PAGE NO.:
DATE: / /

AI61003 Linear Algebra for AI & ML
Assignment 01 - Problem 07

Define $\mathcal{L}_{ij}(X)$ for any $X \in \mathbb{R}^{n \times n}$ (where $1 \leq j < i \leq n$) such that

- $\mathcal{L}_{ij}(X)_{kk} = 1$ for $k = 1 \ldots n$
- $\mathcal{L}_{ij}(X)_{ij} = -X_{ij} / X_{jj}$ $(1 \leq j < i \leq n)$
- $\mathcal{L}_{ij}(X)_{pq} = 0$ $(p \neq q, (p,q) \neq (i,j))$

So given an invertible matrix $A \in \mathbb{R}^{n \times n}$ perform the following steps to construct the matrices L_{ij}
(Total no. of steps - $n(n-1)/2$)

$$\mathcal{L}_{21}(A)\, A = A_{21}$$
$$\mathcal{L}_{31}(A_{21})\, A_{21} = A_{31}$$
$$\vdots$$
$$\mathcal{L}_{n1}(A_{(n-1)1})\, A_{(n-1)1} = A_{n1}$$
$$\mathcal{L}_{32}(A_{n1})\, A_{n1} = A_{32}$$
$$\vdots$$
$$\mathcal{L}_{n(n-2)}(A_{(n-1)(n-2)})\, A_{(n-1)(n-2)} = A_{n(n-2)}$$
$$\mathcal{L}_{n(n-1)}(A_{n(n-2)})\, A_{n(n-2)} = A_{n(n-1)} = U$$
(U computed ✓)

Let all the $\mathcal{L}_{ij}(\cdot)$ matrices on the LHS be called as L_{ij}.
All the operat^ns can be combined as -
$$L_{n(n-1)} L_{n(n-2)} \ldots L_{ij} \ldots L_{31} L_{21} A = U$$
(All L_{ij} matrices are lower-triangular and non-singular).

PAGE NO. :
DATE : / /

$\therefore A = L_{21}^{-1} L_{31}^{-1} \ldots L_{ij}^{-1} \ldots L_{n(n-2)}^{-1} L_{n(n-1)}^{-1} U$
$= L\, U$

Note that

① $L_{ij}^{-1} = 2I - L_{ij}$

② $\prod_{1 \le j < i \le n} L_{ij}^{-1} = \left(\sum\sum_{1 \le j < i \le n} L_{ij}^{-1} \right) - \left(\frac{n^2 - n - 2}{2} \right) I$

Using these observations it becomes fairly easy to compute L since each L_{ij}^{-1} has only one non-zero non-diagonal entry and no two matrices have their non-zero non-diagonal entry unique at the same position.

Hence $L = \prod_{1 \le j < i \le n} L_{ij}^{-1}$ can be computed. ✓

Therefore, LU decomposition for any invertible matrix A can be computed.