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TRIBONACCI MATRICES AND A NEW CODING THEORY

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In this paper, we consider the series of Tribonacci numbers. Thereby, we introduce a new coding theory called Tribonacci coding theory based on Tribonacci numbers and show that in the simplest case, the correct ability of this method is 99.80% whereas the correct ability of the Fibonacci coding/decoding method is 93.33%.

Keywords: Fibonacci numbers; Tribonacci numbers; Fibonacci matrices; Tribonacci matrices; Fibonacci coding; Tribonacci coding.

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1. Introduction

The Fibonacci numbers F_n $(n = 0, \pm 1, \pm 2, \pm 3, \ldots)$ satisfy the recurrence relation

$$F_{n+1} = F_n + F_{n-1}, \quad \text{where } F_0 = 0, \ F_1 = 1. \tag{1.1}$$

Equation (1.1) is known as Cassini formula and generates the Fibonacci numbers F_n $(n = 0, \pm 1, \pm 2, \pm 3, \ldots)$:

Table 1. Fibonacci numbers F_n.

n	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
F_n	-55	34	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21	34	55

In 1999, O. P. Stakov [10] describes generalized Fibonacci matrices. In 2006, A. P. Stakov [9] introduces a new coding theory on Fibonacci matrices. In 2009,

M. Basu *et al.* [1] present the generalized relations among the code matrix elements for Fibonacci coding theory.

In this paper, we discuss Tribonacci numbers t_k $(k = 0, \pm 1, \pm 2, \pm 3, \ldots)$. Also we develop a new coding theory on Tribonacci matrices.

The Tribonacci numbers t_k $(k = 0, 1, 2, 3, \ldots)$ [7] are the generalization of the Fibonacci numbers defined by the recurrence relation

$$t_{k+1} = t_k + t_{k-1} + t_{k-2}, \quad \text{where } t_0 = t_1 = 0, \ t_2 = 1. \tag{1.2}$$

The Tribonacci negetive numbers t_{-k} $(k = 1, 2, 3, \ldots)$ satisfies the recurrence relation

$$t_{-k} = \begin{vmatrix} t_{k+1} & t_{k+2} \\ t_k & t_{k+1} \end{vmatrix} \quad \text{where } t_0 = t_1 = 0, \ t_2 = 1. \tag{1.3}$$

The recurrence relations (1.2) and (1.3) generate the Tribonacci numbers t_k $(k = 0, \pm 1, \pm 2, \pm 3, \ldots)$:

Table 2. Tribonacci numbers t_k.

k	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
t_k	7	5	-8	4	1	-3	2	0	-1	1	0	0	1	1	2	4	7	13	24	44	81

The limit, $\alpha = \lim_{k \to \infty} \frac{t_k}{t_{k-1}}$ exists, called Tribonacci constant and is the one and only real root of the equation $x^3 - x^2 - x - 1 = 0$. Actually $\alpha = 1.83928675$.

Tribonacci numbers t_k $(k = 0, \pm 1, \pm 2, \pm 3, \ldots)$ and the Tribonacci constant α play a very important role in the construction of Tribonacci coding theory.

Introducing the square matrix M of order 3 as:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} t_3 & t_2 + t_1 & t_2 \\ t_2 & t_1 + t_0 & t_1 \\ t_1 & t_0 + t_{-1} & t_0 \end{pmatrix} \tag{1.4}$$

such that $det\,M = 1$.

The inverse of M is as:

$$M^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} t_0^2 - t_{-1}t_1 & t_{-1}t_2 - t_0t_1 & t_1^2 - t_0t_2 \\ t_1^2 - t_0t_2 & t_0t_3 - t_1t_2 & t_2^2 - t_1t_3 \\ t_0t_2 + t_{-1}t_2 - t_1^2 - t_0t_1 & t_1^2 + t_1t_2 - t_0t_3 - t_{-1}t_3 & t_1t_3 + t_0t_3 - t_2^2 - t_1t_2 \end{pmatrix} \tag{1.5}$$

such that $det\,M^{-1} = det\,M = 1$.

Also

$$M^2 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} t_4 & t_3 + t_2 & t_3 \\ t_3 & t_2 + t_1 & t_2 \\ t_2 & t_1 + t_0 & t_1 \end{pmatrix}$$

such that $det\ M^2 = 1$ and

$$M^{-2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} t_1^2 - t_0 t_2 & t_0 t_3 - t_1 t_2 & t_2^2 - t_1 t_3 \\ t_2^2 - t_1 t_3 & t_1 t_4 - t_2 t_3 & t_3^2 - t_2 t_4 \\ t_1 t_3 + t_0 t_3 - t_2^2 - t_1 t_2 & t_2^2 + t_2 t_3 - t_1 t_4 - t_0 t_4 & t_2 t_4 + t_1 t_4 - t_3^2 - t_2 t_3 \end{pmatrix}$$

such that $det\ M^{-2} = 1$.

Theorem 1.1.

$$M^k = \begin{pmatrix} t_{k+2} & t_{k+1} + t_k & t_{k+1} \\ t_{k+1} & t_k + t_{k-1} & t_k \\ t_k & t_{k-1} + t_{k-2} & t_{k-1} \end{pmatrix} \quad where\ M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof. We have,

$$M^1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} t_3 & t_2 + t_1 & t_2 \\ t_2 & t_1 + t_0 & t_1 \\ t_1 & t_0 + t_{-1} & t_0 \end{pmatrix}$$

and

$$M^2 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} t_4 & t_3 + t_2 & t_3 \\ t_3 & t_2 + t_1 & t_2 \\ t_2 & t_1 + t_0 & t_1 \end{pmatrix}.$$

The result is true for $k = 1$ and 2.
Let the result be true for $k = m$. Then

$$M^m = \begin{pmatrix} t_{m+2} & t_{m+1} + t_m & t_{m+1} \\ t_{m+1} & t_m + t_{m-1} & t_m \\ t_m & t_{m-1} + t_{m-2} & t_{m-1} \end{pmatrix}.$$
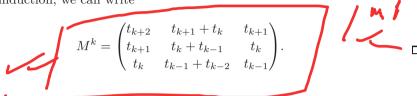
Therefore,

$$M^{m+1} = M^m M^1 = \begin{pmatrix} t_{m+2} & t_{m+1} + t_m & t_{m+1} \\ t_{m+1} & t_m + t_{m-1} & t_m \\ t_m & t_{m-1} + t_{m-2} & t_{m-1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Using (1.1), we have

$$M^{m+1} = \begin{pmatrix} t_{m+3} & t_{m+2} + t_{m+1} & t_{m+2} \\ t_{m+2} & t_{m+1} + t_m & t_{m+1} \\ t_{m+1} & t_m + t_{m-1} & t_m \end{pmatrix}.$$
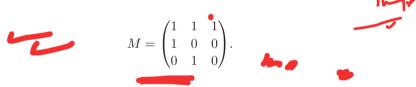
Hence by induction, we can write

$$M^k = \begin{pmatrix} t_{k+2} & t_{k+1} + t_k & t_{k+1} \\ t_{k+1} & t_k + t_{k-1} & t_k \\ t_k & t_{k-1} + t_{k-2} & t_{k-1} \end{pmatrix}. \qquad \square$$

Theorem 1.2.

$$M^{-k} = \begin{pmatrix} t_{k-1}^2 - t_{k-2}t_k & t_{k-2}t_{k+1} - t_{k-1}t_k & t_k^2 - t_{k-1}t_{k+1} \\ t_k^2 - t_{k-1}t_{k+1} & t_{k-1}t_{k+2} - t_k t_{k+1} & t_{k+1}^2 - t_k t_{k+2} \\ t_{k-1}t_{k+1} + t_{k-2}t_{k+1} & t_k^2 + t_k t_{k+1} - t_{k-1}t_{k+2} & t_k t_{k+2} + t_{k-1}t_{k+2} \\ -t_k^2 - t_{k-1}t_k & -t_{k-2}t_{k+2} & -t_{k+1}^2 - t_k t_{k+1} \end{pmatrix}$$

where

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Proof. We have,

$$M^1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore,

$$M^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} t_0^2 - t_{-1}t_1 & t_{-1}t_2 - t_0 t_1 & t_1^2 - t_0 t_2 \\ t_1^2 - t_0 t_2 & t_0 t_3 - t_1 t_2 & t_2^2 - t_1 t_3 \\ t_0 t_2 + t_{-1}t_2 - t_1^2 - t_0 t_1 & t_1^2 + t_1 t_2 - t_0 t_3 - t_{-1}t_3 & t_1 t_3 + t_0 t_3 - t_2^2 - t_1 t_2 \end{pmatrix},$$

$$M^2 = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore,

$$
M^{-2} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & -1 \\ -1 & 2 & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix}
t_1^2 - t_0 t_2 & t_0 t_3 - t_1 t_2 & t_2^2 - t_1 t_3 \\
t_2^2 - t_1 t_3 & t_1 t_4 - t_2 t_3 & t_3^2 - t_2 t_4 \\
t_1 t_3 + t_0 t_3 - t_2^2 - t_1 t_2 & t_2^2 + t_2 t_3 - t_1 t_4 - t_0 t_4 & t_2 t_4 + t_1 t_4 - t_3^2 - t_2 t_3
\end{pmatrix}.
$$

The result is true for $k = 1$ and 2.

Let the result be true for $k = m$. Then

$$
M^{-m} = \begin{pmatrix}
t_{m-1}^2 - t_{m-2} t_m & t_{m-2} t_{m+1} - t_{m-1} t_m & t_m^2 - t_{m-1} t_{m+1} \\
t_m^2 - t_{m-1} t_{m+1} & t_{m-1} t_{m+2} - t_m t_{m+1} & t_{m+1}^2 - t_m t_{m+2} \\
t_{m-1} t_{m+1} & t_m^2 + t_m t_{m+1} & t_m t_{m+2} + t_{m-1} t_{m+2} \\
+ t_{m-2} t_{m+1} & - t_{m-1} t_{m+2} & - t_{m+1}^2 - t_m t_{m+1} \\
- t_m^2 - t_{m-1} t_m & - t_{m-2} t_{m+2} &
\end{pmatrix}.
$$

Therefore,

$$
M^{-(m+1)} = M^{-m} M^{-1}
$$

$$
= \begin{pmatrix}
t_{m-1}^2 - t_{m-2} t_m & t_{m-2} t_{m+1} - t_{m-1} t_m & t_m^2 - t_{m-1} t_{m+1} \\
t_m^2 - t_{m-1} t_{m+1} & t_{m-1} t_{m+2} - t_m t_{m+1} & t_{m+1}^2 - t_m t_{m+2} \\
t_{m-1} t_{m+1} & t_m^2 + t_m t_{m+1} & t_m t_{m+2} \\
+ t_{m-2} t_{m+1} & - t_{m-1} t_{m+2} & + t_{m-1} t_{m+2} \\
- t_m^2 - t_{m-1} t_m & - t_{m-2} t_{m+2} & - t_{m+1}^2 - t_m t_{m+1}
\end{pmatrix}
$$

$$
\times \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix}.
$$

Using (1.1), we have

$$
M^{-(m+1)} = \begin{pmatrix}
t_m^2 - t_{m-1} t_{m+1} & t_{m-1} t_{m+2} - t_m t_{m+1} & t_{m+1}^2 - t_m t_{m+2} \\
t_{m+1}^2 - t_m t_{m+2} & t_m t_{m+3} - t_{m+1} t_{m+2} & t_{m+2}^2 - t_{m+1} t_{m+3} \\
t_m t_{m+2} & t_{m+1}^2 + t_{m+1} t_{m+2} & t_{m+1} t_{m+3} \\
+ t_{m-1} t_{m+2} & - t_m t_{m+3} & + t_m t_{m+3} - t_{m+2}^2 \\
- t_{m+1}^2 - t_m t_{m+1} & - t_{m-1} t_{m+3} & - t_{m+1} t_{m+2}
\end{pmatrix}.
$$

Hence by induction, we can write

$$
M^{-k} = \begin{pmatrix}
t_{k-1}^2 - t_{k-2} t_k & t_{k-2} t_{k+1} - t_{k-1} t_k & t_k^2 - t_{k-1} t_{k+1} \\
t_k^2 - t_{k-1} t_{k+1} & t_{k-1} t_{k+2} - t_k t_{k+1} & t_{k+1}^2 - t_k t_{k+2} \\
t_{k-1} t_{k+1} + t_{k-2} t_{k+1} & t_k^2 + t_k t_{k+1} - t_{k-1} t_{k+2} & t_k t_{k+2} + t_{k-1} t_{k+2} \\
- t_k^2 - t_{k-1} t_k & - t_{k-2} t_{k+2} & - t_{k+1}^2 - t_k t_{k+1}
\end{pmatrix}. \qquad \square
$$

2. Some Properties of M^k Matrix

(1) $M^k = M^{k-1} + M^{k-2} + M^{k-3}$.

Proof.

$$M^k = \begin{pmatrix} t_{k+2} & t_{k+1} + t_k & t_{k+1} \\ t_{k+1} & t_k + t_{k-1} & t_k \\ t_k & t_{k-1} + t_{k-2} & t_{k-1} \end{pmatrix}$$

$$= \begin{pmatrix} t_{k+1} + t_k + t_{k-1} & \begin{array}{c}(t_k + t_{k-1} + t_{k-2}) \\ + (t_{k-1} + t_{k-2} + t_{k-3})\end{array} & t_k + t_{k-1} + t_{k-2} \\ t_k + t_{k-1} + t_{k-2} & \begin{array}{c}(t_{k-1} + t_{k-2} + t_{k-3}) \\ + (t_{k-2} + t_{k-3} + t_{k-4})\end{array} & t_{k-1} + t_{k-2} + t_{k-3} \\ t_{k-1} + t_{k-2} + t_{k-3} & \begin{array}{c}(t_{k-2} + t_{k-3} + t_{k-4}) \\ + (t_{k-3} + t_{k-4} + t_{k-5})\end{array} & t_{k-2} + t_{k-3} + t_{k-4} \end{pmatrix}$$

$$= \begin{pmatrix} t_{k+1} & t_k + t_{k-1} & t_k \\ t_k & t_{k-1} + t_{k-2} & t_{k-1} \\ t_{k-1} & t_{k-2} + t_{k-3} & t_{k-2} \end{pmatrix} + \begin{pmatrix} t_k & t_{k-1} + t_{k-2} & t_{k-1} \\ t_{k-1} & t_{k-2} + t_{k-3} & t_{k-2} \\ t_{k-2} & t_{k-3} + t_{k-4} & t_{k-3} \end{pmatrix}$$

$$+ \begin{pmatrix} t_{k-1} & t_{k-2} + t_{k-3} & t_{k-2} \\ t_{k-2} & t_{k-3} + t_{k-4} & t_{k-3} \\ t_{k-3} & t_{k-4} + t_{k-5} & t_{k-4} \end{pmatrix}$$

$$= M^{k-1} + M^{k-2} + M^{k-3}. \tag{2.1}$$

(2) $M^k M^l = M^l M^k = M^{k+l}$ $(k, l = 0, \pm 1, \pm 2, \pm 3 \dots)$.
(3) $\det M^k = 1$.

The explicit form of the matrix M^k is obtained by (2.1). □

3. Tribonacci Coding/Decoding Method

In this section, we define a new coding theory called Tribonacci coding theory. Let us represent the initial message in the form of the square matrix P of order 3. We take the Tribonacci matrix M^k of order 3 as a coding matrix and its inverse matrix M^{-k} as a decoding matrix for any value of k. We name the transformation $P \times M^k = E$ as Tribonacci coding, the transformation $E \times M^{-k} = P$ as Tribonacci decoding and define E as code matrix.

3.1. Determinant of the code matrix E

We define the code matrix E by the following formula

$$E = P \times M^k.$$

According to matrix theory [8], we have

$$det\ E = det\ (P \times M^k) = det\ P \times det\ M^k = det\ P \times 1 = det\ P. \qquad (3.1)$$

3.2. *Example of Tribonacci coding/decoding method*

Let us represent the initial message in form of square matrix of order 3 as

$$P = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{pmatrix}. \qquad (3.2)$$

Let us assume that all elements of the matrix are nonnegative integer i.e., $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9 \geq 0$. Let us now select for any value of k, the Tribonacci matrix as the coding matrix.

We simply write for $k = 4$. Then

$$M^4 = \begin{pmatrix} t_6 & t_5 + t_4 & t_5 \\ t_5 & t_4 + t_3 & t_4 \\ t_4 & t_3 + t_2 & t_3 \end{pmatrix} = \begin{pmatrix} 7 & 6 & 4 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}. \qquad (3.3)$$

The inverse of M^4 is given by

$$M^{-4} = \begin{pmatrix} t_3^2 - t_2 t_4 & t_2 t_5 - t_3 t_4 & t_4^2 - t_3 t_5 \\ t_4^2 - t_3 t_5 & t_3 t_6 - t_4 t_5 & t_5^2 - t_4 t_6 \\ t_3 t_5 + t_2 t_5 - t_4^2 - t_3 t_4 & t_4^2 + t_4 t_5 - t_3 t_6 - t_2 t_6 & t_4 t_6 + t_3 t_6 - t_5^2 - t_4 t_5 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & -2 & -3 \end{pmatrix}.$$

The Tribonacci coding of the message (3.2) consists of the multiplication of the initial matrix (3.3) i.e.,

$$P \times M^4 = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{pmatrix} \begin{pmatrix} 7 & 6 & 4 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7p_1 + 4p_2 + 2p_3 & 6p_1 + 3p_2 + 2p_3 & 4p_1 + 2p_2 + p_3 \\ 7p_4 + 4p_5 + 2p_6 & 6p_4 + 3p_5 + 2p_6 & 4p_4 + 2p_5 + p_6 \\ 7p_7 + 4p_8 + 2p_9 & 6p_7 + 3p_8 + 2p_9 & 4p_7 + 2p_8 + p_9 \end{pmatrix}$$

$$= \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} = E, \qquad (3.4)$$

where

$$e_1 = 7p_1 + 4p_2 + 2p_3, \quad e_2 = 6p_1 + 3p_2 + 2p_3, \qquad e_3 = 4p_1 + 2p_2 + p_3,$$
$$e_4 = 7p_4 + 4p_5 + 2p_6, \quad e_5 = 6p_4 + 3p_5 + 2p_6, \qquad e_6 = 4p_4 + 2p_5 + p_6,$$
$$e_7 = 7p_7 + 4p_8 + 2p_9, \quad e_8 = 6p_7 + 3p_8 + 2p_9 \quad \text{and} \quad e_9 = 4p_7 + 2p_8 + p_9.$$

Solving these we have,

$$p_1 = -e_1 + 2e_3, \quad p_2 = 2e_1 - e_2 - 2e_3, \qquad p_3 = 2e_2 - 3e_3,$$
$$p_4 = -e_4 + 2e_6, \quad p_5 = 2e_4 - e_5 - 2e_6, \qquad p_6 = 2e_5 - 3e_6,$$
$$p_7 = -e_7 + 2e_9, \quad p_8 = 2e_7 - e_8 - 2e_9 \quad \text{and} \quad p_3 = 2e_8 - 3e_9.$$

Then the code matrix E is sent to a channel. The decoding of the code message E given by (3.4) is performed by the following manner,

$$E \times M^{-4} = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & -2 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -e_1 + 2e_3 & 2e_1 - e_2 - 2e_3 & 2e_2 - 3e_3 \\ -e_4 + 2e_6 & 2e_4 - e_5 - 2e_6 & 2e_5 - 3e_6 \\ -e_7 + 2e_9 & 2e_7 - e_8 - 2e_9 & 2e_8 - 3e_9 \end{pmatrix}$$

$$= \begin{pmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{pmatrix} = P.$$

Example 3.1. Let our message be the sequence of decimal numerals 3567489 in the matrix form

$$P = \begin{pmatrix} 0 & 0 & 3 \\ 5 & 6 & 7 \\ 4 & 8 & 9 \end{pmatrix}.$$

Now we write Tribonacci M matrix of the fourth power as coding matrix

$$M^4 = \begin{pmatrix} 7 & 6 & 4 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

and the inverse of M^4 is given by

$$M^{-4} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & -2 & -3 \end{pmatrix}.$$

The code message

$$E = P \times M^4 = \begin{pmatrix} 0 & 0 & 3 \\ 5 & 6 & 7 \\ 4 & 8 & 9 \end{pmatrix} \begin{pmatrix} 7 & 6 & 4 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 6 & 3 \\ 73 & 62 & 39 \\ 78 & 66 & 41 \end{pmatrix}$$

and the decode message

$$P = E \times M^{-4} = \begin{pmatrix} 6 & 6 & 3 \\ 73 & 62 & 39 \\ 78 & 66 & 41 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 3 \\ 5 & 6 & 7 \\ 4 & 8 & 9 \end{pmatrix}.$$

Example 3.2. We take message 12456274518921 in the sequence of decimal numerals then we can represent this message in the matrix at random as follows:

$$P = \begin{pmatrix} 12 & 4 & 562 \\ 7 & 4 & 51 \\ 8 & 9 & 21 \end{pmatrix}.$$

We write Tribonacci M matrix of the fourth power as a coding matrix

$$M^4 = \begin{pmatrix} 7 & 6 & 4 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

and the inverse of M^4 is given by

$$M^{-4} = \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & -2 & -3 \end{pmatrix}.$$

The code message is:

$$E = P \times M^4 = \begin{pmatrix} 12 & 4 & 562 \\ 7 & 4 & 51 \\ 8 & 9 & 21 \end{pmatrix} \begin{pmatrix} 7 & 6 & 4 \\ 4 & 3 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1224 & 1208 & 618 \\ 167 & 156 & 87 \\ 134 & 117 & 71 \end{pmatrix}$$

and the decode message is:

$$P = E \times M^{-4} = \begin{pmatrix} 1224 & 1208 & 618 \\ 167 & 156 & 87 \\ 134 & 117 & 71 \end{pmatrix} \begin{pmatrix} -1 & 2 & 0 \\ 0 & -1 & 2 \\ 2 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 12 & 4 & 562 \\ 7 & 4 & 51 \\ 8 & 9 & 21 \end{pmatrix}.$$

In this way, we can take the elements of P as odd number of digits or even number of digits as we like.

4. Relations Among the Code Matrix Elements

In this paper, we develop the relations among the code matrix elements. We write the code matrix E and the initial message P as:

$$E = P \times M^k = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{pmatrix} \begin{pmatrix} t_{k+2} & t_{k+1} + t_k & t_{k+1} \\ t_{k+1} & t_k + t_{k-1} & t_k \\ t_k & t_{k-1} + t_{k-2} & t_{k-1} \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}.$$

We choose k in such a manner that $e_i > 0$ for $i = 1, 2, 3$.

Now,

$$P = E \times M^{-k} = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} \begin{pmatrix} t_{k+2} & t_{k+1} + t_k & t_{k+1} \\ t_{k+1} & t_k + t_{k-1} & t_k \\ t_k & t_{k-1} + t_{k-2} & t_{k-1} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}$$

$$\times \begin{pmatrix} t_{k-1}^2 - t_{k-2}t_k & t_{k-2}t_{k+1} - t_{k-1}t_k & t_k^2 - t_{k-1}t_{k+1} \\ t_k^2 - t_{k-1}t_{k+1} & t_{k-1}t_{k+2} - t_k t_{k+1} & t_{k+1}^2 - t_k t_{k+2} \\ \begin{matrix} t_{k-1}t_{k+1} + t_{k-2}t_{k+1} \\ - t_k^2 - t_{k-1}t_k \end{matrix} & \begin{matrix} t_k^2 + t_k t_{k+1} - t_{k-1}t_{k+2} \\ - t_{k-2}t_{k+2} \end{matrix} & \begin{matrix} t_k t_{k+2} + t_{k-1}t_{k+2} \\ - t_{k+1}^2 - t_k t_{k+1} \end{matrix} \end{pmatrix}$$

$$= \begin{pmatrix} p_1 & p_2 & p_3 \\ p_4 & p_5 & p_6 \\ p_7 & p_8 & p_9 \end{pmatrix}.$$

Also,

$$\det M^k = t_{k+2}(t_k t_{k-1} + t_{k-1}^2 - t_k t_{k-1} - t_k t_{k-2}) + (t_{k+1} + t_k)(t_k^2 - t_{k+1}t_{k-1})$$

$$+ t_{k+1}(t_{k+1}t_{k-1} + t_{k+1}t_{k-2} - t_k^2 - t_k t_{k-1}) = 1. \tag{4.1}$$

Since $p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9 \geq 0$, we have

$$p_1 = e_1(t_{k-1}^2 - t_k t_{k-2}) + e_2(t_k^2 - t_{k+1}t_{k-1})$$

$$+ e_3(t_{k-1}t_{k+1} + t_{k-2}t_{k+1} - t_k^2 - t_{k-1}t_k) \geq 0, \tag{4.2}$$

$$p_2 = e_1(t_{k-2}t_{k+1} - t_{k-1}t_k) + e_2(t_{k-1}t_{k+2} - t_k t_{k+1})$$

$$+ e_3(t_k^2 + t_k t_{k+1} - t_{k-1}t_{k+2} - t_{k-2}t_{k+2}) \geq 0, \tag{4.3}$$

$$p_3 = e_1(t_k^2 - t_{k-1}t_{k+1}) + e_2(t_{k+1}^2 - t_k t_{k+2})$$

$$+ e_3(t_k t_{k+2} + t_{k-1}t_{k+2} - t_{k+1}^2 - t_k t_{k+1}) \geq 0, \tag{4.4}$$

$$p_4 = e_4(t_{k-1}^2 - t_k t_{k-2}) + e_5(t_k^2 - t_{k+1}t_{k-1})$$

$$+ e_6(t_{k-1}t_{k+1} + t_{k-2}t_{k+1} - t_k^2 - t_{k-1}t_k) \geq 0, \tag{4.5}$$

$$p_5 = e_4(t_{k-2}t_{k+1} - t_{k-1}t_k) + e_5(t_{k-1}t_{k+2} - t_k t_{k+1})$$

$$+ e_6(t_k^2 + t_k t_{k+1} - t_{k-1}t_{k+2} - t_{k-2}t_{k+2}) \geq 0, \tag{4.6}$$

$$p_6 = e_4(t_k^2 - t_{k-1}t_{k+1}) + e_5(t_{k+1}^2 - t_k t_{k+2})$$

$$+ e_6(t_k t_{k+2} + t_{k-1}t_{k+2} - t_{k+1}^2 - t_k t_{k+1}) \geq 0, \tag{4.7}$$

$$p_7 = e_7(t_{k-1}^2 - t_k t_{k-2}) + e_8(t_k^2 - t_{k+1}t_{k-1})$$

$$+ e_9(t_{k-1}t_{k+1} + t_{k-2}t_{k+1} - t_k^2 - t_{k-1}t_k) \geq 0, \tag{4.8}$$

$$p_8 = e_7(t_{k-2}t_{k+1} - t_{k-1}t_k) + e_8(t_{k-1}t_{k+2} - t_k t_{k+1})$$

$$+ e_9(t_k^2 + t_k t_{k+1} - t_{k-1}t_{k+2} - t_{k-2}t_{k+2}) \geq 0, \tag{4.9}$$

and

$$p_9 = e_7(t_k^2 - t_{k-1}t_{k+1}) + e_8(t_{k+1}^2 - t_k t_{k+2})$$
$$+ e_9(t_k t_{k+2} + t_{k-1}t_{k+2} - t_{k+1}^2 - t_k t_{k+1}) \geq 0. \tag{4.10}$$

From (4.2), we have

$$e_1 t_{k-1}^2 + e_2 t_k^2 + e_3 t_{k+1}^2 \geq e_1 t_k t_{k-2} + e_2 t_{k+1}t_{k-2} + e_3 t_k t_{k+2}. \tag{4.11}$$

From (4.3), we have

$$e_1 t_{k+1}t_{k-2} + e_2 t_{k+2}t_{k-1} + e_3 t_k t_{k+3} \geq e_1 t_k t_{k-1} + e_2 t_k t_{k+1} + e_3 t_{k+1}t_{k+2}. \tag{4.12}$$

From (4.4), we have

$$e_1 t_k^2 + e_2 t_{k+1}^2 + e_3 t_{k+2}^2 \geq e_1 t_{k+1}t_{k-1} + e_2 t_{k+2}t_k + e_3 t_{k+1}t_{k+3}. \tag{4.13}$$

Dividing both sides by $e_1 t_k t_{k-2}$ (>0) of (4.11), $e_1 t_k t_{k-1}$ (>0) of (4.12) and $e_1 t_{k+1}t_{k-1}$ (>0) of (4.13), we have

$$(t_{k+1}^2 - t_k t_{k+2})\frac{e_3}{e_1} \geq (t_{k+1}t_{k-2} - t_k^2)\frac{e_2}{e_1} + (t_k t_{k-2} - t_{k-1}^2), \tag{4.14}$$

$$(t_{k+1}t_{k+2} - t_k t_{k+3})\frac{e_3}{e_1} \leq (t_{k+2}t_{k-1} - t_k t_{k+1})\frac{e_2}{e_1} + (t_{k+1}t_{k-2} - t_k t_{k-1}), \tag{4.15}$$

and

$$(t_{k+2}^2 - t_{k+1}t_{k+3})\frac{e_3}{e_1} \geq (t_{k+2}t_k - t_{k+1}^2)\frac{e_2}{e_1} + (t_{k+1}t_{k-1} - t_k^2). \tag{4.16}$$

Let $a = (t_{k+1}^2 - t_k t_{k+2})$, $b = (t_{k+1}t_{k+2} - t_k t_{k+3})$, $c = (t_{k+2}^2 - t_{k+1}t_{k+3})$.
Now $3^3 = 27$ cases arise for $a \gtrless 0, b \gtrless 0, c \gtrless 0$.

Case 1. When $a > 0, b > 0, c > 0$. Then from (4.14), we have

$$\frac{e_3}{e_1} \geq u, \quad \text{where } u = \frac{e_2}{e_1}\left(\frac{t_{k+1}t_{k-2} - t_k^2}{t_{k+1}^2 - t_k t_{k+2}}\right) + \frac{t_k t_{k-2} - t_{k-1}^2}{t_{k+1}^2 - t_k t_{k+2}}. \tag{4.17}$$

From (4.15), we have

$$\frac{e_3}{e_1} \leq v, \quad \text{where } v = \frac{e_2}{e_1}\left(\frac{t_{k+2}t_{k-1} - t_k t_{k+1}}{t_{k+1}t_{k+2} - t_k t_{k+3}}\right) + \frac{t_{k+1}t_{k-2} - t_k t_{k-1}}{t_{k+1}t_{k+2} - t_k t_{k+3}}. \tag{4.18}$$

From (4.16), we have

$$\frac{e_3}{e_1} \geq w, \quad \text{where } w = \frac{e_2}{e_1}\left(\frac{t_{k+2}t_k - t_{k+1}^2}{t_{k+2}^2 - t_{k+1}t_{k+3}}\right) + \frac{t_{k+1}t_{k-1} - t_k^2}{t_{k+2}^2 - t_{k+1}t_{k+3}}. \tag{4.19}$$

From (4.17) and (4.18), we have

$$\frac{e_1}{e_2} \geq \min\left\{\frac{t_{k+2}}{t_{k+1} + t_k}, \frac{t_{k+1}}{t_k + t_{k-1}}, \frac{t_k}{t_{k-1} + t_{k-2}}\right\}, \quad \text{using (4.1).} \tag{4.20}$$

From (4.18) and (4.19), we have

$$\frac{e_1}{e_2} \leq \max\left\{\frac{t_{k+2}}{t_{k+1} + t_k}, \frac{t_{k+1}}{t_k + t_{k-1}}, \frac{t_k}{t_{k-1} + t_{k-2}}\right\}, \quad \text{using (4.1).} \tag{4.21}$$

From (4.20) and (4.21), we have

$$\min\left\{\frac{t_{k+2}}{t_{k+1} + t_k}, \frac{t_{k+1}}{t_k + t_{k-1}}, \frac{t_k}{t_{k-1} + t_{k-2}}\right\}$$
$$\leq \frac{e_1}{e_2} \leq \max\left\{\frac{t_{k+2}}{t_{k+1} + t_k}, \frac{t_{k+1}}{t_k + t_{k-1}}, \frac{t_k}{t_{k-1} + t_{k-2}}\right\}. \tag{4.22}$$

Similarly, we have

$$\min\left\{\frac{t_{k+1} + t_k}{t_{k+1}}, \frac{t_k + t_{k-1}}{t_k}, \frac{t_{k-1} + t_{k-2}}{t_{k-1}}\right\}$$
$$\leq \frac{e_2}{e_3} \leq \max\left\{\frac{t_{k+1} + t_k}{t_{k+1}}, \frac{t_k + t_{k-1}}{t_k}, \frac{t_{k-1} + t_{k-2}}{t_{k-1}}\right\}$$

and

$$\min\left\{\frac{t_{k+2}}{t_{k+1}}, \frac{t_{k+1}}{t_k}, \frac{t_k}{t_{k-1}}\right\} \leq \frac{e_1}{e_3} \leq \max\left\{\frac{t_{k+2}}{t_{k+1}}, \frac{t_{k+1}}{t_k}, \frac{t_k}{t_{k-1}}\right\}.$$

Case 2. When $a = 0, b > 0, c > 0$.

From (4.14), we have

$$\frac{e_1}{e_2} \geq \min\left\{\frac{t_{k+2}}{t_{k+1} + t_k}, \frac{t_{k+1}}{t_k + t_{k-1}}, \frac{t_k}{t_{k-1} + t_{k-2}}\right\}, \quad \text{since } a = 0. \tag{4.23}$$

From (4.15) and (4.16), we have

$$\frac{e_1}{e_2} \leq \max\left\{\frac{t_{k+2}}{t_{k+1} + t_k}, \frac{t_{k+1}}{t_k + t_{k-1}}, \frac{t_k}{t_{k-1} + t_{k-2}}\right\}, \quad \text{using (4.1) and } a = 0. \tag{4.24}$$

From (4.23) and (4.24), we have

$$\min\left\{\frac{t_{k+2}}{t_{k+1} + t_k}, \frac{t_{k+1}}{t_k + t_{k-1}}, \frac{t_k}{t_{k-1} + t_{k-2}}\right\}$$
$$\leq \frac{e_1}{e_2} \leq \max\left\{\frac{t_{k+2}}{t_{k+1} + t_k}, \frac{t_{k+1}}{t_k + t_{k-1}}, \frac{t_k}{t_{k-1} + t_{k-2}}\right\}. \tag{4.25}$$

Case 3. When $a < 0, b < 0, c < 0$, then from (4.14)

$$\frac{e_3}{e_1} \leq u, \quad \text{where } u = \frac{e_2}{e_1}\left(\frac{t_{k+1}t_{k-2} - t_k^2}{t_{k+1}^2 - t_k t_{k+2}}\right) + \frac{t_k t_{k-2} - t_{k-1}^2}{t_{k+1}^2 - t_k t_{k+2}}. \tag{4.26}$$

From (4.15), we have

$$\frac{e_3}{e_1} \geq v, \quad \text{where } v = \frac{e_2}{e_1}\left(\frac{t_{k+2}t_{k-1} - t_k t_{k+1}}{t_{k+1}t_{k+2} - t_k t_{k+3}}\right) + \frac{t_{k+1}t_{k-2} - t_k t_{k-1}}{t_{k+1}t_{k+2} - t_k t_{k+3}}. \tag{4.27}$$

From (4.16), we have

$$\frac{e_3}{e_1} \leq w, \quad \text{where } w = \frac{e_2}{e_1}\left(\frac{t_{k+2}t_k - t_{k+1}^2}{t_{k+2}^2 - t_{k+1}t_{k+3}}\right) + \frac{t_{k+1}t_{k-1} - t_k^2}{t_{k+2}^2 - t_{k+1}t_{k+3}}. \tag{4.28}$$

From (4.26) and (4.27), we have

$$\frac{e_1}{e_2} \leq \max\left\{\frac{t_{k+2}}{t_{k+1}+t_k}, \frac{t_{k+1}}{t_k+t_{k-1}}, \frac{t_k}{t_{k-1}+t_{k-2}}\right\}, \quad \text{using (4.1).} \tag{4.29}$$

From (4.27) and (4.28), we have

$$\frac{e_1}{e_2} \geq \min\left\{\frac{t_{k+2}}{t_{k+1}+t_k}, \frac{t_{k+1}}{t_k+t_{k-1}}, \frac{t_k}{t_{k-1}+t_{k-2}}\right\}, \quad \text{using (4.1).} \tag{4.30}$$

From (4.29) and (4.30), we have

$$\min\left\{\frac{t_{k+2}}{t_{k+1}+t_k}, \frac{t_{k+1}}{t_k+t_{k-1}}, \frac{t_k}{t_{k-1}+t_{k-2}}\right\}$$
$$\leq \frac{e_1}{e_2} \leq \max\left\{\frac{t_{k+2}}{t_{k+1}+t_k}, \frac{t_{k+1}}{t_k+t_{k-1}}, \frac{t_k}{t_{k-1}+t_{k-2}}\right\}. \tag{4.31}$$

Similarly, we have

$$\min\left\{\frac{t_{k+1}+t_k}{t_{k+1}}, \frac{t_k+t_{k-1}}{t_k}, \frac{t_{k-1}+t_{k-2}}{t_{k-1}}\right\}$$
$$\leq \frac{e_2}{e_3} \leq \max\left\{\frac{t_{k+1}+t_k}{t_{k+1}}, \frac{t_k+t_{k-1}}{t_k}, \frac{t_{k-1}+t_{k-2}}{t_{k-1}}\right\}$$

and

$$\min\left\{\frac{t_{k+2}}{t_{k+1}}, \frac{t_{k+1}}{t_k}, \frac{t_k}{t_{k-1}}\right\} \leq \frac{e_1}{e_3} \leq \max\left\{\frac{t_{k+2}}{t_{k+1}}, \frac{t_{k+1}}{t_k}, \frac{t_k}{t_{k-1}}\right\}.$$

Similarly it can be proved for the rest cases.

Hence, we have

$$\min\left\{\frac{t_{k+2}}{t_{k+1}+t_k}, \frac{t_{k+1}}{t_k+t_{k-1}}, \frac{t_k}{t_{k-1}+t_{k-2}}\right\}$$
$$\leq \frac{e_1}{e_2} \leq \max\left\{\frac{t_{k+2}}{t_{k+1}+t_k}, \frac{t_{k+1}}{t_k+t_{k-1}}, \frac{t_k}{t_{k-1}+t_{k-2}}\right\}, \tag{4.32}$$

$$\min\left\{\frac{t_{k+1}+t_k}{t_{k+1}}, \frac{t_k+t_{k-1}}{t_k}, \frac{t_{k-1}+t_{k-2}}{t_{k-1}}\right\}$$

$$\leq \frac{e_2}{e_3} \leq \max\left\{\frac{t_{k+1}+t_k}{t_{k+1}}, \frac{t_k+t_{k-1}}{t_k}, \frac{t_{k-1}+t_{k-2}}{t_{k-1}}\right\} \qquad (4.33)$$

and

$$\min\left\{\frac{t_{k+2}}{t_{k+1}}, \frac{t_{k+1}}{t_k}, \frac{t_k}{t_{k-1}}\right\} \leq \frac{e_1}{e_3} < \max\left\{\frac{t_{k+2}}{t_{k+1}}, \frac{t_{k+1}}{t_k}, \frac{t_k}{t_{k-1}}\right\}. \qquad (4.34)$$

Therefore, for large value of k we arrive at

$$\frac{e_1}{e_2} \approx \frac{\alpha^2}{\alpha+1}, \quad \frac{e_2}{e_3} \approx \frac{\alpha+1}{\alpha} \quad \text{and} \quad \frac{e_1}{e_3} \approx \alpha, \quad \text{where } \alpha = 1.83928675. \qquad (4.35)$$

Similarly, we have

$$\frac{e_4}{e_5} \approx \frac{\alpha^2}{\alpha+1}, \quad \frac{e_5}{e_6} \approx \frac{\alpha+1}{\alpha} \quad \text{and} \quad \frac{e_4}{e_6} \approx \alpha \qquad (4.36)$$

and

$$\frac{e_7}{e_8} \approx \frac{\alpha^2}{\alpha+1}, \quad \frac{e_8}{e_9} \approx \frac{\alpha+1}{\alpha} \quad \text{and} \quad \frac{e_7}{e_9} \approx \alpha. \qquad (4.37)$$

5. Error Detection and Correction

The most important feature of this method is the error detection and correction ability. The correct ability of the Fibonacci coding/decoding method is 93.33% which exceeds the essentially all well-known correcting codes proved in [9]. In this paper, we develop a correct ability of the Tribonacci coding/decoding method. The Tribonacci coding/decoding method gives a property to detect and correct errors in the code message E. The error detection and correction is based on the property of the determinant of matrix given by (3.1). At first we calculate the determinant of the initial matrix P and then sent to a communication channel right after the code matrix elements. $Det\, P$ is treated as the checking elements of the code matrix E received from the communication channel. After receiving the code matrix and its checking elements $det\, P$, we calculate the determinant of the matrix E and compare it with given $det\, P$ by relation (3.1). If relation (3.1) is true, then we conclude that the elements of the code matrix E were transmitted through the communication channel without errors otherwise there are errors and then we try to correct these errors using relations (3.1), (4.35), (4.36) and (4.37).

If we have single error in the code matrix E, it is clear that there are nine variants of single error in the code matrix E:

$$\text{(i)} \begin{pmatrix} x_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} e_1 & x_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} e_1 & e_2 & x_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix},$$

$$(\text{iv}) \begin{pmatrix} e_1 & e_2 & e_3 \\ x_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}, \qquad (\text{v}) \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & x_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}, \qquad (\text{vi}) \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & x_6 \\ e_7 & e_8 & e_9 \end{pmatrix},$$

$$(\text{vii}) \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ x_7 & e_8 & e_9 \end{pmatrix}, \quad (\text{viii}) \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & x_8 & e_9 \end{pmatrix} \quad \text{and} \quad (\text{ix}) \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & x_9 \end{pmatrix},$$

where $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9$ are possible destroyed elements. For checking the hypotheses (i)–(ix), we write the following algebraic equations based on relations (3.1):

$$x_1(e_5e_9 - e_6e_8) + e_2(e_6e_7 - e_4e_9) + e_3(e_4e_8 - e_5e_7) = det\, P, \qquad (5.1)$$

$$e_1(e_5e_9 - e_6e_8) + x_2(e_6e_7 - e_4e_9) + e_3(e_4e_8 - e_5e_7) = det\, P, \qquad (5.2)$$

$$e_1(e_5e_9 - e_6e_8) + e_2(e_6e_7 - e_4e_9) + x_3(e_4e_8 - e_5e_7) = det\, P, \qquad (5.3)$$

$$x_4(e_3e_8 - e_2e_9) + e_5(e_1e_9 - e_3e_7) + e_6(e_2e_7 - e_1e_8) = det\, P, \qquad (5.4)$$

$$e_4(e_3e_8 - e_2e_9) + x_5(e_1e_9 - e_3e_7) + e_6(e_2e_7 - e_1e_8) = det\, P, \qquad (5.5)$$

$$e_4(e_3e_8 - e_2e_9) + e_5(e_1e_9 - e_3e_7) + x_6(e_2e_7 - e_1e_8) = det\, P, \qquad (5.6)$$

$$x_7(e_2e_6 - e_3e_5) + e_8(e_3e_4 - e_1e_6) + e_9(e_1e_5 - e_2e_4) = det\, P, \qquad (5.7)$$

$$e_7(e_2e_6 - e_3e_5) + x_8(e_3e_4 - e_1e_6) + e_9(e_1e_5 - e_2e_4) = det\, P \qquad (5.8)$$

and

$$e_7(e_2e_6 - e_3e_5) + e_8(e_3e_4 - e_1e_6) + x_9(e_1e_5 - e_2e_4) = det\, P. \qquad (5.9)$$

It follows from (5.1)–(5.9) nine variants of calculations of possible single errors:

$$x_1 = \frac{det\, P - e_2(e_6e_7 - e_4e_9) - e_3(e_4e_8 - e_5e_7)}{e_5e_9 - e_6e_8}, \qquad (5.10)$$

$$x_2 = \frac{det\, P - e_1(e_5e_9 - e_6e_8) - e_3(e_4e_8 - e_5e_7)}{e_6e_7 - e_4e_9}, \qquad (5.11)$$

$$x_3 = \frac{det\, P - e_1(e_5e_9 - e_6e_8) - e_2(e_6e_7 - e_4e_9)}{e_4e_8 - e_5e_7}, \qquad (5.12)$$

$$x_4 = \frac{det\, P - e_5(e_1e_9 - e_3e_7) - e_6(e_2e_7 - e_1e_8)}{e_3e_8 - e_2e_9}, \qquad (5.13)$$

$$x_5 = \frac{det\, P - e_4(e_3e_8 - e_2e_9) - e_6(e_2e_7 - e_1e_8)}{e_1e_9 - e_3e_7}, \qquad (5.14)$$

$$x_6 = \frac{det\, P - e_4(e_3e_8 - e_2e_9) - e_5(e_1e_9 - e_3e_7)}{e_2e_7 - e_1e_8}, \qquad (5.15)$$

$$x_7 = \frac{\det P - e_8(e_3e_4 - e_1e_6) - e_9(e_1e_5 - e_2e_4)}{e_2e_6 - e_3e_5}, \tag{5.16}$$

$$x_8 = \frac{\det P - e_7(e_2e_6 - e_3e_5) - e_9(e_1e_5 - e_2e_4)}{e_3e_4 - e_1e_6} \tag{5.17}$$

and

$$x_9 = \frac{\det P - e_7(e_2e_6 - e_3e_5) - e_8(e_3e_4 - e_1e_6)}{e_1e_5 - e_2e_4}. \tag{5.18}$$

The formulas (5.10)–(5.18) give nine possible variants of single error but we have to choose the correct variant only among the cases of the integer solutions $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9$; which satisfies relations (4.35), (4.36) and (4.37). If the calculations by the formulas (5.10)–(5.18) do not give an integer result, we have to conclude that our hypothesis about single error is incorrect or we have more than single error in the checking element $\det P$. For the latter case, we use the approximate equalities (4.35), (4.36) and (4.37) for checking a correctness of the code matrix E.

By analogy, we check all hypotheses of double errors in the code matrix E. As example, let us consider the following case of double errors in the code matrix E:

$$\begin{pmatrix} x & y & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}. \tag{5.19}$$

Using relation (3.1), we write the algebraic equation for matrix (5.19)

$$x(e_5e_9 - e_6e_8) + y(e_6e_7 - e_4e_9) = e_3(e_5e_7 - e_4e_8) + \det P. \tag{5.20}$$

According to relation (4.35) there is the following relation between x and y:

$$x \approx \frac{\alpha^2}{\alpha + 1} y. \tag{5.21}$$

Equation (5.20) is "Diophantine". As the "Diophantine" equation (5.20) has many solutions, we have to choose such solutions x, y, which satisfy relation (5.21).

It is clear that there are $^9C_2 = 36$ variants of double errors in the code matrix E and by using similar approach we correct all double errors in the code matrix E.

Hence there are

$$^9C_1 + {}^9C_2 + {}^9C_3 + {}^9C_4 + {}^9C_5 + {}^9C_6 + {}^9C_7 + {}^9C_8 + {}^9C_9 = 511$$

cases of errors in the code matrix E.

Similarly, we show by using this approach that there is a possibility to correct all possible triple-fold, four-fold, $\ldots$, eight-fold errors in the code matrix E.

Hence the possibility to correct eight cases of this method is $\frac{510}{511} = 0.9980 = 99.80\%$ since the nine-fold error of the code matrix is not correctable.

6. Conclusion

The Tribonacci coding/decoding method is the main application of the Tribonacci matrices. This coding/decoding method differs from the classical algebraic codes by the following peculiarities and the basic important feature of this method is the correcting capacity.

(1) Tribonacci coding/decoding method converts to matrix multiplication which is a very well-known method and not time consuming in modern computers.

(2) Tribonacci coding/decoding method restore with guarantee all erroneous code 3×3 matrices having single-, double-, ..., eight-fold errors.

(3) The correct ability of this method is 99.80% whereas the correct ability of the Fibonacci coding/decoding method is 93.33% and hence our method exceeds the essentially all well-known correcting codes.

(4) M. S. El Naschine [2] shows that the Fibonacci series and the golden mean plays a very important role in the construction of a relatively new space-time theory, which is referred to as ϵ^∞ Cantorianfractal space-time or E-infinity theory [3, 4]. He [6] also shows that there are relations among E-infinity theory, string theory, exceptional lie symmetry group and various physical quantities. He [5] explains the existence of these relations using the very geometric and topology of space-time as claimed by E-infinity theory. Based on these theories, Tribonacci numbers and Tribonacci constant will give a better result in E-infinity theory and string theory.

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CODING THEORY ON FIBONACCI n-STEP NUMBERS

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We consider the series of Fibonacci n-step numbers and a class of square matrix of order n based on Fibonacci n-step numbers with determinant $+1$ or -1. Thereby, we introduce a new coding theory called Fibonacci n-step coding theory and establish generalized relation among the code elements for all values of n. For $n = 2$, the correct ability of the method is 93.33%; for $n = 3$, the correct ability of the method is 99.80% and for $n = 4$, the correct ability of the method is 99.998%. In general, the correct ability of the method increases as n increases.

Keywords: Fibonacci numbers; Fibonacci n-step numbers; Fibonacci n-step matrices; n-anacci constant; Fibonacci n-step coding; error correction.

Mathematics Subject Classification: 11B39, 15B36, 11C20, 11Z05, 94A15

1. Introduction

The Fibonacci numbers F_k is defined by the second-order linear recurrence relation:

$$F_{k+1} = F_k + F_{k-1} \tag{1.1}$$

with the initial terms

$$F_0 = 0, \quad F_1 = 1.$$

This identity is called "Cassini formula" in honor of the well-known 17th century astronomer Giovanni Cassini (1625–1712) who derived this formula.

The Fibonacci n-step numbers $F_k^{(n)}$ [10] are the generalization of the Fibonacci numbers, defined by the recurrence relation

$$F_k^{(n)} = F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n}^{(n)}, \quad \text{where}$$

$$F_0^{(n)} = F_1^{(n)} = \cdots = F_{n-2}^{(n)} = 0, \quad F_{n-1}^{(n)} = 1 \tag{1.2}$$

for $k = 0, \pm 1, \pm 2, \pm 3, \ldots$, and $n = 1, 2, 3, \ldots$.

The first few sequence of Fibonacci n-step numbers are summarized in Table 1.

For $k \geq 1$, $r_n = \lim_{k \to \infty} \frac{F_k^{(n)}}{F_{k-1}^{(n)}}$ exists, called n-anacci constant and is the real root ≥ 1 of the equation

$$x^n - x^{n-1} - x^{n-2} - \cdots - x - 1 = 0.$$

For even n, there are exactly two real roots, one is >1 and one is <1 and for odd n, there is exactly one real root, which is always > 1, the equality sign holds if and only if $n = 1$.

Actually, $r_1 = 1$, $r_2 = 1.61803$ called Golden mean, $r_3 = 1.83929$ called Tribonacci constant, $r_4 = 1.92756$ called Tetranacci constant, $r_5 = 1.96595$ called Pentanacci constant etc. and $\lim_{n \to \infty} r_n = 2$ [7].

In this paper, we develop a new Fibonacci n-step coding theory where Fibonacci n-step numbers $F_k^{(n)}$ $(k = 0, \pm 1, \pm 2, \pm 3, \ldots)$ and the n-anacci constant r_n play a very important role in the construction of Fibonacci n-step coding theory.

Fibonacci n-step matrix M_n [8] is a square matrix of order n and is given by

$$M_n = \begin{pmatrix} \mathbf{1} & 1 \\ I_{n-1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

Table 1. Fibonacci n-step numbers $F_k^{(n)}$.

k	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	Name
$F_k^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	Degenerate
$F_k^{(2)}$	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21	Fibonacci numbers
$F_k^{(3)}$	-8	4	1	-3	2	0	-1	1	0	0	1	1	2	4	7	13	24	Tribonacci numbers
$F_k^{(4)}$	0	1	-3	2	0	0	-1	1	0	0	0	1	1	2	4	8	15	Tetranacci numbers
$F_k^{(5)}$	1	-3	2	0	0	0	-1	1	0	0	0	0	1	1	2	4	8	Pentanacci numbers
$F_k^{(6)}$	-3	2	0	0	0	0	-1	1	0	0	0	0	0	1	1	2	4	Hexanacci numbers
$F_k^{(7)}$	2	0	0	0	0	0	-1	1	0	0	0	0	0	0	1	1	2	Heptanacci numbers
$F_k^{(8)}$	0	0	0	0	0	0	-1	1	0	0	0	0	0	0	0	1	1	Octanacci numbers

$$= \begin{pmatrix} F_n^{(n)} & F_{n-1}^{(n)} + F_{n-2}^{(n)} + \cdots + F_1^{(n)} & F_{n-1}^{(n)} + F_{n-2}^{(n)} + \cdots + F_2^{(n)} & \cdots & F_{n-1}^{(n)} \\ F_{n-1}^{(n)} & F_{n-2}^{(n)} + F_{n-3}^{(n)} + \cdots + F_0^{(n)} & F_{n-2}^{(n)} + F_{n-3}^{(n)} + \cdots + F_1^{(n)} & \cdots & F_{n-2}^{(n)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ & & & \cdots & \\ & & & \cdots & \\ & & & \cdots & \\ F_2^{(n)} & F_1^{(n)} + F_0^{(n)} + \cdots + F_{-n+3}^{(n)} & F_1^{(n)} + F_0^{(n)} + \cdots + F_{-n+2}^{(n)} & \cdots & F_1^{(n)} \\ F_1^{(n)} & F_0^{(n)} + F_{-1}^{(n)} + \cdots + F_{-n+2}^{(n)} & F_0^{(n)} + F_{-1}^{(n)} + \cdots + F_{-n+1}^{(n)} & \cdots & F_0^{(n)} \end{pmatrix},$$

$$(1.3)$$

where I_{n-1} is the identity matrix of order $n-1$, **1** is the row matrix of order $n-1$ with each element 1 and **0** is the column matrix of order $n-1$ with each element 0 such that

$$\det M_n = (-1)^{n+1} \tag{1.4}$$

and

$$M_n^0 = I_n. \tag{1.5}$$

The inverse of M_n is

$$M_n^{-1} = \begin{pmatrix} \mathbf{0} & I_{n-1} \\ 1 & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & -1 & -1 & \cdots & -1 \end{pmatrix}$$

$$= \begin{pmatrix} F_{n-2}^{(n)} & F_{n-3}^{(n)} + F_{n-4}^{(n)} + \cdots + F_{-1}^{(n)} & F_{n-3}^{(n)} + F_{n-4}^{(n)} + \cdots + F_0^{(n)} & \cdots & F_{n-3}^{(n)} \\ F_{n-3}^{(n)} & F_{n-4}^{(n)} + F_{n-5}^{(n)} + \cdots + F_{-2}^{(n)} & F_{n-4}^{(n)} + F_{n-5}^{(n)} + \cdots + F_{-1}^{(n)} & \cdots & F_{n-4}^{(n)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ & & & \cdots & \\ & & & \cdots & \\ & & & \cdots & \\ F_0^{(n)} & F_{-1}^{(n)} + F_{-2}^{(n)} + \cdots + F_{-n+1}^{(n)} & F_{-1}^{(n)} + F_{-2}^{(n)} + \cdots + F_{-n}^{(n)} & \cdots & F_{-1}^{(n)} \\ F_{-1}^{(n)} & F_{-2}^{(n)} + F_{-3}^{(n)} + \cdots + F_{-n}^{(n)} & F_{-2}^{(n)} + F_{-3}^{(n)} + \cdots + F_{-n-1}^{(n)} & \cdots & F_{-2}^{(n)} \end{pmatrix}.$$

$$(1.6)$$

Theorem 1.1.

$$
M_n^k =
\begin{pmatrix}
F_{k+n-1}^{(n)} & F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_k^{(n)} \\
F_{k+n-2}^{(n)} & F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_{k-1}^{(n)} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
F_{k+1}^{(n)} & F_k^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+2}^{(n)} \\
F_k^{(n)} & F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n+1}^{(n)}
\end{pmatrix}
$$

$$
\begin{matrix}
F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_{k+1}^{(n)} & \cdots & F_{k+n-2}^{(n)} \\
F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_k^{(n)} & \cdots & F_{k+n-3}^{(n)} \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
F_k^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+1}^{(n)} & \cdots & F_k^{(n)} \\
F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n}^{(n)} & \cdots & F_{k-1}^{(n)}
\end{matrix}
$$

where M_n, M_n^{-1} and M_n^0 are given in (1.3), (1.6) and (1.5).

Proof. Case 1: $k > 0$.

$$
M_n^1 =
\begin{pmatrix}
\mathbf{1} & 1 \\
I_{n-1} & \mathbf{0}
\end{pmatrix}
$$

$$
=
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
F_n^{(n)} & F_{n-1}^{(n)} + F_{n-2}^{(n)} + \cdots + F_1^{(n)} & F_{n-1}^{(n)} + F_{n-2}^{(n)} + \cdots + F_2^{(n)} & \ldots & F_{n-1}^{(n)} \\
F_{n-1}^{(n)} & F_{n-2}^{(n)} + F_{n-3}^{(n)} + \cdots + F_0^{(n)} & F_{n-2}^{(n)} + F_{n-3}^{(n)} + \cdots + F_1^{(n)} & \ldots & F_{n-2}^{(n)} \\
. & . & . & \ldots & . \\
. & . & . & \ldots & . \\
. & . & . & \ldots & . \\
. & . & . & \ldots & . \\
F_2^{(n)} & F_1^{(n)} + F_0^{(n)} + \cdots + F_{-n+3}^{(n)} & F_1^{(n)} + F_0^{(n)} + \cdots + F_{-n+2}^{(n)} & \ldots & F_1^{(n)} \\
F_1^{(n)} & F_0^{(n)} + F_{-1}^{(n)} + \cdots + F_{-n+2}^{(n)} & F_0^{(n)} + F_{-1}^{(n)} + \cdots + F_{-n+1}^{(n)} & \ldots & F_0^{(n)}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
F_{1+n-1}^{(n)} & F_{1+n-2}^{(n)} + F_{1+n-3}^{(n)} + \cdots + F_1^{(n)} & F_{1+n-2}^{(n)} + F_{1+n-3}^{(n)} + \cdots + F_{1+1}^{(n)} & \ldots & F_{1+n-2}^{(n)} \\
F_{1+n-2}^{(n)} & F_{1+n-3}^{(n)} + F_{1+n-4}^{(n)} + \cdots + F_{1-1}^{(n)} & F_{1+n-3}^{(n)} + F_{1+n-4}^{(n)} + \cdots + F_1^{(n)} & \ldots & F_{1+n-3}^{(n)} \\
. & . & . & \ldots & . \\
. & . & . & \ldots & . \\
. & . & . & \ldots & . \\
. & . & . & \ldots & . \\
F_{1+1}^{(n)} & F_1^{(n)} + F_{1-1}^{(n)} + \cdots + F_{1-n+2}^{(n)} & F_k^{(n)} + F_{1-1}^{(n)} + \cdots + F_{1-n+1}^{(n)} & \ldots & F_1^{(n)} \\
F_1^{(n)} & F_{1-1}^{(n)} + F_{1-2}^{(n)} + \cdots + F_{1-n+1}^{(n)} & F_{1-1}^{(n)} + F_{1-2}^{(n)} + \cdots + F_{1-n}^{(n)} & \ldots & F_{1-1}^{(n)}
\end{pmatrix}.
$$

The theorem is true for $k = 1$.

Let the theorem is true for $k = m$ then

$$M_n^m = \begin{pmatrix} F_{m+n-1}^{(n)} & F_{m+n-2}^{(n)} + F_{m+n-3}^{(n)} + \cdots + F_m^{(n)} \\ F_{m+n-2}^{(n)} & F_{m+n-3}^{(n)} + F_{m+n-4}^{(n)} + \cdots + F_{m-1}^{(n)} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ F_{m+1}^{(n)} & F_m^{(n)} + F_{m-1}^{(n)} + \cdots + F_{m-n+2}^{(n)} \\ F_m^{(n)} & F_{m-1}^{(n)} + F_{m-2}^{(n)} + \cdots + F_{m-n+1}^{(n)} \end{pmatrix}$$

$$\begin{matrix} F_{m+n-2}^{(n)} + F_{m+n-3}^{(n)} + \cdots + F_{m+1}^{(n)} & \cdots & F_{m+n-2}^{(n)} \\ F_{m+n-3}^{(n)} + F_{m+n-4}^{(n)} + \cdots + F_m^{(n)} & \cdots & F_{m+n-3}^{(n)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ F_m^{(n)} + F_{m-1}^{(n)} + \cdots + F_{m-n+1}^{(n)} & \cdots & F_m^{(n)} \\ F_{m-1}^{(n)} + F_{m-2}^{(n)} + \cdots + F_{m-n}^{(n)} & \cdots & F_{m-1}^{(n)} \end{matrix}$$

Now,

$$M^{m+1} = M^m M^1$$

$$= \begin{pmatrix} F_{m+n-1}^{(n)} & F_{m+n-2}^{(n)} + F_{m+n-3}^{(n)} + \cdots + F_m^{(n)} \\ F_{m+n-2}^{(n)} & F_{m+n-3}^{(n)} + F_{m+n-4}^{(n)} + \cdots + F_{m-1}^{(n)} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ F_{m+1}^{(n)} & F_m^{(n)} + F_{m-1}^{(n)} + \cdots + F_{m-n+2}^{(n)} \\ F_m^{(n)} & F_{m-1}^{(n)} + F_{m-2}^{(n)} + \cdots + F_{m-n+1}^{(n)} \end{pmatrix}$$

$$
\begin{pmatrix}
F_{m+n-2}^{(n)} + F_{m+n-3}^{(n)} + \cdots + F_{m+1}^{(n)} & \cdots & F_{m+n-2}^{(n)} \\
F_{m+n-3}^{(n)} + F_{m+n-4}^{(n)} + \cdots + F_{m}^{(n)} & \cdots & F_{m+n-3}^{(n)} \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
F_{m}^{(n)} + F_{m-1}^{(n)} + \cdots + F_{m-n+1}^{(n)} & \cdots & F_{m}^{(n)} \\
F_{m-1}^{(n)} + F_{m-2}^{(n)} + \cdots + F_{m-n}^{(n)} & \cdots & F_{m-1}^{(n)}
\end{pmatrix}
$$

$$
\times
\begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}.
$$

Using the recurrence relation (1.2), we have

$$
M^{m+1} =
\begin{pmatrix}
F_{m+n}^{(n)} & F_{m+n-1}^{(n)} + F_{m+n-2}^{(n)} + \cdots + F_{m+1}^{(n)} \\
F_{m+n-1}^{(n)} & F_{m+n-2}^{(n)} + F_{m+n-3}^{(n)} + \cdots + F_{m}^{(n)} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
F_{m+2}^{(n)} & F_{m+1}^{(n)} + F_{m}^{(n)} + \cdots + F_{m-n+3}^{(n)} \\
F_{m+1}^{(n)} & F_{m}^{(n)} + F_{m-1}^{(n)} + \cdots + F_{m-n+2}^{(n)}
\end{pmatrix}
$$

$$
\begin{pmatrix}
F_{m+n-1}^{(n)} + F_{m+n-2}^{(n)} + \cdots + F_{m+2}^{(n)} & \cdots & F_{m+n-1}^{(n)} \\
F_{m+n-2}^{(n)} + F_{m+n-3}^{(n)} + \cdots + F_{m+1}^{(n)} & \cdots & F_{m+n-2}^{(n)} \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
F_{m+1}^{(n)} + F_{m}^{(n)} + \cdots + F_{m-n+2}^{(n)} & \cdots & F_{m+1}^{(n)} \\
F_{m}^{(n)} + F_{m-1}^{(n)} + \cdots + F_{m-n+1}^{(n)} & \cdots & F_{m}^{(n)}
\end{pmatrix}.
$$

Hence by induction, for all $k > 0$ we can write

$$M_n^k = \begin{pmatrix} F_{k+n-1}^{(n)} & F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_k^{(n)} \\ F_{k+n-2}^{(n)} & F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_{k-1}^{(n)} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ F_{k+1}^{(n)} & F_k^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+2}^{(n)} \\ F_k^{(n)} & F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n+1}^{(n)} \end{pmatrix}$$

$$\begin{array}{cccc} F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_{k+1}^{(n)} & \cdots & F_{k+n-2}^{(n)} \\ F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_k^{(n)} & \cdots & F_{k+n-3}^{(n)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ F_k^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+1}^{(n)} & \cdots & F_k^{(n)} \\ F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n}^{(n)} & \cdots & F_{k-1}^{(n)} \end{array}\Bigg).$$

Case 2: $k < 0$.

Similarly, we can prove for all $k < 0$.

Hence the theorem. $\qquad\square$

2. Some Properties of M_n^k Matrix

(1) $M_n^k = M_n^{k-1} + M_n^{k-2} + \cdots + M_n^{k-n}$.

Proof.

$$M_n^k = \begin{pmatrix} F_{k+n-1}^{(n)} & F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_k^{(n)} \\ F_{k+n-2}^{(n)} & F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_{k-1}^{(n)} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ F_{k+1}^{(n)} & F_k^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+2}^{(n)} \\ F_k^{(n)} & F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n+1}^{(n)} \end{pmatrix}$$

$n-2$ $\sim \partial \nu.$

$$
\left.\begin{array}{cccc}
F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_{k+1}^{(n)} & \cdots & F_{k+n-2}^{(n)} \\
F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_{k}^{(n)} & \cdots & F_{k+n-3}^{(n)} \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
F_{k}^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+1}^{(n)} & \cdots & F_{k}^{(n)} \\
F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n}^{(n)} & \cdots & F_{k-1}^{(n)}
\end{array}\right).
$$

Now using the recurrence relation (1.2), we can write

$$
F_{k+n-1}^{(n)} = F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_{k-1}^{(n)},
$$

$$
F_{k+n-2}^{(n)} = F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_{k-2}^{(n)},
$$

$$
\vdots
$$

$$
F_{k-1}^{(n)} = F_{k-2}^{(n)} + F_{k-3}^{(n)} + \cdots + F_{k-n}^{(n)}.
$$

(margin handwriting: $F_K^{(n)} = F_{K-1}^{(n)} + F_{K-2}^{(n)} + \cdots + F_{K-n}^{(n)}$ *)*

Hence by using the property of matrix addition, we can write

$$
M_n^k = \left(\begin{array}{cc}
F_{k+n-2}^{(n)} & F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_{k-1}^{(n)} \\
F_{k+n-3}^{(n)} & F_{k+n-4}^{(n)} + F_{k+n-5}^{(n)} + \cdots + F_{k-2}^{(n)} \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
\cdot & \cdot \\
F_{k}^{(n)} & F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n+1}^{(n)} \\
F_{k-1}^{(n)} & F_{k-2}^{(n)} + F_{k-3}^{(n)} + \cdots + F_{k-n}^{(n)}
\end{array}\right.
$$

$$
\left.\begin{array}{cccc}
F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_{k}^{(n)} & \cdots & F_{k+n-3}^{(n)} \\
F_{k+n-4}^{(n)} + F_{k+n-5}^{(n)} + \cdots + F_{k-1}^{(n)} & \cdots & F_{k+n-4}^{(n)} \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
\cdot & \cdots & \cdot \\
F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n}^{(n)} & \cdots & F_{k-1}^{(n)} \\
F_{k-2}^{(n)} + F_{k-3}^{(n)} + \cdots + F_{k-n-1}^{(n)} & \cdots & F_{k-2}^{(n)}
\end{array}\right)
$$

$$+ \begin{pmatrix} F_{k+n-3}^{(n)} & F_{k+n-4}^{(n)} + F_{k+n-5}^{(n)} + \cdots + F_{k-2}^{(n)} \\ F_{k+n-4}^{(n)} & F_{k+n-5}^{(n)} + F_{k+n-6}^{(n)} + \cdots + F_{k-3}^{(n)} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ F_{k-1}^{(n)} & F_{k-2}^{(n)} + F_{k-3}^{(n)} + \cdots + F_{k-n}^{(n)} \\ F_{k-2}^{(n)} & F_{k-3}^{(n)} + F_{k-4}^{(n)} + \cdots + F_{k-n-1}^{(n)} \end{pmatrix}$$

$$\begin{pmatrix} F_{k+n-4}^{(n)} + F_{k+n-5}^{(n)} + \cdots + F_{k-1}^{(n)} & \cdots & F_{k+n-4}^{(n)} \\ F_{k+n-5}^{(n)} + F_{k+n-6}^{(n)} + \cdots + F_{k-2}^{(n)} & \cdots & F_{k+n-5}^{(n)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ F_{k-2}^{(n)} + F_{k-3}^{(n)} + \cdots + F_{k-n-1}^{(n)} & \cdots & F_{k-2}^{(n)} \\ F_{k-3}^{(n)} + F_{k-4}^{(n)} + \cdots + F_{k-n-2}^{(n)} & \cdots & F_{k-4}^{(n)} \end{pmatrix}$$

$$+ \cdots + \begin{pmatrix} F_{k-1}^{(n)} & F_{k-2}^{(n)} + F_{k-3}^{(n)} + \cdots + F_{k-n}^{(n)} \\ F_{k-2}^{(n)} & F_{k-3}^{(n)} + F_{k-4}^{(n)} + \cdots + F_{k-n-1}^{(n)} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ F_{k-n+1}^{(n)} & F_{k-n}^{(n)} + F_{k-n-1}^{(n)} + \cdots + F_{k-2n+2}^{(n)} \\ F_{k-n}^{(n)} & F_{k-n-1}^{(n)} + F_{k-n-2}^{(n)} + \cdots + F_{k-2n+1}^{(n)} \end{pmatrix}$$

$$\begin{pmatrix} F_{k-2}^{(n)} + F_{k-3}^{(n)} + \cdots + F_{k-n+1}^{(n)} & \cdots & F_{k-2}^{(n)} \\ F_{k-3}^{(n)} + F_{k-4}^{(n)} + \cdots + F_{k-n}^{(n)} & \cdots & F_{k-3}^{(n)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ F_{k-n}^{(n)} + F_{k-n-1}^{(n)} + \cdots + F_{k-2n+1}^{(n)} & \cdots & F_{k-n}^{(n)} \\ F_{k-n-1}^{(n)} + F_{k-n-2}^{(n)} + \cdots + F_{k-2n}^{(n)} & \cdots & F_{k-n-1}^{(n)} \end{pmatrix}$$

$$= M_n^{k-1} + M_n^{k-2} + \cdots + M_n^{k-n}. \tag{2.1}$$

So (2.1) is the explicit form of the matrix M_n^k. □

(2) Using the basic property of matrix, we have

$$M_n^k M_n^l = M_n^l M_n^k = M_n^{k+l} \quad (k, l = 0, \pm 1, \pm 2, \pm 3, \ldots). \tag{2.2}$$

(3)

$$\det M_n^k = (\det M_n)^k = ((-1)^{n+1})^k = (-1)^{nk+k}. \tag{2.3}$$

3. Fibonacci n-Step Coding/Decoding Method

Let us represent the initial message in the form of the square matrix P of order n. We take the Fibonacci n-step matrix M_n^k of order n as a coding matrix and its inverse matrix M_n^{-k} as a decoding matrix for an arbitrary positive integer k. We name the transformation $P \times M_n^k = E$ as Fibonacci n-step coding, the transformation $E \times M_n^{-k} = P$ as Fibonacci n-step decoding and define E as code matrix.

3.1. Determinant of the code matrix E

We define the code matrix E by the following formula

$$E = P \times M_n^k.$$

Using the basic property of determinants, we have

$$\det E = \det(P \times M_n^k) = \det P \times \det M_n^k = \det P \times (-1)^{nk+k}. \tag{3.1}$$

3.2. Example of Fibonacci n-step coding/decoding method

The examples for $n = 2$ and $n = 3$ are given in [9, 1] respectively.

We consider $n = 4$, the Tetranacci coding/decoding method.

Let us represent the initial message in the form of square matrix of order 4 as:

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}. \tag{3.2}$$

Let us assume that all elements of the matrix are non-negative integer i.e., $p_{ij} \geq 0$ for $i, j = 1, 2, 3, 4$. Let us now select for any value of k, the matrix M_4^k as the coding matrix.

We simply write for $k = 3$. Then

$$M_4^3 = \begin{pmatrix} F_6^{(4)} & F_5^{(4)} + F_4^{(4)} + F_3^{(4)} & F_5^{(4)} + F_4^{(4)} & F_5^{(4)} \\ F_5^{(4)} & F_4^{(4)} + F_3^{(4)} + F_2^{(4)} & F_4^{(4)} + F_3^{(4)} & F_4^{(4)} \\ F_4^{(4)} & F_3^{(4)} + F_2^{(4)} + F_1^{(4)} & F_3^{(4)} + F_2^{(4)} & F_3^{(4)} \\ F_3^{(4)} & F_2^{(4)} + F_1^{(4)} + F_0^{(4)} & F_2^{(4)} + F_1^{(4)} & F_2^{(4)} \end{pmatrix} = \begin{pmatrix} 4 & 4 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (3.3)$$

The inverse of M_4^3 is given by

$$M_4^{-3} = \begin{pmatrix} F_0^{(4)} & F_{-1}^{(4)} + F_{-2}^{(4)} + F_{-3}^{(4)} & F_{-1}^{(4)} + F_{-2}^{(4)} & F_{-1}^{(4)} \\ F_{-1}^{(4)} & F_{-2}^{(4)} + F_{-3}^{(4)} + F_{-4}^{(4)} & F_{-2}^{(4)} + F_{-3}^{(4)} & F_{-2}^{(4)} \\ F_{-2}^{(4)} & F_{-3}^{(4)} + F_{-4}^{(4)} + F_{-5}^{(4)} & F_{-3}^{(4)} + F_{-4}^{(4)} & F_{-3}^{(4)} \\ F_{-3}^{(4)} & F_{-4}^{(4)} + F_{-5}^{(4)} + F_{-6}^{(4)} & F_{-4}^{(4)} + F_{-5}^{(4)} & F_{-4}^{(4)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}.$$

Then the coding of the message (3.2) consists of the multiplication of the initial matrix (3.3) i.e.,

$$P \times M_4^3 = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} \begin{pmatrix} 4 & 4 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4p_{11} + 2p_{12} + p_{13} + p_{14} & 4p_{11} + 2p_{12} + p_{13} \\ 4p_{21} + 2p_{22} + p_{23} + p_{24} & 4p_{21} + 2p_{22} + p_{23} \\ 4p_{31} + 2p_{32} + p_{33} + p_{34} & 4p_{31} + 2p_{32} + p_{33} \\ 4p_{41} + 2p_{42} + p_{43} + p_{44} & 4p_{41} + 2p_{42} + p_{43} \end{pmatrix}$$

$$\begin{pmatrix} 3p_{11} + 2p_{12} + p_{13} & 2p_{11} + p_{12} + p_{13} \\ 3p_{21} + 2p_{22} + p_{23} & 2p_{21} + p_{22} + p_{23} \\ 3p_{31} + 2p_{32} + p_{33} & 2p_{31} + p_{32} + p_{33} \\ 3p_{41} + 2p_{42} + p_{43} & 2p_{41} + p_{42} + p_{43} \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} = E, \quad (3.4)$$

where **16** $r\gamma^n$

$$e_{11} = 4p_{11} + 2p_{12} + p_{13} + p_{14}, \quad e_{12} = 4p_{11} + 2p_{12} + p_{13}, \quad e_{13} = 3p_{11} + 2p_{12} + p_{13},$$
$$e_{14} = 2p_{11} + p_{12} + p_{13}, \quad e_{21} = 4p_{21} + 2p_{22} + p_{23} + p_{24}, \quad e_{22} = 4p_{21} + 2p_{22} + p_{23},$$
$$e_{23} = 3p_{21} + 2p_{22} + p_{23}, \quad e_{24} = 2p_{21} + p_{22} + p_{23}, \quad e_{31} = 4p_{31} + 2p_{32} + p_{33} + p_{34},$$
$$e_{32} = 4p_{31} + 2p_{32} + p_{33}, \quad e_{33} = 3p_{31} + 2p_{32} + p_{33}, \quad e_{34} = 2p_{31} + p_{32} + p_{33},$$
$$e_{41} = 4p_{41} + 2p_{42} + p_{43} + p_{44}, \quad e_{42} = 4p_{41} + 2p_{42} + p_{43}, \quad e_{43} = 3p_{41} + 2p_{42} + p_{43},$$
$$e_{44} = 2p_{41} + p_{42} + p_{43}.$$

Solving these we have,

$$p_{11} = e_{12} - e_{13}, \quad p_{12} = -e_{12} + 2e_{13} - e_{14}, \quad p_{13} = -e_{12} + e_{14}, \quad p_{14} = e_{11} - e_{12},$$
$$p_{21} = e_{22} - e_{23}, \quad p_{22} = -e_{22} + 2e_{23} - e_{24}, \quad p_{23} = -e_{22} + e_{24}, \quad p_{24} = e_{21} - e_{22},$$
$$p_{31} = e_{32} - e_{33}, \quad p_{32} = -e_{32} + 2e_{33} - e_{34}, \quad p_{33} = -e_{32} + e_{34}, \quad p_{34} = e_{31} - e_{32},$$
$$p_{41} = e_{42} - e_{43}, \quad p_{42} = -e_{42} + 2e_{43} - e_{44}, \quad p_{13} = -e_{42} + e_{44}, \quad p_{44} = e_{41} - e_{42}.$$

The code matrix E is sent to a channel and the decoding of the code message E is given by (3.4), performed by the following manner:

$$E \times M_4^{-3} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} e_{12} - e_{13} & -e_{12} + 2e_{13} - e_{14} & -e_{12} + e_{14} & e_{11} - e_{12} \\ e_{22} - e_{23} & -e_{22} + 2e_{23} - e_{24} & -e_{22} + e_{24} & e_{21} - e_{22} \\ e_{32} - e_{33} & -e_{32} + 2e_{33} - e_{34} & -e_{32} + e_{34} & e_{31} - e_{32} \\ e_{42} - e_{43} & -e_{42} + 2e_{43} - e_{44} & -e_{42} + e_{44} & e_{41} - e_{42} \end{pmatrix}$$

$$= \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} = P.$$

Example 3.1. We take the message 24327156521845, the sequence of decimal numerals. Then we can represent this message in the matrix form at random with maintaining the same order as follows:

$$P = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 4 & 3 & 2 & 7 \\ 1 & 5 & 6 & 2 \\ 1 & 8 & 4 & 5 \end{pmatrix}.$$

Now we write the matrix M_4 of the third power as coding matrix

$$M_4^3 = \begin{pmatrix} 4 & 4 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and the inverse of M_4^3 is given by

$$M_4^{-3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}.$$

The code message

$$E = P \times M_4^3$$

$$= \begin{pmatrix} 0 & 0 & 0 & 2 \\ 4 & 3 & 2 & 7 \\ 1 & 5 & 6 & 2 \\ 1 & 8 & 4 & 5 \end{pmatrix} \begin{pmatrix} 4 & 4 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 31 & 24 & 20 & 13 \\ 81 & 79 & 78 & 72 \\ 29 & 24 & 23 & 14 \end{pmatrix}$$

and the decode message

$$P = E \times M_4^{-3}$$

$$= \begin{pmatrix} 2 & 0 & 0 & 0 \\ 31 & 24 & 20 & 13 \\ 81 & 79 & 78 & 72 \\ 29 & 24 & 23 & 14 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 4 & 3 & 2 & 7 \\ 1 & 5 & 6 & 2 \\ 1 & 8 & 4 & 5 \end{pmatrix}.$$

4. Relations Among The Code Matrix Elements

In this paper, we develop the relations among the code matrix elements of E. Choosing k in such a manner that $e_{ij} > 0$ for $i, j = 1, 2, 3, \ldots, n$, we write the code matrix E as:

$$E = P \times M_n^k$$

$$= \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ . & . & \cdots & . \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

$$\times \begin{pmatrix} F_{k+n-1}^{(n)} & F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_k^{(n)} \\ F_{k+n-2}^{(n)} & F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_{k-1}^{(n)} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ F_{k+1}^{(n)} & F_k^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+2}^{(n)} \\ F_k^{(n)} & F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n+1}^{(n)} \end{pmatrix}$$

$$\begin{pmatrix} F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_{k+1}^{(n)} & \cdots & F_{k+n-2}^{(n)} \\ F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_k^{(n)} & \cdots & F_{k+n-3}^{(n)} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ F_k^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+1}^{(n)} & \cdots & F_k^{(n)} \\ F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n}^{(n)} & \cdots & F_{k-1}^{(n)} \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} & e_{12} & \ldots & e_{1n} \\ e_{21} & e_{22} & \ldots & e_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e_{n1} & e_{n2} & \ldots & e_{nn} \end{pmatrix}$$

and after decoding the initial message

$$P = E \times M_n^{-k}$$

$$= \begin{pmatrix} e_{11} & e_{12} & \ldots & e_{1n} \\ e_{21} & e_{22} & \ldots & e_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ e_{n1} & e_{n2} & \ldots & e_{nn} \end{pmatrix}$$

$$\times \begin{pmatrix} F_{k+n-1}^{(n)} & F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_k^{(n)} \\ F_{k+n-2}^{(n)} & F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_{k-1}^{(n)} \\ . & . \\ . & . \\ . & . \\ . & . \\ F_{k+1}^{(n)} & F_k^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+2}^{(n)} \\ F_k^{(n)} & F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n+1}^{(n)} \end{pmatrix}$$

$$\begin{pmatrix} F_{k+n-2}^{(n)} + F_{k+n-3}^{(n)} + \cdots + F_{k+1}^{(n)} & \ldots & F_{k+n-2}^{(n)} \\ F_{k+n-3}^{(n)} + F_{k+n-4}^{(n)} + \cdots + F_k^{(n)} & \ldots & F_{k+n-3}^{(n)} \\ . & \ldots & . \\ . & \ldots & . \\ . & \ldots & . \\ . & \ldots & . \\ F_k^{(n)} + F_{k-1}^{(n)} + \cdots + F_{k-n+1}^{(n)} & \ldots & F_k^{(n)} \\ F_{k-1}^{(n)} + F_{k-2}^{(n)} + \cdots + F_{k-n}^{(n)} & \ldots & F_{k-1}^{(n)} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} p_{11} & p_{12} & \ldots & p_{1n} \\ p_{21} & p_{22} & \ldots & p_{2n} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ p_{n1} & p_{n2} & \ldots & p_{nn} \end{pmatrix}.$$

Case 1: $n = 2$ [9],

$$\frac{F_{k+1}^{(2)}}{F_k^{(2)}} \lessgtr \frac{e_{11}}{e_{12}} \gtrless \frac{F_k^{(2)}}{F_{k-1}^{(2)}} \quad \text{and} \quad \frac{F_{k+1}^{(2)}}{F_k^{(2)}} \lessgtr \frac{e_{21}}{e_{22}} \gtrless \frac{F_k^{(2)}}{F_{k-1}^{(2)}}.$$

For large k, we have

$$\frac{e_{11}}{e_{12}} \approx r_2 \quad \text{and} \quad \frac{e_{21}}{e_{22}} \approx r_2 \quad \text{where } r_2 = 1.61803.$$

Case 2: $n = 3$.

By [1], we have

$$\min\left\{\frac{F_{k+2}^{(3)}}{F_{k+1}^{(3)} + F_k^{(3)}}, \frac{F_{k+1}^{(3)}}{F_k^{(3)} + F_{k-1}^{(3)}}, \frac{F_k^{(3)}}{F_{k-1}^{(3)} + F_{k+2}^{(3)}}\right\}$$

$$\leq \frac{e_{i1}}{e_{i2}} \leq \max\left\{\frac{F_{k+2}^{(3)}}{F_{k+1}^{(3)} + F_k^{(3)}}, \frac{F_{k+1}^{(3)}}{F_k^{(3)} + F_{k-1}^{(3)}}, \frac{F_k^{(3)}}{F_{k-1}^{(3)} + F_{k-2}^{(3)}}\right\},$$

$$\min\left\{\frac{F_{k+1}^{(3)} + F_k^{(3)}}{F_{k+1}^{(3)}}, \frac{F_k^{(3)} + F_{k-1}^{(3)}}{F_k^{(3)}}, \frac{F_{k-1}^{(3)} + F_{k-2}^{(3)}}{F_{k-1}^{(3)}}\right\}$$

$$\leq \frac{e_{i2}}{e_{i3}} \leq \max\left\{\frac{F_{k+1}^{(3)} + F_k^{(3)}}{F_{k+1}^{(3)}}, \frac{F_k^{(3)} + F_{k-1}^{(3)}}{F_k^{(3)}}, \frac{F_{k-1}^{(3)} + F_{k-2}^{(3)}}{F_{k-1}^{(3)}}\right\},$$

and

$$\min\left\{\frac{F_{k+2}^{(3)}}{F_{k+1}^{(3)}}, \frac{F_{k+1}^{(3)}}{F_k^{(3)}}, \frac{F_k^{(3)}}{F_{k-1}^{(3)}}\right\} \leq \frac{e_{i1}}{e_{i3}} \leq \max\left\{\frac{F_{k+2}^{(3)}}{F_{k+1}^{(3)}}, \frac{F_{k+1}^{(3)}}{F_k^{(3)}}, \frac{F_k^{(3)}}{F_{k-1}^{(3)}}\right\}$$

$$\text{for } i = 1, 2, 3.$$

For large k, we have

$$\frac{e_{i1}}{e_{i2}} \approx \frac{r_3^2}{1 + r_3}, \quad \frac{e_{i2}}{e_{i3}} \approx \frac{1 + r_3}{r_3}, \quad \frac{e_{i1}}{e_{i3}} \approx r_3 \quad \text{where } r_3 = 1.83929.$$

Case 3: $n = 4$.

Choosing k in such a manner that $e_{ij} > 0$ for $i, j = 1, 2, 3, 4$, we write the code matrix E as:

$$E = P \times M_4^k$$

$$= \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix} \begin{pmatrix} F_{k+3}^{(4)} & F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_k^{(4)} & F_{k+2}^{(4)} + F_{k+1}^{(4)} & F_{k+2}^{(4)} \\ F_{k+2}^{(4)} & F_{k+1}^{(4)} + F_k^{(4)} + F_{k-1}^{(4)} & F_{k+1}^{(4)} + F_k^{(4)} & F_{k+1}^{(4)} \\ F_{k+1}^{(4)} & F_k^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)} & F_k^{(4)} + F_{k-1}^{(4)} & F_k^{(4)} \\ F_k^{(4)} & F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)} & F_{k-1}^{(4)} + F_{k-2}^{(4)} & F_{k-1}^{(4)} \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix}.$$

Now,

$$\det M_4^k = \begin{vmatrix} F_{k+3}^{(4)} & F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_k^{(4)} & F_{k+2}^{(4)} + F_{k+1}^{(4)} & F_{k+2}^{(4)} \\ F_{k+2}^{(4)} & F_{k+1}^{(4)} + F_k^{(4)} + F_{k-1}^{(4)} & F_{k+1}^{(4)} + F_k^{(4)} & F_{k+1}^{(4)} \\ F_{k+1}^{(4)} & F_k^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)} & F_k^{(4)} + F_{k-1}^{(4)} & F_k^{(4)} \\ F_k^{(4)} & F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)} & F_{k-1}^{(4)} + F_{k-2}^{(4)} & F_{k-1}^{(4)} \end{vmatrix} = (-1)^k,$$

using (3.1) (4.1)

and

$$M_4^{-k} = \begin{pmatrix} F_{k+3}^{(4)} & F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_k^{(4)} & F_{k+2}^{(4)} + F_{k+1}^{(4)} & F_{k+2}^{(4)} \\ F_{k+2}^{(4)} & F_{k+1}^{(4)} + F_k^{(4)} + F_{k-1}^{(4)} & F_{k+1}^{(4)} + F_k^{(4)} & F_{k+1}^{(4)} \\ F_{k+1}^{(4)} & F_k^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)} & F_k^{(4)} + F_{k-1}^{(4)} & F_k^{(4)} \\ F_k^{(4)} & F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)} & F_{k-1}^{(4)} + F_{k-2}^{(4)} & F_{k-1}^{(4)} \end{pmatrix}^{-1}$$

$$= \frac{1}{\det M_4^k} \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix}, \qquad \text{where } h_{ij} \text{ is the cofactor of } (j,i)th \text{ element of } M_4^k \text{ for } i,j = 1,2,3,4$$

$$= (-1)^k \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix}.$$

The initial message

$$P = E \times M_4^{-k}$$

$$= \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} \begin{pmatrix} F_{k+3}^{(4)} & F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_k^{(4)} & F_{k+2}^{(4)} + F_{k+1}^{(4)} & F_{k+2}^{(4)} \\ F_{k+2}^{(4)} & F_{k+1}^{(4)} + F_k^{(4)} + F_{k-1}^{(4)} & F_{k+1}^{(4)} + F_k^{(4)} & F_{k+1}^{(4)} \\ F_{k+1}^{(4)} & F_k^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)} & F_k^{(4)} + F_{k-1}^{(4)} & F_k^{(4)} \\ F_k^{(4)} & F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)} & F_{k-1}^{(4)} + F_{k-2}^{(4)} & F_{k-1}^{(4)} \end{pmatrix}^{-1}$$

$$= (-1)^k \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix}$$

$$= \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ p_{31} & p_{32} & p_{33} & p_{34} \\ p_{41} & p_{42} & p_{43} & p_{44} \end{pmatrix}.$$

Also, we have $p_{ij} \geq 0$ for $i,j = 1,2,3,4$.

Therefore,

$$p_{11} = (-1)^k(e_{11}h_{11} + e_{12}h_{21} + e_{13}h_{31} + e_{14}h_{41}) \geq 0, \tag{4.2}$$

$$p_{12} = (-1)^k(e_{11}h_{12} + e_{12}h_{22} + e_{13}h_{32} + e_{14}h_{42}) \geq 0, \tag{4.3}$$

$$p_{13} = (-1)^k(e_{11}h_{13} + e_{12}h_{23} + e_{13}h_{33} + e_{14}h_{43}) \geq 0, \tag{4.4}$$

$$p_{14} = (-1)^k(e_{11}h_{14} + e_{12}h_{24} + e_{13}h_{34} + e_{14}h_{44}) \geq 0, \tag{4.5}$$

$$p_{21} = (-1)^k(e_{21}h_{11} + e_{22}h_{21} + e_{23}h_{31} + e_{24}h_{41}) \geq 0, \tag{4.6}$$

$$p_{22} = (-1)^k(e_{21}h_{12} + e_{22}h_{22} + e_{23}h_{32} + e_{24}h_{42}) \geq 0, \tag{4.7}$$

$$p_{23} = (-1)^k(e_{21}h_{13} + e_{22}h_{23} + e_{23}h_{33} + e_{24}h_{43}) \geq 0, \tag{4.8}$$

$$p_{24} = (-1)^k(e_{21}h_{14} + e_{22}h_{24} + e_{23}h_{34} + e_{24}h_{44}) \geq 0, \tag{4.9}$$

$$p_{31} = (-1)^k(e_{31}h_{11} + e_{32}h_{21} + e_{33}h_{31} + e_{34}h_{41}) \geq 0, \tag{4.10}$$

$$p_{32} = (-1)^k(e_{31}h_{12} + e_{32}h_{22} + e_{33}h_{32} + e_{34}h_{42}) \geq 0, \tag{4.11}$$

$$p_{33} = (-1)^k(e_{31}h_{13} + e_{32}h_{23} + e_{33}h_{33} + e_{34}h_{43}) \geq 0, \tag{4.12}$$

$$p_{34} = (-1)^k(e_{31}h_{14} + e_{32}h_{24} + e_{33}h_{34} + e_{34}h_{44}) \geq 0, \tag{4.13}$$

$$p_{41} = (-1)^k(e_{41}h_{11} + e_{42}h_{21} + e_{43}h_{31} + e_{44}h_{41}) \geq 0, \tag{4.14}$$

$$p_{42} = (-1)^k(e_{41}h_{12} + e_{42}h_{22} + e_{43}h_{32} + e_{44}h_{42}) \geq 0, \tag{4.15}$$

$$p_{43} = (-1)^k(e_{41}h_{13} + e_{42}h_{23} + e_{43}h_{33} + e_{44}h_{43}) \geq 0 \tag{4.16}$$

and

$$p_{44} = (-1)^k(e_{41}h_{14} + e_{42}h_{24} + e_{43}h_{34} + e_{44}h_{44}) \geq 0. \tag{4.17}$$

Let k be an even integer.

From (4.2), we have

$$e_{11}h_{11} + e_{12}h_{21} + e_{13}h_{31} + e_{14}h_{41} \geq 0. \tag{4.18}$$

From (4.3), we have

$$e_{11}h_{12} + e_{12}h_{22} + e_{13}h_{32} + e_{14}h_{42} \geq 0. \tag{4.19}$$

From (4.4), we have

$$e_{11}h_{13} + e_{12}h_{23} + e_{13}h_{33} + e_{14}h_{43} \geq 0. \tag{4.20}$$

From (4.5), we have

$$e_{11}h_{14} + e_{12}h_{24} + e_{13}h_{34} + e_{14}h_{44} \geq 0. \tag{4.21}$$

Dividing both sides by e_{11} (>0) of (4.18)–(4.21), we have

$$h_{11} + h_{21}\frac{e_{12}}{e_{11}} + h_{31}\frac{e_{13}}{e_{11}} + h_{41}\frac{e_{14}}{e_{11}} \geq 0, \tag{4.22}$$

$$h_{12} + h_{22}\frac{e_{12}}{e_{11}} + h_{32}\frac{e_{13}}{e_{11}} + h_{42}\frac{e_{14}}{e_{11}} \geq 0, \tag{4.23}$$

$$h_{13} + h_{23}\frac{e_{12}}{e_{11}} + h_{33}\frac{e_{13}}{e_{11}} + h_{43}\frac{e_{14}}{e_{11}} \geq 0 \tag{4.24}$$

and

$$h_{14} + h_{24}\frac{e_{12}}{e_{11}} + h_{34}\frac{e_{13}}{e_{11}} + h_{44}\frac{e_{14}}{e_{11}} \geq 0. \tag{4.25}$$

Now $3^8 = 6561$ cases arise for $h_{ij} \geq = < 0$ for $i = 3, 4$ and $j = 1, 2, 3, 4$.

Case 1: $h_{ij} > 0$ for $i = 3, 4$ and $j = 1, 2, 3, 4$.

From (4.22), we have

$$h_{31}\frac{e_{13}}{e_{11}} + h_{41}\frac{e_{14}}{e_{11}} \geq -\left(h_{11} + h_{21}\frac{e_{12}}{e_{11}}\right). \tag{4.26}$$

From (4.23), we have

$$h_{32}\frac{e_{13}}{e_{11}} + h_{42}\frac{e_{14}}{e_{11}} \geq -\left(h_{12} + h_{22}\frac{e_{12}}{e_{11}}\right). \tag{4.27}$$

From (4.24), we have

$$h_{33}\frac{e_{13}}{e_{11}} + h_{43}\frac{e_{14}}{e_{11}} \geq -\left(h_{13} + h_{23}\frac{e_{12}}{e_{11}}\right). \tag{4.28}$$

From (4.25), we have

$$h_{34}\frac{e_{13}}{e_{11}} + h_{44}\frac{e_{14}}{e_{11}} \geq -\left(h_{14} + h_{24}\frac{e_{12}}{e_{11}}\right). \tag{4.29}$$

From (4.26)–(4.29) and using (4.1), we have

$$\min\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}, \right.$$

$$\left. \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}\right\}$$

$$\leq \frac{e_{11}}{e_{12}} \leq \max\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}, \right.$$

$$\left. \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}\right\}.$$

Similarly, we have

$$\min\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}+F_{k+1}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}+F_k^{(4)}}, \frac{F_{k+1}^{(4)}}{F_k^{(4)}+F_{k-1}^{(4)}}, \frac{F_k^{(4)}}{F_{k-1}^{(4)}+F_{k-2}^{(4)}}\right\}$$

$$\leq \frac{e_{11}}{e_{13}} \leq \max\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}+F_{k+1}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}+F_k^{(4)}}, \frac{F_{k+1}^{(4)}}{F_k^{(4)}+F_{k-1}^{(4)}}, \frac{F_k^{(4)}}{F_{k-1}^{(4)}+F_{k-2}^{(4)}}\right\},$$

$$\min\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_k^{(4)}}, \frac{F_k^{(4)}}{F_{k-1}^{(4)}}\right\} \leq \frac{e_{11}}{e_{14}} \leq \max\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_k^{(4)}}, \frac{F_k^{(4)}}{F_{k-1}^{(4)}}\right\},$$

$$\min\left\{\frac{F_{k+2}^{(4)}+F_{k+1}^{(4)}+F_k^{(4)}}{F_{k+2}^{(4)}+F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}+F_k^{(4)}+F_{k-1}^{(4)}}{F_{k+1}^{(4)}+F_k^{(4)}},\right.$$

$$\left.\frac{F_k^{(4)}+F_{k-1}^{(4)}+F_{k-2}^{(4)}}{F_k^{(4)}+F_{k-1}^{(4)}}, \frac{F_{k-1}^{(4)}+F_{k-2}^{(4)}+F_{k-3}^{(4)}}{F_{k-1}^{(4)}+F_{k-2}^{(4)}}\right\}$$

$$\leq \frac{e_{12}}{e_{13}} \leq \max\left\{\frac{F_{k+2}^{(4)}+F_{k+1}^{(4)}+F_k^{(4)}}{F_{k+2}^{(4)}+F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}+F_k^{(4)}+F_{k-1}^{(4)}}{F_{k+1}^{(4)}+F_k^{(4)}},\right.$$

$$\left.\frac{F_k^{(4)}+F_{k-1}^{(4)}+F_{k-2}^{(4)}}{F_k^{(4)}+F_{k-1}^{(4)}}, \frac{F_{k-1}^{(4)}+F_{k-2}^{(4)}+F_{k-3}^{(4)}}{F_{k-1}^{(4)}+F_{k-2}^{(4)}}\right\},$$

$$\min\left\{\frac{F_{k+2}^{(4)}+F_{k+1}^{(4)}+F_k^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)}+F_k^{(4)}+F_{k-1}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_k^{(4)}+F_{k-1}^{(4)}+F_{k-2}^{(4)}}{F_k^{(4)}},\right.$$

$$\left.\frac{F_{k-1}^{(4)}+F_{k-2}^{(4)}+F_{k-3}^{(4)}}{F_{k-1}^{(4)}}\right\}$$

$$\leq \frac{e_{12}}{e_{14}} \leq \max\left\{\frac{F_{k+2}^{(4)}+F_{k+1}^{(4)}+F_k^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)}+F_k^{(4)}+F_{k-1}^{(4)}}{F_{k+1}^{(4)}},\right.$$

$$\left.\frac{F_k^{(4)}+F_{k-1}^{(4)}+F_{k-2}^{(4)}}{F_k^{(4)}}, \frac{F_{k-1}^{(4)}+F_{k-2}^{(4)}+F_{k-3}^{(4)}}{F_{k-1}^{(4)}}\right\},$$

and

$$\min\left\{\frac{F_{k+2}^{(4)}+F_{k+1}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)}+F_k^{(4)}}{F_{k+1}^{(4)}}, \frac{F_k^{(4)}+F_{k-1}^{(4)}}{F_k^{(4)}}, \frac{F_{k-1}^{(4)}+F_{k-2}^{(4)}}{F_{k-1}^{(4)}}\right\}$$

$$\leq \frac{e_{13}}{e_{14}} \leq \max\left\{\frac{F_{k+2}^{(4)}+F_{k+1}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)}+F_k^{(4)}}{F_{k+1}^{(4)}}, \frac{F_k^{(4)}+F_{k-1}^{(4)}}{F_k^{(4)}}, \frac{F_{k-1}^{(4)}+F_{k-2}^{(4)}}{F_{k-1}^{(4)}}\right\}.$$

Case 2: $h_{31} = 0$, $h_{32} > 0$, $h_{33} > 0$, $h_{34} < 0$, $h_{41} > 0$, $h_{42} = 0$, $h_{43} > 0$, $h_{44} < 0$.

From (4.22), we have

$$h_{41}\frac{e_{14}}{e_{11}} \geq -\left(h_{11} + h_{21}\frac{e_{12}}{e_{11}}\right). \tag{4.30}$$

From (4.23), we have

$$h_{32}\frac{e_{13}}{e_{11}} \geq -\left(h_{12} + h_{22}\frac{e_{12}}{e_{11}}\right). \tag{4.31}$$

From (4.24), we have

$$h_{33}\frac{e_{13}}{e_{11}} + h_{43}\frac{e_{14}}{e_{11}} \geq -\left(h_{13} + h_{23}\frac{e_{12}}{e_{11}}\right). \tag{4.32}$$

From (4.25), we have

$$h_{34}\frac{e_{13}}{e_{11}} + h_{44}\frac{e_{14}}{e_{11}} \leq h_{14} + h_{24}\frac{e_{12}}{e_{11}}. \tag{4.33}$$

From (4.30)–(4.33) and using (4.1), we have

$$\min\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}, \right.$$
$$\left. \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}\right\}$$
$$\leq \frac{e_{11}}{e_{12}} \leq \max\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}, \right.$$
$$\left. \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}\right\}.$$

Similarly, we have

$$\min\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}}\right\}$$
$$\leq \frac{e_{11}}{e_{13}} \leq \max\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}}\right\},$$

$$\min\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)}}\right\} \leq \frac{e_{11}}{e_{14}} \leq \max\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)}}\right\},$$

$$\min \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)}}, \right.$$

$$\left. \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}} \right\}$$

$$\leq \frac{e_{12}}{e_{13}} \leq \max \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)}}, \right.$$

$$\left. \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}} \right\},$$

$$\min \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)}}, \right.$$

$$\left. \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)}} \right\}$$

$$\leq \frac{e_{12}}{e_{14}} \leq \max \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)}}, \right.$$

$$\left. \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)}} \right\},$$

and

$$\min \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k-1}^{(4)}} \right\}$$

$$\leq \frac{e_{13}}{e_{14}} \leq \max \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k-1}^{(4)}} \right\}.$$

Case 3: $h_{ij} < 0$ for $i = 3, 4$ and $j = 1, 2, 3, 4$.

From (4.22), we have

$$h_{31}\frac{e_{13}}{e_{11}} + h_{41}\frac{e_{14}}{e_{11}} \leq -\left(h_{11} + h_{21}\frac{e_{12}}{e_{11}} \right). \tag{4.34}$$

From (4.23), we have

$$h_{32}\frac{e_{13}}{e_{11}} + h_{42}\frac{e_{14}}{e_{11}} \leq -\left(h_{12} + h_{22}\frac{e_{12}}{e_{11}} \right). \tag{4.35}$$

From (4.24), we have

$$h_{33}\frac{e_{13}}{e_{11}} + h_{43}\frac{e_{14}}{e_{11}} \leq -\left(h_{13} + h_{23}\frac{e_{12}}{e_{11}}\right). \tag{4.36}$$

From (4.25), we have

$$h_{34}\frac{e_{13}}{e_{11}} + h_{44}\frac{e_{14}}{e_{11}} \leq -\left(h_{14} + h_{24}\frac{e_{12}}{e_{11}}\right). \tag{4.37}$$

From (4.34)–(4.37) and using (4.1), we have

$$\min\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_k^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_k^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_k^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}},\right.$$
$$\left.\frac{F_k^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}\right\}$$
$$\leq \frac{e_{11}}{e_{12}} \leq \max\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_k^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_k^{(4)} + F_{k-1}^{(4)}},\right.$$
$$\left.\frac{F_{k+1}^{(4)}}{F_k^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}, \frac{F_k^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}\right\}.$$

Similarly, we have

$$\min\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_k^{(4)}}, \frac{F_{k+1}^{(4)}}{F_k^{(4)} + F_{k-1}^{(4)}}, \frac{F_k^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}}\right\}$$
$$\leq \frac{e_{11}}{e_{13}} \leq \max\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_k^{(4)}}, \frac{F_{k+1}^{(4)}}{F_k^{(4)} + F_{k-1}^{(4)}}, \frac{F_k^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}}\right\},$$

$$\min\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_k^{(4)}}, \frac{F_k^{(4)}}{F_{k-1}^{(4)}}\right\} \leq \frac{e_{11}}{e_{14}} \leq \max\left\{\frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_k^{(4)}}, \frac{F_k^{(4)}}{F_{k-1}^{(4)}}\right\},$$

$$\min\left\{\frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_k^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)} + F_k^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)} + F_k^{(4)}}, \frac{F_k^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_k^{(4)} + F_{k-1}^{(4)}},\right.$$
$$\left.\frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}}\right\}$$
$$\leq \frac{e_{12}}{e_{13}} \leq \max\left\{\frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_k^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)} + F_k^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)} + F_k^{(4)}},\right.$$
$$\left.\frac{F_k^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_k^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}}\right\},$$

$$\min \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)}}, \right.$$

$$\left. \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)}} \right\}$$

$$\leq \frac{e_{12}}{e_{14}} \leq \max \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)}}, \right.$$

$$\left. \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)}} \right\},$$

and

$$\min \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k-1}^{(4)}} \right\}$$

$$\leq \frac{e_{13}}{e_{14}} \leq \max \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k-1}^{(4)}} \right\}.$$

Similarly, it can be proved for the rest cases and the case when k is an odd integer.

Hence, we have

$$\min \left\{ \frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}, \right.$$

$$\left. \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}} \right\}$$

$$\leq \frac{e_{11}}{e_{12}} \leq \max \left\{ \frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}, \right.$$

$$\left. \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}} \right\},$$

$$\min \left\{ \frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}} \right\}$$

$$\leq \frac{e_{11}}{e_{13}} \leq \max \left\{ \frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}} \right\},$$

$$\min \left\{ \frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)}} \right\} \leq \frac{e_{11}}{e_{14}} \leq \max \left\{ \frac{F_{k+3}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+2}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k}^{(4)}}{F_{k-1}^{(4)}} \right\},$$

$$\min \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)}}, \right.$$

$$\left. \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}} \right\}$$

$$\leq \frac{e_{12}}{e_{13}} \leq \max \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)} + F_{k+1}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)} + F_{k}^{(4)}}, \right.$$

$$\left. \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)} + F_{k-1}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)} + F_{k-2}^{(4)}} \right\},$$
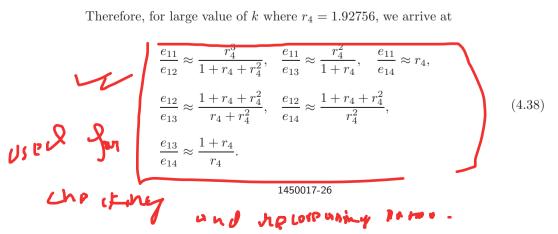
$$\min \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)}}, \right.$$

$$\left. \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)}} \right\}$$

$$\leq \frac{e_{12}}{e_{14}} \leq \max \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k+1}^{(4)}}, \right.$$

$$\left. \frac{F_{k}^{(4)} + F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)} + F_{k-3}^{(4)}}{F_{k-1}^{(4)}} \right\},$$

and

$$\min \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k-1}^{(4)}} \right\}$$

$$\leq \frac{e_{13}}{e_{14}} \leq \max \left\{ \frac{F_{k+2}^{(4)} + F_{k+1}^{(4)}}{F_{k+2}^{(4)}}, \frac{F_{k+1}^{(4)} + F_{k}^{(4)}}{F_{k+1}^{(4)}}, \frac{F_{k}^{(4)} + F_{k-1}^{(4)}}{F_{k}^{(4)}}, \frac{F_{k-1}^{(4)} + F_{k-2}^{(4)}}{F_{k-1}^{(4)}} \right\}.$$

Therefore, for large value of k where $r_4 = 1.92756$, we arrive at

$$\frac{e_{11}}{e_{12}} \approx \frac{r_4^3}{1 + r_4 + r_4^2}, \quad \frac{e_{11}}{e_{13}} \approx \frac{r_4^2}{1 + r_4}, \quad \frac{e_{11}}{e_{14}} \approx r_4,$$

$$\frac{e_{12}}{e_{13}} \approx \frac{1 + r_4 + r_4^2}{r_4 + r_4^2}, \quad \frac{e_{12}}{e_{14}} \approx \frac{1 + r_4 + r_4^2}{r_4^2}, \tag{4.38}$$

$$\frac{e_{13}}{e_{14}} \approx \frac{1 + r_4}{r_4}.$$

Similarly, we have

$$\frac{e_{i1}}{e_{i2}} \approx \frac{r_4^3}{1 + r_4 + r_4^2}, \quad \frac{e_{i1}}{e_{i3}} \approx \frac{r_4^2}{1 + r_4}, \quad \frac{e_{i1}}{e_{i4}} \approx r_4,$$

$$\frac{e_{i2}}{e_{i3}} \approx \frac{1 + r_4 + r_4^2}{r_4 + r_4^2}, \quad \frac{e_{i2}}{e_{i4}} \approx \frac{1 + r_4 + r_4^2}{r_4^2}, \tag{4.39}$$

$$\frac{e_{i3}}{e_{i4}} \approx \frac{1 + r_4}{r_4}.$$

Case 4: Generalized Relations Among The Code Elements.

Similarly, we can establish the generalized relations among the code matrix elements for all n as:

$$\min\left\{\frac{F_{k+r}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}} : r = 0, 1, \ldots, n-1\right\}$$

$$\leq \frac{e_{i1}}{e_{i2}} \leq \max\left\{\frac{F_{k+r}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}} : r = 0, 1, \ldots, n-1\right\},$$

$$\min\left\{\frac{F_{k+r}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}} : r = 0, 1, \ldots, n-1\right\}$$

$$\leq \frac{e_{i1}}{e_{i3}} \leq \max\left\{\frac{F_{k+r}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}} : r = 0, 1, \ldots, n-1\right\},$$

$$\cdots$$

$$\min\left\{\frac{F_{k+r}^{(n)}}{F_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n-1\right\} \leq \frac{e_{i1}}{e_{in}} \leq \max\left\{\frac{F_{k+r}^{(n)}}{F_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n-1\right\},$$

$$\min\left\{\frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+2}^{(n)}} : r = 0, 1, \ldots, n-1\right\}$$

$$\leq \frac{e_{i2}}{e_{i3}} \leq \max\left\{\frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+2}^{(n)}} : r = 0, 1, \ldots, n-1\right\},$$

$$\min\left\{\frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+3}^{(n)}} : r = 0, 1, \ldots, n-1\right\}$$

$$\leq \frac{e_{i2}}{e_{i4}} \leq \max\left\{\frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+3}^{(n)}} : r = 0, 1, \ldots, n-1\right\},$$

$$\cdots$$

$$\min \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}}{F_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n-1 \right\}$$

$$\leq \frac{e_{i2}}{e_{in}} \leq \max \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+1}^{(n)}}{F_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n-1 \right\},$$

$$\min \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+2}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+3}^{(n)}} : r = 0, 1, \ldots, n-1 \right\}$$

$$\leq \frac{e_{i3}}{e_{i4}} \leq \max \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+2}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+3}^{(n)}} : r = 0, 1, \ldots, n-1 \right\},$$

$$\min \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+2}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+r}^{(n)}} : r = 0, 1, \ldots, n-1 \right\}$$

$$\leq \frac{e_{i3}}{e_{i5}} \leq \max \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+2}^{(n)}}{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+r}^{(n)}} : r = 0, 1, \ldots, n-1 \right\},$$

$\cdots$

$$\min \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+2}^{(n)}}{F_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n-1 \right\}$$

$$\leq \frac{e_{i3}}{e_{in}} \leq \max \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)} + \cdots + F_{k+r-n+2}^{(n)}}{F_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n-1 \right\},$$

$\cdots$

$$\min \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)}}{F_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n-1 \right\} \leq \frac{e_{in-1}}{e_{in}}$$

$$\leq \max \left\{ \frac{F_{k+r-1}^{(n)} + F_{k+r-2}^{(n)}}{F_{k+r-1}^{(n)}} : r = 0, 1, \ldots, n-1 \right\}$$

for $i = 1, 2, 3, \ldots, n$.

For large k, we arrive at

$$\frac{e_{i1}}{e_{i2}} \approx \frac{r_n^{n-1}}{1 + r_n + r_n^2 + \cdots + r_n^{n-2}}, \quad \frac{e_{i1}}{e_{i3}} \approx \frac{r_n^{n-2}}{1 + r_n + r_n^2 + \cdots + r_n^{n-3}}, \quad \cdots$$

$$\frac{e_{i1}}{e_{in}} \approx r_n,$$

$$\frac{e_{i2}}{e_{i3}} \approx \frac{1 + r_n + r_n^2 + \cdots + r_n^{n-2}}{r_n + r_n^2 + r_n^3 + \cdots + r_n^{n-2}}, \quad \frac{e_{i2}}{e_{i4}} \approx \frac{1 + r_n + r_n^2 + \cdots + r_n^{n-2}}{r_n^2 + r_n^3 + r_n^4 + \cdots + r_n^{n-2}}, \quad \cdots$$

$$\frac{e_{i2}}{e_{in}} \approx \frac{1 + r_n + r_n^2 + \cdots + r_n^{n-2}}{r_n^{n-2}},$$

$$\frac{e_{i3}}{e_{i4}} \approx \frac{1 + r_n + r_n^2 + \cdots + r_n^{n-3}}{r_n + r_n^2 + r_n^3 + \cdots + r_n^{n-3}}, \qquad \frac{e_{i3}}{e_{i5}} \approx \frac{1 + r_n + r_n^2 + \cdots + r_n^{n-3}}{r_n^2 + r_n^3 + r_n^4 + \cdots + r_n^{n-3}}, \qquad \cdots$$

$$\frac{e_{i1}}{e_{in}} \approx \frac{1 + r_n + r_n^2 + \cdots + r_n^{n-3}}{r_n^{n-3}},$$

$$\vdots$$

$$\frac{e_{in-1}}{e_{in}} \approx \frac{1 + r_n}{r_n} \quad \text{for } i = 1, 2, 3, \ldots, n.$$

5. Error Detection and Correction

For the case $n = 2$ the correct ability of the method is 93.33% [9] and for the case $n = 3$ the correct ability of the method is 99.80% [1]. In this paper, we first develop correct ability of the coding/decoding method for the case $n = 4$ and then for large values of n. The coding/decoding method gives property to detect and correct errors in the code message E. The error detection and correction is based on the property of the determinant of matrix given by (3.1). At first we calculate the determinant of the initial matrix P and then sent to a communication channel right after the code matrix elements. Det P is treated as the checking elements of the code matrix E received from the communication channel. After receiving the code matrix and its checking elements det P, we calculate the determinant of the matrix E and compare it with given det P by relation (3.1). If the relation (3.1) is true then we conclude that the elements of the code matrix E were transmitted through the communication channel without errors otherwise there are errors and then we try to correct these errors using the relations (3.1), (4.38) and (4.39).

If we have single error in the code matrix E, it is clear that there are sixteen variants of single error in the code matrix E. As an example, one of them is

$$\begin{pmatrix} x_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix}, \tag{5.1}$$

where x_{11} is the possible destroyed elements.

Using relation (3.1), we write the following algebraic equation of the matrix (5.1):

$$\begin{vmatrix} x_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{vmatrix} = (-1)^k \det P. \tag{5.2}$$

If x_{ij} are the possible destroyed elements for the elements e_{ij} for $i, j = 1, 2, 3, 4$ then there are sixteen equations like (5.2) for sixteen possible variants of single

error but we have to choose the correct variant only among these cases of the integer solutions x_{ij} for $i, j = 1, 2, 3, 4$ which satisfies the relations (4.38) and (4.39). If these sixteen equations do not give an integer solution, we conclude that our hypothesis about single error is incorrect or we have more than single error in the checking element $\det P$. For the latter case we use the approximate equalities (4.38) and (4.39) for checking a correctness of the code matrix E.

By analogy, we check all hypotheses of double errors in the code matrix E. As an example, let us consider the following case of double errors in the code matrix E:

$$\begin{pmatrix} x_{11} & x_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix}. \tag{5.3}$$

Using the relation (3.1) we write the algebraic equation for the matrix (5.3)

$$\begin{vmatrix} x_{11} & x_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{vmatrix} = (-1)^k \det P. \tag{5.4}$$

According to the relation (4.38) there is the following relation between x_{11} and x_{12}:

$$x_{11} \approx \frac{r_4^3}{1 + r_4 + r_4^2} x_{12}. \tag{5.5}$$

Equation (5.4) give "Diophantine" equation. As the "Diophantine" equation (5.4) has many solutions, we have to choose such solutions x_{11} and x_{12} which satisfy the relation (5.5).

It is clear that there are $^{16}C_2 = 120$ variants of double errors in the code matrix E and by using similar approach we correct all double errors in the code matrix E.

Hence there are

$$\binom{16}{1} + \binom{16}{2} + \binom{16}{3} + \cdots + \binom{16}{16} = 2^{4^2} - 1 = 65535$$

cases of errors in the code matrix E.

Similarly, we show by using this approach there is a possibility to correct all possible triple fold, four fold, ..., fifteen fold errors in the code matrix E. The sixteen fold of error is not correctable.

Hence the possibility to correct fifteen cases of this method is $\frac{65534}{65535} = 0.999980 = 99.998\%$.

In general, for $n = m$ the possibility to correct errors of the method is $\frac{2^{m^2} - 2}{2^{m^2} - 1}$. Therefore, for large value of n the possibility to correct errors of the method is

$$\frac{2^{n^2} - 2}{2^{n^2} - 1} \approx 1 = 100\%.$$

6. Conclusion

The Fibonacci n-step coding/decoding method is the main application of the M_n matrices. This coding/decoding method differs from the classical algebraic codes by the following peculiarities:

(1) Fibonacci n-step coding/decoding method converts to matrix multiplication which is very well-known method and not time consuming in modern computers.
(2) The correct ability of the method for the case $n = 2$ is 93.33% which corrects up to triple errors among four fold errors, for the case $n = 3$ is 98.80% which corrects up to eight fold errors among nine fold errors and for the case $n = 4$ is 99.998% which corrects up to fifteen fold errors among sixteen fold errors.
(3) The correct ability of the method increases as n increases.
(4) El Naschie [2] shows that the Fibonacci series and the golden mean play a very important role in the construction of a relatively new space-time theory, which is referred to as ϵ^∞ Cantorianfractal space-time or E-infinity theory [3, 4]. He [6] also shows that there are relations among E-infinity theory, string theory, exceptional lie symmetry group and various physical quantities. He [5] explains the existence of these relations using the geometric and topology of space-time as claimed by E-infinity theory. It is certain that based on these theories, Fibonacci n-step numbers and n-anacci constants will give a better result in E-infinity theory, string theory etc. in future.

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