3C Exercises

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1 Suppose $T \in \mathcal{L}(V, W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least dim range T nonzero entries.

Let $v_1 \dots v_n$, $w_1 \dots w_m$ bases of V, W respectively.

$$T = \begin{array}{c} v_k \\ \begin{pmatrix} A_{1,k} \\ \vdots \\ A_{m,k} \end{pmatrix} \end{array} \begin{array}{c} w_1 \\ \vdots \\ w_m \end{array} \qquad Tv_k = A_{1,k}w_1 + \dots + A_{m,k}w_m$$

Assume T has z non-zero entries, such that $z <$ dim range T.

Let c the number of columns that have non-zero entries. $c \leq z <$ dim range T

Suppose the non-zero columns are the first c columns (WLOG: vectors in the basis of V can be permutated). Then $Tv_{c+1} = \dots = Tv_n = 0 \Rightarrow$ dim null $T \geq n - c$

We have: dim $V = n =$ dim null T + dim range T

But dim null T + dim range $T > (n-c) + c = n$; contradiction.

Thus T has at least dim range T non-zero entries.

2 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that dim range $T = 1$ if and only if there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

"$\Leftarrow$": $\exists$ basis $v_1 \dots v_n$, $w_1 \dots w_m$ of V, W s.t. $\mathcal{M}(T)_{i,j} = 1 \ \forall i,j$

Let $v \in V$, $v = \sum_{i=1}^{n} a_i v_i$, $a_i \in \mathbb{F} \ \forall i$

$Tv = \sum_{i=1} a_i Tv_i = \sum_{i=1}^{n} a_i \sum_{j=1}^{m} w_j = 0 \Rightarrow a_1 = -\sum_{i=2}^{n} a_i \Rightarrow$ dim null $T = n-1$

$\Rightarrow$ dim range $T = 1$ (as $n =$ dim null T + dim range T)

"$\Rightarrow$": dim range $T = 1$

Let $w_1 \dots w_m$ a basis of W and $v_\Sigma \in V$ s.t. $Tv_\Sigma = \sum_{i=1}^{m} w_i$, $v_\Sigma \neq 0$. (should prove these exist...)

We can extend v_Σ to $v_\Sigma, v_1, \dots v_{n-1}$ to form a basis of V.

dim range T = dim span $\mathcal{M}(T)_{\cdot,1} \dots \mathcal{M}(T)_{\cdot,n} = 1 \Rightarrow \forall i \geq 2, \exists \lambda_i \in \mathbb{F}$ s.t. $\mathcal{M}(T)_{\cdot,i} = \lambda_i \mathcal{M}(T)_{\cdot,1}$

From $\mathcal{M}(T) = \begin{array}{c} \begin{array}{cccc} v_\Sigma & v_1 & \dots & v_{n-1} \end{array} \\ \begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_{n-1} \\ \vdots & \vdots & & \\ 1 & \lambda_1 & \dots & \lambda_{n-1} \end{pmatrix} \end{array} \begin{array}{c} w_1 \\ \vdots \\ w_m \end{array}$ we can construct: $\begin{array}{c} \begin{array}{cccc} v_\Sigma & \frac{v_1}{\lambda_1} & \dots & \frac{v_{n-1}}{\lambda_{n-1}} \end{array} \\ \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \\ 1 & 1 & \dots & 1 \end{pmatrix} \end{array} \begin{array}{c} w_1 \\ \vdots \\ w_m \end{array}$ with a change of base where $v_i \to \frac{v_i}{\lambda_i} \ \forall i$

$\Rightarrow \exists$ basis of V and W s.t. $\mathcal{M}(T) = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \\ 1 & \dots & 1 \end{pmatrix}$

4 Suppose that $D \in \mathcal{L}(\mathcal{P}_3(\mathbf{R}), \mathcal{P}_2(\mathbf{R}))$ is the differentiation map defined by $Dp = p'$. Find a basis of $\mathcal{P}_3(\mathbf{R})$ and a basis of $\mathcal{P}_2(\mathbf{R})$ such that the matrix of D with respect to these bases is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Compare with Example 3.33. The next exercise generalizes this exercise.

Basis of $P_3(\mathbb{R})$: $x^3, x^2, x, 1$

Basis of $P_2(\mathbb{R})$: $\frac{x^2}{3}, \frac{x}{2}, 1$

5 Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exist a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k, column k, equal 1 if $1 \leq k \leq$ dim range T.

Let $v_1 \dots v_n$ basis of V, $w_1 \dots w_m$ basis of W

$Tv_i = \sum_{j=1}^{m} a_{ij} w_j$. We can define $\tilde{w}_i = \sum_{j=1}^{m} a_{ij} w_j$, so $Tv_i = \tilde{w}_i$.

$\tilde{w}_1 \dots \tilde{w}_n$ spans dim range T. We can reduce $\tilde{w}_1 \dots \tilde{w}_n$ until it is linearly independent, s.t. $\tilde{w}_1 \dots \tilde{w}_r$ is a basis of range T, $r =$ dim range T. We can extend it until it spans W with vectors $\bar{w}_{r+1} \dots \bar{w}_m$ (possibly none, if T surjective).

We thus have a basis of W: $\tilde{w}_1 \dots \tilde{w}_r, \bar{w}_{r+1} \dots \bar{w}_m$ with $\tilde{w}_i = Tv_i$ for $i \leq$ dim range T

of range T, $r = \dim \operatorname{range} T$). We can extend it until it spans W with vectors $\overline{w}_{r+1} \dots \overline{w}_m$ (possibly none, if T injective).

We thus have a basis of W: $\tilde{w}_1 \dots \tilde{w}_r, \overline{w}_{r+1} \dots \overline{w}_m$ with $\tilde{w}_i = Tv_i$ for $i = 1 \dots \dim \operatorname{range} T$.

With this basis we have the following matrix:

$$\mathcal{M}(T) = \begin{array}{c} \begin{array}{cc} v_1 \cdots v_r & v_{r+1} \cdots v_m \end{array} \\ \left(\begin{array}{cc} \begin{matrix} 1 & (0) \\ (0) & 1 \end{matrix} & (0) \\ (0) & (0) \end{array}\right) \end{array} \begin{array}{c} \tilde{w}_1 \\ \vdots \\ \tilde{w}_r \\ \overline{w}_{r+1} \\ \vdots \\ \overline{w}_m \end{array}$$

$\mathcal{M}(T)_{ii} = 1$ for $i = 1 \dots \dim \operatorname{range} T$, 0 everywhere else

6 Suppose $v_1, \dots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \dots, w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to the bases $v_1, \dots, v_m$ and $w_1, \dots, w_n$] are 0 except for possibly a 1 in the first row, first column.

In this exercise, unlike Exercise 5, you are given the basis of V instead of being able to choose a basis of V.

Corollary from previous exercise (where we gave ourselves a basis of V).

First column is either $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ if vectors of basis are permutated.

10 Give an example of 2-by-2 matrices A and B such that $AB \neq BA$.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

12 Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Explain why $A(BC)$ makes sense and prove that

$$(AB)C = A(BC).$$

Try to find a clean proof that illustrates the following quote from Emil Artin: "It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

Dimensions making sense and associativity come from considering the matrices as linear maps ($A = \mathcal{M}(T_A)$, with T_A a linear map, etc) and the product of linear maps is associative.

13 Suppose A is an n-by-n matrix and $1 \leq j, k \leq n$. Show that the entry in row j, column k, of A^3 (which is defined to mean AAA) is

$$\sum_{p=1}^{n} \sum_{r=1}^{n} A_{j,p} A_{p,r} A_{r,k}.$$

$$(AAA)_{jk} = \sum_{r=1}^{n} (AA)_{jr} A_{rk} = \sum_{p=1}^{n} \sum_{r=1}^{n} A_{jp} A_{pr} A_{rk}$$

14 Suppose m and n are positive integers. Prove that the function $A \mapsto A^t$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

Let $A, B \in \mathbb{F}^{m,n}$

$[(A+B)^T]_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = A^T_{ij} + B^T_{ij}$ (additivity)

$(\lambda A)^T_{ij} = (\lambda A)_{ji} = \lambda A_{ji} = \lambda A^T_{ij}$ (homogeneity)

15 Prove that if A is an m-by-n matrix and C is an n-by-p matrix, then

$$(AC)^t = C^t A^t.$$

This exercise shows that the transpose of the product of two matrices is the product of the transposes in the opposite order.

$$(AC)^T_{ij} = (AC)_{ji} = \sum_{k=1}^{n} A_{jk} C_{ki} = \sum_{k=1}^{n} C^T_{ik} A^T_{kj} = C^T A^T_{ij}$$

16 Suppose A is an m-by-n matrix with $A \neq 0$. Prove that the rank of A is 1 if and only if there exist $(c_1, \dots, c_m) \in \mathbf{F}^m$ and $(d_1, \dots, d_n) \in \mathbf{F}^n$ such that $A_{j,k} = c_j d_k$ for every $j = 1, \dots, m$ and every $k = 1, \dots, n$.

$\dim \operatorname{range} T = 1 \iff \exists$ basis of U, V s.t. $\mathcal{M}(T) = 1$

"$\Rightarrow$": $\operatorname{rank} A = 1 \Rightarrow \exists C$ m-by-1 matrix and R 1-by-n matrix s.t. $A = CR$

Then we can see that $A_{jk} = c_j d_k$ $\forall j, k$.

"$\Leftarrow$": Suppose $\exists (c_1 \dots c_m) \in \mathbb{F}^m, (d_1 \dots d_n) \in \mathbb{F}^n$ s.t. $A_{jk} = c_j d_k$.

$(c_1 \dots c_m) \neq 0, (d_1 \dots d_n) \neq 0$ as $A \neq 0$. Suppose $c_1 \neq 0, d_1 \neq 0$.

$A = c\, d^T \Rightarrow \operatorname{rank} A = \operatorname{rank}(c\, d^T) = \dim \operatorname{range} T_{c d^T}$.

$(c_1 \dots c_m) \neq 0$, $(d_1 \dots d_n) \neq 0$ as $A \neq 0$. Suppose $c_1 \neq 0$, $d_1 \neq 0$.

$A = c\, d^T \Rightarrow \operatorname{rank} A = \operatorname{rank}(c\, d^T) = \dim \operatorname{range} T_c T_{d^T}$,

where $M(T_c) = c$, $M(T_{d^T}) = d^T$, $T_{d^T} \in \mathcal{L}(V,W)$, $T_c \in \mathcal{L}(W,X)$,

with $\dim V = n$, $\dim W = 1$, $\dim X = m$

Let $v_1 \dots v_n$ basis of V, w_1 basis of W wrt which $M(T_{d^T})$ is defined.

$T_{d^T} v_1 = d_1 w_1 \neq 0$, as $d_1 \neq 0 \Rightarrow \dim \operatorname{range} T_{d^T} = 1$ (as $\dim W = 1$ and $\dim \operatorname{range} T_{d^T} \neq 0$) (1)

Let $x_1 \dots x_m$ basis of X wrt which $M(T_c)$ is defined.

$T_c w_1 = \sum_{i=1}^{m} c_i x_i \neq 0$ as $c_1 \neq 0$. Also, $\dim W = \dim \operatorname{null} T_c + \dim \operatorname{range} T_c$

$\dim W = 1$, and $\dim \operatorname{range} T_c \neq 0$. Then $\dim \operatorname{range} T_c = 1$ (2)

$\overset{(1),(2)}{\Longrightarrow} \dim \operatorname{range} T_c T_{d^T} \leq \min(\dim \operatorname{range} T_c, \dim \operatorname{range} T_{d^T}) = 1$ (*)

$T_c T_{d^T} v_1 = d_1 T_c w_1 = d_1 \sum_{i=1}^{m} c_i x_i \neq 0$ as $d_1 \neq 0, c_1 \neq 0$

$\Longrightarrow \dim \operatorname{range} T_c T_{d^T} \geq 1$ (**)

$\overset{(*),(**)}{\Longrightarrow} \dim \operatorname{range} T_c T_{d^T} = 1$

$\Longrightarrow \operatorname{rank} A = 1$