6B Exercises

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1 Suppose $e_1, \dots, e_m$ is a list of vectors in V such that

$$\|a_1 e_1 + \cdots + a_m e_m\|^2 = |a_1|^2 + \cdots + |a_m|^2$$

for all $a_1, \dots, a_m \in \mathbf{F}$. Show that $e_1, \dots, e_m$ is an orthonormal list.

This exercise provides a converse to 6.24.

Let $i = 1 \dots m$. Take $a_k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otw} \end{cases}$: $\|\sum_{k=1}^{m} a_k e_k\| = \|e_i\| = \sum_{i=1}^{n} |a_i|^2 = |a_i|^2 = 1$

$\Rightarrow \|e_i\| = 1 \ \forall i = 1 \dots m$ (1)

Let $i, j = 1 \dots m$ s.t $i \neq j$. Take $a_k = \begin{cases} 1 & \text{if } k = i \text{ or } k = j \\ 0 & \text{otw} \end{cases}$

$\Rightarrow \|e_i + e_j\|^2 = \underbrace{\|e_i\|^2}_{1} + \underbrace{\|e_j\|^2}_{1} + \langle e_i, e_j \rangle = |1|^2 + |1|^2 = 2$

$\Rightarrow \langle e_i, e_j \rangle = 0$ (2)

(1), (2) $\Rightarrow e_1 \dots e_m$ orthonormal

2 (a) Suppose $\theta \in \mathbf{R}$. Show that both

$$(\cos\theta, \sin\theta), (-\sin\theta, \cos\theta) \quad \text{and} \quad (\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$$

are orthonormal bases of $\mathbf{R}^2$.

(b) Show that each orthonormal basis of $\mathbf{R}^2$ is of the form given by one of the two possibilities in (a).

a) $\langle (\cos\theta, \sin\theta), (-\sin\theta, \cos\theta) \rangle = -\cos\theta \sin\theta + \sin\theta\cos\theta = 0$

$\|(\cos\theta, \sin\theta)^2\| = \cos^2\theta + \sin^2\theta = 1 \Rightarrow \|(\cos\theta, \sin\theta)\| = 1$ (same with $(-\sin\theta, \cos\theta)$)

Same reasoning for $(\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)$.

b) Let $(x_1, x_2), (y_1, y_2)$ an orthonormal basis of $\mathbb{R}^2$. We can show it has form $(x_1, x_2), (-x_2, x_1)$ or $(x_1, x_2), (x_2, -x_1)$

$$\begin{cases} \langle (x_1, x_2), (y_1, y_2) \rangle = 0 \Rightarrow x_1 y_1 + x_2 y_2 = 0 \\ \|(x_1, x_2)\|^2 = 1 \Rightarrow x_1^2 + x_2^2 = 1 \\ \|(y_1, y_2)\|^2 = 1 \Rightarrow y_1^2 + y_2^2 = 1 \end{cases}$$

$\overset{(y_1 \neq 0)}{\Rightarrow} x_1 = -\frac{x_2 y_2}{y_1} = \pm\frac{\sqrt{1-x_1^2}\, y_2}{\sqrt{1-y_2^2}} \overset{(x_1 \neq 1)}{\Rightarrow} \frac{x_1}{\sqrt{1-x_1^2}} = \pm\frac{y_2}{\sqrt{1-y_2^2}}$ (if $x_1 = 1$, then $x_2 = 0$, $y_1 = 0$, $y_2 = 1$, satisfies both forms) if $y_1 = 0$

$\left(\frac{1}{\sqrt{1-x^2}}\right)' = -\frac{\left(\sqrt{1-x^2}\right)'}{1-x^2} = -\frac{\left(\frac{1}{2\sqrt{1-x^2}}\right)}{1-x^2} = -\frac{1-x^2}{2\sqrt{1-x^2}}$

$= -\frac{\sqrt{1-x^2}}{2} < 0 \ \forall x \in]0, 1]$

$\Rightarrow \frac{1}{\sqrt{1-x^2}}$ strictly decreasing ($\Rightarrow$ bijection)

$\Rightarrow \boxed{x_1 = \pm y_2}$

$x_1 = -y_2 \overset{(y_2 \neq 0)}{\Rightarrow} -y_1 + x_2 = 0 \Rightarrow y_1 = x_2$ (if $y_2 = 0$, then $y_1 = 1$, $x_1 = 0$ and $x_2 = 1$, satisfies both forms)

$x_1 = y_2 \overset{(y_2 \neq 0)}{\Rightarrow} y_1 = -x_2$

3 Suppose $e_1, \dots, e_m$ is an orthonormal list in V and $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2 \iff v \in \operatorname{span}(e_1, \dots, e_m).$$

Let $e_1 \dots e_m, e_{m+1} \dots e_n$ an orthonormal basis of V.

$$\|v\|^2 = \sum_{i=1}^{m} |\langle v, e_i\rangle|^2, \text{ using } (6.30b)$$

ith component is $\langle v, e_i\rangle$ in the basis $e_1 \dots e_n$

$$= \sum_{i=1}^{m} |\langle v, e_i\rangle|^2 \iff \sum_{i=m+1}^{n} |\langle v, e_i\rangle|^2 = 0 \iff \langle v, e_i\rangle = 0 \ \forall i = m+1 \dots n \iff v \in \operatorname{span}(e_1 \dots e_m)$$

5 Suppose $f: [-\pi, \pi] \to \mathbf{R}$ is continuous. For each nonnegative integer k, define

$$a_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx \quad \text{and} \quad b_k = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx.$$

Prove that

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \le \int_{-\pi}^{\pi} f^2.$$

The inequality above is actually an equality for all continuous functions $f: [-\pi, \pi] \to \mathbf{R}$. However, proving that this inequality is an equality involves Fourier series techniques beyond the scope of this book.

From ex 4, $\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}} \dots \frac{\sin nx}{\sqrt{\pi}}$ orthonormal in $C[-\pi,\pi]$ with inner prod $\int_{-\pi}^{\pi} fg$.

Call these vectors e_i, $i = 1 \dots m$, $m = 2n+1$.

Using Bessel inequality, we have:

$$\sum_{i=1}^{m} |\langle f, e_i\rangle|^2 \le \|f\|^2 \Rightarrow |\langle f, \tfrac{1}{\sqrt{2\pi}}\rangle|^2 + \sum_{i=1}^{n} |\langle f, \tfrac{\cos ix}{\sqrt{\pi}}\rangle|^2 + \sum_{i=1}^{n} |\langle f, \tfrac{\sin ix}{\sqrt{\pi}}|^2 \le \int_{-\pi}^{\pi} f^2$$

$$|\langle f, \tfrac{1}{\sqrt{2\pi}}\rangle|^2 = \frac{1}{2\pi} \langle f, 1\rangle^2 = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} f(x)\, dx\right)^2 = \frac{1}{2} a_0^2$$

For any $j > 1$: $|\langle f, \frac{\cos jx}{\sqrt{\pi}}\rangle|^2 = \left(\int_{-\pi}^{\pi} f(x) \frac{\cos(jx)}{\sqrt{\pi}}\right)^2 = \left(\frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} f(x) \cos(jx)\right)^2 = a_j^2$

Similarly: $|\langle f, \frac{\sin jx}{\sqrt{\pi}}\rangle|^2 = b_j^2$

$$\Rightarrow \frac{a_0^2}{2} + \sum_{i=1}^{\infty} a_k^2 + b_k^2 \le \int_{-\pi}^{\pi} f^2$$

6 Suppose $e_1, \dots, e_n$ is an orthonormal basis of V.

(a) Prove that if $v_1, \dots, v_n$ are vectors in V such that

$$\|e_k - v_k\| < \frac{1}{\sqrt{n}}$$

for each k, then $v_1, \dots, v_n$ is a basis of V.

(b) Show that there exist $v_1, \dots, v_n \in V$ such that

$$\|e_k - v_k\| \le \frac{1}{\sqrt{n}}$$

for each k, but $v_1, \dots, v_n$ is not linearly independent.

This exercise states in (a) that an appropriately small perturbation of an orthonormal basis is a basis. Then (b) shows that the number $1/\sqrt{n}$ on the right side of the inequality in (a) cannot be improved upon.

a) Let $v_1 \dots v_n \in V$ s.t. $\|e_k - v_k\| < \frac{1}{\sqrt{n}}$ (1)

Assume $v_1 \dots v_n$ is not a basis of V. Then $\exists w \in V$ s.t. $w \notin \operatorname{span}(v_1 \dots v_n)$ (so $w \neq 0$).

$$\|e_i - v_i\|^2 \|w\|^2 \geq |\langle e_i - v_i, w\rangle|^2$$
$$= |\langle e_i, w\rangle - \langle v_i, w\rangle|^2$$

Assume $\langle v_i, w\rangle = 0 \ \forall i$. This can be achieved since $\operatorname{span}(v_1 \dots v_n) \neq V$, implying $\exists e_1 \dots e_m, m < n$ s.t. $\operatorname{span}(v_1 \dots v_n) = \operatorname{span}(e_1 \dots e_m)$. Then just pick w from $\operatorname{span}(e_{m+1} \dots e_n) \setminus \{0\}$.

$$\Rightarrow \|e_i - v_i\|^2 \|w\|^2 \geq |\langle e_i, w\rangle|^2 \ \forall i = 1 \dots n$$

$$\Rightarrow \sum_{i=1}^{n} \|e_i - v_i\|^2 \|w\|^2 \geq \sum_{i=1}^{n} |\langle e_i, w\rangle|^2. \text{ From (1): } \sum_{i=1}^{n} \|e_i - v_i\|^2 < 1$$

$\Longrightarrow \quad \|w\|^2 > \|w\|^2$

This is a contradiction, hence $\text{span}(v_1 \ldots v_n) = V$ and $v_1 \ldots v_n$ is thus a basis of V.

7 Suppose $T \in \mathcal{L}(\mathbf{R}^3)$ has an upper-triangular matrix with respect to the basis $(1,0,0)$, $(1,1,1)$, $(1,1,2)$. Find an orthonormal basis of $\mathbf{R}^3$ with respect to which T has an upper-triangular matrix.

$Tv_k \in \text{span}(v_1 \ldots v_k) = \text{span}(e_1 \ldots e_k)$

$\text{span}(v_1 \ldots v_k)$ invariant under T

Applying the Gram-Schmidt procedure to vectors $v_1 \ldots v_m$ gives orthonormal vectors $e_1 \ldots e_n$ s.t. for all $i = 1 \ldots m$, we have $\text{span}(v_1 \ldots v_i) = \text{span}(e_1 \ldots e_i)$, meaning if T has an upper tri. matrix wrt. $v_1 \ldots v_m$, then T has an upper tri. matrix wrt $e_1 \ldots e_m$ ($\text{span}(e_1 \ldots e_i)$ invariant under T).

So apply G.S. to $(1,0,0), (1,1,1), (1,1,2)$:

$e_1 = (1,0,0) = f_1$

$$f_2 = (1,1,1) - \frac{\langle (1,1,1),(1,0,0)\rangle}{\|(1,0,0)\|^2}(1,0,0) = (1,1,1) - (1,0,0) = (0,1,1)$$

$$f_3 = (1,1,2) - \frac{\langle (1,1,2),(1,0,0)\rangle}{\|(1,0,0)\|^2}(1,0,0) - \frac{\langle (1,1,2),(0,1,1)\rangle}{\|(0,1,1)\|^2}(0,1,1) = (1,1,2) - (1,0,0) - \frac{3}{2}(0,1,1)$$

$$= \frac{1}{2}(0,-1,1)$$

$$\Longrightarrow e_1 = (1,0,0), \quad e_2 = \sqrt{2}^{-1}(0,1,1), \quad e_3 = \frac{\sqrt{2}}{2}(0,-1,1)$$

8 Make $\mathcal{P}_2(\mathbf{R})$ into an inner product space by defining $\langle p, q\rangle = \int_0^1 pq$ for all $p, q \in \mathcal{P}_2(\mathbf{R})$.

(a) Apply the Gram–Schmidt procedure to the basis $1, x, x^2$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

(b) The differentiation operator (the operator that takes p to p') on $\mathcal{P}_2(\mathbf{R})$ has an upper-triangular matrix with respect to the basis $1, x, x^2$, which is not an orthonormal basis. Find the matrix of the differentiation operator on $\mathcal{P}_2(\mathbf{R})$ with respect to the orthonormal basis produced in (a) and verify that this matrix is upper triangular, as expected from the proof of 6.37.

a) $f_1 = 1$

$$f_2 = x - \frac{\langle x, 1\rangle}{\|1\|^2} 1 \qquad \langle x, 1\rangle = \int_0^1 x \, dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$= x - \frac{1}{2}$$

$$f_3 = x^2 - \frac{\langle x^2, 1\rangle}{\|1\|^2} 1 - \frac{\langle x^2, x - \frac{1}{2}\rangle}{\|x - \frac{1}{2}\|^2}\left(x - \frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - \frac{\frac{1}{4} - \frac{1}{6}}{\frac{1}{12}}\left(x - \frac{1}{2}\right)$$

$$= x^2 - \frac{1}{3} - x + \frac{1}{2} = x^2 - x + \frac{1}{6}$$

$$\langle x^2, 1\rangle = \int_0^1 x^2 \, dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\langle x^2, x\rangle = \int_0^1 x^3 \, dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$\langle x^2, -\frac{1}{2}\rangle = -\frac{1}{2}\langle x^2, 1\rangle = -\frac{1}{6}$$

$$\left\|x - \frac{1}{2}\right\|^2 = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx = \int_0^1 x^2 + \frac{1}{4} - x \, dx = \left[\frac{x^3}{3}\right]_0^1 + \frac{1}{4}[x]_0^1 - \left[\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}$$

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$$\Longrightarrow e_1 = 1, \quad e_2 = \frac{x - \frac{1}{2}}{\|x - \frac{1}{2}\|} = \sqrt{12}\left(x - \frac{1}{2}\right)$$

$$e_3 = \frac{x^2 - x + \frac{1}{6}}{\|x^2 - x + \frac{1}{6}\|} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)$$

b) $\mathcal{M}(D, (1, x, x^2)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

$D(1) = 0$

$D\left(2\sqrt{3}\left(x - \frac{1}{2}\right)\right) = 2\sqrt{3} = 2\sqrt{3} \times 1$

$D\left(6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\right) = 6\sqrt{5}(2x - 1) = 12\sqrt{5} \times \left(x - \frac{1}{2}\right)$

$D(2\sqrt{3}(x-\frac{1}{2})) = 2\sqrt{3} = 2\sqrt{3}\times 1$

$D(6\sqrt{5}(x^2-x+\frac{1}{6})) = 6\sqrt{5}(2x-1) = 12\sqrt{5}\,x(x-\frac{1}{2})$

$\mathcal{M}(D,(1, 2\sqrt{3}(x-\frac{1}{2}), 6\sqrt{5}(x^2-x+\frac{1}{6}))) = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 12\sqrt{5} \\ 0 & 0 & 0 \end{pmatrix}$ (upper triangular)

11 Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that $p\left(\frac{1}{2}\right) = \int_0^1 pq$ for every $p \in \mathcal{P}_2(\mathbf{R})$.

Define $\varphi \in \mathcal{P}_2(\mathbb{R})'$, $\varphi p = p(\frac{1}{2})$.

By the Riesz representation theorem, $\exists! q \in \mathcal{P}_2(\mathbb{R})$ s.t. $\varphi p = \langle p, q\rangle \ \forall p \in \mathcal{P}_2(\mathbb{R})$, with $\langle r,s\rangle = \int_0^1 rs \ \forall r,s \in \mathcal{P}_2(\mathbb{R})$ inner product on $\mathcal{P}_2(\mathbb{R})$.

$q(x) = \sum_{i=1}^{3} \varphi(e_i) e_i$, with e_1, e_2, e_3 an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$, like $1, 2\sqrt{3}(x-\frac{1}{2}), 6\sqrt{5}(x^2-x+\frac{1}{6})$

$= 1 + 6\sqrt{5}(\frac{1}{4}-\frac{1}{2}+\frac{1}{6})[6\sqrt{5}(x^2-x+\frac{1}{6})]$

$= 1 - 15(x^2-x+\frac{1}{6}) = -15(x^2-x) - \frac{3}{2}$

13 Show that a list $v_1, \dots, v_m$ of vectors in V is linearly dependent if and only if the Gram–Schmidt formula in 6.32 produces $f_k = 0$ for some $k \in \{1, \dots, m\}$.

This exercise gives an alternative to Gaussian elimination techniques for determining whether a list of vectors in an inner product space is linearly dependent.

Suppose $v_1 \dots v_m$ linearly independent.

$\iff \exists k \in \{1..m\}$ s.t. $\exists a_1 \dots a_{k-1}$ s.t. $v_k = \sum_{i=1}^{k-1} a_i v_i$ (reorder $v_1 \dots v_m$ s.t. it is the case)

$f_k = v_k - \sum_{i=1}^{k-1} \frac{\langle v_k, f_i\rangle}{\|f_i\|^2} f_i = \sum_{i=1}^{k-1} a_i v_i - \sum_{i=1}^{k-1} \langle v_k, e_i\rangle e_i$.

$\operatorname{span}(v_1 \dots v_{k-1}) = \operatorname{span}(e_1 \dots e_{k-1})$

$\iff v_k \in \operatorname{span}(e_1 \dots e_{k-1})$

$\iff v_k = \sum_{i=1}^{k-1} \langle v_k, e_i\rangle e_i$

$\iff f_k = 0$

14 Suppose V is a real inner product space and $v_1, \dots, v_m$ is a linearly independent list of vectors in V. Prove that there exist exactly 2^m orthonormal lists $e_1, \dots, e_m$ of vectors in V such that

$$\operatorname{span}(v_1, \dots, v_k) = \operatorname{span}(e_1, \dots, e_k)$$

for all $k \in \{1, \dots, m\}$.

There exists at least 2^m orthonormal lists $e_1 \dots e_m$ s.t. $\operatorname{span}(v_1 \dots v_k) = \operatorname{span}(e_1 \dots e_k) \ \forall k \in \{1..m\}$ since each element e_i can be replaced by $-e_i$, meaning there are m choices of 2 options, 2^m.

Exactly 2^m:

Suppose $\exists$ another basis $e'_1 \dots e'_m$ s.t. $\operatorname{span}(v_1 \dots v_k) = \operatorname{span}(e'_1 \dots e'_k) \ \forall k = \{1..m\}$ and $\exists e'_i$ s.t. $e'_i \neq e_i$ and $e'_i \neq -e_i$, where e_i is obtained via the Gram-Schmidt procedure.

We can show by induction that $e_i = \pm e'_i \ \forall i = 1..m$ by induction.

Init: Since $e_1 = \pm \frac{v_1}{\|v_1\|}$. $e'_1 \in \operatorname{span}(v_1) \Rightarrow e'_1 = k v_1$,

Init: Since $e_1 = \pm \frac{v_1}{\|v_1\|}$. $e_1' \in \operatorname{span}(v_1) \Rightarrow e_1' = k v_1$,
with $\|e_1'\| = 1$
$\Rightarrow e_1' = \pm \frac{v_1}{\|v_1\|}$
$\Rightarrow e_1' = \pm e_1$

Step: Suppose $e_i' = \pm e_i$ for $i = 1 \ldots k-1$.

$e_k' \in \operatorname{span}(e_1 \ldots e_k)$

$\Rightarrow e_k' = \sum_{i=1}^{k} \langle e_k', e_i \rangle e_i = \sum_{i=1}^{k-1} \langle e_k', e_i \rangle e_i + \langle e_k', e_k \rangle e_k$

$= \langle e_k', e_k \rangle e_k \;(1)$

$\Rightarrow \|e_k'\| = |\langle e_k', e_k \rangle| \|e_k\| \Rightarrow |\langle e_k', e_k \rangle| = 1 \Rightarrow \langle e_k', e_k \rangle = \pm 1$ (real inner prod. space)

$\overset{(1,2)}{\Rightarrow} e_k' = \pm e_k$

$\Rightarrow e_i' = e_i \;\forall i = 1 \ldots m$

$\Rightarrow$ only 2^m orthonormal bases such that $\operatorname{span}(v_1 \ldots v_k) = \operatorname{span}(e_1 \ldots e_k) \;\forall k \in \{1 \ldots m\}$

17 Suppose $\mathbf{F} = \mathbf{C}$ and V is finite-dimensional. Prove that if T is an operator on V such that 1 is the only eigenvalue of T and $\|Tv\| \le \|v\|$ for all $v \in V$, then T is the identity operator.

$\mathbb{F} = \mathbb{C} \Rightarrow \exists$ orthonormal basis $e_1 \ldots e_m$ of V s.t. $\mathcal{M}(T, (e_1 \ldots e_m))$ is upper triangular

1 is the only eigenvalue of $T \Rightarrow \mathcal{M}(T, (e_1 \ldots e_m))_{ii} = 1$

$Te_k \in \operatorname{span}(e_1 \ldots e_k) \Rightarrow Te_k = \sum_{i=1}^{k-1} a_i e_i + e_k$ and we have $\|Te_k\| \le \|e_k\|$.

$\Rightarrow \|\sum_{i=1}^{k-1} a_i e_i + e_k\|^2 \le \|e_k\|^2 \Rightarrow \sum_{i=1}^{k-1} |a_i|^2 + 1 \le 1 \Rightarrow \sum_{i=1}^{k-1} |a_i|^2 \le 0 \Rightarrow a_i = 0 \;\forall i \in \{1 \ldots k-1\}$.

$\Rightarrow Te_k = e_k \;\forall k \in \{1 \ldots m\} \Rightarrow T$ is the identity operator

21 Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dimensional, $T \in \mathcal{L}(V)$, and all eigenvalues of T have absolute value less than 1. Let $\epsilon > 0$. Prove that there exists a positive integer m such that $\|T^m v\| \le \epsilon \|v\|$ for every $v \in V$.

$\mathbb{F} = \mathbb{C} \Rightarrow \exists e_1 \ldots e_m$ orth. basis of V s.t. $\mathcal{M}(T)$ upper tri.

let $v \in V$, $v = \sum_{i=1}^{m} a_i e_i$. $Te_i = \sum_{j=1}^{i} b_{ij} e_j$ ($\mathcal{M}(T)$ upper triangular). let $k \ge 1$.

$\|T^k v\|^2 = \|T^{k-1}(\sum_{i=1}^{m} a_i \sum_{j=1}^{i} b_{ij} e_j)\|^2 = \|T^{k-2}(\sum_{i=1}^{m} a_i \sum_{j=1}^{i} b_{ij} \sum_{p=1}^{j} b_{jp} e_p)\|^2$

$= \|\sum_{i=1}^{m} a_i \sum_{i_1=1}^{i} \cdots \sum_{i_k=1}^{i_{k-1}} b_{i i_1} \cdots b_{i_{k-1} i_k} e_{i_k}\|^2$

$\le \|\sum_{i=1}^{m} a_i b_{ii}^{k} e_i\|^2 + C^2$ (triangle inequality)

$\le \sum_{i=1}^{m} |a_i b_{ii}^{k}|^2$ ($e_1 \ldots e_m$ orthonormal)

$= \sum^{m} |a_i|^2 |b_{ii}|^{2k} \le \max |b_{ii}|^{2k} \sum^{m} |a_i|^2 = \max |b_{ii}|^{2k} \|v\|^2$

$$= \sum_{i=1}^{n} |a_i|^2 |b_{ii}|^{2k} \leq \max_i |b_{ii}|^{2k} \sum_{i=1}^{n} |a_i|^2 = \max_i |b_{ii}|^{2k} \|v\|^2$$

$\Rightarrow \|T^k v\| \leq \max_{i=1..n} |b_{ii}|^k \|v\| \quad \forall k \geq 1$ (1)

$\lim_{k\to\infty} \max_{i=1..n} |b_{ii}|^k = 0$, since all e.va. ($b_{ii}$'s)'s absolute values are inferior to 1.

Hence $\forall \varepsilon > 0$, no matter how small, $\exists m \in \mathbb{N}^+$ s.t. $\max |b_{ii}|^m \leq \varepsilon$ (2)

The result follows from (1) and (2).