3D Exercises

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1 Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T.$$

let $T \in L(V,W)$ invertible. $TT^{-1} = I_W$ and $T^{-1}T = I_V$, with $T^{-1} \in L(W,V)$.

We can see that T^{-1} is invertible with inverse T, thus $(T^{-1})^{-1} = T$

2 Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

$(T^{-1}S^{-1})ST = T^{-1}(S^{-1}S)T = T^{-1}T = I$. Same reasoning for TS.

This proves ST is invertible and $(ST)^{-1} = T^{-1}S^{-1}$

3 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.
(a) T is invertible.
(b) $Tv_1, \dots, Tv_n$ is a basis of V for every basis $v_1, \dots, v_n$ of V.
(c) $Tv_1, \dots, Tv_n$ is a basis of V for some basis $v_1, \dots, v_n$ of V.

3B Ex 9 : T injective and $v_1 \dots v_k$ linearly independent

$\Rightarrow Tv_1, \dots Tv_k$ linearly independent

3A Ex 4 : $Tv_1, \dots Tv_k$ linearly independent $\Rightarrow v_1 \dots v_k$ linearly independent

(a) $\Rightarrow T$ injective $\overset{3B\ ex\ 9}{\Rightarrow} Tv_1 \dots Tv_n$ linearly independent, with $v_1 \dots v_n$ any basis of V

$\Rightarrow Tv_1 \dots Tv_n$ basis of V ($\dim V$ linearly independent vectors)

$\Rightarrow$ (b) $\Rightarrow$ (c)

(c) $\Rightarrow \exists$ basis $v_1 \dots v_n$ of V s.t $Tv_1 \dots Tv_n$ is a basis of V

$\Rightarrow$ range $T = V \Rightarrow T$ injective $\overset{3.65}{\Rightarrow}$ (a)

Thus all three statements are equivalent.

4 Suppose V is finite-dimensional and $\dim V > 1$. Prove that the set of noninvertible linear maps from V to itself is not a subspace of $\mathcal{L}(V)$.

Let $T_1, T_2 \in L(V)$ invertible. let $v_1 \dots v_n$ a basis of V, $n > 1$.

let $T_1 v_1 = v_2$ and $T_1 v_2 = v_1$, $T_1 v_i = v_i\ \forall i > 2$

let $T_2 v_1 = v_1 - v_2$ and $T_2 v_2 = v_1$, $T_2 v_i = v_i\ \forall i > 2$

T_1 and T_2 are injective linear maps from V to V, hence invertible.

$(T_1 + T_2) v_1 = v_2 + (v_1 - v_2) = v_1$, $(T_1 + T_2) v_2 = 2v_1$, $(T_1 + T_2) v_i = 2v_i$

$\Rightarrow \dim$ range $T_1 + T_2 = n - 1 \Rightarrow T_1 + T_2$ not surjective $\Rightarrow T_1 + T_2$ not invertible

5 Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

"$\Rightarrow$" : $\exists T \in L(V)$ s.t $Tu = Su\ \forall u \in U$, and T invertible.

let $u_1, u_2 \in U$, s.t $Su_1 = Su_2 \Rightarrow Tu_1 = Tu_2 \Rightarrow u_1 = u_2$, as T is injective

$\Rightarrow S$ is injective

"$\Leftarrow$" : let $u_1 \dots u_p$ basis of U. let $s_1 \dots s_p$ basis of range S (S injective so $p = \dim U$ vectors).

$u_1 \dots u_p, s_1 \dots s_p$ can be reduced so that it is linearly independent in V, until it reaches the

size p. We choose to remove additional vectors from the list $s_1 \dots s_p$. This gives us

the following linearly independent list : $u_1 \dots u_p, s_1 \dots s_k$, with $k \in \{0 \dots p\}$.

We can then extend this list to a base of V with vectors $v_1 \dots v_{n-p-k}$ ($n = \dim V$)

let $T \in L(V)$ s.t $Tu_i = Su_i$, $Ts_i = s_i$

and $Tv_i = v_i\ \forall i$, T is a linear map.

let $w \in V$. $w = u + s + v$, $u \in U$, $s \in$ range $S \setminus U$, $v \in V \setminus (U \cup \text{range } S)$

$Tw = Tu + Ts + Tv = Su + s + v$ (linearly independent vectors)

$$\Rightarrow \begin{cases} Sv = 0 \\ S = 0 \\ v = 0 \end{cases} \Rightarrow \begin{cases} v = 0 \text{ (as } S \text{ injective)} \\ S = 0 \\ v = 0 \end{cases} \Rightarrow w = 0$$

$\Rightarrow T$ injective $\Rightarrow T$ invertible

6 Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that null S = null T if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

"$\Leftarrow$": Suppose $\exists E \in \mathcal{L}(W)$ s.t. $S = ET$

$\left(x \in \operatorname{null} S \Leftrightarrow Sx = 0 \Leftrightarrow ETx = 0 \overset{(E \text{ injective})}{\Leftrightarrow} Tx = 0 \Leftrightarrow x \in \operatorname{null} T\right)$

$\Rightarrow \operatorname{null} S = \operatorname{null} T$

"$\Rightarrow$": $\operatorname{null} S = \operatorname{null} T \Rightarrow \dim \operatorname{range} S = \dim \operatorname{range} T = p$

$s_1 \dots s_p$, $t_1 \dots t_p$, $v_1 \dots v_m$ basis of range S, range T and V respectively.

3B ex. 25: $\operatorname{null} T \subseteq \operatorname{null} S \Rightarrow \exists E_1 \in \mathcal{L}(W)$ s.t. $S = E_1 T$

We can define $E_2 \in \mathcal{L}(\operatorname{range} T, W)$, $\forall t \in \operatorname{range} T$, $E_2 t = E_1 t$.

Let $t \in \operatorname{range} T$

$E_2 t = 0 \Rightarrow E_1 t = 0 \Rightarrow E_1 T v_t = 0$ (with $T v_t = t$)

$\Rightarrow S v_t = 0 \Rightarrow v_t \in \operatorname{null} S \Rightarrow v_t \in \operatorname{null} T$ (as $\operatorname{null} S = \operatorname{null} T$)

$\Rightarrow T v_t = 0 \Rightarrow t = 0 \Rightarrow \operatorname{null} T = \{0\}$

$\Rightarrow E_2$ injective

From previous exercise:

$\exists E \in \mathcal{L}(W)$ s.t. E is invertible and $Et = E_2 t$ $\forall t \in \operatorname{range} T$, as $E_2 \in \mathcal{L}(\operatorname{range} T, W)$ is injective.

Let $v \in V$. $E \underbrace{Tv}_{\in \operatorname{range} T} = E_2 T v = E_1 T v = Sv$

8 Suppose V and W are finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that there exist invertible $E_1 \in \mathcal{L}(V)$ and $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$ if and only if dim null S = dim null T.

We can show there is an isomorphism $E_1 \in \mathcal{L}(V)$ s.t. $\operatorname{null} TE_1 = \operatorname{null} S$.

$\dim \operatorname{null} S = \dim \operatorname{null} T \Rightarrow \exists E' \in \mathcal{L}(\operatorname{null} S, \operatorname{null} T)$ isomorphism,

and $\exists E'' \in \mathcal{L}(U_S, U_T)$ isomorphism, with U_S, U_T s.t. $U_S \oplus \operatorname{null} S = V$, $U_T \oplus \operatorname{null} T = V$

Define $E_1 \in \mathcal{L}(V)$ with $E_1 s = E' s$ $\forall s \in \operatorname{null} S$, and $E_1 \bar{s} = E'' \bar{s}$ $\forall \bar{s} \in U_S$

Let $v \in V$, $v = \underbrace{s}_{\in \operatorname{null} S} + \underbrace{\bar{s}}_{\in U_S}$, $E_1 v = \underbrace{E' s}_{\in \operatorname{null} T} + \underbrace{E'' \bar{s}}_{\in U_T} = 0 \Rightarrow v = 0 \Rightarrow E_1$ injective $\Rightarrow E_1$ invertible

Furthermore, let $v \in \operatorname{null} TE_1$. $TE_1 v = 0 \Rightarrow TE_1(s + \bar{s}) = 0 \Rightarrow \cancel{T\underbrace{E's}_{\in \operatorname{null} T}}^{0} + T\underbrace{E''\bar{s}}_{\in U_T} = 0 \Rightarrow E''\bar{s} \in U_T \cap \operatorname{null} T \Rightarrow E''\bar{s} = 0 \Rightarrow \bar{s} = 0$ (as E'' invertible) $\Rightarrow v = s \in \operatorname{null}$

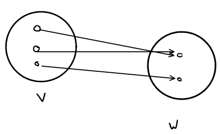
Let $v \in \operatorname{null} S$. $TE_1 v = T\underbrace{E's}_{\in \operatorname{null} T} = 0 \Rightarrow v \in \operatorname{null} TE_1$

$\Rightarrow \operatorname{null} S = \operatorname{null} TE_1$

Then, we can apply ex. 6: $\exists E_2 \in \mathcal{L}(W)$ invertible s.t. $S = E_2 T E_1$. This concludes the proof.

9 Suppose V is finite-dimensional and $T: V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

Here $T|_U$ means the function T restricted to U. Thus $T|_U$ is the function whose domain is U, with $T|_U$ defined by $T|_U(u) = Tu$ for every $u \in U$.



Let $w_1 \dots w_m$ basis of W.

T surjective $\Rightarrow \exists v_1 \dots v_m \in V$ s.t. $Tv_i = w_i$.

From ex. 4 3A, since $w_1 \dots w_m$ is linearly independent, $v_1 \dots v_m$ is linearly independent.

$v_1 \dots v_m$ spans to a subspace U of V, of which it is a basis, so $\dim U = m = \dim W$. $T|_U$ is surjective by construction,

therefore $T|_U$ is an isomorphism of U onto W.

10 Suppose V and W are finite-dimensional and U is a subspace of V. Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \operatorname{null} T\}.$$

(a) Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.

(b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V$, $\dim W$, and $\dim U$.

Hint: Define $\Phi: \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. What is null Φ? What is range Φ?

a) · $\operatorname{null} 0 = V$, thus $U \subseteq \operatorname{null} 0$, so $0 \in \mathcal{E}$

· let $S, T \in \mathcal{E}$. let $u \in U$. $u \in \operatorname{null} S$ and $u \in \operatorname{null} T \Rightarrow Su + Tu = 0 \Rightarrow (S+T)u = 0 \Rightarrow u \in \operatorname{null} S+T \Rightarrow U \subseteq \operatorname{null} S+T$
$\Rightarrow S+T \in \mathcal{E}$

· let $\lambda \in \mathbb{F}$, $S \in \mathcal{E}$. let $u \in U$. $u \in \operatorname{null} S \Rightarrow Su = 0 \Rightarrow \lambda S u = 0 \Rightarrow u \in \operatorname{null} \lambda S \Rightarrow U \subseteq \operatorname{null} \lambda S$
$\Rightarrow \lambda S \in \mathcal{E}$

$\Rightarrow \mathcal{E}$ subspace of $\mathcal{L}(V,W)$

b) let $\Phi : \mathcal{L}(V,W) \to \mathcal{L}(U,W)$ s.t. $\Phi(T) = T|_U$

$\operatorname{null} \Phi = \{T \in \mathcal{L}(V,W) \text{ s.t } T|_U = 0\} = \mathcal{E} \Rightarrow \dim \operatorname{null} \Phi = \dim \mathcal{E}$

$\operatorname{range} \Phi = \{T \in \mathcal{L}(U,W) \text{ s.t } \exists S \in \mathcal{L}(V,W) : S|_U = T\} = \dim \mathcal{L}(U,W)$

$\dim \mathcal{L}(V,W) = \dim \operatorname{null} \Phi + \dim \operatorname{range} \Phi$
$\Rightarrow \dim V \dim W = \dim \mathcal{E} + \dim U \dim W$
$\Rightarrow \dim \mathcal{E} = \dim W (\dim V - \dim U)$

11 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that
$$ST \text{ is invertible} \iff S \text{ and } T \text{ are invertible.}$$

"$\Leftarrow$": $ST T^{-1} S^{-1} = I \Rightarrow ST$ invertible with inverse $T^{-1}S^{-1}$

"$\Rightarrow$": 3B ex 23: $\dim \operatorname{range} ST = \dim V \leq \min(\operatorname{range} S, \operatorname{range} T) \Rightarrow \operatorname{range} S = \operatorname{range} T = \dim V$

$\Rightarrow S$ and T surjective and thus invertible (since $S, T \in \mathcal{L}(V)$)

12 Suppose V is finite-dimensional and $S, T, U \in \mathcal{L}(V)$ and $STU = I$. Show that T is invertible and that $T^{-1} = US$.

$(ST)U = I \Rightarrow U(ST) = I$ (all linear maps are in $\mathcal{L}(V)$)

$\Rightarrow ST$ is invertible with inverse U.
From previous exercise, T and S are then invertible, and $(ST)^{-1} = T^{-1}S^{-1}$.
$\Rightarrow U = T^{-1}S^{-1} \Rightarrow T^{-1}S^{-1}S = US \Rightarrow T^{-1} = US$

13 Show that the result in Exercise 12 can fail without the hypothesis that V is finite-dimensional.

let $V = \mathbb{R}^\infty$, $U \in \mathcal{L}(\mathbb{R}^\infty)$ with $U(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$
$T \in \mathcal{L}(\mathbb{R}^\infty)$ with $T(x_1, x_2, \dots) = (x_2, x_3, \dots)$
$S = I$
Then $STU(x_1, x_2, \dots) = T(0, x_1, x_2, \dots) = (x_1, x_2, \dots) \Rightarrow STU = I$

However, T is not invertible, as $\operatorname{null} T = 1 \neq 0$.

14 Prove or give a counterexample: If V is a finite-dimensional vector space and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective, then S is injective.

Suppose RST is surjective.
Then $\dim V = \dim \operatorname{range} RST \leq \min \dim \operatorname{range} RS, \dim \operatorname{range} T \Rightarrow \dim \operatorname{range} RS = \dim V$
$\dim \operatorname{range} RS \leq \min \dim \operatorname{range} R, \dim \operatorname{range} S \Rightarrow \dim \operatorname{range} S = \dim V$
$\Rightarrow S$ is surjective and in $\mathcal{L}(V)$, thus invertible, and injective.

15 Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_m$ is a list in V such that $Tv_1, \dots, Tv_m$ spans V. Prove that $v_1, \dots, v_m$ spans V.

$Tv_1, \dots, Tv_m$ span $V \Rightarrow$ It can be reduced to a basis $Tv_1, \dots, Tv_n$ of V, with $n = \dim V$
Then by ex. 4 in 3A, $v_1, \dots, v_n$ is also a linearly independent list of vectors in V, and hence a basis, so it spans V.

17 Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by
$$\mathcal{A}(T) = ST$$
for $T \in \mathcal{L}(V)$.
(a) Show that $\dim \operatorname{null} \mathcal{A} = (\dim V)(\dim \operatorname{null} S)$.
(b) Show that $\dim \operatorname{range} \mathcal{A} = (\dim V)(\dim \operatorname{range} S)$.

a) $\operatorname{null} \mathcal{A} = \{T \in \mathcal{L}(V) \text{ s.t } \mathcal{A}(T) = ST = 0\}$

let $s_1, \dots, s_k$ a basis of $\operatorname{null} S$. It can be extended to a basis of V: $s_1, \dots, s_k, \bar{s}_1, \dots, \bar{s}_{n-k}$ with $n = \dim V$.
let $v_1, \dots, v_n$ a basis of V.

let $T \in \operatorname{null} \mathcal{A}$, $ST = 0$. This implies every vector of $v_1, \dots, v_n$ is mapped to one of $s_1, \dots, s_k$.

There are $n \times k$ such possibilities, hence $\dim \operatorname{null} \mathcal{A} = nk = \dim V \dim \operatorname{null} S$.

b) $\dim \mathcal{L}(V) = \dim \operatorname{null} \mathcal{A} + \dim \operatorname{range} \mathcal{A}$
$\Rightarrow (\dim V)^2 = \dim V \dim \operatorname{null} S + \dim \operatorname{range} \mathcal{A}$
$\Rightarrow (\dim V)^2 - \dim V (\dim V - \dim \operatorname{range} S) = \dim \operatorname{range} \mathcal{A}$
$\Rightarrow \dim \operatorname{range} \mathcal{A} = \dim V \dim \operatorname{range} S$

20 Suppose $q \in \mathcal{P}(\mathbf{R})$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{R})$ such that

$$q(x) = (x^2 + x)p''(x) + 2xp'(x) + p(3)$$

for all $x \in \mathbf{R}$.

let $T \in \mathcal{L}(P_m(\mathbb{R}), P_m(\mathbb{R}))$, such that $Tp = (x^2+x)p''(x) + 2Xp'(X) + p(3)$.

It is easy to show T is a linear map. We can show T is injective. Then $\forall q \in P_m(\mathbb{R})$, $\exists p \in P_m(\mathbb{R})$ s.t. $q = Tp$, which would conclude the proof.

let $p \in \operatorname{null} T$, $p(X) = \sum_{i=0}^{m} a_i X^i$. $p'(X) = \sum_{i=1}^{m} a_i X^{i-1}$, $p''(X) = \sum_{i=2}^{m} a_i X^{i-2}$

$$Tp = (x^2+x)p''(x) + 2Xp'(x) + p(3) = (x^2+X)\sum_{i=2}^{m} a_i X^{i-2} + 2X\sum_{i=1}^{m} a_i X^{i-1} + p(3)$$

$$= \sum_{i=2}^{m} a_i X^i + \sum_{i=2}^{m} a_i X^{i-1} + \sum_{i=1}^{m} 2a_i X^i + c = 0$$

$$\Rightarrow \begin{cases} p(3) = 0 & (X^0) \\ 2a_1 + a_2 = 0 & (X^1) \\ 3a_2 + a_3 = 0 & (X^2) \\ \vdots & \vdots \\ 3a_m = 0 \Rightarrow a_m = 0 & (X^m) \end{cases} \Rightarrow \begin{cases} a_0 = 0 \\ a_1 = 0 \\ \vdots \\ a_m = 0 \end{cases} \Rightarrow T \text{ injective} \Rightarrow T \text{ surjective } (3.65)$$

$\Rightarrow \forall q \in P(\mathbb{R})$, $\exists p \in P(\mathbb{R})$ (with $\deg p = \deg q$) s.t. $q = Tp$

21 Suppose n is a positive integer and $A_{j,k} \in \mathbf{F}$ for all $j, k = 1, \ldots, n$. Prove that the following are equivalent (note that in both parts below, the number of equations equals the number of variables).

(a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\sum_{k=1}^{n} A_{1,k} x_k = 0$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{n,k} x_k = 0.$$

(b) For every $c_1, \ldots, c_n \in \mathbf{F}$, there exists a solution to the system of equations

$$\sum_{k=1}^{n} A_{1,k} x_k = c_1$$
$$\vdots$$
$$\sum_{k=1}^{n} A_{n,k} x_k = c_n.$$

"$\Rightarrow$": $x_1 = \ldots = x_m = 0$ is the only solution to the homogeneous system of equations.

$(Ax = 0 \Rightarrow x = 0) \Rightarrow T \in \mathcal{L}(\mathbb{F}^m)$ is injective, with $A = \mathcal{M}(T)$ for some basis of $\mathbb{F}^m$

$\Rightarrow T$ is surjective (3.65)

$\Rightarrow \forall c \in \mathbb{F}^m$, $\exists x \in \mathbb{F}^m$ s.t. $Ax = c$

"$\Leftarrow$": Same reasoning except we start from T being surjective, we infer it is injective and conclude.