2B Exercises

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1 Find all vector spaces that have exactly one basis.

The only vector space that has exactly one basis is $\{0\}$.
Indeed, if $v_1 \dots v_m$ is a basis, then $\lambda v_1, \dots, \lambda v_m$ is a basis too (easy to show). Thus, a unique basis would have to be empty or only include 0's.

0 in a list of vectors makes it not linearly Ind, so it cannot be a basis, so only the empty list possibility remains, which is the basis of $\{0\}$.

2 Verify all assertions in Example 2.27.

2.27 example: *bases*

(a) The list $(1, 0, \dots, 0), (0, 1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$ is a basis of $\mathbf{F}^n$, called the *standard basis* of $\mathbf{F}^n$.

(b) The list $(1, 2), (3, 5)$ is a basis of $\mathbf{F}^2$. Note that this list has length two, which is the same as the length of the standard basis of $\mathbf{F}^2$. In the next section, we will see that this is not a coincidence.

(c) The list $(1, 2, -4), (7, -5, 6)$ is linearly independent in $\mathbf{F}^3$ but is not a basis of $\mathbf{F}^3$ because it does not span $\mathbf{F}^3$.

(d) The list $(1, 2), (3, 5), (4, 13)$ spans $\mathbf{F}^2$ but is not a basis of $\mathbf{F}^2$ because it is not linearly independent.

(e) The list $(1, 1, 0), (0, 0, 1)$ is a basis of $\{(x, x, y) \in \mathbf{F}^3 : x, y \in \mathbf{F}\}$.

(f) The list $(1, -1, 0), (1, 0, -1)$ is a basis of

$$\{(x, y, z) \in \mathbf{F}^3 : x + y + z = 0\}.$$

(g) The list $1, z, \dots, z^m$ is a basis of $\mathcal{P}_m(\mathbf{F})$, called the *standard basis* of $\mathcal{P}_m(\mathbf{F})$.

a) • Let $v \in \mathbb{F}^m$ $v = (x_1, \dots, x_m) = (1, 0, \dots, 0)x_1 + \dots + (0, \dots, 0, 1)x_m$

$\Rightarrow$ This list spans $\mathbb{F}^m$

• Let $\lambda_1 \dots \lambda_m \in \mathbb{F}$. $\lambda_1(1, 0, \dots, 0) + \dots + (0, \dots, 0, 1)\lambda_m = 0$

$\Rightarrow (\lambda_1, \dots, \lambda_m) = 0 \Rightarrow \lambda_1 = \dots = \lambda_m = 0$

$\Rightarrow$ This list is linearly independent

$\Rightarrow (\alpha_1, \dots, \alpha_m) = 0 \Rightarrow \alpha_1 = \dots = \alpha_m = 0$

$\Rightarrow$ This list is linearly independent

$\Rightarrow$ Basis of $\mathbb{F}^m$

b) Let $v \in \mathbb{F}^2$. $v = (x, y)$

$$\begin{cases} x = 1a_1 + 3a_2 \\ y = 2a_1 + 5a_2 \end{cases} \Rightarrow \begin{cases} a_1 = x - 3a_2 \\ y = 2x - a_2 \end{cases} \Rightarrow \begin{cases} a_2 = 2x - y \\ a_1 = 3y - 5x \end{cases}$$

$\Rightarrow v = (x, y) = (3y - 5x)(1, 2) + (2x - y)(3, 5)$

$\Rightarrow$ The list spans $\mathbb{F}^2$

Let $a_1, a_2 \in \mathbb{F}$ s.t $a_1(1,2) + a_2(3,5) = 0$

$$\Rightarrow \begin{cases} a_1 + 3a_2 = 0 \\ 2a_1 + 5a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -3a_2 \\ a_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases}$$

$\Rightarrow$ The list is linearly independent

$\Longrightarrow$ Basis of $\mathbb{F}^2$

3 (a) Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

(b) Extend the basis in (a) to a basis of $\mathbf{R}^5$.

(c) Find a subspace W of $\mathbf{R}^5$ such that $\mathbf{R}^5 = U \oplus W$.

a) $(3,1,0,0,0), (0,0,7,1,0), (0,0,0,0,1)$

Easy to show it is a basis of U

b) We can add $(2,1,0,0,0)$ and $(0,0,2,1,0)$

Let $a_1 \dots a_5 \in \mathbb{F}$ s.t

$a_1(3,1,0,0,0) + a_2(0,0,7,1,0) + a_3(0,0,0,0,1) + a_4(2,1,0,0,0) + a_5(0,0,2,1,0) = 0$

$$\Rightarrow \begin{cases} 3a_1 + 2a_4 = 0 \\ a_1 + a_4 = 0 \\ 7a_2 + 2a_5 = 0 \\ a_2 + a_5 = 0 \\ a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -\frac{2}{3}a_4 \\ a_1 = -a_4 \\ a_2 = -\frac{2}{7}a_5 \\ a_2 = -a_5 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ a_4 = 0 \\ a_5 = 0 \end{cases}$$

$$\begin{cases} a_2 + a_5 = 0 \\ a_3 = 0 \end{cases} \quad \begin{cases} a_2 = -a_5 \\ a_3 = 0 \end{cases} \quad \begin{cases} a_4 = 0 \\ a_5 = 0 \end{cases}$$

=> linearly ⫫, and of size 5 ($\dim \mathbb{R}^5 = 5$) so basis of $\mathbb{R}^5$

(with the use of ~~the~~ the next chapter)

c) $\{(x_1, \frac{1}{2}x_1, x_2, \frac{1}{2}x_2, 0)$ is such a subspace

5 Suppose V is finite-dimensional and U, W are subspaces of V such that $V = U + W$. Prove that there exists a basis of V consisting of vectors in $U \cup W$.

Let v be a basis of V, of size m.

Since $V = U + W$, $\forall v_i \in v, \exists u_i \in U, \exists w_i \in W$ s.t.

$u_i = v_i + w_i$.

This implies $\forall x \in V, \exists a_1 \ldots a_m \in \mathbb{F}$ s.t.:

$$x = \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} a_i (u_i + w_i) = \sum_{i=1}^{m} a_i u_i + \sum_{i=1}^{m} a_i w_i$$

Thus $u_1 \ldots u_m, w_1 \ldots w_m$ span V.

According to 2.30, we can reduce this list to be a basis of V. This list would only contain v_i's ($\in U$) and w_i's ($\in W$), so vectors in $U \cup W$.

6 Prove or give a counterexample: If p_0, p_1, p_2, p_3 is a list in $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2, then p_0, p_1, p_2, p_3 is not a basis of $\mathcal{P}_3(\mathbf{F})$.

Let p_0, p_1, p_2, p_3 a list in $\mathcal{P}_3(\mathbb{F})$ such that $\deg(p_i) \neq 2 \ \forall i$

$\Rightarrow \forall i \ \exists k_{i0}, k_{i1}, k_{i3} \in \mathbb{F}$ s.t $p_i(X) = k_{i0} + k_{i1} X + k_{i3} X^3$

Let $p \in \mathcal{P}_3(\mathbb{F})$ s.t. $p(X) = X^2$

We can show $p \notin \operatorname{span}(p_0, \dots, p_3)$ and thus this list does not span $\mathcal{P}_3(\mathbb{F})$ and is not a basis of it.

Suppose $\exists a_0, a_1, a_2, a_3 \in \mathbb{F}$ s.t:

$$X^2 = \sum_{i=0}^{3} a_i p_i(X) = \sum_{i=0}^{3} k_{i0} + k_{i1} X + k_{i3} X^3$$

The polynom on the right has degree 0, 1 or 3, whereas $\deg(X^2) = 2$, so they are different. This is a contradiction.

7 Suppose v_1, v_2, v_3, v_4 is a basis of V. Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V.

Let $a_1 \dots a_4 \in \mathbb{F}$ s.t :

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4 v_4 = 0$$

$\Rightarrow a_1 v_1 + (a_1 + a_2) v_2 + (a_2 + a_3) v_3 + (a_3 + a_4) v_4 = 0$

v_i linearly ⫫

$$\Rightarrow \begin{cases} a_1 = 0 \\ a_1 + a_2 = 0 \\ a_2 + a_3 = 0 \\ a_3 + a_4 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \\ a_4 = 0 \end{cases} \Rightarrow$$ This list is linearly independent.

$\dim(V) = 4 \Rightarrow$ This list is a basis of V (see next chapter)

8 Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3 \notin U$ and $v_4 \notin U$, then v_1, v_2 is a basis of U.

$\underbrace{(1,0,0,0)}_{v_1}, \underbrace{(0,1,0,0)}_{v_2}, \underbrace{(0,0,2,1)}_{v_3}, \underbrace{(0,0,1,2)}_{v_4}$

$U = \{(x, y, z, 0) : x, y, z \in \mathbb{R}\}$ subspace of $\mathbb{R}^4$

We have $v_1, v_2 \in U$ and $v_3, v_4 \notin U$.

We see that v_1, v_2 is not a base of U

as it cannot span it.

9 Suppose $v_1, \ldots, v_m$ is a list of vectors in V. For $k \in \{1, \ldots, m\}$, let

$$w_k = v_1 + \cdots + v_k.$$

Show that $v_1, \ldots, v_m$ is a basis of V if and only if $w_1, \ldots, w_m$ is a basis of V.

From previous exercises we know:

- $\operatorname{span}(w_1 \ldots w_k) = \operatorname{span}(v_1 \ldots v_k)$
- $v_1 \ldots v_k$ linearly independent $\Leftrightarrow w_1 \ldots w_k$ linearly independent

- $v_1 \dots v_k$ linearly independent $\iff w_1 \dots w_k$ linearly independent

We can conclude $v_1 \dots v_m$ basis of V iff $w_1 \dots w_m$ basis of V.

10 Suppose U and W are subspaces of V such that $V = U \oplus W$. Suppose also that $u_1, \dots, u_m$ is a basis of U and $w_1, \dots, w_n$ is a basis of W. Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V.

Let $a_1 \dots a_{m+n} \in \mathbb{F}$

$$\sum_{i=1}^{m} a_i v_i + \sum_{i=m+1}^{m+n} a_i w_{i-m} = 0$$

$$\Longrightarrow \sum_{i=1}^{m} a_i v_i = \sum_{i=m+1}^{m+n} -a_i w_{i-m}$$

This equation is only true if both sides are 0, otherwise an element of U would be a linear combination of elements in W which is not possible as $U \cap W = \{0\}$.

So we have $\sum_{i=1}^{m} a_i v_i = 0$ and $\sum_{i=m+1}^{m+n} a_i w_{i-m} = 0$

Since v_i's and w_i's are bases they are linearly independent, so $a_i = 0 \ \forall i$.

$\Longrightarrow v_1 \dots v_m, w_1 \dots w_n$ linearly independent. (1)

$\Rightarrow v_1 \dots v_m, w_1 \dots w_n$ linearly independent. (1)

Since $V = U \oplus W$, $\forall v \in W$ $\exists! u \in U, w \in W$

s.t: $v = u + w$

$\Rightarrow \exists! a_1 \dots a_{m+n} \in \mathbb{F}$ s.t:

$v = (a_1 v_1 + \dots + a_m v_m) + (a_{m+1} w_1 + \dots + a_{m+n} w_n)$

This implies $\operatorname{span}(v_1 \dots v_m, w_1 \dots w_n) = V$ (2)

(1), (2) $\Rightarrow v_1 \dots v_m, w_1 \dots w_n$ basis of V

11 Suppose V is a real vector space. Show that if $v_1, \dots, v_n$ is a basis of V (as a real vector space), then $v_1, \dots, v_n$ is also a basis of the complexification $V_{\mathbf{C}}$ (as a complex vector space).

See Exercise 8 in Section 1B for the definition of the complexification $V_{\mathbf{C}}$.

- let $u + iv \in V_c$, $u \in V$, $v \in V$

$\Rightarrow \exists a_1 \dots a_m, b_1 \dots b_m \in \mathbb{F}$ s.t:

$u = \sum_{i=1}^{m} a_i v_i$, $v = \sum_{i=1}^{m} b_i v_i$

$\Rightarrow u + iv = \sum_{k=1}^{m} a_k v_k + i \sum_{k=1}^{m} b_k v_k$

$= \sum_{k=1}^{m} (a_k + i b_k) v_k$

$\Rightarrow v_1 \dots v_m$ spans V_c

- Let $a_1 \ldots a_n, b_1 \ldots b_n \in \mathbb{R}$

$$\sum (a_k + i b_k) v_k = 0$$

$$\Rightarrow \sum a_k v_k + i \sum b_k v_k = 0$$

$$\Rightarrow \begin{cases} \sum a_k v_k = 0 \\ \sum b_k v_k = 0 \end{cases} \Rightarrow \begin{cases} a_k = 0 \ \forall k \\ b_k = 0 \ \forall k \end{cases}$$

$$\Rightarrow a_k + i b_k = 0 \quad \forall k$$

$\Rightarrow v_1 \ldots v_n$ linearly independent

$\Rightarrow v_1 \ldots v_n$ basis of $V_{\mathbb{C}}$