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AI61003 Linear Algebra for AI & ML
Assignment 01 - Problem 01

(a) Define addition operation
'+' : $P_n(\mathbb{R}) \times P_n(\mathbb{R}) \mapsto P_n(\mathbb{R})$
Define scalar multiplication operation
'.' : $\mathbb{R} \times P_n(\mathbb{R}) \mapsto P_n(\mathbb{R})$

$P_n(\mathbb{R})$ is a real vector space because the following properties are satisfied.

① $\forall\, p_1, p_2 \in P_n(\mathbb{R})$, $(p_1 + p_2) \in P_n(\mathbb{R})$
($\because$ addition of two polynomials is also a polynomial)

② $\forall\, p_1, p_2 \in P_n(\mathbb{R})$, $(p_1 + p_2) = (p_2 + p_1)$
($\because$ addition on polynomials is commutative)

③ $\forall\, p_1, p_2, p_3 \in P_n(\mathbb{R})$, $p_1 + (p_2 + p_3) = (p_1 + p_2) + p_3$ ($\because$ addition on polynomials is associative).

④ $\exists$ a unique $p_0 \in P_n(\mathbb{R})$ s.t. $p_0 + p = p + p_0 = p \quad \forall\, p \in P_n(\mathbb{R})$
* $p_0 = 0$ (zero polynomial)

⑤ $\forall\, p \in P_n(\mathbb{R})$ $\exists$ a unique $p' \in P_n(\mathbb{R})$ s.t. $p + p' = p' + p = p_0 = 0$.
* If $p = \sum_{i=0}^{n} \alpha_i x^i$, then choose $p' = \sum_{i=0}^{n} (-\alpha_i) x^i$.

⑥ $\forall\, p \in P_n(\mathbb{R})$ $\forall\, k \in \mathbb{R}$, $(k.p) \in P_n(\mathbb{R})$
$k.p = k \sum_{i=0}^{n} \alpha_i x^i = \sum_{i=0}^{n} (k\alpha_i) x^i \in P_n(\mathbb{R})$

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(7) $\exists\, 1 \in \mathbb{R}$ s.t. $1 \cdot p = p \quad \forall\, p \in P_n(\mathbb{R})$

$$1 \cdot \sum_{i=0}^{n} \alpha_i x^i = \sum_{i=0}^{n} (1 \cdot \alpha_i) x^i = \sum_{i=0}^{n} (\alpha_i) x^i$$

(8) $\forall (\alpha, \beta) \in \mathbb{R}^2, \quad p \in P_n(\mathbb{R})$

$$(\alpha\beta)\, p = \alpha(\beta p)$$

$$(\alpha\beta) \sum_{i=0}^{n} t_i x^i = \alpha \sum_{i=0} (\beta t_i) x^i = \alpha(\beta p)$$

(9) $\forall (\alpha, \beta) \in \mathbb{R}^2, \quad p \in P_n(\mathbb{R})$

$$(\alpha + \beta)\, p = \alpha p + \beta p$$

$$(\alpha+\beta) \sum_{i=0}^{n} t_i x^i = \sum_{i=0}^{n} (\alpha+\beta) t_i x^i$$

$$= \sum_{i=0}^{n} (\alpha t_i x^i + \beta t_i x^i)$$

$$= \sum_{i=0}^{n} (\alpha t_i) x^i + \sum_{i=0}^{n} (\beta t_i) x^i = \alpha p + \beta p$$

(10) $\forall\, p_1, p_2 \in P_n(\mathbb{R}) \quad \forall\, k \in \mathbb{R}$

$$k(p_1 + p_2) = k p_1 + k p_2$$

$$p_1 + p_2 = \sum_{i=0}^{n} \alpha_i x^i + \sum_{i=0}^{n} \beta_i y^i$$

$$= \sum_{i=0}^{n} (\alpha_i x^i + \beta_i y^i)$$

$$k(p_1 + p_2) = k \sum_{i=0}^{n} (\alpha_i x^i + \beta_i y^i)$$

$$k(p_1 + p_2) = \sum_{i=0}^{n} (k \alpha_i x^i + k \beta_i y^i)$$

$$k(p_1 + p_2) = \sum_{i=0}^{n} (k\alpha_i) x^i + \sum_{i=0}^{n} (k\beta_i) y^i$$

$$k(p_1 + p_2) = k p_1 + k p_2.$$

(b) F is a linear functional because the following properties are satisfied.

① Additivity

Consider $p_x, q_x \in P_n(\mathbb{R})$

Let $p_x = \sum_{i=0}^{n} \alpha_i x^i$, $q_x = \sum_{i=0}^{n} \beta_i x^i$

$$\frac{d}{dx} p_x = \sum_{i=1}^{n} \alpha_i \, i x^{i-1}$$

$$\frac{d}{dx} q_x = \sum_{i=1}^{n} \beta_i \, i x^{i-1}$$

$$F(p_x + q_x) = \frac{d}{dx}(p_x + q_x)\Big|_{x=0}$$

$$= \left(\frac{d}{dx} p_x + \frac{d}{dx} q_x\right)\Big|_{x=0}$$

$$= \left(\frac{d}{dx} p_x\right)\Big|_{x=0} + \left(\frac{d}{dx} q_x\right)\Big|_{x=0}$$

$$= F(p_x) + F(q_x)$$

Hence proved.

② Homogeneity

Consider $p_x \in P_n(\mathbb{R})$ (defined in the same way) and $\lambda \in \mathbb{R}$

$$F(\lambda p_x) = \frac{d}{dx}(\lambda p_x)\Big|_{x=0}$$

* Apply chain rule

$$= \left[p_x \frac{d}{dx}(\lambda) + \lambda \frac{d}{dx}(p_x) \right] \Big|_{x=0}$$

$$= \lambda \frac{d}{dx}(p_x) \Big|_{x=0} \quad (\because \lambda \text{ is constt.})$$

$$= \lambda F(p_x)$$

Hence proved.

(c) Any polynomial $p \in P_n(\mathbb{R})$ can be represented by $p_c \in \mathbb{R}^{n+1}$ where p_c is the vector of coefficients in p.

If $p = \sum_{i=0}^{n} \alpha_i x^i$, then

$p_c \in \mathbb{R}^{n+1}$ s.t. $e_k^T p_c = \alpha_{k-1} \quad \forall k = 1 \dots (n+1)$

As already stated

$$\frac{d}{dx} p = \sum_{i=1}^{n} \alpha_i \, i \, x^{i-1}$$

$$\Rightarrow F(p) = \frac{d}{dx} p \Big|_{x=0} = \alpha_1$$

$$\Rightarrow F(p) = \alpha_1 = e_2^T p_c$$

So the linear functional F has the following representat$^{\underline{n}}$.

$F(p) = e_2^T p_c$, where $e_2, p_c \in \mathbb{R}^{n+1}$

Precisely $e_2^T = [0 \ 1 \ 0 \dots 0 \dots 0 \ 0]_{1 \times (n+1)}$