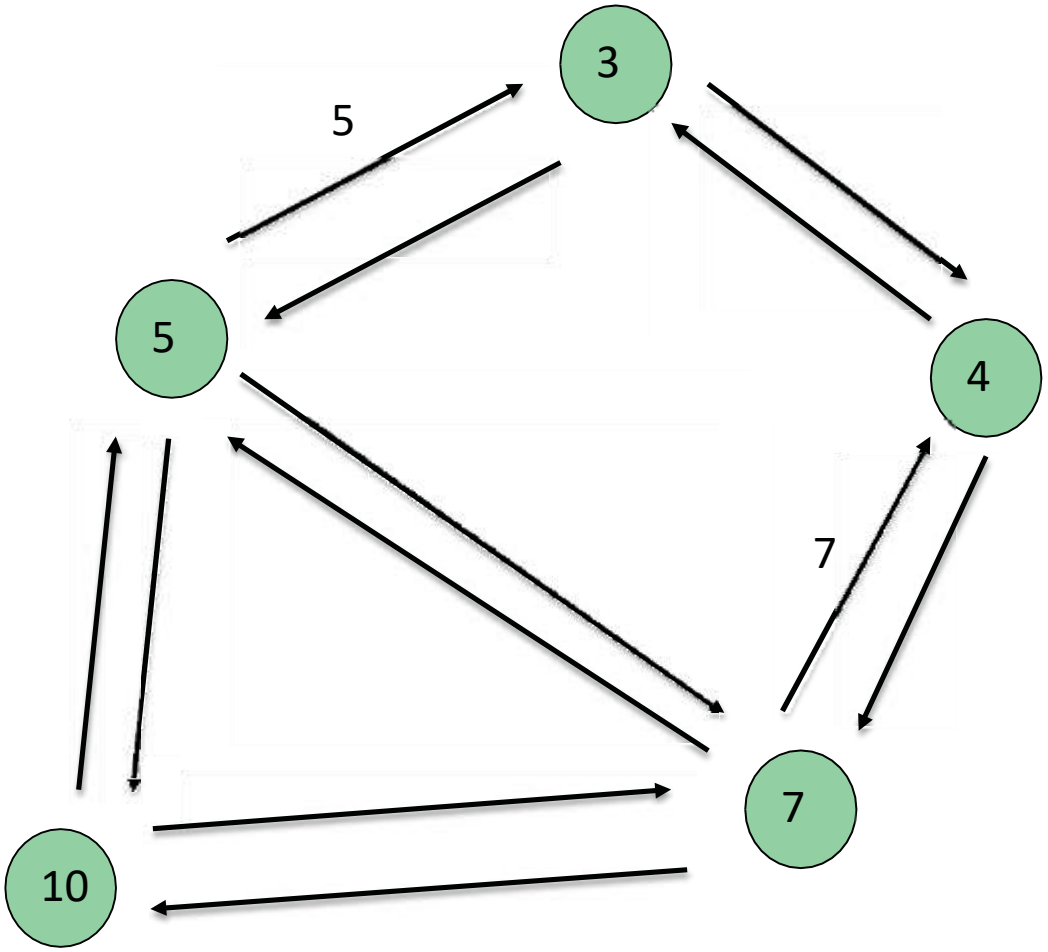

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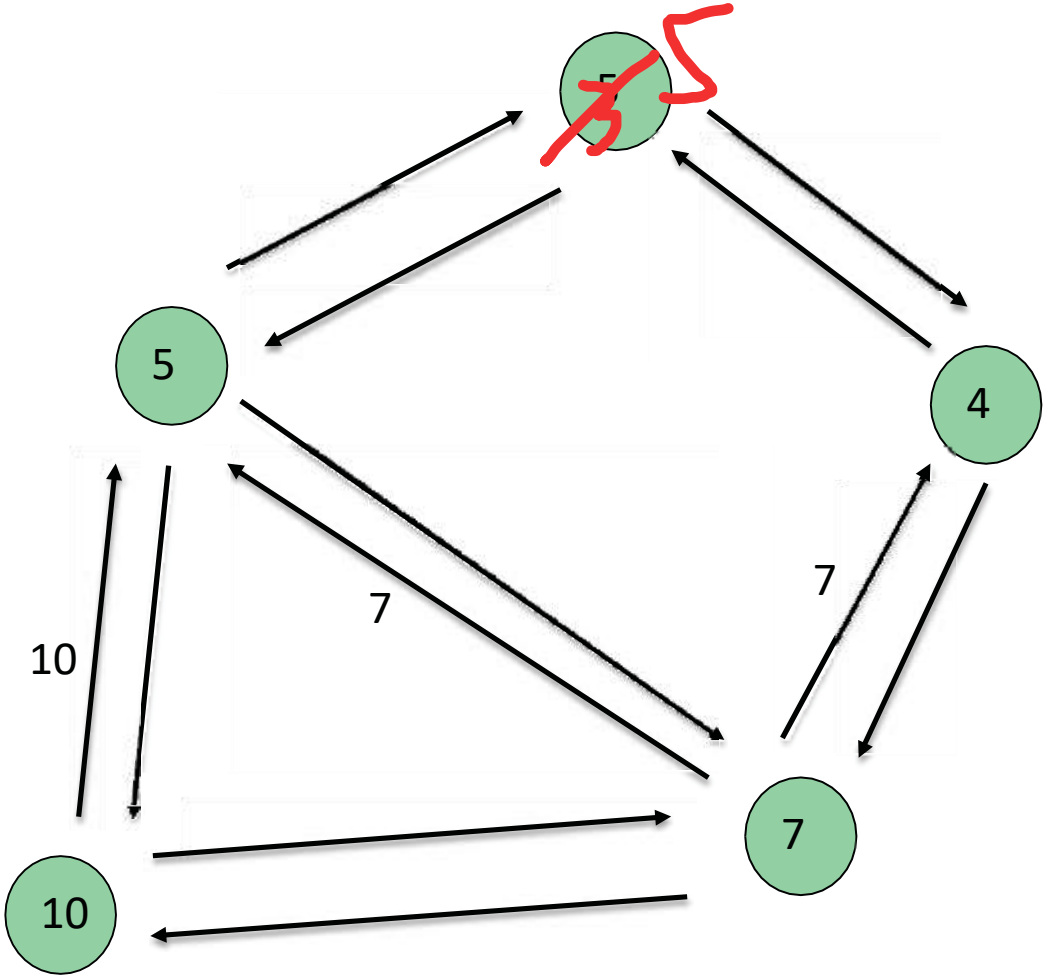

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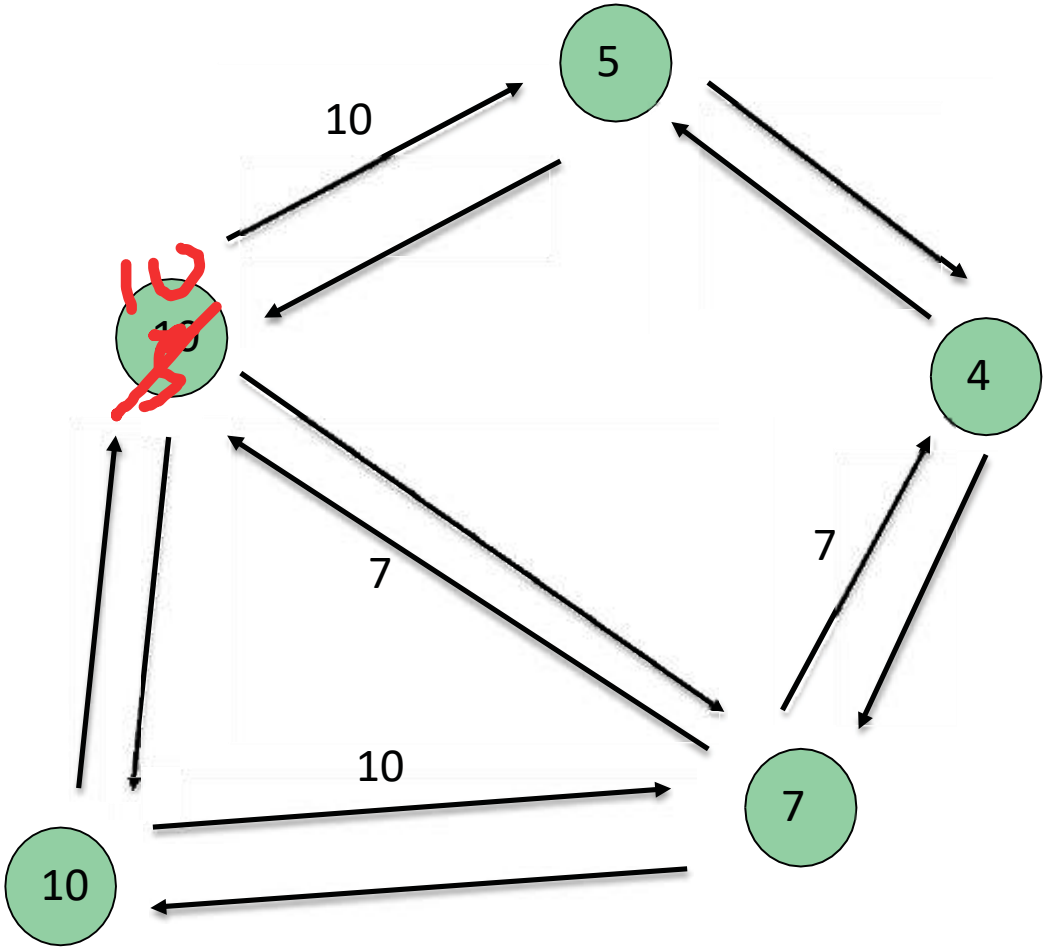

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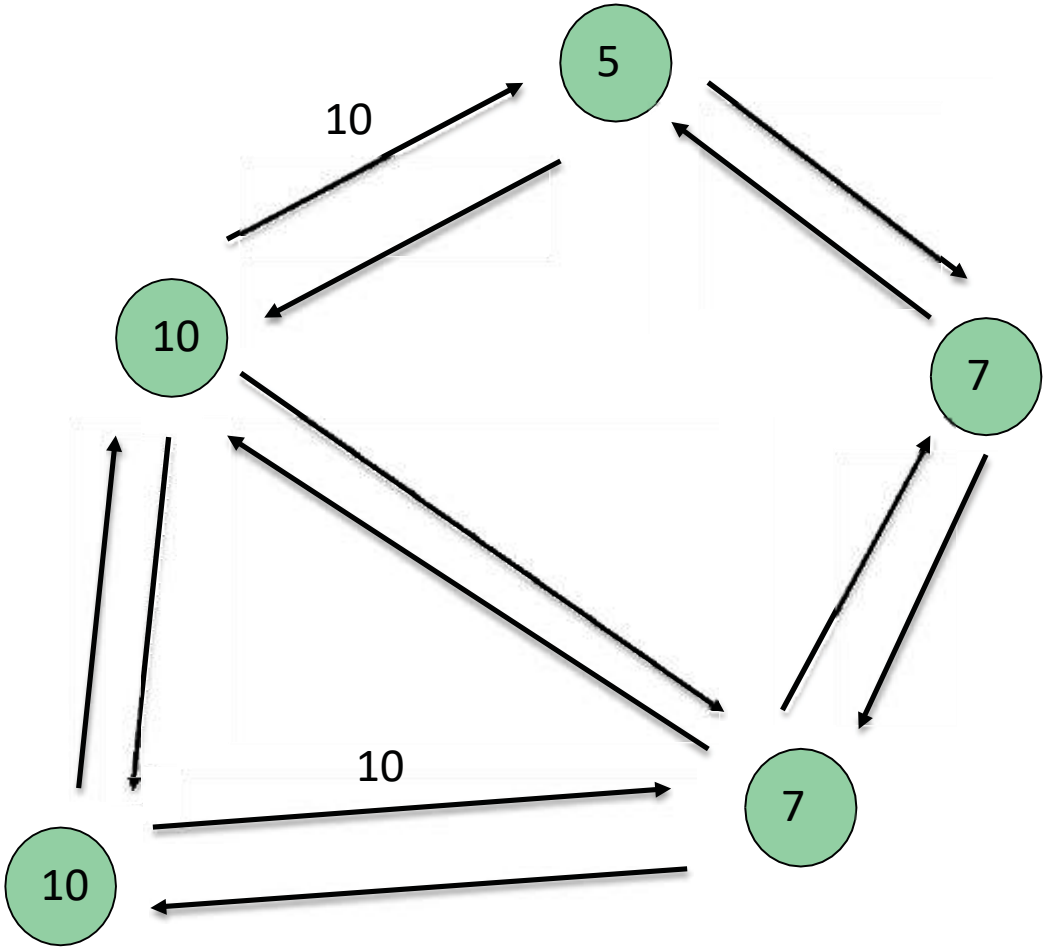

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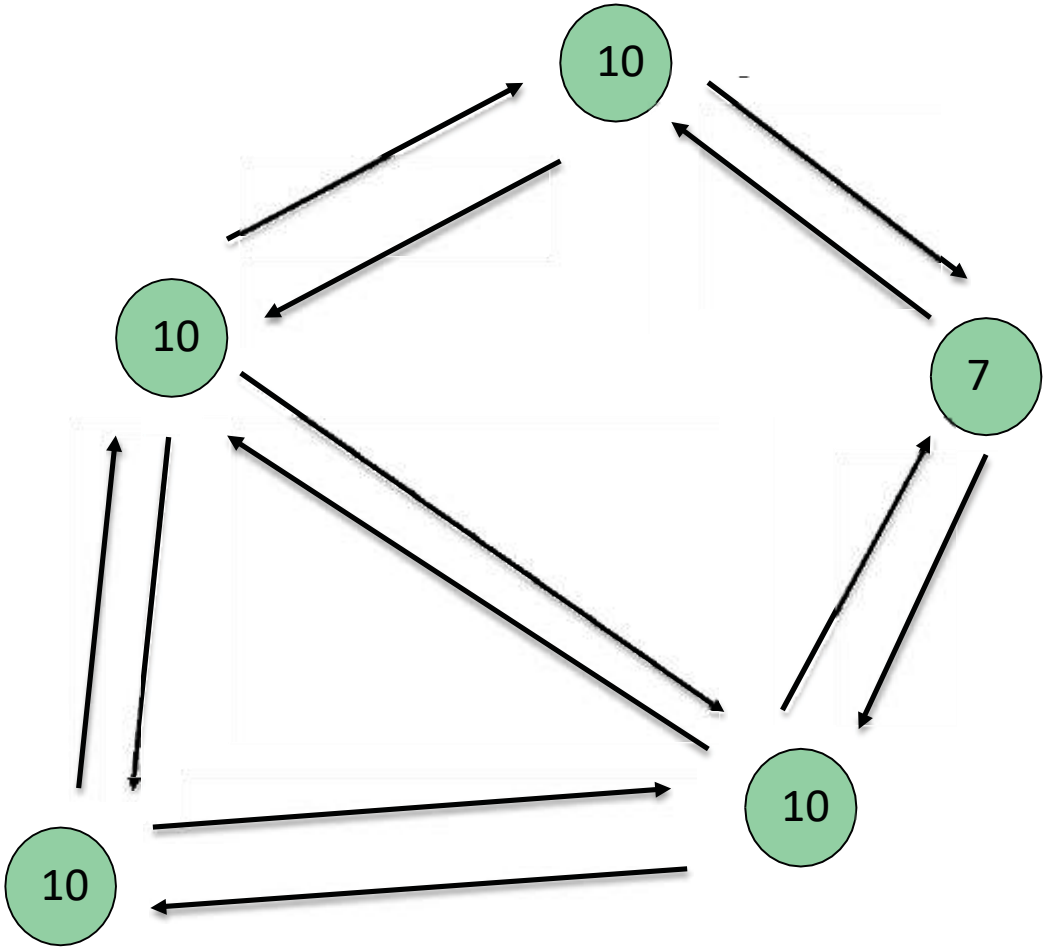

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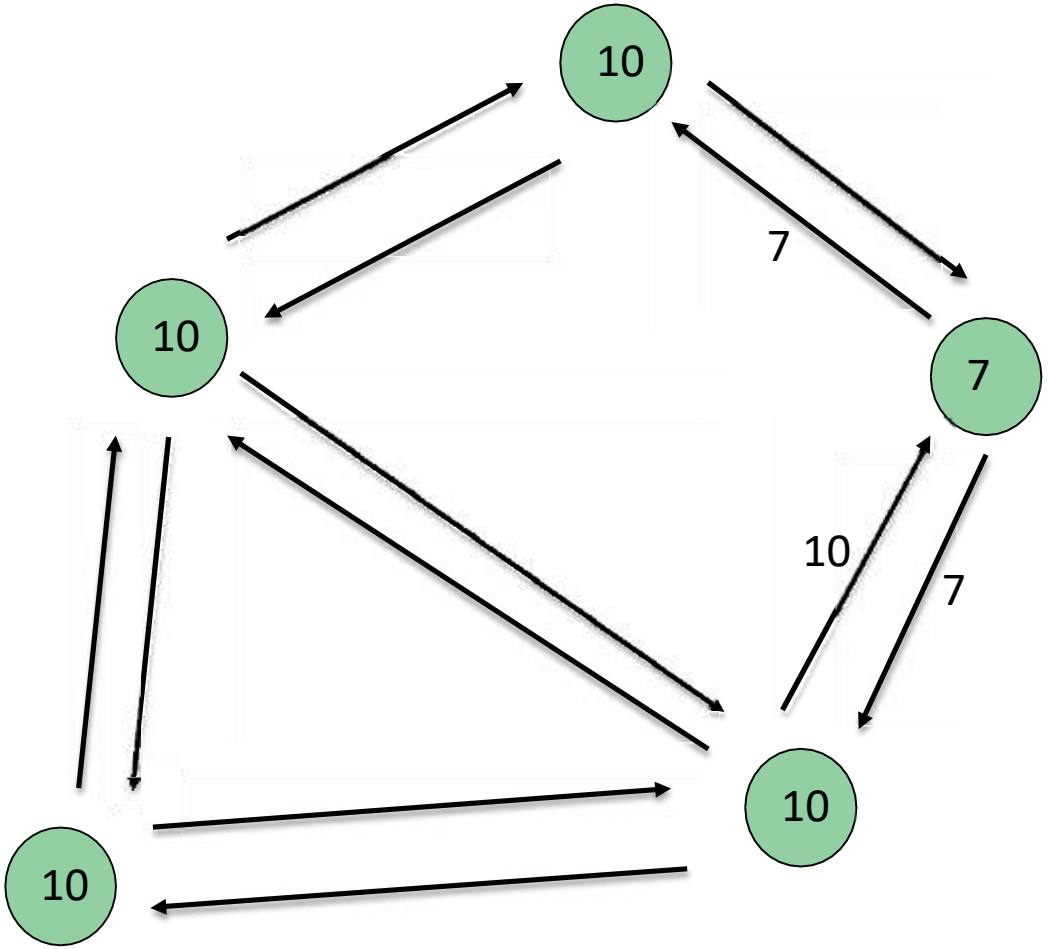

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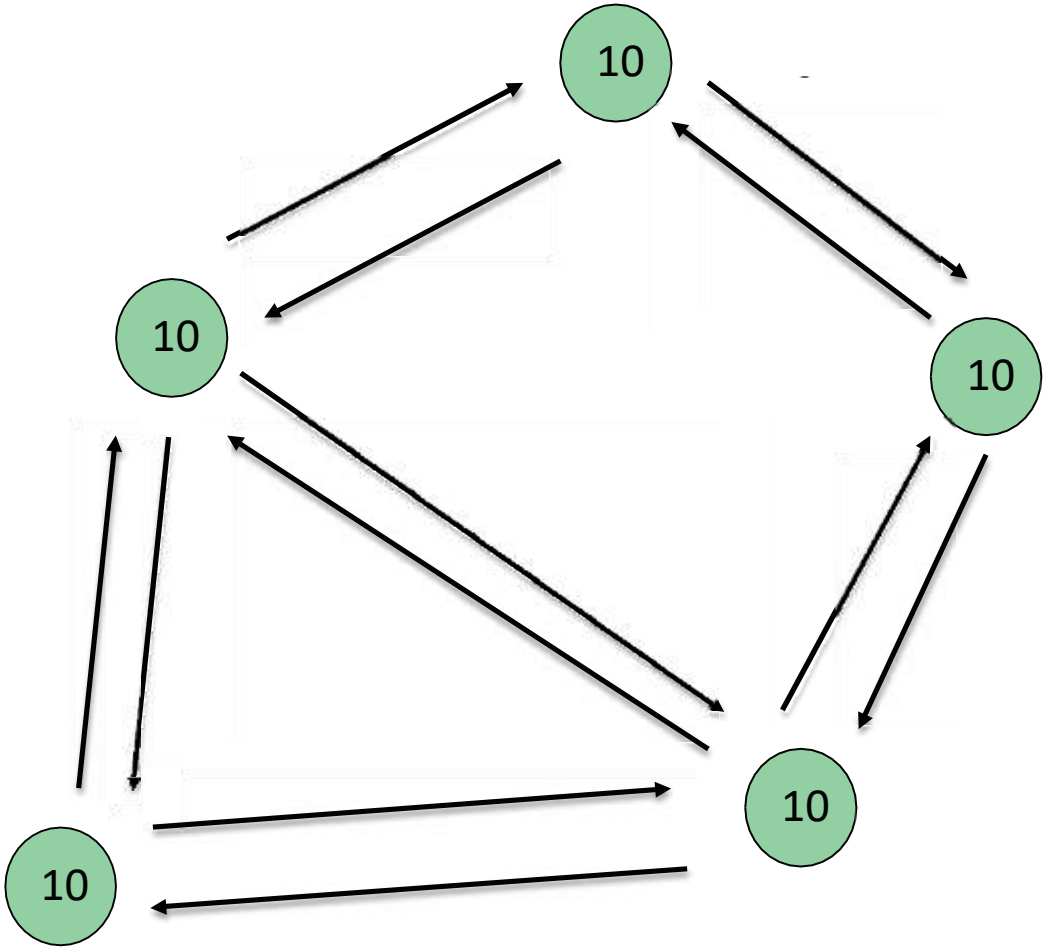


Complexity

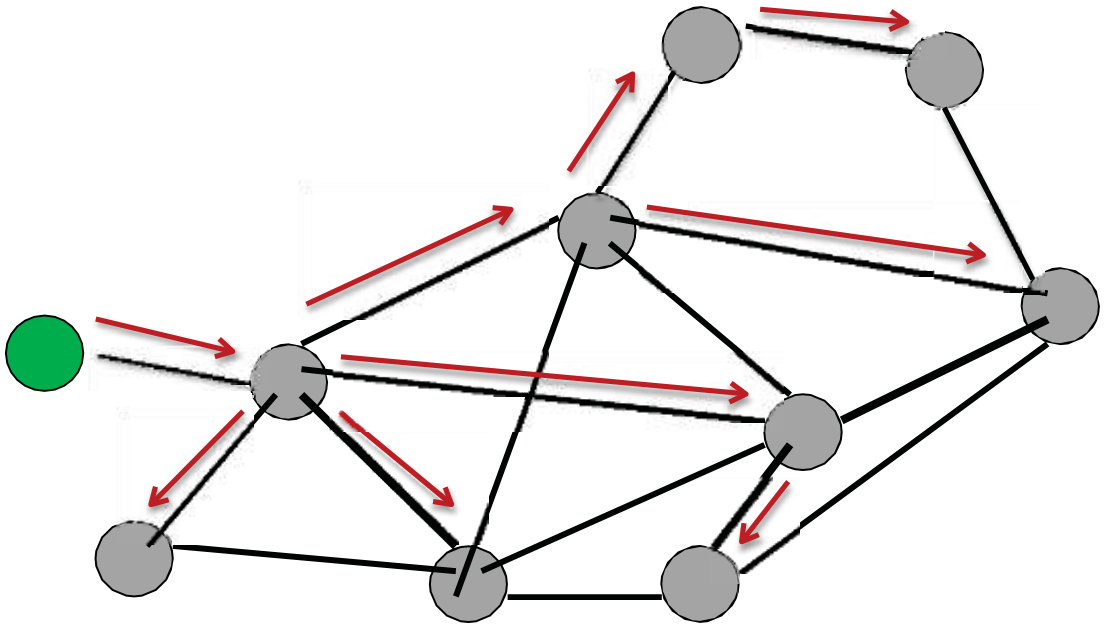
- Message complexity:
  - Number of messages sent by all processes during the entire execution.
  - $O(n \cdot |E|)$
- Time complexity:
  - Q: What should we measure?
  - Not obvious, because the various components are taking steps in arbitrary orders---no "rounds".
  - A common approach:
    - Assume real-time upper bounds on the time to perform basic steps:
      - d for a channel to deliver the next message, and
      - l for a process to perform its next step.
    - Infer a real-time upper bound for solving the overall problem.

Complexity

- Time complexity:
  - Assume real-time upper bounds on the time to perform basic steps:
    - d for a channel to deliver the next message, and
    - l for a process to perform its next step.
  - Infer a real-time upper bound for solving the problem.

- For the Max system:
  - Ignore local processing time ($l = 0$), consider only channel sending time.
  - Straightforward upper bound: $O(diam \cdot n \cdot d)$
    - Consider the time for the max to reach any particular vertex n, along a shortest path in the graph.
    - At worst, it waits in each channel on the path for every other value, which is at most time $n \cdot d$ for that channel.

Breadth-First Spanning Trees



Breadth-First Spanning Trees

- Problem: Compute a Breadth-First Spanning Tree in an asynchronous network.
- Connected graph $G = (V, E)$.
- Distinguished root vertex v_a.
- Processes have no knowledge about the graph.
- Processes have UIDs
  - i_a is the UID of the root v_a.
  - Processes know UIDs of their neighbors, and know which ports are connected to each neighbor.
- Processes must produce a BFS tree rooted at v_a.
- Each process $i = i_a$ should output $\mathrm{parent}(j)$, meaning that j's vertex is the parent of i's vertex in the BFS tree.

First Attempt

- Just run the simple synchronous BFS algorithm asynchronously.
- Process i_a sends search messages, which everyone propagates the first time they receive it.
- Everyone picks the first node from which it receives a search message as its parent.

- Nondeterminism:
  - No longer any nondeterminism in process decisions.
  - But plenty of new nondeterminism: orders of message deliveries and process steps.

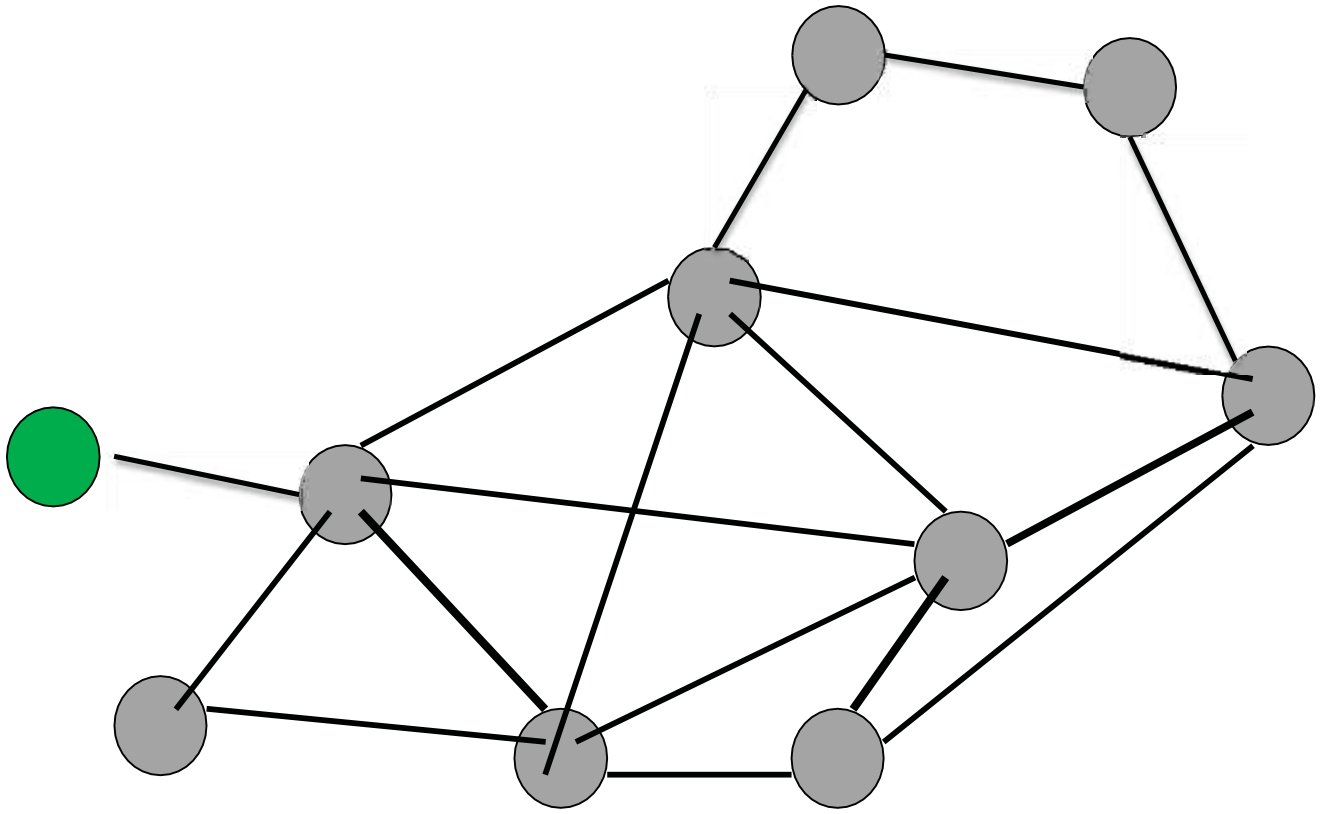
Process Automaton P_u

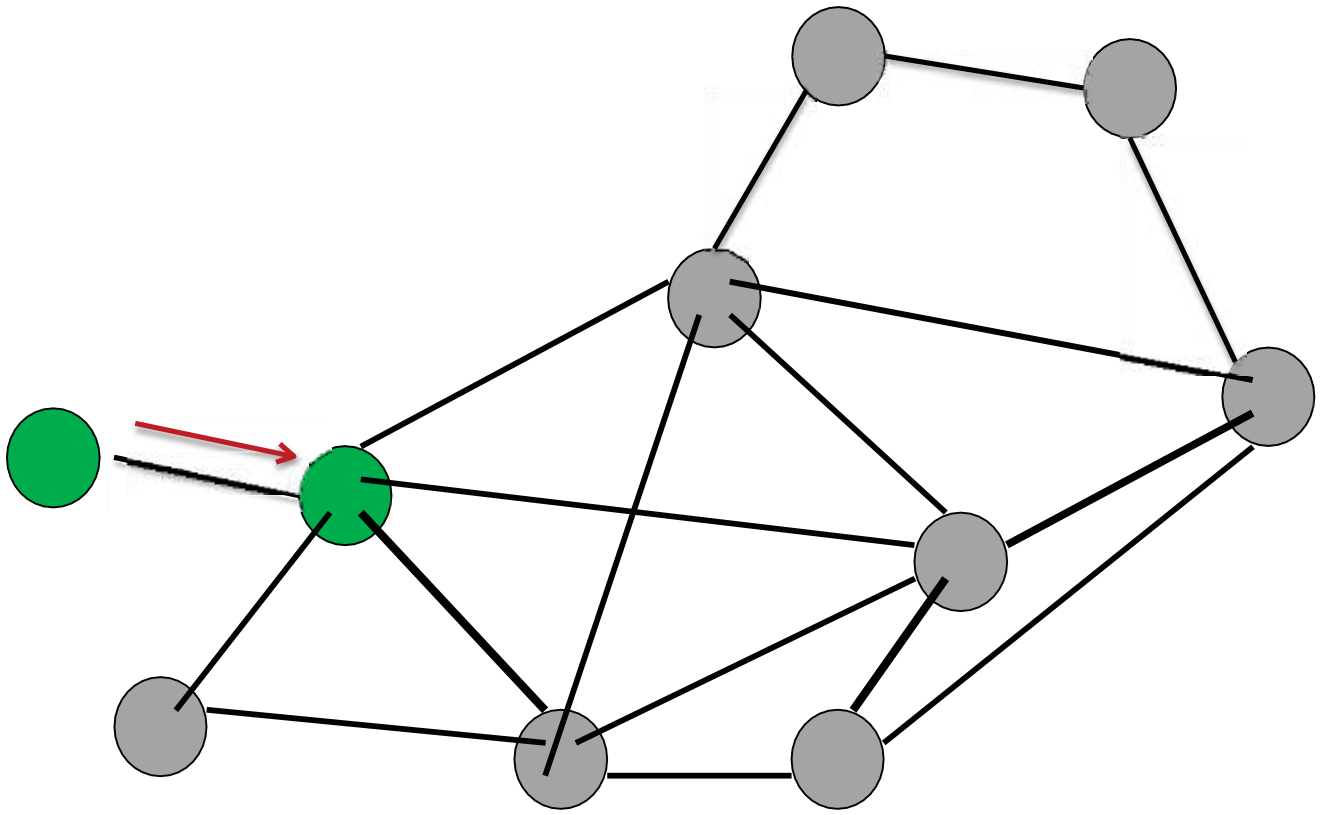
- Input actions: $receive(search)_{v,u}$
- Output actions: $send(search)_{u,v}$; $parent(v)_u$
- State variables:
  - parent: $T(n) \cup \{1\}$, initially 1
  - reported: Boolean, initially false
  - For every $v \in T(n)$:
    - $send(v) \in \{search, 1\}$, initially search if $n = v_a$, else 1
- Transitions:
  - $receive(search)_{v,u}$
    - Effect: if $n = v_a$ and $parent = 1$ then
      - parent := v
      - for every w, $send(w)$:= search

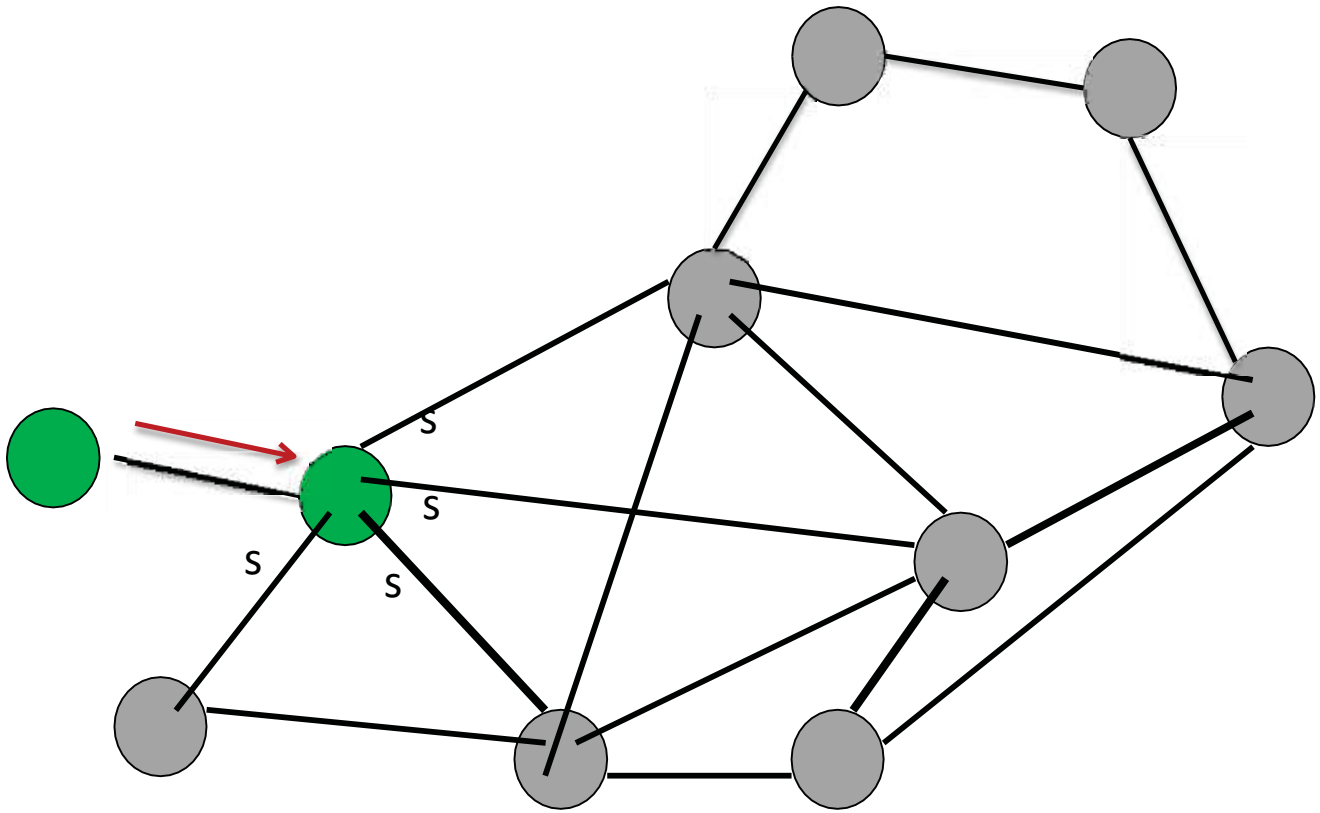
Process Automaton P_u

- Transitions:
  - $\text{receive}(\text{search})_{v,u}$
    - Effect: if $n = v_a$ and $\text{parent} = 1$ then
      - $\text{parent} := v$
      - for every w, $\text{send}(w) := \text{search}$
  - $\text{send}(\text{search})_{u,v}$
    - Precondition: $\text{send}(v) = \text{search}$
    - Effect: $\text{send}(v) : 1$
  - $\text{parent}(v)_u$
    - Precondition: $\text{parent} = v$ and $\text{reported} = \text{false}$
    - Effect: $\text{reported} : \text{tr}n\text{e}$

Running Simple BFS Asynchronously



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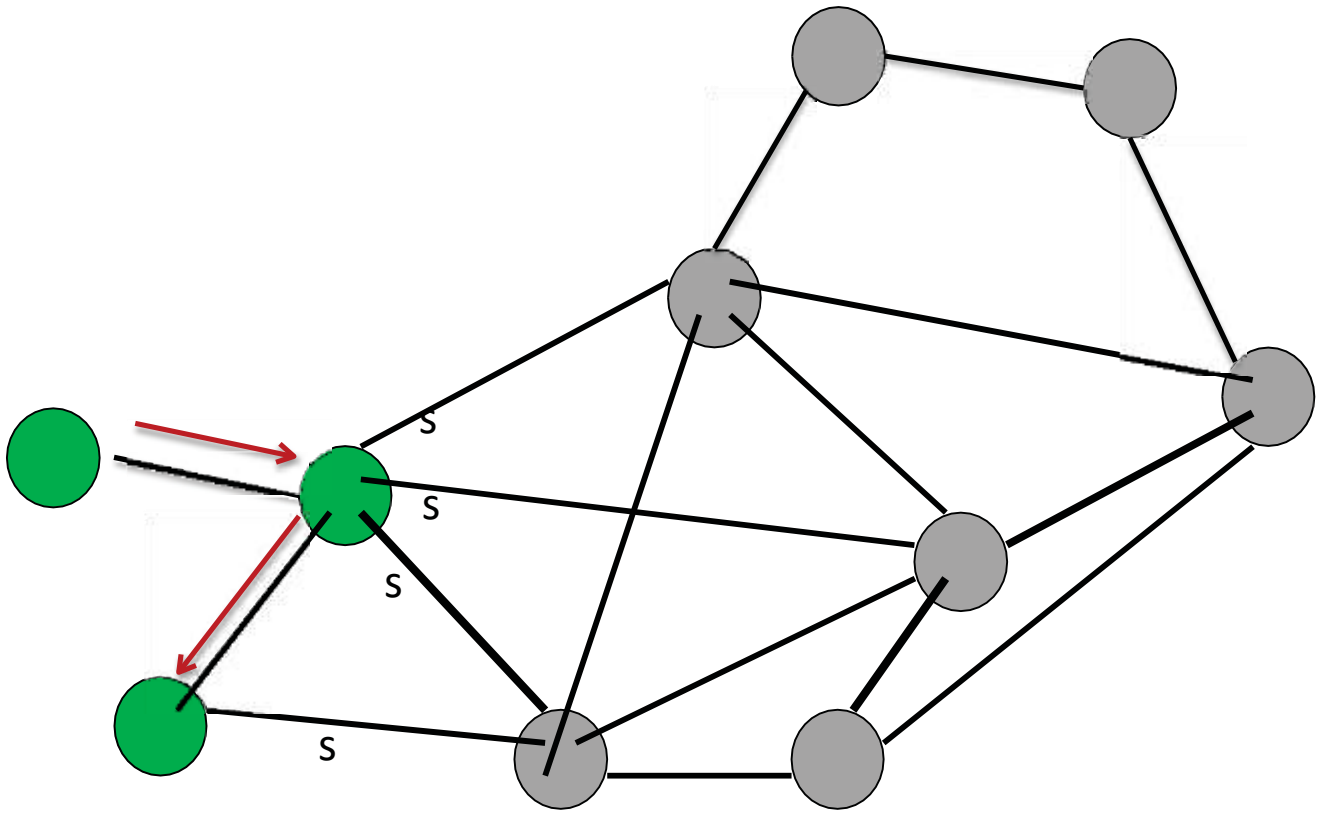
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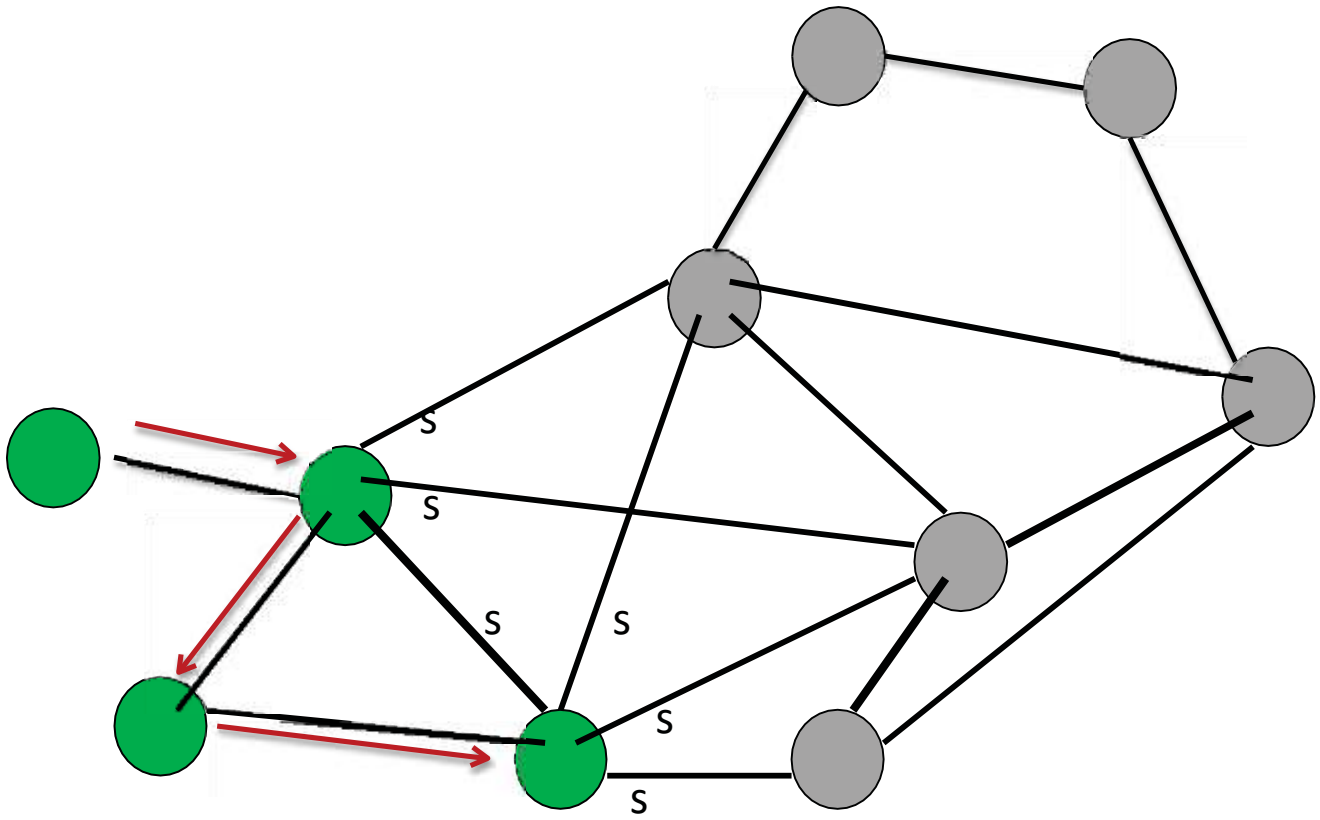
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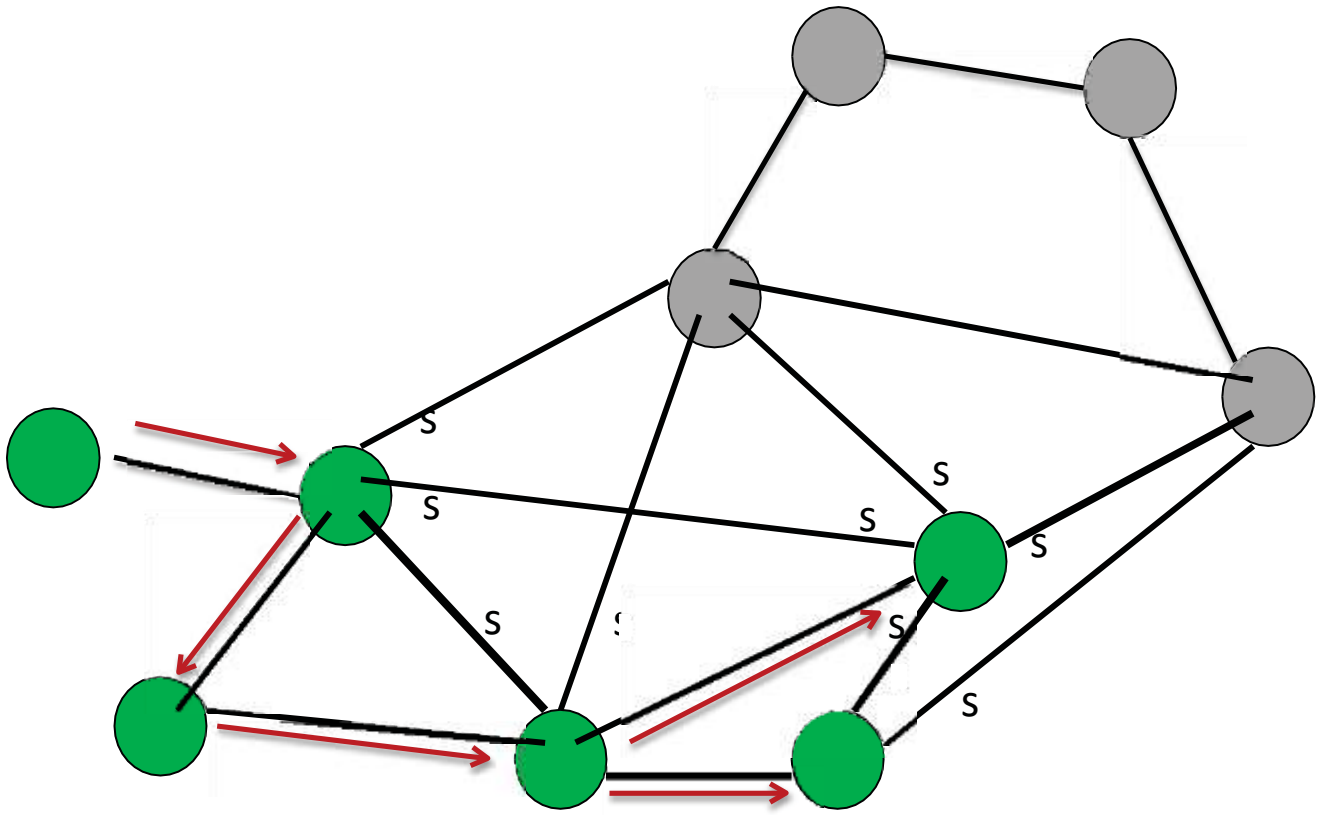
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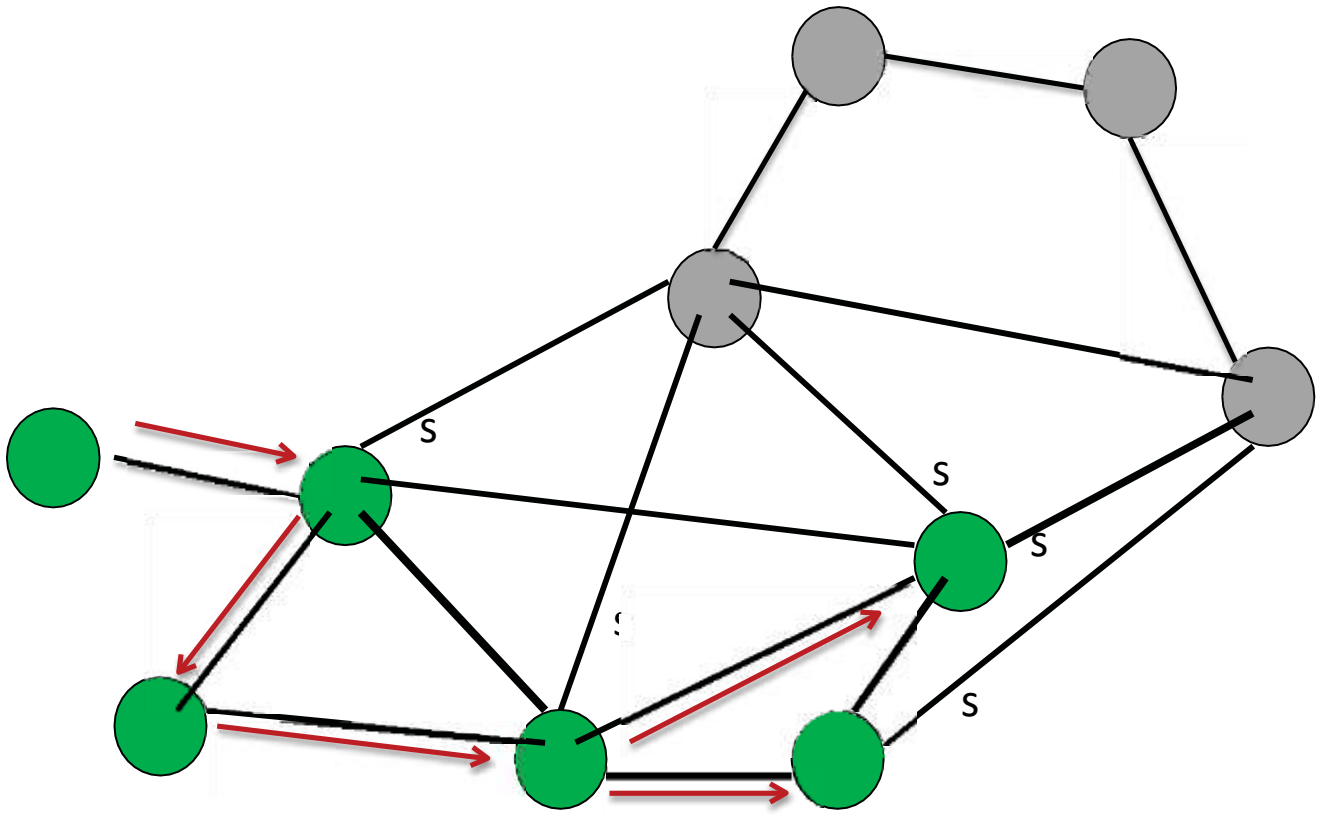
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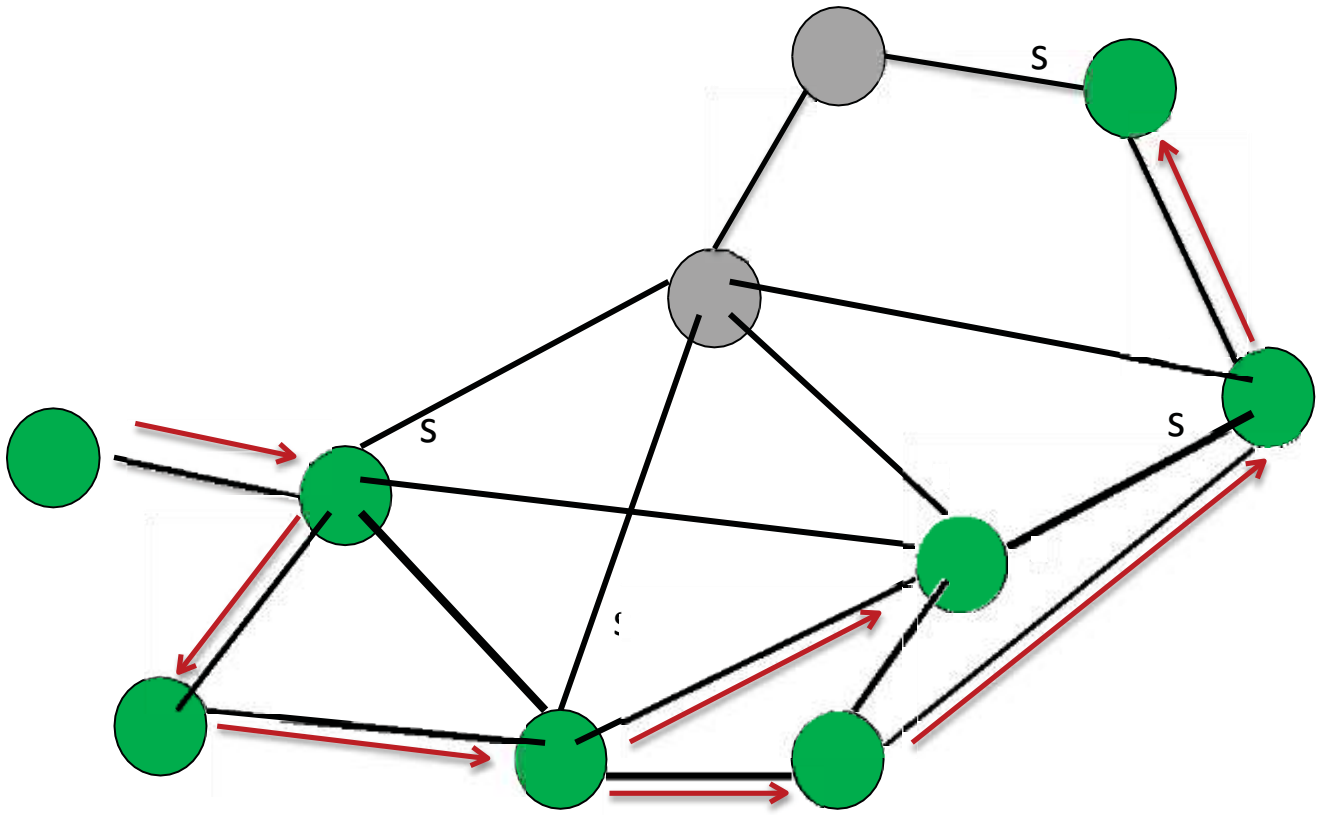
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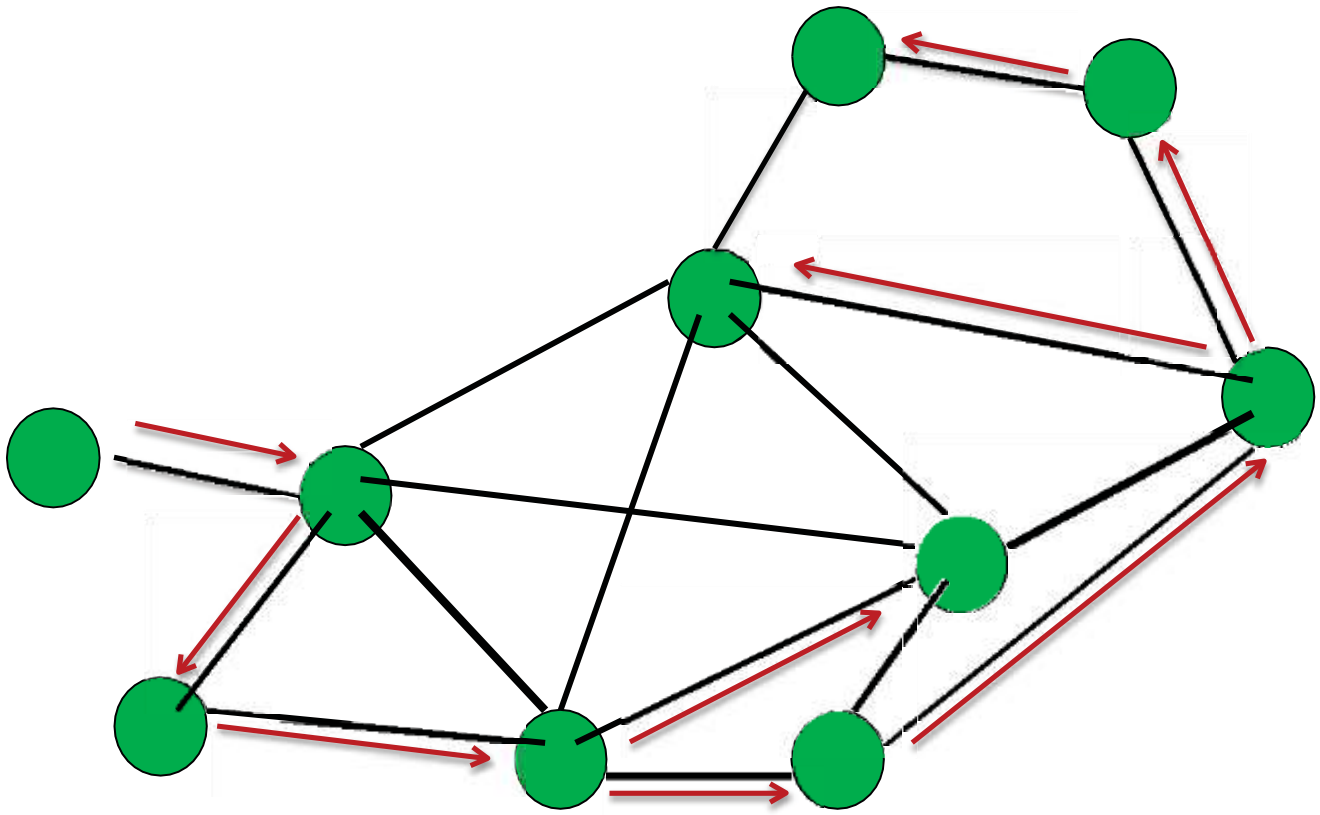
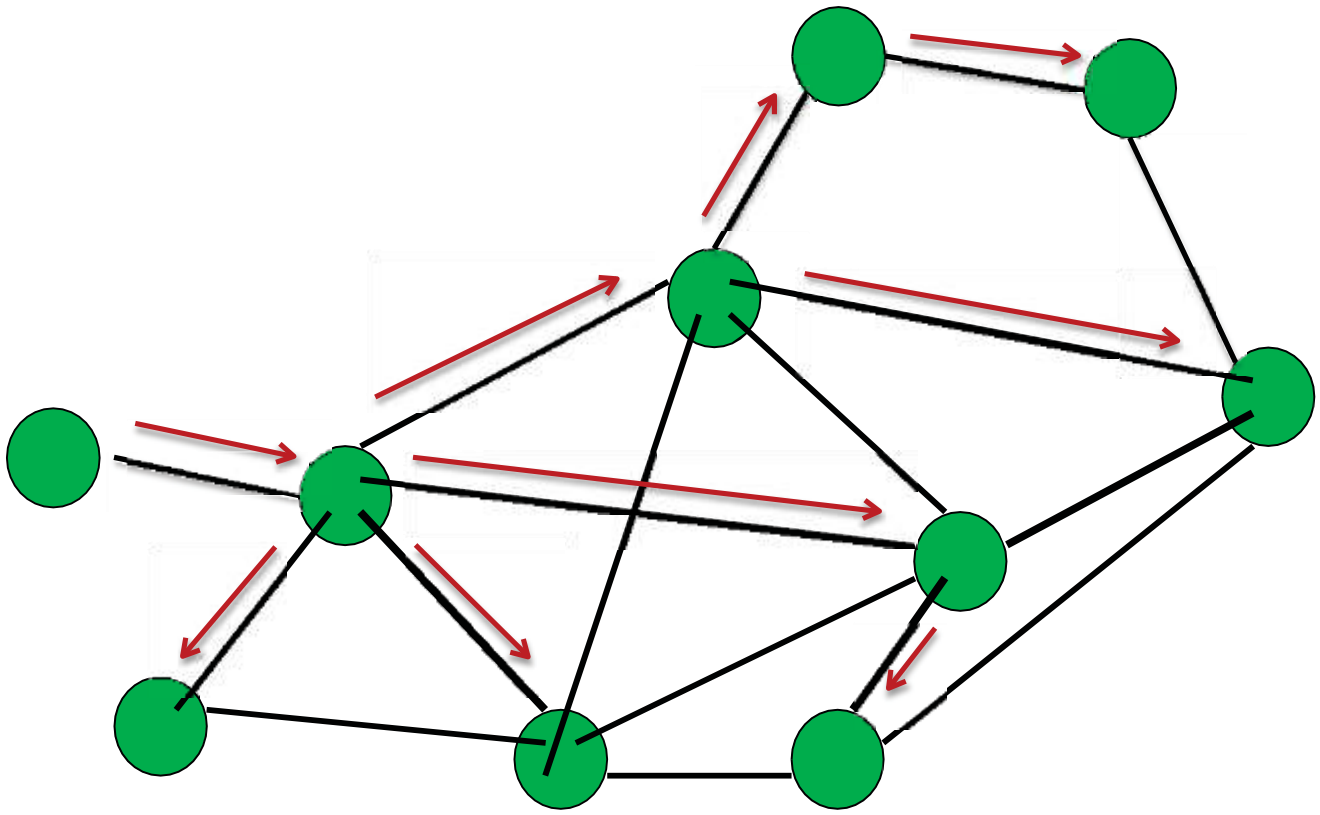
Final Spanning Tree

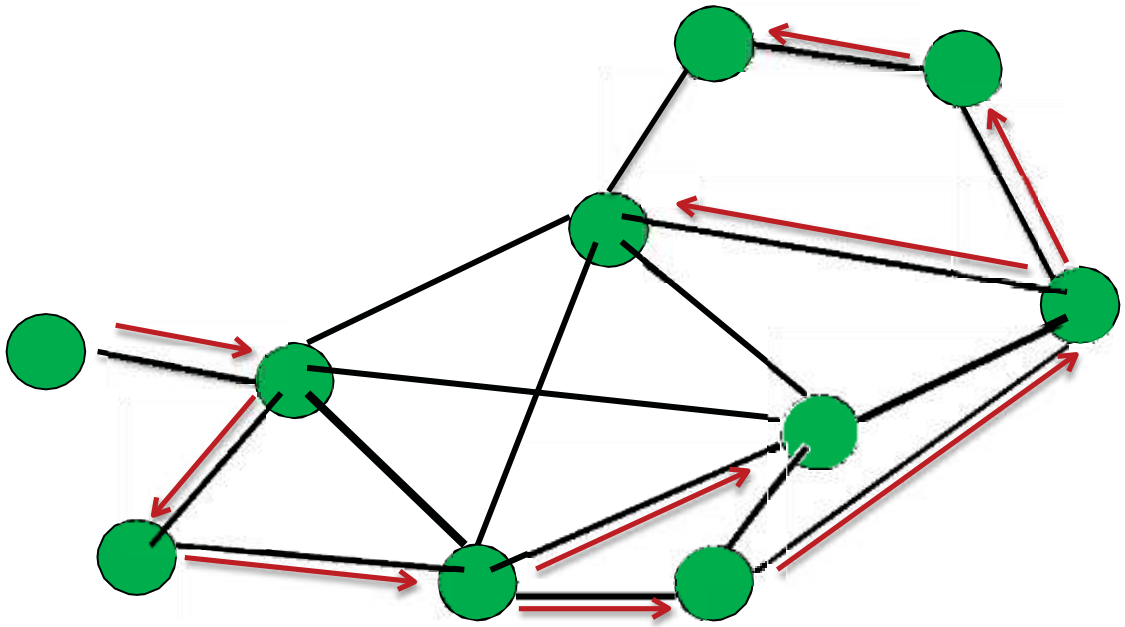


Actual BFS



Anomaly

- Paths produced by the algorithm may be longer than the shortest paths.
- Because in asynchronous networks, messages may propagate faster along longer paths.



Complexity

- Message complexity:
  - Number of messages sent by all processes during the entire execution.
  - $O(|E|)$

- Time complexity:
  - Time until all processes have chosen their parents.
  - Neglect local processing time.
  - $O(\text{diam} \cdot d)$ — all nodes reached.
  - Q: Why diam, when some of the paths are longer?
  - The time until a node receives a search message is at most the time it would take on a shortest path.

Extensions

- Child pointers:
  - As for synchronous BFS.
  - Everyone who receives a `search` message sends back a `parent` or `nonparent` response.
- Termination:
  - After a node has received responses to all its `search` its messages, it knows who its children are, and knows they are marked.
  - The leaves of the tree learn who they are.
  - Use a convergecast strategy, as before.
  - Time complexity: After the tree is done, it takes time $O(n \cdot d)$ for the done information to reach i_a.
  - Message complexity: $O(n)$

Applications

- Message broadcast:
  - Process i_a can use the tree (with child pointers) to broadcast a message.
  - Takes $O(n \cdot d)$ time and n messages.
- Global computation:
  - Suppose every process starts with some initial value, and process i_a should determine the value of some function of the set of all processes' values.
  - Use convergecast on the tree.
  - Takes $O(n \cdot d)$ time and n messages.

Second Attempt

- A relaxation algorithm, like synchronous Bellman-Ford.
- Before, we corrected for paths with many hops but low weights.
- Now, instead, correct for errors caused by asynchrony.
- Strategy:
  - Each process keeps track of the hop distance, changes its parent when it learns of a shorter path, and propagates the improved distances.
  - Eventually stabilizes to a breadth-first spanning tree.

Process Automaton P_u

- Input actions: $receive(m)_{v,u}$, m a nonnegative integer
- Output actions: $send(m)_{u,v}$, m a nonnegative integer

- State variables:
  - parent: $T(n) \cup \{1\}$, initially 1
  - dist $\in N \cup \{0\}$, initially 0 if $n = v_a$, 0 otherwise
  - For every $v \in T(n)$
    - $send(v)$, a FIFO queue of N, initially (0) if $n = v_a$, else empty

- Transitions:
  - $receive(m)_{v,u}$
    - Effect: if $m + 1 < dist$ then
      - $dist := m + 1$
      - $parent := v$
      - for every w, add $dist$ to $send(w)$

m = no. of hops

Process Automaton P_u

- Transitions:
  - $\text{receive}(m)_{v,u}$
    - Effect: if $m + 1 < \text{dist}$ then
      - $\text{dist} := m + 1$
      - $\text{parent} := v$
      - for every w, add $m + 1$ to $\text{send}(w)$
  - $\text{send}(m)_{u,v}$
    - Precondition: $m = \text{head}(\text{send}(v))$
    - Effect: remove head of $\text{send}(v)$

- No terminating actions…

Correctness

- For synchronous BFS, we characterized precisely the situation after r rounds.
- We can't do that now.
- Instead, state abstract properties, e.g., invariants and timing properties, e.g.:
- Invariant: At any point, for any node $n = v_a$, if its $\mathrm{dist} = 0$, then it is the actual distance on some path from v_a to n, and its parent is n's predecessor on such a path.
- Timing property: For any node n, and any r, $0 < r < \mathrm{diam}$, if there is an at-most-r-hop path from v_a to n, then by time $r \cdot n \cdot d$, node n's dist is $< r$.

Complexity

- Message complexity:
  - Number of messages sent by all processes during the entire execution.
  - $O(n\,|E|)$
- Time complexity:
  - Time until all processes' $dist$ and $parent$ values have stabilized.
  - Neglect local processing time.
  - $O(diam \cdot n \cdot d)$
    - Time until each node receives a message along a shortest path, counting time $O(n \cdot d)$ to traverse each link.

Termination

- Q: How can processes learn when the tree is completed?
- Q: How can a process know when it can output its own dist and parent?
- Knowing a bound on n doesn't help here: can't use it to count rounds.

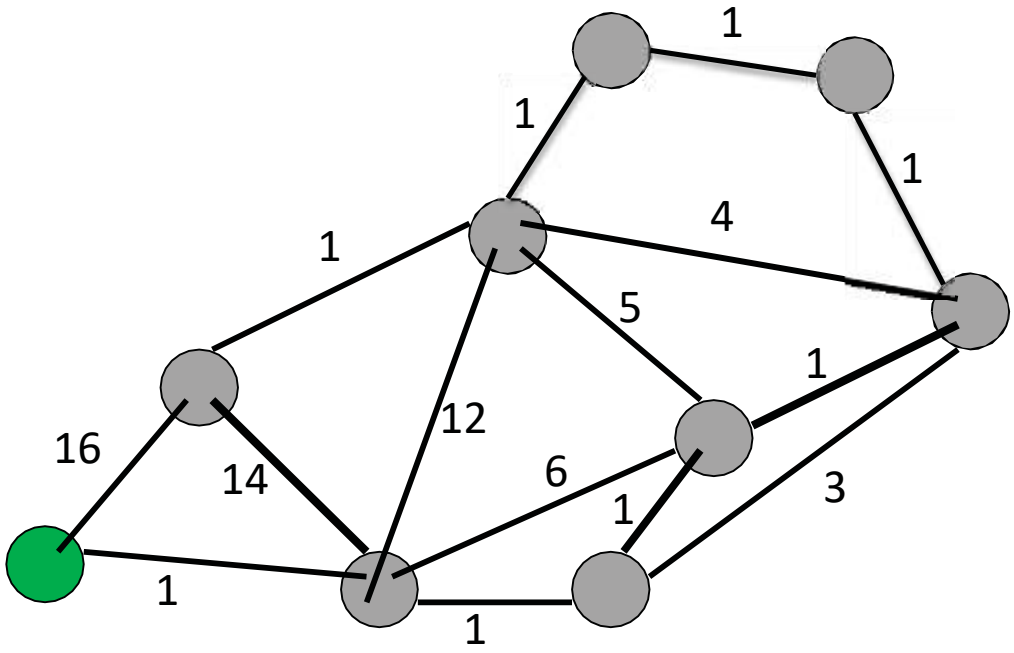
- Can use convergecast, as for synchronous Bellman-Ford:
  - Compute and recompute child pointers.
  - Process $= v_a$ sends done to its current parent after:
    - It has received responses to all its messages, so it believes it knows all its children, and
    - It has received done messages from all of those children.

  The same process may be involved several times, based on improved estimates.

Uses of Breadth-First Spanning Trees

- Same as in synchronous networks, e.g.:
  - Broadcast a sequence of messages
  - Global function computation
- Similar costs, but now count time d instead of one round.

Shortest Paths Trees



Shortest Paths

- Problem: Compute a Shortest Paths Spanning Tree in an asynchronous network.
- Connected weighted graph, root vertex v_a.
- $\text{weight}_{\{u,v\}}$ for edge $\{n, v\}$.
- Processes have no knowledge about the graph, except for weights of incident edges.
- UIDs

- Processes must produce a Shortest Paths spanning tree rooted at v_a.
- Each process $n = v_a$ should output its distance and parent in the tree.

Shortest Paths

- Use a relaxation algorithm, once again.
- Asynchronous Bellman-Ford.

- Now, it handles two kinds of corrections:
  - Because of long, small-weight paths (as in synchronous Bellman-Ford).
  - Because of asynchrony (as in asynchronous Breadth-First search).
- The combination leads to surprisingly high message and time complexity, much worse than either type of correction alone (exponential).

Asynch Bellman-Ford, Process P_u

- Input actions: $receive(m)_{v,u}$, m a nonnegative integer
- Output actions: $send(m)_{u,v}$, m a nonnegative integer

- State variables:
  - parent: $T(n)$ U $\{1\}$, initially 1
  - dist E N U $\{0\}$, initially 0 if $n = v_a$, 0 otherwise
  - For every v E $T(n)$
    - $send(v)$, a FIFO queue of N, initially (0) if $n = v_a$, else empty

- Transitions:
  - $receive(m)_{v,u}$
    - Effect: if $m + weight_{\{v,u\}} < dist$ then
      - $dist := m + weight_{\{v,u\}}$
      - $parent := v$
      - for every w, add dist to $send(w)$

Asynch Bellman-Ford, Process P_u

- Transitions:
  - $\text{receive}(m)_{v,u}$
    - Effect: if $m + \text{weight}_{\{v,u\}} < \text{dist}$ then
      - $\text{dist} := m + \text{weight}_{\{v,u\}}$
      - $\text{parent} := v$
      - for every w, add dist to send(w)
  - $\text{send}(m)_{u,v}$
    - Precondition: $m = \text{head}(\text{send}(v))$
    - Effect: remove head of send(v)

- No terminating actions…

Correctness: Invariants and Timing Properties

- Invariant: At any point, for any node $n = \mathrm{v_a}$, if its $\mathtt{dist} = 0$, then it is the actual distance on some path from $\mathrm{v_a}$ to n, and its $\mathtt{parent}$ is n's predecessor on such a path.

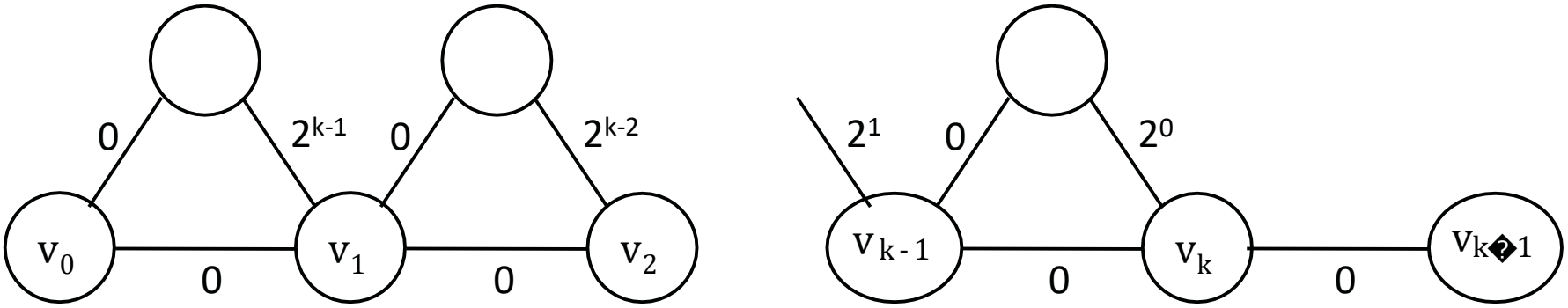
- Timing property: For any node n, and any r, $0 < \mathrm{r} < \mathrm{diam}$, if p is any at-most-r-hop path from $\mathrm{v_a}$ to n, then by time ???, node n's $\mathtt{dist}$ is $<$ total weight of p.

- Q: What is ??? ?
- It depends on how many messages might pile up in a channel.
- This can be a lot!

Complexity

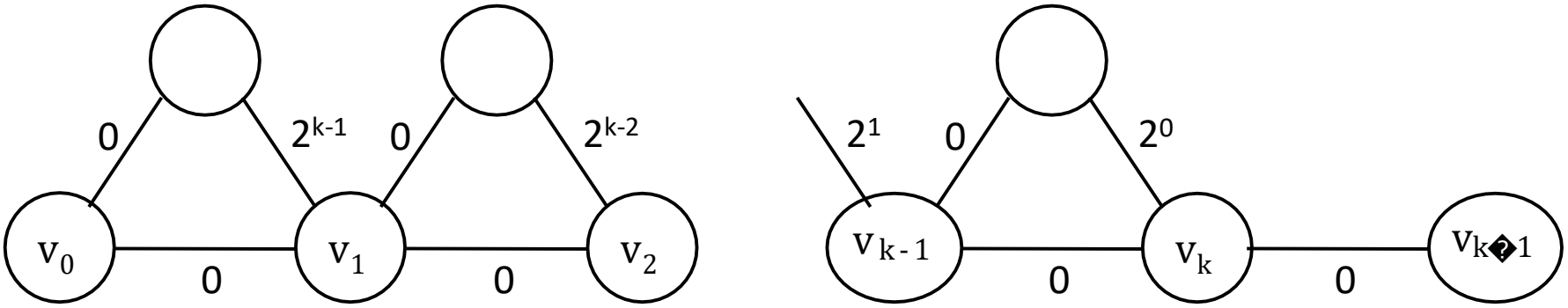
- $O(n!)$ simple paths from v_0 to any other node n, which is $O(n^n)$.
- So the number of messages sent on any channel is $O(n^n)$.
- Message complexity: $O(n^n |E|)$.
- Time complexity: $O(n^n \cdot n \cdot d)$.

- Q: Are such exponential bounds really achievable?



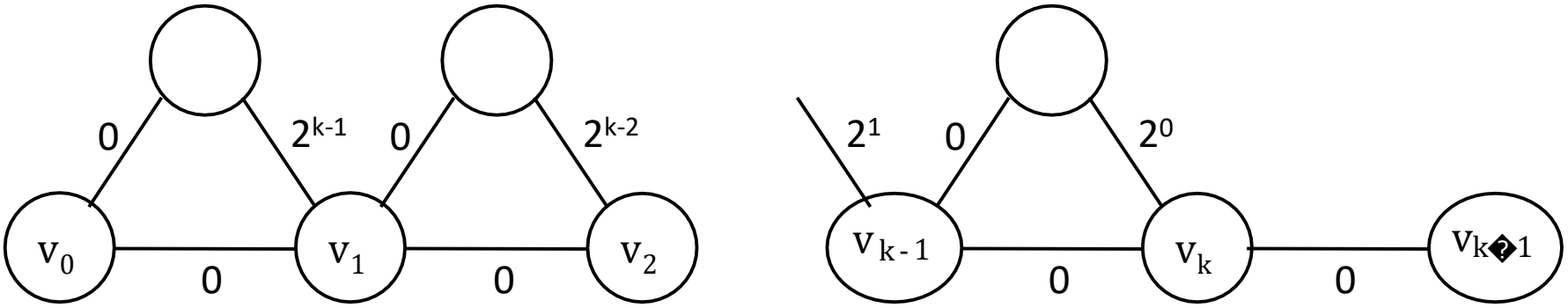
Complexity

- Q: Are such exponential bounds really achievable?
- Example:
  - There is an execution of the network below in which node v_k sends 2^k $\diamond$ $2^{n/2}$ messages to node v_{k+1}.
  - Message complexity is $O(2^{n/2})$.
  - Time complexity is $O(2^{n/2} d)$.



Complexity

- Execution in which node v_k sends 2^k messages to node v_{k+1}.
- Possible distance estimates for v_k are $2^k - 1, 2^k - 2, \ldots, 0$.
- Moreover, v_k can take on all these estimates in sequence:
  - First, messages traverse upper links, $2^k - 1$.
  - Then last lower message arrives at v_k, $2^k - 2$.
  - Then lower message $v_{k-2} \rightarrow v_{k-1}$ arrives, reduces v_{k-1}'s estimate by 2, message $v_{k-1} \rightarrow v_k$ arrives on upper links, $2^k - 3$.
  - Etc. Count down in binary.
  - If this happens quickly, get pileup of 2^k search messages in $C_{k,k-1}$.



Termination

- Q: How can processes learn when the tree is completed?
- Q: How can a process know when it can output its own dist and parent?

- Convergecast, once again
  - Compute and recompute child pointers.
  - Process $= v_a$ sends done to its current parent after:
    - It has received responses to all its messages, so it believes it knows all its children, and
    - It has received done messages from all of those children.
  - The same process may be involved several (many) times, based on improved estimates.