5D Exercises

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1 Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$.

(a) Prove that if $T^4 = I$, then T is diagonalizable.
(b) Prove that if $T^4 = T$, then T is diagonalizable.
(c) Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^2)$ such that $T^4 = T^2$ and T is not diagonalizable.

a) let $p \in \mathcal{P}(\mathbb{F})$, $p(z) = z^4 - 1$: $p(T) = T^4 - I = 0 \Rightarrow p$ is a multiple of the minimal polynomial of T. We can notice $z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$, then by 5.62, T is diagonalizable.

b) let $p \in \mathcal{P}(\mathbb{F})$, $p(z) = z^4 - z$: $p(T) = T^4 - T = 0 \Rightarrow p$ is a multiple of the minimal polynomial of T. $z(z^3 - 1)$ has roots $0, 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.
$z^4 - z = z(z-1)(z + \frac{1}{2} + i\frac{\sqrt{3}}{2})(z + \frac{1}{2} - i\frac{\sqrt{3}}{2})$. By 5.62, T is diagonalizable.

c) $z^4 - z^2 = z^2(z^2 - 1) = z^2(z-1)(z+1) = 0$, so multiple of minimal polynomial of T.
We can find T s.t. $T^2 = 0$. let $T \in \mathcal{L}(\mathbb{C}^2)$: $T(x, y) = (0, x)\ \forall (x,y) \in \mathbb{C}^2$.
We have $T^2 = 0$, and z^2 is the minimal polynomial of T. There is no distinct λ_1, λ_2 s.t $z^2 = (z-\lambda_1)(z-\lambda_2)$, so T is not diagonalizable.

2 Suppose $T \in \mathcal{L}(V)$ has a diagonal matrix A with respect to some basis of V. Prove that if $\lambda \in \mathbf{F}$, then λ appears on the diagonal of A precisely $\dim E(\lambda, T)$ times.

let $v_1 \dots v_m$ basis of V s.t. $A = \mathcal{M}(T, (v_1 \dots v_m))$ is a diagonal matrix.
A is an upper triangular matrix, so by 5.41 the entries on its diagonal are the e.va. of T. let $\lambda \in \mathbb{F}$, λ not an e.va. of T. λ appears $0 = \dim E(\lambda, T)$ times on the diagonal of A.
$\forall i = 1 \dots m$, v_i is an e.ve. of T corresponding to e.va. A_{ii} by definition of A : $Tv_i = A_{ii} v_i$, so every e.va. λ of T must appear $\dim E(\lambda, T)$ times on the diagonal of A, like there are $\dim E(\lambda, T)$ e.ve. corresponding to λ in $v_1 \dots v_m$.

3 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that if the operator T is diagonalizable, then $V = \operatorname{null} T \oplus \operatorname{range} T$.

T diagonalizable $\Rightarrow \exists v_1 \dots v_m$ basis of V s.t. all v_i's are e.ve., corresponding to e.va. λ_i.
let $v_1 \dots v_k$ the e.ve. wrt e.va. that are non-zero.
Then we have $Tv_i = \lambda v_i \Rightarrow v_i = T\lambda^{-1} v_i \Rightarrow v_i \in \operatorname{range} T\ \forall i = 1 \dots k$
e.va. $v_{k+1} \dots v_m$ are such that $Tv_i = 0 \Rightarrow v_i \in \operatorname{null} T\ \forall i = k+1 \dots m$.
$\Rightarrow V = \operatorname{null} T \oplus \operatorname{range} T$

4 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.

(a) $V = \operatorname{null} T \oplus \operatorname{range} T$.
(b) $V = \operatorname{null} T + \operatorname{range} T$.
(c) $\operatorname{null} T \cap \operatorname{range} T = \{0\}$.

a) $\Rightarrow$ b), c). We can show b) $\Leftrightarrow$ c), since b)$\wedge$c) $\Rightarrow$ a.
"$\Rightarrow$": Suppose $V = \operatorname{null} T + \operatorname{range} T$
$\dim \operatorname{null} T + \operatorname{range} T = \dim \operatorname{null} T + \dim \operatorname{range} T - \dim \operatorname{range} T \cap \operatorname{null} T$

"$\Rightarrow$": Suppose $V = \text{null}\, T + \text{range}\, T$

$\dim \text{null}\, T + \text{range}\, T = \dim \text{null}\, T + \dim \text{range}\, T - \dim \text{range}\, T \cap \text{null}\, T$

Also: $\overset{\shortparallel}{\dim V} = \dim \text{null}\, T + \dim \text{range}\, T$ (linear map to V)

$\Rightarrow \dim \text{range}\, T \cap \text{null}\, T = 0 \Rightarrow \text{range}\, T \cap \text{null}\, T = \{0\}$.

"$\Leftarrow$": Suppose $\text{null}\, T \cap \text{range}\, T = \{0\}$.

$\dim V = \dim \text{null}\, T + \dim \text{range}\, T$

$\dim \text{null}\, T + \text{range}\, T = \dim \text{null}\, T + \dim \text{range}\, T - \underbrace{\dim \text{null}\, T \cap \text{range}\, T}_{=0}$

$\Rightarrow \dim V = \dim \text{null}\, T + \text{range}\, T$

$\Rightarrow V = \text{null}\, T + \text{range}\, T$ (as $\text{null}\, T + \text{range}\, T$ subspace of V)

5 Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if

$$V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$$

for every $\lambda \in \mathbf{C}$.

"$\Rightarrow$": Suppose T is diagonalizable.

Let $v_1 \in \text{null}(T - \lambda I) \cap \text{range}(T - \lambda I)$

If $\text{null}(T - \lambda I) = \{0\}$, then $\text{null}(T - \lambda I) \cap \text{range}(T - \lambda I) = \{0\}$, implying according to previous exercise $V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$.

Otherwise, $\exists v \in V$ s.t. $(T - \lambda I)v = 0 \Rightarrow Tv = \lambda v \Rightarrow \lambda$ e.va. of T.

Let $v, v_1 \ldots v_{m-1}$ a basis of V s.t. all elements are eigenvectors, which exists as T is diagonalizable, associated to e.va. $\lambda, \lambda_1 \ldots \lambda_{m-1}$.

$v \in \text{range}(T - \lambda I) \Rightarrow \exists w \in V$ s.t. $(T - \lambda I) w = v$, $w = av + \sum_{i=1}^{m-1} a_i v_i$

$\Rightarrow Tw - \lambda w = v \Rightarrow a\underbrace{Tv}_{\lambda v} + \sum_{i=1}^{m-1} a_i \underbrace{Tv_i}_{\lambda_i v_i} - \lambda a v - \sum_{i=1}^{m-1} \lambda a_i v_i - v = 0$

$\Rightarrow v = \sum_{i=1}^{m-1} a_i (\lambda_i - \lambda) v_i$. Since $v, v_1 \ldots v_{m-1}$ is linearly independent,

this implies $v = 0$. Thus $\text{null}(T - \lambda I) \cap \text{range}(T - \lambda I) = \{0\}$, so

by previous exercise, $V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$

"$\Leftarrow$": Suppose $V = \text{null}(T - \lambda I) \oplus \text{range}(T - \lambda I)$ $\forall \lambda \in \mathbb{C}$

V complex v.s. $\Rightarrow$ minimal polynomial of T can be written:

$p(z) = \prod_{i=1}^{m} (z - \lambda_i)^{m_i}$, where $\lambda_i \neq \lambda_j$ $\forall i, j$, $(m_i)_i \in \mathbb{N}^*$

$\forall v \in V$ $\prod_{i=1}^{m} (T - \lambda_i)^{m_i} v = 0$. Suppose $\exists m_k$ s.t. $m_k \neq 1$.

Let $v = m + n \in V$, $m \in \text{null}\, T - \lambda_k I$, $n \in \text{range}\, T - \lambda_k I$.

$\cdot$ $(T - \lambda_k)^{m_k - 1} \prod_{\substack{i=1 \\ i \neq k}}^{m} (T - \lambda_i)^{m_i} v = 0$

• $(T-\lambda_k)^{m_k-1}\prod_{\substack{i=1\\ i\neq k}}(T-\lambda_i)^{m_i}v = 0$

• $p(T)w = 0 \Rightarrow (T-\lambda_k)(T-\lambda_k)^{m_k-1}\prod_{\substack{i=1\\ i\neq k}}^{m}(T-\lambda_i)^{m_i}w = 0$

$\Rightarrow (T-\lambda_k)^{m_k-1}\prod_{\substack{i=1\\ i\neq k}}^{m}(T-\lambda_i)^{m_i}v = 0$

$\Rightarrow (z-\lambda_k)^{m_k-1}\prod_{\substack{i=1\\ i\neq k}}^{m}(z-\lambda_i)^{m_i}$ is the 0 polynomial for $z=T$ and

has lower degree than T's minimal polynomial: contradiction.

$\Rightarrow m_i = 1 \ \forall i \Rightarrow T$ diagonalizable by 5.62.

6 Suppose $T \in \mathcal{L}(\mathbf{F}^5)$ and $\dim E(8,T) = 4$. Prove that $T-2I$ or $T-6I$ is invertible.

T has at most one e.va. other than 8, as e.ve. of different e.va. are linearly independent and $\dim V - \dim E(8,T) = 1$.

If this e.va. is 2, then $\operatorname{null} T - 6I = \{0\}$, and hence $T-6I$ is invertible.

Same reasoning for 6.

Therefore $T-6I$ or $T-2I$ is invertible.

7 Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

$$E(\lambda,T) = E\left(\tfrac{1}{\lambda},T^{-1}\right)$$

for every $\lambda \in \mathbf{F}$ with $\lambda \neq 0$.

In ex. 5A.21, we showed that T and T^{-1} have inversed e.va. ($\neq 0$) associated with the same e.ve. Therefore $E(\lambda,T) = E(\frac{1}{\lambda},T^{-1})$.

8 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1,\dots,\lambda_m$ denote the distinct nonzero eigenvalues of T. Prove that

$$\dim E(\lambda_1,T) + \cdots + \dim E(\lambda_m,T) \leq \dim \operatorname{range} T.$$

$\sum_{i=1}^{m} \dim E(\lambda_i,T) + \dim E(0,T) \leq \dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

$\Rightarrow \sum_{i=1}^{m} \dim E(\lambda_i,T) \leq \dim \operatorname{range} T$, as $E(0,T) = \operatorname{null} T$ by definition.

9 Suppose $R,T \in \mathcal{L}(\mathbf{F}^3)$ each have 2, 6, 7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbf{F}^3)$ such that $R = S^{-1}TS$.

R, T have 3 e.va. in $\mathbb{F}^3$

$\Rightarrow$ basis of e.ve of R, T, respectively v_1, v_2, v_3 and w_1, w_2, w_3.

Define $S \in \mathcal{L}(V)$ s.t. $Sv_i = w_i$ (change of basis)

$S^{-1}TSv_i = S^{-1}Tw_i = S^{-1}2w_i = 2v_i \Rightarrow R = S^{-1}TS$

13 Suppose A is a diagonal matrix with distinct entries on the diagonal and B is a matrix of the same size as A. Show that $AB = BA$ if and only if B is a diagonal matrix.

$A = \operatorname{diag}(\lambda_1 \dots \lambda_m)$

"$\Rightarrow$": Suppose $AB = BA$

$$(AB)_{ij} = \sum_{\ell=1}^{m} A_{i\ell} B_{\ell j} = A_{ii} B_{ij} = \lambda_i B_{ij}$$

$$(BA)_{ij} = \sum_{\ell=1}^{m} B_{i\ell} A_{\ell j} = B_{ij} A_{jj} = \lambda_j B_{ij} \overset{AB=BA}{=} \lambda_i B_{ij} \quad (1)$$

For $i \neq j$, since $\lambda_i \neq \lambda_j$, $(1) \Rightarrow B_{ij} = 0 \Rightarrow B$ diagonal matrix

"$\Leftarrow$": AB is a diagonal matrix with entries $A_{ii} B_{ii}$ like BA.

14 (a) Give an example of a finite-dimensional complex vector space and an operator T on that vector space such that T^2 is diagonalizable but T is not diagonalizable.

(b) Suppose $\mathbf{F} = \mathbf{C}$, k is a positive integer, and $T \in \mathcal{L}(V)$ is invertible. Prove that T is diagonalizable if and only if T^k is diagonalizable.

a) Let T the operator s.t. $\mathcal{M}(T) = A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for the standard basis.

$A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, diagonal matrix $\Rightarrow T^2$ diagonalizable

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} \Rightarrow \begin{cases} y = \lambda x \\ 0 = \lambda y \end{cases} \Rightarrow \begin{cases} \lambda = 0 \\ y = 0, x \in \mathbb{F} \end{cases}$$

$\Rightarrow$ not diagonalizable, no basis of e.ve. of T.

b) "$\Rightarrow$": Suppose T diagonalizable. Let $v_1 \dots v_n$ basis of e.ve. of T.

$\forall k > 0$, $T^k v_i = \lambda_i^k v_i$, with λ_i e.va. associated with v_i.

$\Rightarrow \exists$ basis of e.ve. of $T^k \Rightarrow T^k$ diagonalizable

"$\Leftarrow$": Suppose T^k diagonalizable. Let $p \in \mathcal{P}(\mathbb{F})$ minimal polynomial of T^k.

$$p(z) = \prod_{i=1}^{m} (z - \lambda_i), \text{ with } \{\lambda_i\}_{i=1 \dots m} \text{ distinct e.va. of } T^k.$$

Since T is invertible, T^k is invertible and $\forall i = 1 \dots m$ $\lambda_i \neq 0$.

$$q(T) = p(T^k) = \prod_{i=1}^{m} (T^k - \lambda_i) = \prod_{i=1}^{m} s_i(T^k), \quad s_i(z) = z^k - \lambda_i$$

We can check whether q has roots of multiplicity > 1.

- Common roots between s_i and s_j, $i \neq j$:

  Let z_0 a root of s_i: $z_0^k - \lambda_i = 0$

  $s_j(z_0) = z_0^k - \lambda_j = z_0^k - \lambda_i + \lambda_i - \lambda_j = \lambda_i - \lambda_j \neq 0$ ($\lambda_i \neq \lambda_j$)

  $\Rightarrow$ no roots in common

- Root of multiplicity > 1 in s_i:

- Root of multiplicity > 1 in s_i:

  $s_i(z)' = k z^{k-1}$.
  Suppose z_0 is a zero of s_i: $s_i(z_0) = k z_0^{k-1} = 0$
  $s_i(z_0) = z_0^k - \lambda_i = k^{-1} z_0 (k z_0^{k-1}) - \lambda_i = -\lambda_i \neq 0$,
  otherwise the constant term of p would be 0, implying
  by 5.32 T^k is not invertible.
  By 4 ex 8, this implies s_i has m distinct zeros $\forall i$.

q is a multiple of the minimal polynomial of T, and has no multiple roots, implying T's minimal polynomial does not either. Thus, T is diagonalizable (as $\mathbb{F} = \mathbb{C}$).

15 Suppose V is a finite-dimensional complex vector space, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Prove that the following are equivalent.
(a) T is diagonalizable.
(b) There does not exist $\lambda \in \mathbf{C}$ such that p is a polynomial multiple of $(z-\lambda)^2$.
(c) p and its derivative p' have no zeros in common.
(d) The greatest common divisor of p and p' is the constant polynomial 1.

*The **greatest common divisor** of p and p' is the monic polynomial q of largest degree such that p and p' are both polynomial multiples of q. The Euclidean algorithm for polynomials (look it up) can quickly determine the greatest common divisor of two polynomials, without requiring any information about the zeros of the polynomials. Thus the equivalence of (a) and (d) above shows that we can determine whether T is diagonalizable without knowing anything about the zeros of p.*

$b \to a \to c \to d \to b$

4 ex 8

$a \iff \exists \lambda_1 \dots \lambda_m, \lambda_i \neq \lambda_j \ \forall i \neq j$ s.t. $p(z) = \prod_{i=1}^{m} (z - \lambda_i) \iff c$

$[\neg d \implies \exists$ monic polynomial q s.t. $\exists r_1, r_2 : p = q r_1, p' = q r_2$, with degree > 1, then it has at least one zero in $\mathbb{C}$, which is a shared root of p and $p' \implies \neg c] \implies [c \implies d]$

$[\neg b \implies z - \lambda$ is a common divisor of p and p', and $\deg(z-\lambda) > \deg(1)$
$\implies \neg d] \implies [d \implies b]$

$b \implies p(z) = \prod_{i=1}^{m} (z - \lambda_i)$ with distinct λ_i's, otw it would be divisible by $(z-\lambda_i)^2$.
$\implies a$

$\implies$ all the propositions are equivalent

16 Suppose that $T \in \mathcal{L}(V)$ is diagonalizable. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T. Prove that a subspace U of V is invariant under T if and only if there exist subspaces $U_1, \dots, U_m$ of V such that $U_k \subseteq E(\lambda_k, T)$ for each k and $U = U_1 \oplus \dots \oplus U_m$.

T diag. T invariant under $U \implies T|_U$ diagonalizable

"$\Leftarrow$": Suppose $\exists U_1 \dots U_m$ subspaces of V s.t. $U_k \subseteq E(\lambda_k, T) \ \forall k$ and $U = U_1 \oplus \dots \oplus U_m$.

each k and $U = U_1 \oplus \cdots \oplus U_m$.

"$\Leftarrow$": Suppose $\exists U_1 \dots U_m$ subspaces of V s.t. $U_k \subseteq E(\lambda_k, T)\ \forall k$ and $U = U_1 \oplus \dots \oplus U_m$.

Let $v \in U$. $v = \sum_{i=1}^{m} v_i$, $v_i \in U_i\ \forall i = 1 \dots m$.

$$Tv = \sum_{i=1}^{m} Tv_i = \sum_{i=1}^{m} \lambda_i v_i \quad (\text{since } U_k \subseteq E(\lambda_k, T))$$

$\Longrightarrow Tv \in U_1 \oplus \dots \oplus U_m = U \Longrightarrow U$ invariant under T

"$\Rightarrow$": Suppose U subspace of V is invariant under T.

Let λ e.va. of $T|_U$, with e.ve v.

We can immediately see that λ is then an e.va. of T, and $v \in E(\lambda, T)$.

Thus e.va. of $T|_U$ is a subset of those of T and $E(\lambda, T|_U) \subseteq E(\lambda, T)\ \forall \lambda$.

From 5.65, $T|_U$ is diagonalizable, implying: $U = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_k, T)$

where $\lambda_1 \dots \lambda_k$ are the e.va. of T that are e.va. of $T|_U$.

For λ_i's that are not e.va. of $T|_U$, we have $E(\lambda_i, T|_U) = \{0\}$.

Thus: $U = E(\lambda_1, T) \oplus \dots \oplus E(\lambda_m, T)$.

This concludes the proof.

17 Suppose V is finite-dimensional. Prove that $\mathcal{L}(V)$ has a basis consisting of diagonalizable operators.

Define $\forall i = 1 \dots m$ $\phi_i \in \mathcal{L}(V)$ s.t. $\phi_i v_i = v_i$, $\phi_i v_j = 0\ \forall j \neq i$, where $v_1 \dots v_m$ basis of V.

Let $a_1 \dots a_m \in \mathbb{F}$ s.t.: $\sum_{i=1}^{m} a_i \phi_i = 0 \Longrightarrow \begin{cases} \sum a_i \phi_i v_1 = 0 \Rightarrow a_1 v_1 = 0 \Rightarrow a_1 = \dots = a_m = 0 \\ \vdots \\ \sum a_i \phi_i v_m = 0 \Rightarrow a_m v_m = 0 \end{cases}$

$\Longrightarrow \phi_i$ are linearly independent (1)

(1) $\wedge$ $\dim \mathcal{L}(V) = \dim V = m \Longrightarrow \phi_1 \dots \phi_m$ is a basis of $\mathcal{L}(V)$.

$$\left(\mathcal{M}(\phi_k, (v_1 \dots v_m))\right)_{ij} = \begin{cases} 1 & \text{if } i = j = k \\ 0 & \text{otw} \end{cases}$$ for $k = 1 \dots m$, so they are diagonal matrices,

therefore $\phi_1 \dots \phi_m$ are all diagonalizable.

18 Suppose that $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Prove that the quotient operator T/U is a diagonalizable operator on V/U.

The quotient operator T/U was defined in Exercise 38 in Section 5A.

From 5B ex. 25, the minimal polynomial of T is a polynomial multiple of the minimal operator on V/U.

The quotient operator T/U was defined in Exercise 38 in Section 5A.

From 5B ex. 25, the minimal polynomial of T is a polynomial multiple of the minimal polynomial of T/U. Since T is diagonalizable, the minimal polynomial of T is of the form: $\prod_{i=1}^{m}(z-\lambda_i)$ with λ_i distinct values. Therefore the minimal polynomial of T/U is of the form $\prod_{i=1}^{k}(z-\lambda_i)$ with λ_i distinct values, and thus T/U is diagonalizable.

19 Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and there exists a subspace U of V that is invariant under T such that $T|_U$ and T/U are both diagonalizable, then T is diagonalizable.

See Exercise 13 in Section 5C for an analogous statement about upper-triangular matrices.

Should design T s.t. $\exists v \in \overline{U}, v \neq 0, Tv \in U$, and U e.va. of T/U

let $\mathbb{F} = \mathbb{R}$, $V = \mathbb{R}^2$. Define $T \in \mathcal{L}(\mathbb{R}^2)$: $Te_1 = e_2$, $Te_2 = 0$

$T(x,y) = (0,x) = \lambda(x,y) \longrightarrow \begin{cases} \lambda x = 0 \\ \lambda y = x \end{cases}$

$\Longrightarrow T$ has one e.va., 0 with $E(0,T) = \{(0,y) \in \mathbb{R}^2\} \neq V$

$\Longrightarrow T$ is not diagonalizable.

Consider $U = \operatorname{span}(e_2)$. $Te_2 \in U \Rightarrow U$ invariant under U (1)

- We have $T|_U = 0$, as $Te_2 = 0 \Longrightarrow T|_U$ diagonalizable (2)
- $v_1 + U$ is a basis of V/U. Furthermore, $(T/U)(v_1+U) = v_2 + U = U$
  $\Longrightarrow v_1$ e.ve. of $T/U \Rightarrow v_1$ is a basis of e.ve. of T/U
  $\Longrightarrow T/U$ diagonalizable (3)

We thus have (1), (2), (3) but T not diagonalizable.

20 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if the dual operator T' is diagonalizable.

From 5B ex. 20, the minimal polynomial p of T equals that of T'.

Hence T diagonalizable $\Leftrightarrow p(z) = \prod_{i=1}^{m} z-\lambda_i$, with distinct λ_i's.

$\Leftrightarrow T'$ diagonalizable

21 The *Fibonacci sequence* $F_0, F_1, F_2, \dots$ is defined by

$$F_0 = 0,\ F_1 = 1,\ \text{and } F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 2.$$

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(x,y) = (y, x+y)$.

(a) Show that $T^n(0,1) = (F_n, F_{n+1})$ for each nonnegative integer n.
(b) Find the eigenvalues of T.
(c) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of T.
(d) Use the solution to (c) to compute $T^n(0,1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

21 The *Fibonacci sequence* $F_0, F_1, F_2, \ldots$ is defined by

$$F_0 = 0,\ F_1 = 1, \text{ and } F_n = F_{n-2} + F_{n-1} \text{ for } n \geq 2.$$

Define $T \in \mathcal{L}(\mathbf{R}^2)$ by $T(x, y) = (y, x + y)$.

(a) Show that $T^n(0, 1) = (F_n, F_{n+1})$ for each nonnegative integer n.
(b) Find the eigenvalues of T.
(c) Find a basis of $\mathbf{R}^2$ consisting of eigenvectors of T.
(d) Use the solution to (c) to compute $T^n(0, 1)$. Conclude that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for each nonnegative integer n.

(e) Use (d) to conclude that if n is a nonnegative integer, then the Fibonacci number F_n is the integer that is closest to

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

Each F_n is a nonnegative integer, even though the right side of the formula in (d) does not look like an integer. The number

$$\frac{1+\sqrt{5}}{2}$$

is called the ***golden ratio***.

a) • $T^0(0,1) = (0,1) = (F_0, F_1)$.

• Suppose $T^{m-1}(0,1) = (F_{m-1}, F_m)$, for $m \geq 1$

Then $T^m(0,1) = T(F_{m-1}, F_m) = (F_m, F_{m-1} + F_m) = (F_m, F_{m+1})$

By induction, $T^n(0,1) = (F_n, F_{n+1})$ $\forall n \geq 0$

b) $T(x,y) = \lambda(x,y) \Rightarrow \begin{cases} y = \lambda x \\ x + y = \lambda y \end{cases}$

$x = \frac{1}{\lambda}$

$y = 1$

$\frac{1}{\lambda} + 1 = \lambda$

$-\lambda^2 + \lambda + 1 = 0$

$\Rightarrow x(1+\lambda) = \lambda y$

$\Rightarrow x(1+\lambda) = \lambda^2 x$

$\Rightarrow \begin{cases} x \neq 0 \\ 1 + \lambda = \lambda^2 \Rightarrow 1 + \lambda - \lambda^2 = 0 \end{cases}$

$\Rightarrow \lambda_1 = \frac{1+\sqrt{5}}{2},\ \lambda_2 = \frac{1-\sqrt{5}}{2}$

c) $v_1 = \left(1, \frac{1+\sqrt{5}}{2}\right)$ is an e.ve. associated with e.va. $\frac{1+\sqrt{5}}{2}$

$v_2 = \left(\frac{2}{1-\sqrt{5}}, 1\right)$ is an e.ve. associated with e.va. $\frac{1-\sqrt{5}}{2}$.

d) c $\Rightarrow$ T is diagonalizable, and has a diagonal matrix in base v_1, v_2.

$(0,1) = a_1\left(1, \frac{1+\sqrt{5}}{2}\right) + a_2\left(\frac{2}{1-\sqrt{5}}, 1\right) = \left(a_1 + \frac{2a_2}{1-\sqrt{5}},\ a_2 + a_1\frac{1+\sqrt{5}}{2}\right)$

$\Rightarrow \begin{cases} a_1 = \sqrt{5}^{-1} \end{cases}$

$$\Longrightarrow \begin{cases} a_1 = \sqrt{5}^{-1} \\ a_2 = 10^{-1}(5-\sqrt{5}) \end{cases}$$

$$\Longrightarrow T^n(0,1) = T^n\left(\sqrt{5}^{-1} v_1 + 10^{-1}(5-\sqrt{5}) v_2\right) \overset{a)}{=} (F_n, F_{n+1})$$

$$\Longrightarrow F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n 1 + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n \frac{2}{1-\sqrt{5}}$$

$$= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

22 Suppose $T \in \mathcal{L}(V)$ and A is an n-by-n matrix that is the matrix of T with respect to some basis of V. Prove that if

$$|A_{j,j}| > \sum_{\substack{k=1 \\ k \neq j}}^{n} |A_{j,k}|$$

for each $j \in \{1, \ldots, n\}$, then T is invertible.

This exercise states that if the diagonal entries of the matrix of T are large compared to the nondiagonal entries, then T is invertible.

$|A_{jj}| > \sum_{\substack{k=1 \\ k \neq j}}^{n} |A_{jk}| \ \forall j = 1..n \Longrightarrow$ There is no Gershgorin disk with 0 in it (by definition)

By 5.67, each e.va. of T wrt some basis of V is contained in a Gershgorin disk.

$\Longrightarrow$ 0 is not an e.va. of $T \Longrightarrow T$ invertible