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AI61003 Linear Algebra for AI & ML
Assignment 01 - Problem 03

$$\|x\|_w = \sqrt{\sum_{i=1}^{n} w_i x_i^2} \;, \quad x, w \in \mathbb{R}^n \text{ and } w_i > 0 \;\forall\, i \in \{1 \dots n\}$$

Let for a vector $x \in \mathbb{R}^n$, y be defined as $(\sqrt{w_1}\, x_1, \sqrt{w_2}\, x_2, \dots \sqrt{w_n}\, x_n)^T$.

$$\therefore y(x) = \left\{ v \in \mathbb{R}^n \;\middle|\; v_i = \sqrt{w_i}\, x_i \text{ for } i = 1 \dots n \right\}$$

This definition is valid because $w_i > 0 \;\; \forall\; i = 1 \dots n$.

$$\therefore \|x\|_w = \sqrt{\sum_{i=1}^{n} (\sqrt{w_i}\, x_i)^2} = \sqrt{y(x)^T y(x)}$$

$$\therefore \|x\|_w = \|y(x)\|_2$$

① Definiteness : $\|x\|_w = 0$ iff $x = 0$.

if $x = 0$,

$$\|x\|_w = \sqrt{\sum_{i=1}^{n} w_i 0^2} = 0$$

if $\|x\|_w = 0$

$$\|y(x)\|_2 = 0$$

Since $\|.\|_2$ is a norm, it must be satisfying property of definiteness.

$$\therefore y(x) = 0 \Rightarrow \sqrt{w_i}\, x_i = 0 \;\; \forall\; i = 1 \dots n$$

$$\therefore x_i = 0 \;\; \forall\; i = 1 \dots n$$

$$\Rightarrow x = 0$$

Hence proved.

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(2) Non-negative homogeneity:
$\|\alpha x\|_w = |\alpha| \, \|x\|_w \quad \forall \alpha \in R, x \in \mathbb{R}^n$

$$\|\alpha x\|_w = \sqrt{\sum_{i=\phi}^{n} w_i (\alpha x_i)^2}$$

$$= \sqrt{\alpha^2 \sum_{i=1}^{n} w_i x_i^2}$$

$$= |\alpha| \sqrt{\sum_{i=1}^{n} w_i x_i^2} = |\alpha| \, \|x\|_w .$$

(3) Non-negativety : $\|x\|_w \geq 0 \quad \forall x \in \mathbb{R}^n$

$$\|x\|_w = \sqrt{\sum_{i=1}^{n} w_i x_i^2} \geq 0$$

($f(x) = \sqrt{x}$ function is always non(-)ve)

(4) Triangle Inequality :
$\left(\|u\|_w + \|v\|_w \geq \|u+v\|_w \quad \forall u, v \in \mathbb{R}^n \right)$

$$\|u+v\|_w = \|y(u+v)\|_2$$

☆ Since $\|.\|_2$ is a norm it satisfied the triangle inequality.
☆ Also $y(u+v) = y(u) + y(v)$

$\cancel{\sum_{i=1}^{n} w_i (u_i + v_i)^2}$

$\because (\sqrt{w_1}(u_1+v_1), \sqrt{w_2}(u_2+v_2) \ldots, \sqrt{w_n}(u_n+v_n))^T$
$= \left((\sqrt{w_1} u_1, \sqrt{w_2} u_2 \ldots \sqrt{w_n} u_n)^T + (\sqrt{w_1} v_1, \sqrt{w_2} v_2 \ldots \sqrt{w_n} v_n)^T \right)$

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$$\|u+v\|_w = \|y(u)+y(v)\|_2$$
$$\leq \|y(u)\|_2 + \|y(v)\|_2$$

By definition,

$$\|u+v\|_w \leq \|u\|_w + \|v\|_w$$

Hence, $\|\cdot\|_w$ satisfies all the properties of norm & $\therefore$ it is a norm.

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