7C Exercises

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1 Suppose $T \in \mathcal{L}(V)$. Prove that if both T and $-T$ are positive operators, then $T = 0$.

Let $v \in V$. We have: $\begin{cases} \langle Tv, v\rangle \geq 0 \\ \langle -Tv, v\rangle \geq 0 \end{cases} \Rightarrow \begin{cases} \langle Tv, v\rangle \geq 0 \\ \langle Tv, v\rangle \leq 0 \end{cases} \Rightarrow Tv = 0 \Rightarrow T = 0$

2 Suppose $T \in \mathcal{L}(\mathbf{F}^4)$ is the operator whose matrix (with respect to the standard basis) is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Show that T is an invertible positive operator.

We can find the e.va. of T. After some computation, we find:
$\lambda_1 = \frac{\sqrt{5}+3}{2}$, $\lambda_2 = \frac{-\sqrt{5}+5}{2}$, $\lambda_3 = \frac{\sqrt{5}+5}{2}$, $\lambda_4 = \frac{-\sqrt{5}+3}{2}$.

They are all non-negative, and $M(T) = M(T)^T$ (self-adjoint), hence T is positive.
Moreover, they are positive (>0), hence T is invertible.

3 Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is the operator whose matrix (with respect to the standard basis) consists of all 1's. Show that T is a positive operator.

$M(T) = M(T)^T \Rightarrow T$ self-adjoint

$$\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{pmatrix}\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \vdots \\ \lambda a_n \end{pmatrix} \Rightarrow \begin{cases} \sum_{i=1}^n a_i = \lambda a_1 \\ \vdots \\ \sum_{i=1}^n a_i = \lambda a_n \end{cases} \Rightarrow \begin{cases} \lambda a_1 = \dots = \lambda a_n \\ \sum a_i = \lambda a_1 \end{cases} \Rightarrow \begin{cases} \lambda = n \\ a_1 = \dots = a_n \end{cases} \text{ or } \begin{cases} \lambda = 0 \\ \sum_{i=1}^n a_i = 0 \end{cases}$$

$\Rightarrow$ E.va. are ≥ 0 and T self-adjoint $\Rightarrow T$ positive

4 Suppose n is an integer with $n > 1$. Show that there exists an n-by-n matrix A such that all of the entries of A are positive numbers and $A = A^*$, but the operator on $\mathbf{F}^n$ whose matrix (with respect to the standard basis) equals A is not a positive operator.

$-\lambda^2 + \lambda + 1$ $\quad 1 - 4 \times (-1) \times 1 = 1 + 4 = 5 = \Delta$ $\quad \frac{-1+\sqrt{5}}{-2}$ $\quad \frac{-1-\sqrt{5}}{-2}$ *

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \end{pmatrix} \Rightarrow \begin{cases} a_1 + a_2 = \lambda a_1 \\ a_1 = \lambda a_2 \end{cases} \xRightarrow{\lambda \neq 0} \begin{cases} a_2 = \lambda^{-1} a_1 \\ a_1 + \lambda^{-1} a_1 = \lambda a_1 \end{cases} \Rightarrow \begin{cases} \overbrace{(1 + \frac{1}{\lambda} - \lambda)} a_1 = 0 \end{cases}$$

5 Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that T is a positive operator if and only if for every orthonormal basis $e_1, \dots, e_n$ of V, all entries on the diagonal of $\mathcal{M}(T, (e_1, \dots, e_n))$ are nonnegative numbers.

"$\Rightarrow$": Suppose $\exists$ orth. basis of V s.t. $M(T)_{ij} = a_{ij}$, with $a_{ii} < 0$.

$\langle Te_i, e_i\rangle = \langle \sum_{j=1}^n a_{ij} e_j, e_i\rangle = a_{ii} < 0 \Rightarrow T$ is not a positive operator

By contradiction, the part is proven.

"$\Leftarrow$": T self-adjoint $\Rightarrow$ T diagonalizable in an orth. basis $\Rightarrow$ Diagonal matrix has nonnegative numbers $\Rightarrow$ T positive

6 Prove that the sum of two positive operators on V is a positive operator.

Let S, T positive operators. Let $e_1, \dots, e_n$ an orth. basis of V.

$\mathcal{M}(S+T)_{ii} = \underbrace{\mathcal{M}(S)_{ii}}_{\geq 0} + \underbrace{\mathcal{M}(T)_{ii}}_{\geq 0} \geq 0 \Rightarrow$ For all orth. basis of V, $S+T$ has non-negative elements on their matrix's diagonal, hence by ex. 5 this one is positive.

(by ex. 5) (by ex 5.)

7 Suppose $S \in \mathcal{L}(V)$ is an invertible positive operator and $T \in \mathcal{L}(V)$ is a positive operator. Prove that $S + T$ is invertible.

Let $v \in V$ s.t. $(S+T)v = 0$.

$\langle (S+T)v, v \rangle = 0 \Rightarrow \langle Sv, v \rangle + \langle Tv, v \rangle = 0$ (1)

$\langle Sv, v \rangle \geq 0$ since S is positive.

- If $\langle Sv, v \rangle = 0$, then $Sv = 0$ (7.43) $\Rightarrow v = 0$ (since S invertible)

- If $\langle Sv, v \rangle > 0$, then $\langle Tv, v \rangle < 0$ by (1), but T is positive so there is a contradiction.

Hence $v = 0 \Rightarrow \operatorname{null}(S+T) = \{0\} \Rightarrow S+T$ invertible

8 Suppose $T \in \mathcal{L}(V)$. Prove that T is a positive operator if and only if the pseudoinverse $T^\dagger$ is a positive operator.

T positive $\overset{7.38 c)}{\Leftrightarrow} \exists$ some orth. basis of V s.t. $\mathcal{M}(T) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix}$

$\overset{7B\ ex\ 25}{\Leftrightarrow}$ ——— $\mathcal{M}(T^\dagger) = \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_k) \end{pmatrix}$, $f(x) = \begin{cases} \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

$\overset{7.38 c)}{\Leftrightarrow} T^\dagger$ positive

9 Suppose $T \in \mathcal{L}(V)$ is a positive operator and $S \in \mathcal{L}(W, V)$. Prove that S^*TS is a positive operator on W.

- $(S^*TS)^* = S^*TS \Rightarrow S^*TS$ self-adjoint
- $\langle S^*TSv, v \rangle = \langle TSv, Sv \rangle \geq 0$ (because T positive)

$\Rightarrow S^*TS$ positive

10 Suppose T is a positive operator on V. Suppose $v, w \in V$ are such that

$$Tv = w \quad \text{and} \quad Tw = v.$$

Prove that $v = w$.

$\langle T(v-w), v-w \rangle = \langle Tv - Tw, v-w \rangle = \langle w-v, v-w \rangle = -\|v-w\|^2$, and $\langle T(v-w), v-w \rangle \geq 0$ (T positive)

$\Rightarrow \|v-w\|^2 = 0 \Rightarrow v-w = 0 \Rightarrow v = w$

11 Suppose T is a positive operator on V and U is a subspace of V invariant under T. Prove that $T|_U \in \mathcal{L}(U)$ is a positive operator on U.

By 7B ex 19 b), $T|_U$ is self-adjoint on U.

$\langle T|_U u, u \rangle = \langle Tu, u \rangle \geq 0$

$\Rightarrow T|_U$ positive operator on U

12 Suppose $T \in \mathcal{L}(V)$ is a positive operator. Prove that T^k is a positive operator for every positive integer k.

$\langle T^k v, w\rangle = \langle T^{k-1} v, T^* w\rangle = \cdots = \langle v, (T^*)^k w\rangle = \langle v, T^k w\rangle \Rightarrow T^k$ self-adjoint

$$\langle T^k v, v\rangle = \begin{cases} \langle T^{k/2} v, T^{k/2} v\rangle = \|T^{k/2} v\| \geq 0 & \text{if } k \text{ even} \\ \langle T(T^{\lfloor k/2\rfloor} v), T^{\lfloor k/2\rfloor} v\rangle \geq 0 \text{ using } T \text{ positive} & \text{if } k \text{ odd} \end{cases}$$

13 Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $\alpha \in \mathbf{R}$.

(a) Prove that $T - \alpha I$ is a positive operator if and only if α is less than or equal to every eigenvalue of T.

(b) Prove that $\alpha I - T$ is a positive operator if and only if α is greater than or equal to every eigenvalue of T.

a) Let λ an e.va. of T, with e.ve. e_1 s.t. $\|e_1\| = 1$

$\langle (T - \alpha I)e_1, e_1\rangle = \langle Te_1, e_1\rangle - \langle \alpha e_1, e_1\rangle = \lambda - \alpha \quad (1)$

$T - \alpha I$ positive $\overset{(1)}{\Leftrightarrow} \lambda - \alpha \geq 0 \ \forall \lambda$ e.va. of T ("$\Leftarrow$" true because T self-adjoint)

$\Leftrightarrow \alpha \leq \lambda \ \forall \lambda$ e.va. of T

b) Similar reasoning.

14 Suppose T is a positive operator on V and $v_1, \ldots, v_m \in V$. Prove that

$$\sum_{j=1}^{m}\sum_{k=1}^{m} \langle Tv_k, v_j\rangle \geq 0.$$

$$\sum_{j=1}^{m}\sum_{k=1}^{m} \langle Tv_k, v_j\rangle \overset{T \text{ linear}}{\underset{\text{ineq}}{=}} \langle T\sum_{k=1}^{m} v_k, \sum_{k=1}^{m} v_k\rangle \geq 0 \text{ since } T \text{ positive}$$

15 Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Prove that there exist positive operators $A, B \in \mathcal{L}(V)$ such that

$$T = A - B \quad \text{and} \quad \sqrt{T^*T} = A + B \quad \text{and} \quad AB = BA = 0.$$

Let $e_1 \ldots e_m$ the basis s.t. $\mathcal{M}(T)$ diagonal (spectral theorem)

Let A, B the operators s.t. $\mathcal{M}(A) = \operatorname{diag}\{\lambda \in \mathbb{F} \text{ s.t. } \lambda \geq 0 \text{ and } \lambda \text{ e.va. of } T\}$

$\mathcal{M}(B) = \operatorname{diag}\{-\lambda \in \mathbb{F} \text{ s.t. } \lambda < 0 \text{ and } \lambda \text{ e.va. of } T\}$

$\Rightarrow A, B$ positive since $\exists$ orth. basis s.t. $\mathcal{M}(A)_{ii}, \mathcal{M}(B)_{ii} \geq 0 \ \forall i \in \{1 \ldots m\}$ and $\mathcal{M}(A), \mathcal{M}(B)$ diagonal

It is easy to see that $\mathcal{M}(T) = \mathcal{M}(A) - \mathcal{M}(B)$ hence $T = A - B$.

$\sqrt{\cdot}$ is unique and we see that $\mathcal{M}(A+B) = \begin{pmatrix} |\lambda| & & \\ & \ddots & \\ & & |\lambda| \end{pmatrix} = \mathcal{M}(\sqrt{T^2}) = \mathcal{M}(\sqrt{T^*T})$ hence $A + B = \sqrt{T^*T}$

Finally, $AB = BA = 0$.

16 Suppose T is a positive operator on V. Prove that

$$\operatorname{null}\sqrt{T} = \operatorname{null} T \quad \text{and} \quad \operatorname{range}\sqrt{T} = \operatorname{range} T.$$

$\sqrt{T}$ positive $\overset{7A,\text{ ex }27}{\Rightarrow} \operatorname{null}\sqrt{T} = \operatorname{null}\sqrt{T}^2 = \operatorname{null} T$ (T positive so $\sqrt{T}^2 = T$)

$\sqrt{T}$ positive, hence self-adjoint, hence normal

Same for range $\sqrt{T}$.

17 Suppose that $T \in \mathcal{L}(V)$ is a positive operator. Prove that there exists a polynomial p with real coefficients such that $\sqrt{T} = p(T)$.

Positive $\Rightarrow \exists$ basis of V s.t. $\mathcal{M}(T) = \operatorname{diag}(a_1 \ldots a_m)$, with $a_i \geq 0 \ \forall i \in \{1 \ldots m\}$. $\mathcal{M}(\sqrt{T}) = \operatorname{diag}(\sqrt{a_1} \ldots \sqrt{a_m})$.

By interpolation, we can find a polynomial p s.t. $\sqrt{a_i} = p a_i$, and hence $\mathcal{M}(\sqrt{T}) = p(\mathcal{M}(T))$.

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By interpolation, we can find a polynomial p s.t. $\sqrt{a_i} = p(a_i)$, and hence $M(\sqrt{T}) = p(M(T))$.

$\Rightarrow \exists p$ s.t. $\sqrt{T} = p(T)$

18 Suppose S and T are positive operators on V. Prove that ST is a positive operator if and only if S and T commute.

ST positive $\Rightarrow \exists$ orth. basis s.t. $M(ST)_{ii} \geq 0 \ \forall i \in \{1 \dots m\}$

ST commute $\overset{5.76}{\Rightarrow} \exists$ basis s.t. $M(S) = \text{diag}\{\lambda_1 \dots \lambda_m\}$, $M(T) = \text{diag}\{\mu_1 \dots \mu_m\}$,

with $\lambda_1 \dots \lambda_m, \mu_1 \dots \mu_m \geq 0$ since S, T are positive.

$\Rightarrow ST$ positive operator ($M(ST) = \text{diag}\{\mu_1\lambda_1 \dots \mu_m\lambda_m\}$, with $\mu_i\lambda_i \geq 0 \ \forall i \in \{1 \dots m\}$)

$STe_i = S\mu_i e_i = \mu_i Se_i = \lambda_i\mu_i e_i$

19 Show that the identity operator on $\mathbf{F}^2$ has infinitely many self-adjoint square roots.

$$\begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} a^2 + b^2 = 1 \\ b^2 + d^2 = 1 \\ ab + bd = 0 \end{cases} \xRightarrow{a=-d} \begin{cases} d^2 + b^2 = 1 \\ a = -d \end{cases} \quad \infty\text{-ly many solutions}$$

20 Suppose $T \in \mathcal{L}(V)$ and $e_1, \dots, e_n$ is an orthonormal basis of V. Prove that T is a positive operator if and only if there exist $v_1, \dots, v_n \in V$ such that

$$\langle Te_k, e_j \rangle = \langle v_k, v_j \rangle$$

for all $j, k = 1, \dots, n$.

The numbers $\{\langle Te_k, e_j \rangle\}_{j,k=1,\dots,n}$ are the entries in the matrix of T with respect to the orthonormal basis $e_1, \dots, e_n$.

"$\Leftarrow$": Suppose $\exists v_1 \dots v_m \in V$ s.t. $\langle v_k, v_j \rangle = \langle Te_k, e_j \rangle \ \forall j, k = 1 \dots m$.

- Then $\langle Te_i, e_i \rangle = \langle v_i, v_i \rangle \geq 0 \ \forall i = 1 \dots m$
- $\langle Te_k, e_j \rangle = \langle v_k, v_j \rangle = \overline{\langle v_j, v_k \rangle} = \overline{\langle Te_j, e_k \rangle} = \overline{\overline{\langle e_k, Te_j \rangle}} = \langle e_k, Te_j \rangle \ \forall k, j \Rightarrow T = T^*$

"$\Rightarrow$": ???

22 Suppose $T \in \mathcal{L}(V)$ is a positive operator and $u \in V$ is such that $\|u\| = 1$ and $\|Tu\| \geq \|Tv\|$ for all $v \in V$ with $\|v\| = 1$. Show that u is an eigenvector of T corresponding to the largest eigenvalue of T.

let $\lambda = \max_{i=1 \dots m} \{\lambda_i \in \mathbb{F} \mid \lambda_i \text{ e.va. of } T\}$, and $v \in V$ s.t. $Tv = \lambda v$ and $\|v\| = 1$

$\|Tu\|^2 \geq \|Tv\|^2 = |\lambda|^2 \|v\|^2 = |\lambda|^2$. $\quad u = \sum_{i=1}^m a_i e_i$, with $e_1 \dots e_m$ orth. basis s.t.:

$\Rightarrow \|\sum_{i=1}^m a_i \lambda_i e_i\|^2 = \sum_{i=1}^m |a_i\lambda_i|^2 = \sum_{i=1}^m |a_i|^2|\lambda_i|^2 \quad M(T) = \text{diag}(\lambda_1 \dots \lambda_m)$, $\lambda_i \geq 0$

$\Rightarrow |\lambda|^2 \underbrace{\sum_{i=1}^m |a_i|^2}_{= \|u\| = 1} \geq \|Tu\|^2 \geq |\lambda|^2 \Rightarrow \|Tu\| = |\lambda|$

23 For $T \in \mathcal{L}(V)$ and $u, v \in V$, define $\langle u, v \rangle_T$ by $\langle u, v \rangle_T = \langle Tu, v \rangle$.

(a) Suppose $T \in \mathcal{L}(V)$. Prove that $\langle \cdot, \cdot \rangle_T$ is an inner product on V if and only if T is an invertible positive operator (with respect to the original inner product $\langle \cdot, \cdot \rangle$).

(b) Prove that every inner product on V is of the form $\langle \cdot, \cdot \rangle_T$ for some positive invertible operator $T \in \mathcal{L}(V)$.

a) "$\Rightarrow$": Suppose $\langle \cdot,\cdot \rangle_T$ inner-prod on V (1)

- let $v \in V$. $\langle v,v \rangle_T \geq 0$ by (1) $\Rightarrow \langle Tv,v \rangle \geq 0$
- $\langle u,v \rangle_T = \overline{\langle v,u \rangle_T} \Rightarrow \langle Tu,v \rangle = \overline{\langle Tv,u \rangle} = \langle u,Tv \rangle \Rightarrow T$ self-adjoint

$\Big\}$ T positive

- let u s.t. $Tu = 0$: $\langle u,u \rangle_T = \langle Tu,u \rangle = 0 \Rightarrow u = 0 \Rightarrow \text{null}\,T = \{0\} \Rightarrow T$ invertible

"$\Leftarrow$":
- let $u \in V$. $\langle u,u \rangle_T = \langle Tu,u \rangle \geq 0$ as T positive, T invertible
- $\langle u,u \rangle_T = 0 \iff \langle Tu,u \rangle = 0 \overset{7.43}{\iff} Tu = 0 \iff u = 0$
- $\langle u,v \rangle_T = \langle Tu,v \rangle \overset{T\text{ self-adj.}}{=} \langle u,Tv \rangle = \overline{\langle Tv,u \rangle} = \overline{\langle v,u \rangle_T}$
- $\langle u+v,w \rangle_T = \langle T(u+v),w \rangle = \langle Tu,w \rangle + \langle Tv,w \rangle = \langle u,w \rangle_T + \langle v,w \rangle_T$
- $\langle \lambda u,v \rangle_T = \langle T\lambda u,v \rangle = \lambda \langle Tu,v \rangle = \lambda \langle u,v \rangle_T$

$\Big\}$ $\langle \cdot,\cdot \rangle_T$ norm

24 Suppose S and T are positive operators on V. Prove that

$$\text{null}(S + T) = \text{null}\,S \cap \text{null}\,T.$$

"$\supseteq$": Straightforward even if S,T not positive.

"$\subseteq$": let $v \in \text{null}(S+T)$. $S+T$ is positive by ex. 6

$\langle (S+T)v,v \rangle = 0 \Rightarrow \langle Sv,v \rangle + \langle Tv,v \rangle = 0 \Rightarrow \underbrace{\langle Sv,v \rangle}_{\geq 0} = \underbrace{-\langle Tv,v \rangle}_{\leq 0} \Rightarrow \langle Sv,v \rangle = \langle Tv,v \rangle = 0$

$\Rightarrow v \in \text{null}\,S \cap T$