5A Exercises

dimanche 18 août 2024 22:38

1 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.

a) $u \in U \Rightarrow u \in \operatorname{null} T \Rightarrow Tu = 0 \in U$ (as U is a subspace of V) $\Rightarrow U$ invariant under T

b) Let $u \in U$. $Tu \in \operatorname{range} T \Rightarrow Tu \in U \Rightarrow U$ invariant under T

2 Suppose that $T \in \mathcal{L}(V)$ and $V_1, \dots, V_m$ are subspaces of V invariant under T. Prove that $V_1 + \cdots + V_m$ is invariant under T.

Let $v = \sum_{i=1}^{m} v_i \in V_1 + \dots + V_m$. $Tv = \sum_{i=1}^{m} Tv_i$, where for each i, $Tv_i \in V_i$, as V_i's are invariant under T. Therefore $Tv \in V_1 + \dots + V_m \Rightarrow V_1 + \dots + V_m$ is invariant under T.

3 Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Let $A = \bigcap_{i=1}^{m} V_i$, where $m \in \mathbb{N}^* \cup \{\infty\}$ and V_i is invariant under T for each i.

Let $v \in A$. For each i, $Tv \in V_i$, which implies $Tv \in \bigcap_{i=1}^{m} V_i$, so $\bigcap_{i=1}^{m} V_i$ is invariant under T.

4 Prove or give a counterexample: If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

By contraposition, suppose U is a subspace of V s.t. $U \neq \{0\}$ and $U \neq V$. Let $u_1 \dots u_m$ a basis of U, and $u_1, \dots u_m, v_1 \dots v_{n-m}$ a basis of V. We can define an operator T s.t.

$Tu_1 = v_1$ and T is the identity for every other vectors in the base.

Then U is not invariant under T.

5 Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

$$(-3y, x) = \lambda(x, y) \Rightarrow \begin{cases} \lambda x = -3y \\ \lambda y = x \end{cases} \Rightarrow \begin{cases} \lambda^2 y = -3y \\ \lambda y = x \end{cases} \overset{y \neq 0, \text{ otw } x = y = 0}{\Rightarrow} \begin{cases} \lambda^2 = -3 \\ \lambda y = x \end{cases} \text{ no solution in } \mathbb{R}^2$$

$\Rightarrow$ no eigenvalues of T.

6 Define $T \in \mathcal{L}(\mathbf{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T.

$$(z, w) = \lambda(w, z) \Rightarrow \begin{cases} z = \lambda w \\ w = \lambda z \end{cases} \Rightarrow \begin{cases} z = \lambda^2 z \\ w = \lambda z \end{cases} \overset{z \neq 0 \text{ as otw } w = z = 0}{\Rightarrow} \begin{cases} \lambda^2 = 1 \Rightarrow \lambda = 1 \text{ or } \lambda = -1 \\ w = \lambda z \end{cases}$$

$\Rightarrow$ 2 eigenvalues, 1 with e.ve. span$(1,1)$, and -1 with e.ve. span$(1,-1)$

7 Define $T \in \mathcal{L}(\mathbf{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T.

$$(2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3) \Rightarrow \begin{cases} \lambda z_1 = 2z_2 \\ \lambda z_2 = 0 \\ 5z_3 = \lambda z_3 \end{cases}$$

$$\Rightarrow \left(\lambda = 0 \text{ and } \begin{cases} z_2 = 0 \\ z_3 = 0 \\ z_1 \in \mathbb{F} \end{cases}\right) \text{ or } \left(\begin{cases} z_2 = 0 \\ z_1 = 0 \\ 5z_3 = \lambda z_3 \Rightarrow \lambda = 5 \end{cases}\right.$$

E.va: $\lambda = 0$ with e.ve. span$(x, 0, 0)$

$\lambda = 5$ with e.ve. span$(0, 0, x)$

8 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P, then $\lambda = 0$ or $\lambda = 1$.

Let $\lambda \in \mathbb{F}$ a e.va. of P, with $v \in V$ an associated e.ve.

$P^2 v = P(Pv) = P(\lambda v) = \lambda P v = \lambda^2 v$

$P^2 v = Pv$ (as $P = P^2$)

$= \lambda v$

$\Rightarrow \lambda^2 v = \lambda v \Rightarrow \lambda(\lambda v - v) = 0 \Rightarrow \lambda = 0$ or $(\lambda v = v \Rightarrow \lambda = 1)$

$$= \lambda v$$

$$\Rightarrow \lambda^2 v = \lambda v \Rightarrow \lambda(\lambda v - v) = 0 \Rightarrow \lambda = 0 \text{ or } (\lambda v = v \Rightarrow \lambda = 1)$$

9 Define $T: \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

$p' = \lambda p$. $\deg p' = \deg p - 1$, which is only true if $p = 0$. An eigenvector can't be 0 so T has no eigenvalues.

10 Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbf{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbf{R}$. Find all eigenvalues and eigenvectors of T.

$$Tp = \lambda p \Rightarrow (Tp)(x) = (\lambda p)(x) \ \forall x \in \mathbb{R}$$
$$\Rightarrow x p'(x) = \lambda p(x)$$
$$\Rightarrow x \sum_{k=1}^{4} a_k k x^{k-1} = \lambda \sum_{k=0}^{4} a_k x^k$$
$$\Rightarrow \sum_{k=1}^{4} a_k k x^k = \sum_{k=0}^{4} \lambda a_k x^k$$

$$\Rightarrow \lambda a_0 = 0,\ a_1 = \lambda a_1,\ 2a_2 = \lambda a_2,\ 3a_3 = \lambda a_3,\ 4a_4 = \lambda a_4$$

E.va. with associated e.ve.: $\begin{cases} k \in \{0,1,2,3,4\} \text{ (e.va)} \\ \text{span}(p_k), \text{ with } p_k(x) = a_k x^k,\ a_k \in \mathbb{R} \end{cases}$

11 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbf{F}$. Prove that there exists $\delta > 0$ such that $T - \lambda I$ is invertible for all $\lambda \in \mathbf{F}$ such that $0 < |\alpha - \lambda| < \delta$.

T has at most $n = \dim V$ eigenvalues. If there is none, then any $\delta > 0$ can be chosen s.t. $T - \lambda I$ is invertible, as $T - \lambda I$ invertible is equivalent to λ is not an eigenvalue of T.

Let A the set of e.va of T. Suppose $A \neq \emptyset$.

Let $\delta = \min_{\lambda \in A} |\alpha - \lambda|$. Let $\lambda \in \mathbb{F}$ s.t. $0 < |\alpha - \lambda| < \delta$.

By definition of δ, λ is not an e.va of T, therefore $T - \lambda I$ is not invertible. If $\alpha \in A$, any value of $\delta > 0$ works.

12 Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$. Find all eigenvalues and eigenvectors of P.

Let $v = u + w \in V$, $u \in U$, $w \in W$

$$Pv = \lambda v \Rightarrow u = \lambda(u + w) \Rightarrow \underbrace{u - \lambda u}_{\in U} = \underbrace{w}_{\in W}$$

$$\Rightarrow u - \lambda u = w = 0$$

$$\Rightarrow \begin{cases} u - \lambda u = 0 \\ w = 0 \end{cases} \Rightarrow \begin{cases} u = \lambda u \\ w = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 1 \\ w = 0 \end{cases}$$

E.va: 1, with e.ve. U.

13 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

a) • let λ e.va. of T. $Tv = \lambda v \Rightarrow S^{-1}Tv = \lambda S^{-1}v$. $v \in V$ and S invertible implies $\exists w$ s.t. $Sw = v$. This yields: $S^{-1}T(Sw) = \lambda S^{-1}(Sw)$

$\Rightarrow S^{-1}TSw = \lambda w \Rightarrow \lambda$ e.va of S

• let λ e.va of $S^{-1}TS$. $S^{-1}TSv = \lambda v$. $\exists w \in V$ s.t $v = S^{-1}w$

$\Rightarrow S^{-1}Tw = \lambda S^{-1}w \Rightarrow SS^{-1}Tw = \lambda SS^{-1}w \Rightarrow Tw = \lambda w$

$\Rightarrow \lambda$ e.va. of T

b) v e.ve of T wrt $\lambda \iff Sv$ e.ve of $S^{-1}TS$ wrt λ

14 Give an example of an operator on $\mathbf{R}^4$ that has no (real) eigenvalues.

$$T(\underbrace{x_1, x_2, x_3, x_4}_{x \in \mathbb{R}^4}) = (-x_2, x_1, 0, 0) = \lambda(x_1, x_2, x_3, x_4)$$

$$\Rightarrow \begin{cases} -x_2 = \lambda x_1 \\ x_1 = \lambda x_2 \\ x_3 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} -x_2 = \lambda^2 x_2 \\ x_1 = \lambda x_2 \\ x_3 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_2(\lambda^2 + 1) = 0 \\ x_1 = \lambda x_2 \\ x_3 = 0 \\ x_4 = 0 \end{cases}$$

If $x_2 = 0$, $x = 0$.

If $\lambda^2 + 1 = 0$, then $\lambda^2 = -1$, which has no real solution.

$\Rightarrow$ No real eigenvalues.

15 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

λ e.va. of T

$\text{null}(T - \lambda I) \neq 0 \iff \text{range}(T - \lambda I) \neq V \iff \exists \varphi \in V', \varphi \neq 0$ s.t $\forall v \in \text{range}(T - \lambda I), (\varphi \circ (T - \lambda I))v = 0$

$\iff$ ————————————, $((T - \lambda I)'(\varphi))v = 0$

$\iff$ ————————————, $((T' - \underbrace{\lambda I}_{\text{id. in } V'})(\varphi))v = 0$

$\iff \exists \varphi \in V', \varphi \neq 0$ s.t $\forall v \in V, ((T' - \lambda I)(\varphi))v = 0$ (by setting $\varphi v_i = 0$ for v_i's completing a basis of range$(T - \lambda I)$ to form a basis of V)

$\iff \text{null}(T' - \lambda I) \neq \{0\}$

$\iff \lambda$ e.va. of T'

16 Suppose $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T, then

$$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\},$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row j, column k of the matrix of T with respect to the basis $v_1, \ldots, v_n$.

See Exercise 19 in Section 6A for a different bound on $|\lambda|$.

let λ e.va. of T, with $v \in V$ e.ve., $v = \sum_{i=1} a_i v_i$, $v \neq 0$.

$$Tv = \sum_{i=1}^{n} a_i T v_i = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \mathcal{M}(T)_{ij} v_j = \lambda v = \lambda \sum_{i=1}^{n} a_i v_i$$

$$\Rightarrow \Big(\sum^{n} a_i \mathcal{M}(T)_{i1} = \lambda a_1 \quad (1)$$

$$\Rightarrow \begin{cases} \sum_{i=1}^{m} a_i M(T)_{i1} = \lambda a_1 \ (1) \\ \vdots \\ \sum_{i=1}^{n} a_i M(T)_{im} = \lambda a_m \end{cases}$$

$$\Rightarrow \sum_{i=1}^{m} a_i M(T)_{ik} = \lambda a_k, \text{ with } k = \arg\max_{i=1..m} |a_i|$$

$$\Rightarrow \left|\sum_{i=1}^{m} a_i M(T)_{ik}\right| = |\lambda a_k| \Rightarrow \sum_{i=1}^{m} |a_i||M(T)_{ik}| \geq |\lambda||a_k|$$

$$\Rightarrow m\,|a_k|\,|\max_{\substack{i=1..m \\ j=1..m}} M(T)_{ij}| \geq |\lambda||a_k|$$

$$\Rightarrow |\lambda| \leq m\,|\max_{\substack{i=1..m \\ j=1..m}} M(T)_{ij}| \quad (\text{as } |a_k| \neq 0, \text{ since } v \neq 0).$$

17 Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{R}$. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of the complexification $T_{\mathbf{C}}$.

See Exercise 33 in Section 3B for the definition of $T_{\mathbf{C}}$.

33 Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$. Define $T_{\mathbf{C}}: V_{\mathbf{C}} \to W_{\mathbf{C}}$ by

$$T_{\mathbf{C}}(u + iv) = Tu + iTv$$

"$\Rightarrow$": let λ e.va of T, with e.ve. $v \in V$, $v \neq 0$.

$T_C(v + i0) = Tv = \lambda v = \lambda(v + i0) \Rightarrow \lambda$ e.va of T_C

"$\Leftarrow$": let λ e.va of T_C, with e.ve. $(v, w) \in V_C$.

$T_C(v + iw) = Tv + iTw = \lambda v + i\lambda w \overset{\lambda \in \mathbb{R}}{\Rightarrow} \begin{cases} Tv = \lambda v \\ Tw = \lambda w \end{cases} \Rightarrow v$ e.va of T

18 Suppose $\mathbf{F} = \mathbf{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbf{C}$. Prove that λ is an eigenvalue of the complexification $T_{\mathbf{C}}$ if and only if $\overline{\lambda}$ is an eigenvalue of $T_{\mathbf{C}}$.

let λ e.va. of T_C, with e.ve. $v + iw \in V$

$T_C(v + iw) = Tv + iTw = \lambda(v + iw) \Leftrightarrow Tv - iTw = \bar{\lambda}(v - iw)$

$\Leftrightarrow T_C(v - iw) = \bar{\lambda}(v - iw) \Leftrightarrow \bar{\lambda}$ e.va. of T_C

19 Show that the forward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by

$$T(z_1, z_2, \ldots) = (0, z_1, z_2, \ldots)$$

has no eigenvalues.

Suppose $\exists \lambda \in \mathbb{F}$ s.t. $\exists z = (z_1, z_2, \ldots) \in \mathbb{F}^\infty$ with $Tz = \lambda z$ and $z \neq 0$

$\Rightarrow (0, z_1, z_2 \ldots) = \lambda(z_1, z_2, \ldots) \Rightarrow \begin{cases} 0 = \lambda z_1 \\ z_k = \lambda z_{k+1} \text{ for } k \geq 1 \end{cases} \Rightarrow \lambda = 0 \text{ or } z_1 = 0$

In both cases, this leads to $z = (0, 0, \ldots)$.

Therefore λ is not an e.va. of T, so T has no e.va.

20 Define the backward shift operator $S \in \mathcal{L}(\mathbf{F}^\infty)$ by

$$S(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots).$$

(a) Show that every element of $\mathbf{F}$ is an eigenvalue of S.
(b) Find all eigenvectors of S.

a) let $\lambda \in \mathbb{F}$. let $z = (\lambda, \lambda^2, \ldots) \in \mathbb{F}^\infty$.

$S(\lambda, \lambda^2, \lambda^3 \ldots) = (\lambda^2, \lambda^3, \ldots) = \lambda(\lambda, \lambda^2, \ldots) \Rightarrow \lambda$ e.va. of S

b) $S(z_1, z_2 \ldots) = \lambda(z_1 \ldots z_2) \Rightarrow (z_2, z_3 \ldots) = \lambda(z_1, z_2, \ldots)$

$$\Rightarrow \begin{cases} z_2 = \lambda z_1 \\ z_3 = \lambda z_2 \\ \vdots \end{cases} \Rightarrow \begin{cases} z_2 = \lambda z_1 \\ z_3 = \lambda^2 z_1 \\ \vdots \end{cases} \Rightarrow z = (z_1, \lambda z_1, \lambda^2 z_1, \ldots) \ (1)$$
$$= z_1(\lambda, \lambda^2, \ldots) \text{ for } z_1 \in \mathbb{F}$$

... for each $\lambda \in \mathbb{F}$ (easy to show

$\Rightarrow$ e.ve. of S are $\operatorname{span}((\lambda, \lambda^2, \ldots))$ for each $\lambda \in \mathbf{F}$ (easy to show the other inclusion of the equality).

21 Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbf{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

a) let λ e.va. of T with e.ve v $\iff Tv = \lambda v$

$\iff T^{-1}Tv = \lambda T^{-1} v$

$\iff \lambda^{-1} v = T^{-1} v \quad (\lambda \neq 0)$

$\iff \lambda^{-1}$ e.va. of T^{-1}, with e.ve. v.

b) We proved in a) that all of T's nonzero e.va. are inverses of T^{-1}'s and reciprocally, and that they share the same e.ve..
Furthermore $\lambda = 0$ is not an e.va. of T or T^{-1}, else $\exists v \neq 0$ s.t $Tv = 0$, meaning T would not be injective and hence not invertible.

22 Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors u and w in V such that

$$Tu = 3w \quad \text{and} \quad Tw = 3u.$$

Prove that 3 or -3 is an eigenvalue of T.

$Tu + Tw = 3w + 3u$	$Tu - Tw = 3w - 3u$
$\Rightarrow T(u+w) = 3(u+w)$	$\Rightarrow T(u-w) = -3(u-w)$
$\Rightarrow$ 3 is an e.va of T if $u+w \neq 0$	$\Rightarrow -3$ is an e.va. of T if $u-w \neq 0$

$u+w$ and $u-w$ cannot both be 0 since $u \neq 0$ and $w \neq 0$, hence 3 or -3 is e.va. of T.

23 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

λ e.va. of ST with e.ve $v \neq 0 \Rightarrow STv = \lambda v \ (1)$

$\Rightarrow TS(Tv) = \lambda Tv$

If $Tv \neq 0$, then λ is an e.va of TS with e.ve. Tv.

If $Tv = 0$, $(1) \Rightarrow STv = 0 \Rightarrow \lambda = 0$, 0 e.va. of ST.

We then have to show 0 is also an e.va. of TS.

Either $\exists w \in V$ s.t $Sw = v$, or S is not surjective, meaning S is not injective since V is finite dimensional. In the first case, it implies $\exists w \in V$ s.t $T\underbrace{Sw}_{=0} = 0$, hence 0 is an e.va. of TS. In the second case, it implies $\exists u, w$ s.t $Su = Sw$, $u \neq w$, so $S(u-w) = 0$, with $u - w \neq 0$. Then $(TS)(u-w) = T(S(u-w)) = 0$, implying 0 is an e.va. of TS.

24 Suppose A is an n-by-n matrix with entries in $\mathbf{F}$. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $Tx = Ax$, where elements of $\mathbf{F}^n$ are thought of as n-by-1 column vectors.

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.

ex15: λ e.va. of $T \iff \lambda$ e.va. of $T' \in \mathcal{L}(V')$

a) $T((1 \ldots 1)) = A(1 \ldots 1) = \left(\sum_{i=1}^{n} a_{1i} \ldots \sum_{i=1}^{n} a_{ni}\right) = (1 \ldots 1)$

$\Rightarrow 1$ e.va. of T with e.ve. $(1 \ldots 1)$

b) By definition $A = \mathcal{M}(T)$ for the basis of $\mathbf{F}^n$.
We have $\mathcal{M}(T') = \mathcal{M}(T)^T$, hence $T'\varphi = A^T\varphi$. The sum of entries in each row of A^T equals 1,

b) By definition $A = \mathcal{M}(T)$ for one basis of $\mathbb{F}^n$.
We have $\mathcal{M}(T') = \mathcal{M}(T)^T$, hence $T'\varphi = A^T\varphi$. The sum of entries in each row of A^T equals 1, hence using a), 1 is an e.va. of T'. By ex. 15, 1 is also an e.va. of T.

25 Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

Suppose $Tu = \lambda u$, $Tw = \mu w$, $T(u+w) = \varepsilon(u+w)$

$$\left.\begin{array}{l} T(u+w) = \varepsilon(u+w) \\ T(u+w) = Tu + Tw = \lambda u + \mu w \end{array}\right\} \Rightarrow \varepsilon u + \varepsilon w = \lambda u + \mu w \Rightarrow (\varepsilon - \lambda) u + (\varepsilon - \mu) w = 0$$

If u and w are e.ve. to different e.va., $\varepsilon - \lambda = 0$ and $\varepsilon - \mu = 0$, meaning $\lambda = \mu$: contradiction.
This implies u and w are e.ve. to the same e.va. (and $u+w$ too).

26 Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Assume T has two e.va. $\lambda, \mu \in \mathbb{F}$ s.t $\lambda \neq \mu$: $Tv = \lambda v$, $Tw = \mu w$. $v+w$ or $v-w$ is an e.ve., and $-w$ has the same e.va. as w. Therefore using previous exercise's result, v and w have the e.va.. This is true for any two e.ve. of T, so any non-zero vector of V, and $T0 = 0$. Thus $\forall v \in V, Tv = \lambda v, \lambda \in \mathbb{F} \Rightarrow T = \lambda I$

27 Suppose that V is finite-dimensional and $k \in \{1, \ldots, \dim V - 1\}$. Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V of dimension k is invariant under T. Prove that T is a scalar multiple of the identity operator.

28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has at most $1 + \dim \operatorname{range} T$ distinct eigenvalues.

Let $\lambda_1 \ldots \lambda_m$ distinct eigenvalues of T, with e.ve. $v_1 \ldots v_m$, $v_1 \ldots v_m$ linearly independent.
$Tv_k = \lambda_k v_k \Rightarrow T\lambda_k^{-1} v_k = v_k$ if $\lambda_k \neq 0$. If all e.va. are non-zero, then T has at most $\dim \operatorname{range} T$ e.va., as $v_1 \ldots v_m$ are linearly independent. If 0 is an e.va. of T, T has at most $\underbrace{\dim \operatorname{range} T}_{\text{non zero ones}} + \underbrace{1}_{\text{the 0 e.va.}}$.

29 Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ and -4, 5, and $\sqrt{7}$ are eigenvalues of T. Prove that there exists $x \in \mathbf{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

T has 3 e.va.: $-4, 5$ and $\sqrt{7}$, and 0 is not among them, so T is invertible. We can show this implies $T - 9I$ is also invertible. Let $v \in \operatorname{null} T - 9I$: $(T-9I)v = 0 \Rightarrow Tv = 9v \Rightarrow v = 0$ as 9 is not an e.va. of $T \Rightarrow \operatorname{null} T - 9I = \{0\} \Rightarrow T - 9I$ invertible $\Rightarrow \exists w \in V$ s.t $(T - 9I)w = (-4, 5, \sqrt{7})$
$\Rightarrow Tw - 9w = (-4, 5, \sqrt{7})$.

30 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T. Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Let λ e.va. of T, with e.ve. v: $Tv = \lambda v$

$$\begin{aligned}(T-2I)(T-3I)(T-4I)v = 0 &\Rightarrow (T-2I)v = 0 \text{ or } (T-3I)v = 0 \text{ or } (T-4I)v = 0 \\ &\Rightarrow Tv = 2v \text{ or } Tv = 3v \text{ or } Tv = 4v \\ &\Rightarrow \lambda = 2 \text{ or } \lambda = 3 \text{ or } \lambda = 4 \quad (\text{as } v \neq 0)\end{aligned}$$

31 Give an example of $T \in \mathcal{L}(\mathbf{R}^2)$ such that $T^4 = -I$.

We can define T s.t. it "applies" a rotation of $\frac{\pi}{4}$ degrees to vectors.

$$\begin{aligned} T(x_1, x_2) &= \left(x_1 \cos\frac{\pi}{4} - x_2 \sin\frac{\pi}{4},\ x_1 \sin\frac{\pi}{4} + x_2 \cos\frac{\pi}{4}\right) \\ &= \frac{\sqrt{2}}{2}(x_1 - x_2, x_1 + x_2) \end{aligned}$$

$T(x_1, x_2) =$

$= \frac{\sqrt{2}}{2}(x_1 - x_2, x_1 + x_2)$

$T^2(x_1, x_2) = \frac{1}{2}(x_1 - x_2 - (x_1 + x_2), x_1 - x_2 + x_1 + x_2) = \frac{1}{2}(-2x_2, 2x_1) = (-x_2, x_1)$ (rotation of $\frac{\pi}{2}$)

$T^3(x_1, x_2) = \frac{\sqrt{2}}{2}(-x_2 - x_1, -x_2 + x_1)$

$T^4(x_1, x_2) = \frac{1}{2}(-x_2 - x_1 - (-x_2 + x_1), -x_2 - x_1 + (-x_2 + x_1)) = \frac{1}{2}(-2x_1, -2x_2) = (-x_1, -x_2)$

$= -(x_1, x_2)$

32 Suppose $T \in \mathcal{L}(V)$ has no eigenvalues and $T^4 = I$. Prove that $T^2 = -I$.

T has no e.va. $\Rightarrow$ 0 is not an e.va. of T $\Rightarrow$ T is invertible.

$T^4 = I \Rightarrow T^{-2} \cdot T^4 = T^{-2} \cdot I \Rightarrow T^2 = T^{-2} \Rightarrow T = I$ or $T = -I$

I has 1 as an e.va., therefore $T = -I$.

33 Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

(a) Prove that T is injective if and only if T^m is injective.

(b) Prove that T is surjective if and only if T^m is surjective.

a) "$\Rightarrow$": Suppose T is injective. Let $v \in \operatorname{null} T^m$.

$T^m v = T(T^{m-1} v) = 0 \Rightarrow T^{m-1} v = 0$, as T is injective.

Repeating this procedure $m-1$ times yield $Tv = 0 \Rightarrow v = 0$.

"$\Leftarrow$": Suppose T^m is injective. Let $v \in \operatorname{null} T$.

$Tv = 0 \Rightarrow T^2 v = 0 \Rightarrow \dots \Rightarrow T^m v = 0 \Rightarrow v = 0$, as T^m injective.

b) T surjective $\Leftrightarrow$ T injective $\Leftrightarrow$ T^m injective $\Leftrightarrow$ T^m surjective

34 Suppose V is finite-dimensional and $v_1, \dots, v_m \in V$. Prove that the list $v_1, \dots, v_m$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \dots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

"$\Leftarrow$": It is immediate using result 5.11.

"$\Rightarrow$": Suppose $v_1 \dots v_m$ are linearly independent. This list can be completed to form $v_1 \dots v_n$ a basis of V.

Define $T \in \mathcal{L}(V)$ by: $Tv_i = \lambda_i v_i$, with $\lambda_1, \dots, \lambda_m = 1, \dots, m$

$\Rightarrow v_1 \dots v_m$ are eigenvectors of T corresponding to distinct e.va..

35 Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbf{R}$.

Hint: Let $V = \operatorname{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$, and define an operator $D \in \mathcal{L}(V)$ by $Df = f'$. Find eigenvalues and eigenvectors of D.

Let $V = \operatorname{span}(e^{\lambda_1 x} \dots e^{\lambda_m x})$. Let $D \in \mathcal{L}(V)$, $Df = f'$. $De^{\lambda_i} = \lambda_i e^{\lambda_i}$ (with $(e^{\lambda_i})(x) = e^{\lambda_i x}$)

Eigenvalues of D are any $\lambda \in \mathbb{R}$, with associated e.ve. e^{λ}. There exists an operator D s.t $e^{\lambda_1 x} \dots e^{\lambda_m x}$ are e.ve. of D corresponding to distinct e.va. (as $\lambda_1 \dots \lambda_m$ are distinct).

Therefore using previous exercise's result, $e^{\lambda_1 x} \dots e^{\lambda_m x}$ is linearly independent.

36 Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct positive numbers. Prove that the list $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ is linearly independent in the vector space of real-valued functions on $\mathbf{R}$.

Let $V = \operatorname{span}(\cos(\lambda_1 x) \dots \cos(\lambda_m x))$. Let $D^2 \in \mathcal{L}(V)$, $D^2 f = f''$. $D^2 \cos(\lambda_i) = -\lambda_i^2 \cos(\lambda_i)$

Eigenvalues of D are any $-\lambda^2 \in \mathbb{R}$ with associated e.ve. $\cos \lambda$. $\lambda_1 \dots \lambda_m$ is a list of distinct positive numbers, meaning $-\lambda_i^2$'s are distinct as well. There exists an operator s.t. $\cos \lambda_1 \dots \cos \lambda_m$ are e.ve. of D^2 corresponding to distinct e.va.. Therefore using ex 34's result, $\cos \lambda_1 \dots \cos \lambda_m$ is linearly independent.

37 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(S) = TS$$

for each $S \in \mathcal{L}(V)$. Prove that the set of eigenvalues of T equals the set of eigenvalues of $\mathcal{A}$.

"$\subseteq$": Let λ e.va. of T, with e.ve. v. Let $v, v_1 \dots v_{m-1}$ basis of V. (as $v \neq 0$ and V finite dim)

Let $S \in \mathcal{L}(V)$ s.t. $Sv = v$ and $Sv_i = 0 \ \forall i = 1 \dots m-1$.

"$\subseteq$": let λ e.va. of T, with e.ve. v. Let $v, v_1, \dots, v_{m-1}$ basis of V ...

let $S \in \mathcal{L}(V)$ s.t. $Sv = v$ and $Sv_i = 0 \ \forall i = 1 \dots m-1$.

let $w \in V$, $w = av + \sum_{i=1}^{m-1} a_i v_i$.

$(\mathcal{A}(S))(w) = (TS)(w) = Tav = aTv = a\lambda v = \lambda S av = \lambda S w$

$\Rightarrow \forall w \in V$, $\mathcal{A}(S) = \lambda S \Rightarrow \lambda$ e.va. of $\mathcal{A}$

"$\supseteq$": let λ e.va. of $\mathcal{A}$, with e.ve. $S \in \mathcal{L}(V)$: $\mathcal{A}(S) = \lambda S$, $S \neq 0$

$\Rightarrow \forall v \in V$, $(\mathcal{A}(S))(v) = (TS)v = \lambda S v$

$\Rightarrow T(Sv) = \lambda(Sv)$ (1)

Since $S \neq 0$, $\exists v \in V$ s.t. $Sv \neq 0$. Using this v in (1) yields that λ is an eigenvalue of T.

39 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.

"$\Leftarrow$": let U of dim $V-1$ invariant under T. Let $u_1, \dots, u_{m-1}$ basis of U, and $u_1, \dots, u_{m-1}, v$ basis of V.

- If $Tv \notin U$, then $Tv = \lambda v$ ($\lambda \neq 0$), and $v \neq 0$ since it is in a basis of V. Therefore T has an e.va..
- If $Tv \in U$:
  - If T is not surjective: then T is not injective, ie. $\exists w \in V$ s.t. $Tw = 0$, $w \neq 0$, meaning T has an e.va ($\lambda = 0$).
  - If T is surjective: suppose $Tv = u + \lambda v$, $u \in U$, $u \neq 0$. Since T is surjective, $\exists x \in U$ s.t. $Tx = -u + \lambda x$, since $-u + \lambda x \in U$.

$\Rightarrow T(v + x) = u + \lambda v - u + \lambda x = \lambda(v + x)$, with $v + x \neq 0$, (else $v = -x \Rightarrow v = x = 0$)
$\Rightarrow \lambda$ e.va. of T

"$\Rightarrow$": let λ e.va. of T, with e.ve. v.

Let $p(T) = T - \lambda I$. $p(T) \in \mathcal{L}(V) \Rightarrow \dim V = \dim \operatorname{null} p(T) + \dim \operatorname{range} p(T)$

If null $p(T) = 1$, then dim range $p(T) = \dim V - 1$, and range $p(T)$ is invariant under T (5.18). If dim null $p(T) = m > 1$, we can add $m-1$ linearly independent elements $w_1, \dots, w_{m-1}$ of null $p(T)$ to a basis of range $p(T)$ to form a subspace of dimension $\dim V - 1$ that will still be invariant under T as added vectors satisfy: $Tw = \lambda w$

40 Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbf{F})$ is a polynomial. Prove that

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

$p(STS^{-1}) = \sum_{i=0}^{m} a_i (STS^{-1})^i$

$(STS^{-1})^0 = I \quad (STS^{-1})^1 = STS^{-1}$

$(STS^{-1})^2 = (ST\underbrace{S^{-1})(S}_{I}TS^{-1}) = ST^2S^{-1}$

...

$(STS^{-1})^i = ST^iS^{-1}$

$= \sum_{i=0}^{m} a_i ST^iS^{-1}$

$= S\left(\sum_{i=0}^{m} a_i T^i\right)S^{-1}$

$= S\,p(T)\,S^{-1}$

41 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbf{F})$.

Let $u \in U$. $p(T)(u) = \left(\sum_{i=0}^{m} a_i T^i\right)(u) = \sum_{i=0}^{m} a_i T^i(u)$. $T^i(u) \in U$ for any i.

U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbf{F})$.

Let $v \in U$. $p(T)(v) = (\sum_{i=0}^{m} a_i T^i)(v) = \sum_{i=0}^{m} a_i T^i(v)$. $T^i(v) \in U$ for any i.

Thus $p(T)(v)$ is a linear combination of elements of U, therefore $p(T)(v) \in U$

$\Rightarrow U$ is invariant under $p(T)$.

42 Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$.

(a) Find all eigenvalues and eigenvectors of T.

(b) Find all subspaces of $\mathbf{F}^n$ that are invariant under T.

a) $(x_1, 2x_2, 3x_3, \dots, nx_n) = \lambda(x_1, \dots x_n) \Rightarrow k x_k = \lambda x_k$ for each $k = 1 \dots n$

$\Rightarrow \begin{cases} x_1 = \lambda x_1 \\ 2x_2 = \lambda x_2 \end{cases} \Rightarrow \begin{cases} \lambda = 1 \\ \lambda = 2 \end{cases} \Rightarrow T$ has no e.va..

b) T has no e.va. $\Rightarrow$ There are no invariant subspaces of dimension $n-1$.

Let U subspace of size $n-1$ of $\mathbb{F}^n$. T has no e.va. in U, therefore there are no invariant subspaces of dimension $n-2$ in U. This is true for any U, therefore there are no subspaces of dimension $n-2$ in V. Repeating this process, we conclude the only invariant subspaces of $\mathbb{F}^n$ under T are $\{0\}$ and $\mathbb{F}^n$.