DATE
PAGE

AI61003 Linear Algebra for AI & ML
Assignment 02 - Problem 03

(a)
$$A = \begin{bmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1 & 1 \end{bmatrix}$$

Let $(a, b)^T \in \mathbb{R}^2$ s.t. $a^2 + b^2 = 1$.

$$\begin{bmatrix} -1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$x = -a/\sqrt{2} \Rightarrow a = -\sqrt{2}x$$
$$y = -b/\sqrt{2} \Rightarrow b = -\sqrt{2}y$$
$$z = b - a \Rightarrow z = \sqrt{2}(x - y)$$

$$a^2 + b^2 = 1 \Rightarrow x^2 + y^2 = 1/2$$

$\therefore$ Image of the unit circle in $\mathbb{R}^2$ is the region Im in $\mathbb{R}^3$ s.t.

$$Im = \left\{ (x, y, z) \;\middle|\; \begin{matrix} x^2 + y^2 = 1/2, \\ z = \sqrt{2}(x - y) \end{matrix} \right\}$$

Im is a cylinder cut by an oblique plane which produces a 2D ellipse at the intersection.

Let
$$f = x^2 + y^2 + z^2$$
$$f = 1/2 + z^2$$
$$f = 1/2 + 2(1/2 - 2xy)$$

$f = 3/2 - 4xy$

$x^2 + y^2 = 1/2$; so use the parametric eq^{n} for x and y coordinates.

$x = (1/\sqrt{2})\cos\theta \quad , \quad y = (1/\sqrt{2})\sin\theta$

$0 \leq \theta < 2\pi$

$f = 3/2 - 4\left(\frac{1}{2}\cos\theta\sin\theta\right) = 3/2 - \sin 2\theta$

$f(\theta = \pi/4) = 1/2 = f_{min}$

$f(\theta = 3\pi/4) = 5/2 = f_{max}$

If $(\alpha, \beta, \gamma) \in Im$ $(\alpha, \beta, \gamma \in \mathbb{R})$ then $(-\alpha, -\beta, -\gamma) \in Im$.

This implies that $O(0,0,0)$ is the center of the ellipse traced by Im.

semi-major axis length

$$= \max_{(x,y,z)\in Im} (x^2 + y^2 + z^2)^{1/2} = \sqrt{f_{max}}$$

$$= \sqrt{5/2}$$

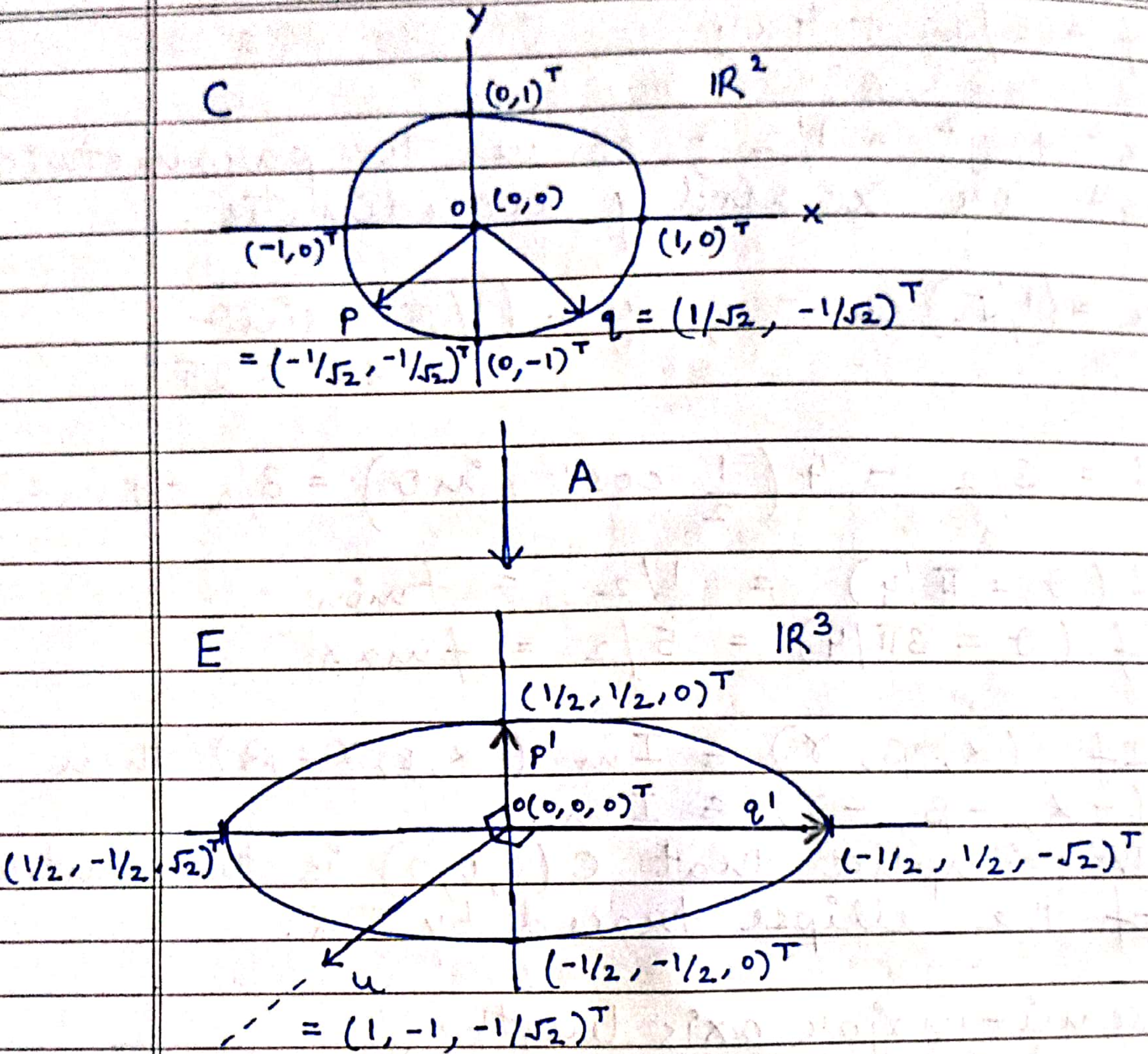
semi-minor axis length

$$= \min_{(x,y,z)\in Im} (x^2 + y^2 + z^2)^{1/2} = \sqrt{f_{min}}$$

$$= \sqrt{1/2}$$

$$K_2(A) = \frac{\text{semi-major axis length}}{\text{semi-minor axis length}}$$

$$= \sqrt{5} = 2.236$$



The ellipse E lies on the plane
$z = \sqrt{2}(x-y)$.
p' is the image of p.
q' is the image of q.

DATE
PAGE

(b) $A = \begin{bmatrix} -2 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

Let $(a, b, c)^T \in \mathbb{R}^3$ s.t. $a^2 + b^2 + c^2 = 1$

$$\begin{bmatrix} -2 & 1 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = -2a + b + 2c$
$y = 2b$

$b = y/2$
$c - a = x/2 - y/4$

$b^2 = y^2/4$
$a^2 + c^2 - 2ac = \frac{x^2}{4} + \frac{y^2}{16} - \frac{xy}{4}$
$(a^2 + b^2 + c^2) - 2ac = \frac{x^2}{4} + \frac{5y^2}{16} - \frac{xy}{4}$
$ac = \frac{1}{2} + \frac{xy}{8} - \frac{x^2}{8} - \frac{5y^2}{32}$

$a^2 + b^2 + c^2 = 1$; so use the parametric eq^n for a and c.
$a = \sin\phi \cos\theta$, $c = \cos\phi$
$b = \sin\phi \sin\theta$
$0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$.

$ac = \sin\phi \cos\phi \cos\theta = (1/2) \sin 2\phi \cos\theta$
$\Rightarrow -1/2 \le ac \le 1/2$

① $ac \leq 1/2$

$\Rightarrow \; 1/2 + xy/8 - x^2/8 - 5y^2/32 \leq 1/2$

$\Rightarrow \; x^2 - xy + \dfrac{5y^2}{4} \geq 0$

$\Rightarrow \; (x - y/2)^2 + y^2 \geq 0$ (trivial).

$\therefore$ $(ac \leq 1/2)$ is redundant.

② $ac \geq -1/2$

$\Rightarrow \; 1/2 + xy/8 - x^2/8 - 5y^2/32 \geq -1/2$

$\Rightarrow \; 4x^2 + 5y^2 - 4xy - 32 \leq 0$

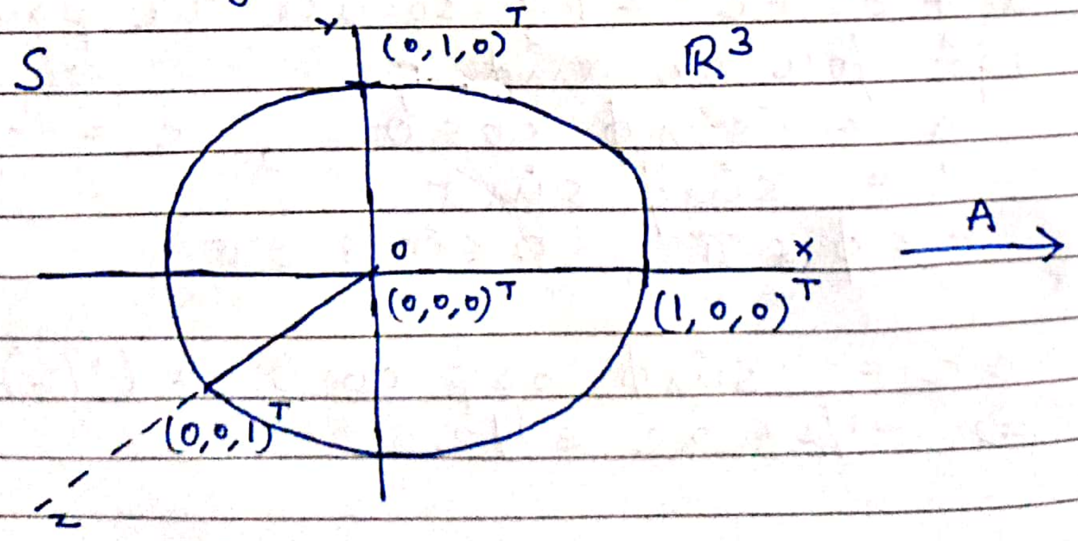
$\therefore$ Image of the unit sphere in $\mathbb{R}^3$ is the region Im in $\mathbb{R}^2$ s.t.

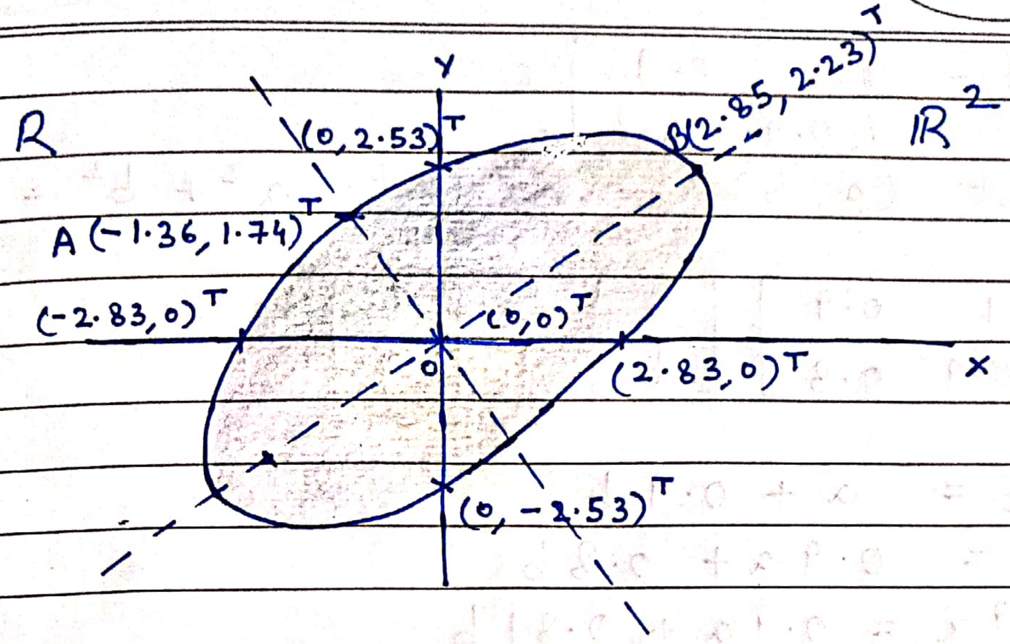
$\text{Im} = \{ (x,y) \mid 4x^2 + 5y^2 - 4xy - 32 \leq 0 \}$

$f(x,y) = 4x^2 + 5y^2 - 4xy - 32$.

$f(x,y) = 0$ is an ellipse.

$\therefore$ $f(x,y) \leq 0$ is the region enclosed by the ellipse, i.e. interior & the boundary of the ellipse.





The region R is the region encompas-sed by the boundary & interior of the ellipse $f(x,y) = 0$.

A : co-vertex, B - vertex.

$$A \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \end{pmatrix}^T$$

$\Rightarrow$ $\text{minmag}(A) = 0$

$k_2(A) = \text{max mag}(A) / \text{minmag}(A)$

$\Rightarrow$ $k_2(A)$ is infinite.

Scanned with CamScanner

DATE
PAGE

(c) $A = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 0.8 \end{bmatrix}$

Let $(a, b)^T \in \mathbb{R}^2$ s.t. $a^2 + b^2 = 1$

$$\begin{bmatrix} 1 & 0.9 \\ 0.9 & 0.8 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = a + 0.9b$

$y = 0.9a + 0.8b$

$0.9x = 0.9a + 0.81b$

$0.9x - y = 0.01b$

$b = 90x - 100y$

$a = x - 81x + 90y = 90y - 80x$

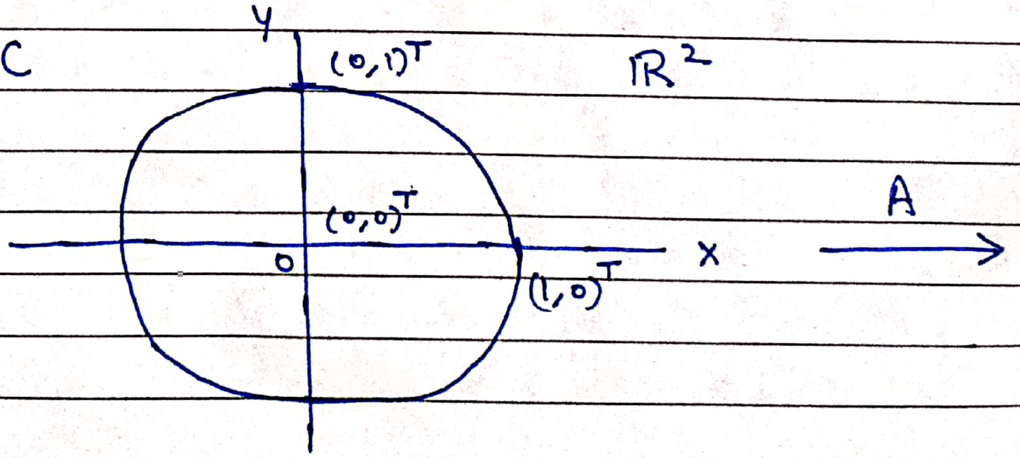
$a^2 + b^2 = 1$

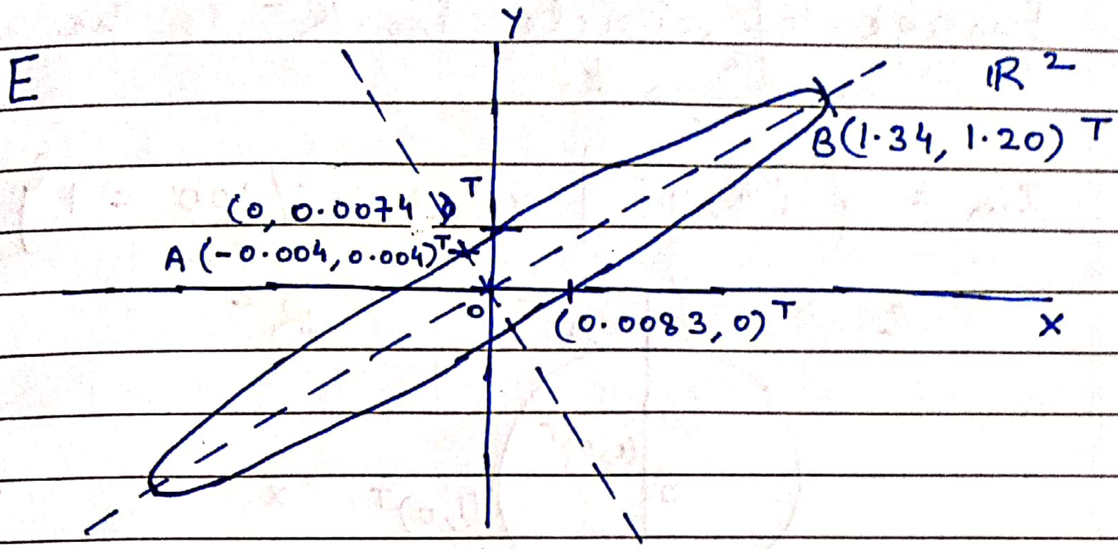
$\Rightarrow (90y - 80x)^2 + (90x - 100y)^2 = 1$

$\Rightarrow 145x^2 + 181y^2 - 324xy = 1/100$

$\therefore$ Image of unit circle in $\mathbb{R}^2$ is the region Im in $\mathbb{R}^2$ s.t.

$$Im = \left\{ (x,y) \mid 145x^2 + 181y^2 - 324xy = 1/100 \right\}$$





A is the co-vertex and B is the vertex of the ellipse E.

semi-major axis length of $E = 1.8055\ldots$

semi-minor axis length of $E = 0.0055\ldots$

$$k_2(A) = \frac{\text{semi-major axis length}}{\text{semi-minor axis length}}$$

$$= 1.8055/0.0055 \approx 326.$$

$\det(A) = 1 \times 0.8 - 0.9 \times 0.9 = -0.01 \neq 0$

$\det(A) \neq 0 \Rightarrow A$ is non-singular

$\therefore$ A is invertible.

(d) $A = \begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix}$

Let $(a, b)^T \in \mathbb{R}^2$ s.t. $a^2 + b^2 = 1$

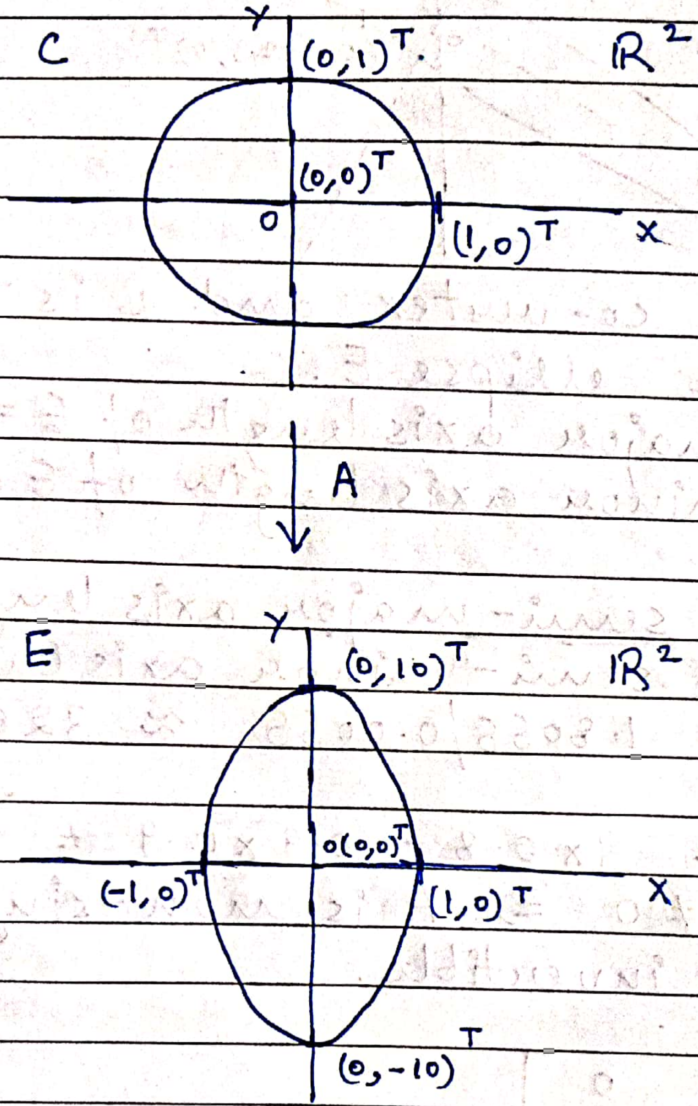
$$\begin{bmatrix} 1 & 0 \\ 0 & -10 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = a, \quad y = -10b$

$a^2 + b^2 = 1 \Rightarrow x^2 + \frac{y^2}{100} = 1$

$\therefore$ Image of unit circle in $\mathbb{R}^2$ is the region I_m in $\mathbb{R}^2$ s.t.

$$I_m = \{ (x, y) \mid x^2 + y^2/100 = 1 \}$$



semi-major axis length of E = 10
semi-minor axis length of E = 1

$$k_2(A) = \frac{\text{semi-major axis length}}{\text{semi-minor axis length}}$$

$$= 10/1 = 10$$

$\det(A) = 1 \times (-10) - 0 \times 0 = -10 \neq 0$

$\det(A) \neq 0 \Rightarrow A$ is non-singular

$\therefore$ A is invertible.

(e) $A = \begin{bmatrix} 1 & 1 \\ 1 & \varepsilon \end{bmatrix}$, $\varepsilon (\in \mathbb{R})$ is a parameter.

Let $(a, b)^T \in \mathbb{R}^2$ s.t. $a^2 + b^2 = 1$

$$\begin{bmatrix} 1 & 1 \\ 1 & \varepsilon \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$x = a + b$

$y = a + b\varepsilon$

$b = (x - y) / (1 - \varepsilon)$

$a = x - b = (y - x\varepsilon) / (1 - \varepsilon)$

$a^2 + b^2 = 1$

$\Rightarrow (x-y)^2 + (y - x\varepsilon)^2 = (1-\varepsilon)^2$

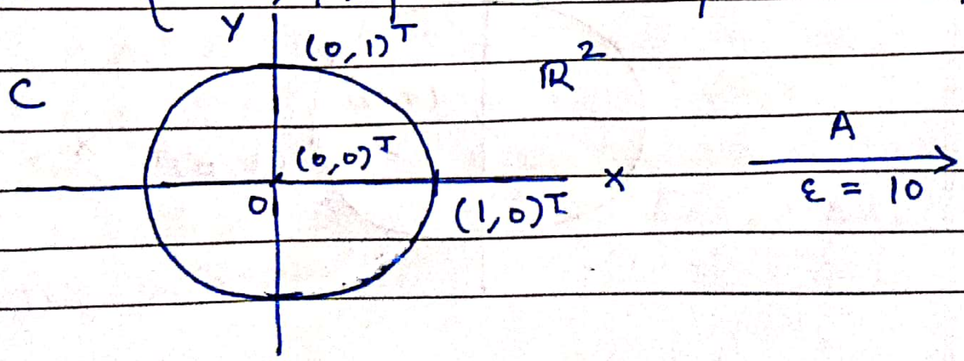
$x^2(1+\varepsilon^2) + 2y^2 - 2xy(1+\varepsilon) = (1-\varepsilon)^2 \quad (\bigstar)$

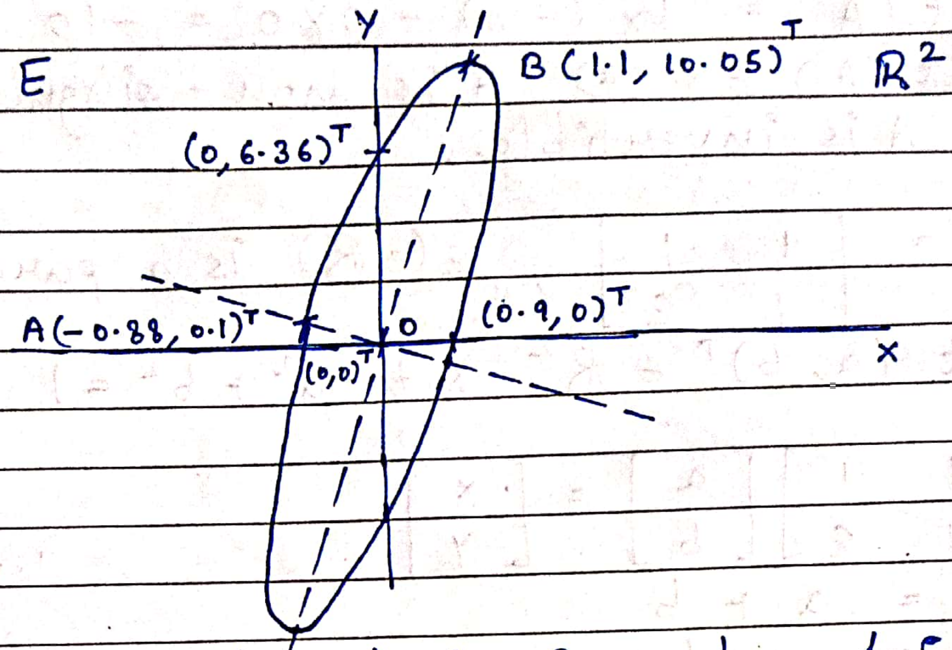
$\therefore$ Image of unit circle in $\mathbb{R}^2$ is the region $Im(\varepsilon)$ in $\mathbb{R}^2$ s.t.

$$Im(\varepsilon) = \left\{ (x, y) \,\middle|\, x^2(1+\varepsilon^2) + 2y^2 - 2xy(1+\varepsilon) = (1-\varepsilon)^2 \right\}$$

① $\varepsilon = 10$

$$Im(10) = \left\{ (x, y) \,\middle|\, 101x^2 + 2y^2 - 22xy = 81 \right\}$$





A : co-vertex of E, B: vertex of E

α = semi-major axis length of E = 10.1097...

β = semi-minor axis length of E = 0.8902...

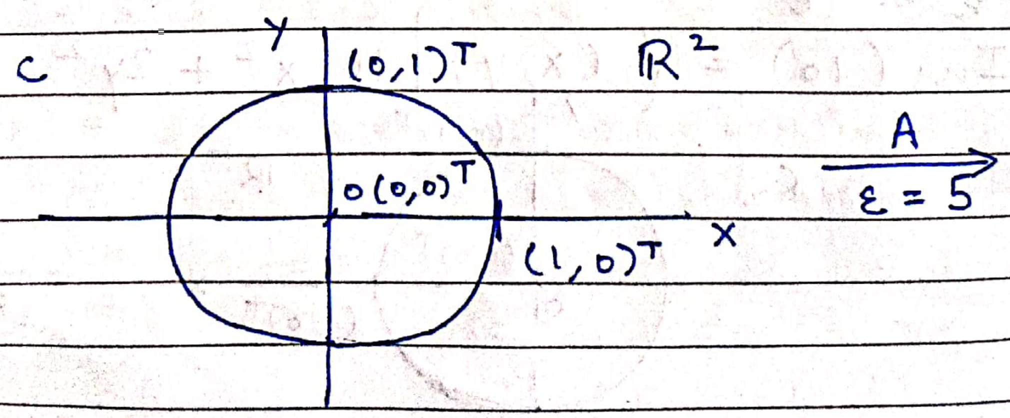
$$K_2(A)_{\varepsilon=10} = \alpha/\beta = 11.36$$

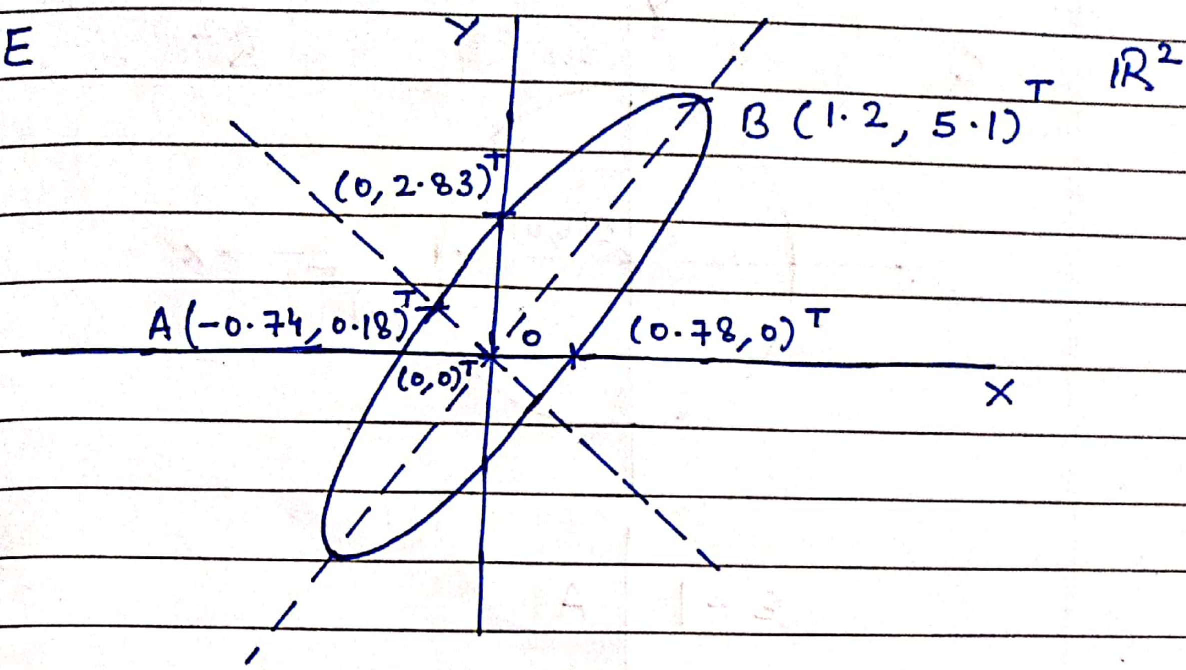
$\det(A)_{\varepsilon=10} = 1\times 10 - 1\times 1 = 9 \neq 0$

$\det(A)_{\varepsilon=10} \neq 0 \Rightarrow A(\varepsilon=10)$ is non-singular. $\therefore$ $A(\varepsilon=10)$ is invertible.

② $\varepsilon = 5$

$$I_m(5) = \{(x,y) \mid 13x^2 + y^2 - 6xy = 8\}$$





A : co-vertex of E, B : vertex of E

α = semi-major axis length of $E = 5.2360\ldots$

β = semi-minor axis length of $E = 0.7639\ldots$

$$k_2(A)_{\varepsilon=5} = \alpha/\beta = 6.85$$

$$\det(A)_{\varepsilon=5} = 1\times 5 - 1\times 1 = 4 \neq 0$$

$\det(A)_{\varepsilon=5} \neq 0 \Rightarrow A(\varepsilon=5)$ is non-singu-lar. $\therefore A(\varepsilon=5)$ is invertible.

③ $\varepsilon = 1$

Put $\varepsilon = 1$ in eqn ($\bigstar$)

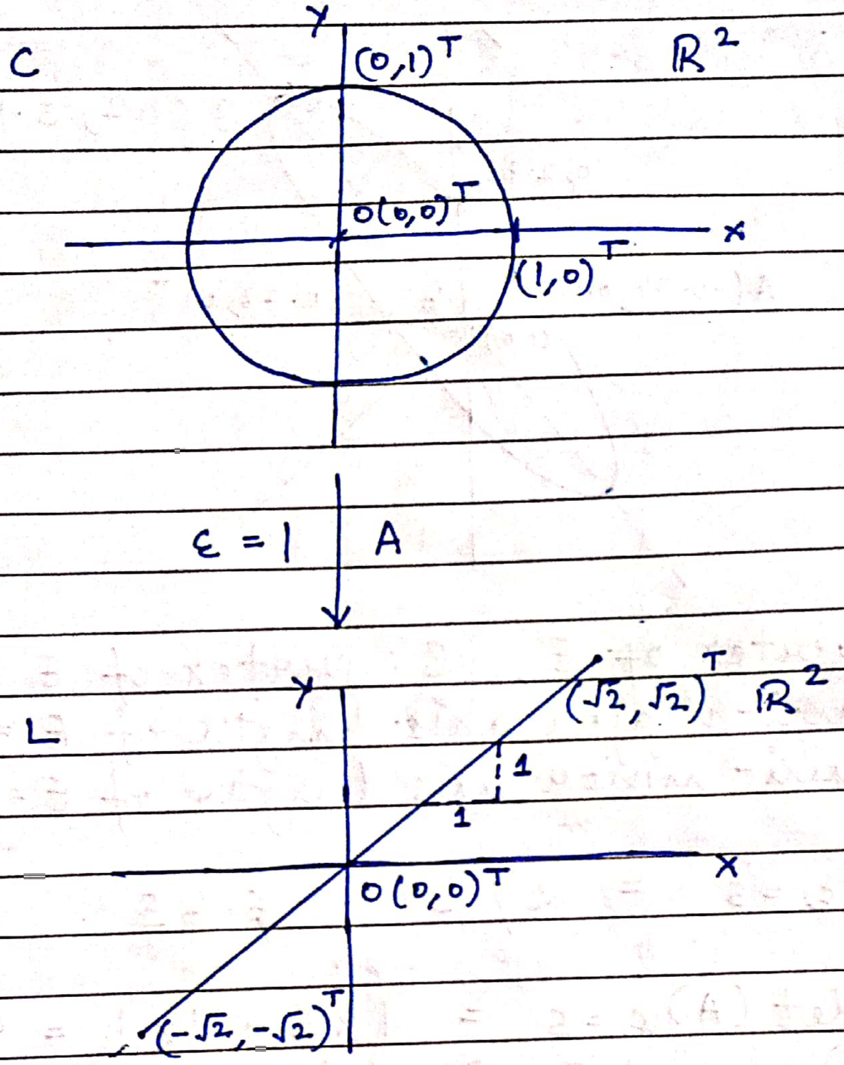
$$x^2(2) + 2y^2 - 4xy = 0$$

$\Rightarrow (x-y)^2 = 0 \Rightarrow x = y$ (eqn of line).

Also $x = y = a+b$ & $-\sqrt{2} \leq a+b \leq \sqrt{2}$ ($\because a^2+b^2=1$)

$\therefore \operatorname{Im}(1) = \{(x,y) \mid x = y,\ -\sqrt{2} \leq x \leq \sqrt{2}\}$

★ A ~~straight line~~ line segment can be treated as an ellipse with 0 semi-minor axis length.



$$\text{maxmag}(A(\varepsilon=1)) = 2$$
$$\text{minmag}(A(\varepsilon=1)) = 0$$

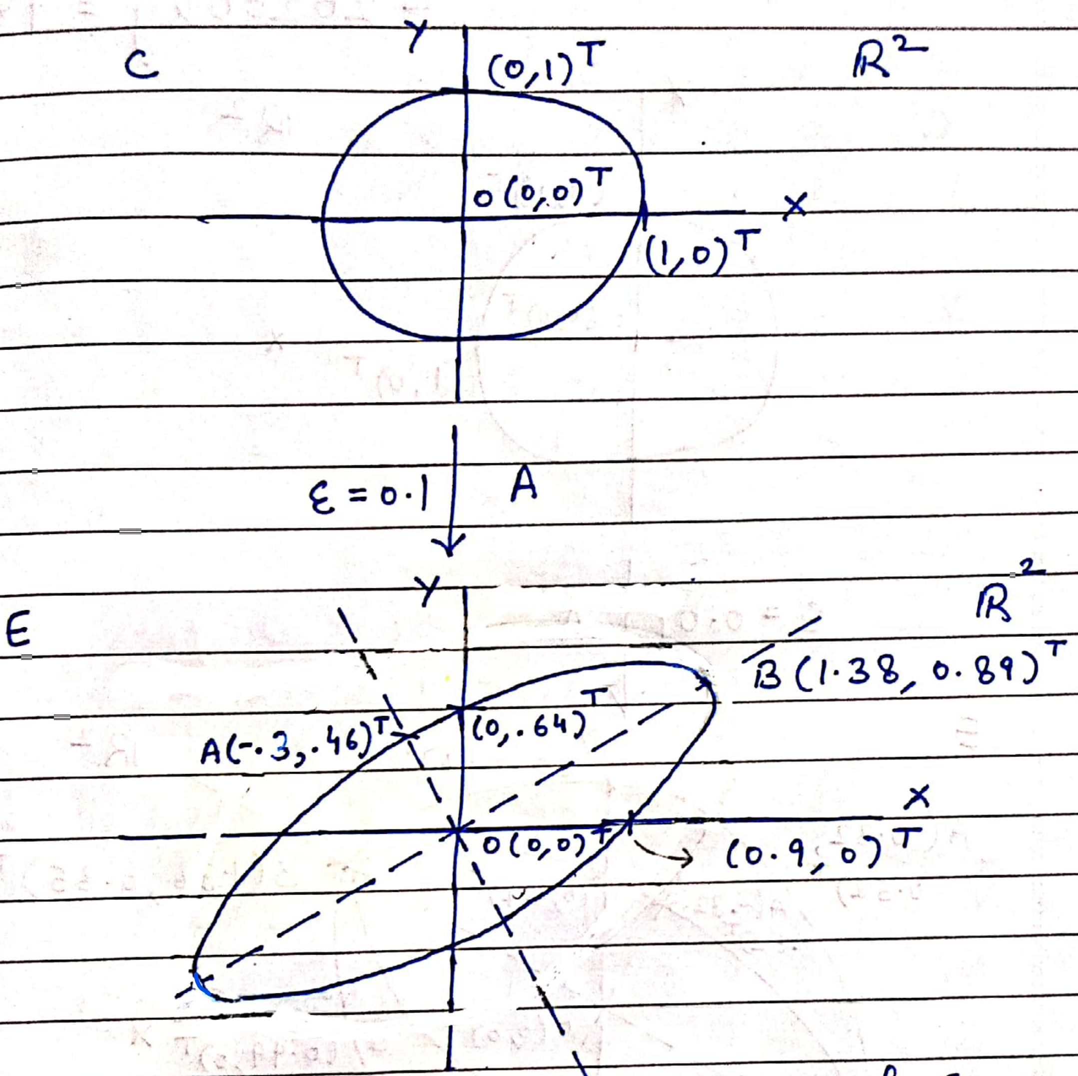
$$k_2(A)_{\varepsilon=1} = \frac{\text{maxmag}(A(\varepsilon=1))}{\text{minmag}(A(\varepsilon=1))} = \text{infinite.}$$

$$\det(A)_{\varepsilon=1} = 1\times 1 - 1\times 1 = 0$$
$\det(A)_{\varepsilon=1} = 0 \Rightarrow A(\varepsilon=1)$ is singular

$\therefore$ $A(\varepsilon=1)$ is non-invertible.

④ $\varepsilon = 0.1$

$Im(0.1) = \{(x,y) \mid 101x^2 + 200y^2 - 220xy = 81\}$



A: co-vertex of E B: vertex of E.

α = semi-major axis length of E = 1.6465...

β = semi-minor axis length of E = 0.5465...

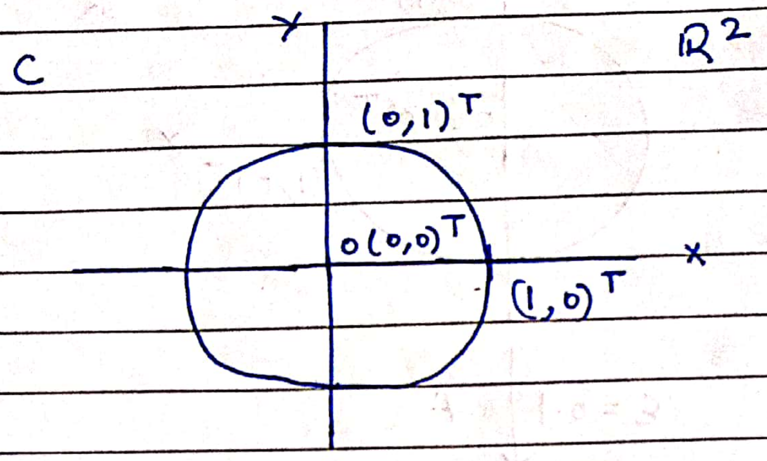
$K_2(A)_{\varepsilon=0.1} = \alpha/\beta = 3.01$

$\det(A)_{\varepsilon=0.1} = 1\times 0.1 - 1\times 1 = -0.9 \neq 0$

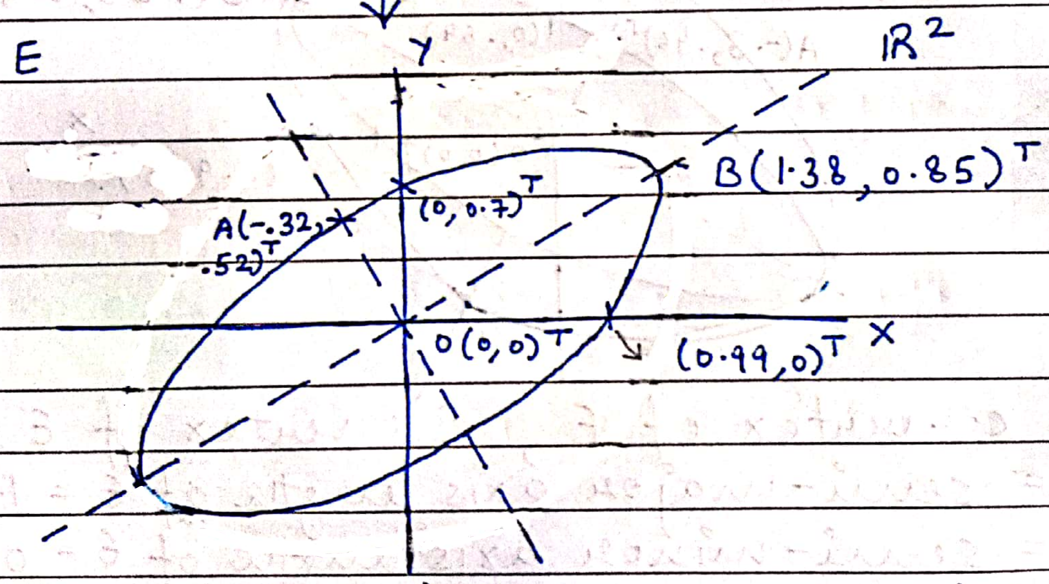
$\det(A)_{\varepsilon=0.1} \neq 0 \Rightarrow A(\varepsilon=0.1)$ is non-singular $\therefore A(\varepsilon = 0.1)$ is invertible.

(5) $\varepsilon = 10^{-2}$

$I_m(10^{-2}) = \{(x,y) \mid 10001x^2 + 20000y^2 - 20200xy = 9801\}$



$\varepsilon = 0.01$ A



A: co-vertex of E B: vertex of E

α = semi-major axis length of E = 1.6208...

β = semi-minor axis length of E = 0.6108...

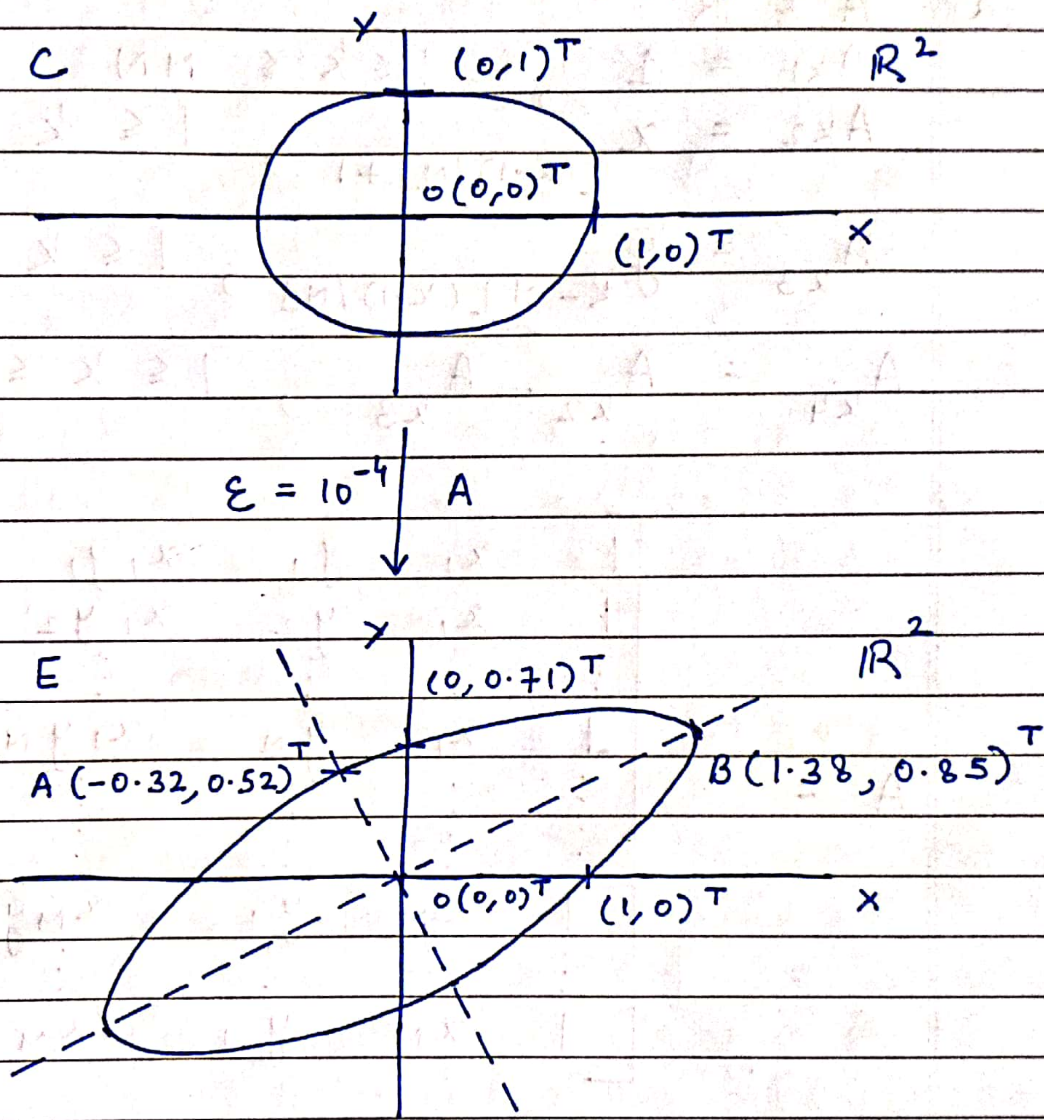
$K_2(A)_{\varepsilon=10^{-2}} = \alpha/\beta = 2.65$

$\det(A)_{\varepsilon=10^{-2}} = 1 \times 10^{-2} - 1 \times 1 = -0.99 \neq 0$

$\det(A)_{\varepsilon=10^{-2}} \neq 0 \Rightarrow A(\varepsilon = 10^{-2})$ is non-singular. $\therefore A(\varepsilon = 10^{-2})$ is invertible.

⑥ $\varepsilon = 10^{-4}$

$$Im(10^{-4}) = \{(x,y) \mid (10^8+1)x^2 + 2\times10^8 y^2 - 2.0002\times10^8 xy = 99980001\}$$



A: co-vertex of E, B: vertex of E

α = semi-major axis length of E = 1.6180...

β = semi-minor axis length of E = 0.6179...

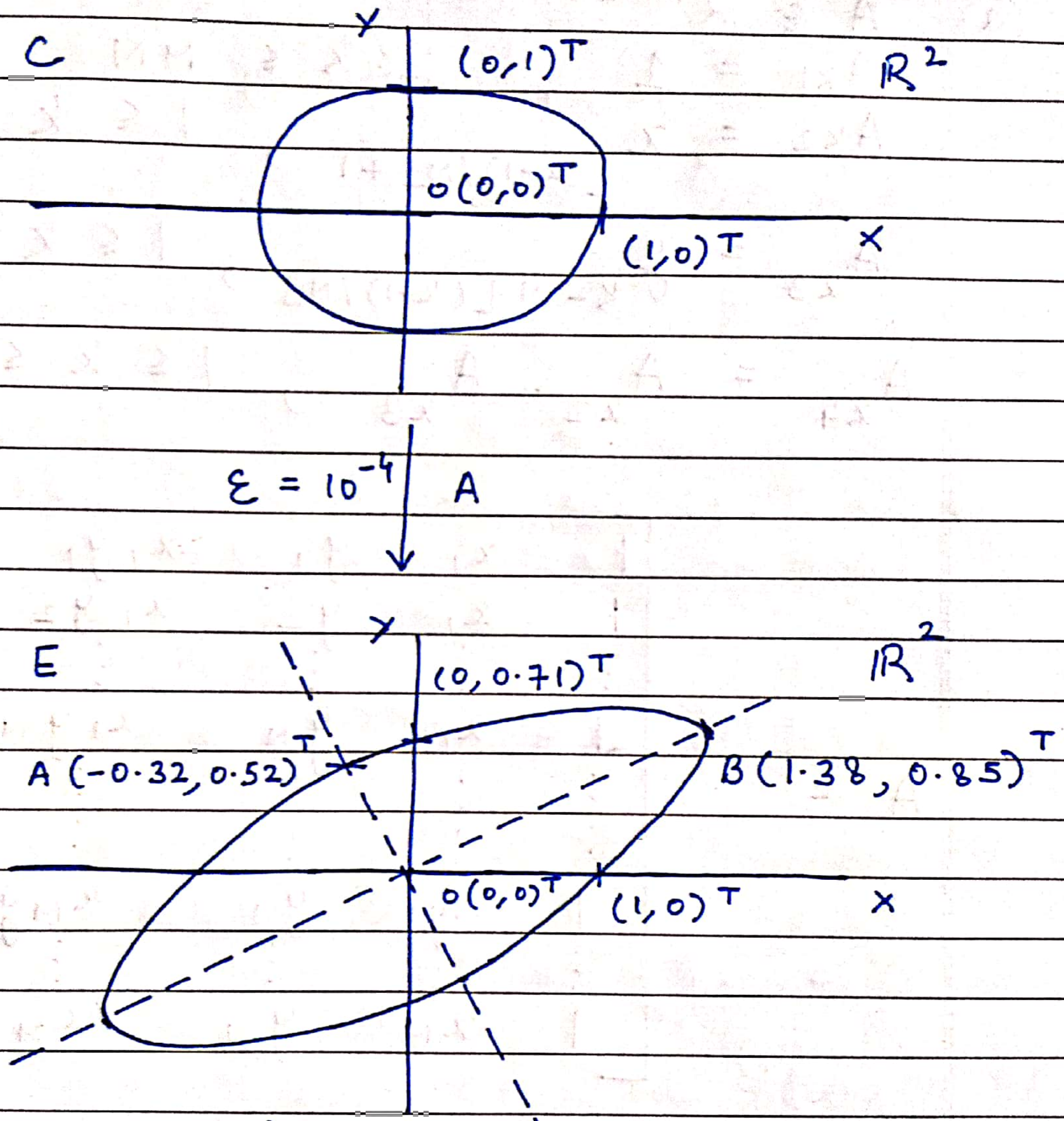
$k_2(A)_{\varepsilon=10^{-4}} = \alpha/\beta = 2.62$

$\det(A)_{\varepsilon=10^{-4}} = 1\times10^{-4} - 1\times1 = -0.9999 \neq 0$

$\det(A)_{\varepsilon=10^{-4}} \neq 0 \Rightarrow A(\varepsilon=10^{-4})$ is non-singular. $\therefore A(\varepsilon=10^{-4})$ is invertible.

(7) $\varepsilon = 0$

$\mathrm{Im}(0) = \{(x,y) \mid x^2 + 2y^2 - 2xy = 1\}$



A: co-vertex of E, B: vertex of E

α = semi-major axis length of E = 1.6180...

β = semi-minor axis length of E = 0.6180...

$k_2(A)_{\varepsilon=0} = \alpha/\beta = 2.62$

$\det(A)_{\varepsilon=0} = 1 \times 0 - 1 \times 1 = -1 \neq 0$

$\det(A)_{\varepsilon=0} \neq 0 \Rightarrow A(\varepsilon=0)$ is non-singular. $\therefore A(\varepsilon=0)$ is invertible.

DATE
PAGE

A large condition number indicates that minmag of the matrix (say A) is ~~large~~ small. This implies that $\exists$ a linear combinatⁿ of columns of A that is very close 0 that inturn implies that columns of A are "almost" linearly dependent. For a square matrix, if columns are "almost" linearly dependent, then the matrix is "almost" non-invertible and hence "almost" singular that implies a low absolute determinant value.

For example in part (c), determinant value was -0.01 and the conditⁿ number was as high as 326.

✓ Hence, the relationship b/w det(A) and $k_2(A)$ for a square matrix A is that high $k_2(A)$ implies low $|\det(A)|$. Contrapositively high $|\det(A)|$ implies low $k_2(A)$. ($|\det(A)|$ is absolute value of determinant)

★ Note that the terms "high" and "low" are used in relative terms.

★ Note that matrix with low det(A) can have high (or not low) $k_2(A)$.

Eg - $A = 0.001\, I_{5\times5}$; $\det(A) = 10^{-15}$ and $k_2(A) = 1$.

In this case $|\det(A)| <<< k_2(A)$