6C Exercises

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1 Suppose $v_1, \dots, v_m \in V$. Prove that
$$\{v_1, \dots, v_m\}^\perp = (\operatorname{span}(v_1, \dots, v_m))^\perp.$$

"$\subseteq$": Let $u \in \{v_1 \dots v_m\}^\perp$: $\langle u, v_i \rangle = 0 \ \forall i \in \{1 \dots m\}$. Let $v \in \operatorname{span}(v_1 \dots v_m)$. We have $\langle u, v \rangle = \langle u, \sum_{i=1}^{m} a_i v_i \rangle = 0$

$\Rightarrow u \in (\operatorname{span}(v_1 \dots v_m))^\perp$

"$\supseteq$": Let $u \in (\operatorname{span}(v_1 \dots v_m))^\perp$: $\langle u, \sum_{i=1}^{m} a_i v_i \rangle = 0 \ \forall a_1 \dots a_m \in \mathbb{F}^m$. In particular, for $a_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$ for all $i \in \{1 \dots m\}$

we have $\langle u, v_i \rangle = 0 \ \forall i \in \{1 \dots m\}$, implying $u \in \{v_1 \dots v_m\}^\perp$.

$\Rightarrow \{v_1 \dots v_m\}^\perp = (\operatorname{span}(v_1 \dots v_m))^\perp$

2 Suppose U is a subspace of V with basis $u_1, \dots, u_m$ and
$$u_1, \dots, u_m, v_1, \dots, v_n$$
is a basis of V. Prove that if the Gram–Schmidt procedure is applied to the basis of V above, producing a list $e_1, \dots, e_m, f_1, \dots, f_n$, then $e_1, \dots, e_m$ is an orthonormal basis of U and $f_1, \dots, f_n$ is an orthonormal basis of $U^\perp$.

Gram Schmidt: $v_1 \dots v_m \to e_1 \dots e_m$ s.t. $\begin{cases} e_1 \dots e_m \text{ orth.} \\ \operatorname{span}(e_1 \dots e_k) = \operatorname{span}(v_1 \dots v_k) \ \forall k \end{cases}$

We have $\operatorname{span}(u_1 \dots u_m) = \operatorname{span}(e_1 \dots e_m) = U$ (by property of G-S procedure). And since $e_1 \dots e_m$ are orth.,

then $e_1 \dots e_m$ is an orth. basis of U.

Furthermore, we have from 6.49: $V = U^\perp \oplus U$. Hence $f_1 \dots f_n$ is an orth. basis of $U^\perp$.

4 Suppose $e_1, \dots, e_n$ is a list of vectors in V with $\|e_k\| = 1$ for each $k = 1, \dots, n$ and
$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \dots + |\langle v, e_n \rangle|^2$$
for all $v \in V$. Prove that $e_1, \dots, e_n$ is an orthonormal basis of V.

This exercise provides a converse to 6.30(b).

We can show $\langle e_i, e_j \rangle = 0 \ \forall i,j \in \{1 \dots m\}, i \neq j$ (orthonormal)

Let $j \in \{1 \dots m\}$.

$\|e_j\|^2 = 1 = \sum_{i=1}^{m} |\langle e_j, e_i \rangle|^2 = 1 + \sum_{\substack{i=1 \\ i \neq j}}^{m} |\langle e_j, e_i \rangle|^2 \Rightarrow \langle e_j, e_i \rangle = 0 \ \forall j \neq i$

$\Rightarrow e_1 \dots e_m$ orthonormal list.

By 6B ex 3, $\operatorname{span}(e_1 \dots e_m) = V$.

5 Suppose that V is finite-dimensional and U is a subspace of V. Show that $P_{U^\perp} = I - P_U$, where I is the identity operator on V.

Let $v \in V$, s.t. $v = u + u^\perp$, with $u \in U$, $u^\perp \in U^\perp$ (by 6.49)

$P_{U^\perp} v = P_{U^\perp}(u + u^\perp) = u^\perp = v - u = (I - P_U) v$

6 Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Show that
$$T = T P_{(\operatorname{null} T)^\perp} = P_{\operatorname{range} T} T.$$

Let $v \in V$.

$P_{\operatorname{range} T}(Tv) = Tv$ (since $Tv \in \operatorname{range} T$)

$T P_{(\operatorname{null} T)^\perp} v = T P_{(\operatorname{null} T)^\perp}(u + w)$ (with $u \in \operatorname{null} T$, $w \in (\operatorname{null} T)^\perp$ (6.49))

$= Tw = Tw + Tu$ (since $Tu = 0$)

$= T(w + u) = Tv$

7 Suppose that X and Y are finite-dimensional subspaces of V. Prove that $P_X P_Y = 0$ if and only if $\langle x, y \rangle = 0$ for all $x \in X$ and all $y \in Y$.

"$\Rightarrow$": Suppose $P_X P_Y = 0$. Let $x \in X$, $y \in Y$.

$\langle x, y\rangle = \langle x, P_X y + P_{X^\perp} y\rangle$ (6.49)

$= \langle x, P_X P_Y y + P_{X^\perp} y\rangle$ ($y = P_Y y$)

$= \langle x, P_{X^\perp} y\rangle$ ($P_X P_Y = 0$)

$= 0$ ($\langle x, v\rangle = 0\ \forall v \in X^\perp$ since $x \in X$)

"$\Leftarrow$": Suppose $\forall x \in X, \forall y \in Y$: $\langle x, y\rangle = 0$ (1)

Let $v \in V$.

$P_X P_Y v = P_X P_Y (\underbrace{v_Y}_{\in Y} + \underbrace{v_{Y^\perp}}_{\in Y^\perp}) = P_X v_Y = P_X(\underbrace{v_{Y_X}}_{\in Y\cap X} + \underbrace{v_{Y_{X^\perp}}}_{\in Y \cap X^\perp}) = v_{Y_X}$, where $v_{Y_X} \in X \cap Y$.

But then $\langle x, x\rangle = 0$ (using (1)), thus $x = 0$, and $P_X P_Y = 0$.

8 Suppose U is a finite-dimensional subspace of V and $v \in V$. Define a linear functional $\varphi: U \to \mathbf{F}$ by

$$\varphi(u) = \langle u, v\rangle$$

for all $u \in U$. By the Riesz representation theorem (6.42) as applied to the inner product space U, there exists a unique vector $w \in U$ such that

$$\varphi(u) = \langle u, w\rangle$$

for all $u \in U$. Show that $w = P_U v$.

$\langle u, P_U v\rangle = \langle u, v - P_{U^\perp} v\rangle = \langle u, v\rangle - \cancel{\langle u, P_{U^\perp} v\rangle}^{=0} = \langle u, v\rangle$

Thus $w = P_U v$.

9 Suppose V is finite-dimensional. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in null P is orthogonal to every vector in range P. Prove that there exists a subspace U of V such that $P = P_U$.

$x \in$ null T, $x \in$ range T $\Rightarrow \langle x, x\rangle = 0 \Rightarrow$ null $T \cap$ range $T = \{0\} \Rightarrow V =$ null $T \oplus$ range T

Let $x \in$ null $P \cap$ range P: $\langle \underbrace{x}_{\in \text{null } P}, \underbrace{x}_{\in \text{range } P}\rangle = 0 \overset{\text{Ex 4.50}}{\Longrightarrow} V = \text{null } P \oplus \text{range } P \Longrightarrow \text{null } P = (\text{range } P)^\perp$

Let $v \in V$, $v = m + n$, $m \in$ null P, $n \in$ range P s.t. $Pu = n$, $u \in V$.

$Pv = P(m+n) = Pn = P^2 u = Pu$

$P_{\text{range } P}(\underbrace{m}_{\in (\text{range } P)^\perp} + n) = P_{\text{range } P}\, n = n = Pu$

$\Longrightarrow$ range P is a subspace of V s.t. $P = P_{\text{range } P}$

11 Suppose $T \in \mathcal{L}(V)$ and U is a finite-dimensional subspace of V. Prove that

$$U \text{ is invariant under } T \iff P_U T P_U = T P_U.$$

"$\Rightarrow$": Suppose U is invariant under T. Let $v \in V$, $v = u + u_\perp$, $u \in U$, $u_\perp \in U^\perp$.

$P_U T P_U v = P_U T P_U (u + u_\perp) = P_U \underbrace{Tu}_{\in U \text{ (invariant under } T)} = Tu$

$T P_U v = T P_U (u + u_\perp) = Tu$

$\Longrightarrow P_U T P_U = T P_U$

"$\Leftarrow$": Suppose $P_U T P_U = T P_U$ (1)

Let $u \in U$. $T P_U u = Tu \overset{(1)}{=} P_U T P_U u \Longrightarrow Tu = P_U(T P_U u) \in U$

$\Longrightarrow U$ invariant under T

12 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V. Prove that

$$U \text{ and } U^\perp \text{ are both invariant under } T \iff P_U T = T P_U.$$

"$\Rightarrow$": Suppose U and $U^\perp$ invariant under T.

$U^\perp$ invariant under T

U and $U^\perp$ are both invariant under $T \iff P_U T = T P_U$.

"$\Rightarrow$": Suppose U and $U^\perp$ invariant under T.

Let $v \in V$, $v = u + u_\perp$, $u \in U$, $u_\perp \in U^\perp$. $=0$ since $U^\perp$ invariant under T

$P_U T v = P_U T(u + u_\perp) = P_U T u + P_U T u_\perp = T u$

$T P_U v = T P_U (u + u_\perp) = T P_U u + T P_U u_\perp = T u$

$\Rightarrow P_U T = T P_U$

"$\Leftarrow$": Suppose $P_U T = T P_U$.

Let $u \in U$: $Tu = T P_U u = P_U T u \in U \Rightarrow U$ invariant under T

Let $u_\perp \in U^\perp$: $P_U T u_\perp = T P_U u_\perp = 0 \Rightarrow T u_\perp \in U^\perp \Rightarrow U^\perp$ invariant under T

13 Suppose $\mathbf{F} = \mathbf{R}$ and V is finite-dimensional. For each $v \in V$, let φ_v denote the linear functional on V defined by

$$\varphi_v(u) = \langle u, v \rangle$$

for all $u \in V$.

(a) Show that $v \mapsto \varphi_v$ is an injective linear map from V to V'.

(b) Use (a) and a dimension-counting argument to show that $v \mapsto \varphi_v$ is an isomorphism from V onto V'.

The purpose of this exercise is to give an alternative proof of the Riesz representation theorem (6.42 and 6.58) when $\mathbf{F} = \mathbf{R}$. Thus you should not use the Riesz representation theorem as a tool in your solution.

a) Let $\Psi \in \mathcal{L}(V, V')$ s.t. $\Psi v = \varphi_v$.

We can check Ψ is a linear map: let $u, v, w \in V$, $\lambda \in \mathbf{R}$:

- $[\Psi(v+w)](u) = \varphi_{v+w}(u) = \langle v+w, u\rangle = \langle v, u\rangle + \langle w, u\rangle = \varphi_v u + \varphi_w u = [\Psi(v)](u) + [\Psi(w)](u)$
- $[\Psi(\lambda v)](u) = \varphi_{\lambda v}(u) = \langle \lambda v, u\rangle = \lambda\langle v, u\rangle = \lambda \varphi_v u = \lambda[\Psi(v)](u)$.

Ψ is injective:

Let $v \in V$, s.t. $\varphi_v = 0$. Then $\varphi_v(v) = \langle v, v\rangle = 0 \Rightarrow v = 0 \Rightarrow \operatorname{null}\Psi = \{0\}$

$\Rightarrow \Psi$ injective linear map

b) $\dim V = \dim V'$, so an injective linear map from V to V' is an isomorphism.

Thus Ψ is an isomorphism of V onto V'.

14 Suppose that $e_1, \dots, e_n$ is an orthonormal basis of V. Explain why the dual basis (see 3.112) of $e_1, \dots, e_n$ is $e_1, \dots, e_n$ under the identification of V' with V provided by the Riesz representation theorem (6.58).

Let $\varphi_1 \dots \varphi_n$ the dual basis of $e_1 \dots e_n$.

$$\varphi_i e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

$\varphi_i e_j = \langle v_i, e_j \rangle$ (6.58)

$\forall i$: $\langle v_i, e_i\rangle = 1$, $\langle v_i, e_j\rangle = 0 \Rightarrow v_i = e_i$

15 In $\mathbf{R}^4$, let

$$U = \operatorname{span}((1,1,0,0), (1,1,1,2)).$$

Find $u \in U$ such that $\|u - (1,2,3,4)\|$ is as small as possible.

We can find an orth. basis of U.

$e_1 = \dfrac{(1,1,0,0)}{\|(1,1,0,0)\|} = \sqrt{2}^{-1}(1,1,0,0)$

$$e_1 = \frac{(1,1,0,0)}{\|(1,1,0,0)\|} = \sqrt{2}^{-1}(1,1,0,0)$$

$$f_2 = v_2 - \frac{\langle v_2, f_1 \rangle}{\|f_1\|^2} f_1 = (1,1,1,2) - \frac{\langle (1,1,1,2), (1,1,0,0) \rangle}{2}(1,1,0,0) = (1,1,1,2) - (1,1,0,0) = (0,0,1,2)$$

$$\Rightarrow e_2 = \frac{(0,0,1,2)}{\|(0,0,1,2)\|} = \sqrt{5}^{-1}(0,0,1,2)$$

$$v = \underbrace{\langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2}_{\in U} + \underbrace{v - \langle v, e_1 \rangle e_1 - \langle v, e_2 \rangle e_2}_{\in U^\perp}$$

$$P_U(1,2,3,4) = \langle (1,2,3,4), \sqrt{2}^{-1}(1,1,0,0) \rangle \sqrt{2}^{-1}(1,1,0,0) + \langle (1,2,3,4), \sqrt{5}^{-1}(0,0,1,2) \rangle \sqrt{5}^{-1}(0,0,1,2)$$

$$= \frac{1}{2}(3,3,0,0) + \frac{1}{5}(0,0,11,22)$$

$$= \left(\frac{3}{2}, \frac{3}{2}, \frac{11}{5}, \frac{22}{5}\right)$$

this value minimizes $\|u - (1,2,3,4)\|$ in U.

16 Suppose $C[-1,1]$ is the vector space of continuous real-valued functions on the interval $[-1,1]$ with inner product given by

$$\langle f, g \rangle = \int_{-1}^{1} fg$$

for all $f, g \in C[-1,1]$. Let U be the subspace of $C[-1,1]$ defined by

$$U = \{f \in C[-1,1] : f(0) = 0\}.$$

(a) Show that $U^\perp = \{0\}$.

(b) Show that 6.49 and 6.52 do not hold without the finite-dimensional hypothesis.

a) Let $f \in U^\perp$. Define $g \in U$ s.t. $g(x) = x^2 f(x)$

$\langle f, g \rangle = \int_{-1}^{1} (x f(x))^2 \, dx = 0$. However $[x f(x)]^2 \geq 0 \ \forall x \in [-1,1]$.

This implies $x f(x) = 0 \ \forall x \in [-1,1]$, so $f(x) = 0$ for $x \in [-1,1] \setminus \{0\}$.

$f \in C[-1,1]$, so $f(0) = 0$. Thus we have $f = 0$, and $U^\perp = \{0\}$.

b) Here U is infinite dimensional, and we have $U \oplus U^\perp \neq C[-1,1]$, as $U^\perp = \{0\}$ and U does not include $f(x) = 1$, and $f \in C[-1,1]$. Thus 6.49 does not hold.

Furthermore, we have $(U^\perp)^\perp = \{0\}^\perp = C[-1,1] \neq U$, thus 6.52 does not hold either in this case.

17 Find $p \in \mathcal{P}_3(\mathbf{R})$ such that $p(0) = 0$, $p'(0) = 0$, and $\int_0^1 |2 + 3x - p(x)|^2 \, dx$ is as small as possible.

Define $U = \{p \in \mathcal{P}_3(\mathbb{R}) : p(0) = 0 \wedge p'(0) = 0\}$

Consider the inner-product space $\mathcal{P}_3(\mathbb{R})$ with $\langle p, q \rangle = \int_0^1 p(x) q(x) \, dx$.

$\int_0^1 (2+3x-p(x))^2 \, dx$ is minimized for $\|2+3x-p(x)\|$ minimized. We can thus find $P_U(2+3x)$.

A basis of U: x^2, x^3. We can apply the G-S procedure on it to find an orth. basis:

$f_1 = x^2$

$f_1 = x^2$

$f_2 = x^3 - \frac{\langle x^3, x^2\rangle}{\|x^2\|^2} x^2$

$= x^3 - \frac{5}{6} x^2$

$\Longrightarrow e_1 = 5x^2, \quad e_2 = 252\left(x^3 - \frac{5}{6}x^2\right)$

$\int_0^1 x^5\, dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6} \qquad \|x^2\|^2 = \int_0^1 x^4\, dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$

$\|x^3 - \frac{5}{6}x^2\|^2 = \int_0^1 \left(x^3 - \frac{5}{6}x^2\right)^2 dx = \int_0^1 x^6 + \frac{25}{36}x^4 - \frac{5}{3}x^5\, dx$

$= \left[\frac{x^7}{7}\right]_0^1 + \frac{25}{36}\left[\frac{x^5}{5}\right]_0^1 - \frac{5}{3}\left[\frac{x^6}{6}\right]_0^1$

$= \frac{1}{7} + \frac{5}{36} - \frac{5}{18} = \frac{1}{7} - \frac{5}{36} = \frac{36}{252} - \frac{35}{252} = \frac{1}{252}$

$p = \langle e_1, 2+3x\rangle e_1 + \langle e_2, 2+3x\rangle e_2$

$= 10 \langle x^2, 2+3x\rangle x^2 + 252^2 \langle x^3 - \frac{5}{6}x^2, 2+3x\rangle$

...

19 Suppose V is finite-dimensional and $P \in \mathcal{L}(V)$ is an orthogonal projection of V onto some subspace of V. Prove that $P^\dagger = P$.

Let U the subspace s.t. $P = P_U$.

We have $\operatorname{null} P = U^\perp \Rightarrow (\operatorname{null} P)^\perp = U$ (6.52)

$\Rightarrow P|_{(\operatorname{null} P)^\perp} = P|_U = I_U$, hence $\left(P|_{(\operatorname{null} P)^\perp}\right)^{-1} = I_U$.

By definition: $P^\dagger = \left(P|_{(\operatorname{null} P)^\perp}\right)^{-1} P_{\operatorname{range} P} = I_U P_U = P$ (range $P = U$ as it is a projection on U).

22 Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$TT^\dagger T = T \quad \text{and} \quad T^\dagger T T^\dagger = T^\dagger.$$

Both formulas above clearly hold if T is invertible because in that case we can replace $T^\dagger$ with T^{-1}.

$TT^\dagger T = P_{\operatorname{range} T} T = T$ (6.69 b))

$\Rightarrow TT^\dagger T = T$

$T^\dagger T T^\dagger = T^\dagger P_{\operatorname{range} T} = \left(T|_{(\operatorname{null} T)^\perp}\right)^{-1} \overbrace{P_{\operatorname{range} T} P_{\operatorname{range} T}}^{= P_{\operatorname{range} T}} = T^\dagger$

23 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$(T^\dagger)^\dagger = T.$$

The equation above is analogous to the equation $(T^{-1})^{-1} = T$ that holds if T is invertible.

$(T^\dagger)^\dagger = \left[\left(T|_{(\operatorname{null} T)^\perp}\right)^{-1} P_{\operatorname{range} T}\right]^\dagger$

$\operatorname{null}\left[\left(T|_{(\operatorname{null} T)^\perp}\right)^{-1} P_{\operatorname{range} T}\right]^\perp = \left[(\operatorname{range} T)^\perp\right]^\perp = \operatorname{range} T$

$\operatorname{range}\left[\left(T|_{(\operatorname{null} T)^\perp}\right)^{-1} P_{\operatorname{range} T}\right] = (\operatorname{null} T)^\perp$

$\left[\left[\left(T|_{(\operatorname{null} T)^\perp}\right)^{-1} P_{\operatorname{range} T}\right]\Big|_{\operatorname{range} T}\right]^{-1} P_{(\operatorname{null} T)^\perp} = \left(T|_{(\operatorname{null} T)^\perp}\right)^{-1^{-1}} P_{(\operatorname{null} T)^\perp}$

$$\Longrightarrow (T^{\dagger})^{\dagger} = T\big|_{(\text{null}\,T)^{\perp}} P_{(\text{null}\,T)^{\perp}}$$

let $v \in V$: $(T^{\dagger})^{\dagger} v = T\big|_{(\text{null}\,T)^{\perp}} P_{(\text{null}\,T)^{\perp}} (n + n_{\perp})$ $\quad (n \in \text{null}\,T,\ n_{\perp} \in (\text{null}\,T)^{\perp},\ n + n_{\perp} = v)$

$P_{(\text{null}\,T)^{\perp}} n = 0$

$P_{(\text{null}\,T)^{\perp}} n_{\perp} = n_{\perp}$

$$= T\big|_{(\text{null}\,T)^{\perp}} n_{\perp} = T n_{\perp} = T v \quad (\text{since } T n = 0)$$

$$\Longrightarrow (T^{\dagger})^{\dagger} = T$$