3E Exercises

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1 Suppose T is a function from V to W. The *graph* of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Formally, a function T from V to W is a subset T of $V \times W$ such that for each $v \in V$, there exists exactly one element $(v, w) \in T$. In other words, formally a function is what is called above its graph. We do not usually think of functions in this formal manner. However, if we do become formal, then this exercise could be rephrased as follows: Prove that a function T from V to W is a linear map if and only if T is a subspace of $V \times W$.

"$\Rightarrow$": Suppose T is a linear map.

- $T0 = 0 \Rightarrow (0,0) \in$ graph of T
- Let $x = (v, Tv)$, $y = (w, Tw) \in$ graph of T

  $(v, Tv) + (w, Tw) = (v+w, Tv+Tw) = (v+w, T(v+w)) \in$ graph of T
- Let $\lambda \in \mathbb{F}$:

  $\lambda(v, Tv) = (\lambda v, \lambda Tv) = (\lambda v, T(\lambda v)) \in$ graph of T

$\Rightarrow$ graph of T subspace of $V \times W$

"$\Leftarrow$": Suppose graph of T is a subspace of $V \times W$

- Let $v, w \in V$

  $(v, Tv) + (w, Tw)$

  $= (v+w, Tv+Tw) \in V \times W$

  $\Rightarrow Tv + Tw = T(v+w)$
- Let $\lambda \in \mathbb{F}$

  $\lambda(v, Tv) \in V \times W$

  $= (\lambda v, \lambda Tv) \in V \times W$

$\Rightarrow \lambda Tv = T\lambda v$

2 Suppose that $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that V_k is finite-dimensional for each $k = 1, \ldots, m$.

By contraposition, suppose $\exists\, k \in \{1 \ldots m\}$ s.t. V_k is infinite dimensional. We can show $V_1 \times \ldots \times V_m$ is then infinite dimensional.

V_k infinite dimensional $\Rightarrow \exists\, v_1^k, v_2^k, \ldots$ s.t. $\forall m$ $v_1^k \ldots v_m^k$ is linearly independent.

Then we can construct a list of vectors of $V_1 \times \ldots \times V_m$ s.t. all elements are 0 except for the k^{th} element, which are $v_1^k, v_2^k, \ldots$. Any list of such vectors are linearly independent in $V_1 \times \ldots \times V_m$, therefore $V_1 \times \ldots \times V_m$ is infinite dimensional.

3 Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Those are indeed vector spaces, as $\mathcal{L}(V_1 \times \ldots \times V_m, W)$ is the linear maps between two vector spaces and $\mathcal{L}(V_1, W) \times \ldots \times \mathcal{L}(V_m, W)$ is a product of vector spaces.

We can define an isomorphism between these spaces to show they are isomorphic.

Let $E : \mathcal{L}(V_1 \times \ldots \times V_m, W) \longrightarrow \mathcal{L}(V_1, W) \times \ldots \times \mathcal{L}(V_m, W)$ s.t.

$\forall T \in \mathcal{L}(V_1 \times \ldots \times V_m, W)$, $ET = (T_1 \ldots T_m)$, with $T_i \in \mathcal{L}(V_i, W)$,

$T_i v = T 0_i^v$, where 0_i^v is all 0's except v at the i th position.

E is a linear map:

- Let $T, T' \in \mathcal{L}(V_1 \times \ldots \times V_m, W)$.

  $E(T + T') = ((T_1 + T_1') \ldots (T_m + T_m'))$

  $= (T_1 \ldots T_m) + (T_1' \ldots T_m')$ (T, T' linear maps)

  $= ET + ET'$
- Let $\lambda \in \mathbb{F}$. $\lambda ET = \lambda(T_1 \ldots T_m)$

  $= (\lambda T_1 \ldots \lambda T_m)$

  $= E \lambda T$

  $\Rightarrow \lambda ET = E\lambda T$

$= E\lambda T$

$\Rightarrow \lambda E T = E \lambda T$

let $T \in \operatorname{null} E$.

$ET = 0 \Rightarrow (T_1 \dots T_m) = (0 \dots 0)$

$\Rightarrow \begin{cases} T_1 = 0 \\ \vdots \\ T_m = 0 \end{cases}$ let $(v_1 \dots v_m) \in V_1 \times \dots \times V_m$ $\begin{cases} T_1 v_1 = 0 \Rightarrow T O_1^{v_1} = 0 \\ \vdots \\ T_m v_m = 0 \Rightarrow T O_m^{v_m} = 0 \end{cases}$

$\Rightarrow T O_1^{v_1} + \dots + T O_m^{v_m} = 0$

$\Rightarrow T(v_1 \dots v_m) = 0 \ \forall v_1 \dots v_m$

$\Rightarrow T = 0$

$\Rightarrow \operatorname{null} E = \{0\} \Rightarrow E$ is injective

let $T_1 \times \dots \times T_m \in \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$

Define $T \in \mathcal{L}(V_1 \times \dots \times V_m, W)$ by $Tv = T_1 v_1 + \dots + T_m v_m$

$ET = (T'_1 \dots T'_m)$, with $T'_i v = T O_i^v \ \forall v \in V_i$

$\Rightarrow T'_i v = T_i v \ \forall v \in V_i$

$\Rightarrow T'_i = T_i$

$\Rightarrow ET = (T_1 \dots T_m)$

$\Rightarrow E$ surjective

$\Rightarrow \mathcal{L}(V_1 \times \dots \times V_m, W)$ isomorphic to $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$

5 For m a positive integer, define V^m by

$$V^m = \underbrace{V \times \cdots \times V}_{m \text{ times}}.$$

Prove that V^m and $\mathcal{L}(\mathbb{F}^m, V)$ are isomorphic vector spaces.

let $E \in \mathcal{L}(V^m, \mathcal{L}(\mathbb{F}^m, V))$ s.t. $\forall v \in V^m$, $Ev = T_v$, with $T_v \in \mathcal{L}(\mathbb{F}^m, V)$, s.t $T_v(a_1 \dots a_m) = \sum_{i=1}^{m} a_i v_i$

E linear map:

- let $v, w \in V^m$, $(a_1 \dots a_m) \in \mathbb{F}^m$. $(E(v+w))(a_1 \dots a_m) = T_{v+w}(a_1 \dots a_m) = \sum_{i=1}^{m} a_i (v_i + w_i) = \sum_{i=1}^{m} a_i v_i + \sum_{i=1}^{m} a_i w_i$

  $= T_v(a_1 \dots a_m) + T_w(a_1 \dots a_m) = Ev(a_1 \dots a_m) + Ew(a_1 \dots a_m)$

  $\Rightarrow E(v+w) = Ev + Ew$

- let $\lambda \in \mathbb{F}$. $(E \lambda v)(a_1 \dots a_m) = \sum_{i=1}^{m} a_i \lambda v_i = \lambda \sum_{i=1}^{m} a_i v_i = \lambda (Ev) \Rightarrow E \lambda v = \lambda E v$

let $v = (v_1 \dots v_m) \in \operatorname{null} E$:

$Ev = 0 \Rightarrow T_v = 0 \Rightarrow \forall a_1 \dots a_m,\ T_v(a_1 \dots a_m) = 0$

$\Rightarrow \forall a_1 \dots a_m \ \sum_{i=1}^{m} a_i v_i = 0$

[illegible] $v = 0$

$$\Longrightarrow \forall a_1 \dots a_m \quad \sum_{i=1} a_i v_i = 0$$

In particular, for $(a_1 \dots a_m) = (1, 0 \dots 0) : v_1 = 0$

$(a_1 \dots a_m) = (0, 1, 0 \dots 0) : v_2 = 0$

$\vdots$

$(a_1 \dots a_m) = (0 \dots 0, 1) : v_m = 0$

$\Longrightarrow v = 0 \Longrightarrow \text{null } E = \{0\} \Longrightarrow E$ injective

Let $T \in \mathcal{L}(\mathbb{F}^m, V)$. Define $v_1 = T(1, 0 \dots 0), \dots, v_m = T(0 \dots 0, 1)$, and $v = (v_1 \dots v_m)$

$$(Ev)(a_1 \dots a_m) = T_v(a_1 \dots a_m) = \sum_{i=1}^{m} a_i v_i = \sum_{i=1}^{m} a_i T(0 \dots 0, 1, 0 \dots 0) = T(a_1 \dots a_m) \quad \forall (a_1 \dots a_m) \in \mathbb{F}^m$$

$\Longrightarrow E$ surjective

E isomorphism from V^m to $\mathcal{L}(\mathbb{F}^m, V)$, so these spaces are isomorphic.

6 Suppose that v, x are vectors in V and that U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

$v + U = x + W \Longrightarrow \exists w_0 \in W$ s.t. $v + 0 = x + w_0 \Longrightarrow v = x + w_0$ (U subspace so $0 \in U$)

"$\subseteq$": let $u \in U$. $v + U \subseteq x + W \Longrightarrow \exists w \in W$ s.t. $v + u = x + w \Longrightarrow (x + w_0) + u = x + w$

$\Longrightarrow u = x + w - x - w_0 = w - w_0 \in W$ (as W subspace of W)

$\Longrightarrow u \in W \Longrightarrow U \subseteq W$

The other inclusion can be proven in the same manner.

7 Let $U = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = 0\}$. Suppose $A \subseteq \mathbf{R}^3$. Prove that A is a translate of U if and only if there exists $c \in \mathbf{R}$ such that

$$A = \{(x, y, z) \in \mathbf{R}^3 : 2x + 3y + 5z = c\}.$$

We can show that $\forall a \in \mathbb{R}^3$, $\{a + u : u \in U\} = \{\overbrace{\{(x, y, z) \in \mathbb{R}^3 : 2x + 3y + 5z = c\}}^{A^c} : c \in \mathbb{R}\}$

"$\subseteq$": let $a = (x_a, y_a, z_a) \in \mathbb{R}^3$. Let $v \in \{a + u : u \in U\}$, s.t. $v = a + w$, $w = (x_w, y_w, z_w) \in U$

$v = (x_a + x_w, y_a + y_w, z_a + z_w)$

$2(x_a + x_w) + 3(y_a + y_w) + 5(z_a + z_w) = 0 \Longrightarrow 2x_w + 3y_w + 5z_w = c \Longrightarrow v \in A$

$\Longrightarrow \{a + U : a \in \mathbb{R}^3\} \subseteq \{A^c : c \in \mathbb{R}\}$

"$\supseteq$": let $c \in \mathbb{R}$, $A^c = \{(x, y, z) \in \mathbb{R}^3 : 2x + 3y + 5z = c\}$

$= \{(x, y, z) \in \mathbb{R}^3 : 2(x - \frac{c}{10}) + 3(y - \frac{c}{10}) + 5(z - \frac{c}{10}) = 0\}$

$= \{-\frac{c}{10}(1,1,1) + u : u \in U\} \subseteq \{a + U : a \in \mathbb{R}^3\}$

$\Longrightarrow \{a + U : a \in \mathbb{R}^3\} = \{A^c : c \in \mathbb{R}\}$

8 (a) Suppose $T \in \mathcal{L}(V, W)$ and $c \in W$. Prove that $\{x \in V : Tx = c\}$ is either the empty set or is a translate of null T.

(b) Explain why the set of solutions to a system of linear equations such as 3.27 is either the empty set or is a translate of some subspace of $\mathbf{F}^n$.

a) Let $c \notin \text{range } T$. In that case, $\{x \in V : Tx = c\} = \emptyset$

Let $c \in \text{range } T$. Then $\{x \in V : Tx = c\} \neq \emptyset$. Let $x^* \in \{x \in V : Tx = c\}$.

Let $v \in x^* + \text{null } T$. $\exists t \in \text{null } T$ s.t. $v = x^* + t \Longrightarrow Tv = Tx^* + Tt = c$

$\Longrightarrow v \in \{x \in V : Tx = c\} \Longrightarrow x^* + \text{null } T \subseteq \{x \in V : Tx = c\}$

Let $y \in \{x \in V : Tx = c\}$. $Ty = c = Tx^* \Longrightarrow T(y - x^*) = 0 \Longrightarrow y - x^* \in \text{null } T$

$\Longrightarrow y \in x^* + \text{null } T$

$\Longrightarrow \{x \in V : Tx = c\} = x^* + \text{null } T$ so it is a translate of null T.

b) A set of linear equations like 3.27 can be represented by a linear map $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$, with n the number of variables and m of equations. According to the previous question, the set of elements of $\mathbb{F}^n$ pointing via T to a given element of $\mathbb{F}^m$ is either empty (which means no solution to the linear equations) or a translate of null T, which is a subspace of $\mathbb{F}^m$

(which means no solution to the linear equations) or a translate of null T, which is a subspace of $\mathbb{F}^m$.

9 Prove that a nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1-\lambda)w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbf{F}$.

"$\Rightarrow$": Let $v, w \in A$, $\lambda \in \mathbb{F}$. Suppose $A = a + U$, $a \in V$, U subspace of V.

$v = a + u_v$, $u_v \in U \Rightarrow \lambda v = \lambda a + \lambda u_v$

$w = a + u_w$, $u_w \in U \Rightarrow (1-\lambda) w = (1-\lambda) a + (1-\lambda) u_w$

$\Rightarrow \lambda v + (1-\lambda) w = a + \underbrace{\lambda u_v + (1-\lambda) u_w}_{\in U \text{ (subspace)}}$

$\Rightarrow \lambda v + (1-\lambda) w \in A \ \forall v, w \in A, \lambda \in \mathbb{F}$.

"$\Leftarrow$": Suppose $\lambda v + (1-\lambda) w \in A \ \forall v, w \in A, \lambda \in \mathbb{F}$

Let $v \in A$. Define $U = -v + A$. We can show U is a subspace of V:

- $v \in A \Rightarrow -v + v = 0 \Rightarrow 0 \in U$
- Let $w_1, w_2 \in U$. $w_1 + w_2 = -v + a_1 - v + a_2$, $a_1, a_2 \in A$

$= -v + (a_1 - v + a_2)$

$a_1 - v + a_2 = \underbrace{2(\underbrace{\tfrac{1}{2} a_1 + \tfrac{1}{2} a_2}_{\in A\ (\lambda = \frac{1}{2})}) - v}_{\in A\ (\lambda = 2)} \in A \Rightarrow w_1 + w_2 \in -v + A$

- Let $\lambda \in \mathbb{F}$, $w \in U$. $\lambda w = -\lambda v + \lambda a$, $a \in A$

$= v - v - \lambda v + \lambda a$

$= -v + \underbrace{(1-\lambda) v + \lambda a}_{\in A} \in -v + A$

$\Rightarrow \lambda w \in -v + A$

$\Rightarrow U$ subspace of V

Thus A is a translation of a subspace of V

10 Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V. Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is the empty set.

Suppose $A_1 \cap A_2 \neq \emptyset$.

Let $x, y \in A_1 \cap A_2$: $x = v + u_1^x = w + u_2^x$, $y = v + u_1^y = w + u_2^y$, $u_1^x, u_1^y \in U_1$, $u_2^x, u_2^y \in U_2$

Let $\lambda \in \mathbb{F}$.

$\lambda x + (1-\lambda) y = \lambda (v + u_1^x) + (1-\lambda)(v + u_1^y) = v + (\underbrace{\lambda u_1^x + u_2^y - \lambda u_2^y}_{\in U}) \in v + U_1$

$\lambda x + (1-\lambda) y = \lambda (w + u_2^x) + (1-\lambda)(w + u_2^y) \in w + U_2$

$\Rightarrow \lambda x + (1-\lambda) y \in A_1 \cap A_2 \ \forall x, y \in A_1 \cap A_2, \lambda \in \mathbb{F}$

$\Rightarrow A_1 \cap A_2$ is a translate of some subspace of V (see previous exercise)

11 Suppose $U = \{(x_1, x_2, \dots) \in \mathbf{F}^\infty : x_k \neq 0$ for only finitely many $k\}$.

(a) Show that U is a subspace of $\mathbf{F}^\infty$.

(b) Prove that $\mathbf{F}^\infty / U$ is infinite-dimensional.

a)
- $(0, 0, \dots) \in U$ ($x_k = 0$ for all k)
- Let $x, y \in U$, $\bar{O}_x = \{k \in \mathbb{N} : x_k \neq 0\}$. By hypothesis $\bar{O}_x$ and $\bar{O}_y$ are finite.

$x_k + y_k \neq 0 \Rightarrow x_k \neq 0$ or $y_k \neq 0$

$\Rightarrow k \in \bar{O}_x \cup \bar{O}_y$

Union of two finite sets is finite, so $\bar{O}_{x+y}$ is finite and $x + y \in U$

- Let $x \in U$, $\lambda \in \mathbb{F}$

$\lambda x_k \neq 0 \Rightarrow x_k \neq 0 \Rightarrow k \in \bar{O}_x \Rightarrow \bar{O}_{\lambda x}$ is finite and $\lambda x \in U$

$\Rightarrow U$ is a subspace of $\mathbb{F}^\infty$

b) $\mathbb{F}^\infty / U = \{v + U : v \in \mathbb{F}^\infty\}$

We can show $\exists v_1, v_2 \dots \in \mathbb{F}^\infty$ s.t. $v_1 + U, \dots v_m + U$ is linearly independent $\forall m$.

$v_{ik} = 1$ if $k = 0 \bmod P(i)$ where $P(i)$ is the ith prime number.

[illegible] $\Rightarrow \sum^m \lambda_i v_i \in U$

$v_{ik} = 1$ if $k = 0 \mod P(i)$ where $P(i)$ is the ith prime number.

$$x = \sum_{i=1}^{m} \lambda_i (v_i + U) = \left(\sum_{i=1}^{m} \lambda_i v_i\right) + U = U \Rightarrow \sum_{i=1}^{m} \lambda_i v_i \in U$$

Suppose one of the λ_i is different from 0, say λ_{i^*}.

Then the i^*th element of x is equal to λ_{i^*}, and the $i^* \times i^*$ element, etc ...

so there is an infinite number of elements of x different than 0, and $x \notin U$.

This implies all λ_i are equal to 0, so these v_i's are linearly independent $\forall m$.

Thus $\mathbb{F}^\infty / U$ is infinite dimensional.

12 Suppose $v_1, \dots, v_m \in V$. Let

$$A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \lambda_1, \dots, \lambda_m \in \mathbf{F} \text{ and } \lambda_1 + \cdots + \lambda_m = 1\}.$$

(a) Prove that A is a translate of some subspace of V.
(b) Prove that if B is a translate of some subspace of V and $\{v_1, \dots, v_m\} \subseteq B$, then $A \subseteq B$.
(c) Prove that A is a translate of some subspace of V of dimension less than m.

a) let $v, w \in A$, $\lambda \in \mathbb{F}$.

$$\lambda v + (1-\lambda) w = \lambda \sum_{i=1}^{m} \lambda_i^v v_i + (1-\lambda) \sum_{i=1}^{m} \lambda_i^w v_i = \sum_{i=1}^{m} (\lambda \lambda_i^v + (1-\lambda)\lambda_i^w) v_i$$

$$\sum_{i=1}^{m} \lambda\lambda_i^v + (1-\lambda)\lambda_i^w = \lambda \underbrace{\sum_{i=1}^{m} \lambda_i^v}_{=1} + (1-\lambda) \underbrace{\sum_{i=1}^{m} \lambda_i^w}_{=1} = 1 \Rightarrow \lambda v + (1-\lambda) w \in A \Rightarrow$$ A translate of some subspace of V (3E ex. 9)

b) B is a translate of some subspace of V

$\Rightarrow \lambda_2 v_2 + (1-\lambda_2) v_1 \in B \;\; \forall \lambda \in \mathbb{F} \Rightarrow \lambda_3 v_3 + (1-\lambda_3)(\lambda_2 v_2 + (1-\lambda_2) v_1) \in B$

$\Rightarrow \lambda_4 v_4 + (1-\lambda_4)(\lambda_3 v_3 + (1-\lambda_3)(\lambda_2 v_2 + (1-\lambda_2) v_1)) \Rightarrow \dots$

$\Rightarrow \sum_{i=1}^{m} \lambda_i v_i \prod_{j=i+1}^{m} (1-\lambda_j) \in B$ (1)

Fix $\sum_{i=1}^{m} \lambda_i \prod_{j=i+1}^{m} (1-\lambda_j) = 1$ (special case of (1)).

c)

13 Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that V is isomorphic to $U \times (V/U)$.

V/U finite dimensional, $\dim V/U = p$

let $v_1 + U, \dots, v_p + U$ basis of V/U.

Suppose $v \in \operatorname{span}(v_1, \dots, v_p) \cap U$.

$\exists \lambda_1, \dots, \lambda_p \in \mathbb{F}$, $v = \sum_{i=1}^{p} \lambda_i v_i$ and $v + U = U$

$\Rightarrow \sum_{i=1}^{p} \lambda_i v_i + U = U \Rightarrow \sum_{i=1}^{p} \lambda_i (\overbrace{v_i + U}^{\text{basis}}) = U \Rightarrow \lambda_1 = \dots = \lambda_p = 0$

$\Rightarrow v = 0 \Rightarrow \operatorname{span}(v_1 \dots v_p) \cap U = \{0\}$ (1)

let $v \in V$. $\exists \lambda_1 \dots \lambda_p \in \mathbb{F}$ s.t. $v + U = \sum_{i=1}^{p} \lambda_i v_i + U$

$v \in v + U$, as $v = v + 0$, so $v \in \sum_{i=1}^{p} \lambda_i v_i + U$. Thus, $\exists u \in U$ s.t. $v = \sum_{i=1}^{p} \lambda_i v_i + u$

$\Rightarrow V = \operatorname{span}(v_1 \dots v_p) + U$ (2)

(1), (2) $\Rightarrow V = \operatorname{span}(v_1 \dots v_p) \oplus U$

$\Rightarrow \forall v \in V, \exists! s \in \operatorname{span}(v_1 \dots v_p), u \in U$ s.t. $v = s + u$

Define $T \in \mathcal{L}(V, U \times (V/U))$ s.t. $\forall v = s + u \in V$, $Tv = (u, s + U)$

T is a linear map:

· let $x = s^x + u^x$, $y = s^y + u^y \in V$

$T(x+y) = T(\underbrace{s^x + s^y}_{\in \operatorname{span}(v_1 \dots v_p)} + \underbrace{u^x + u^y}_{\in U}) = (u^x + u^y, s^x + s^y + U) = Tx + Ty$

· let $\lambda \in \mathbb{F}$.

$T(\lambda x) = T(\lambda s^x + \lambda u^x) = (\lambda u^x, \lambda s^x + U) = \lambda Tx$

T is injective:

let $x \in$ null T, $x = s_1$ $s \in U \cap \text{span}(v_1 \dots v_p) = \{0\}$

$Tx = 0 \Rightarrow (u, s+U) = (0, U) \Rightarrow \begin{cases} u = 0 \\ s+U = U \end{cases} \Rightarrow \begin{cases} u = 0 \\ s \in U \end{cases} \Rightarrow \begin{cases} u = 0 \\ s = 0 \end{cases} \Rightarrow x = 0$

T is surjective:

let $(u, v+U) \in U \times (V/U)$, with $v = s+u$. $v+U = s+u+U = s+U$

let $w = s+u$.

Then $Tw = (u, s+U) = (u, v+U)$

$\Rightarrow$ T is an isomorphism between V and $U \times (V/U)$, thus they are isomorphic.

14 Suppose U and W are subspaces of V and $V = U \oplus W$. Suppose $w_1, \dots, w_m$ is a basis of W. Prove that $w_1 + U, \dots, w_m + U$ is a basis of V/U.

let $\lambda_1 \dots \lambda_m \in \mathbb{F}$.

$\sum_{i=1}^m \lambda_i (w_i + U) = (\sum_{i=1}^m \lambda_i w_i) + U = U \Rightarrow \sum_{i=1}^m \lambda_i w_i \in U \Rightarrow \sum_{i=1}^m \lambda_i w_i \in \overbrace{U \cap W}^{=\{0\}} \Rightarrow \sum_{i=1}^m \lambda_i w_i = 0$

$\Rightarrow \lambda_1 = \dots = \lambda_m = 0$, as $w_1 \dots w_m$ basis of W

$\Rightarrow w_1 + U \dots w_m + U$ linearly independent

let $v \in V$. $v = u + w$, $u \in U$, $w = \sum_{i=1}^m \lambda_i w_i \in W$

$v + U = (u+w) + U = w + U = (\sum_{i=1}^m \lambda_i w_i) + U = \sum_{i=1}^m \lambda_i (w_i + U)$

$\Rightarrow w_1 + U \dots w_m + U$ spans V/U

$\Rightarrow w_1 \dots w_m$ basis of V/U

15 Suppose U is a subspace of V and $v_1 + U, \dots, v_m + U$ is a basis of V/U and $u_1, \dots, u_n$ is a basis of U. Prove that $v_1, \dots, v_m, u_1, \dots, u_n$ is a basis of V.

We proved in ex 13 that $V = \text{span}(v_1 \dots v_m) \oplus U \Rightarrow \dim V = m+n$, and every element of V can be expressed as a sum of linear combinations of v_i's and u_i's, therefore $v_1 \dots v_m, u_1 \dots u_n$ basis of V.

16 Suppose $\varphi \in \mathcal{L}(V, \mathbf{F})$ and $\varphi \neq 0$. Prove that $\dim V/(\text{null } \varphi) = 1$.

$\dim V = \dim \text{range } \varphi + \dim \text{null } \varphi$

$\quad = 1 + \dim \text{null } \varphi$ (as $\varphi \neq 0$)

$\Rightarrow \dim \text{null } \varphi = \dim V - 1$

$\dim V/\text{null } \varphi = \dim V - \dim \text{null } \varphi$

$\quad = \dim V - (\dim V - 1)$

$\quad = 1$

17 Suppose U is a subspace of V such that $\dim V/U = 1$. Prove that there exists $\varphi \in \mathcal{L}(V, \mathbf{F})$ such that null $\varphi = U$.

let $v + U \in V/U$, $v \neq 0$. $v + U$ is a basis of V/U.

We showed in ex 13: $V = \text{span}(v) \oplus U$

Define $\varphi \in \mathcal{L}(V, \mathbb{F})$ s.t. $\forall w = \lambda v + u \in W$, $\lambda \in \mathbb{F}$, $u \in U$

$\varphi w = \lambda v$

φ is a linear map: $\varphi(x+y) = (\lambda^x + \lambda^y) v = \lambda^x v + \lambda^y v = \varphi x + \varphi y$

$\varphi(\lambda x) = \lambda \lambda^x v = \lambda \varphi x$

let $x \in$ null φ, $x = \lambda v + u$

$\varphi x = \lambda v = 0 \Rightarrow \lambda = 0 \Rightarrow x = u \in U \Rightarrow$ null $\varphi \subseteq U$

let $u \in U$.

$\varphi u = 0 \Rightarrow u \in$ null φ

$\Rightarrow$ null $\varphi = U$

18 Suppose that U is a subspace of V such that V/U is finite-dimensional.
(a) Show that if W is a finite-dimensional subspace of V and $V = U + W$, then $\dim W \geq \dim V/U$.
(b) Prove that there exists a finite-dimensional subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

$U + W = \{u + w \mid u \in U, w \in W\}$

a) let $w_1 \dots w_p$ basis of W.

let $v + U \in \dim V/U$, $v \in V$

$V = U + W \Rightarrow \exists u \in U, w \in W$ s.t. $v = u + w$

a) Let $w_1, \dots, w_p$ basis of W.

Let $v+U \in V/U$, $v \in V$

$V = U + W \Rightarrow \exists u \in U, w \in W$ s.t. $v = u + w$

$\Rightarrow v + U = u + w + U = w + U$

$\Rightarrow \exists \lambda_1, \dots, \lambda_p \in \mathbb{F}$ s.t. $v + U = \sum_{i=1}^{p} \lambda_i w_i + U$

$\Rightarrow w_1, \dots, w_p$ span $V/U \Rightarrow \dim W \geq \dim V/U$

b) We showed in ex 13 that $\operatorname{span}(w_1, \dots, w_p) \oplus U = V$, with $w_1, \dots, w_p$ basis of W.

19 Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq \operatorname{null} T$.

"$\Rightarrow$": Let $u \in U$. $Tu = (S \circ \pi)u = SU = 0 \Rightarrow u \in \operatorname{null} T \Rightarrow U \subseteq \operatorname{null} T$

"$\Leftarrow$": Suppose $U \subseteq \operatorname{null} T$.

Define $S \in \mathcal{L}(V/U, W)$, by $S(v+U) = Tv \ \forall v \in V$.

This definition is only appropriate if all representation of $v+U$ map to the same element of W.

Suppose $\exists v, w \in V$ s.t. $v + U = w + U$. Then $v - w \in U$, and $v - w \in \operatorname{null} T$ (since $U \subseteq \operatorname{null} T$).

$\Rightarrow T(v-w) = 0 \Rightarrow Tv = Tw. \Rightarrow S(v+U) = S(w+U) = Tv = Tw$

It is easy to check S is a linear map and $T = S \circ \pi$.