3A Exercises

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1 Suppose $b, c \in \mathbf{R}$. Define $T: \mathbf{R}^3 \to \mathbf{R}^2$ by

$$T(x,y,z) = (2x - 4y + 3z + b, 6x + cxyz).$$

Show that T is linear if and only if $b = c = 0$.

"$\Rightarrow$": T linear map $\Rightarrow T(0) = 0$

$T(0) = (b, 0) = 0 \Rightarrow b = 0$

T linear map $\Rightarrow T((1,1,1)+(1,0,0)) = T(1,1,1) + T(1,0,0)$

$T(2,1,1) = (3, 12+2c) = (1, 6+c) + (2, 6) = T(1,1,1) + T(1,0,0)$

$\Rightarrow (3, 12+2c) = (3, 12+c)$

$\Rightarrow c = 0$

"$\Leftarrow$": We can verify additivity and homogeneity with $b = c = 0$

- Additivity: let $(x,y,z), (x',y',z') \in \mathbb{R}^3$

$T((x,y,z)+(x',y',z'))$

$= T(x+x', y+y', z+z')$
$= (2(x+x') - 4(y+y') + 3(z+z'), 6(x+x'))$

$= (2x - 4y + 3z, 6x) + (2x' - 4y' + 3z', 6x')$

$= T(x,y,z) + T(x',y',z')$

- Homogeneity: let $\lambda \in \mathbb{F}$, $(x,y,z) \in \mathbb{R}^3$

$T(\lambda(x,y,z)) = (2\lambda x - 4\lambda y + 3\lambda z, 6\lambda x)$

$= \lambda(2x - 4y + 3z, 6x)$

$= \lambda T(x,y,z)$

$\Rightarrow T$ linear map

3 Suppose that $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbf{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbf{F}^n$.

This exercise shows that the linear map T has the form promised in the second to last item of Example 3.3.

$T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. let $(x_1 \ldots x_m) \in \mathbb{F}^n$.

$$T(x_1 \dots x_m) = T(x_1(1,0\dots 0) + \dots + x_m(0\dots 0,1))$$
$$= x_1 T(1,0\dots 0) + \dots + x_m T(0\dots 0,1)$$
$$= x_1(A_{1,1}, \dots, A_{m,1}) + \dots + x_m(A_{1,m} \dots A_{m,m}) \quad (\text{for } A_{ij} \in \mathbb{F})$$
$$= (A_{11}x_1 + \dots + A_{1m}x_m, \dots, A_{m1}x_1 + \dots + A_{mm}x_m)$$

4 Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_m$ is a list of vectors in V such that $Tv_1, \dots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \dots, v_m$ is linearly independent.

Let $a_1 \dots a_m \in \mathbb{F}$

$\sum_{i=1}^{m} a_i v_i = 0 \Rightarrow T(\sum_{i=1}^{m} a_i v_i) = T(0)$

Using additivity, homogeneity and $T(0) = 0$:

$\sum_{i=1}^{m} a_i T v_i = 0 \Rightarrow a_1 = \dots = a_m = 0$ as $Tv_1 \dots Tv_m$ linearly ind.

$\Rightarrow v_1 \dots v_m$ linearly independent

6 Prove that multiplication of linear maps has the associative, identity, and distributive properties asserted in 3.8.

- Associativity comes from the associativity of functions composition
- let $T: U \to V$, and I_U identity on U, I_V id on V

  $\forall u,\ T I_U u = Tu = I_V T u$
- let $S_1, S_2 : U \to V$, $T: V \to W$, $v \in V$

  $((S_1 + S_2)T)(v) = (S_1 + S_2)(Tv)$

  (addition in $\mathcal{L}(U,V)$) $= S_1 T v + S_2 T v = (S_1 T)(v) + (S_2 T)(v)$

  Same reasoning for $S(T_1 + T_2)$.

7 Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbf{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Using Ex. 3's result on a linear map T from a one dimensional vector space V onto itself:

Using Ex. 7 S seen as a linear map T from a one dimensional vector space V onto itself:

$\exists \lambda \in \mathbb{F}$, s.t $Tx = \lambda x \ \forall x \in V$

8 Give an example of a function $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$ such that

$$\varphi(av) = a\varphi(v)$$

for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but φ is not linear.

This exercise and the next exercise show that neither homogeneity nor additivity alone is enough to imply that a function is a linear map.

$\varphi(x, y) = x + y + 1$

$\varphi(a(x,y)) = \varphi(ax, ay) = ax + ay + a$

$a\varphi(x,y) = a(x + y + 1) = ax + ay + a$

However, $\varphi(0,0) = 1 \neq 0$, hence φ is not linear.

9 Give an example of a function $\varphi\colon \mathbf{C} \to \mathbf{C}$ such that

$$\varphi(w + z) = \varphi(w) + \varphi(z)$$

for all $w, z \in \mathbf{C}$ but φ is not linear. (Here $\mathbf{C}$ is thought of as a complex vector space.)

There also exists a function $\varphi\colon \mathbf{R} \to \mathbf{R}$ such that φ satisfies the additivity condition above but φ is not linear. However, showing the existence of such a function involves considerably more advanced tools.

$\varphi(a + bi) = b$

$\varphi((a+bi) + (c+di)) = \varphi((a+c) + (b+d)i) = b + d$

$\varphi(a+bi) + \varphi(c+di) = b + d$

However, $\varphi(i(a+bi)) = \varphi(ai - b) = a$

and $i\varphi(a+bi) = bi$ $\neq$

φ is not homogeneous in the complex vector space $\mathbb{C}$, so it is not linear.

10 Prove or give a counterexample: If $q \in \mathcal{P}(\mathbf{R})$ and $T\colon \mathcal{P}(\mathbf{R}) \to \mathcal{P}(\mathbf{R})$ is defined by $Tp = q \circ p$, then T is a linear map.

The function T defined here differs from the function T defined in the last bullet point of 3.3 by the order of the functions in the compositions.

If $q(X) = 1$, then $(q \circ p)(X) = 1 \ \forall X \in \mathbb{R}$

If $q(x)=1$, then $(q\circ p)(x)=1 \ \forall x \in \mathbb{R}$

In particular $(q\circ p)(0)=1\neq 0$, hence T is not a linear map.

12 Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U,W)$ and $S \neq 0$ (which means that $Su \neq 0$ for some $u \in U$). Define $T\colon V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U, \\ 0 & \text{if } v \in V \text{ and } v \notin U. \end{cases}$$

Prove that T is not a linear map on V.

let $u \in U$ s.t. $Su \neq 0$, and $\bar{u} \in V \setminus U$

$T(u+\bar{u}) = 0$, as $u+\bar{u} \notin U$

However, $Tu + \underbrace{T\bar{u}}_{0} = Su \neq 0$

$\Rightarrow T(u+\bar{u}) \neq Tu + T\bar{u}$, T is not a linear map

13 Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U,W)$, then there exists $T \in \mathcal{L}(V,W)$ such that $Tu = Su$ for all $u \in U$.

The result in this exercise is used in the proof of 3.125.

let $u_1, \dots, u_m$ be a basis of U.

This basis can be extended to a basis of V by adding some vectors $w_{m+1} \dots w_n$, as it is a linearly independent list.

let $v \in V$. $v = \sum_{i=1}^{m} a_i u_i + \sum_{i=m+1}^{n} a_i w_i$

Given $S \in \mathcal{L}(U,W)$, define T as: $Tv = \sum_{i=1}^{m} a_i S u_i$

let $u = \sum^{m} a_i u_i + \sum^{m} a_i w_i$ and $v = \sum^{m} b_i u_i + \sum^{m} b_i w_i$

Let $v = \sum_{i=1}^{m} a_i v_i + \sum_{i=m+1}^{n} a_i w_i$ and $u = \sum_{i=1}^{m} b_i v_i + \sum_{i=m+1}^{n} b_i w_i$

$$T(v+u) = T\left(\sum_{i=1}^{m} (a_i+b_i) v_i + \sum_{i=m+1}^{n} (a_i+b_i) w_i\right)$$

$$= \sum_{i=1}^{m} (a_i+b_i) S v_i$$

$$= \sum_{i=1}^{m} a_i S v_i + \sum_{i=1}^{m} b_i S v_i$$

$$= Tv + Tu$$

Homogeneity of T is easy to show

14 Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Let $v_1 \ldots v_m$ be a base of V.

W infinite dimensional $\Rightarrow \exists w_1, w_2, \ldots \in W$ s.t $\forall k$ integer, $w_1 \ldots w_k$ is linearly independent.

Let $(T_k)_{k \in \mathbb{N}}$ the sequence of elements of $\mathcal{L}(V,W)$ s.t :

$$T_k v_i = w_{km+i} \quad \forall i = 1 \ldots m$$

We only need to show that $\forall m$ integer, $T_1 \ldots T_m$ is linearly independent to show $\mathcal{L}(V,W)$ is infinite dimensional.

Let $a_1 \ldots a_m \in \mathbb{F}$ s.t :

$$\sum_{i=1}^{m} a_i T_i = 0$$

$$\sum_{i=1}^{l} a_i T_i = 0$$

$$\Rightarrow \forall v \in V, \sum_{i=1}^{m} a_i T_i v = 0$$

In particular, for $v = v_1$:

$$\sum_{i=1}^{m} a_i T_i v_1 = \sum_{i=1}^{m} a_i w_{im+1} = 0$$

We defined $w_1, w_2, \dots$ to be a linearly independent list of vectors.

We can thus conclude $a_i = 0 \; \forall i = 1 \dots m$, $T_1 \dots T_m$ is linearly independent.

This implies $L(V, W)$ is infinite dimensional.

15 Suppose $v_1, \dots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \dots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \dots, m$.

We have to prove:

$v_1 \dots v_m$ linearly dependent

$$\Rightarrow \exists w_1 \dots w_m \in W, \nexists T \in L(V, W) \text{ s.t. } T v_k = w_k \; \forall k = 1 \dots m$$

The contraposition is:

$$\forall w_1 \dots w_m, \exists T \in L(V, W) \text{ s.t. } T v_k = w_k \; \forall k \quad (1)$$

$$\Rightarrow v_1 \dots v_m \text{ linearly independent}$$

Let $a_1 \dots a_m \in \mathbb{F}$.

$$\sum_{i=1}^{m} a_i v_i = 0$$

We can apply (1) with $w_1 \neq 0$ (this assumes $W \neq \{0\}$), $w_i = 0 \; \forall i \geq 2$

We then have:

$$\sum_{i=1}^{m} a_i v_i = 0 \Rightarrow \sum_{i=1}^{m} a_i T v_i = T(0)$$

$$\sum_{i=1}^{} a_i v_i = 0 \Rightarrow \sum_{i=1}^{} a_i T v_i = T(0)$$

$$\Rightarrow a_1 w_1 = 0$$

$$\Rightarrow a_1 = 0, \text{ as } w_1 \neq 0$$

We can repeat this process with T_i s.t $Tv_i = w_i \neq 0$, $Tv_j = 0$ $\forall j \neq i$

This leads to $a_i = 0$ $\forall i$

$\Rightarrow v_1 \dots v_m$ linearly independent.

As stated at the begining, taking the contraposition back gives the result.

16 Suppose V is finite-dimensional with $\dim V > 1$. Prove that there exist $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

let $v_1 \dots v_m$ be a basis of V ($\dim V = m > 1$)

We can define $S \in \mathcal{L}(V)$ s.t. $Sv_1 = v_2$, and $Sv_2 = v_1$, and $Sv_i = v_i$ $\forall i \neq 1,2$

and $T \in \mathcal{L}(V)$ s.t. $Tv_1 = v_1 + v_2$, $Tv_i = v_i$ $\forall i \neq 1$

$$STv_1 = S(v_1 + v_2) = v_2 + v_1$$

$$TSv_1 = Tv_2 = v_2 \neq STv_1$$

$$\Rightarrow TS \neq ST$$