Problem 1.

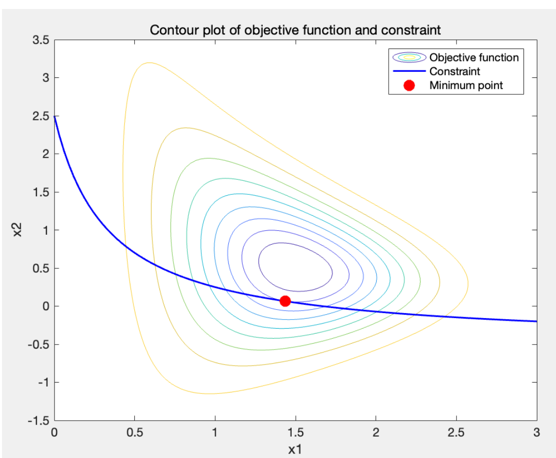
- The Lagrangian is the following:

$$L(x_1, x_2, \lambda) = -\exp\left(-\frac{1}{2}(x_1x_2 - \frac{3}{4})^2 - 2(x_1 - \frac{3}{2})^2\right) + \lambda\left(\frac{1}{x_1 + \frac{1}{3}} - x_2 - \frac{1}{2}\right).$$

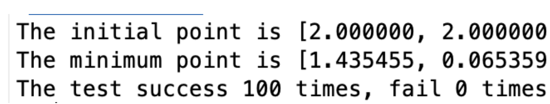
- The elements are listed below

$$\frac{\partial L}{\partial x_1} = -\frac{\lambda}{(x_1 + \frac{1}{3})^2} + \left(x_2(x_1x_2 - \frac{3}{4}) + 4(x_1 - \frac{3}{2})\right)\exp\left(-\frac{1}{2}(x_1x_2 - \frac{3}{4})^2 - 2(x_1 - \frac{3}{2})^2\right)$$
$$\frac{\partial L}{\partial x_2} = -\lambda + x_1(x_1x_2 - \frac{3}{4})\exp\left(-\frac{1}{2}(x_1x_2 - \frac{3}{4})^2 - 2(x_1 - \frac{3}{2})^2\right)$$
$$\frac{\partial L}{\partial \lambda} = \frac{1}{x_1 + \frac{1}{3}} - x_2 - \frac{1}{2}$$

- Here the test function is **test_P1.m**. The result should be like figure 1a. The success and fail times stand for the comparison of each point on the constraint comparing to the minimum. The plotting should be like figure 1b.



(a) Problem 1 plotting



```
The initial point is [2.000000, 2.000000]
The minimum point is [1.435455, 0.065359]
The test success 100 times, fail 0 times.
```

(b) Problem 1 result

Figure 1: Problem 1

Problem 2.

- The function is the following:

$$-\exp\left(-\frac{1}{2}(x_1x_2 - \frac{3}{4})^2 - 2(x_1 - \frac{3}{2})^2\right) + \rho\left(\frac{1}{x_1 + \frac{1}{3}} - x_2 - \frac{1}{2}\right)^2.$$

- Let:

$$A = \left(-\frac{1}{2}(x_1x_2 - \frac{3}{4})^2 - 2(x_1 - \frac{3}{2})^2\right) \quad B = \left(\frac{1}{x_1 + \frac{1}{3}} - x_2 - \frac{1}{2}\right)$$

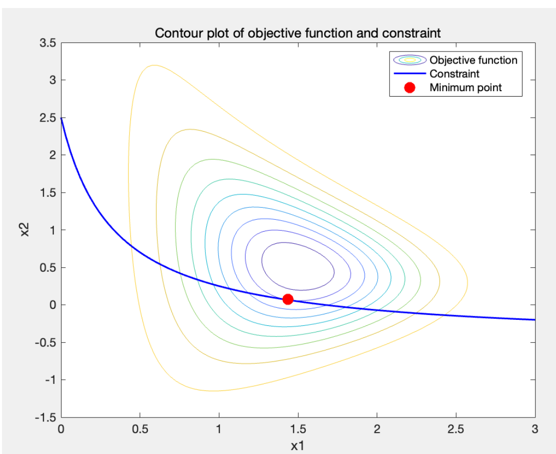
  The gradient are the following:

$$\frac{\partial f}{\partial x_1} = \left(x_2(x_1x_2 - \frac{3}{4}) + 4(x_1 - \frac{3}{2})\right)\exp(A) - \frac{2\rho B}{(x_1 + \frac{1}{3})^2}$$
$$\frac{\partial f}{\partial x_2} = x_1(x_1x_2 - \frac{3}{4})\exp(A) - 2\rho B$$

  The Hessian are the following:

$$\frac{\partial^2 f}{\partial x_1^2} = -\exp(A)\cdot\left[x_2(x_1x_2 - \frac{3}{4}) + 4(x_1 - \frac{3}{2})\right]^2 + \exp(A)\cdot(x_2^2 + 4) + 2\rho\cdot\frac{1}{(x_1 + \frac{1}{3})^4} + 4\rho\cdot\frac{B}{(x_1 + \frac{1}{3})^3}$$
$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \exp(A)\cdot\left[-x_1(x_1x_2 - \frac{3}{4})\cdot\left(x_2(x_1x_2 - \frac{3}{4}) + 4(x_1 - \frac{3}{2})\right) + 2x_1(x_1x_2 - \frac{3}{4})\right] + \frac{2\rho}{\left(x_1 + \frac{1}{3}\right)^2}$$
$$\frac{\partial^2 f}{\partial x_2^2} = \exp(A)\cdot x_1^2\left[1 - \left(x_1x_2 - \frac{3}{4}\right)^2\right] + 2\rho$$

- Here the test function is **test_P2.m**. Here I use 0.15 as the initial ρ. It works quite well. The result should be like figure 2b. The success and fail times stand for the comparison of each point on the constraint comparing to the minimum. The plotting should be like figure 2a. The result is the same as P1.



(a) Problem 2 plotting

```
i = 8, rho = 11796.875000, Optimal [L,K] = [1.435359, 0.065421]
Terminating after 1 iterations because norm(delta) < tol.
i = 9, rho = 58984.375000, Optimal [L,K] = [1.435577, 0.065326]
Terminating after 1 iterations because norm(delta) < tol.
i = 10, rho = 294921.875000, Optimal [L,K] = [1.435331, 0.065400]
Terminating after 1 iterations because norm(delta) < tol.
i = 11, rho = 1474609.375000, Optimal [L,K] = [1.435586, 0.065317]
Terminating after 1 iterations because norm(delta) < tol.
i = 12, rho = 7373046.875000, Optimal [L,K] = [1.435328, 0.065400]
Terminating after 1 iterations because norm(delta) < tol.
i = 13, rho = 36865234.375000, Optimal [L,K] = [1.435548, 0.065329]
Terminating after 1 iterations because norm(delta) < tol.
i = 14, rho = 184326171.875000, Optimal [L,K] = [1.435437, 0.065365]
Terminating after 1 iterations because norm(delta) < tol.
i = 15, rho = 921630859.375000, Optimal [L,K] = [1.435457, 0.065358]
Terminating after 1 iterations because norm(delta) < tol.
i = 16, rho = 4608154296.875000, Optimal [L,K] = [1.435454, 0.065359]
Terminating after 1 iterations because norm(delta) < tol.
i = 17, rho = 23040771484.375000, Optimal [L,K] = [1.435457, 0.065358]
Terminating after 1 iterations because norm(delta) < tol.
i = 18, rho = 115203857421.875000, Optimal [L,K] = [1.435454, 0.065359]
Terminating after 1 iterations because norm(delta) < tol.
i = 19, rho = 576019287109.375000, Optimal [L,K] = [1.435456, 0.065358]
Terminating after 1 iterations because norm(delta) < tol.
i = 20, rho = 2880096435546.875000, Optimal [L,K] = [1.435455, 0.065359]
The minimum point is [1.435455, 0.065359]
The test success 100 times, fail 0 times.
```

(b) Problem 2 result

Figure 2: Problem 2

Problem 3.

1. When three plains intersect with each other, there's three conditions. The first is they are not parallel to each other, than we get a point, the optimization is meaningless. The second is they intersect all on a line. In this case, we intersect the line with the paraboloid, we got two or one point, which is still meaningless. The third is they are parallel to each other, and the intersection is nothing. Base on these case, we can see that the it is either meaningless or no solution.

2. When we have 3 planes, the Jacobian is the following:

$$J_n = \begin{pmatrix} 2a(1+\lambda_1+\lambda_2+\lambda_3) & c(1+\lambda_1+\lambda_2+\lambda_3) & 2ax+cy+d-\alpha_1 & 2ax+cy+d-\alpha_2 & 2ax+cy+d-\alpha_3 \\ c(1+\lambda_1+\lambda_2+\lambda_3) & 2b(1+\lambda_1+\lambda_2+\lambda_3) & cx+2by+e-\beta_1 & cx+2by+e-\beta_2 & cx+2by+e-\beta_3 \\ 2ax+cy+d-\alpha_1 & cx+2by+e-\beta_1 & 0 & 0 & 0 \\ 2ax+cy+d-\alpha_2 & cx+2by+e-\beta_2 & 0 & 0 & 0 \\ 2ax+cy+d-\alpha_3 & cx+2by+e-\beta_3 & 0 & 0 & 0 \end{pmatrix}$$

We can find out that the last three row is a linear combination of each other by solving the following equation:

$$p(2ax+cy+d-\alpha_1, cx+2by+e-\beta_1) + q(2ax+cy+d-\alpha_2, cx+2by+e-\beta_2) + \\ r(2ax+cy+d-\alpha_3, cx+2by+e-\beta_3) = 0.$$

It can be solve by the following:

$$\begin{aligned} p+q+r &= 0 \\ p\alpha_1 + q\alpha_2 + r\alpha_3 &= 0 \\ p\beta_1 + q\beta_2 + r\beta_3 &= 0 \end{aligned}$$

It is solvable given that the three planes are not parallel to each other. Therefore, the Jacobian is not invertible, so the Newton's method can't go on.