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AI61003 Linear Algebra for AI & ML
Assignment 02 – Problem 02

Consider $A \in \mathbb{R}^{n \times n}$ where A is an invertible matrix.

$$\text{maxmag}(A) = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2}$$

Equivalently, $\text{maxmag}(A) = \max_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} \|Ax\|_2$

$$\text{minmag}(A) = \min_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2}$$

Equivalently, $\text{minmag}(A) = \min_{\substack{x \in \mathbb{R}^n \\ \|x\|_2 = 1}} \|Ax\|_2$

So, maxmag(A) is the maximum magnificat$^{\underline{n}}$ that is produced by $A \in \mathbb{R}^{n \times n}$ on $x \in \mathbb{R}^n$ when A linearly transforms x to Ax.
minmag(A) is the minimum magnifi-cation that is produced by $A \in \mathbb{R}^{n \times n}$ on $x \in \mathbb{R}^n$ when A linearly transforms x to Ax.

cond(A) is the condition number of matrix A that measures how much the output of linear transformat$^{\underline{n}}$ by A changes for a small change in

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input.

For a matrix (invertible) $A \in \mathbb{R}^{n\times n}$ and matrix norm $\|\cdot\|$

$$\text{cond}(A) = \|A\| \, \|A^{-1}\|$$

For $\|\cdot\|_2$ norm (operator norm)

$$\text{cond}_2(A) \; (\text{or } k_2(A)) = \|A\|_2 \, \|A^{-1}\|_2$$

Equivalently $k_2(A) = \dfrac{\text{max mag}(A)}{\text{min mag}(A)}$

(a) $$\text{maxmag}(A) = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{\|Ax\|_2}{\|x\|_2}$$

$A^{-1} \in \mathbb{R}^{n\times n}$, $y \in \mathbb{R}^n$. So replace x by $A^{-1}y$ $(\in \mathbb{R}^n)$ in the definit$^{\underline{n}}$.

$$\text{maxmag}(A) = \max_{\substack{y \in \mathbb{R}^n \\ A^{-1}y \neq 0}} \frac{\|y\|_2}{\|A^{-1}y\|_2}$$

Since A^{-1} is invertible, $A^{-1}y = 0 \iff y = 0$

$$\therefore \text{maxmg}(A) = \max_{\substack{y \in \mathbb{R}^n \\ y \neq 0}} \frac{\|y\|_2}{\|A^{-1}y\|_2}$$

$(\because y \neq 0$ in the definit$^{\underline{n}})$

$$\text{maxmag}(A) = \left(\min_{\substack{y \in \mathbb{R}^n \\ y \neq 0}} \frac{\|A^{-1}y\|_2}{\|y\|_2} \right)^{-1}$$

$$= \left(\text{minmag}(A^{-1})\right)^{-1}$$

$$\therefore \text{maxmag}(A) = \left(\text{minmag}(A^{-1})\right)^{-1}$$

(b) $\text{cond}(A) = \|A\|_2 \, \|A^{-1}\|_2$
$= \text{maxmag}(A) \cdot \text{maxmag}(A^{-1})$

From (a),
$\text{maxmag}(A) = 1/\text{minmag}(A^{-1})$
Replace A by A^{-1}.
$\text{maxmag}(A^{-1}) = 1/\text{minmag}((A^{-1})^{-1})$
$= 1/\text{minmag}(A)$

$\Rightarrow \text{cond}(A) = \dfrac{\text{maxmag}(A)}{\text{minmag}(A)}$

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