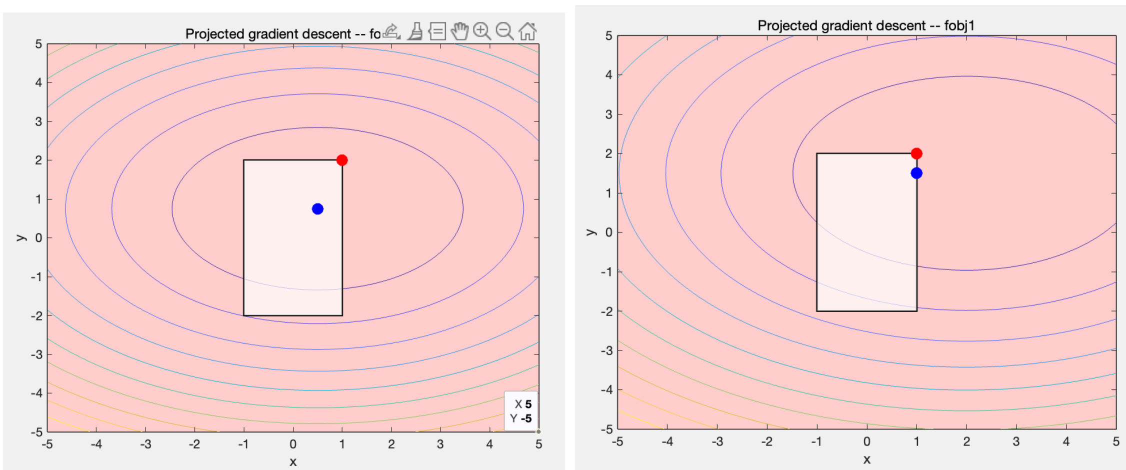
Problem 1.

The main function is the projected_gradient_descent.m. The test function is the **test_P1().m**. The plotting of the first function is figure 1a and the second are figure 1b. The red dot are the starting point, the blue point are the optimal point. The results are like figure 2.
From the definition we know that the first function gets its global minimal point which is $[0.5, 0.75]$ but the second can only get its local minimal point which is $[1.0, 1.5]$. My results are correct.



(a) Problem 1 plotting 1

(b) Problem 1 plotting 2

Figure 1: Problem 1 plotting

```
>> test_p1
start Xn = [ 1.00000,  2.00000]
Terminating after 22 iterations because norm(delta) < tol.
Found minimum xstar = [ 0.503689; 0.750016 ]
start Xn = [ 1.00000,  2.00000]
Terminating after 12 iterations because norm(delta) < tol.
Found minimum xstar = [ 1.000000; 1.501088 ]
```

Figure 2: Problem 1 result

Problem 2.

The first result is $[0.5, 0.75]$. We have L is the following:

$$\begin{aligned} L = (x-\frac{1}{2})^2 + 2(y-\frac{1}{2})^2 + \lambda_1(x-1) + \lambda_2(-1-x) \\ + \lambda_3(y-2) + \lambda_4(-2-y) \end{aligned}$$

$$\begin{gathered} \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0 \\ 0.5 - 1 \leq 0 \quad -1 - 0.5 \leq 0 \quad 0.75 - 2 \leq 0 \quad -2 - 0.75 \leq 0 \end{gathered}$$

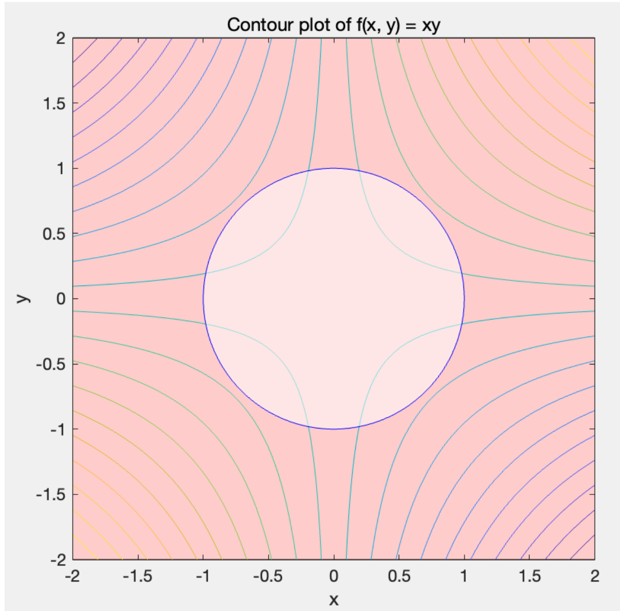


Figure 3: Plotting of P3

We get “complimentary slackness” immediately by all $\lambda = 0$

$$\nabla L = \left(2(x - \frac{1}{2}) + \lambda_1 - \lambda_2,\ 4(y - \frac{1}{2}) + \lambda_3 - \lambda_4\right) = (0, 0)$$

The second result is $[1, 1.5]$.

$$\begin{aligned} L = c(x-2)^2 + 2(y - \frac{3}{2})^2 + \lambda_1(x-1) + \lambda_2(-1-x) \\ + \lambda_3(y-2) + \lambda_4(-2-y) \end{aligned}$$

$$\begin{aligned} \lambda_1 &= 2 \quad \lambda_2 = \lambda_3 = \lambda_4 = 0 \\ 1 - 1 &\le 0 \quad -1 - 1 \le 0 \quad 1.5 - 2 \le 0 \quad -2 - 1.5 \le 0 \\ \lambda_i &\ge 0 \\ \lambda_1(x-1) &= 0, \quad \lambda_2 = \lambda_3 = \lambda_4 = 0 \\ \nabla L &= \left(2(x-2) + \lambda_1 - \lambda_2,\ 4(y - \frac{3}{2}) + \lambda_3 - \lambda_4\right) = (0, 0) \end{aligned}$$

Problem 3.

The plotting of the objective function and the feasible region is 3. The Lagrangian is the following:

$$L = xy + \lambda(x^2 + y^2 - 1)$$

The General statement of the four KKT conditions are the following:

$$\begin{aligned} x_*^2 + y_*^2 - 1 &\leq 0 \\ \lambda &\geq 0 \\ \lambda(x_*^2 + y_*^2 - 1) &= 0 \\ \begin{pmatrix} y_* + 2\lambda x_* \\ x_* + 2\lambda y_* \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y_* = -2\lambda x_* \\ x_* = -2\lambda y_* \end{cases} \end{aligned}$$

The following are the tests:

Point 1: $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Condition (1):

$$\left(-\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 - 1 = \frac{1}{2} + \frac{1}{2} - 1 = 0 \leq 0$$

Condition (1) is satisfied.
Condition (3):
Since $x_*^2 + y_*^2 - 1 = 0$, we have:

$$\lambda \cdot 0 = 0$$

Condition (3) holds for any λ.
Condition (4):

$$\begin{cases} \frac{\sqrt{2}}{2} + 2\lambda\left(-\frac{\sqrt{2}}{2}\right) = 0 \\ -\frac{\sqrt{2}}{2} + 2\lambda\left(\frac{\sqrt{2}}{2}\right) = 0 \end{cases}$$

Simplify the equations:
First equation:

$$\frac{\sqrt{2}}{2} - \sqrt{2}\lambda = 0 \implies \lambda = \frac{1}{2}$$

Second equation:

$$-\frac{\sqrt{2}}{2} + \sqrt{2}\lambda = 0 \implies \lambda = \frac{1}{2}$$

Condition (2):

$$\lambda = \frac{1}{2} \geq 0$$

Condition (2) is satisfied.
Conclusion: Point 1 satisfies all the conditions.

Point 2: $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Condition (1):

$$\left(\frac{\sqrt{2}}{2}\right)^2 + \left(-\frac{\sqrt{2}}{2}\right)^2 - 1 = 0 \leq 0$$

Condition (1) is satisfied.
Condition (3):

$$\lambda \cdot 0 = 0$$

Condition (3) holds for any λ.
Condition (4):

$$\begin{cases} -\frac{\sqrt{2}}{2} + 2\lambda\left(\frac{\sqrt{2}}{2}\right) = 0 \\ \frac{\sqrt{2}}{2} + 2\lambda\left(-\frac{\sqrt{2}}{2}\right) = 0 \end{cases}$$

Simplify the equations:
First equation:

$$-\frac{\sqrt{2}}{2} + \sqrt{2}\lambda = 0 \implies \lambda = \frac{1}{2}$$

Second equation:

$$\frac{\sqrt{2}}{2} - \sqrt{2}\lambda = 0 \implies \lambda = \frac{1}{2}$$

Condition (2):

$$\lambda = \frac{1}{2} \geq 0$$

Condition (2) is satisfied.
Conclusion: Point 2 satisfies all the conditions.

Point 3: $(0, 0)$

Condition (1):

$$0^2 + 0^2 - 1 = -1 \leq 0$$

Condition (1) is satisfied.
Condition (3):

$$\lambda(-1) = 0 \implies \lambda = 0$$

Condition (4):

$$\begin{cases} 0 + 2 \cdot 0 \cdot 0 = 0 \\ 0 + 2 \cdot 0 \cdot 0 = 0 \end{cases}$$

Both equations are satisfied.
Condition (2):

$$\lambda = 0 \geq 0$$

Condition (2) is satisfied.
Conclusion: Point 3 satisfies all the conditions.

Point 4: $(1, 0)$

Condition (1):

$$1^2 + 0^2 - 1 = 0 \leq 0$$

Condition (1) is satisfied.
Condition (3):

$$\lambda \cdot 0 = 0$$

Condition (3) holds for any λ.
Condition (4):

$$\begin{cases} 0 + 2\lambda \cdot 1 = 0 \implies 2\lambda = 0 \implies \lambda = 0 \\ 1 + 2\lambda \cdot 0 = 0 \implies 1 = 0 \end{cases}$$

The second equation leads to $1 = 0$, which is a contradiction.
Conclusion: Point 4 does not satisfy all the conditions.

From the tests above, we know that $(0, 0)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ satisfy the condition. But it turn out that $(0, 0)$ is the saddle point, so the only true answers are

$$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$