CSCI203

Algorithms and Data Structures

Graphs (II)

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Tweaking Dijkstra

- As we have already seen, Dijkstra's algorithm provides an effective solution strategy for solving the single source-all destinations version of the shortest path problem.
- We will now look at a few simple modification of Dijkstra that will:
  - Improve its practical performance;
  - and/or extend its range of applicability.

Recall Dijkstra's Algorithm

```
Procedure Dijkstra(G: array[1..n, 1..n]): array [2..n]
   D: array[2..n], P: array[2..n]
   C: set = {2, 3, …, n}
   for i = 2 to n do
        D[i] = G[1, i]
        P[i]=1
   od
   repeat
        v = the index of the minimum D[v] not yet selected
        remove v from C // and implicitly add v to S
        for each u ∈ C do
           D[u] = min(D[u], D[v] + G[v, u])
           p[u] = v
        rof
   until C contains no reachable nodes
   return D
end Dijkstra
```

Overall Efficiency

- The algorithm as you have seen it so far has efficiency $O(|V|^2 + |E|)$.
- But it was $O(|V| * \log|V| + |E|)$, How come?
- The answer is simple:
  - As presented, we find the next vertex to select by searching a list of candidate vertices and selecting the vertex with minimum D value.
    - This is a linear search process; $O(|V|)$.
    - We do this for each vertex, also $O(|V|)$.
    - This is $O|V^2|$.
- So, how do we improve on this?

Reaching Peak Efficiency

- The answer is surprisingly simple.
- Replace the candidate list/array C with...
- ...a priority queue (or a heap), ordered on $D(v)$.
- Now:
  - Finding the best candidate is $O(1)$;
  - Updating C is $O(\log |V|)$.
- Now, over all vertices, we have $|V| * \log |V|$.

All Sources - Single Destination

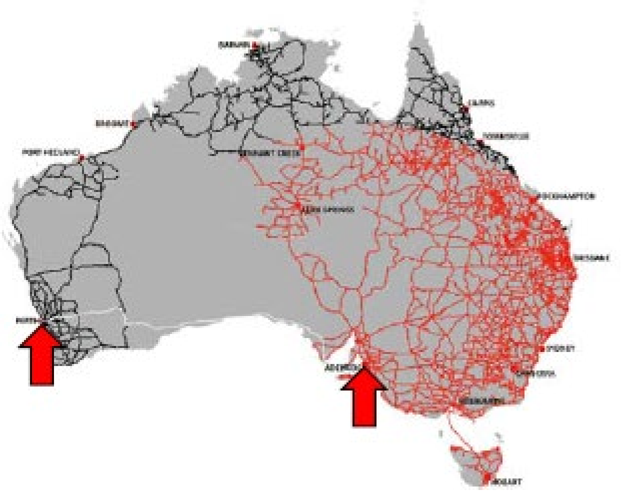
- In this case, rather than finding paths from a starting vertex, s, to all other reachable vertices we are looking for the shortest paths to some goal vertex, g, from all possible starting vertices.
- How do we do this?
  - Run *Dijkstra backwards*.
- Specifically:
  - Redefine $Adj(v)$ to be the list of set of vertices leading to vertex v…
    - …instead of reachable from v;
  - Start with $D(g) = 0$…
    - …instead of $D(s) = 0$;
  - Let $P(v)$ indicate the next vertex in the path…
    - …instead of the prior vertex;
  - Let the selected set, S, start at $\{g\}$…
    - …instead of $\{s\}$.

Single Source - Single Destination

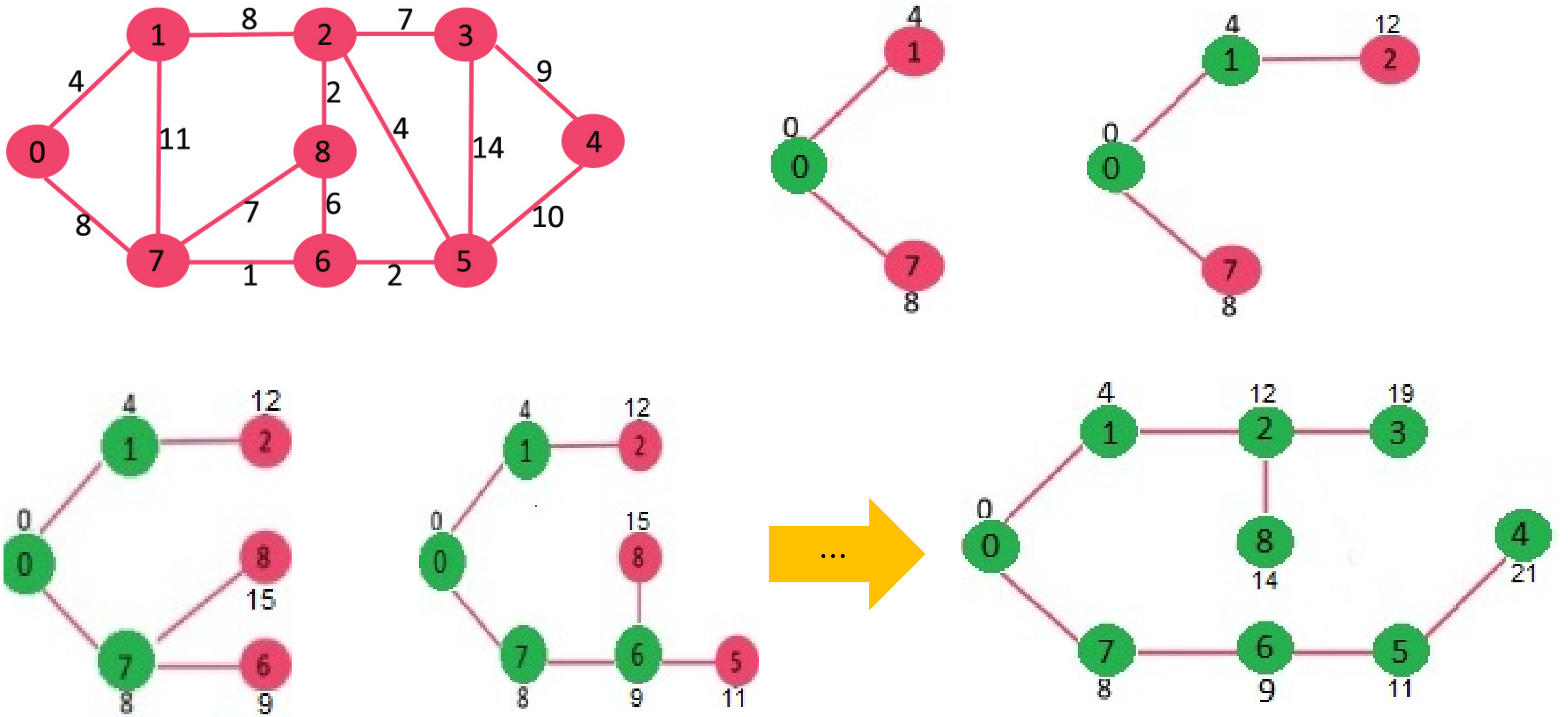
- What do we do if, rather than looking for the shortest paths from a start vertex, s, to all other vertices, we wish to find the shortest path from s to a specific goal vertex, g?
- The answer is easy:
  - Stop when vertex g becomes a member of S, the selected set.
- This means that we do not waste time with any vertex further away from s than g.
- This usually reduces the total running time of the algorithm.
  - Why?

The Problem with Dijkstra's

- There is a big problem with using Dijkstra on the single source/single destination problem:
  - The order in which the vertices are added.
- Consider the following graph:
  - Say we want to get from Adelaide…
  - …to Perth.
- Dijkstra will add all of the closer vertices first:
- Before we ever get close to the path we seek.



Another Example

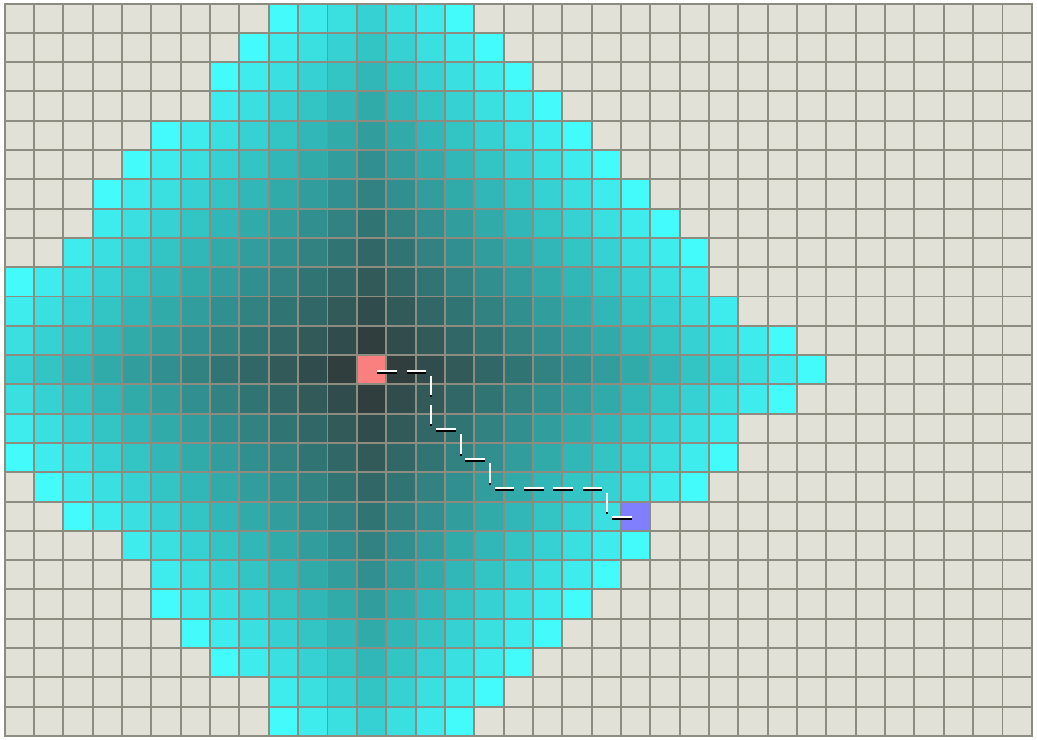


$$v_0 \to v_4$$

Another Example

Dijkstra Algorithm

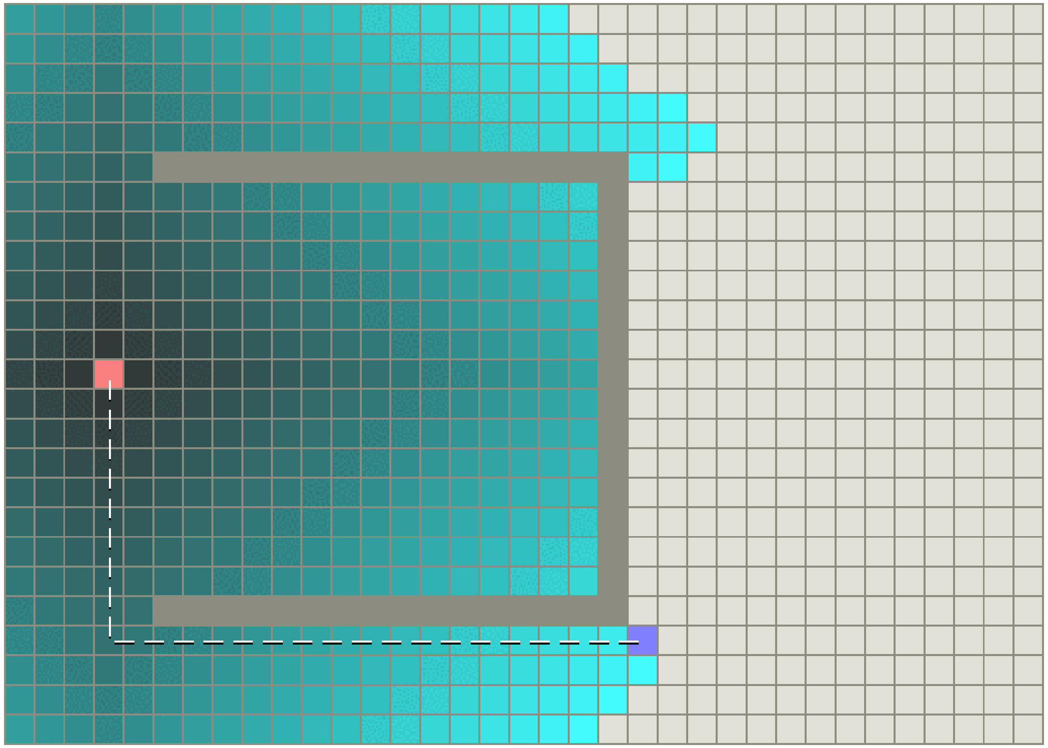
- Repeatedly examines the closest not-examined vertex
- Expands outwards until reach the goal
- Guarantee to find a shortest path
- But have a large teal area



Another Example

Dijkstra Algorithm

- The shortest path is not a straight line when has obstacles
- Works harder but is guaranteed to find a shortest path



Fixing the Problem

- So, how do we remedy this?
- Before we get to the answer let us take a step back.
- Let us generalize Dijkstra's algorithm.
- The key step in the algorithm is on the process by which we select the next vertex.
  - Specifically, we select the vertex in the candidate set, C, for which the overall distance to the vertex from the source vertex, s, is minimized.
  - $D(v) = P(s, v)$.
- Note that this does not involve the goal vertex, g.

Eyes on the Goal

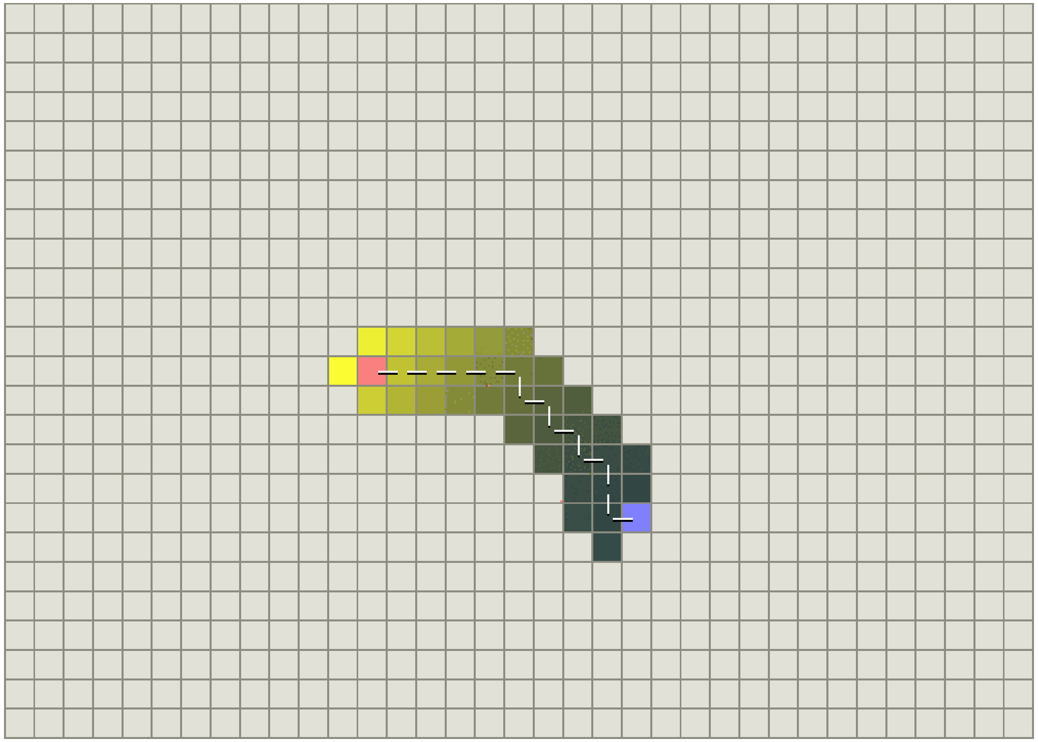
- What if we could bias the selection towards the goal in some way.
- We can!
- Essentially, we select the minimum not simply of $D(s, v)$ but, instead of $D(s, v) + H(v, g)$.
- This new function, H, is a *heuristic*; an estimate of the remaining distance from each candidate vertex, v, to the goal vertex, g.
- What is a good estimator?

Greedy Best-First-Search Algorithm

- Similar as the Dijkstra algorithm but with a heuristic
- Instead of selecting the vertex closest to the starting point, it selects the vertex closets to the goal.
- Not guarantee to find a

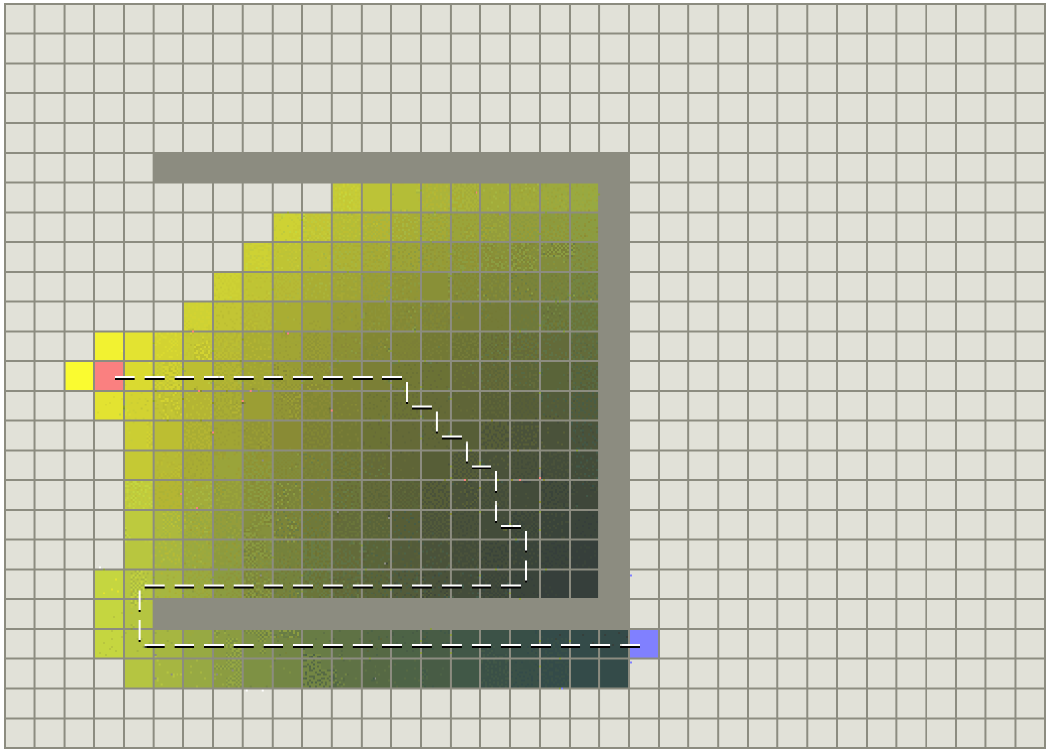
shortest path

- But it is much quicker



Greedy Best-First-Search Algorithm

- Does less work and faster
- But the path is not the shortest



Can we combine the Dijkstra and the Greedy algorithms?

Greedy Best-First-Search Algorithm

```
procedure GBS(start, target) is:
    mark start as visited
    add start to queue
    while queue is not empty do:
        current_node ← vertex of queue with min distance to
          target
        remove current_node from queue
        foreach neighbor n of current_node do:
            if n not in visited then:
                if n is target:
                    return n
                else:
                    mark n as visited
                    add n to queue
return failure
```

A Good Heuristic

- We require $H(v, g)$ to have one key property:
  - $H(v, g) \le P(v, g)$;
  - The heuristic estimate must never exceed the actual shortest path length.
  - This requirement guarantees that the final path we find is still the correct answer.
- In our example we have a ready-made heuristic…
- …The Euclidean (straight-line) distance between v and g.
- Provided the graph behaves according the rules of geometry there can never be a shorter path than this.

A^*

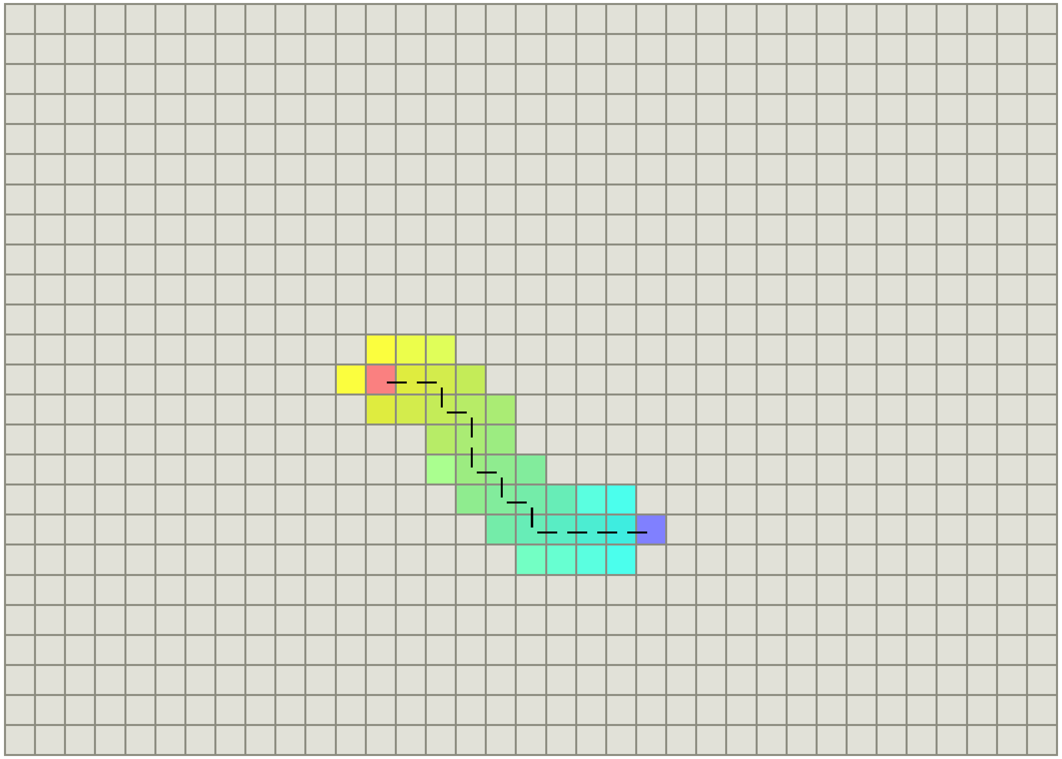
- The heuristic modification to the vertex selection rule changes Dijkstra's algorithm into an example of what is called the A^* algorithm.
- Although, in the worst case, A^* is no faster than Dijkstra in practice it will generally represent a substantial improvement.
- Note: the trick to A^* is finding a good heuristic.
- The nearer that $H(v, g)$, the estimated minimum path length from v to g, is to $P(v, g)$, the actual minimum path length, the faster A^* will find the solution.

Through the Looking Glass

- Because of the order in which we saw them, it is easy to think of A^* as a generalization of Dijkstra's algorithm.
- This is not the only, or perhaps even the best, way to view this.
- Consider instead this viewpoint:
- Dijkstra's algorithm is simply a special case of A^*:
- The one with the worst possible choice of H.
- Specifically, $H(v, g) = 0$.

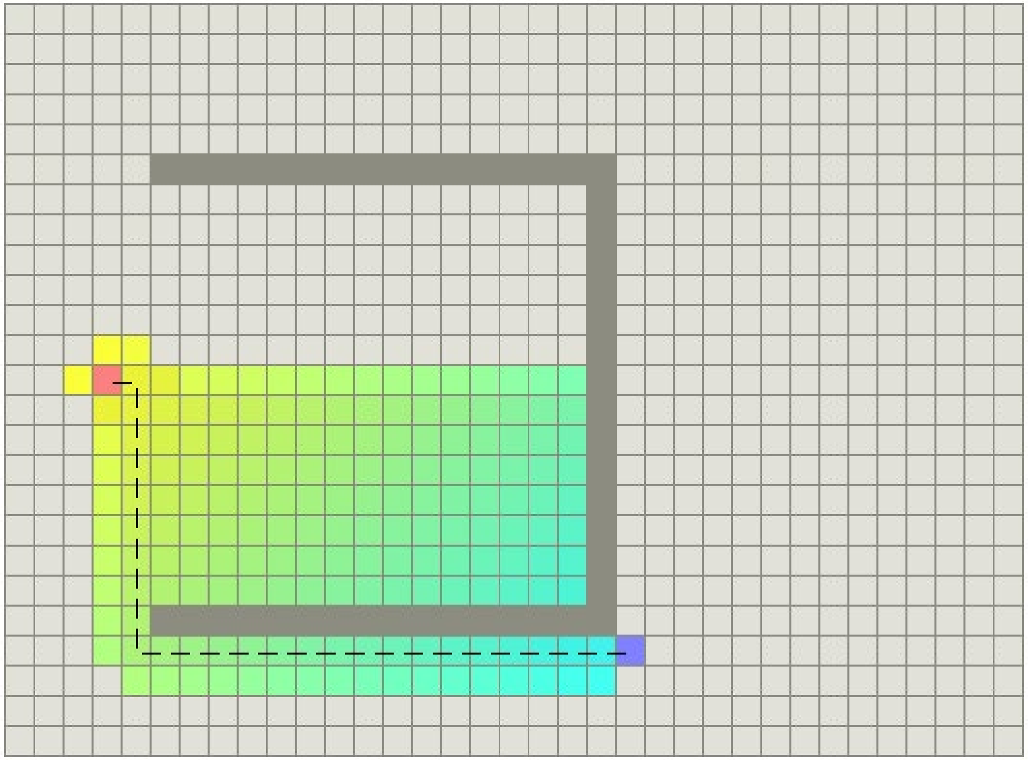
A^*

- A^* algorithm is like Dijkstra's algorithm to find the shortest path
- A^* algorithm also is like Greedy Best-First-Search in that it can use a heuristic to guide itself
- In short, A^* algorithm is as reliable as Dijkstra's algorithm and as fast as Greedy Best-First-Search algorithm

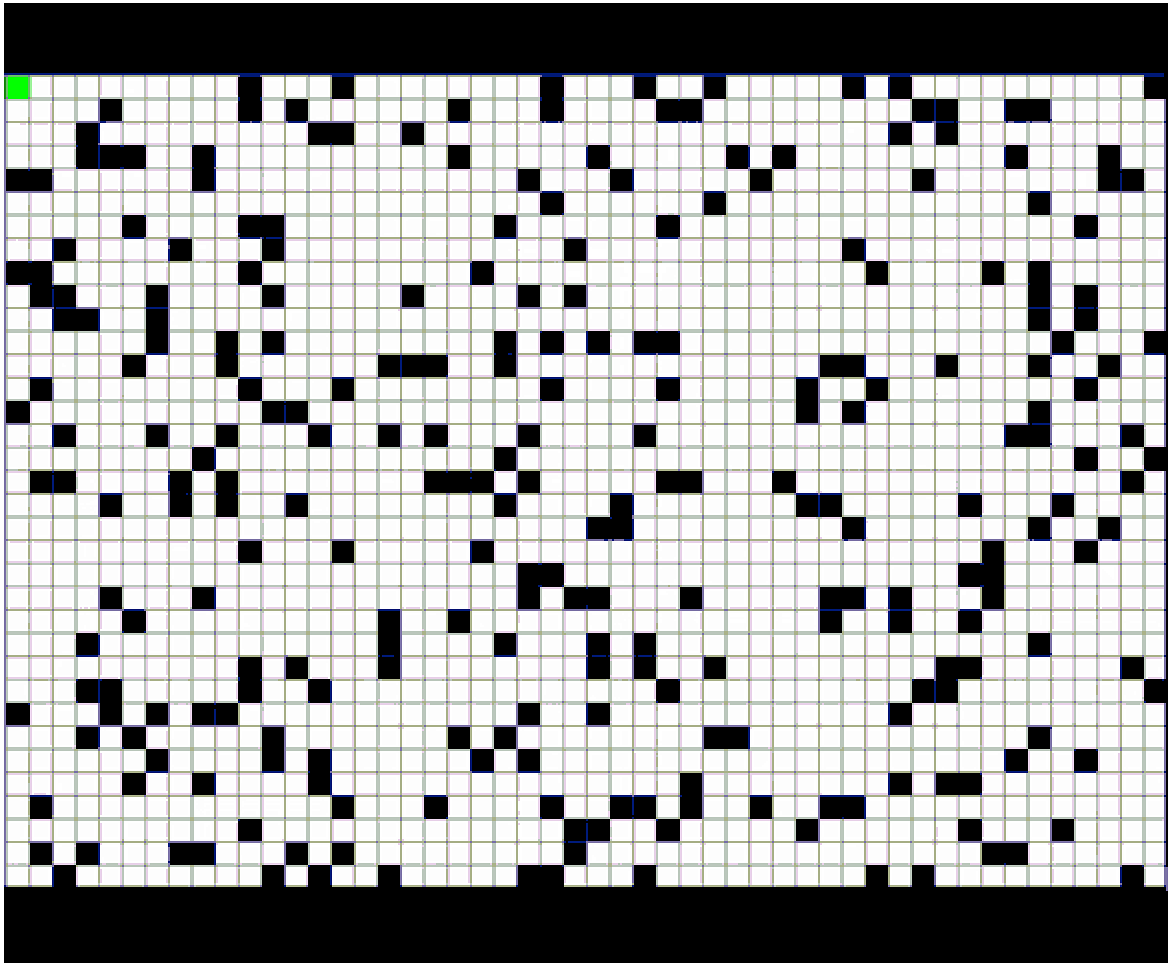


A^*

- A^* algorithm finds a path as good as what Dijkstra's algorithm found
- But A^* algorithm is faster.
- Because it combines the Dijkstra's algorithm with the heuristics.



A^*

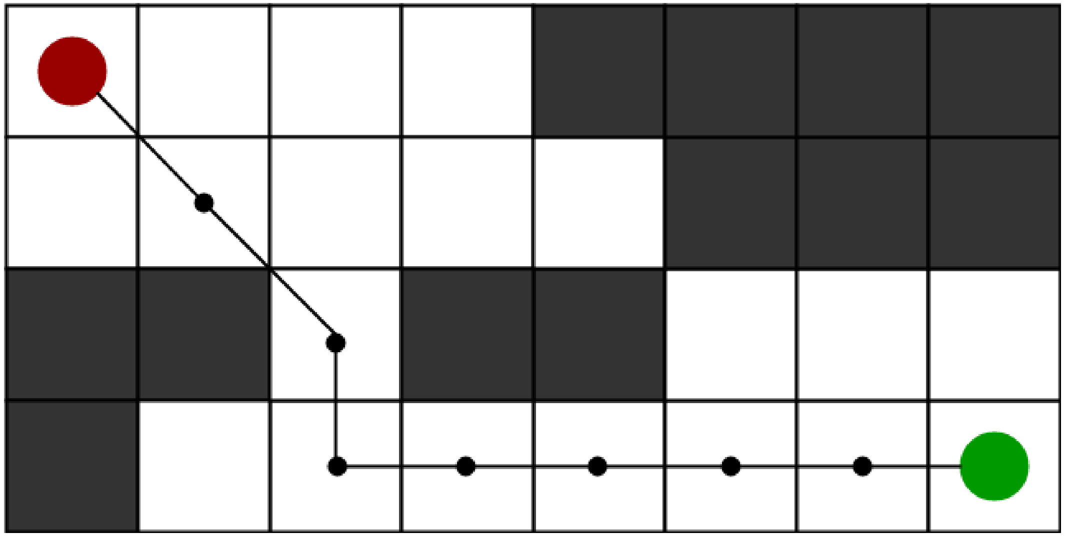


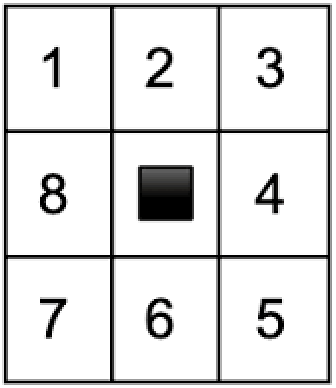
1	2	3
8		4
7	6	5

```
// A* Search Algorithm
1. Initialize the open list - a list of nodes to be explored
2. Initialize the closed list - a list of nodes in the path so far
   put the starting node on the open list (you can leave its f at zero)
3. while the open list is not empty
     a) find the node with the least f on the open list, call it "q"
     b) pop q off the open list
     c) generate q's non-blocked successors from the 8 ones and set their parents to q
     d) for each successor
         i) if successor is the goal, stop search
         successor.g = q.g + distance between successor and q
         successor.h = distance from goal to successor
         // This can be done using many ways, we will discuss three heuristics-
         // Manhattan, Diagonal and Euclidean Heuristics
         successor.f = successor.g + successor.h
         ii) if a node with the same position as successor is in the OPEN list
             which has a lower f than successor, skip this successor
         iii) if a node with the same position as  successor is in the CLOSED list
              which has a lower f than successor, skip this successor otherwise,
              add the node to the open list
      end (for loop)
      e) push q on the closed list
   end (while loop)
```

An Example

- We can consider a 2D Grid having several obstacles and we start from a source cell (green), s, to reach towards a goal cell (red), z
- We want to reach the target cell (if possible) from the starting cell as quickly as possible.





https://www.geeksforgeeks.org/a-search-algorithm/

An Example

In the A^* Algorithm

- At each step, it picks the node (while cell), x according to a value $f(s, z) = g(s, x) + h(x, z)$
  - $g(s, x)$ – cost/distance to moved from s to x
  - $h(x, z)$ – estimated cost/distance, heuristic, to move from x to z
- We don't know the actual $f(s, z)$ until we find the path
- There are many ways to calculate $h(\,)$

Heuristic h

- We can calculate g but how to calculate h ?
- We can do things.
  - Either calculate the exact value of h (which is certainly time consuming). OR
  - Approximate the value of h using some heuristics (less time consuming).

Heuristic h - Exact Heuristic

- We can find exact values of h, but that is generally very time consuming.

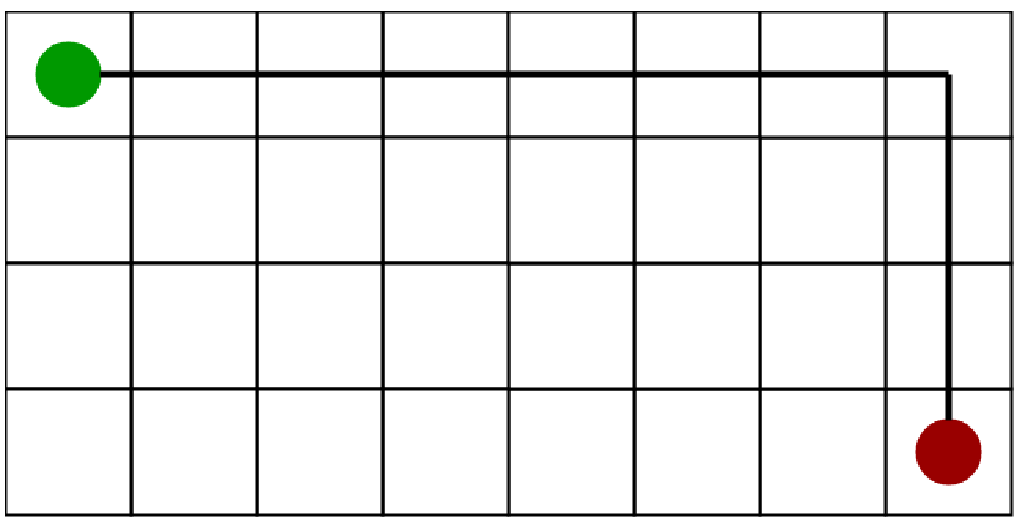
- e.g. Pre-compute the distance between each pair of cells before running the A* Search Algorithm.

Heuristic h -Approximation

- Manhattan Distance –

$$h = abs\,(current\_cell.x - goal.x) + abs\,(current\_cell.y - goal.y)$$

  - The sum of absolute values of differences in the goal's x and y coordinates and the current cell's x and y coordinates respectively,
  - When to use this heuristic? – When we are allowed to move only in four directions only (right, left, top, bottom)

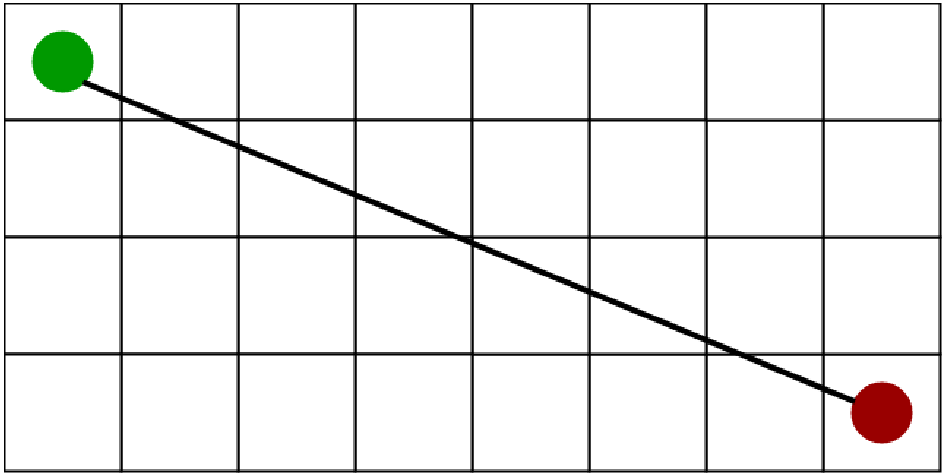


Heuristic…Approximation

- **Euclidean Distance-**
  - the distance between the current cell and the goal cell using the distance formula

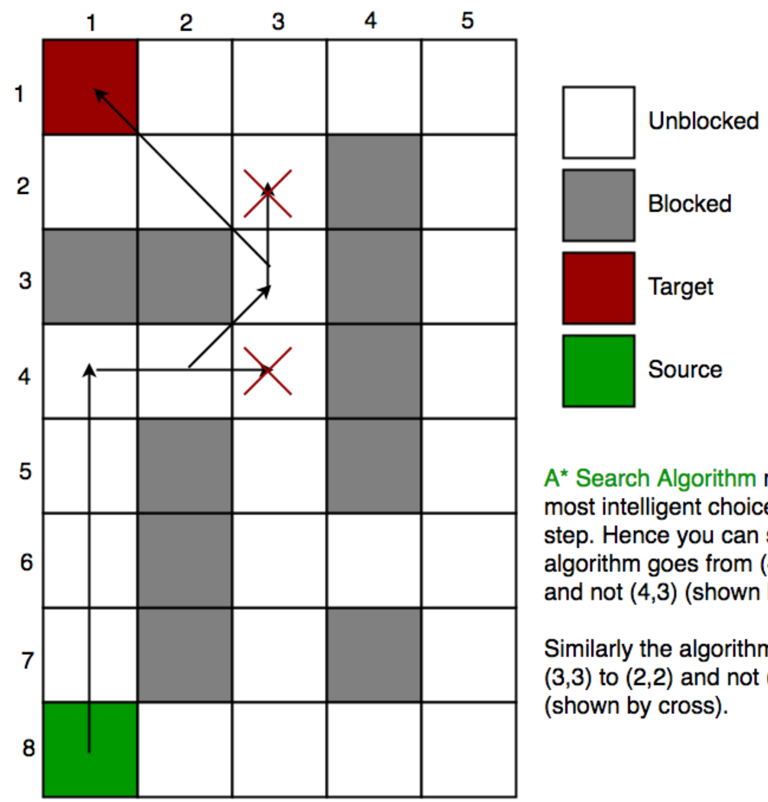
    $$h = sqrt\ (\ (current\_cell.x - goal.x)^2 + (current\_cell.y - goal.y)^2\ )$$

  - When to use this heuristic? – When we are allowed to move in any directions.



An Example

- A* Search algorithm would follow path as shown below if Euclidean Distance is chosen as a heuristics.



A* Search Algorithm makes the most intelligent choice at each step. Hence you can see that algorithm goes from (4,2) to (3,3) and not (4,3) (shown by cross).

Similarly the algorithm goes from (3,3) to (2,2) and not (2,3) (shown by cross).

Related References

- Introduction to the Design and Analysis of Algorithms, A. Levitin, 3rd Ed., Pearson 2011.
  - Chapters 9.3
- Introduction to Algorithms, T. H. Cormen, 3rd Ed, MIT Press 2009.
  - Chapters 24.1 and 24.3