7B Exercises

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1 Prove that a normal operator on a complex inner product space is self-adjoint if and only if all its eigenvalues are real.

This exercise strengthens the analogy (for normal operators) between self-adjoint operators and real numbers.

let T a normal operator on V, $\mathbb{F} = \mathbb{C}$.

"$\Rightarrow$": Holds for any self-adjoint operator.

"$\Leftarrow$": Suppose all e.va. of T are real.

Then by the complex spectral theorem, there is an orth. basis of V s.t. $\mathcal{M}(T)$ is a diagonal matrix, where the diagonal holds T's e.va.. Since they are real, $\mathcal{M}(T) = \mathcal{M}(T^*)$, and thus $T = T^*$.

2 Suppose $\mathbf{F} = \mathbf{C}$. Suppose $T \in \mathcal{L}(V)$ is normal and has only one eigenvalue. Prove that T is a scalar multiple of the identity operator.

By the complex spectral theorem, there is an orth. basis of V of e.ve. of T, $e_1 \dots e_n$.

For all e_i's, we have $Te_i = \lambda e_i$, λ being the only e.va. of T. Hence $T = \lambda I$

3 Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$ is normal. Prove that the set of eigenvalues of T is contained in $\{0, 1\}$ if and only if there is a subspace U of V such that $T = P_U$.

"$\Rightarrow$": Suppose e.va. of $T \in \{0,1\}$. let $e_1 \dots e_n$ the orth. basis obtained by the spectral theorem s.t. all are e.ve. of T, and $Te_i = \begin{cases} e_i & \text{if } i \leq k \\ 0 & \text{if } i > k \end{cases}$, where $k \in [\![0 \dots n]\!]$.

Then one can see $T = P_{\operatorname{span}\{e_1 \dots e_k\}}$ (or $T = 0$ if $k = 0$)

"$\Leftarrow$": Suppose there is a subspace of U s.t. $T = P_U$.

$e_1 \dots e_n$ orth. basis of e.ve. of T (complex spectral theorem)

$Te_i = \lambda_i e_i = P_U e_i$. If $e_i \in U$, then $\lambda_i = 1$, otw $\lambda_i = 0$

Thus all e.va. of T are either 0 or 1.

4 Prove that a normal operator on a complex inner product space is skew (meaning it equals the negative of its adjoint) if and only if all its eigenvalues are purely imaginary (meaning that they have real part equal to 0).

let T normal operator on V, $\mathbb{F} = \mathbb{C}$

$e_1 \dots e_n$ orth. basis of e.ve. of T (complex spectral theorem) with associated e.va $\lambda_1 \dots \lambda_n$.

$$T^* = -T \iff Te_i = -T^* e_i \ \forall i \in \{1 \dots n\}$$
$$\iff \lambda_i e_i = -\bar{\lambda}_i e_i \quad (\text{using } 7.21, e))$$
$$\iff \lambda_i = -\bar{\lambda}_i \left(\iff a + bi = -(a - bi) = -a + bi \Rightarrow \begin{cases} a = 0 \\ b \in \mathbb{R} \end{cases}\right)$$
$$\iff \lambda_i \in \mathbb{C} \setminus \mathbb{R}$$

5 Prove or give a counterexample: If $T \in \mathcal{L}(\mathbf{C}^3)$ is a diagonalizable operator, then T is normal (with respect to the usual inner product).

$(1,0,0), (0,1,0), (0,1,1)$ not orth. basis of $\mathbb{C}^3$

5 Prove or give a counterexample: If $T \in \mathcal{L}(\mathbf{C}^3)$ is a diagonalizable operator, then T is normal (with respect to the usual inner product).

$\underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(0,1,1)}$: not orth. basis of $\mathbb{C}^3$

$Tv_1 = v_1$, $Tv_2 = 0$, $Tv_3 = v_3$: T diagonalizable in $(1,0,0),(0,1,0),(0,1,1)$

$$\langle Tv, w\rangle = \langle a_1v_1 + a_3v_3, b_1v_1 + b_2v_2 + b_3v_3\rangle$$
$$= a_1\overline{b_1} + a_3(\overline{b_2} + \overline{b_3})$$
$$= \langle a_1v_1 + a_2v_2 + a_3v_3, b_1T^*v_1 + b_2T^*v_2 + b_3T^*v_3\rangle$$
$$= a_1\overline{b_1}\langle v_1, \underbrace{T^*v_1}_{=v_1}\rangle + a_2\overline{b_2}\langle v_2, \underbrace{T^*v_2}_{(0,0,1)}\rangle + a_2\overline{b_3}\langle v_2, \underbrace{T^*v_3}_{(0,0,1)}\rangle + a_3\overline{b_2}\langle v_3, \underbrace{T^*v_2}\rangle + a_3\overline{b_3}\langle v_3, T^*v_3\rangle$$
$$= \langle v, b_1v_1 + (b_2+b_3)(0,0,1)\rangle \qquad (0,0,1) = v_3 - v_2$$
$$T^*v_1 = v_1, \; T^*v_2 = v_3 - v_2, \; T^*v_3 = v_3 - v_2$$

$TT^*v_2 = T(v_3 - v_2) = v_3$, $T^*Tv_2 = 0 \implies T$ not normal

Tv
$= T^9v$

6 Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

Let $v \in V$. Since T is normal, from 7A ex. 27 : $\exists u \in V$ s.t. $T^9v = Tu$
$\implies T^8v = Tu \implies T^9v = T^2u \implies \dots \implies T^9v = T^9u \implies v - u \in \operatorname{null} T^9$
$\implies v - u \in \operatorname{null} T$ (7A, ex. 27) $\implies Tv = Tu = T^9v$. Thus $T = T^9 = T^8 = (T^8)T = T^2$.

Using spectral theorem, $M(T)$ diagonal in some orth. basis. Then $M(T^*) = M(T)^*$. This implies all e.va. of T (which are on $M(T)$'s diagonal) are equal to their complex conjugate, meaning they are real.

$$\langle Tv, v\rangle = \langle \sum_{i=1}^m z_i \overbrace{Te_i}^{\lambda e_i}, \sum_{i=1}^m z_i e_i\rangle = \sum_{i=1}^m \sum_{j=1}^m z_i\overline{z_j}\lambda_i\langle e_i, e_j\rangle = \sum_{i=1}^m z_i\overline{z_i}\lambda_i = \sum_{i=1}^m \underbrace{|z_i|^2}_{\in\mathbb{R}}\underbrace{\lambda_i}_{\in\mathbb{R}} \in \mathbb{R}$$

$\implies \langle Tv, v\rangle \in \mathbb{R} \; \forall v \in V \implies T$ self-adjoint by 7.14

7 Give an example of an operator T on a complex vector space such that $T^9 = T^8$ but $T^2 \neq T$.

Consider the complex vector space $\mathbb{C}^2$, and $T \in \mathcal{L}(\mathbb{C}^2)$ s.t. $T(1,0) = (0,0)$, $T(0,1) = (1,0)$.
We have $T^9 = T^8 = T^2 = 0$, but $T \neq 0$.

8 Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if every eigenvector of T is also an eigenvector of T^*.

"$\Rightarrow$": Proven in 7.21 e).
"$\Leftarrow$": Let $e_1 \dots e_n$ e.ve. of T that form a basis of V in which $M(T)$ is diagonal.
$e_1 \dots e_n$ are also e.ve. of T^*, so $M(T^*)$ is also diagonal.
Thus $M(T)M(T^*) = M(T^*)M(T)$ (both diagonal), and $TT^* = T^*T$.

9 Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if there exists a polynomial $p \in \mathcal{P}(\mathbf{C})$ such that $T^* = p(T)$.

"$\Leftarrow$": $TT^* = Tp(T) = p(T)T = T^*T$

"$\Rightarrow$": Suppose T normal. By complex spectral theorem, $\exists$ basis of V s.t. $M(T)$ diagonal in an orth. basis of V. $M(T^*) = M(T)^*$ (orth basis), so $M(T^*)$ is also diagonal in this basis. Let a_i, b_i the elements of $M(T), M(T^*)$ on the diagonal, $i \in \{1 \dots n\}$. We then have n "data points" (a_i, b_i) and we can find by interpolation a polynomial of max degree $n-1$ that transforms any a_i into b_i counterpart.

10 Suppose V is a complex inner product space. Prove that every normal operator on V has a square root.

*An operator $S \in \mathcal{L}(V)$ is called a **square root** of $T \in \mathcal{L}(V)$ if $S^2 = T$. We will discuss more about square roots of operators in Sections 7C and 8C.*

$S^2 = T \qquad T^* = p(T)$

Consider base given by spectral theorem for T $M(T)$ diagonal. Let $S \in \mathcal{L}(V)$ s.t. $M(S) = \begin{pmatrix} \sqrt{\lambda_1} & & (0) \\ & \ddots & \\ (0) & & \sqrt{\lambda_m} \end{pmatrix}$, where $\sqrt{\cdot}$ exists as $\mathbb{F} = \mathbb{C}$ (not necessarily unique solution but one exists for all $z \in \mathbb{C}$.

11 Prove that every self-adjoint operator on V has a cube root.

*An operator $S \in \mathcal{L}(V)$ is called a **cube root** of $T \in \mathcal{L}(V)$ if $S^3 = T$.*

Self-adjoint $\Rightarrow$ normal, so in all cases a spectral theorem can be applied.

Same reasoning as before. Here works if $\mathbb{F} = \mathbb{R}$ as $\sqrt[3]{\cdot}$ is a bijection on $\mathbb{R}$.

12 Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is normal. Prove that if S is an operator on V that commutes with T, then S commutes with T^*.

The result in this exercise is called Fuglede's theorem.

$ST = TS \Rightarrow ST^* = T^*S$?

From ex. 9, $\exists p \in \mathcal{P}(\mathbb{C})$ s.t. $T^* = p(T)$.

Hence $ST^* = Sp(T) = p(T)S = T^*S$

13 Without using the complex spectral theorem, use the version of Schur's theorem that applies to two commuting operators (take $\mathcal{E} = \{T, T^*\}$ in Exercise 20 in Section 6B) to give a different proof that if $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$ is normal, then T has a diagonal matrix with respect to some orthonormal basis of V.

$ST = TS \ \forall S, T \in \mathcal{E} \Rightarrow \exists$ basis s.t. S upper tri $\forall S \in \mathcal{E}$

Let T normal operator on V. Hence T, T^* commute.

Let $\mathcal{E} = \{T, T^*\}$. By Schur's theorem, $\exists$ orth. basis of V s.t. $M(T), M(T^*)$ are upper triangular.

We also have $M(T^*) = M(T)^*$. This implies $M(T)$ (and $M(T^*)$) diagonal matrices, and hence all vectors of the basis are e.ve. of T (and T^*).

14 Suppose $\mathbf{F} = \mathbf{R}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, where $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T.

We can construct a basis of V s.t. $M(T)$ is diagonal with its e.va. on the diagonal and all elements of the basis are e.ve. of T, hence by the spectral theorem T is self-adjoint.

Same for "only if".

15 Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that T is normal if and only if all pairs of eigenvectors corresponding to distinct eigenvalues of T are orthogonal and $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, where $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T.

Same reasoning.

16 Suppose $\mathbf{F} = \mathbf{C}$ and $\mathcal{E} \subseteq \mathcal{L}(V)$. Prove that there is an orthonormal basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if S and T are commuting normal operators for all $S, T \in \mathcal{E}$.

This exercise extends the complex spectral theorem to the context of a collection of commuting normal operators.

let $\mathcal{E}^* = \mathcal{E} \cup \{T^* \mid \forall T \in \mathcal{E}\}$. All elements still commute (see ex. 12).

$$[M(S)M(T)]_{ij} = [M(T)M(S)]_{ij}$$

$$\sum_{k=1}^{m} a_{ik} b_{kj} = \sum_{k=1}^{m} b_{ik} a_{kj}$$

$$a_{ii} b_{ij} = a_{jj} b_{ij}$$

18 Give an example of a real inner product space V, an operator $T \in \mathcal{L}(V)$, and real numbers b, c with $b^2 < 4c$ such that

$$T^2 + bT + cI$$

is not invertible.

This exercise shows that the hypothesis that T is self-adjoint cannot be deleted in 7.26, even for real vector spaces.

$$\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} -b & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} -b-1+c & 0 \\ 0 & c \end{pmatrix}$$

Set $b = 0, c = 1$, we obtain $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow 0$ is an eva. so it is not invertible.

19 Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T.

(a) Prove that $U^\perp$ is invariant under T.
(b) Prove that $T|_U \in \mathcal{L}(U)$ is self-adjoint.
(c) Prove that $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

$T = T^*$ $\quad Tu \in U$

a) let $u_\perp \in U^\perp$, $u \in U$.

$$\langle Tu_\perp, u\rangle = \langle u_\perp, T^* u\rangle \overset{(T=T^*)}{=} \langle u_\perp, Tu\rangle \overset{(Tu\in U)}{=} 0 \Rightarrow Tu_\perp \in U^\perp \Rightarrow U^\perp \text{ invariant under } T$$

b) let $u, w \in U$.

$$\langle T|_U u, w\rangle = \langle Tu, w\rangle = \langle u, T^* w\rangle = \langle u, Tw\rangle = \langle u, T|_U w\rangle \Rightarrow T|_U^* = T|_U \Rightarrow T|_U \text{ self-adjoint}$$

c) let $u, w \in U^\perp$

$$\langle T|_{U^\perp} u, w\rangle = \langle Tu, w\rangle = \langle u, T^* w\rangle = \langle u, Tw\rangle = \langle u, T|_{U^\perp} w\rangle \overset{\text{By a)}}{\Rightarrow} T|_{U^\perp} \text{ self-adjoint}$$

20 Suppose $T \in \mathcal{L}(V)$ is normal and U is a subspace of V that is invariant under T.

(a) Prove that $U^\perp$ is invariant under T.
(b) Prove that U is invariant under T^*.
(c) Prove that $(T|_U)^* = (T^*)|_U$.
(d) Prove that $T|_U \in \mathcal{L}(U)$ and $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ are normal operators.

This exercise can be used to give yet another proof of the complex spectral theorem (use induction on $\dim V$ and the result that T has an eigenvector).

$U \oplus U^\perp = V$

a) $U \oplus U^\perp = V$. let $e_1 \dots e_m, f_1 \dots f_k$ orthonormal basis of V with $e_1 \dots e_m$ basis of U, $f_1 \dots f_k$ of $U^\perp$.

$$M(T) = \begin{array}{c|cc} & e_1 \dots e_m & f_1 \dots f_k \\ \hline e_1 \dots e_m & A & B \\ f_1 \dots f_k & 0 & C \end{array} \qquad M(T^*) = M(T)^* = \begin{array}{c|cc} & e_1 \dots e_m & f_1 \dots f_k \\ \hline e_1 \dots e_m & A^* & 0 \\ f_1 \dots f_k & \overline{B} & C^* \end{array}$$

$$\begin{matrix} \vdots \\ f_k \end{matrix}\begin{pmatrix} 0 & C \end{pmatrix} \qquad \begin{matrix} f_1 \\ \vdots \\ f_k \end{matrix}\begin{pmatrix} \overline{B} & C^* \end{pmatrix}$$

T normal $\Rightarrow \|Tv\|^2 = \|T^*v\|^2 \ \forall v \in V$

$\Rightarrow \sum_{i=1}^{m} \|Te_i\|^2 = \sum_{i=1}^{m} \|T^*e_i\|^2$

$\Rightarrow \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ji}|^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ji}|^2 + \sum_{j=1}^{k} |b_{ji}|^2$

$\Rightarrow B = 0$

$\Rightarrow U^\perp$ invariant under T

b) let $u \in U$, $u^\perp \in U^\perp$.

$\langle T^*u, u^\perp \rangle = \langle u, Tu^\perp \rangle = 0$, since $U^\perp$ invariant under T (see a)).

Hence $T^*u \in U$, and thus U invariant under T^*.

$(T|_U)^*$, $(T^*)|_U$

c) let $u, v \in U$.

$\langle (T|_U)^* u, v \rangle = \langle u, T|_U v \rangle = \langle u, Tv \rangle = \langle T^* u, v \rangle = \langle (T^*)|_U u, v \rangle$

$\Rightarrow (T|_U)^* = (T^*)|_U$

d) $T|_U (T|_U)^* u = T|_U (T^*)|_U = T|_U T^* u$

$= TT^*u$ (since U invariant under T^* by b))

$= T^*Tu$

$= (T^*)|_U T|_U u$

$= T|_U)^* T|_U u$

21 Suppose that T is a self-adjoint operator on a finite-dimensional inner product space and that 2 and 3 are the only eigenvalues of T. Prove that

$$T^2 - 5T + 6I = 0.$$

Apply spectral theorem so that $M(T)$ diagonal.

$M(T^2) - 5M(T) + 6I = \begin{pmatrix} 4 & & & \\ & 4 & & \\ & & 9 & \\ & & & 9 \end{pmatrix} - 5\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 3 & \\ & & & 3 \end{pmatrix} + 6\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

$= \Big(-6$ . . . $\Big(6$ . . . $\Big(\quad 0$

$$= \begin{pmatrix} -6 & & & \\ & -6 & & \\ & & -6 & \\ & & & -6 \end{pmatrix} + \begin{pmatrix} 6 & \\ & 6 \end{pmatrix} = 0$$

$\bullet \ \| \sum_{i=1}^{m} a_i e_i \|^2 = \sum_{i=1}^{m} |a_i|^2$ ($e_1 \dots e_m$ orth.)

23 Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbf{F}$, and $\epsilon > 0$. Suppose there exists $v \in V$ such that $\|v\| = 1$ and

$$\|Tv - \lambda v\| < \epsilon.$$

Prove that T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

This exercise shows that for a self-adjoint operator, a number that is close to satisfying an equation that would make it an eigenvalue is close to an eigenvalue.

T self-adjoint $\Rightarrow$ V has an orth. basis of e.ve. of T

$Tv = \sum a_i T e_i$

$$\| \sum_{i=1}^{m} a_i T e_i - \lambda \sum_{i=1}^{m} a_i e_i \|^2 = \| \sum_{i=1}^{m} a_i \lambda_i e_i - \lambda \sum_{i=1}^{m} a_i e_i \|^2 = \| \sum_{i=1}^{m} a_i (\lambda_i - \lambda) e_i \|^2$$

$$= \sum_{i=1}^{m} |a_i(\lambda_i - \lambda)|^2 < \varepsilon^2$$

$$= \sum_{i=1}^{m} |a_i|^2 |\lambda - \lambda_i|^2 < \varepsilon^2 \quad (1)$$

Suppose $|\lambda - \lambda_i| \geq \varepsilon \ \forall \lambda'$: $\sum_{i=1}^{m} |a_i|^2 |\lambda - \lambda_i|^2 \geq \varepsilon^2 \overbrace{\sum_{i=1}^{m} |a_i|^2}^{\|v\|=1} = \varepsilon^2 < \varepsilon^2$ by (1), which is a contradiction.

Hence $\exists \lambda_j$ e.va. of T s.t. $|\lambda - \lambda_i| < \varepsilon$.

24 Suppose U is a finite-dimensional vector space and $T \in \mathcal{L}(U)$.

(a) Suppose $\mathbf{F} = \mathbf{R}$. Prove that T is diagonalizable if and only if there is a basis of U such that the matrix of T with respect to this basis equals its transpose.

(b) Suppose $\mathbf{F} = \mathbf{C}$. Prove that T is diagonalizable if and only if there is a basis of U such that the matrix of T with respect to this basis commutes with its conjugate transpose.

This exercise adds another equivalence to the list of conditions equivalent to diagonalizability in 5.55.

$M(T)$ $\quad M(T)$ $\quad Tv_1$

$M(T^*) = M(T)^* = M(T)$

$M(T)^* = M(T)^T$ ($\mathbb{F} = \mathbb{R}$)

a) "$\Rightarrow$": T diagonalizable $\Rightarrow$ $\exists$ basis s.t. $M(T)$ diagonal. Diagonal matrices equal their transpose.

"$\Leftarrow$": Suppose $\exists v_1 \dots v_n$ basis of $\underset{\sim}{V}$ s.t. $M(T) = M(T)^T$. Here $\mathbb{F} = \mathbb{R}$ hence $M(T)^T = M(T)^* = M(T^*)$

$$\Rightarrow \begin{cases} \langle Te_i, e_j \rangle = \langle \sum_{k=1} a_{ki} e_k, e_j \rangle = a_{ji} \\ \qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel \\ \langle e_i, Te_j \rangle = \langle e_i, \sum_{k=1}^{n} a_{kj} e_j \rangle = a_{ij} \end{cases} \Rightarrow T = T^* \overset{\text{Spectral thm}}{\Rightarrow} T \text{ diagonalizable}$$

b) "$\Rightarrow$": T diagonalizable $\Rightarrow$ $M(T)M(T)^* = M(T)^* M(T)$ $\Rightarrow$ $M(T)$ commutes with its conjugate transpose

"$\Leftarrow$": $M(T)M(T)^* = M(T)^* M(T) \Rightarrow TT^* = T^*T \overset{\text{c. spectral thm}}{\Rightarrow} T$ diagonalizable

25 Suppose that $T \in \mathcal{L}(V)$ and there is an orthonormal basis $e_1, \dots, e_n$ of V consisting of eigenvectors of T, with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$. Show that if $k \in \{1, \dots, n\}$, then the pseudoinverse $T^\dagger$ satisfies the equation

$$T^\dagger e_k = \begin{cases} \frac{1}{\lambda_k} e_k & \text{if } \lambda_k \neq 0, \\ 0 & \text{if } \lambda_k = 0. \end{cases}$$

$T^\dagger e_k = (T|_{(\text{null}\, T)^\perp})^{-1} P_{\text{range}\, T}\, e_k$

$= \left(T|_{(\mathrm{null}\, T)^\perp}\right)^{-1} e_k$

$= (T|_V)^{-1} e_k$ There is a basis of e.ve. of $T \Rightarrow \mathrm{null}\, T = 0 \Rightarrow (\mathrm{null}\, T)^\perp = V$

$= T^{-1} e_k$ $Te_k = \lambda_k e_k \Rightarrow e_k = \lambda_k T^{-1} e_k$. If $\lambda_k = 0$ then $e_k = 0$

$= \begin{cases} \lambda_k^{-1} e_k & \text{if } \lambda_k \neq 0 \\ 0 & \text{if } \lambda_k = 0 \end{cases}$ and $T^{-1} e_k = T^{-1} 0 = 0$. Otw, $\lambda_k^{-1} e_k = T^{-1} e_k$