1\. a. $\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0$ The answer should be like

$$\theta = e^{rt} \quad \theta' = re^{rt} \quad \theta'' = r^2 e^{rt} \Rightarrow r^2 + \frac{g}{L} = 0$$

$r = \pm\sqrt{-\frac{g}{L}}$, the answer should be like

$$e^{\alpha t}(C_1 \cos\sqrt{\tfrac{g}{L}}t + C_2 \sin\sqrt{\tfrac{g}{L}}t) \text{ and } e^{\alpha t}(C_1\cos\sqrt{\tfrac{g}{L}}t + C_2 \sin\sqrt{\tfrac{g}{L}}t)$$

$\theta = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$ take derivative

$$\dot\theta = (\alpha C_1 \cos\beta t + \alpha C_2 \sin\beta t)e^{\alpha t} + (-\beta C_1 \sin\beta t + \beta C_2 \cos\beta t)e^{\alpha t}$$

$$= [(\alpha C_1 + \beta C_2)\cos\beta t + (\alpha C_2 - \beta C_1)\sin\beta t]e^{\alpha t}$$

$$\ddot\theta = \alpha[(\alpha C_1 + \beta C_2)\cos\beta t + (\alpha C_2 - \beta C_1)\sin\beta t]e^{\alpha t} + \beta[-(\alpha C_1 + \beta C_2)\sin\beta t + (\alpha C_2 - \beta C_1)\cos\beta t]e^{\alpha t}$$

$$= [\alpha^2 C_1 + \alpha\beta C_2 + \alpha\beta C_2 - \beta^2 C_1]\cos\beta t\, e^{\alpha t} + [\alpha^2 C_2 - \alpha\beta C_1 - \alpha\beta C_1 - \beta^2 C_2]\sin\beta t\, e^{\alpha t}$$

$$\frac{g}{L}\theta = \frac{g}{L}C_1\cos\beta t\, e^{\alpha t} + \frac{g}{L}C_2 \sin\beta t\, e^{\alpha t} \qquad \begin{cases} \alpha^2 - \beta^2 = -\frac{g}{L} \\ \alpha\beta = 0 \end{cases}$$

α, β are real numbers, $g, L > 0$. Then $\Rightarrow \alpha = 0 \quad \beta = \pm\sqrt{\frac{g}{L}}$

$$\theta = C_1\cos(\sqrt{\tfrac{g}{L}}t) + C_2 \sin(\sqrt{\tfrac{g}{L}}t) \text{ or } \theta = C_1 \cos(-\sqrt{\tfrac{g}{L}}t) + C_2 \sin(-\sqrt{\tfrac{g}{L}}t)$$

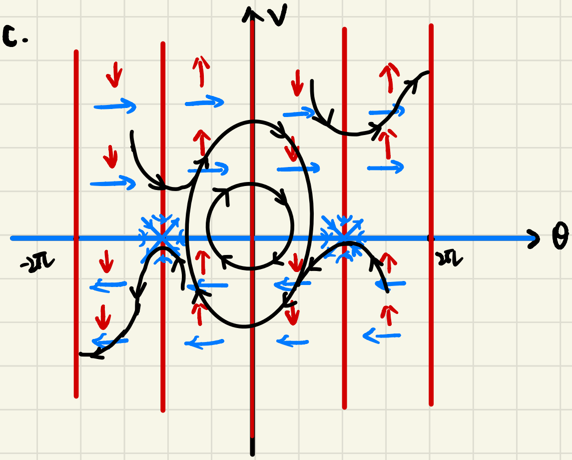
b. $\tau = \sqrt{\frac{g}{L}}t$ then $\frac{d\theta}{dt} = \frac{d\theta}{d\tau}\frac{d\tau}{dt} = \frac{d\theta}{d\tau}\sqrt{\frac{g}{L}}$

$$\frac{d^2\theta}{dt^2} = \frac{d}{dt}\left(\frac{d\theta}{dt}\right) = \frac{d}{d\tau}\left(\frac{d\theta}{dt}\right)\cdot\frac{d\tau}{dt} = \frac{d}{d\tau}\left(\frac{d\theta}{d\tau}\cdot\sqrt{\tfrac{g}{L}}\right)\cdot\sqrt{\tfrac{g}{L}}$$

$= \frac{d^2\theta}{d\tau^2}\cdot\frac{g}{L}$ Then the formula becomes

$$\frac{d^2\theta}{d\tau^2}\cdot\frac{g}{L} + \frac{g}{L}\sin\theta = 0 \Rightarrow \frac{d^2\theta}{d\tau^2} + \sin\theta = 0.$$

c.



steady states: $\begin{cases} V=0 \\ \sin\theta=0 \end{cases}$

$\Rightarrow \begin{cases} V=0 \\ \theta=k\pi \ (k\in\mathbb{Z}) \end{cases}$ $\qquad \frac{d\theta}{dt}=V$ $\qquad \frac{dv}{dt}=-\sin\theta$

Let $G=-\sin\theta$

$G_\theta = -\cos\theta = \begin{cases} 1 & \theta=(2k+1)\pi \\ -1 & \theta=2k\pi \end{cases} \quad (k\in\mathbb{Z})$

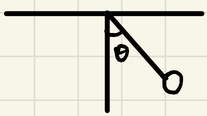
The matrix looks like $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $(\theta=(2k+1)\pi)$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\theta=2k\pi$

Both trace are 0.

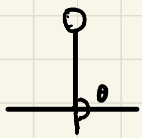
$\theta=(2k+1)\pi$ $\Delta=-1$ $\lambda=\pm1$ eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ It's a saddle point.

$\theta=2k\pi$, $\Delta=1$ It's a center. The phase plain portrait is ploted above.

d.

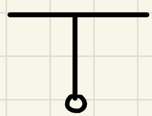


$\theta=\pi$ it looks like
$V=0$



The ball will not swing, it will just bounce repeatedly.

$\theta=0$
$V=0$



The ball will just stay the position

e. Matching the physic scenario, the steady point will stay the same.

However, the behavior will change near each point.

$\theta=2k\pi$
$V=0$
Should like

$\theta=(2k+1)\pi$
$V=0$

It is still a saddle point but the eigen vector may change.

For the phase plane, all the initial value will go to $\theta=2k\pi$, and gradually sink to the steady point.

2. Let $x^{(k)} = x(t_0)$

$$x^{(k+2)} = x(t_0+2\Delta t) = x(t_0) + 2\Delta t\, x'(t_0) + \frac{(2\Delta t)^2}{2!} x''(t_0) + \frac{(2\Delta t)^3}{3!} x'''(t_0)$$
$$+ \frac{(2\Delta t)^4}{4!} x''''(t_0) + O(\Delta t^5)$$

$$x^{(k+1)} = x(t_0+\Delta t) = x(t_0) + \Delta t\, x'(t_0) + \frac{(\Delta t)^2}{2!} x''(t_0) + \frac{\Delta t^3}{3!} x'''(t_0)$$
$$+ \frac{(\Delta t)^4}{4!} x''''(t_0) + O(\Delta t^5)$$

$$x^{(k-1)} = x(t_0-\Delta t) = x(t_0) - \Delta t\,(x'(t_0)) + \frac{(-\Delta t)^2}{2!} x''(t_0) + \frac{(-\Delta t)^3}{3!} x'''(t_0)$$
$$+ \frac{(-\Delta t)^4}{4!} x''''(t_0) + O(\Delta t^5)$$

Let $$D_+ D^2 x(t) = \frac{x^{(k+2)} - 3x^{(k+1)} + 3x^{(k)} - x^{(k-1)}}{(\Delta t)^3}$$
$$= \frac{1}{(\Delta t)^3}\left(k_0 x(t_0) + k_1 \Delta t\, x'(t_0) + k_2 (\Delta t)^2 x''(t_0) + k_3 (\Delta t)^3 x'''(t_0)\right.$$
$$\left. + k_4 (\Delta t)^4 x''''(t_0) + O(\Delta t^5)\right)$$

$k_0 = 1-3+3-1 = 0 \qquad k_1 = 2 - 3\times 1 - (-1) = 0$

$k_2 = \frac{2^2}{2!} - 3\times\frac{1}{2!} - \frac{1}{2!} = 0 \qquad k_3 = \frac{2^3}{3!} - 3\times\frac{1}{3!} - \frac{(-1)^3}{3!} = 1$

$k_4 = \frac{2^4}{4!} - 3\times\frac{1}{4!} - \frac{(-1)^4}{4!} = \frac{1}{2}$

$$\left| D_+ D^2 x(t) - \frac{d^3x}{dt^3} \right| = \left| x'''(t_0) - \frac{d^3x}{dt^3}(t_0) + \frac{1}{2}\Delta t\, x''''(t_0) + O(h^2) \right|$$

$= \left| \frac{1}{2}\Delta t\, x''''(t_0) + O(h^2) \right|$ so it's first order accurate.