1C Exercises

vendredi 28 juin 2024 01:12

1 For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$.

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$

(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

let V denote the candidate subspace for a), b), c), d)

a) • $0 + 2.0 + 3.0 = 0 \Rightarrow 0 \in V$

• let $u, v \in V$ s.t. $u = (x_1, x_2, x_3)$, $v = (y_1, y_2, y_3)$

$u + v = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

$x_1 + y_1 + 2(x_2 + y_2) + 3(x_3 + y_3)$

$= \underbrace{x_1 + 2x_2 + 3x_3}_{= 0\ (u \in V)} + \underbrace{y_1 + 2y_2 + 3y_3}_{= 0\ (v \in V)} = 0 \Rightarrow u + v \in V$

• let $\lambda \in \mathbb{F}$, $v \in V$

$\lambda . v = (\lambda x_1, \lambda x_2, \lambda x_3)$

$\lambda x_1 + 2\lambda x_2 + 3\lambda x_3 = \lambda(\underbrace{x_1 + 2x_2 + 3x_3}_{= 0\ (x \in V)}) = \lambda 0 = 0$

$\Rightarrow \lambda v \in V$

$\Rightarrow V$ subspace of $\mathbb{F}^3$

b) $0 + 2.0 + 3.0 = 0 \neq 4 \Rightarrow 0 \notin V$

$\Rightarrow V$ <u>not</u> a subspace of $\mathbb{F}^3$

c) $u = (1, 0, 0) \in V$ as $1.0.0 = 0$

$v = (0, 1, 1) \in V$ as $0.1.1 = 1$

$u + v = (1, 1, 1) \notin V$ as $1.1.1 = 1$

$\Rightarrow V$ <u>not</u> a subspace of $\mathbb{F}^3$

d) • $0 \in V$

• $x_1 + y_1 = 5x_3 + 5y_3 = 5(x_3 + y_3) \in V$

• $\lambda x_1 = \lambda 5 x_3 = 5\lambda x_3 \in V$

$\cdot\ \lambda x_1 = \lambda 5 x_3 = 5\lambda x_3 \in V$

$\Longrightarrow V$ is a subspace of $\mathbb{F}^3$

2 Verify all assertions about subspaces in Example 1.35.

1.35 example: *subspaces*

(a) If $b \in \mathbf{F}$, then
$$\{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_3 = 5x_4 + b\}$$
is a subspace of $\mathbf{F}^4$ if and only if $b = 0$.

(b) The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbf{R}^{[0,1]}$.

(c) The set of differentiable real-valued functions on $\mathbf{R}$ is a subspace of $\mathbf{R}^{\mathbf{R}}$.

(d) The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbf{R}^{(0,3)}$ if and only if $b = 0$.

(e) The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbf{C}^\infty$.

a) "$\Leftarrow$": same reasoning as 1. d) to show V subspace of $\mathbb{F}^4$

"$\Rightarrow$": V subspace $\Rightarrow 0 \in V$

$\Rightarrow 0 = 5 . 0 + b$

$\Rightarrow b = 0$ ∎

b) 0 in $\mathbb{R}^{[0,1]}$ is defined as: $\forall x \in [0,1], 0(x) = 0$

· 0 is a constant function, hence continuous $\Rightarrow 0 \in V$

· The sum of two continuous functions is a continuous function

· The product of a scalar with a continuous function is a continuous function

$\Longrightarrow V$ subspace of $\mathbb{R}^{[0,1]}$

c) · $0(x) = 0\ \forall x \in \mathbb{R}$ is differentiable in $\mathbb{R}$ $(0'(x) = 0)$ hence $0 \in V$

· $f \in V, g \in V$ sum of diff. fct on $\mathbb{R}$ is diff on $\mathbb{R}$ and of derivative
$(f+g)'(x) = f'(x) + g'(x)\ \forall x \in \mathbb{R}$

· Same with $\lambda f'(x) = (\lambda f)'(x)$

$\Longrightarrow V$ subspace of $\mathbb{R}^{\mathbb{R}}$

d) "$\Rightarrow$": V subspace of $\mathbb{R}^{(0,3)} \Rightarrow 0 \in V$

$\Longrightarrow 0'(2) = b$, but $0'(2) = 0$ (constant fct)

$\Longrightarrow b = 0$

"$\Leftarrow$": Assume $b = 0$:

"$\Leftarrow$": Assume $b = 0$.

- 0 differentiable on $\mathbb{R}$ so differentiable on $(0,3)$, and of derivative $0'(x) = 0 \ \forall x \in \mathbb{R}$, hence $0'(2) = 0$
  $\Longrightarrow 0 \in V$
- let $f, g \in V$. Sum is diff on $(0,3)$, and
  $$(f+g)'(2) = f'(2) + g'(2) = 0 + 0 = 0$$
  $\Rightarrow (f+g)' \in V$
- let $\lambda \in \mathbb{R}$, $f \in V$. λf diff on $(0,3)$, and
  $$(\lambda f)'(2) = \lambda f'(2) = \lambda 0 = 0$$
  $\Rightarrow (\lambda f)' \in V$

$\Rightarrow \left(V \text{ subspace of } \mathbb{R}^{(0,3)} \Longleftrightarrow b = 0 \right)$

e) - 0 in $\mathbb{C}^\infty$ is $(0, 0, \dots)$

$(0,0,\dots)$ is a sequence of complex numbers $(0_n)_{n \in \mathbb{N}}$ s.t

$0_n = 0 \ \forall n \in \mathbb{N}$. $\lim\limits_{n \to \infty} 0_n = 0 \Rightarrow 0 \in V$

- let $u, v \in V$
  $$\lim_{n \to \infty} (u+v)_n = \lim_{n \to \infty} u_n + \lim_{n \to \infty} v_n \text{ as } u_n, v_n \text{ converge}$$
  $$= 0 + 0 = 0 \Rightarrow u + v \in V$$

- let $u \in V, \lambda \in \mathbb{C}$
  $$\lim_{n \to \infty} (\lambda u)_n \overset{(\lambda \text{ constant})}{=} \lambda \lim_{n \to \infty} u_n = \lambda 0 = 0 \Rightarrow \lambda u \in V$$

$\Longrightarrow V$ subspace of $\mathbb{C}^\infty$

3 Show that the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$.

- $0'(-1) = 0 = 3\,0(2) \Rightarrow 0 \in V$
- let $f, g \in V$ sum is differentiable and real valued on $(-4,4)$

$(f+g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2)$

$= 3(f(2)+g(2)) = 3(f+g)(2)$

$\Rightarrow f+g \in V$

- let $\lambda \in \mathbb{R}$, $f \in V$ λf differentiable and real valued in $(-4,4)$

$(\lambda f)'(-1) = \lambda f'(-1) = \lambda 3 f(2) = 3\lambda f(2)$

$= 3(\lambda f)(2) \Rightarrow \lambda f \in V$

$\Rightarrow V$ subspace of $\mathbb{R}^{(-4,4)}$

4 Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$.

" $\Rightarrow$ ": V subspace of $\mathbb{R}^{[0,1]}$

$\Rightarrow \int_0^1 0 = b \Rightarrow [c]_0^1 = b \Rightarrow 0 = b$ (c some constant)

" $\Leftarrow$ ": $b = 0$

- $0(x) = 0$ $\forall x \in [0,1]$ is a continuous real-valued fct on $[0,1]$

$\int_0^1 0 = 0 \Rightarrow 0 \in V$

- let $f, g \in V$. Sum of f and g is continuous and real-valued on $[0,1]$.

$\int_0^1 (f+g) \overset{(\int \text{ linear operator})}{=} \int_0^1 f + \int_0^1 g = 0 + 0 = 0 \Rightarrow f+g \in V$

- let $\lambda \in \mathbb{R}$, $f \in V$. Product is continuous and real-valued on $[0,1]$.

$\lambda f \in V$

on $[0,1]$.

$$\int_0^1 (\lambda f) \overset{(\int \text{ linearity prop.})}{=} \lambda \int_0^1 f = \lambda 0 = 0 \Rightarrow \lambda f \in V$$

$\Longrightarrow V$ is a subspace of $\mathbb{R}^{[0,1]}$

5 Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

let $\lambda \in \mathbb{C}$, $\lambda = bi$ with $b \in \mathbb{R}^*$

let $x \in \mathbb{R}^2$, $x = (x_1, 0)$, $x_1 \in \mathbb{R}^*$

$\lambda x = (\lambda x_1, 0) = (bix_1, 0)$

Since $b, x_1 \neq 0$, bix_1 is complex, s.t $\lambda x \notin \mathbb{R}^2$

$\Rightarrow \exists \lambda \in \mathbb{C}, x \in \mathbb{R}^2$ s.t $\lambda x \notin \mathbb{R}^2$

$\Longrightarrow \mathbb{R}^2$ is <u>not</u> a subspace of the <u>complex</u> vector space $\mathbb{C}^2$

6 (a) Is $\{(a,b,c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?
(b) Is $\{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?

a) $\cdot$ $0 = (0,0,0) \in V$

$\cdot$ $a^3 = b^3 \Rightarrow a = b$ for $a, b \in \mathbb{R}$ (if it needs to be proven, $f: x \mapsto x^3$ is strictly increasing, as $3x^2 > 0 \; \forall x \in \mathbb{R}^*$, hence injective)

let $u = (a_1, b_1, c_1)$, $v = (a_2, b_2, c_2)$

$u + v = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

$= (\underbrace{a_1 + b_2}, \underbrace{a_1 + b_2}, c_1 + c_2)$
$=$

$\Rightarrow u + v \in V$

$\cdot$ $\lambda u = \lambda (a_1, b_1, c_1) = (\underbrace{\lambda a_1}, \underbrace{\lambda a_1}, c_1) \Rightarrow \lambda u \in V$
$=$

$\Longrightarrow V$ subspace of $\mathbb{R}^3$

b) In $\mathbb{C}$, $a^3 = b^3 \not\Rightarrow a = b$, as a^3 has 3 roots

z_1, z_2, z_3 s.t $z_1 \neq z_2 \neq z_3$

For instance $z^3 = -1 \Rightarrow \begin{cases} z = -1 \\ \text{or} \\ z = \frac{1}{2} + \frac{\sqrt{3}}{2} i \\ \text{or} \\ z = \frac{1}{2} - \frac{\sqrt{3}}{2} i \end{cases}$

$u = (-1, \frac{1}{2} + \frac{\sqrt{3}}{2} i, 0) \in V$

$z^a = \frac{1}{2} - \frac{\sqrt{3}}{2} i$

$u = \left(-1, \frac{1}{2} + \frac{\sqrt{3}}{2} i, 0\right) \in V$

$v = \left(-1, \frac{1}{2} - \frac{\sqrt{3}}{2} i, 0\right) \in V$

$u + v = (-2, 1, 0) \notin V$ (as $-2^3 \neq 1^3$)

$\Longrightarrow V$ is not a subspace of $\mathbb{C}^3$

7 Prove or give a counterexample: If U is a nonempty subset of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), then U is a subspace of $\mathbf{R}^2$.

let $U = \mathbb{Z}^2$

U is closed under addition:

let $a, b, c, d \in \mathbb{Z}$:

$$(a,b) + (c,d) = (\underbrace{a+c}_{\in \mathbb{Z}}, \underbrace{b+d}_{\in \mathbb{Z}}) \in \mathbb{Z}^2$$

U is closed under additive inverse:

let $a, b \in \mathbb{Z}$:

$$(-a,-b) + (a,b) = (0,0) \text{ and } (-a,-b) \in \mathbb{Z}^2$$

However, U is <u>not</u> a subspace of $\mathbb{R}^2$.

Indeed, U is not necessarily closed under scalar multiplication. If the field is $\mathbb{Q}$ (or $\mathbb{R}$ or $\mathbb{C}$):

$$\underbrace{0.1}_{\in \mathbb{Q}} \underbrace{(1,0)}_{\in U} = (\underbrace{0.1}_{\notin \mathbb{Z}}, 0) \notin \mathbb{Z}^2$$

8 Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

$$U = \{(x,y) \in \mathbb{R}^2 \mid \underset{(1)}{x = y} \vee \underset{(2)}{x = 2y}\}$$

let $u \in U$. $\lambda u = (\lambda x, \lambda y)$

If u satisfies (1): $\lambda x = \lambda y \Rightarrow \lambda u \in U$

If u satisfies (2): $\lambda x = \lambda 2y = 2(\lambda y) \Rightarrow \lambda u \in U$

$\Rightarrow U$ closed under scalar multiplication

If U satisfies (2): $\lambda x = \lambda \lambda y = \lambda(\lambda y)$ ...

$\Rightarrow U$ closed under scalar multiplication

However, U is not a subspace of $\mathbb{R}^2$.

Indeed, let $u, v \in U$ s.t $u = (1,1)$, $v = (1,2)$.

$u + v = (1+1, 1+2) = (2,3) \notin U$

$\Rightarrow U$ is not closed under addition

$\Rightarrow U$ is not a subspace of $\mathbb{R}^2$

9 A function $f\colon \mathbf{R} \to \mathbf{R}$ is called *periodic* if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

The sum of periodic functions is not necessarily periodic, which makes addition not closed on V.

Example: $[\sin(x) + \sin(\pi x),\ x \in \mathbb{R}]$

(Here is a proof for this, easier/more straightforward proves may exist...)

Suppose $\exists c \in \mathbb{R}^{+*}$ s.t:

$\sin(x+c) + \sin(\pi(x+c)) = \sin(x) + \sin(\pi x)$

For $x=0$: $\sin(c) + \sin(\pi c) = 0 \Rightarrow \sin(c) = -\sin(\pi c)$ (1)

For $x=1$: $\sin(1+c) - \sin(\pi c) = \sin(1)$ (2)

$\Rightarrow \sin(1+c) + \sin(c) = \sin(1)$

For $x=-1$: $\sin(c-1) - \sin(\pi c) = \sin(-1)$

$\Rightarrow \sin(c-1) + \sin(c) = -\sin(1)$

$\left.\begin{array}{l} \sin(1+c) + \sin(c) = \sin(1) \\ \sin(c-1) + \sin(c) = -\sin(1) \end{array}\right\} \stackrel{+}{\Rightarrow} \sin(1+c) + \sin(c-1) + 2\sin(c) = 0$

$\Rightarrow \cancel{\sin(1)\cos(c)} + \cos(1)\sin(c) + \sin(c)\cos(1) - \cancel{\cos(c)\sin(1)} + 2\sin(c) = 0$

$\Rightarrow \sin(c) + \cos(1)\sin(c) = 0$

$\Rightarrow \sin(c)\underbrace{(1 + \cos(1))}_{\neq 0} = 0$

$\Rightarrow \sin(c) = 0$

$\Rightarrow c \in \{0 + k\pi, k \in \mathbb{Z}\}$

$\Rightarrow \sin(c) = 0$

$\Rightarrow c \in \{0 + k\pi, k \in \mathbb{Z}\}$

$\Rightarrow \sin(x + k\pi) + \sin(\pi x + k\pi^2) = \sin(x) + \sin(\pi x)$

$\Rightarrow (-1)^k (\sin x) - \sin(x) + \sin(\pi x + k\pi^2) - \sin(\pi x) = 0$

$x = 0 : \sin(k\pi^2) = 0$

$\Rightarrow \exists k' \in \mathbb{Z}$ s.t $k\pi^2 = k'\pi$

$\Rightarrow \underbrace{k\pi}_{\text{irrational if } k \neq 0} = \underbrace{k'}_{\text{rational } (\in \mathbb{Z})}$

$\Rightarrow \boxed{k = 0}$

$\Rightarrow$ $c = 0$, which contradicts the initial statement.

Hence $\sin x + \sin \pi x$ is not periodic.

10 Suppose V_1 and V_2 are subspaces of V. Prove that the intersection $V_1 \cap V_2$ is a subspace of V.

- V_1 and V_2 subspaces $\Rightarrow \begin{cases} 0 \in V_1 \\ 0 \in V_2 \end{cases} \Rightarrow 0 \in V_1 \cap V_2$

- let $u, v \in V_1 \cap V_2$.

  $u \in V_1 \cap V_2 \Rightarrow u \in V_1 \wedge u \in V_2$ (same for v)

$\begin{cases} u \in V_1 \\ v \in V_1 \end{cases} \Rightarrow u + v \in V_1$ (V_1 subspace) $\Rightarrow u + v \in V_1 \cap V_2$

$\begin{cases} v \in V_2 \\ v \in V_2 \end{cases} \Rightarrow u + v \in V_2$ (V_2 subspace)

- let $\lambda \in \mathbb{R}$, $u \in V_1 \cap V_2$

  $\lambda u \in V_1$ (V_1 subspace), $\lambda u \in V_2$ (V_2 subspace)

  $\Rightarrow \lambda u \in V_1 \cap V_2$

  $\Rightarrow V_1 \cap V_2$ subspace

11 Prove that the intersection of every collection of subspaces of V is a subspace of V.

Same reasoning as 10.

of V.

Same reasoning as 10.

12 Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Let U_1, U_2 be subspaces of V

"$\Leftarrow$" Suppose WLOG $U_1 \subset U_2$

Then $U_1 \cup U_2 = U_2$ which is a subspace of V

$\Rightarrow U_1 \cup U_2$ subspace of V

"$\Rightarrow$" By contradiction: suppose $U_1 \not\subset U_2$ and $U_2 \not\subset U_1$

$\Rightarrow \begin{cases} \exists u_1 \in U_1 \text{ s.t. } u_1 \notin U_2 \\ \exists u_2 \in U_2 \text{ s.t. } u_2 \notin U_1 \quad (*) \end{cases}$

$u_1 + u_2 \in U_1 \cup U_2$ (as we supposed $U_1 \cup U_2$ subspace)

If $u_1 + u_2 \in U_1$:

$\underbrace{(u_1 + u_2)}_{\in U_1} - \underbrace{u_1}_{\in U_1} = u_2 \in U_1$, as U_1 subspace, contradiction with $(*)$

The symmetric contradiction can be found if $u_1 + u_2 \in U_2$.

13 Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two.

This exercise is surprisingly harder than Exercise 12, possibly because this exercise is not true if we replace **F** *with a field containing only two elements.*

(I hope it is correct I am pretty happy with this proof!)

"$\Leftarrow$" Same reasoning as "$\Leftarrow$" of Ex. 12.

"$\Rightarrow$" Let U_1, U_2, U_3 be subspaces of V.

Suppose $U_1 \not\subset U_2, U_3$ and $U_1 \cup U_2 \cup U_3$ subspace of V

$U_2 \not\subset U_1, U_3 \quad (*)$

$U_2 \not\subseteq U_1, U_3$ $(*)$
$U_3 \not\subseteq U_1, U_2$

$\Rightarrow \exists u_1 \in U_1$ s.t $u_1 \notin U_2, U_3$
$\exists u_2 \in U_2$ s.t $u_2 \notin U_1, U_3$ $(**)$
$\exists u_3 \in U_3$ s.t $u_3 \notin U_1, U_2$

We can show that $\forall i \in \{1,2,3\}, \forall \lambda \in \mathbb{F} \setminus \{0\}$:
$\lambda u_i \in U_i$ s.t. $\lambda u_i \notin U_j \; \forall j \in \{1,2,3\} \setminus \{i\}$ (1)

Indeed, if $\lambda u_i \in U_j$ for some $j \in \{1,2,3\} \setminus \{i\}$,
then as U_j is a subspace we would have:
$\frac{1}{\lambda} \cdot \lambda u_i \in U_j \Rightarrow u_i \in U_j$
which contradicts assumptions $(*)$

We can also show that $\forall i,j,k \in \{1,2,3\}$ with $i \neq j, j \neq k, i \neq k$:
$u_i + u_j \in U_k$ and $u_i + u_j \notin U_i, U_j$ (2)

Indeed if $u_i + u_j \in U_i$, then $\underbrace{(u_i + u_j)}_{\in U_i} - \underbrace{u_i}_{\in U_i} = u_j \in U_i$
which contradicts assumptions $(**)$. Same reasoning for U_j.

Let us consider $u_1 + u_2 + u_3$.
Since $U_1 \cup U_2 \cup U_3$ subspace, $u_1 + u_2 + u_3 \in U_1 \cup U_2 \cup U_3$

Since $U_1 \cup U_2 \cup U_3$ subspace, $u_1+u_2+u_3 \in U_1 \cup U_2 \cup U_3$

However, $\underbrace{u_1+}_{\in U_3 \text{ (by (2))}} u_2 + \underbrace{u_3}_{\in U_3} \in U_3$, and the same reasoning can be made to show $u_1+u_2+u_3 \in U_1, U_2$.

Thus $\boxed{u_1+u_2+u_3 \in U_1 \cap U_2 \cap U_3}$,

let $\lambda \in \mathbb{F} \setminus \{0,1\}$. We showed in (1) that $\lambda u_1 \in U_1$ and $\lambda u_1 \notin U_2, U_3$. We can thus apply (2) with λu_1 instead of u_1.

$\Rightarrow \underbrace{\lambda u_1 +}_{\in U_3} u_2 + \underbrace{u_3}_{\in U_3} \in U_3$. With the same reasoning, we find that:

$\lambda u_1 + u_2 + u_3 \in U_1 \cap U_2 \cap U_3$.

As $U_1 \cup U_2 \cup U_3$ subspace of V,

$$\boxed{-\lambda u_1 - u_2 - u_3 \in U_1 \cap U_2 \cap U_3}$$

Thus $(u_1+u_2+u_3) - (\lambda u_1 - u_2 - u_3) \in U_1 \cup U_2 \cup U_3$

$= u_1(1-\lambda) \in U_1 \cap U_2 \cap U_3$

$= \mu u_1 \in U_1 \cup U_2 \cup U_3$, with $\mu \neq 0$ as $\lambda \neq 1$

But we showed in (1) that since $u_1 \in U_1 \setminus U_2, U_3$, But we showed in (1) that since $v_1 \in U_1 \setminus U_2, U_3$,

$\mu v_1 \in U_1 \setminus U_2, U_3 \ \forall \mu \in \mathbb{F} \setminus \{0\}$.

Thus (*) and $U_1 \cup U_2 \cup U_3$ subspace leads to a contradiction.

We can conclude that $U_1 \cup U_2 \cup U_3 \Rightarrow \neg(*)$, meaning that one of the subspaces must include the two others for $U_1 \cup U_2 \cup U_3$ to be a subspace of V (at least when $\exists \lambda \in \mathbb{F}$ s.t $\lambda \neq 0, \lambda \neq 1$, meaning $\mathbb{F}$ has more than two elements)

14 Suppose

$$U = \{(x, -x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\} \quad \text{and} \quad W = \{(x, x, 2x) \in \mathbf{F}^3 : x \in \mathbf{F}\}.$$

Describe $U + W$ using symbols, and also give a description of $U + W$ that uses no symbols.

$$U + W = \{u + w, u \in U, w \in W\}$$

$$= \{(x, -x, 2x) + (y, y, 2y), x \in \mathbb{F}, y \in \mathbb{F}\}$$

$$= \{(x+y, -x+y, 2(x+y)), x \in \mathbb{F}, y \in \mathbb{F}\}$$

$$\overset{(1)}{=} \{(x, y, 2x) : x \in \mathbb{F}, y \in \mathbb{F}\} = U$$

$U + W$ is the set of elements of $\mathbb{F}^3$ such that the first coordinate is half the third one.

To prove (1):

"$\subseteq$": $(x+y, -x+y, 2(x+y))$ can be written

"$\subseteq$" : $(x+y, -x+y, 2x+y)$, can be written
$(z, -x+y, 2z) \in U$

"$\supseteq$" : let $x, y \in \mathbb{F}$.

Then $(x, y, 2x) = \left(\frac{x-y}{2}, \frac{y-x}{2}, x-y\right)$ $(\in U)$

$+ \left(\frac{x+y}{2}, \frac{x+y}{2}, x+y\right)$ $(\in W)$

15 Suppose U is a subspace of V. What is $U + U$?

$U + U = \{u + v,\ u, v \in U\}$

- let $w \in U + U$. $w = u + v$, $u, v \in U$. As U subspace, $w \in U$.
- let $u \in U$. $u = u + 0 \in U + U$ ($0 \in U$ as U subspace).

Thus, $U + U = U$

16 Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

- let $v \in U + W$, $v = u + w$, $u \in U$, $w \in W$

$\Rightarrow v = w + u$ (commutativity of + in a vector field)

$\Rightarrow v \in W + U \Rightarrow U + W \subseteq W + U$

Same reasoning for the other side of the equality.

17 Is the operation of addition on the subspaces of V associative? In other words, if V_1, V_2, V_3 are subspaces of V, is

$$(V_1 + V_2) + V_3 = V_1 + (V_2 + V_3)?$$

let $v \in (V_1 + V_2) + V_3$, $v = (v_1 + v_2) + v_3$, $v_i \in V_i$ for $i \in \{1, 2, 3\}$

$\Rightarrow v = v_1 + (v_2 + v_3)$ (associativity of V)

$\Rightarrow v \in V_1 + (V_2 + V_3) \Rightarrow (V_1 + V_2) + V_3 \subseteq V_1 + (V_2 + V_3)$

$\Rightarrow v \in V_1 + (V_2 + V_3) \Rightarrow (V_1 + V_2) + V_3 \subseteq V_1 + (V_2 + V_3)$

Same reasoning for the other side of the equality.

18 Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

An additive identity O in the subspaces of V would be a subspace s.t. if U is a subspace of V, then

$U + O = U (= O + U$ by commutativity$)$.

The subspace $O = \{0\}$ consisting of the additive identity in V is such a subspace. Indeed:

- let $v \in U + O$. $v = u + 0$, $v \in U \Rightarrow U + O \subseteq U$
- let $u \in U$. $u = u + 0 \in U + O \Rightarrow U \subseteq U + O$

$\Rightarrow U = U + O$

let U be a subspace of V.

Suppose $\exists W$ s.t. $U + W = O$

let $v \in U + W$. $v = u + w$, $u \in U$, $w \in W$

$u + w = 0 \Rightarrow u = -w$

This should hold $\forall u, \forall w$.

For a fixed u, $\exists! w$ s.t. $u = -w$ and reciprocally, implying U and W should hold only one element (1).

However, if one element x is present and different than 0, since U and W are subspaces $-x$ should also be in U and W, which contradicts (1).

Therefore, only $\{x\}$ can have an additive

Therefore, only $\{0\}$ can have an additive inverse, itself (we can easily check that $\{0\}+\{0\}=\{0\}$).

19 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V_1 + U = V_2 + U,$$

then $V_1 = V_2$.

Counterexample:

$V_1 = \{(0, y) \in \mathbb{R}^2\}$ (subspace of $\mathbb{R}^2$)

$V_2 = \{(x, 2x) \in \mathbb{R}^2\}$ (subspace of $\mathbb{R}^2$)

$U = \{(x, 0) \in \mathbb{R}^2\}$ (subspace of $\mathbb{R}^2$)

We can show $V_1 + U = V_2 + U = \mathbb{R}^2$

It is obvious that $V_1 + U = \mathbb{R}^2$:

- $\forall (x,y) \in \mathbb{R}^2$, $(x,y) = (\underbrace{x,0}_{\in V_1}) + (\underbrace{0,y}_{\in U}) \in V_1 + U \Rightarrow \mathbb{R}^2 \subseteq V_1 + U$
- $V_1 + U \subseteq \mathbb{R}^2$ (sum of subspaces)

$\Rightarrow V_1 + U = \mathbb{R}$

We can also show that $V_2 + U = \mathbb{R}^2$:

- $\forall (x,y) \in \mathbb{R}^2$, $(x,y) = \underbrace{(2y, y)}_{\in V_2} + (\underbrace{x-2y}_{\in U}, 0) \Rightarrow \mathbb{R}^2 \subseteq V_2 + U$
- Same as for $V_1 + U$

Thus we have $V_1 + U = V_2 + U = \mathbb{R}^2$ but $V_1 \neq V_2$

20 Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$.

$\{(x, y, 2y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\} = W$ (subspace of $\mathbb{F}^4$)

$\{(x, y, 2y, y) \in \mathbb{F}^4 : x, y \in \mathbb{F}\} = W$ (subspace of $\mathbb{F}^4$)

Let $(a,b,c,d) \in U \cap W$, $(a,b,c,d) = (x,x,y,y)$
$(a,b,c,d) = (w,z,2z,z)$

$$\begin{cases} a = x \\ b = x \\ c = y \\ d = y \\ a = w \\ b = z \\ c = 2z \\ d = z \end{cases} \Rightarrow \begin{cases} d = y = z \\ c = y = 2z \end{cases} \Rightarrow z = 0$$

$\Rightarrow (a,b,c,d) = (0,0,0,0)$

$\Rightarrow U \cap W = \{0\}$

$\Rightarrow \mathbb{F}^4 = U \oplus W$

21 Suppose

$$U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$.

$W = \{(0, x, x, x, x) \in \mathbb{F}^5\}$

Let $(a,b,c,d,e) \in W \cap U$

$$\Rightarrow \begin{cases} a = x \\ b = y \\ c = x + y \\ d = x - y \\ e = 2x \\ a = 0 \\ b = z \\ c = z \\ d = z \\ e = z \end{cases} \Rightarrow \begin{cases} a = x = 0 \\ e = 2x = 0 \\ e = b = c = d = 0 \end{cases}$$

$\Rightarrow (a,b,c,d,e) = (0,0,0,0)$

$\Rightarrow W \cap U = \{0\}$

$\Rightarrow \mathbb{F}^5 = W \oplus U$

22 Suppose

$$U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}.$$

Find three subspaces W_1, W_2, W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

$W_1 = \{(0, x, 0, 0, 0) \mid x \in \mathbb{F}\}$

$W_2 = \{(0, 0, x, 0, 0) \mid x \in \mathbb{F}\}$

$W_3 = \{(0, 0, 0, x, 0) \mid x \in \mathbb{F}\}$

$(a,b,c,d,e) \in U \cap W_1 \cap W_2 \cap W_3$

$$\begin{cases} a = x & a = 0 & a = 0 & a = 0 \\ b = y & b = w & b = 0 & b = 0 \\ c = x+y & c = 0 & c = z & c = 0 \\ d = x-y & d = 0 & d = 0 & d = v \\ e = 2x & e = 0 & e = 0 & e = 0 \\ \Rightarrow (a,b,c,d,e) = (0,0,0,0,0) \\ \Rightarrow U \cap W_1 \cap W_2 \cap W_3 = \{0\} \Rightarrow \end{cases}$$

$$U \oplus W_1 \oplus W_2 \oplus W_3 = \mathbb{F}^5$$

23 Prove or give a counterexample: If V_1, V_2, U are subspaces of V such that

$$V = V_1 \oplus U \quad \text{and} \quad V = V_2 \oplus U,$$

then $V_1 = V_2$.

Hint: When trying to discover whether a conjecture in linear algebra is true or false, it is often useful to start by experimenting in $\mathbf{F}^2$.

Our counterexample in Ex. 19 can be used here, as we chose V_1 and V_2 s.t $V = V_1 \oplus U$ and $V = V_2 \oplus U$.

24 A function $f\colon \mathbf{R} \to \mathbf{R}$ is called *even* if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$. A function $f\colon \mathbf{R} \to \mathbf{R}$ is called *odd* if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$. Let V_e denote the set of real-valued even functions on $\mathbf{R}$ and let V_o denote the set of real-valued odd functions on $\mathbf{R}$. Show that $\mathbf{R}^{\mathbf{R}} = V_e \oplus V_o$.

Easy to check V_e and V_o are subspaces of $\mathbb{R}^{\mathbb{R}}$.

Let $f \in \mathbb{R}^{\mathbb{R}}$ such that $f \in V_e, V_o$.

$$\forall x \in \mathbb{R}, \begin{cases} f(-x) = f(x) & (f \in V_e) \\ f(-x) = -f(x) & (f \in V_o) \end{cases}$$

$\Rightarrow f(x) = -f(x) \Rightarrow f(x) = 0$

$\Rightarrow f = 0$ (additive identity for $\mathbb{R}^{\mathbb{R}}$)

$\Rightarrow V_e \cap V_o = \{0\}$

$V_e \oplus V_o = \mathbb{R}^{\mathbb{R}}$

$\Longrightarrow V_e \perp V_\sigma$ ([illegible])

$\Longrightarrow V_e \oplus V_\sigma = \mathbb{R}^2$