CSCI203

Algorithms and Data Structures

Hashing

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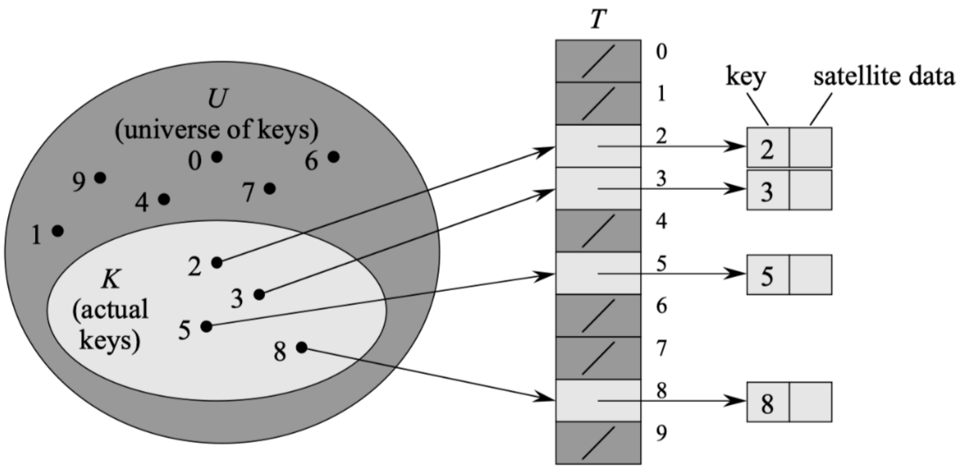
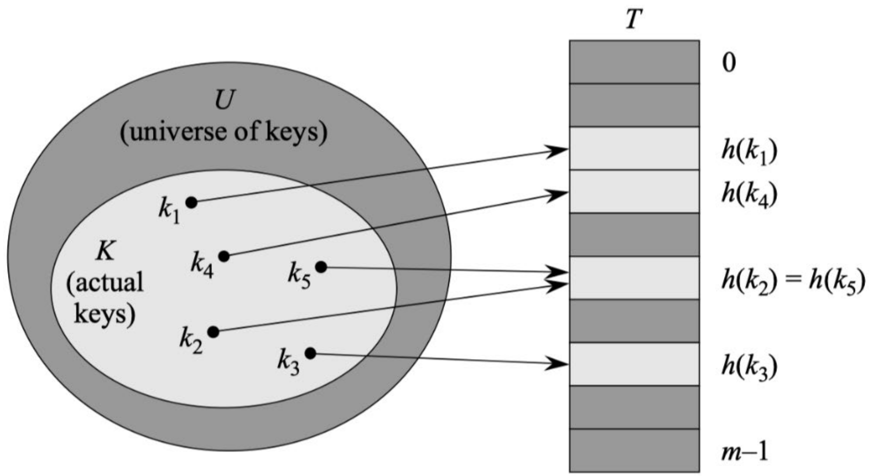
Direct Access vs Hash Table



Direct Access Table

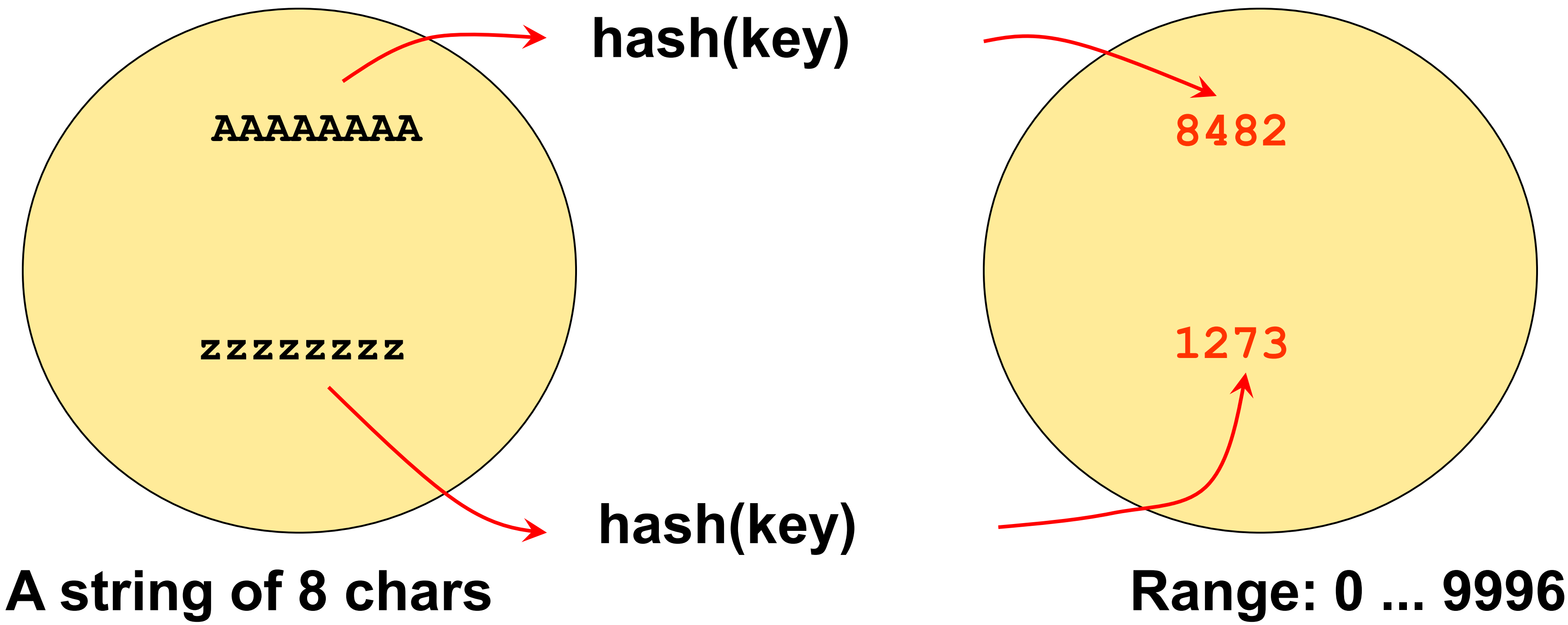
Hash Table



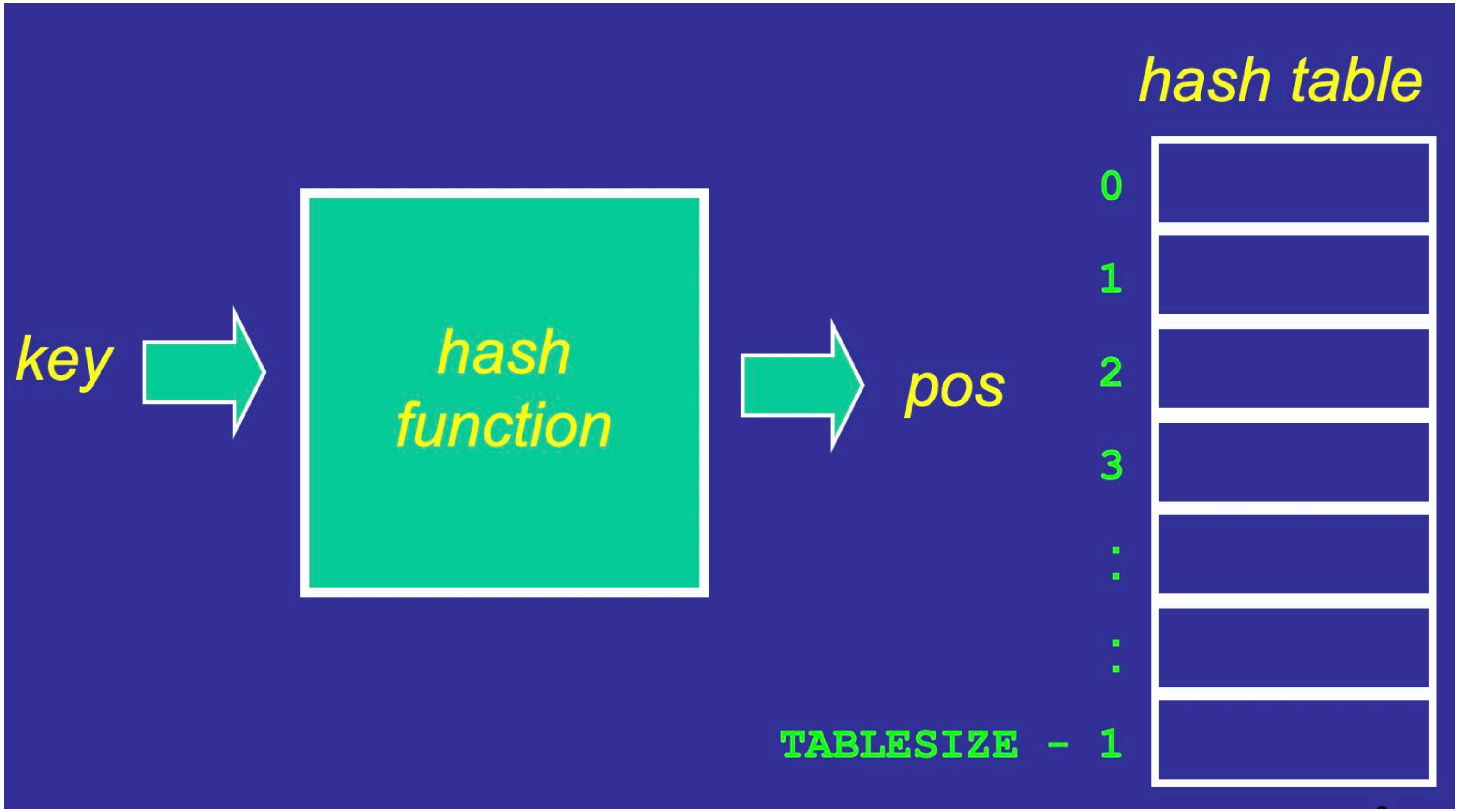
Hash method works something like...

Convert a String key into an integer that will be in the range of 0 through the maximum capacity-1

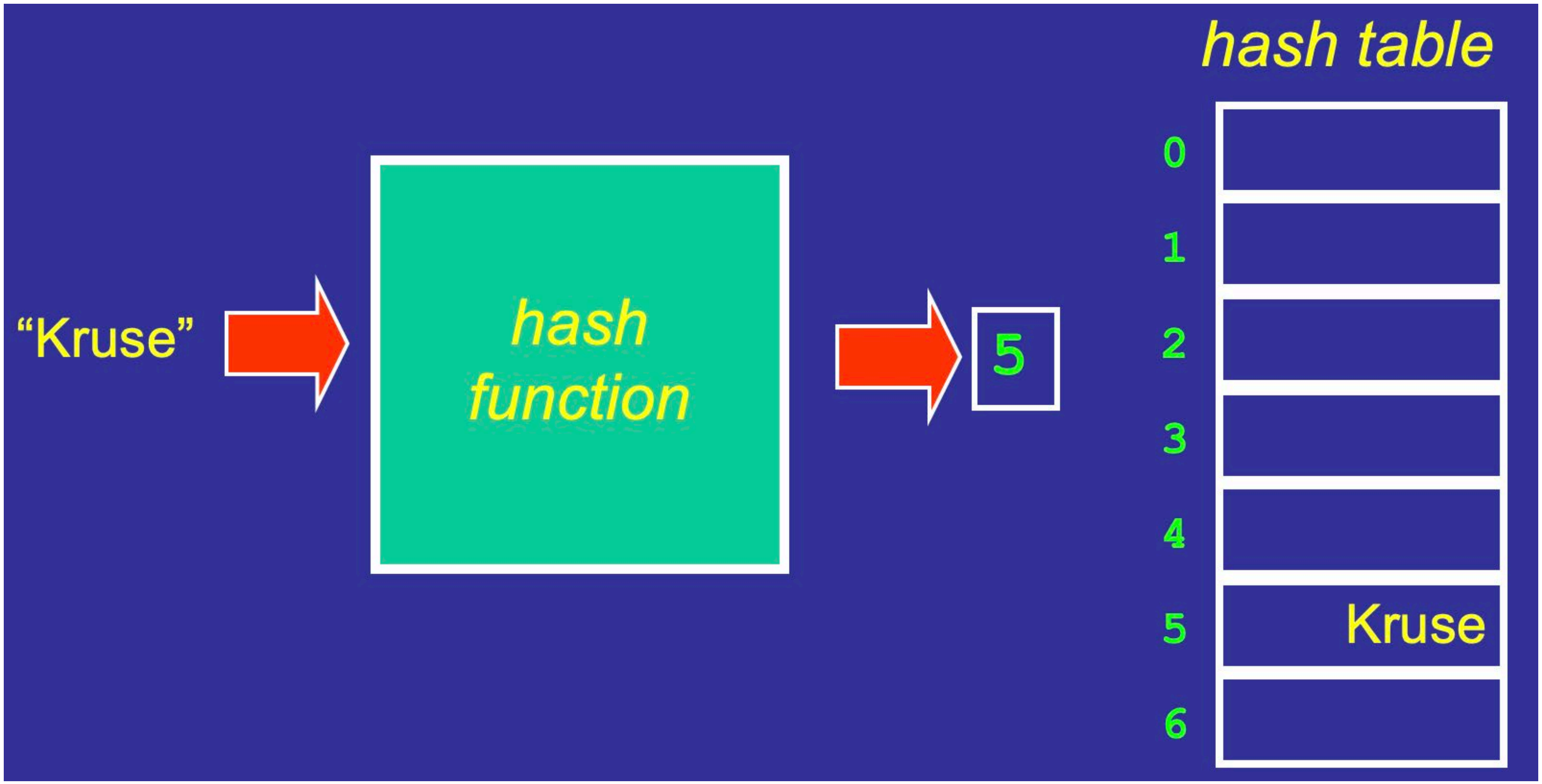
Assume the array capacity is 9997



Hashing



Hashing



Example: use ASCII code

Mia	M	77	i	105	a	97	279	4
Tim	T	84	i	105	m	109	298	1
Bea	B	66	e	101	a	97	264	0
Zoe	Z	90	o	111	e	101	302	5
Jan	J	74	a	97	n	110	281	6
Ada	A	65	d	100	a	97	262	9
Leo	L	76	e	101	o	111	288	2
Sam	S	83	a	97	m	109	289	3
Lou	L	76	o	111	u	117	304	7
Max	M	77	a	97	x	120	294	8
Ted	T	84	e	101	d	100	285	10

Bea	Tim	Leo	Sam	Mia	Zoe	Jan	Lou	Max	Ada	Ted
0	1	2	3	4	5	6	7	8	9	10

Hashing

- Each item has a unique key.
- Use a large array called a Hash Table.
- Use a Hash Function.

Operations

- Initialize
  - all locations in Hash Table are empty.
- Insert
- Search
- Delete

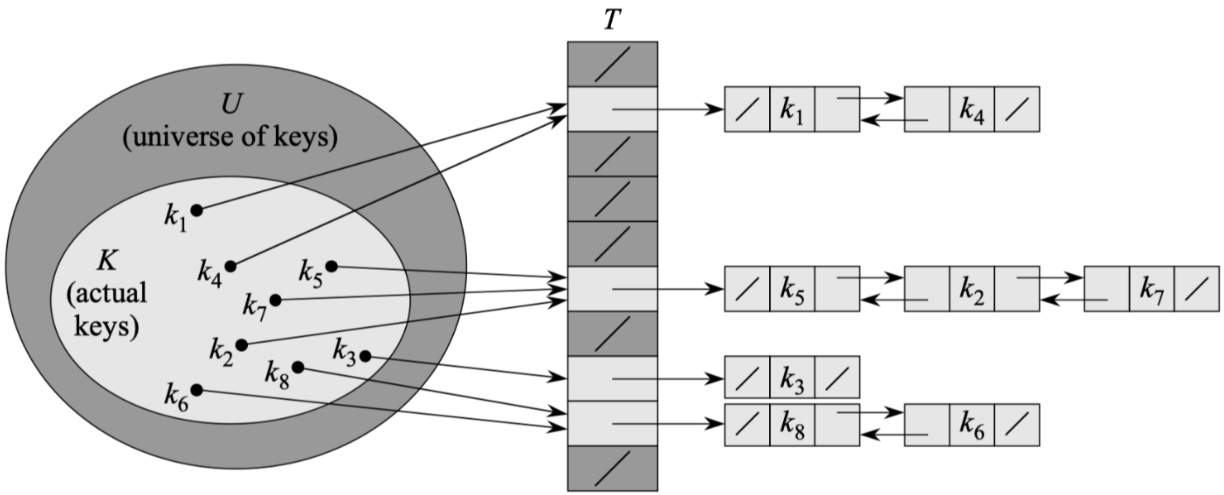
Hash Function

- Maps keys to positions in the Hash Table.
- Be easy to calculate.
- Use all of the key.
- Spread the keys uniformly.

Hashing

- Ideally, if we have n keys with associated values, we would like $m \in \Theta(n)$.
  - $m = 2n, m = 3n$.
- This presents a problem:
  - Although $m > n$, the number of keys we are storing, it is far smaller that the number of possible keys.
  - There will always be circumstances where $key_1 \neq key_2$ but $h(key_1) = h(key_2)$.
  - This leads to a collision:
    - Two different keys with the same hash value;
    - Two different keys with the same location in the table.
  - How do we fix this?

Hashing with Chaining



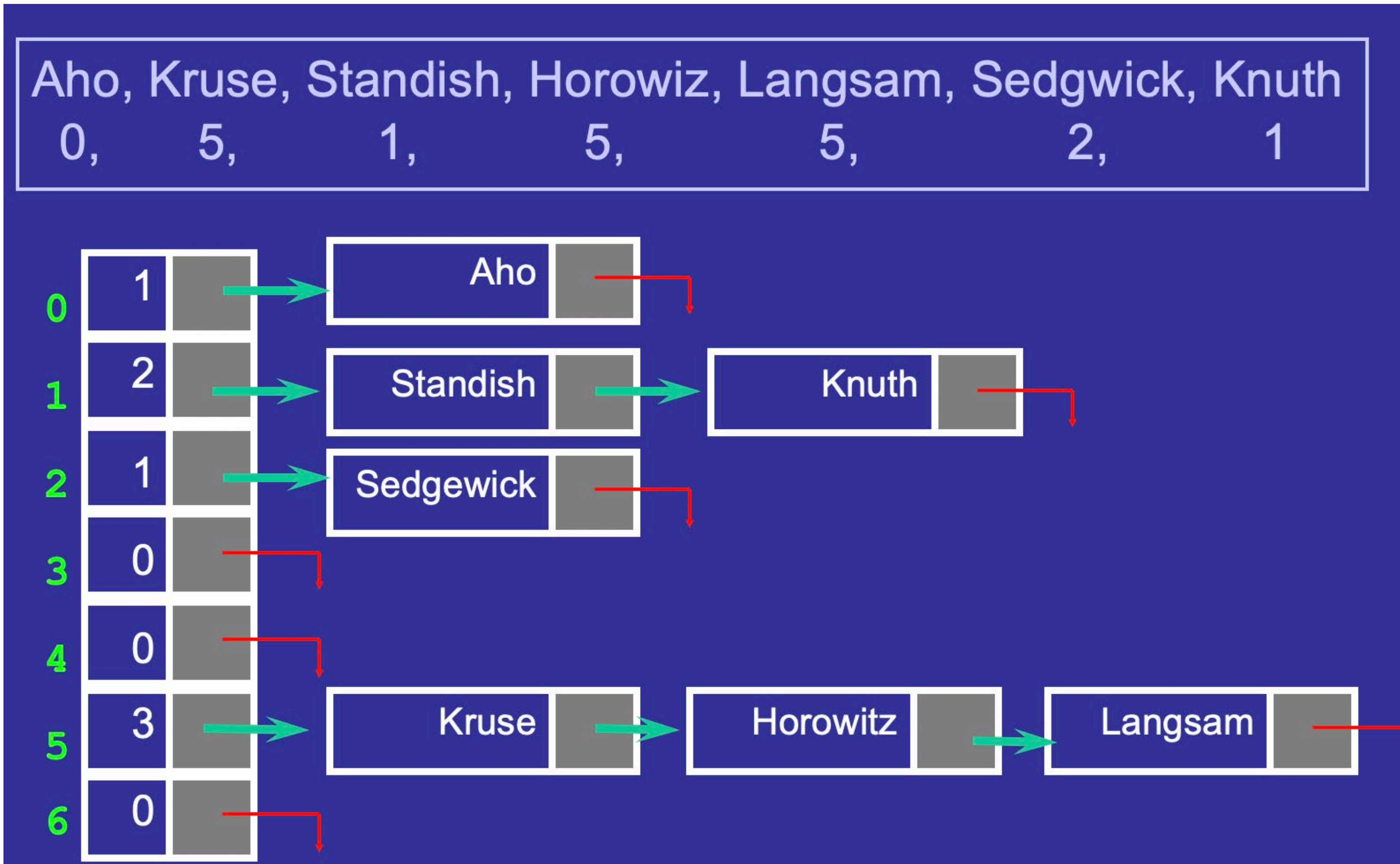
m number of slots in T

- We call h s hash function
- $h : U \rightarrow \{0, 1, \ldots, m - 1\}$, so that $h(k)$ is a legal slot number in T .
- We say that k hashes to slot $h(k)$.
- Collision
  - When two or more keys hash to the same slot

Chaining

- Uses a Linked List at each position in the Hash Table.

Chaining



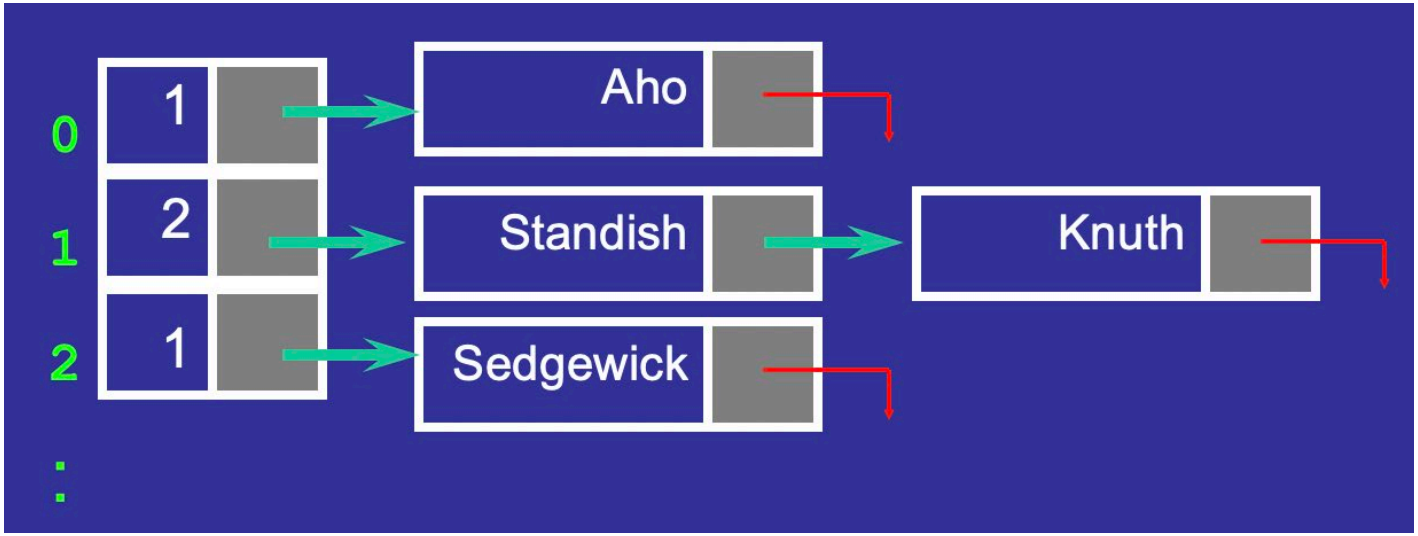
Aho, Kruse, Standish, Horowiz, Langsam, Sedgwick, Knuth
0, 5, 1, 5, 5, 2, 1
0
1
2
3
4
5
6
1
2
1
0
0
3
0
Aho
Standish
Knuth
Sedgewick
Kruse
Horowitz
Langsam

Insert with Chaining

- Apply hash function to get a position in the array.
- Insert key into the Linked List at this position in the array.

Insert with Chaining

```
module InsertChaining(item)
{
    posHash = hash(key of item)
    insert (hashTable[posHash], item);
}
```

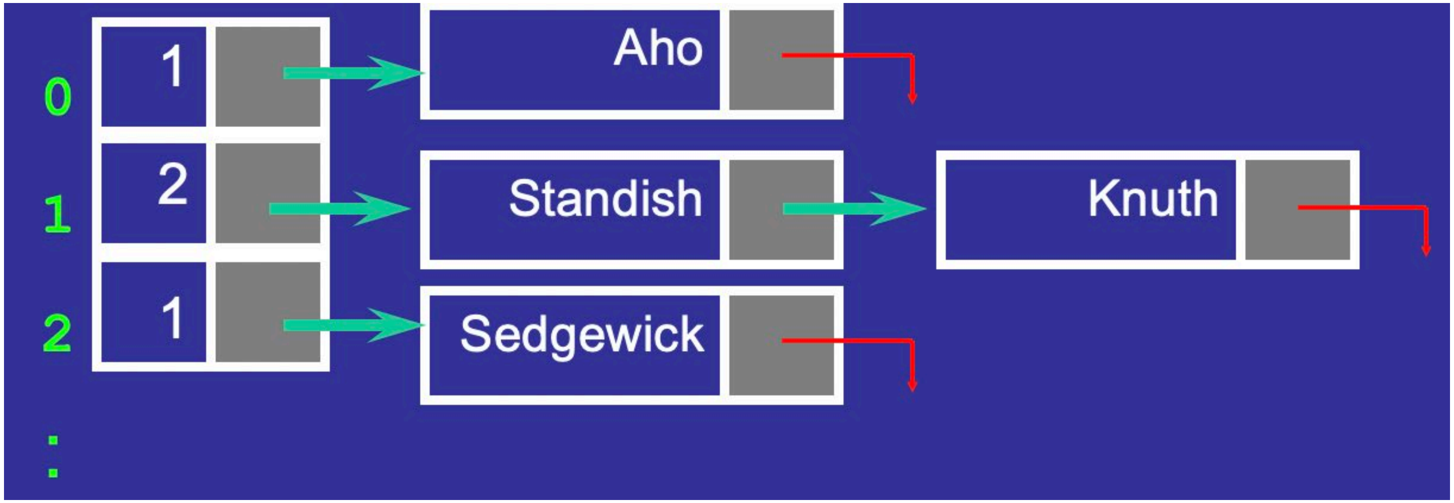


Search with Chaining

- Apply hash function to get a position in the array.
- Search the Linked List at this position in the array.

Search with Chaining

```
/* module returns NULL if not found, or the address of the
 *  node if found */
module SearchChaining(item){
   posHash = hash(key of item)
   Node* found;
   found = searchList (hashTable[posHash], item);
   return found;
}
```

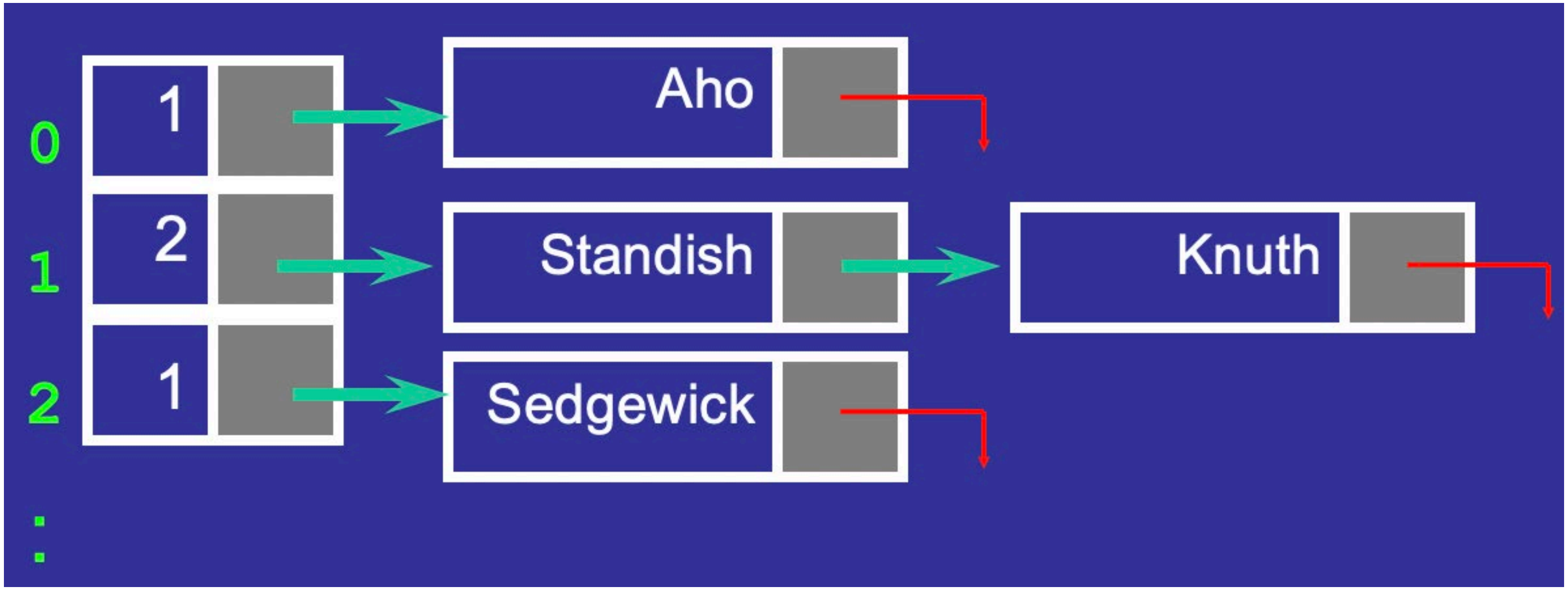


Delete with Chaining

- Apply hash function to get a position in the array.
- Delete the node in the Linked List at this position in the array.

Delete with Chaining

```
/* module uses the Linked list delete function to delete an item
 *inside that list, it does nothing if that item isn't there.  */
module DeleteChaining(item){
   posHash = hash(key of item)
   deleteList (hashTable[posHash], item);
}
```



Advantages of Chaining

- Insertions and Deletions are easy and quick.
- Allows more records to be stored.
- Naturally resizable, allows a varying number of records to be stored.

Disadvantages of Chaining

- Uses more space.
- More complex to implement.
  - A linked list at every element in the array

Collision Frequency

- Birthdays *or* the von Mises paradox
  - There are 365 days in a normal year
    - Birthdays on the same day unlikely?
  - How many people do I need before "it's an even bet" (*ie* the probability is > 50%) that two have the same birthday?
  - View
    - the days of the year as the slots in a hash table
    - the "birthday function" as mapping people to slots
  - Answering von Mises' question answers the question about the probability of collisions in a hash table

Distinct Birthdays

- Let $Q(n)$ = probability that n people have distinct birthdays
- $Q(1) = 1$
- With two people, the 2nd has only 364 "free" birthdays

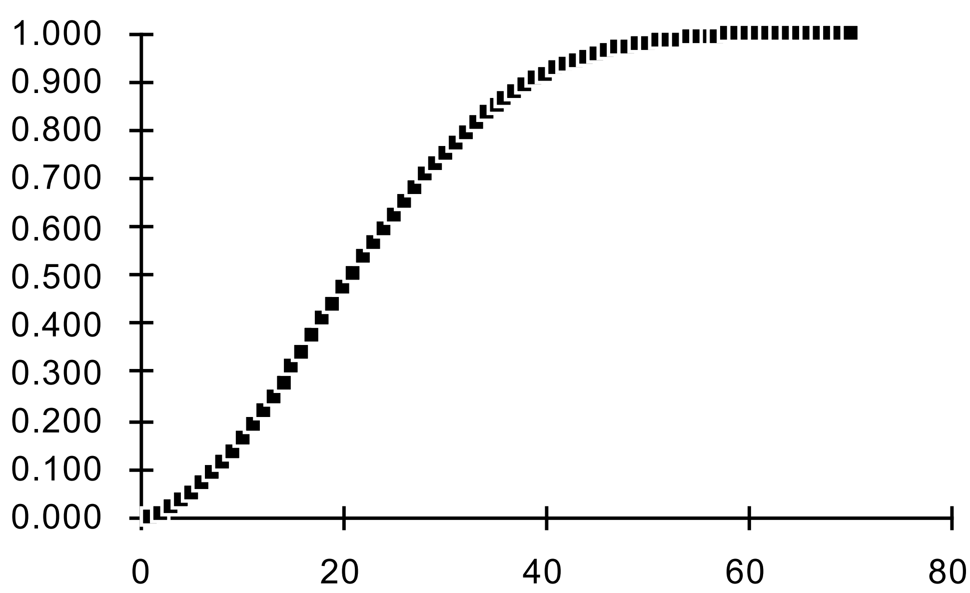
$$Q(2) = Q(1) * \frac{364}{365}$$

- The 3rd has only 363, and so on:

$$Q(n) = Q(1) * \frac{364}{365} * \frac{363}{365} * \ldots * \frac{365-n+1}{365}$$

Coincident Birthdays

- Probability of having two identical birthdays
- $P(n) = 1 - Q(n)$
- $P(23) = 0.507$



- With 23 entries, table is only 23/365 = 6.3% full!

Hash Tables - Load factor

- Collisions are very probable!
- Table load factor

$$\alpha = \frac{n}{m}$$

***n* = number of items**
***m* = number of slots**

must be kept low

- Detailed analyses of the average chain length (or number of comparisons/search) are available
- Separate chaining
  - linked lists attached to each slot

gives best performance

  - but uses more space!

Hash Tables - General Design

- Choose the table size
  - Large tables reduce the probability of collisions!
  - Table size, m
  - n items
  - Collision probability $\alpha = n / m$
- Choose a table organisation
  - Does the collection keep growing?
    - Linked lists (....... but consider a tree!)
  - Size relatively static?
    - Overflow area *or*
    - Re-hash

Hash Tables - General Design

- Choose a hash function
  - A simple (and fast) one may well be fine ...
  - Read your text for some ideas!
- Check the hash function against your data
  - Fixed data
    - Try various h, m until the maximum collision chain is acceptable
    - Known performance
  - Changing data
    - Choose some representative data
    - Try various h, m until collision chain is OK
    - Usually predictable performance

Hashing Functions

- The following are simple approaches which often work reasonably well:
- The Division method:
  - $h(k) = k \; mod \; m$
  - Good if m is prime and is not close to a power of 2 or a power of 10.
- The Multiplication method:
  - $h(k) = \lfloor m \, (kA \bmod 1) \rfloor$
  - A is a constant in the range $0 < A < 1$
  - $kA \; mod \; 1$ means the fractional part of kA, that is $kA - \lfloor kA \rfloor$.
  - $m = 2^p \; for \; some \; integer \; p$.

Hashing Functions…

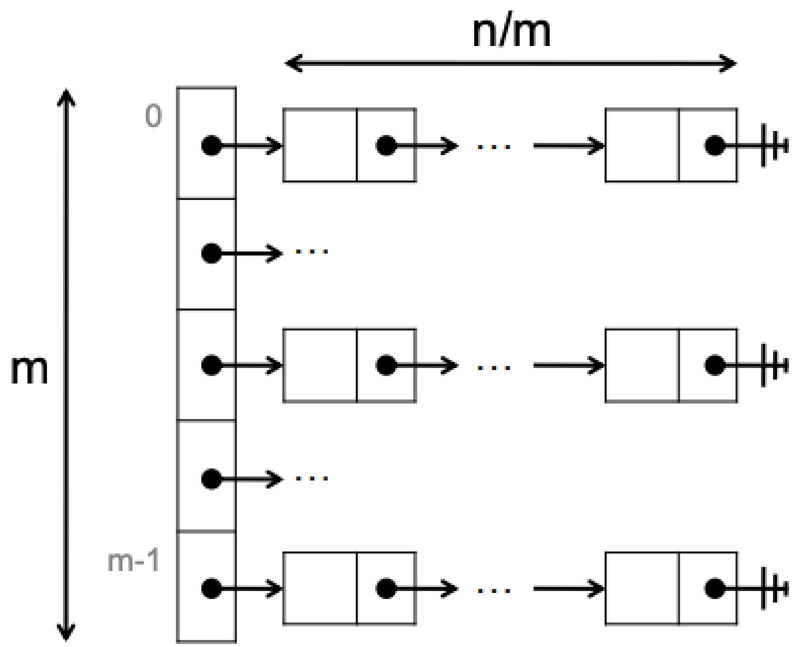
- Universal Hashing:
  - $h(k) = (a \times k + b \bmod p) \bmod m$
  - p is a prime number, bigger than $|U|$, the number of all possible keys.
    - Yes, p is BIG!
  - a and b are random integers between 0 and $p - 1$.
- This is an excellent hash function.
- The worst-case probability of two keys colliding is $1/m$.
- This means that, even if a and b are poorly chosen, this hash function will always work well.
- The problems of finding a large prime and performing arithmetic on big integers will be left for now.

Worst Case

- What if $h(key)$ has the same value for all the keys in our set?
- Our hash table has just become a complicated way of storing a single linked list!
- Access to a given $key{:}\,value$ pair is now $O(n)$.
- So, should we give up on hashing?
  - No!
  - In practice this does not happen.

Best Possible Layout

- All slots have the same number of keys
- Each chain has the same length $\frac{n}{m}$ (load factor)
- Operations on a dictionary are $O(1 + n/m)$,



Picking up m

- m is the number of slots in the dictionary, to be $\Theta(n)$, where n is the number of entries in the dictionary.
- Operations on a dictionary are $O(1 + n/m)$, so if n grows too large we get less and less efficient.
- The problem we face is that, often, we do not know how many records n we will need to store.
  - If m is too small, the dictionary becomes inefficient.
  - If m is too large, we waste storage (memory or disc).
- How do we get the right value for m?
- Let's say we want $m \geq n$ at all times.

Lucky Guess?

- If we have no knowledge of the ultimate size of n, what can we do?
  - Guess.
  - Pick m based on an optimistic assessment of the likely size of n.
  - No idea?
    - Pick your favourite small number.
    - m = 8, say.
  - Now what?
    - What if n turns out to be greater than 8?
  - Make m bigger.
    - How much bigger?

Changing m

- When n becomes large, change m
- If we change m we have problems:
  - Our hash array is too small.
  - Our hashed keys will be wrong.
    - They depend on the value of m.
- Does this mean that we have to recreate the hash table from scratch?
  - It sure does.
  - Isn't this a BAD THING™?

Growing(Resizing) a Hash Table

- What exactly has to happen if we change m?
  - Let's say the new table size is m'.
- We now need a new array with m' elements.
  - We also need to move all of the existing elements from the old table to the new one.
- Build a new hash function h'.
  - Remember, the hash function depends on m'.
- Insert the existing data into the new table.
  - This involves re-hashing every key.
- So, the first question is:
  - How much do we grow m?

$m' =?$

- Let's look at some options:
  - $m' = m + 1.$
  - What is the cost of n insertions?
    - $\Theta(1)$ for the first m insertions.
    - $\Theta(m')$ for each insertion after that.
  - Overall $\Theta(n^2)$

$m' = ?$

- $m' = 2m$
  - $\Theta(1)$ for the first m insertions.
  - $\Theta(m)$ for the next insertion.
  - $\Theta(1)$ for the next $m-1$ insertions.
  - $\Theta(2m)$ for the next insertion.
  - $\Theta(1)$ for the next $2m-1$ insertions.
- Overall $\Theta(n + (n/2) + (n/4) + \cdots) = \Theta(2n) = \Theta(n)$
- The cost of expanding the table gets spread over the extra elements we are making room for.
- This is known as Amortized cost.
- Note: an amortized cost of $\Theta(1)$ per operation does not mean that every operation has this cost.
  - Just that this is the average cost per operation.

Amortized Cost

- We say an operation has a cost of "$T(k)$ Amortized" if k operations take a total of $k \times T(k)$ time.
- Table doubling takes $\Theta(n)$ operations for n insertions so the amortized cost is $\Theta(1)$.
- This is, actually, a GOOD THING™.
- Note: we can use table doubling to implement any solution where we do not know the size of the data structure in advance and it grows in a "well behaved" way.
- Table doubling minimizes the cost associated with dynamic data structures.

Deletions

- What about deletions?
  - Each deletion is still $\Theta(1)$.
  - They simply increase the number of operations (insertions and deletions) we can perform between doublings.
- What if it's all deletions?
  - In this case the table becomes progressively less and less full.
  - Solution: Shrink the table.
- How, exactly?

Shrinking a Hash Table

- What should our strategy for reducing the size of the table be?
- How about "if $n < m/2$ make $m' = m/2$"?
- What if the next operation is an insertion?
  - Double the table size!
  - Then a deletion?
  - Halve the table!
  - Insertion?
  - Double…
- We now have $\Theta(n)$ operations for each change in the data.
- Instead use "if $n < m/4$ make $m' = m/2$".

Hashing With Chaining Considered Bad

- There is still one small issue with this method.
  - We have a hybrid data structure—an array of linked lists.
- A second approach uses just a simple array.
- Clearly, we still have a potential problem with collision
  - two keys which hash to the same value.
- We resolve this with a technique known as Open Addressing.

Open Addressing

- An Alternative to Chaining
- We wish to hash n items into an array with m slots.
- We may only store one item per slot.
- Clearly, $m \geq n$.
- We insert an item into the table using an iterative technique known as probing.

Probing

- This process works as follows: (for insertion)

```
Set hash function to starting value, h0
repeat
    calculate probe = hash(key)
    if table(probe) contains data then
            go to the next hash function
    else
    store the item in table(probe)
    fi
until we have stored the item
```

- This means we must have a sequence of hash functions, h0, h1, h2 …
- … or a hash function which produces a sequence of values.

The Hash Function

- Our new hash function requires two arguments:
  - The key;
  - The iteration count.
- Thus: $probe = OpenHash(key, count)$
- Here:
  - key is a valid element of U, the universe of keys;
  - count is a non-negative integer.
  - As usual, $0 \leq probe < m - 1$.

The Hash Function…

- In addition, we want our hash function to have the following property:
- For any arbitrary key k the sequence of m probes:
  - $h(k,0), h(k,1), h(k,2), \ldots, h(k,m-1);$
- Must be a permutation of the integers:
  - $0, 1, 2, \ldots, m-1.$
- This property guarantees that we must eventually find a vacant slot to insert the item into.
- Clearly, the sequence of probes must be different for different keys.
- We can see this with an example.

Example: Insertion with Open Addressing

- Consider the following table:

k	h(0,k)	h(1,k)	h(2,k)	h(3,k)	h(4,k)	h(5,k)	h(6,k)	h(7,k)	h(8,k)	h(9,k)
899	9	8	5	6	0	7	8	2	4	1
950	5	7	4	9	2	3	1	6	8	0
12	3	8	7	2	5	9	1	6	0	4
367	7	1	2	3	4	5	6	8	9	0
359	2	1	9	5	6	7	3	8	0	4
980	4	7	1	8	9	3	0	5	2	6
229	0	8	2	7	1	6	3	9	4	5
598	8	6	3	5	0	7	9	1	4	2
838	6	2	6	7	1	3	8	2	0	2
549	9	8	4	6	7	5	0	1	2	3

- Let us insert the keys into our hash table in order

Example: Insertion with Open Addressing…

k	h(0,k)	h(1,k)	h(2,k)	h(3,k)	h(4,k)	h(5,k)	h(6,k)	h(7,k)	h(8,k)	h(9,k)
899	9	8	5	6	0	7	8	2	4	1
950	5	7	4	9	2	3	1	6	8	0
12	3	8	7	2	5	9	1	6	0	4
367	7	1	2	3	4	5	6	8	9	0
359	2	1	9	5	6	7	3	8	0	4
980	3	7	1	8	9	4	0	5	2	6
229	0	8	2	7	1	6	3	9	4	5
598	8	6	3	5	0	7	9	1	4	2
838	6	2	4	7	1	3	8	2	0	2
549	9	8	4	6	7	5	0	1	2	3

0	1	2	3	4	5	6	7	8	9
229	980	359	12	549	950	838	367	598	899

Search with Open Addressing

- The procedure used to search using open addressing is similar to insertion.

```
count=0
repeat
    probe=hash(key, count)
    if table(probe)==key then
        return item
    else
        count++
    fi
until table(probe)==empty or count==n
return not found
```

- This is pretty straightforward.

Deletion with Open Addressing

- When we get to deletion we have a new problem.

```
count=0
repeat
    probe=hash(key, count)
    if table(probe)==key then
        delete item
        return
    else
        count++
    fi
until table(probe)==empty or count==n
return not found
```

- How, exactly, do we delete the item?

Deletion…

- If we simply replace the item with our empty value we will have an issue:
  - What if the key we next search for is after the probe corresponding to the deleted key's location.
  - If, in our previous example, we delete 899, where h(899,0)=9, and then search for 549, where the sequence of hash values are 9, 8, 4…
    - We test D(9) and discover it has the value empty.
    - We conclude that 549 is not in the table.
    - Wrong! It is in D(4).
- To fix this we need a second special value, deleted.

0	1	2	3	4	5	6	7	8	9
229	980	359	12	549	950	838	367	598	899

Deletion...

- Our deletion process becomes:

```
count=0
repeat
    probe=hash(key, count)
    if table(probe)==key then
        table(probe)==deleted
        return
    else
        count++
    fi
until table(probe)==empty or count==n
return not found
```

- This fixes search but introduces a problem with insertion.

Insertion Revisited

- We note that we can insert a new item into the dictionary in two circumstances:

```
D(i)==empty
D(i)==deleted
```

- We modify our insert process as follows:

```
count=0
repeat
    probe=hash(key, count)
    if table(probe)==empty or table(probe)==deleted then
        store item in table(probe)
        return
    else
        count++
    fi
until count==n
return no room
```

- Now we can insert into the first vacant slot, empty or deleted, that we find in the table.

Search Revisited

- Because empty and deleted are different, we do not have to modify our search procedure.

- The search will skip over deleted records because they do not match the key but will still terminate when it reaches an empty record.

Open Addressing Hash Functions

- One question remains.
- Can we find a function $h(k, i)$ which is:
  - Easy to compute;
  - Produces a permutation of $\{0, 1, \ldots, m-1\}$ as i varies over $\{0, 1, \ldots, m-1\}$?

- Let us examine two possible strategies.

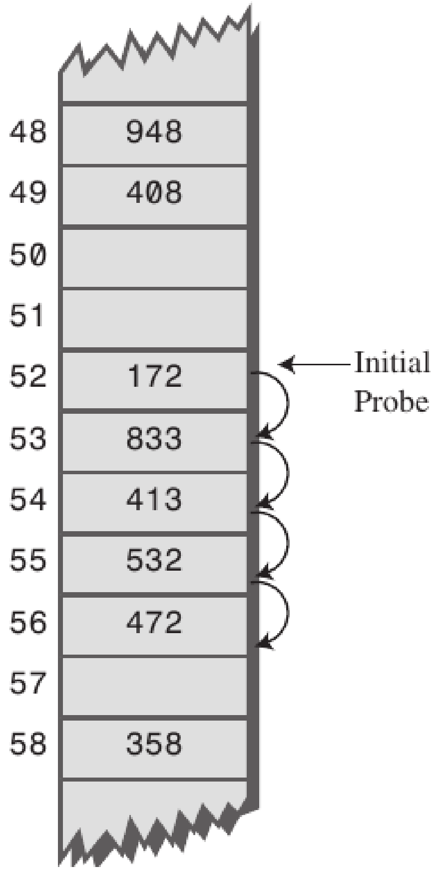
Strategy I : Linear Probing

- In this approach we simply take a standard hash function, $h(k)$ and compute the probe $p(k, i)$ as follows:
  - $p(k, i) = (h(k) + i)\ mod\ m$

- In other words, we simply look at sequential entries in the dictionary starting at the entry corresponding to $h(k)$.
  - This is certainly easy to compute.
  - It does satisfy the permutation.
- There are $\Theta(m)$ distinct probing sequences

Strategy I ; Linear Probing

- Is it any good?
  - No! It produces sets of consecutive occupied slots.
  - Primary Clustering – tendency to create long runs of filled slots near the hash position of keys
- The bigger the cluster, the more likely it is to be hit..
  - ...and it gets even bigger!



Strategy II: Double Hashing

- In this strategy we have two standard hash functions, $h_1(k)$ and $h_2(k)$.
- We compute $p(k)$, our probe value as follows:
  - $p(k,i) = (h1(k) + i \times h2(k))\, mod\, m.$
- Do we still satisfy our requirements?
  - This is still easy to compute.
  - Do we always get a permutation?
    - No.
    - Unless we are clever in how we define h_2.

Choose h_2

- We need $h_2(k)$ to be relatively prime to m.
  - i.e. $h_2(k)$ and m must have no common factors except 1.
- This is easy in many cases.
- If we select m to be a power of 2; say $m = 2^r$ then all we need is for $h_2(k)$ to always be an odd number.
- For example, if we have a standard hash function $h'(k)$, we can create $h_2(k)$ as follows:
  - $h_2(k) = (2h'(k) + 1)\ mod\ m$
- There are $\Theta(m^2)$ probing sequences
  - Each possible $(h_1(k), h_2(k))$ pair yields a distinct probe sequence

Table Doubling

- Once again, we need to expand the dictionary whenever it becomes too full.
- What does "too full" mean in this case?
- We define the occupancy of a table, α, to be the ratio of n, the number of entries to m, the number of slots.
  - $\alpha = n/m$
  - $0 \leq \alpha \leq 1$
- We can show that the average cost of an operation on a table with occupancy α is in $\Theta(1/(1-\alpha))$.
- In practice we want this value to be reasonably close to 1 so we double as soon as α exceeds 0.5 or thereabouts.
- This keeps operations between $\Theta(1)$ and $\Theta(2)$.

An Important Note on α

- When calculating the occupancy value, α, we must count slots with a value of deleted as containing data.
- This is because some operations, notably searching, treat deleted records as still containing data.
- Slots containing deleted may be removed in two ways:
  - Being overwritten with valid data as a result of an insert operation;
  - Being cleaned up when the table is expanded.
- If we did not count deleted records in calculating α we could have a notionally empty table in which every slot was deleted.
- Search (and delete) in this table would be $\Theta(m)$, not $\Theta(1)$, as we might expect.

Chaining vs. Open Addressing

- So, which is the better scheme?
- Open Addressing:
  - Uses less memory—no need for pointers;
  - Is faster—provided α is kept below 0.5;
  - Is a little harder to implement and understand.
  - Is clean—one data structure, the array.
- Chaining:
  - Uses more memory;
  - Is faster—if we are not careful with open addressing.
  - Is a little easier to implement and understand.
  - Is a bit messy—arrays of linked lists.

Related References

- Introduction to the Design and Analysis of Algorithms, A. Levitin, 3rd Ed., Pearson 2011.
  - Chapters 7.3
- Introduction to Algorithms, T. H. Cormen, 3rd Ed, MIT Press 2009.
  - Chapters 11