AI61003 Linear Algebra for AI & ML
Assignment 01 - Problem 05

(a) Prove associativity of matrix multiplicⁿ

Notation: If $A \in \mathbb{R}^{m \times n}$ then A_{ij} for $i = 1 \dots m$, $j = 1 \dots n$ denotes (i,j)th entry in matrix A.

Consider $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times q}$

$$(BC)_{ij} = \sum_{k=1}^{p} (B_{ik} C_{kj})$$

$$(A(BC))_{ij} = \sum_{t=1}^{n} A_{it} (BC)_{tj}$$

$$= \sum_{t=1}^{n} A_{it} \sum_{k=1}^{p} (B_{tk} C_{kj})$$

$$= \sum_{t=1}^{n} \sum_{k=1}^{p} (A_{it} B_{tk} C_{kj})$$

||ly $(AB)_{ij} = \sum_{k=1}^{n} (A_{ik} B_{kj})$

$$((AB)C)_{ij} = \sum_{t=1}^{p} (AB)_{it} C_{tj}$$

$$= \sum_{t=1}^{p} \left(\sum_{k=1}^{n} (A_{ik} B_{kt}) \right) C_{tj}$$

$$= \sum_{t=1}^{p} \sum_{k=1}^{n} (A_{ik} B_{kt} C_{tj})$$

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$$= \sum_{k=1}^{n} \sum_{t=1}^{p} (A_{ik} B_{kt} C_{tj})$$

$$\Rightarrow (A(BC))_{ij} = ((AB)C)_{ij}$$
$$\forall\, i = 1 \ldots m,\ j = 1 \ldots q$$

$$\Rightarrow (A(BC)) = ((AB)C)$$

$\Rightarrow$ Matrix multiplicatⁿ is associative.

(b) Disprove commutativity of matrix multiplicatⁿ.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$\because AB \neq BA$ matrix multiplicatⁿ is not commutative.

(c) Lemma: Consider $A \in \mathbb{R}^{l \times m}$, $B \in \mathbb{R}^{m \times n}$. Then the number of computations reqd to calculate AB is lmn.

Let com(AB) = no. of computatⁿs reqd. to calculate AB.

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$$com((AB)C) = com(AB) + com(DC)$$
(where $D \in \mathbb{R}^{p \times r}$, $D = AB$)
$$= pqr + prt$$

$$com(A(BC)) = com(BC) + com(AE)$$
(where $E \in \mathbb{R}^{q \times t}$, $E = BC$)
$$= qrt + pqt$$

If $(AB)C$ is computationally more efficient than $A(BC)$, then
$$com((AB)C) < com(A(BC))$$

$$\Rightarrow \quad pqr + prt < qrt + pqt$$

$$(\because p, q, r, t \in \mathbb{Z}^+)$$

$$\Rightarrow \quad \frac{1}{t} + \frac{1}{q} < \frac{1}{p} + \frac{1}{r}$$