5E Exercises

1 Give an example of two commuting operators S, T on $\mathbf{F}^4$ such that there is a subspace of $\mathbf{F}^4$ that is invariant under S but not under T and there is a subspace of $\mathbf{F}^4$ that is invariant under T but not under S.

$S(a,b,c,d) = (a,b,0,0)$
$T(a,b,c,d) = (a,0,c,0)$

Subspace $U = \text{span}(1,1,0,0)$ invariant under S not under T
Subspace $W = \text{span}(1,0,1,0)$ invariant under T not under S

3 Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbf{F})$.

(a) Prove that null $p(S)$ is invariant under T.

(b) Prove that range $p(S)$ is invariant under T.

See 5.18 for the special case $S = T$.

a) Let $v \in$ null $p(S)$.

$p(S)(Tv) = (p(S)T)v = (Tp(S))v = T(p(S)v) = 0$

$\Rightarrow Tv \in$ null $p(S) \Rightarrow$ null $p(S)$ invariant under T

b) Let $v \in$ range $p(S)$.

$\Rightarrow \exists w \in V$ s.t. $p(S)w = v$

$Tv = T(p(S)w) = (Tp(S))w = (p(S)T)w = p(S)(Tw)$

$\Rightarrow Tv \in$ range $p(S) \Rightarrow$ range $p(S)$ invariant under T

4 Prove or give a counterexample: If A is a diagonal matrix and B is an upper-triangular matrix of the same size as A, then A and B commute.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

5 Prove that a pair of operators on a finite-dimensional vector space commute if and only if their dual operators commute.

See 3.118 for the definition of the dual of an operator.

"$\Leftarrow$": Suppose S', T' commute. $S'T' = T'S'$

$(S'T')\varphi = (T'\varphi)S = \varphi TS$ $\Big\}$ $\Rightarrow \varphi TS = \varphi ST$ $\underline{\forall \varphi}$ $\Rightarrow TS = ST$

$= (T'S')\varphi \quad (S'\varphi)T = \varphi ST$

"$\Rightarrow$": Suppose S, T commute.

$(S'T')\varphi \overset{\text{(see above)}}{=} \varphi ST = T'(\varphi S) = T'(S'(\varphi)) = (T'S')\varphi$

$\Rightarrow S', T'$ commute

6 Suppose V is a finite-dimensional complex vector space and $S, T \in \mathcal{L}(V)$ commute. Prove that there exist $\alpha, \lambda \in \mathbf{C}$ such that

$$\text{range}(S - \alpha I) + \text{range}(T - \lambda I) \neq V.$$

$$\operatorname{range}(S - \alpha I) + \operatorname{range}(T - \lambda I) \neq V.$$

By 5.78, S and T have a common e.ve. v_1, and e.va α, λ for S, T respectively.

Suppose $v_1 \in \operatorname{range}(S-\alpha I) + \operatorname{range}(T-\lambda I)$, $v_1 \dots v_n$ a basis of V.

$\Rightarrow \exists\, s \in \operatorname{range}(S-\alpha I),\ t \in \operatorname{range}(T-\lambda I)$ s.t.: $v_1 = (S-\alpha I)s + (T-\lambda I)t$

$\Rightarrow v_1 = S\sum a_i v_i - \alpha \sum a_i v_i + T \sum b_i v_i - \lambda \sum b_i v_i$ (with $s = \sum a_i v_i$, $t = \sum b_i v_i$)

$\Rightarrow 1 = \alpha a_1 - \alpha a_1 + \lambda b_1 - \lambda b_1 \Rightarrow 1 = 0\ \perp$

$\Rightarrow v \notin \operatorname{range}(S-\alpha I) + \operatorname{range}(T-\lambda I)$

$\Rightarrow \operatorname{range}(S-\alpha I) + \operatorname{range}(T-\lambda I) \neq V$

7 Suppose V is a complex vector space, $S \in \mathcal{L}(V)$ is diagonalizable, and $T \in \mathcal{L}(V)$ commutes with S. Prove that there is a basis of V such that S has a diagonal matrix with respect to this basis and T has an upper-triangular matrix with respect to this basis.

S diagonalizable $\Rightarrow \exists$ basis of e.ve. $v_1 \dots v_m$ with corresponding e.va. $\lambda_1 \dots \lambda_m$

S, T commute $\Rightarrow S, T$ share an e.ve., WLOG say v_1.

Define $U_1 = \operatorname{span} v_1$. $v_2 \dots v_m$ is a basis of $V \setminus U_1$. $S|_{V\setminus U_1}, T|_{V \setminus U_1}$ commute,

so they share an e.ve., say v_2. Continue this process for $v_3, \dots v_m$.

$M(S, v_1 \dots v_m)$ is diagonal, and $M(T, v_1 \dots v_m)$ is upper triangular.

Indeed, we have for each $k>1$ that v_k is an e.ve. of $T|_{V \setminus U_1 \dots U_{k-1}}$

$$T|_{V\setminus U_1 \dots U_{k-1}} v_k = \mu_k v_k \Rightarrow T v_k = \mu_k v_k + \sum_{i=1}^{k-1} a_i v_i$$

$\Rightarrow T v_k \in \operatorname{span}(v_1 \dots v_k)$ for each $k > 1$

For $k = 1$, $T v_1 = \mu_1 v_1$.

$\Rightarrow T v_k \in \operatorname{span}(v_1 \dots v_k)$ for each k

$\Rightarrow M(T, v_1 \dots v_m)$ upper triangular.

8 Suppose $m = 3$ in Example 5.72 and D_x, D_y are the commuting partial differentiation operators on $\mathcal{P}_3(\mathbf{R}^2)$ from that example. Find a basis of $\mathcal{P}_3(\mathbf{R}^2)$ with respect to which D_x and D_y each have an upper-triangular matrix.

Let $1, x, y, xy, x^2, y^2, x^2y, xy^2, x^3, y^3$ a basis of $P_3(\mathbb{R}^2)$.

$D_x 1 = 0 \qquad D_y 1 = 0$

$D_x x = 1$ $\quad$ $D_y x = 0$

$D_x y = 0$ $\quad$ $D_y y = 1$

$D_x xy = y$ $\quad$ $D_y xy = x$

$D_x x^2 = 2x$ $\quad$ $D_y x^2 = 0$

$D_x y^2 = y^2$ $\quad$ $D_y y^2 = 2y$

$D_x x^2y = 2xy$ $\quad$ $D_y x^2y = x^2$

$D_x xy^2 = y^2$ $\quad$ $D_y xy^2 = 2xy$

$D_x x^3 = 3x^2$ $\quad$ $D_y x^3 = 0$

$D_x y^3 = 0$ $\quad$ $D_y y^3 = 3y^2$

For each k, $D_x p_k \in \operatorname{span}(p_1 \dots p_k)$, $D_y p_k \in \operatorname{span}(p_1 \dots p_k)$

$\Rightarrow$ D_x and D_y have an upper-triangular matrix wrt this basis.