3B Exercises

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1 Give an example of a linear map T with $\dim \operatorname{null} T = 3$ and $\dim \operatorname{range} T = 2$.

$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$
$= 3 + 2 = 5$ (1)

$T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^5)$, $Tx = (x_1, x_2, 0, 0, 0)$

$\dim \{(x_1, x_2, 0, 0, 0) \in \mathbb{R}^5\} = 2$

$\dim \mathbb{R}^5 = 5 \Rightarrow \dim \operatorname{null} T = 3$

2 Suppose $S, T \in \mathcal{L}(V)$ are such that $\operatorname{range} S \subseteq \operatorname{null} T$. Prove that $(ST)^2 = 0$.

$\operatorname{range} S \subseteq \operatorname{null} T \Rightarrow TS = 0$, as $\forall v \in V$ $T \underbrace{Sv}_{\in \operatorname{null} T} = 0$

$(ST)^2 = (ST)(ST) = S(TS)T$ (associativity of linear map multiplication)

$= S0T = 0$ (as $S0 = 0T = 0 \ \forall S, T \in \mathcal{L}(V)$)

3 Suppose $v_1, \dots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbf{F}^m, V)$ by

$$T(z_1, \dots, z_m) = z_1 v_1 + \cdots + z_m v_m.$$

(a) What property of T corresponds to $v_1, \dots, v_m$ spanning V?
(b) What property of T corresponds to the list $v_1, \dots, v_m$ being linearly independent?

a) T surjective $\Leftrightarrow v_1 \dots v_m$ spanning V

$v_1 \dots v_m$ spanning $V \Leftrightarrow \forall v \in V \ \exists z_1 \dots z_m \in \mathbb{F}$ s.t. $v = \sum_{i=1}^m z_i v_i$

$\Leftrightarrow \forall v \in V \ \exists (z_1 \dots z_m) \in \mathbb{F}^m$ s.t. $\sum_{i=1}^m z_i v_i = T(z_1 \dots z_m)$

$\Leftrightarrow T$ surjective

b) T injective $\Leftrightarrow v_1 \dots v_m$ linearly independent

$v_1 \dots v_m$ linearly independent $\Leftrightarrow \forall v \in \operatorname{span}(v_1 \dots v_m) \ \exists! z_1 \dots z_m$ s.t. $v = \sum_{i=1}^m z_i v_i$

$\Leftrightarrow (T(z_1 \dots z_m) = T(z'_1 \dots z'_m) \Rightarrow (z_1 \dots z_m) = (z'_1 \dots z'_m))$

4 Show that $\{T \in \mathcal{L}(\mathbf{R}^5, \mathbf{R}^4) : \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbf{R}^5, \mathbf{R}^4)$.

$T(x_1 \dots x_5) = (x_1, x_2, 0, 0)$

$S(x_1 \dots x_5) = (0, 0, x_3, x_4)$

similar reasoning

$T(x_1 \dots x_5) = 0 \Rightarrow \begin{cases} x_1 = x_2 = 0 \\ x_3, x_4, x_5 \in \mathbb{R} \end{cases} \Rightarrow \dim \operatorname{null} T = 3 = \dim \operatorname{null} S$

$(T+S)(x_1 \dots x_5) = 0 \Rightarrow (x_1, x_2, x_3, x_4) = 0 \Rightarrow \begin{cases} x_1 = \dots = x_4 = 0 \\ x_5 \in \mathbb{R} \end{cases} \Rightarrow \dim \operatorname{null} T+S = 1 \leq 2$

Addition is not closed on this subset, so it is <u>not</u> a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

5 Give an example of $T \in \mathcal{L}(\mathbf{R}^4)$ such that $\operatorname{range} T = \operatorname{null} T$.

We must have $\dim \operatorname{range} T = \dim \operatorname{null} T = \frac{\dim \mathbb{R}^4}{2} = \frac{4}{2} = 2$

$T(x_1, x_2, x_3, x_4) = (x_3, x_4, 0, 0)$

Let $x \in \operatorname{range} T$. $x = (x_1, x_2, 0, 0)$ $Tx = 0$, so $x \in \operatorname{null} T \Rightarrow \operatorname{range} T \subseteq \operatorname{null} T$

Let $x \in \operatorname{null} T$. $Tx = 0 \Rightarrow (x_3, x_4, 0, 0) = 0 \Rightarrow x_3 = x_4 = 0 \Rightarrow x \in \operatorname{range} T \Rightarrow \operatorname{null} T \subseteq \operatorname{range} T$

$\Rightarrow \operatorname{range} T = \operatorname{null} T$

6 Prove that there does not exist $T \in \mathcal{L}(\mathbf{R}^5)$ such that $\operatorname{range} T = \operatorname{null} T$.

let $T \in \mathcal{L}(\mathbb{R}^5)$, with $m = \dim \operatorname{range} T = \dim \operatorname{null} T$

$\dim \mathbb{R}^5 = \dim \operatorname{null} T + \dim \operatorname{range} T$

$\Rightarrow 5 = 2m \Rightarrow m = 5/2 \notin \mathbb{N}$

A number of dimension cannot not be an integer, therefore $\dim \operatorname{range} T \neq \dim \operatorname{null} T$ and therefore $\operatorname{range} T \neq \operatorname{null} T$.

7 Suppose V and W are finite-dimensional with $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Let $U = \{T \in \mathcal{L}(V,W) : T \text{ is not injective}\}$

Let $v_1 \dots v_n$ basis of V, $w_1 \dots w_m$ basis of W, with $2 \leq n \leq m$

$Tv_1 = w_2$, $Tv_i = w_i \ \forall i \geq 2$ (using $2 \leq n \leq m$)

$Sv_2 = -w_1$, $Sv_i = w_i \ \forall i \neq 2$

$Tv_1 = Tv_2 = w_2$, and $v_1 \neq v_2$ (linearly independent). Similar reasoning for S

$\Rightarrow T, S \in U$

$(T+S)v = Tv + Sv = (a_1 + a_2)w_2 + \sum_{i=3}^{n} a_i w_i + (a_1 - a_2)w_1 + \sum_{i=3}^{n} a_i w_i = 0$

$v_1 \dots w_n$ linearly independent $\Rightarrow \begin{cases} a_1 + a_2 = 0 \\ a_1 - a_2 = 0 \\ a_3 = 0 \\ \vdots \\ a_n = 0 \end{cases} \Rightarrow \begin{cases} a_1 = a_2 \\ 2a_1 = 0 \\ a_3 = 0 \\ \vdots \\ a_n = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ \vdots \\ a_n = 0 \end{cases} \Rightarrow v = 0$

$\Rightarrow \operatorname{null} T+S = \{0\} \Rightarrow T+S$ is injective, hence $T+S \notin U$ despite $T, S \in U$

$\Rightarrow U$ is not a subspace of $\mathcal{L}(V,W)$

8 Suppose V and W are finite-dimensional with $\dim V \geq \dim W \geq 2$. Show that $\{T \in \mathcal{L}(V, W) : T \text{ is not surjective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

$Tv_i = w_i \ \forall i = 1 \dots m-1$, $Tv_i = 0, \forall i \geq m$ (not surjective, w_m is not in the range)

$Sv_m = w_m$, $Sv_i = 0 \ \forall i \neq m$ (not surjective, w_1 is not in the range)

$\Rightarrow T, S \in U = \{T \in \mathcal{L}(V,W) : T \text{ is not surjective}\}$

Let $w \in W$

$w = \sum_{i=1}^{m} a_i w_i$, for some $a_i \in \mathbb{F} \ \forall i$

$= \sum_{i=1}^{m-1} a_i w_i + a_m w_m$

$= Tv + Sv$, with $v = \sum_{i=1}^{m} a_i v_i + \sum_{i=m+1}^{n} b_i v_i$, for some b_i's $\in \mathbb{F}$

$\Rightarrow w \in \operatorname{range}(T+S)$

$\Rightarrow T+S$ is surjective

$\Rightarrow +$ is not closed on $T+S$, meaning U is not a subspace of $\mathcal{L}(V,W)$

9 Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \dots, v_n$ is linearly independent in V. Prove that $Tv_1, \dots, Tv_n$ is linearly independent in W.

T injective $\Rightarrow Tv_1, \dots, Tv_m$ linearly independent

$\Leftrightarrow$ ($Tv_1, \dots, Tv_m$ linearly dependent $\Rightarrow T$ not injective)

$Tv_1, \dots, Tv_m$ linearly dependent $\Rightarrow \exists\, v_i, v_j\ \exists \lambda \in \mathbb{F}$ s.t. $Tv_i = \lambda T v_j$

$\Rightarrow Tv_i = T\lambda v_j$, but $v_i \neq \lambda v_j$ as $v_1, \dots, v_m$ is linearly independent.

$\Rightarrow T$ is <u>not</u> injective

10 Suppose $v_1, \dots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Show that $Tv_1, \dots, Tv_n$ spans range T.

let $w \in$ range T. $\exists\, v \in V$ s.t. $Tv = w$, $v = \sum_{i=1}^{m} a_i v_i$, with $a_1 \dots a_m \in \mathbb{F}$, $m = \dim V$

$\Rightarrow w = T \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} a_i T v_i$

$\Rightarrow$ Every element of range T can be written as a linear combination of $Tv_1 \dots Tv_m$, thus this list spans range T

11 Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that

$$U \cap \operatorname{null} T = \{0\} \quad \text{and} \quad \operatorname{range} T = \{Tu : u \in U\}.$$

let U a subspace of V. $\exists\, \bar{U}$ subspace of s.t $U \oplus \bar{U} = V$, and $U \cap \bar{U} = \{0\}$.

let $u_1, \dots u_\ell$ a basis of U, $\bar{u}_1, \dots \bar{u}_\ell$, $\ell = \dim U$, and $\bar{u}_1, \dots \bar{u}_p$ a basis of $\bar{U}$, $p = \dim \bar{U}$

We can define the map $T \in \mathcal{L}(V, W)$ s.t. $Tv = \sum_{i=1}^{k} a_i u_i$, with $v = \sum_{i=1}^{k} a_i u_i + \sum_{i=1}^{p} b_i \bar{u}_i$

null $T = \bar{U}$

"$\supseteq$": let $\bar{u} \in \bar{U}$. $T\bar{u} = 0$ by definition of T.

"$\subseteq$": let $v \in$ null T. $Tv = 0 \overset{(U \oplus \bar{U} = V)}{\Rightarrow} T(u + \bar{u}) = 0$, $u \in U$, $\bar{u} \in \bar{U}$

$\Rightarrow Tu + \underbrace{T\bar{u}}_{=0} = 0 \Rightarrow Tu = 0 \Rightarrow u = 0 \Rightarrow v = \bar{u} \in \bar{U}$

$\Rightarrow U \cap$ null $T = U \cap \bar{U} = \{0\}$

range $T = \{Tu : u \in U\}$

"$\subseteq$": let $w \in$ range T. $\exists\, v \in V$ s.t. $w = Tv = T(u + \bar{u})$, with $u \in U$, $\bar{u} \in \bar{U}$

$\Rightarrow w = Tu + \underbrace{T\bar{u}}_{=0} = Tu \Rightarrow w \in \{Tu : u \in U\}$

"$\supseteq$": let $w \in \{Tu : u \in U\} \Rightarrow \exists\, u \in U\ Tu = w \Rightarrow w \in$ range T

12 Suppose T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that

$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\}.$$

Prove that T is surjective.

$\dim \mathbb{F}^4 = \dim$ null $T + \dim$ range T

$\Rightarrow 4 = 2 + \dim$ range $T \Rightarrow \dim$ range $T = 2 = \dim \mathbb{F}^2$

[illegible]

$\Rightarrow 4 = 2 + \dim \operatorname{range} T \Rightarrow \dim \operatorname{range} T = 2 = \dim \mathbb{F}^2$

$\Rightarrow \operatorname{range} T = \mathbb{F}^2 \Rightarrow T$ is surjective

13 Suppose U is a three-dimensional subspace of $\mathbf{R}^8$ and that T is a linear map from $\mathbf{R}^8$ to $\mathbf{R}^5$ such that null $T = U$. Prove that T is surjective.

$\dim \mathbb{R}^8 = \dim \operatorname{null} T + \dim \operatorname{range} T$

$\Rightarrow 8 = 3 + \dim \operatorname{range} T \Rightarrow \dim \operatorname{range} T = 5 = \dim \mathbb{R}^5 \Rightarrow \operatorname{range} T = \mathbb{R}^5$

$\Rightarrow T$ is surjective

14 Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals $\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}$.

Such a subspace would be of dimension 1. We would then have:

$\dim \mathbb{F}^5 = 5 = 1 + \dim \operatorname{range} T \Rightarrow \dim \operatorname{range} T = 4$

However, the dimension of the arrival space is $2 < 4$, so it is not possible.

16 Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

"$\Leftarrow$": $\exists$ linear map from V to $W \Rightarrow \dim V \leq \dim W$ (by contraposition of 3.22)

"$\Rightarrow$": $\dim V \leq \dim W$. Let $v_1 \dots v_m$ basis of V, $w_1 \dots w_m$ basis of W

Define $Tv_i = w_i$ for $i = 1 \dots m$. Let $v = \sum_{i=1}^{m} a_i v_i \in V$, $a_i \in \mathbb{F}\ \forall i$

$Tv = \sum_{i=1}^{m} a_i T v_i = \sum_{i=1}^{m} a_i w_i = 0 \Rightarrow \begin{cases} a_1 = 0 \\ \vdots \\ a_m = 0 \end{cases}$, as $w_1 \dots w_m$ linearly independent, and thus $w_1 \dots w_m$ linearly ind.

$\Rightarrow v = 0 \Rightarrow \operatorname{null} T = \{0\} \Rightarrow T$ injective.

18 Suppose V and W are finite-dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = U$ if and only if $\dim U \geq \dim V - \dim W$.

"$\Rightarrow$" $\operatorname{null} T = U \Rightarrow \dim \operatorname{null} T = \dim U$

$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

$\Rightarrow \dim U = \dim V - \dim \operatorname{range} T$

$\Rightarrow \dim U \geq \dim V - \dim W$ (as $\dim W \geq \dim \operatorname{range} T$)

"$\Leftarrow$": Let $\bar{U}$ subspace of V s.t. $U \oplus \bar{U} = V$, $u_1 \dots u_n$ basis of U, $\bar{u}_1 \dots \bar{u}_m$ basis of W

Let $T \in \mathcal{L}(V, W)$ s.t. $Tu_i = 0\ \forall i$, $T\bar{u}_i = w_i$.

Easy to check $\operatorname{null} T = U$ (done previously)

19 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V.

"$\Leftarrow$": $\forall v \in V$, $STv = v$

Suppose $v_1, v_2 \in V$ s.t. $Tv_1 = Tv_2$

$\Rightarrow STv_1 = STv_2 \xRightarrow{(ST \text{ identity})} v_1 = v_2$

$\Rightarrow T$ is injective

"$\Rightarrow$": Suppose T is injective

$\forall w \in \operatorname{range} T$, $\exists!\, v \in V$ s.t. $Tv = w$

$\forall w \in \text{range}\,T, \exists v \in V$ s.t. $Tv = w$

let $R \in \mathcal{L}(\text{range}\,T, V)$ s.t. $Rw = v$ $\forall w = Tv \in \text{range}\,T$

R can be extended to a linear map for W to V (following a previous exercise), that we can call S.

let $v \in V$. $STv = Sw = v$

$\Rightarrow ST$ is the identity on V

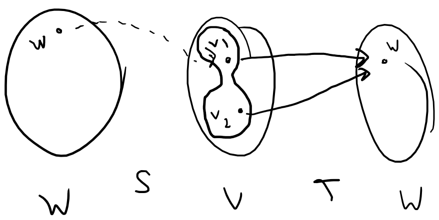
20 Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity operator on W.

"$\Leftarrow$": $\forall w \in W$, $TSw = w$

let $w \in W$. $TSw = w \Rightarrow T(Sw) = w$

$\Rightarrow w \in \text{range}\,T \Rightarrow T$ is surjective

"$\Rightarrow$":



T surjective $\Rightarrow \forall w \in W, \exists U^w \subseteq V$ s.t. $Tu = w$ $\forall u \in U^w$

let's construct a list $v_1 \dots v_m$ following this algorithm:

$\forall w_i \in$ basis of W, add $v_i \in V$ s.t. $Tv = w$ and $v_1 \dots v_i$ is linearly independent.

let's prove such a v_i exists. For $i = 1$, it is straightforward; T being surjective, $\exists v_1 \in V$ s.t. $Tv_1 = w_1$ and the list v_1 is linearly independent.

Now assume $v_1 \dots v_{i-1}$ is linearly independent. T is surjective so $\exists v_i \in V$ s.t. $Tv_i = w_i$. (T surjective) $v_1 \dots v_i$ linearly independent?

let $a_1 \dots a_i$ s.t. $\sum_{j=1}^{i} a_j v_j = 0 \Rightarrow \sum_{j=1}^{i} a_j T v_j = T0 \Rightarrow \sum_{j=1}^{i} a_j w_j = 0$

$\Rightarrow \begin{cases} a_1 = 0 \\ \vdots \\ a_i = 0 \end{cases} \Rightarrow v = 0 \Rightarrow v_1 \dots v_i$ linearly independent

We thus have a linearly independent list $v_1 \dots v_m \in V$, $m = \dim W$, s.t. $Tv_i = w_i$

We can then just define $S \in \mathcal{L}(W, V)$ with: $Sw_i = v_i$.

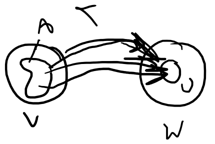
let $w \in W$, $w = \sum_{i=1}^{m} a_i w_i$, $a_i \in \mathbb{F}\ \forall i$

$(TS)w = T(Sw) = T \sum_{i=1}^{m} a_i S w_i = T \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} a_i T v_i = \sum_{i=1}^{m} a_i w_i = w$

$\Rightarrow \exists S \in \mathcal{L}(W, V)$ s.t. TS is the identity operator

21 Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and U is a subspace of W. Prove that $\{v \in V : Tv \in U\}$ is a subspace of V and

$$\dim\{v \in V : Tv \in U\} = \dim \operatorname{null} T + \dim(U \cap \operatorname{range} T).$$



$A = \{v \in V : Tv \in U\}$

- U subspace of $W \Rightarrow 0 \in U$. $T0 = 0 \Rightarrow 0 \in A$
- let $x, y \in A$. $T(x+y) = \underbrace{Tx}_{\in U} + \underbrace{Ty}_{\in U} \in U$ (as U subspace of W) $\Rightarrow x+y \in A$
- let $\lambda \in \mathbb{F}$, $x \in A$ $T(\lambda x) = \lambda \underbrace{Tx}_{\in U} \in U$ (as U subspace of W) $\Rightarrow \lambda x \in A$

$\Rightarrow A$ subspace of V

let $\bar{U}$ subspace of W s.t $U \oplus \bar{U} = W$, and $\bar{A}$ subspace of V s.t $A \oplus \bar{A} = V$.
let $T' \in L(V,W)$ s.t $\forall v = a + \bar{a} \in V$, $T'v = Ta$

$\dim A + \dim \bar{A} = \dim V = \dim \operatorname{null} T' + \dim \operatorname{range} T'$

- $\operatorname{range} T' = \{w \in W \mid \exists v \in V, T'v = w\} = \{w \in W \mid \exists a \in A, Ta = w\} = U \cap \operatorname{range} T$

Now we just need to prove $\dim \operatorname{null} T' = \dim \operatorname{null} T + \dim \bar{A}$

We can prove $\operatorname{null} T' = \operatorname{null} T + \bar{A}$
let $x \in \operatorname{null} T'$. $T'x = 0 \Rightarrow T'(a + \bar{a}) = Ta = 0 \Rightarrow a \in \operatorname{null} T$
$\Rightarrow x = a + \bar{a}$, with $a \in \operatorname{null} T$, $\bar{a} \in \bar{A}$,
so $x \in \operatorname{null} T + \bar{A}$

let $\underbrace{u}_{\in \operatorname{null} T} + \underbrace{\bar{a}}_{\in \bar{A}} \in \operatorname{null} T + \bar{A}$. $T'(u + \bar{a}) = Tu = 0 \Rightarrow u + \bar{a} \in \operatorname{null} T'$

Thus $\operatorname{null} T' = \operatorname{null} T + \bar{A}$. We can also show $\operatorname{null} T \cap \bar{A} = \{0\}$.
let $v \in V$, s.t $v \in \operatorname{null} T \cap \bar{A}$, so $Tv = 0$, and $(\underbrace{Tv \notin U}_{v \in \bar{A} \setminus A}$ or $v = 0)$
$Tv = 0 \in U$ (subspace), so $v = 0$.
Hence $\operatorname{null} T \cap \bar{A} = \{0\}$.

We have: $\dim A + \dim \bar{A} = \dim \operatorname{null} T' + \dim \operatorname{range} T'$
$\Rightarrow \dim A + \dim \bar{A} = \dim \operatorname{null} T + \bar{A} + \dim U \cap \operatorname{range} T$
$\Rightarrow \dim A + \dim \bar{A} = \dim \operatorname{null} T + \dim \bar{A} - \dim \cancel{\operatorname{null} T \cap \bar{A}}_{=0} + \dim U \cap \operatorname{range} T$
$\Rightarrow \dim A = \dim \operatorname{null} T + \dim U \cap \operatorname{range} T$

23 Suppose U and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{range} ST \le \min(\dim \operatorname{range} S, \dim \operatorname{range} T).$$

- $y \in$ range $ST \Rightarrow \exists u \in U$, $S\underbrace{Tu}_{\in V} = y$
  $\Rightarrow y \in$ range S
  $\Rightarrow$ range $ST \subseteq$ range S

$\Rightarrow$ dim range $ST \le$ dim range S (as both subspaces) $(*)$

- $x \in$ null $T \Rightarrow Tx = 0 \Rightarrow STx = 0 \Rightarrow x \in$ null $ST \Rightarrow$ null $T \subseteq$ null ST
  $\Rightarrow$ dim null $T \le$ dim null ST (1)

  dim U = dim null T + dim range T = dim null ST + dim range ST (2)

  (1), (2) $\Rightarrow$ dim range $T \ge$ dim range ST $(**)$

$\Rightarrow (*), (**) \Rightarrow$ dim range $ST \le$ min(dim range T, dim range S)

24 (a) Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \operatorname{range} TS \le 2$.
(b) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^5)$ with $ST = 0$ and $\dim \operatorname{range} TS = 2$.

a) $ST = 0 \Rightarrow$ dim null ST = dim V = 5

From previous exercises:

dim null $ST \le$ dim null S + dim null T

$\Rightarrow 5 \le$ dim null S + dim null T

$\Rightarrow 5 \le 5 -$ dim range S + $5 -$ dim range T

$\Rightarrow$ dim range S + dim range $T \le 5$

$\Rightarrow$ min (dim range S, dim range T) ≤ 2

And from previous exercise:

dim range $TS \le$ min(dim range S, dim range T)

$\Rightarrow$ dim range $TS \le 2$

b) $ST = 0$, dim range $TS = 2$

$Tx = (x_3, x_4, 0, 0, 0)$

$Sx = (0, 0, x_3, x_4, 0)$

$STx = S(x_3, x_4, 0, 0, 0) = 0$

$TSx = T(0, 0, x_3, x_4, 0) = (x_3, x_4, 0, 0, 0)$, so dim range $TS = 2$