6B Exercises

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1 Suppose $e_1, \dots, e_m$ is a list of vectors in V such that
$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$
for all $a_1, \dots, a_m \in \mathbf{F}$. Show that $e_1, \dots, e_m$ is an orthonormal list.

This exercise provides a converse to 6.24.

Let $i = 1..m$. Take $a_k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otw} \end{cases}$: $\|\sum_{k=1}^{m} a_k e_k\| = \|e_i\| = \sum_{i=1}^{n} |a_i|^2 = |a_i|^2 = 1$

$\Rightarrow \|e_i\| = 1 \ \forall i = 1..m$ (1)

Let $i, j = 1..m$ s.t $i \neq j$. Take $a_k = \begin{cases} 1 & \text{if } k = i \text{ or } k = j \\ 0 & \text{otw} \end{cases}$

$\Rightarrow \|e_i + e_j\|^2 = \underbrace{\|e_i\|^2}_{1} + \underbrace{\|e_j\|^2}_{1} + \langle e_i, e_j\rangle = |1|^2 + |1|^2 = 2$

$\Rightarrow \langle e_i, e_j\rangle = 0$ (2)

(1), (2) $\Rightarrow e_1 \dots e_m$ orthonormal

2 (a) Suppose $\theta \in \mathbf{R}$. Show that both
$$(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta) \quad \text{and} \quad (\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$$
are orthonormal bases of $\mathbf{R}^2$.

(b) Show that each orthonormal basis of $\mathbf{R}^2$ is of the form given by one of the two possibilities in (a).

a) $\langle (\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\rangle = -\cos\theta\sin\theta + \sin\theta\cos\theta = 0$

$\|(\cos\theta, \sin\theta)\|^2 = \cos^2\theta + \sin^2\theta = 1 \Rightarrow \|(\cos\theta, \sin\theta)\| = 1$ (same with $(-\sin\theta, \cos\theta)$)

Same reasoning for $(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$.

b) Let $(x_1, x_2), (y_1, y_2)$ an orthonormal basis of $\mathbb{R}^2$. We can show it has form $(x_1, x_2), (-x_2, x_1)$ or $(x_1, x_2), (x_2, -x_1)$

$$\begin{cases} \langle (x_1, x_2), (y_1, y_2)\rangle = 0 \Rightarrow x_1 y_1 + x_2 y_2 = 0 \\ \|(x_1, x_2)\|^2 = 1 \Rightarrow x_1^2 + x_2^2 = 1 \\ \|(y_1, y_2)\|^2 = 1 \Rightarrow y_1^2 + y_2^2 = 1 \end{cases}$$

$\overset{(y_1 \neq 0)}{\Rightarrow} x_1 = -\dfrac{x_2 y_2}{y_1} = \pm\dfrac{\sqrt{1-x_1^2}\, y_2}{\sqrt{1-y_2^2}} \overset{(x_1 \neq 1)}{\Rightarrow} \dfrac{x_1}{\sqrt{1-x_1^2}} = \pm\dfrac{y_2}{\sqrt{1-y_2^2}}$ (if $x_1 = 1$, then $x_2 = 0$, $y_1 = 0$, $y_2 = 1$, satisfies both forms)

if $y_1 = 0$

$$\left(\frac{1}{\sqrt{1-x^2}}\right)' = -\frac{\left(\sqrt{1-x^2}\right)'}{1-x^2} = -\frac{\left(\frac{1}{2\sqrt{1-x^2}}\right)}{1-x^2} = -\frac{1-x^2}{2\sqrt{1-x^2}}$$
$$= -\frac{\sqrt{1-x^2}}{2} < 0 \ \forall x \in]0,1]$$

$\Rightarrow \dfrac{1}{\sqrt{1-x^2}}$ strictly decreasing ($\Rightarrow$ bijection)

$\Rightarrow \boxed{x_1 = \pm y_2}$

$x_1 = -y_2 \overset{(y_2 \neq 0)}{\Rightarrow} -y_1 + x_2 = 0 \Rightarrow y_1 = x_2$ (if $y_2 = 0$, then $y_1 = 1$, $x_1 = 0$ and $x_2 = 1$, satisfies both forms)

$x_1 = y_2 \overset{(y_2 \neq 0)}{\Rightarrow} y_1 = -x_2$

3 Suppose $e_1, \dots, e_m$ is an orthonormal list in V and $v \in V$. Prove that
$$\|v\|^2 = |\langle v, e_1\rangle|^2 + \cdots + |\langle v, e_m\rangle|^2 \iff v \in \operatorname{span}(e_1, \dots, e_m).$$

Let $e_1 \dots e_m, e_{m+1} \dots e_n$ an orthonormal basis of V.

$$\|v\|^2 = \sum_{i=1}^{m} |\langle v, e_i\rangle|^2, \text{ using } (6.30b)$$

ith component is $\langle v, e_i\rangle$ in the basis $e_1 \ldots e_n$

$$= \sum_{i=1}^{m} |\langle v, e_i\rangle|^2 \iff \sum_{i=m+1}^{n} |\langle v, e_i\rangle|^2 = 0 \iff \langle v, e_i\rangle = 0 \ \forall i = m+1 \ldots n \iff v \in \operatorname{span}(e_1 \ldots e_m)$$

5 Suppose $f\colon [-\pi,\pi] \to \mathbf{R}$ is continuous. For each nonnegative integer k, define

$$a_k = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx \quad \text{and} \quad b_k = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx.$$

Prove that

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2 + b_k^2) \le \int_{-\pi}^{\pi} f^2.$$

The inequality above is actually an equality for all continuous functions $f\colon [-\pi,\pi] \to \mathbf{R}$. However, proving that this inequality is an equality involves Fourier series techniques beyond the scope of this book.

From ex 4, $\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}} \ldots \frac{\sin nx}{\sqrt{\pi}}$ orthonormal in $C[-\pi,\pi]$ with inner prod $\int_{-\pi}^{\pi} fg$.

Call these vectors e_i, $i = 1 \ldots m$, $m = 2n+1$.

Using Bessel inequality, we have:

$$\sum_{i=1}^{m} |\langle f, e_i\rangle|^2 \le \|f\|^2 \Rightarrow |\langle f, \tfrac{1}{\sqrt{2\pi}}\rangle|^2 + \sum_{i=1}^{n} |\langle f, \tfrac{\cos ix}{\sqrt{\pi}}\rangle|^2 + \sum_{i=1} |\langle f, \tfrac{\sin ix}{\sqrt{\pi}}|^2 \le \int_{-\pi}^{\pi} f^2$$

$$\left[\begin{array}{l} |\langle f, \frac{1}{\sqrt{2\pi}}\rangle|^2 = \frac{1}{2\pi}\langle f, 1\rangle^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx^2 = \frac{1}{2}a_0^2 \\ \text{For any } j > 1: \ |\langle f, \frac{\cos jx}{\sqrt{\pi}}\rangle|^2 = \left(\int_{-\pi}^{\pi} f(x)\frac{\cos(jx)}{\sqrt{\pi}}\right)^2 = \left(\frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi} f(x)\cos(jx)\right)^2 = a_j^2 \\ \text{Similarly}: |\langle f, \frac{\sin jx}{\sqrt{\pi}}\rangle|^2 = b_j^2 \end{array}\right.$$

$$\Rightarrow \frac{a_0^2}{2} + \sum_{i=1}^{\infty} a_k^2 + b_k^2 \le \int_{-\pi}^{\pi} f^2$$

6 Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V.

(a) Prove that if $v_1, \ldots, v_n$ are vectors in V such that

$$\|e_k - v_k\| < \frac{1}{\sqrt{n}}$$

for each k, then $v_1, \ldots, v_n$ is a basis of V.

(b) Show that there exist $v_1, \ldots, v_n \in V$ such that

$$\|e_k - v_k\| \le \frac{1}{\sqrt{n}}$$

for each k, but $v_1, \ldots, v_n$ is not linearly independent.

This exercise states in (a) that an appropriately small perturbation of an orthonormal basis is a basis. Then (b) shows that the number $1/\sqrt{n}$ on the right side of the inequality in (a) cannot be improved upon.

a) let $v_1 \ldots v_n \in V$ s.t. $\|e_k - v_k\| < \frac{1}{\sqrt{n}}$ (1)

Assume $v_1 \ldots v_n$ is not a basis of V. Then $\exists w \in V$ s.t. $w \notin \operatorname{span}(v_1 \ldots v_n)$ (so $w \neq 0$).

$$\|e_i - v_i\|^2 \|w\|^2 \ge |\langle e_i - v_i, w\rangle|^2$$
$$= |\langle e_i, w\rangle - \langle v_i, w\rangle|^2$$

Assume $\langle v_i, w\rangle = 0 \ \forall i$. This can be achieved since $\operatorname{span}(v_1 \ldots v_n) \neq V$, implying $\exists e_1 \ldots e_m, m < n$ s.t. $\operatorname{span}(v_1 \ldots v_n) = \operatorname{span}(e_1 \ldots e_m)$. Then just pick w from $\operatorname{span}(e_{m+1} \ldots e_n) \setminus \{0\}$.

$$\Rightarrow \|e_i - v_i\|^2 \|w\|^2 \ge |\langle e_i, w\rangle|^2 \ \forall i = 1 \ldots n$$

$$\Rightarrow \sum_{i=1}^{n} \|e_i - v_i\|^2 \|w\|^2 \ge \sum_{i=1}^{n} |\langle e_i, w\rangle|^2. \text{ From (1)}: \sum_{i=1}^{n} \|e_i - v_i\|^2 < 1$$

$\Rightarrow \|w\|^2 > \|w\|^2$

This is a contradiction, hence span $(v_1 \ldots v_n) = V$ and $v_1 \ldots v_n$ is thus a basis of V.

7 Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ has an upper-triangular matrix with respect to the basis $(1,0,0)$, $(1,1,1)$, $(1,1,2)$. Find an orthonormal basis of $\mathbf{R}^3$ with respect to which T has an upper-triangular matrix.

$Tv_k \in$ span $(v_1 \ldots v_k)$ = span $(e_1 \ldots e_k)$

span $(v_1 \ldots v_k)$ invariant under T

Applying the Gram-Schmidt procedure to vectors $v_1 \ldots v_m$ gives orthonormal vectors $e_1 \ldots e_n$ s.t. for all $i = 1 \ldots m$, we have span $(v_1 \ldots v_i)$ = span $(e_1 \ldots e_i)$, meaning if T has an upper tri. matrix wrt. $v_1 \ldots v_m$, then T has an upper tri. matrix wrt $e_1 \ldots e_m$ (span $(e_1 \ldots e_i)$ invariant under T).

So apply G.S. to $(1,0,0)$, $(1,1,1)$, $(1,1,2)$:

$e_1 = (1,0,0) = f_1$

$$f_2 = (1,1,1) - \frac{\langle (1,1,1),(1,0,0)\rangle}{\|(1,0,0)\|^2}(1,0,0) = (1,1,1) - (1,0,0) = (0,1,1)$$

$$f_3 = (1,1,2) - \frac{\langle (1,1,2),(1,0,0)\rangle}{\|(1,0,0)\|^2}(1,0,0) - \frac{\langle (1,1,2),(0,1,1)\rangle}{\|(0,1,1)\|^2}(0,1,1) = (1,1,2) - (1,0,0) - \frac{3}{2}(0,1,1)$$

$$= \frac{1}{2}(0,-1,1)$$

$$\Rightarrow e_1 = (1,0,0),\ e_2 = \sqrt{2}^{-1}(0,1,1),\ e_3 = \frac{\sqrt{2}}{2}(0,-1,1)$$

8 Make $\mathcal{P}_2(\mathbf{R})$ into an inner product space by defining $\langle p,q\rangle = \int_0^1 pq$ for all $p, q \in \mathcal{P}_2(\mathbf{R})$.

(a) Apply the Gram–Schmidt procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

(b) The differentiation operator (the operator that takes p to p') on $\mathcal{P}_2(\mathbf{R})$ has an upper-triangular matrix with respect to the basis $1, x, x^2$, which is not an orthonormal basis. Find the matrix of the differentiation operator on $\mathcal{P}_2(\mathbf{R})$ with respect to the orthonormal basis produced in (a) and verify that this matrix is upper triangular, as expected from the proof of 6.37.

a) $f_1 = 1$

$$f_2 = x - \frac{\langle x, 1\rangle}{\|1\|^2}1 \qquad \langle x,1\rangle = \int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$= x - \frac{1}{2}$$

$$f_3 = x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2}1 - \frac{\langle x^2, x-\frac{1}{2}\rangle}{\|x-\frac{1}{2}\|^2}\left(x - \frac{1}{2}\right)$$

$$\langle x^2, 1\rangle = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\langle x^2, x\rangle = \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$\langle x^2, -\frac{1}{2}\rangle = -\frac{1}{2}\langle x^2, 1\rangle = -\frac{1}{6}$$

$$\|x - \frac{1}{2}\|^2 = \int_0^1 \left(x-\frac{1}{2}\right)^2 dx = \int_0^1 x^2 + \frac{1}{4} - x\,dx = \left[\frac{x^3}{3}\right]_0^1 + \frac{1}{4}[x]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}$$

$$= x^2 - \frac{1}{3} - \frac{\frac{1}{4} - \frac{1}{6}}{\frac{1}{12}}\left(x - \frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}$$

$$\Rightarrow e_1 = 1,\ e_2 = \frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} = \sqrt{12}\left(x - \frac{1}{2}\right)$$

$$e_3 = \frac{x^2 - x + \frac{1}{6}}{\|x^2 - x + \frac{1}{6}\|} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)$$

b) $\mathcal{M}(D, (1, x, x^2)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$D(1) = 0$

$D\left(2\sqrt{3}\left(x - \frac{1}{2}\right)\right) = 2\sqrt{3} = 2\sqrt{3} \times 1$

$D\left(6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\right) = 6\sqrt{5}(2x - 1) = 12\sqrt{5} \times \left(x - \frac{1}{2}\right)$

$$D(2\sqrt{3}(x-\tfrac{1}{2})) = 2\sqrt{3} = 2\sqrt{3}\times 1$$

$$D(6\sqrt{5}(x^2-x+\tfrac{1}{6})) = 6\sqrt{5}(2x-1) = 12\sqrt{5}\,x(x-\tfrac{1}{2})$$

$$M(D,(1, 2\sqrt{3}(x-\tfrac{1}{2}), 6\sqrt{5}(x^2-x+\tfrac{1}{6}))) = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 12\sqrt{5} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{(upper triangular)}$$

11 Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that $p\left(\frac{1}{2}\right) = \int_0^1 pq$ for every $p \in \mathcal{P}_2(\mathbf{R})$.

Define $\varphi \in \mathcal{P}_2(\mathbb{R})'$, $\varphi p = p(\frac{1}{2})$.

By the Riesz representation theorem, $\exists!\, q \in \mathcal{P}_2(\mathbb{R})$ s.t. $\varphi p = \langle p, q\rangle\ \forall p \in \mathcal{P}_2(\mathbb{R})$, with $\langle r,s\rangle = \int_0^1 rs\ \forall r,s \in \mathcal{P}_2(\mathbb{R})$ inner product on $\mathcal{P}_2(\mathbb{R})$.

$q(x) = \sum_{i=1}^{3} \overline{\varphi(e_i)}\, e_i$, with e_1, e_2, e_3 an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$, like $1, 2\sqrt{3}(x-\frac{1}{2}), 6\sqrt{5}(x^2-x+\frac{1}{6})$

$= 1 + 6\sqrt{5}\left(\frac{1}{4}-\frac{1}{2}+\frac{1}{6}\right)\left[6\sqrt{5}(x^2-x+\frac{1}{6})\right]$

$= 1 - 15(x^2-x+\frac{1}{6}) = -15(x^2-x) - \frac{3}{2}$

13 Show that a list $v_1, \dots, v_m$ of vectors in V is linearly dependent if and only if the Gram–Schmidt formula in 6.32 produces $f_k = 0$ for some $k \in \{1, \dots, m\}$.

This exercise gives an alternative to Gaussian elimination techniques for determining whether a list of vectors in an inner product space is linearly dependent.

Suppose $v_1 \dots v_m$ linearly independent.

$\iff \exists k \in \{1..m\}$ s.t. $\exists a_1 \dots a_{k-1}$ s.t. $v_k = \sum_{i=1}^{k-1} a_i v_i$ (reorder $v_1 \dots v_m$ s.t. it is the case)

$f_k = v_k - \sum_{i=1}^{k-1} \frac{\langle v_k, f_i\rangle}{\|f_i\|^2} f_i = \sum_{i=1}^{k-1} a_i v_i - \sum_{i=1}^{k-1} \langle v_k, e_i\rangle e_i$.

$\operatorname{span}(v_1 \dots v_{k-1}) = \operatorname{span}(e_1 \dots e_{k-1})$

$\iff v_k \in \operatorname{span}(e_1 \dots e_{k-1})$

$\iff v_k = \sum_{i=1}^{k-1} \langle v_k, e_i\rangle e_i$

$\iff f_k = 0$

14 Suppose V is a real inner product space and $v_1, \dots, v_m$ is a linearly independent list of vectors in V. Prove that there exist exactly 2^m orthonormal lists $e_1, \dots, e_m$ of vectors in V such that

$$\operatorname{span}(v_1, \dots, v_k) = \operatorname{span}(e_1, \dots, e_k)$$

for all $k \in \{1, \dots, m\}$.

There exists at least 2^m orthonormal lists $e_1 \dots e_m$ s.t. $\operatorname{span}(v_1 \dots v_k) = \operatorname{span}(e_1 \dots e_k)\ \forall k \in \{1..m\}$ since each element e_i can be replaced by $-e_i$, meaning there are m choices of 2 options, 2^m.

Exactly 2^m:

Suppose $\exists$ another basis $e_1' \dots e_m'$ s.t. $\operatorname{span}(v_1 \dots v_k) = \operatorname{span}(e_1' \dots e_k')\ \forall k = \{1..m\}$ and $\exists e_i'$ s.t. $e_i' \neq e_i$ and $e_i' \neq -e_i$, where e_i is obtained via the Gram-Schmidt procedure.

We can show by induction that $e_i = \pm e_i'\ \forall i = 1..m$ by induction.

Init: Since $e_1 = \pm \frac{v_1}{\|v_1\|}$. $e_1' \in \operatorname{span}(v_1) \Rightarrow e_1' = k v_1$,

Init: Since $e_1 = \pm \frac{v_1}{\|v_1\|}$. $e_1' \in \operatorname{span}(v_1) \Rightarrow e_1' = k v_1$,
with $\|e_1'\| = 1$
$\Rightarrow e_1' = \pm \frac{v_1}{\|v_1\|}$

$\Rightarrow e_1' = \pm e_1$

Step: Suppose $e_i' = \pm e_i$ for $i = 1 \ldots k-1$.

$e_k' \in \operatorname{span}(e_1 \ldots e_k)$

$$\Rightarrow e_k' = \sum_{i=1}^{k} \langle e_k', e_i \rangle e_i = \sum_{i=1}^{k-1} \langle e_k', e_i' \rangle e_i + \langle e_k', e_k \rangle e_k$$

$$= \langle e_k', e_k \rangle e_k \ (1)$$

$\Rightarrow \|e_k'\| = |\langle e_k', e_k \rangle| \|e_k\| \Rightarrow |\langle e_k', e_k \rangle| = 1 \Rightarrow \langle e_k', e_k \rangle = \pm 1$ (real inner prod. space)

$\overset{(1)(2)}{\Rightarrow} e_k' = \pm e_k$

$\Rightarrow e_i' = e_i \ \forall i = 1 \ldots m$

$\Rightarrow$ only 2^m orthonormal bases such that $\operatorname{span}(v_1 \ldots v_k) = \operatorname{span}(e_1 \ldots e_k) \ \forall k \in \{1 \ldots m\}$

17 Suppose $\mathbf{F} = \mathbf{C}$ and V is finite-dimensional. Prove that if T is an operator on V such that 1 is the only eigenvalue of T and $\|Tv\| \leq \|v\|$ for all $v \in V$, then T is the identity operator.

$\mathbb{F} = \mathbb{C} \Rightarrow \exists$ orthonormal basis $e_1 \ldots e_m$ of V s.t. $\mathcal{M}(T, (e_1 \ldots e_m))$ is upper triangular

1 is the only eigenvalue of $T \Rightarrow \mathcal{M}(T, (e_1 \ldots e_m))_{ij} = 1$

$T e_k \in \operatorname{span}(e_1 \ldots e_k) \Rightarrow T e_k = \sum_{i=1}^{k-1} a_i e_i + e_k$ and we have $\|T e_k\| \leq \|e_k\|$.

$$\Rightarrow \left\| \sum_{i=1}^{k-1} a_i e_i + e_k \right\|^2 \leq \|e_k\|^2 \Rightarrow \sum_{i=1}^{k-1} |a_i|^2 + 1 \leq 1 \Rightarrow \sum_{i=1}^{k-1} |a_i|^2 \leq 0 \Rightarrow a_i = 0 \ \forall i \in \{1 \ldots k-1\}.$$

$\Rightarrow T e_k = e_k \ \forall k \in \{1 \ldots m\} \Rightarrow T$ is the identity operator