Fractional Gamma Function via Fractional Derivatives

18 Sep 2025

Introduction

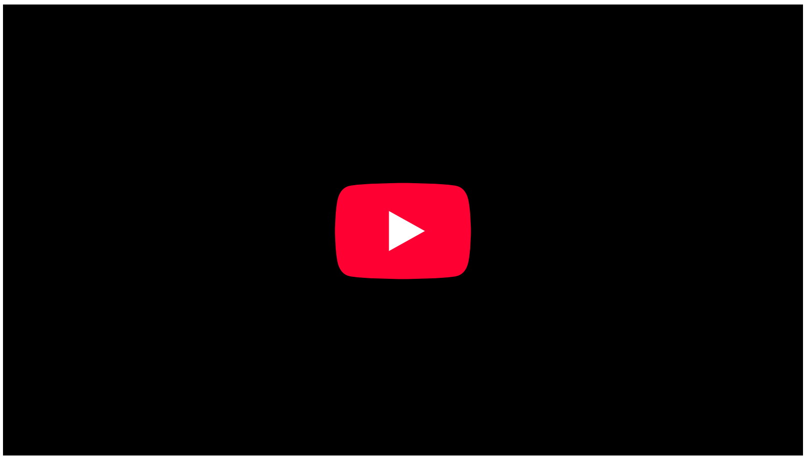
The classical gamma function $\Gamma(z)$ is one of the most important special functions in mathematics, serving as a continuous extension of the factorial function. It satisfies the fundamental recurrence relation $\Gamma(z+1) = z \cdot \Gamma(z)$, which represents a "unit step" in the argument.

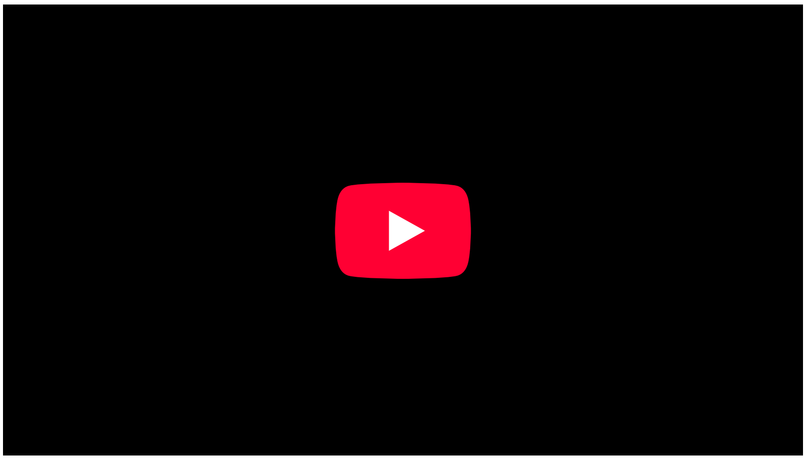
But what if we could generalize this to fractional steps? What if instead of moving from z to $z+1$, we could move in increments of α, where α is any positive real number? This question leads us into the fascinating world of fractional calculus and opens up new mathematical structures with applications ranging from fractional differential equations to mathematical physics.

In this post, we'll explore how fractional derivatives naturally give rise to a generalized gamma function with fractional step sizes. This isn't just a mathematical curiosity – these functions appear naturally in solutions to fractional differential equations and systems with memory effects or fractional dimensionality.

The key insight is that just as the classical gamma function emerges from integer-order derivatives of $x^n e^{-x}$, we can use fractional derivatives to create a new family of functions that satisfy fractional recurrence relations of the form $\Gamma_\alpha(z+\alpha) = z \cdot \Gamma_\alpha(z)$.

Recommended Watching





Classical Setup

Starting with the function family:

$$g_n(x) = x^n e^{-x}$$

The classical gamma function arises from:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx = \int_0^\infty g_n(x)\, dx$$

With the recurrence relation:

$$\Gamma(n+1) = n \cdot \Gamma(n)$$

Fractional Derivative Foundation

Riemann-Liouville Fractional Derivative

For a function $f(x)$ and fractional order $\alpha > 0$, the Riemann-Liouville fractional derivative is:

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-\alpha-1} f(t)\, dt$$

where $n = \lceil \alpha \rceil$.

Fractional Derivative of Power Functions

For $f(x) = x^\beta$ with $\beta > -1$:

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}$$

Fractional Extension of $g_n(x)$

Step 1: Define Fractional Order Functions

For any real $\nu \geq 0$, define:

$$g_\nu(x) = x^\nu e^{-x}$$

Step 2: Fractional Derivative Relation

The fractional derivative of order α gives us:

$$D^\alpha g_\nu(x) = D^\alpha[x^\nu e^{-x}]$$

Using the fractional product rule (generalized Leibniz rule):

$$D^\alpha[x^\nu e^{-x}] = \sum_{k=0}^{\infty} \binom{\alpha}{k} D^k[x^\nu] \cdot D^{\alpha-k}[e^{-x}]$$

For $\nu > \alpha$, the fractional derivative gives:

$$D^\alpha g_\nu(x) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)} x^{\nu-\alpha} e^{-x} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)} g_{\nu-\alpha}(x)$$

Fractional Gamma Function Definition

Definition 1: Direct Fractional Integration

$$\Gamma_\alpha(z) = \int_0^\infty x^{z-1} e^{-x^\alpha}\, dx$$

This reduces to the classical gamma function when $\alpha = 1$.

Definition 2: Fractional Step Recurrence

Define the fractional gamma function via:

$$\Gamma_\alpha(z+\alpha) = \frac{\Gamma(z+\alpha)}{\Gamma(z)} \cdot \Gamma_\alpha(z)$$

More explicitly:

$$\Gamma_\alpha(z) = \frac{\Gamma(z)}{\Gamma(z-\alpha)} \quad \text{for } z > \alpha$$

Definition 3: Integral Transform Approach

$$\Gamma_\alpha(z) = \frac{1}{\alpha} \int_0^\infty t^{z/\alpha - 1} e^{-t}\, dt$$

Properties of the Fractional Gamma Function

Fractional Recurrence Relation

$$\Gamma_\alpha(z+\alpha) = z \cdot \Gamma_\alpha(z)$$

This is the key property - instead of unit steps, we have fractional steps of size α.

Relationship to Classical Gamma

$$\Gamma_1(z) = \Gamma(z)$$

$$\Gamma_\alpha(z) = \frac{\Gamma(z)}{\Gamma(z-\alpha)} \quad \text{when well-defined}$$

Functional Equation

For positive integer n:

$$\Gamma_\alpha(z+n\alpha) = z(z+\alpha)(z+2\alpha)\cdots(z+(n-1)\alpha) \cdot \Gamma_\alpha(z)$$

Special Values

At Integer Multiples

For integer $n \geq 1$:

$$\Gamma_\alpha(n\alpha) = \frac{\Gamma(n\alpha)}{\Gamma((n-1)\alpha)} \cdot \frac{\Gamma((n-1)\alpha)}{\Gamma((n-2)\alpha)} \cdots \frac{\Gamma(\alpha)}{\Gamma(0)}$$

This telescopes to:

$$\Gamma_\alpha(n\alpha) = \frac{\Gamma(n\alpha)}{\Gamma(0)}$$

However, since $\Gamma(0)$ is undefined, we use the recurrence relation instead.

Half-Integer Case ($\alpha = 1/2$)

$$\Gamma_{1/2}(z) = \frac{\Gamma(z)}{\Gamma(z-1/2)}$$

For $z = 3/2$:

$$\Gamma_{1/2}(3/2) = \frac{\Gamma(3/2)}{\Gamma(1)} = \frac{\sqrt{\pi}/2}{1} = \frac{\sqrt{\pi}}{2}$$

Applications

This fractional gamma function naturally appears in:

1. **Fractional Differential Equations**: Solutions to $D^\alpha u + \lambda u = 0$
2. **Fractional Probability Distributions**: Generalizations of gamma and beta distributions
3. **Special Functions**: Connections to Mittag-Leffler functions and Fox H-functions
4. **Mathematical Physics**: Systems with fractional dimensionality or memory effects

Acknowledgements

Here is the chat link

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