Clarification on the typos in Lemmas 5.6 and 5.7

1. The origin of the typo

To clarify how this typo arose, we first present he ambitious sample-based design and explain why it fails:

$$Q_{t+1} = (Q_t - \langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x_{t-1}) \rangle + 2\langle \pi_t, \breve{\mathbf{g}}_t(x_t) \rangle)^+.$$

For this update, we have the following results for the drift term:

$$\Delta_t \leq Q_t \langle \pi_t, \breve{\mathbf{g}}_t(x_t) \rangle - \langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x_{t-1}) \rangle \langle \pi_t, \breve{\mathbf{g}}_t(x_t) \rangle + \langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle^2,$$

Again, to handle the cross term, we introduce Lemma 5.7 without expectation $\mathbb{E}_x$:

Lemma 1.1. *The following inequality holds for $t \in [T]$,*

$$\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x_{t-1}) \rangle \langle \pi_t, \breve{\mathbf{g}}_t(x_t) \rangle \geq \frac{\langle \pi_t, \breve{\mathbf{g}}_t(x_t) \rangle^2}{2} + \frac{\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x_{t-1}) \rangle^2}{2} - \|\breve{\mathbf{g}}_{t-1}(x_{t-1}) - \breve{\mathbf{g}}_t(x_t)\|^2 - \|\pi_t - \pi_{t-1}\|_1^2$$

While the remaining terms can be bounded via telescoping summation and Pinsker's inequality, the difference term $\|\breve{\mathbf{g}}_{t-1}(x_{t-1}) - \breve{\mathbf{g}}_t(x_t)\|^2$ poses huge analytical challenges since it captures the complexity of context-dependent variations and introduces substantial variance that ultimately prevents the algorithm from attaining sublinear regret guarantees.

While we believe this challenge could be addressed through more refined analytical techniques (which we defer to future work), we instead adopt a mild assumption of known context distributions as an effective solution to circumvent this issue. The typo originated during a rushed revision process when introducing the expectation operator $\mathbb{E}_x$ throughout the proof.

2. Clarification on Lemma 5.6 with expectation $\mathbb{E}_x$

Then, we present the manuscript's current dual update and analysis, incorporating all necessary corrections. We sincerely apologize for the confusion caused by such typos and confirm that these **do not compromise the theoretical validity** of our results. We begin by fixing the typo in Lemma 5.6, with all modifications clearly highlighted in red to ensure transparency.

Lemma 2.1 (Restatement of Lemma 5.6)**.** *Under the $\texttt{Optimistic}^3$ framework, the following inequality holds for the drift term*

$$\begin{aligned}
\mathbb{E}_x[\Delta_t] \leq & q_t \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle] + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle]^2 \\
= & Q_t \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle] - \mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x) \rangle] \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle] \\
& + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle]^2
\end{aligned}$$

Proof. Recall the definition of q_t and the optimistic dual update:

$$Q_{t+1} = (Q_t - \mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x) \rangle] + 2\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle])^+.$$

We can derive the update rule of q_{t+1}:

$$\begin{aligned}
q_{t+1} + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle] &= (q_t + 2\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle])^+ \\
q_{t+1} &= \max(-\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle], q_t + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle]),
\end{aligned}$$

To facilitate the analysis of q_t, we define the following $\phi(\pi_t)$:

$$\phi(\pi_t) = \begin{cases} \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle], & \text{if } q_t + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle] \geq -\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle] \\ -q_t - \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) \rangle], & \text{else.} \end{cases}$$

Then we can rewrite the update of q_{t+1} as

$$q_{t+1} = q_t + \phi(\pi_t).$$

Recall the definition of the drift term, we have

$$\begin{aligned}
\Delta_t =& \frac{\phi(\pi_t)^2}{2} + q_t\phi(\pi_t) \\
=& \frac{\phi(\pi_t)^2}{2} + q_t\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle] + q_t(\phi(\pi_t) - \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]) \\
=& \frac{\phi(\pi_t)^2}{2} + q_t\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle] - (\phi(\pi_t) + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle])(\phi(\pi_t) - \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]) \\
=& q_t\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle] + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]^2 - \frac{\phi(\pi_t)^2}{2} \\
\le& q_t\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle] + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]^2,
\end{aligned}$$

where the third equality follows from the fact that $q_t(\phi(\pi_t) - \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]) = (\phi(\pi_t) + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle])(\phi(\pi_t) - \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle])$, which can be proven by considering the cases $\phi(\pi_t) = \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]$ and $\phi(\pi_t) \neq \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]$. Finally, we substitute $q_t = Q_t - \mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle]$, which gives

$$\mathbb{E}_x[\Delta_t] \le Q_t\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle] - \mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle]\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle] + \mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]^2.$$

□

The correct expression in our analysis is $\mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle]\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]$, not $\mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]$ as previously stated.

3. Clarification on Lemma 5.7 with expectation $\mathbb{E}_x$

The concern of the reviewer arises from the same typo propagated in Lemma 5.7. Below, we restate the correct Lemma 5.7 and fix the typo. The typo correction is also explicitly highlighted in red.

Lemma 3.1 (Restatement of Lemma 5.7)**.** *The following inequality holds for $t \in [T]$,*

$$\mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle]\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle] \ge \frac{\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]^2}{2} + \frac{\mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle]^2}{2} - \mathbb{E}_x[\|\breve{\mathbf{g}}_{t-1}(x) - \breve{\mathbf{g}}_t(x)\|^2] - \|\pi_t - \pi_{t-1}\|_1^2$$

Proof.

$$\begin{aligned}
(\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle - \langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle])^2 =& (\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x) - \breve{\mathbf{g}}_{t-1}(x)\rangle] + \mathbb{E}_x[\langle \breve{\mathbf{g}}_{t-1}(x), \pi_t - \pi_{t-1}\rangle])^2 \\
\le& (\mathbb{E}_x[\|\pi_t\|\|\breve{\mathbf{g}}_t(x) - \breve{\mathbf{g}}_{t-1}(x)\|] + \mathbb{E}_x[\|\breve{\mathbf{g}}_{t-1}(x)\|_\infty\|\pi_t - \pi_{t-1}\|_1])^2 \\
\le& (\mathbb{E}_x[\|\breve{\mathbf{g}}_t(x) - \breve{\mathbf{g}}_{t-1}(x)\|] + \mathbb{E}_x[\|\breve{\mathbf{g}}_{t-1}(x)\|_\infty\|\pi_t - \pi_{t-1}\|_1])^2 \\
\le& 2\mathbb{E}_x[\|\breve{\mathbf{g}}_t(x) - \breve{\mathbf{g}}_{t-1}(x)\|^2] + 2\|\pi_t - \pi_{t-1}\|_1^2,
\end{aligned}$$

where the first inequality comes from Hölder's inequality, the second and third inequality hold since $\|\pi_t\| \le 1$ and $\|\breve{\mathbf{g}}_t(x)\|_\infty \le 1$, the last inequality holds since $(a+b)^2 \le 2a^2 + 2b^2$ and $\mathbb{E}[X]^2 \le \mathbb{E}[X^2]$. Then we can obtain

$$\begin{aligned}
&\frac{\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle]^2}{2} + \frac{\mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle]^2}{2} - \mathbb{E}_x[\langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle]\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle] \\
\le& \mathbb{E}_x[\|\breve{\mathbf{g}}_{t-1}(x) - \breve{\mathbf{g}}_t(x)\|^2] + \|\pi_t - \pi_{t-1}\|_1^2.
\end{aligned}$$

Rearranging these terms and then we complete the proof. □

The typo occurs in the expansion of $(\mathbb{E}_x[\langle \pi_t, \breve{\mathbf{g}}_t(x)\rangle - \langle \pi_{t-1}, \breve{\mathbf{g}}_{t-1}(x)\rangle])^2$, but the remaining analysis remains valid and unchanged.

4. Summary

The reviewer's primary concern centers on the incompatibility of Lemmas 5.6 and 5.7. However, with the aforementioned correction, we can correctly derive the following key inequality:

$$\begin{aligned}\mathbb{E}_x[\Delta_t] \leq & Q_t\mathbb{E}_x[\langle \pi_t, \check{\mathbf{g}}_t(x)\rangle] - \mathbb{E}_x[\langle \pi_{t-1}, \check{\mathbf{g}}_{t-1}(x)\rangle]\mathbb{E}_x[\langle \pi_t, \check{\mathbf{g}}_t(x)\rangle] + \mathbb{E}_x[\langle \pi_t, \check{\mathbf{g}}_t(x)\rangle]^2 \\ \leq & Q_t\mathbb{E}_x[\langle \pi_t, \check{\mathbf{g}}_t(x)\rangle] + \mathbb{E}_x[\|\check{\mathbf{g}}_{t-1}(x) - \check{\mathbf{g}}_t(x)\|^2] + \|\pi_t - \pi_{t-1}\|_1^2 + \left(\frac{\mathbb{E}_x[\langle \pi_{t-1}, \check{\mathbf{g}}_{t-1}(x)\rangle]^2}{2} - \frac{\mathbb{E}_x[\langle \pi_t, \check{\mathbf{g}}_t(x)\rangle]^2}{2}\right),\end{aligned}$$

where the final inequality follows from the application of the fixed Lemma 5.7, thereby preserving the validity of our results.

Based on the clarification above, we sincerely hope the reviewer's concerns have been resolved. The typo originated from a hasty revision under time limits, but crucially, it **does not affect the validity of our theoretical results**. We deeply apologize for any confusion caused by this typo and thank you very much for your time in reading the correction.