2A Exercises

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1 Find a list of four distinct vectors in $\mathbf{F}^3$ whose span equals

$$\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}. = V$$

At least 2 vectors of the list should be linearly dependent to the 2 remaining ones (as $\dim(\mathbb{F}^3) = 3$ and V does not span $\mathbb{F}^3$).

$U = (1, 0, -1), (0, 1, -1), (2, 0, -2), (0, 2, -2)$

let $v \in \{(x,y,z) \in \mathbb{F}^3 : x+y+z = 0\}$

$v = (x, y, z) = (x, y, -x-y)$

$= x(1, 0, -1) + y(0, 1, -1) + 0(2, 0, -2) + 0(0, 2, -2)$

Let $u \in \operatorname{span}(U)$, $a_1 \dots a_4 \in \mathbb{F}$

$u = a_1(1, 0, -1) + a_2(0, 1, -1) + a_3(2, 0, -2) + a_4(0, 2, -2)$

$= (a_1 + 2a_3, a_2 + 2a_4, -a_1 - a_2 - 2a_3 - 2a_4)$

$a_1 + 2a_3 + a_2 + 2a_4 - a_1 - a_2 - 2a_3 - 2a_4 = 0$

$\Rightarrow \operatorname{span}(U) = \{(x, y, z) : x + y + z = 0\}$

2 Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V, then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V.

Let $v \in V$. $v_1 \dots v_4$ span $V \Rightarrow \exists\, a_1 \dots a_4$ s.t:

$v = a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4$

$= a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 - a_1 v_2 + a_1 v_2 - (a_1 + a_2) v_3 + (a_1 + a_2) v_3 - (a_1 + a_2 + a_3) v_4 + (a_1 + a_2 + a_3) v_4$

$= a_1(v_1 - v_2) + (a_2 + a_1)(v_2 - v_3) + (a_3 + a_1 + a_2)(v_3 - v_4) + (a_4 + a_1 + a_2 + a_3) v_4$

Thus $v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ spans V

3 Suppose $v_1, \dots, v_m$ is a list of vectors in V. For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $\operatorname{span}(v_1, \dots, v_m) = \operatorname{span}(w_1, \dots, w_m)$.

$w_k = \sum_{i=1}^{k} v_i \Rightarrow v_k = w_k - \sum_{i=1}^{k-1} v_i = w_k - w_{k-1}, \forall k \geq 2$

"$\subseteq$": let $v \in \operatorname{span}(v_1, \dots, v_m)$. $\exists\, a_1 \dots a_m \in \mathbb{F}$ s.t

$v = \sum_{i=1}^{m} a_i v_i = a_1 w_1 + \sum_{i=2}^{m} a_i (w_i - w_{i-1})$

This is a linear combination of w_i's, hence $v \in \operatorname{span}(w_1 \dots w_m)$

"$\supseteq$" let $w \in \operatorname{span}(w_1, \dots, w_m)$ $\exists\, a_1 \dots a_m \in \mathbb{F}$ s.t

This is a linear combination of w_i's

"$\supseteq$" let $w \in \operatorname{span}(w_1, \dots, w_m)$ $\exists a_1 \dots a_m \in \mathbb{F}$ s.t

$$w = \sum_{i=1}^{m} a_i w_i = \sum_{i=1}^{m} a_i (v_1 + \dots + v_i)$$

This is a linear combination of v_i's, hence $w \in \operatorname{span}(v_1, \dots v_m)$

4 (a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.

(b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

a) let V a vector space, and v a vector in V. let $a \in \mathbb{F}$.

$av = 0 \iff a = 0$ or $v = 0$, as shown in previous exercises.

If $v \neq 0$, then $a = 0$, and the list consisting of v is linearly independent by definition.

If $v = 0$, then the vector in the list is 0.

b) let $u, v \in V$

u, v linearly independent $\iff (\forall a_1, a_2 \in \mathbb{F}, a_1 u + a_2 v = 0 \Rightarrow a_1 = a_2 = 0)$

$\iff (\forall a_1, a_2 \in \mathbb{F}, a_1 \neq 0 \vee a_2 \neq 0 \Rightarrow a_1 u + a_2 v \neq 0)$

$$\iff \begin{cases} \forall a_1, a_2 \in \mathbb{F}, a_1 \neq 0 \vee a_2 \neq 0 \Rightarrow u \neq -\frac{a_2}{a_1} v \\ \text{or} \\ \forall a_1, a_2 \in \mathbb{F}, a_1 \neq 0 \vee a_2 \neq 0 \Rightarrow v \neq -\frac{a_2}{a_1} u \end{cases}$$

$$\iff \begin{cases} \forall \lambda \in \mathbb{F}, \lambda \neq 0 \Rightarrow u \neq \lambda v \\ \text{or} \\ \forall \lambda \in \mathbb{F}, \lambda \neq 0 \Rightarrow v \neq \lambda u \end{cases}$$

5 Find a number t such that

$$(3, 1, 4), (2, -3, 5), (5, 9, t)$$

is not linearly independent in $\mathbf{R}^3$.

We are looking for t s.t $\exists a_1, a_2, a_3 \in \mathbb{F}$ with $a_1 \neq 0$ or $a_2 \neq 0$ or $a_3 \neq 0$ s.t:

$$\begin{cases} 3a_1 + 2a_2 = a_3 5 \\ a_1 - 3a_2 = a_3 9 \\ 4a_1 + 5a_2 = a_3 t \end{cases} \Rightarrow \begin{cases} 9a_2 + 27a_3 + 2a_2 = a_3 5 \Rightarrow 11a_2 = -22a_3 \Rightarrow a_2 = -2a_3 \\ a_1 - 3a_2 = a_3 9 \\ \underbrace{4a_1}_{12a_3} + \underbrace{5a_2}_{-10a_3} = a_3 t \end{cases}$$

$$\Rightarrow \begin{cases} a_2 = -2a_3 \\ a_1 = 3a_3 \\ (2-t)a_3 = 0 \end{cases}$$

We can set $t = 2$ to make the vectors linearly dependent.

6 Show that the list $(2,3,1), (1,-1,2), (7,3,c)$ is linearly dependent in $\mathbf{F}^3$ if and only if $c = 8$.

linearly dependent in $\mathbb{F}^3 \iff \exists a_1, a_2, a_3 \in \mathbb{F}$ with $a_1 \neq 0$ or $a_2 \neq 0$ or $a_3 \neq 0$ s.t.:

$$\begin{cases} 2a_1 + a_2 = 7a_3 \\ 3a_1 - a_2 = 3a_3 \\ a_1 + 2a_2 = a_3 c \end{cases} \iff \begin{cases} 5a_1 = 10a_3 \Rightarrow a_1 = 2a_3 \\ 3a_3 = a_2 \\ 8a_3 = a_3 c \end{cases} \overset{a_3 \neq 0 \text{ else } a_1 = a_2 = a_3 = 0}{\iff} c = 8$$

7 (a) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{R}$, then the list $1+i, 1-i$ is linearly independent.

(b) Show that if we think of $\mathbf{C}$ as a vector space over $\mathbf{C}$, then the list $1+i, 1-i$ is linearly dependent.

a) Let $a_1, a_2 \in \mathbb{R}$

$$a_1(1+i) + a_2(1-i) = 0$$

$$\Rightarrow a_1 + a_1 i + a_2 - a_2 i = 0$$

$$\Rightarrow (a_1 + a_2) + (a_1 - a_2)i = 0$$

$$\Rightarrow \begin{cases} a_1 + a_2 = 0 \\ a_1 - a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = a_2 \\ 2a_1 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases}$$

$\Rightarrow$ $i+1, 1-i$ linearly independent

b) Let $a_1, a_2 \in \mathbb{C}$, $a_1 = a + bi$, $a_2 = c + di$

$$(a+bi)(i+1) + (c+di)(1-i) = 0$$

$$\iff (a+b)i + a - b + d + c + i(d-c) = 0$$

$$\iff (a+b+d-c)i + a - b + d + c = 0$$

$$\iff \begin{cases} a+b+d-c = 0 \\ a-b+d+c = 0 \end{cases} \iff \begin{cases} c = a+b+d \\ a+d = 0 \end{cases}$$

$$\iff \begin{cases} c = b \\ a+d = 0 \end{cases}$$

For instance $c = b = 1 \neq 0$, $a = -d = 1 \neq 0$

$\Rightarrow$ linearly dependent

8 Suppose v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

Let $a_1, a_2, a_3, a_4 \in \mathbb{F}$

$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4 v_4 = 0$

$\Rightarrow a_1 v_1 + (a_2 - a_1) v_2 + (a_3 - a_2) v_3 + (a_4 - a_3) v_4 = 0$

v linearly $\perp\!\!\!\perp$

$\Rightarrow \begin{cases} a_1 = 0 \\ a_2 - a_1 = 0 \\ a_3 - a_2 = 0 \\ a_4 - a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ a_4 = 0 \end{cases} \Rightarrow v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$ is linearly independent

9 Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V, then

$$5v_1 - 4v_2, v_2, v_3, \ldots, v_m$$

is linearly independent.

Let $a_1 \ldots a_m \in \mathbb{F}$

$a_1(5v_1 - 4v_2) + a_2 v_2 + \ldots + a_m v_m = 0$

$\Rightarrow 5a_1 v_1 + (a_2 - 4) v_m + \ldots + a_m v_m = 0$

$v_1 - v_m$ linearly $\perp\!\!\!\perp$

$\Rightarrow \begin{cases} 5a_1 = 0 \\ a_2 - 4 = 0 \\ a_3 = 0 \\ \vdots \\ a_m = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 4 \neq 0 \\ a_3 = 0 \\ \vdots \\ a_m = 0 \end{cases} \Rightarrow$ Picking $a_i = 0 \; \forall i \neq 2$ and $a_2 = 4$ yields 0, thus this list is not linearly $\perp\!\!\!\perp$

10 Prove or give a counterexample: If $v_1, v_2, \ldots, v_m$ is a linearly independent list of vectors in V and $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\lambda v_1, \lambda v_2, \ldots, \lambda v_m$ is linearly independent.

Let $a_1 \ldots a_m \in \mathbb{F}$

$a_1 \lambda v_1 + \cdots + a_m \lambda v_m = 0$

$\Rightarrow \lambda(a_1 v_1 + \ldots + a_m v_m) = 0$

$\Rightarrow a_1 v_1 + \ldots + a_m v_m = 0$ (as $\lambda \neq 0$)

$\Rightarrow \begin{cases} a_1 = 0 \\ \vdots \\ a_m = 0 \end{cases}$ (as $v_1 \ldots v_m$ linearly $\perp\!\!\!\perp$)

$\Rightarrow \lambda v_1 \ldots \lambda v_m$ linearly $\perp\!\!\!\perp$

11 Prove or give a counterexample: If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then the list $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

Counter-example: $(1,0),(0,1)$ and $(0,1),(1,0)$ in $\mathbb{R}^2$

$(1,1) = (0,1) + (1,0)$

Counter-example : $(1,0),(0,1)$ and $(0,1),(1,0)$ " "

$(1,0)+(0,1)=(1,1)=(0,1)+(1,0)$

$(1,1)=1(1,1)$ so the list $(1,1),(1,1)$ is not linearly ⫫

12 Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, ..., v_m + w$ is linearly dependent, then $w \in \operatorname{span}(v_1, ..., v_m)$.

By contraposition: let $w \notin \operatorname{span}(v_1, ..., v_m)$

We will show $v_1+w, ..., v_m+w$ is linearly ⫫

let $a_1 ... a_m \in \mathbb{F}$:

$$\sum_{i=1}^{m} a_i(v_i+w) = 0 \Rightarrow \sum_{i=1}^{m} a_i v_i + w\sum_{i=1}^{m} a_i = 0$$

$$\Rightarrow \sum_{i=1}^{m} a_i v_i = -w\sum_{i=1}^{m} a_i$$

But $w \notin \operatorname{span}(v_1 ... v_m)$, so this equation

is only true if $\sum_{i=1}^{m} a_i v_i = -w\sum_{i=1}^{m} a_i = 0$

$$\Rightarrow \sum_{i=1}^{m} a_i v_i = 0 \Rightarrow a_i = 0 \ \forall i = 1 ... m$$

$\Rightarrow v_1+w ... v_m+w$ linearly independent

13 Suppose $v_1, ..., v_m$ is linearly independent in V and $w \in V$. Show that

$$v_1, ..., v_m, w \text{ is linearly independent} \iff w \notin \operatorname{span}(v_1, ..., v_m).$$

$v_1, ..., v_m, w$ linearly ⫫ $\iff \left(\sum_{i=1}^{m} a_i v_i + a_w w = 0 \Rightarrow a_1 = ... = a_m = a_w = 0\right)$

$\iff \left(\sum_{i=1}^{m} a_i v_i = -a_w w \Rightarrow a_1 = ... = a_m = a_w = 0\right)$

$\iff \left(\nexists\, a_1, ..., a_m \in \mathbb{F} \text{ s.t. } w = \sum_{i=1}^{m} a_i v_i\right)$

$\iff w \notin \operatorname{span}(v_1 ... v_m)$

14 Suppose $v_1, ..., v_m$ is a list of vectors in V. For $k \in \{1, ..., m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that the list $v_1, ..., v_m$ is linearly independent if and only if the list $w_1, ..., w_m$ is linearly independent.

"$\Rightarrow$": let $a_1 ... a_m \in \mathbb{F}$

$$\sum_{i=1}^{m} a_i w_i = 0 \Rightarrow \sum_{i=1}^{m} a_i \sum_{j=1}^{i} v_j = 0$$

$$\Rightarrow \sum^{m} v_i \sum^{i} a_i = 0$$

$$\Rightarrow \sum_{i=1}^{m} v_i \sum_{j=1}^{i} a_i = 0$$

$$\Rightarrow \begin{cases} a_1 = 0 \\ a_1 + a_2 = 0 \\ \vdots \\ a_1 + \dots + a_m = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_m = 0 \end{cases} \Rightarrow w_1 \dots w_m \text{ linearly } \perp\!\!\!\perp$$

"$\Leftarrow$": let $a_1 \dots a_m \in \mathbb{F}$

$$\sum_{i=1}^{m} a_i v_i = \sum_{i=2}^{m} a_i (w_i - w_{i-1}) + a_1 w_1$$

$$= a_1 w_1 + \sum_{i=2}^{m} a_i w_i - \sum_{i=2}^{m} a_i w_{i-1}$$

$$= \sum_{i=1}^{m} a_i w_i - \sum_{i=1}^{m-1} a_{i+1} w_i = \sum_{i=1}^{m-1} (a_i - a_{i+1}) w_i + a_m w_m = 0$$

$$\xRightarrow{w_1 \dots w_m \text{ linearly } \perp\!\!\!\perp} \begin{cases} a_1 - a_2 = 0 \\ a_2 - a_3 = 0 \\ \vdots \\ a_m - a_{m-1} = 0 \\ a_m = 0 \end{cases} \Rightarrow \begin{cases} a_m = 0 \\ a_{m-1} = 0 \\ \vdots \\ a_1 = 0 \end{cases} \Rightarrow v_1 \dots v_m \text{ linearly } \perp\!\!\!\perp$$

15 Explain why there does not exist a list of six polynomials that is linearly independent in $\mathcal{P}_4(\mathbf{F})$.

A spanning list of $P_4(\mathbb{F})$ is $1, X, X^2, X^3, X^4$

Any linearly $\perp\!\!\!\perp$ list of vectors thus has to be 5 or less, according to 2.22. A list of 6 polynomials cannot be linearly $\perp\!\!\!\perp$ in $P_4(\mathbb{F})$.

16 Explain why no list of four polynomials spans $\mathcal{P}_4(\mathbf{F})$.

$1, X, X^2, X^3, X^4$ contains 5 elements and is linearly $\perp\!\!\!\perp$.

Thus according to 2.22, any list of vectors spanning $P_4(\mathbb{F})$ must contain 5 elements or more.

17 Prove that V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \dots$ of vectors in V such that $v_1, \dots, v_m$ is linearly independent for every positive integer m.

"$\Rightarrow$" First we can prove:

"⟹" First we can prove:

$$\boxed{\begin{aligned}&\forall k,\ v_1 \dots v_k \text{ linearly ind.},\ \operatorname{span}(v_1 \dots v_k) \neq V \\ &\iff \exists v \in V \text{ s.t. } v_1 \dots v_k v \text{ linearly ind.}\end{aligned}} \quad (1)$$

"⟹" Indeed, let $v \in V$ and $v \notin \operatorname{span}(v_1 \dots v_k)$

Let $a_1 \dots a_{k+1} \in \mathbb{F}$.

$$a_1 v_1 + \dots + a_k v_k + a_{k+1} v = 0$$

$$\Rightarrow a_1 v_1 + \dots + a_k v_k = -a_{k+1} v$$

$$\Rightarrow \begin{cases} a_{k+1} = 0, \text{ which implies } \sum_{i=1}^{k} a_i v_i = 0 \Rightarrow a_i = 0\ \forall i \text{ as } v_1 \dots v_k \text{ are linearly ind.} \\ a_{k+1} \neq 0 \Rightarrow v = \sum_{i=1}^{k} -\frac{a_i}{a_{k+1}} v_i, \text{ but } v \notin \operatorname{span}(v_1 \dots v_k) \text{ so it is not possible} \end{cases}$$

$$\Rightarrow a_1 = \dots = a_k = a_{k+1} = 0$$

$$\Rightarrow v_1, \dots, v_k, v \text{ linearly ind.}$$

"⟸" Length of linearly ind. lists ≤ length of spanning lists

length$(v_1 \dots v_k, v) = k+1 \leq$ length of spanning lists

$\Rightarrow k <$ length of spanning lists

$\Rightarrow$ length$(v_1 \dots v_k) <$ length of spanning lists

$\Rightarrow \operatorname{span}(v_1 \dots v_k) \neq V$

V infinite dim $\Rightarrow \nexists v_1 \dots v_m$ s.t. $\operatorname{span}(v_1 \dots v_m) = V$

We can construct $v_1, v_2, \dots$ such ~~that~~ $\forall m$, $v_1 \dots v_m$ is linearly ind.:

Start with $v_1 \in V \setminus \{0\}$. v_1 is linearly ind.

$\operatorname{span}(v_1) \neq V \Rightarrow \exists v_2 \in V$ s.t. v_1 and v_2 are linearly ind. (using (1))

$\operatorname{span}(v_1, v_2) \neq V \Rightarrow \exists v_3 \in V$ s.t. v_1, v_2, v_3 are linearly ind. (using (1))

$\mathrm{span}(v_1, v_2) \neq V \Rightarrow \exists v_3 \in V$ s.t. v_1, v_2, v_3 are linearly $\perp\!\!\!\perp$ (using (1))

This process can be repeated indefinitely (for any arbitrary number m of vectors) as no finite list can span V, with $v_1 \ldots v_m$ linearly $\perp\!\!\!\perp$ $\forall m$.

"$\Leftarrow$" Let $v_1, v_2 \ldots$ be a sequence s.t. $v_1 \ldots v_m$ linearly $\perp\!\!\!\perp$ $\forall m$

We can apply (1) to deduce $\mathrm{span}(v_1 \ldots v_m) \neq V$ $\forall m$,

so V is infinite dimensional.

18 Prove that $\mathbf{F}^\infty$ is infinite-dimensional.

Let the sequence $v_1, v_2, \ldots$ of vectors in $\mathbb{F}^\infty$ s.t. the jth element of v_i is 1 if $j = i$, and 0 otherwise.

For all integer m, $v_1, \ldots, v_m$ is linearly independent.

According to the result in Ex. 17, this implies $\mathbb{F}^\infty$ is infinite-dimensional.

19 Prove that the real vector space of all continuous real-valued functions on the interval $[0, 1]$ is infinite-dimensional.

$$\mathbb{R}^{[0,1]} = \{ f : [0,1] \to \mathbb{R} \}$$

Let the sequence of functions $f_1, f_2, \ldots$ on $[0,1]$ s.t.

$f_i = x^{i-1}$ $\forall m \in \mathbb{N}$, $v_1, \ldots, v_m$ is linearly independent.

According to the result in Ex. 17, this implies $\mathbb{R}^{[0,1]}$ is infinite-dimensional.

20 Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_k(2) = 0$ for each $k \in \{0, \ldots, m\}$. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

$$U = \{ p(x) \in \mathcal{P}_m(\mathbb{F}) : p(2) = 0 \}$$

U is a subspace over $\mathcal{P}_m(\mathbb{F})$:

U is a subspace over $\mathcal{P}_m(\mathbb{F})$:

- $0 \in U$
- let $p, q \in U$
  $(p+q)(2) = p(2)+q(2) = 0$
  $\Rightarrow p+q \in U$
- let $\lambda \in \mathbb{F}$
  $(\lambda p)(2) = \lambda p(2) = 0$
  $\Rightarrow \lambda p \in U$

$\exists v \in \mathcal{P}_m(\mathbb{F})$ s.t $v \notin U$, thus the minimum number of vectors in a list to span U is strictly inferior to the amount required to span $\mathcal{P}_m(\mathbb{F})$, which is $m+1$.

Also, a list of independent vectors in U should be $\leq$ to the nb of vectors in a spanning list, which is at most m as previously shown.

$p_0, p_1, \dots, p_m$ has $m+1$ vectors; thus it is not linearly independent in U.