# 1A

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*Exercises 1A*

**1** Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbf{C}$.

**2** Show that $(\alpha + \beta) + \lambda = \alpha + (\beta + \lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

**3** Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$ for all $\alpha, \beta, \lambda \in \mathbf{C}$.

**4** Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbf{C}$.

**5** Show that for every $\alpha \in \mathbf{C}$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha + \beta = 0$.

**6** Show that for every $\alpha \in \mathbf{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \mathbf{C}$ such that $\alpha\beta = 1$.

**7** Show that

$$\frac{-1 + \sqrt{3}i}{2}$$

is a cube root of 1 (meaning that its cube equals 1).

**8** Find two distinct square roots of $i$.

**9** Find $x \in \mathbf{R}^4$ such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

**10** Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$





**11** Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

**12** Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

**13** Show that $1x = x$ for all $x \in \mathbf{F}^n$.

**14** Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

**15** Show that $(a + b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

1. Let $\lambda, \alpha, \beta \in \mathbb{C}$, $\alpha = a + bi$, $\beta = c + di$, $\lambda = f + gi$

1. $\alpha + \beta = (a + c) + (b + d)i$ (definition of + in $\mathbb{C}$)
$= (c + a) + (d + b)i$ (+ commutative in $\mathbb{R}$)
$= \beta + \alpha$ (definition of + in $\mathbb{C}$)

2. $(\alpha + \beta) + \lambda = (a + c) + (b + d)i + \lambda$ (definition of + in $\mathbb{C}$)
$= ((a + c) + f) + ((b + d) + g)i$ "
$= (a + (c + f)) + (b + (d + g))i$ (+ associative in $\mathbb{R}$)
$= \alpha + (\beta + \lambda)$ (definition of + in $\mathbb{C}$)

3. $(\alpha\beta)\lambda = ((ac - bd) + (ad + bc)i)\lambda$ (definition of $\cdot$ in $\mathbb{C}$)
$= ((ac - bd)f - (ad + bc)g) + ((ac - bd)g + (ad + bc)f)i$ "
$= (a(cf - dg) - b(cg + df)) + (a(cg + df) + b(cf - dg))i$ (assoc + comm in $\mathbb{R}$)
$= \alpha(\beta\lambda)$ (definition of $\cdot$ in $\mathbb{C}$)

4. $\lambda(\alpha + \beta)$
$= \lambda((a + c) + (b + d)i)$
$= (f(a + c) - g(b + d))$
$+ (f(b + d) + g(a + c))i$
$= [(fa - gb) + (fb + ga)i]$
$+ [(fc - gd) + (fd + gc)i]$
$= \lambda\alpha + \lambda\beta$

5. Let $\alpha = a + bi \in \mathbb{C}$.
Let $\beta = c + di \in \mathbb{C}$, $\lambda = f + gi$ s.t. $\underset{(1)}{\alpha + \beta = 0} \wedge \underset{(2)}{\alpha + \lambda = 0}$.

$\alpha + \beta = 0 \implies \lambda + (\alpha + \beta) = \lambda$

comm. + assoc.

$\implies (\alpha + \lambda) + \beta = 0 + \beta = \lambda$

$\implies \beta = \lambda$

6. Let $\alpha \in \mathbb{C}$ s.t. $\alpha \neq 0$, $\alpha = a + bi$
Let $\beta, \lambda \in \mathbb{C}$ s.t. $\underset{(1)}{\alpha\beta = 1} \wedge \underset{(2)}{\alpha\lambda = 1}$

comm. + assoc.

$\beta = \lambda$

$$\alpha\beta = 1 \Rightarrow \lambda(\alpha\beta) = \lambda \overset{\text{comm. + assoc.}}{\Longrightarrow} (\alpha\lambda)\beta = \lambda$$

$$\Rightarrow 1\beta = \lambda \Rightarrow \beta = \lambda$$

7. $$\left(\frac{-1+\sqrt{3}i}{2}\right)^3 = \frac{[-2-2\sqrt{3}i](-1+\sqrt{3}i)}{8} = \frac{(2+6)}{8} = 1$$

8. Let $a, b \in \mathbb{R}$ s.t.:

$$\sqrt{i} = a+bi \Rightarrow i = (a+bi)^2$$

$$\Rightarrow i = (a^2 - b^2) + 2abi$$

$$\Rightarrow \begin{cases} a^2 - b^2 = 0 \\ 2ab = 1 \end{cases} \Rightarrow \begin{cases} (a-b)(a+b) = 0 \\ 2ab = 1 \end{cases}$$

$$\Rightarrow \begin{cases} a = b \\ 2a^2 = 1 \end{cases} \text{ or } \begin{cases} a = -b \\ -2a^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} a = b \\ a = \frac{1}{\sqrt{2}} \text{ or } a = -\frac{1}{\sqrt{2}} \end{cases} \text{ or } \begin{cases} a = -b \\ \underbrace{a^2 = -\frac{1}{2}}_{\text{impossible for } a \in \mathbb{R}} \end{cases}$$

$\Rightarrow$ $i$ has two square roots: $\frac{1}{\sqrt{2}}(1+i)$ and $-\frac{1}{\sqrt{2}}(1+i)$

9. $(4, -3, 1, 7) + 2x = (5, 9, -6, 8)$

$$\Rightarrow \begin{cases} 4 + 2x_1 = 5 \\ -3 + 2x_2 = 9 \\ 1 + 2x_3 = -6 \\ 7 + 2x_4 = 8 \end{cases} \Rightarrow \begin{cases} x_1 = 1/2 \\ x_2 = 6 \\ x_3 = -7/2 \\ x_4 = 1/2 \end{cases}$$

10. $\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$

10. $\lambda(2-3i, 5+4i, -6+7i) = (12-5i, 7+22i, -32-9i)$

$\lambda(2-3i) = 12-5i \Rightarrow (a+bi)(2-3i) = 12-5i$

$\Rightarrow \begin{cases} 2a+3b = 12 \ (1) \\ -3a+2b = -5 \ (2) \end{cases} \Rightarrow \begin{cases} -a+5b = 7 \ (1+2) \\ -3a+2b = -5 \ (2) \end{cases} \Rightarrow \begin{cases} a = 5b-7 \\ -15b+21+2b = -5 \end{cases}$

$\Rightarrow \begin{cases} a = 5b-7 \\ -13b = -26 \end{cases} \Rightarrow \begin{cases} b = 2 \\ a = 3 \end{cases}$

$(3+2i)(5+4i) = 7+22i$

$(3+2i)(-6+7i) = \boxed{-32+9i \neq -32-9i}$

11. let $x, y, z \in \mathbb{F}^n$

$(x+y)+z = (x_1+y_1, \ldots, x_n+y_n)+z = ((x_1+y_1)+z_1, \ldots, (x_n+y_n)+z_n)$

$= (x_1+(y_1+z_1), \ldots, x_n+(y_n+z_n))$ (associativity of + in $\mathbb{F}$)

$= x + (y_1+z_1, \ldots, y_n+z_n) = x+(y+z)$

12. let $a, b \in \mathbb{F}$

(associativity of • in $\mathbb{F}$)

$(ab)x = ((ab)x_1, \ldots, (ab)x_n) = (a(bx_1), \ldots, a(bx_n))$

$= a(bx_1, \ldots, bx_n) = a(bx)$

13. $1x = (1x_1, \ldots, 1x_n) = (x_1, \ldots, x_n) = x$

14. $\lambda(x+y) = \lambda(x_1+y_1, \ldots, x_n+y_n) = (\lambda(x_1+y_1), \ldots, \lambda_n(x_n+y_n))$

distributivity of • over + in a field

$= (\lambda x_1 + \lambda y_1, \ldots, \lambda x_n + \lambda y_n) = (\lambda x_1, \ldots, \lambda x_n) + (\lambda y_1, \ldots, \lambda y_n) = \lambda x + \lambda y$

15. $(a+b)x = ((a+b)x_1, \ldots, (a+b)x_n)$

dist. of • over + in $\mathbb{F}$

$(a+b)x = ($ ...

dist. of $\cdot$ over $+$ in $\mathbb{F}$

$$= (ax_1 + bx_1, \dots, ax_n + bx_n)$$

$$= (ax_1, \dots, ax_n) + (bx_1, \dots, bx_n)$$

$$= ax + bx$$