

5A Exercises

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- 1 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V .

- (a) Prove that if $U \subseteq \text{null } T$, then U is invariant under T .
 (b) Prove that if $\text{range } T \subseteq U$, then U is invariant under T .

a) $v \in U \Rightarrow v \in \text{null } T \Rightarrow Tv = 0 \in U$ (as U is a subspace of V) $\Rightarrow U$ invariant under T

b) $\exists v \in U. Tv \in \text{range } T \Rightarrow Tv \in U \Rightarrow U$ invariant under T

- 2 Suppose that $T \in \mathcal{L}(V)$ and V_1, \dots, V_m are subspaces of V invariant under T .
 Prove that $V_1 + \dots + V_m$ is invariant under T .

Let $v = \sum_{i=1}^m v_i \in V_1 + \dots + V_m$. $Tv = \sum_{i=1}^m Tv_i$, where for each i , $Tv_i \in V_i$, as V_i 's are invariant under T . Therefore $Tv \in V_1 + \dots + V_m \Rightarrow V_1 + \dots + V_m$ is invariant under T .

- 3 Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T .

Let $A = \bigcap_{i=1}^m V_i$, where $m \in \overline{\mathbb{N}}$ and V_i invariant under T for each i .

$\forall v \in A$. For each i , $Tv \in V_i$, which implies $Tv \in \bigcap_{i=1}^m V_i$, so $\bigcap_{i=1}^m V_i$ is invariant under T .

- 4 Prove or give a counterexample: If V is finite-dimensional and U is a subspace of V that is invariant under every operator on V , then $U = \{0\}$ or $U = V$.

By contradiction, suppose U is a subspace of V s.t. $U \neq \{0\}$ and $U \neq V$. Let v_1, \dots, v_m a basis of U , and $u_1, \dots, u_m, v_1, \dots, v_{m-m}$ a basis of V . We can define an operator T s.t.

$Tv_1 = v_1$, and T is the identity for every other vectors in the base.

Then U is not invariant under T .

- 5 Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues

of T .

$$(-3y, x) = \lambda(x, y) \Rightarrow \begin{cases} \lambda x = -3y \\ \lambda y = x \end{cases} \Rightarrow \begin{cases} \lambda^2 y = -3y \\ \lambda y = x \end{cases} \Rightarrow \begin{cases} \lambda^2 = -3 \\ \lambda y = x \end{cases} \text{ in } \mathbb{R}^2$$

\Rightarrow no eigenvalues of T .

- 6 Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(w, z) = (z, w)$. Find all eigenvalues and eigenvectors of T .

$$(z, w) = \lambda(w, z) \Rightarrow \begin{cases} z = \lambda w \\ w = \lambda z \end{cases} \Rightarrow \begin{cases} z = \lambda^2 z \\ w = \lambda z \end{cases} \Rightarrow \begin{cases} \lambda^2 = 1 \Rightarrow \lambda = 1 \text{ or } \lambda = -1 \\ w = \lambda z \end{cases}$$

\Rightarrow 2 eigenvalues, 1 with e.v.e. $\text{span}(1, 1)$, and -1 with e.v.e. $\text{span}(1, -1)$

- 7 Define $T \in \mathcal{L}(\mathbb{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T .

$$(2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3) \Rightarrow \begin{cases} 2z_2 = \lambda z_1 \\ 0 = \lambda z_2 \\ 5z_3 = \lambda z_3 \end{cases} \Rightarrow \begin{cases} \lambda z_1 = 2z_2 \\ \lambda z_2 = 0 \\ 5z_3 = \lambda z_3 \end{cases}$$

$$\Rightarrow (\lambda = 0 \text{ and } \begin{cases} z_2 = 0 \\ z_3 = 0 \\ z_1 \in \mathbb{F} \end{cases}) \text{ or } (\begin{cases} z_2 = 0 \\ z_1 = 0 \\ 5z_3 = \lambda z_3 \end{cases} \Rightarrow \lambda = 5)$$

E.v.e.: $\lambda = 0$ with e.v.e. $\text{span}(1, 0, 0)$

$\lambda = 5$ with e.v.e. $\text{span}(0, 0, 1)$

- 8 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that if λ is an eigenvalue of P , then $\lambda = 0$ or $\lambda = 1$.

Let $\lambda \in \overline{\mathbb{N}}$ an e.v.a. of P , with $v \in V$ an associated e.v.e.

$$P^2 v = P(Pv) = P(\lambda v) = \lambda Pv = \lambda^2 v$$

$$P^2 v = Pv \text{ (as } P = P^2)$$

$$= \lambda v$$

$$\Rightarrow \lambda^2 v = \lambda v \Rightarrow \lambda(\lambda v - v) = 0 \Rightarrow \lambda = 0 \text{ or } (\lambda v = v \Rightarrow \lambda = 1)$$

$$\begin{aligned} 1) v &= \lambda v \Leftrightarrow v - \lambda v = 0 \\ &\Rightarrow \lambda v = \lambda v \Rightarrow \lambda(\lambda v - v) = 0 \Rightarrow \lambda = 0 \text{ or } (\lambda v = v \Rightarrow \lambda = 1) \end{aligned}$$

- 9 Define $T: \mathcal{P}_k(\mathbb{R}) \rightarrow \mathcal{P}_k(\mathbb{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T .

$p' = \lambda p$. $\deg p' = \deg p - 1$, except if p is constant, in which case $p' = 0$. Therefore, T has eva. of 0 with e.v.e. the non-zero constant polynomials.

- 10 Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$ by $(Tp)(x) = xp'(x)$ for all $x \in \mathbb{R}$. Find all eigenvalues and eigenvectors of T .

$$\begin{aligned} Tp = \lambda p &\Rightarrow (Tp)(x) = (\lambda p)(x) \quad \forall x \in \mathbb{R} \\ &\Rightarrow xp'(x) = \lambda p(x) \\ &\Rightarrow x \sum_{k=1}^4 a_k x^k = \lambda \sum_{k=0}^4 a_k x^k \\ &\Rightarrow \sum_{k=1}^4 a_k x^k = \sum_{k=0}^4 \lambda a_k x^k \\ &\Rightarrow a_0 = 0, a_1 = \lambda a_1, 2a_2 = \lambda a_2, 3a_3 = \lambda a_3, 4a_4 = \lambda a_4 \end{aligned}$$

E.v.a. with associated e.v.e. : $\begin{cases} k \in \{0, 1, 2, 3, 4\} \text{ (e.v.a)} \\ p_k(x), \text{ with } p_k(x) = a_k x^k, a_k \in \mathbb{R} \end{cases}$

- 11 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $a \in \mathbb{F}$. Prove that there exists $\delta > 0$ such that $T - \lambda I$ is invertible for all $\lambda \in \mathbb{F}$ such that $0 < |a - \lambda| < \delta$.

T has at most $m = \dim V$ eigenvalues. If there is none, then any $\delta > 0$ can be chosen s.t. $T - \lambda I$ is invertible, as $T - \lambda I$ will be equivalent to λ is not an eigenvalue of T .

Let A the set of e.v.a. of T . Suppose $A \neq \emptyset$.

Let $\delta = \min_{\lambda \in A} |\alpha - \lambda|$. Let $\lambda \in \mathbb{F}$ s.t. $0 < |\alpha - \lambda| < \delta$.

By definition of δ , λ is not an e.v.a. of T , therefore

$T - \lambda I$ is not invertible. If $\alpha \in A$, any value of $\delta > 0$ works.

- 12 Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V . Define $P \in \mathcal{L}(V)$ by $P(u + w) = u$ for each $u \in U$ and each $w \in W$. Find all eigenvalues and eigenvectors of P .

Let $v = u + w \in V$, $u \in U$, $w \in W$

$$Pv = \lambda v \Rightarrow u = \lambda(u + w) \Rightarrow \underbrace{\frac{u - \lambda u}{\in U}}_{=0} = \underbrace{\frac{w}{\in W}}_{\neq 0}$$

$$\Rightarrow u - \lambda u = w = 0$$

$$\Rightarrow \begin{cases} u - \lambda u = 0 \\ w = 0 \end{cases} \Rightarrow \begin{cases} u = \lambda u \\ w = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 1 \\ w = 0 \end{cases}$$

E.v.a.: 1, with e.v.e. U .

- 13 Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.
- Prove that T and $S^{-1}TS$ have the same eigenvalues.
 - What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

a) Let λ e.v.a. of T . $Tv = \lambda v \Rightarrow S^{-1}Tv = \lambda S^{-1}v$. $v \in V$ and S invertible implies $\exists w \in V$ s.t. $Sw = v$. This yields: $S^{-1}T(Sw) = \lambda S^{-1}(Sw)$
 $\Rightarrow S^{-1}TSw = \lambda w \Rightarrow \lambda$ e.v.a. of S

• Let λ e.v.a. of $S^{-1}TS$. $S^{-1}TSv = \lambda v$. $\exists w \in V$ s.t. $v = S^{-1}w$
 $\Rightarrow S^{-1}Tw = \lambda S^{-1}w \Rightarrow SS^{-1}Tw = \lambda SS^{-1}w \Rightarrow Tw = \lambda w$
 $\Rightarrow \lambda$ e.v.a. of T

b) v.e.w. of T w.r.t $\lambda \Leftrightarrow$ Sv e.v.e. of $S^{-1}TS$ w.r.t λ

- 14 Give an example of an operator on \mathbb{R}^4 that has no (real) eigenvalues.

$$\begin{aligned} T(\underbrace{\alpha_1, \alpha_2, \alpha_3, \alpha_4}_{\text{re } \in \mathbb{R}^4}) &= (-\alpha_2, \alpha_1, 0, 0) = \lambda (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \\ \Rightarrow \begin{cases} -\alpha_2 = \lambda \alpha_1 \\ \alpha_1 = \lambda \alpha_2 \\ \alpha_3 = 0 \\ \alpha_4 = 0 \end{cases} &\Rightarrow \begin{cases} -\alpha_2 = \lambda^2 \alpha_2 \\ \alpha_1 = \lambda \alpha_2 \\ \alpha_3 = 0 \\ \alpha_4 = 0 \end{cases} \Rightarrow \begin{cases} \alpha_2(\lambda^2 + 1) = 0 \\ \alpha_1 = \lambda \alpha_2 \\ \alpha_3 = 0 \\ \alpha_4 = 0 \end{cases} \end{aligned}$$

If $\alpha_2 = 0$, $\alpha_1 = 0$.

If $\lambda^2 + 1 = 0$, then $\lambda^2 = -1$, which has no real solution.

\Rightarrow No real eigenvalues.

- 15 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T^* \in \mathcal{L}(V^*)$.

λ e.v.a. of T
 $\text{null}(T - \lambda I) \neq 0 \Leftrightarrow \text{range}(T - \lambda I) \neq V \Leftrightarrow \exists \psi \in V^*, \psi \neq 0$ s.t. $\forall v \in \text{range}(T - \lambda I), ((T - \lambda I)^*(\psi))_v = 0$
 $\Leftrightarrow \text{null}((T - \lambda I)^*(\psi)) \neq 0$
 $\Leftrightarrow \text{null}((T^* - \lambda I)(\psi)) \neq 0$
 $\Leftrightarrow \text{id. in } V^*$
 $\Leftrightarrow \exists \psi \in V^*, \psi \neq 0$ s.t. $\forall v \in V, ((T^* - \lambda I)(\psi))_v = 0$ (by taking $\psi_{V_i} = 0$ for V_i 's
 choosing a basis of $\text{range}(T - \lambda I)$ to form a basis of V)
 $\Leftrightarrow \text{null}(T^* - \lambda I) \neq \{0\}$
 $\Leftrightarrow \lambda$ e.v.a. of T^*

- 16 Suppose v_1, \dots, v_n is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T , then

$$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| : 1 \leq j, k \leq n\},$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row j , column k of the matrix of T with respect to the basis v_1, \dots, v_n .

See Exercise 19 in Section 6A for a different bound on $|\lambda|$.

Let λ e.v.a. of T , with $v \in V$ e.v.e., $v = \sum_{i=1}^n a_i v_i$, $v \neq 0$.

$$Tv = \sum_{i=1}^n a_i T v_i = \sum_{i=1}^n a_i \sum_{j=1}^m M(T)_{ij} v_j = \lambda v = \lambda \sum_{i=1}^n a_i v_i$$

$$\Rightarrow \left(\sum_{i=1}^n a_i M(T)_{i,j} \right) = \lambda a_j \quad (1)$$

$$\begin{aligned}
& \Rightarrow \left\{ \begin{array}{l} \sum_{i=1}^m a_i M(\tau)_{ii} = \lambda a_1 (1) \\ \vdots \\ \sum_{i=1}^m a_i M(\tau)_{im} = \lambda a_m \end{array} \right. \\
& \Rightarrow \sum_{i=1}^m a_i M(\tau)_{ik} = \lambda a_k, \text{ with } k = \arg \max_{i=1 \dots m} |a_i| \\
& \Rightarrow \left| \sum_{i=1}^m a_i M(\tau)_{ik} \right| = |\lambda a_k| \Rightarrow \sum_{i=1}^m |a_i| |M(\tau)_{ik}| \geq |\lambda| |a_k| \\
& \Rightarrow m |a_k| \max_{\substack{i=1 \dots m \\ j=1 \dots m}} |M(\tau)_{ij}| \geq |\lambda| |a_k| \\
& \Rightarrow |\lambda| \leq m \max_{\substack{i=1 \dots m \\ j=1 \dots m}} |M(\tau)_{ij}| \quad (\text{as } |a_k| \neq 0, \text{ since } v \neq 0)
\end{aligned}$$

17 Suppose $F = \mathbb{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{R}$. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of the complexification T_C .

See Exercise 33 in Section 3B for the definition of T_C .

33 Suppose that V and W are real vector spaces and $T \in \mathcal{L}(V, W)$. Define $T_C : V_C \rightarrow W_C$ by

$$T_C(u + iv) = Tu + iTv$$

\Leftarrow : let λ e.v.a. of T , with e.v.e. $v \in V, v \neq 0$.

$$T_C(v + i0) = T v = \lambda v = \lambda(v + i0) \Rightarrow \lambda \text{ e.v.a. of } T_C$$

\Leftarrow : let λ e.v.a. of T_C , with e.v.e. $(v, w) \in V_C$.

$$T_C(v + iw) = T v + iTw = \lambda v + i\lambda w \stackrel{\lambda \in \mathbb{R}}{=} \begin{cases} T v = \lambda v \\ T w = \lambda w \end{cases} \Rightarrow v \text{ e.v.a. of } T$$

18 Suppose $F = \mathbb{R}$, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{C}$. Prove that λ is an eigenvalue of the complexification T_C if and only if λ is an eigenvalue of $T_{\bar{C}}$.

\Leftarrow : let λ e.v.a. of T_C , with e.v.e. $v + iw \in V$

$$T_C(v + iw) = T v + iTw = \lambda(v + iw) \Leftrightarrow T v - iTw = \overline{\lambda}(v - iw)$$

$$\Leftrightarrow T_C(v - iw) = \overline{\lambda}(v - iw) \Leftrightarrow \overline{\lambda} \text{ e.v.a. of } T_C$$

19 Show that the forward shift operator $T \in \mathcal{L}(F^\infty)$ defined by

$$T(z_1, z_2, \dots) = (0, z_1, z_2, \dots)$$

has no eigenvalues.

Suppose $\exists \lambda \in F$, s.t. $\exists z = (z_1, z_2, \dots) \in F^\infty$ with $Tz = \lambda z$ and $z \neq 0$

$$\Rightarrow (0, z_1, z_2, \dots) = \lambda(z_1, z_2, \dots) \Rightarrow \begin{cases} 0 = \lambda z_1 \\ z_2 = \lambda z_{k+1} \text{ for } k \geq 1 \end{cases} \Rightarrow \lambda = 0 \text{ or } z_1 = 0$$

$$z_1 = 0 \Rightarrow z_2 = z_3 = \dots = 0, \text{ but } z \neq 0 \perp$$

$$\lambda = 0 \Rightarrow z_1 = z_2 = \dots = 0, \text{ but } z \neq 0 \perp \Rightarrow T \text{ has no e.v.a.}$$

20 Define the backward shift operator $S \in \mathcal{L}(F^\infty)$ by

$$S(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

(a) Show that every element of F is an eigenvalue of S .

(b) Find all eigenvectors of S .

a) Let $\lambda \in F$. Let $z = (\lambda, \lambda^2, \lambda^3, \dots) \in F^\infty$.

$$S(\lambda, \lambda^2, \lambda^3, \dots) = (\lambda^2, \lambda^3, \dots) = \lambda(\lambda, \lambda^2, \lambda^3, \dots) \Rightarrow \lambda \text{ e.v.a. of } S$$

$$\text{b) } S(z_1, z_2, \dots) = \lambda(z_1, \dots, z_1) \Rightarrow (z_2, z_3, \dots) = \lambda(z_1, z_2, \dots)$$

$$\Rightarrow \begin{cases} z_2 = \lambda z_1 \\ z_3 = \lambda z_2 \\ \vdots \end{cases} \Rightarrow \begin{cases} z_2 = \lambda z_1 \\ z_3 = \lambda^2 z_1 \\ \vdots \end{cases} \Rightarrow z = (z_1, \lambda z_1, \lambda^2 z_1, \dots) \quad (1) \\ = z_1(\lambda, \lambda^2, \dots) \text{ for } z_1 \in F$$

and $\dots \dots \dots \perp$ in each $\lambda \in F$ (easy to show)

\Rightarrow e.v.e. of S are open $(\lambda, \lambda^2, \dots)$ for each $\lambda \in \mathbb{F}$ (say to show the other inclusion of the equality).

21 Suppose $T \in \mathcal{L}(V)$ is invertible.

- (a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1} .
- (b) Prove that T and T^{-1} have the same eigenvectors.

$$\begin{aligned} \text{a)} \text{ Let } \lambda \text{ e.v.a. of } T \text{ with e.v.e. } v &\Leftrightarrow Tv = \lambda v \\ &\Leftrightarrow T^{-1}Tv = \lambda T^{-1}v \\ &\Leftrightarrow \lambda^{-1}v = T^{-1}v \quad (\lambda \neq 0) \\ &\Leftrightarrow \lambda^{-1} \text{ e.v.a. of } T^{-1}, \text{ with e.v.e. } v. \end{aligned}$$

b) We proved in a) that all of T 's nonzero e.v.a. are inverses of T^{-1} 's and reciprocally, and that they share the same e.v.e..
 Furthermore $\lambda = 0$ is not an e.v.a. of T or T^{-1} , i.e. $\exists v \neq 0$ s.t. $Tv = 0$, meaning T would not be injective and hence not invertible.

22 Suppose $T \in \mathcal{L}(V)$ and there exist nonzero vectors u and w in V such that

$$Tu = 3w \quad \text{and} \quad Tw = 3u.$$

Prove that 3 or -3 is an eigenvalue of T .

$$\begin{aligned} Tu + Tw &= 3w + 3u \\ \Rightarrow T(u+w) &= 3(u+w) \\ \Rightarrow 3 \text{ is an e.v.a. of } T \text{ if } u+w \neq 0 &\left| \begin{array}{l} Tu - Tw = 3w - 3u \\ \Rightarrow T(u-w) = -3(u-w) \\ \Rightarrow -3 \text{ is an e.v.a. of } T \text{ if } u-w \neq 0 \end{array} \right. \\ u+w \text{ and } u-w \text{ cannot both be } 0 \text{ since } u \neq 0 \text{ and } w \neq 0, \text{ hence } 3 \text{ or } -3 \text{ is e.v.a. of } T. \end{aligned}$$

23 Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

$$\begin{aligned} \lambda \text{ e.v.a. of } ST \text{ with e.v.e. } v \neq 0 &\Rightarrow STv = \lambda v \quad (1) \\ &\Rightarrow TS(Tv) = \lambda T v \end{aligned}$$

If $Tv \neq 0$, then λ is an e.v.a. of TS with e.v.e. Tv .

If $Tv = 0$, (1) $\Rightarrow STv = 0 \Rightarrow \lambda = 0$, 0 e.v.a. of ST .

We then have to show 0 is also an e.v.a. of TS .

Either $\exists w \in V$ s.t. $Sw = v$, or S is not injective, meaning S is not injective since V is finite dimensional. In the first case, it implies $\exists w \in V$ s.t. $TSw = 0$, hence 0 is an e.v.a. of TS . In the second case, it implies $\exists v, w \in V$ s.t. $Sw = Sv$, $v \neq w$, so $S(v-w) = 0$, with $v-w \neq 0$. Then $(T-S)(v-w) = T(S(v-w)) = 0$, implying 0 is an e.v.a. of TS .

24 Suppose A is an n -by- n matrix with entries in \mathbb{F} . Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $Tx = Ax$, where elements of \mathbb{F}^n are thought of as n -by-1 column vectors.

- (a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T .
- (b) Suppose the sum of the entries in each column of A equals 1. Prove that 1 is an eigenvalue of T .

$$\begin{aligned} \text{a)} \quad T(1_{\dots 1}) &= A(1_{\dots 1}) = \left(\sum_{i=1}^n a_{1i} \dots \sum_{i=1}^n a_{ni} \right) = (1_{\dots 1}) \\ &\Rightarrow 1 \text{ e.v.a. of } T \text{ with e.v.e. } (1_{\dots 1}) \end{aligned}$$

b) By definition $A = M(T)$ for one basis of \mathbb{F}^n .
 We have $M(T') = M(T)^T$, hence $T'^T = A^T$. The sum of entries in each row of A^T equals 1,

$$\text{e.g.: } \lambda \text{ e.v.a. of } T \Leftrightarrow \lambda \text{ e.v.a. of } T' \in \mathcal{L}(V')$$

b) By definition $A = M(T)$ for some basis of \mathbb{F}^n .

We have $M(T') = M(T)^T$, hence $T'v = A^T v$. The no. of entries in each row of A^T equals 1, hence wlog v_1 , 1 is an e.v.a. of T' . By ex. 15, 1 is also an e.v.a. of T .

- 25 Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T . Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

Suppose $Tu = \lambda u$, $Tw = \mu w$, $T(u+w) = \xi(u+w)$

$$T(u+w) = \xi(u+w) \Rightarrow \xi u + \xi w = \lambda u + \mu w \Rightarrow (\xi - \lambda)u + (\xi - \mu)w = 0$$

$$Tu + Tw = \lambda u + \mu w$$

If u and w are e.v.e. to different e.v.a., $\xi - \lambda = 0$ and $\xi - \mu = 0$, which $\lambda = \mu$: contradiction.

This implies u and w are e.v.e. to the same e.v.a. (and $u+w$ too).

- 26 Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T . Prove that T is a scalar multiple of the identity operator.

Assume T has two e.v.a. $\lambda, \mu \in \mathbb{F}$, s.t. $\lambda \neq \mu$: $Tu = \lambda u$, $Tw = \mu w$. $u+w$ or $u-w$ is an e.v.e., and $-w$ has the same e.v.a. as w . Therefore wlog previous exercise's result, u and w have the same e.v.a.. This is true for any two e.v.e. of T , so any nonzero vector of V is an e.v.a. of T . Thus $\forall v \in V, T_v = \lambda v, \lambda \in \mathbb{F} \Rightarrow T = \lambda I$

- 27 Suppose that V is finite-dimensional and $k \in \{1, \dots, \dim V - 1\}$. Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V of dimension k is invariant under T . Prove that T is a scalar multiple of the identity operator.

- 28 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has at most $1 + \dim \text{range } T$ distinct eigenvalues.

Let $\lambda_1, \dots, \lambda_m$ distinct eigenvalues of T , with e.v.e. $v_1, \dots, v_m, v_1, \dots, v_m$ linearly independent.

$Tv_k = \lambda_k v_k \Rightarrow T^{-1} \lambda_k v_k = v_k$ if $\lambda_k \neq 0$. If all e.v.a. are non-zero, then T has m distinct e.v.a., as v_1, \dots, v_m are linearly independent. If 0 is a e.v.a. of T , T has at most $m-1$ non-zero e.v.a. (the 0 e.v.a.).

- 29 Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ and $-4, 5$, and $\sqrt{7}$ are eigenvalues of T . Prove that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (-4, 5, \sqrt{7})$.

T has 3 e.v.a.: $-4, 5$ and $\sqrt{7}$, and 0 is not among them, so T is invertible. We can show this implies $T - 9I$ is also invertible. Let $v \in \text{null } T - 9I : (T - 9I)v = 0 \Rightarrow Tv = 9v \Rightarrow v = 0$ as 9 is not an e.v.a. of $T \Rightarrow \text{null } T - 9I = \{0\} \Rightarrow T - 9I$ invertible $\Rightarrow \exists w \in V$, s.t. $(T - 9I)w = (-4, 5, \sqrt{7})$ $\Rightarrow Tw - 9w = (-4, 5, \sqrt{7})$.

- 30 Suppose $T \in \mathcal{L}(V)$ and $(T - 2I)(T - 3I)(T - 4I) = 0$. Suppose λ is an eigenvalue of T . Prove that $\lambda = 2$ or $\lambda = 3$ or $\lambda = 4$.

Let λ e.v.a. of T , with e.v.e. v : $Tv = \lambda v$

$$(T - 2I)(T - 3I)(T - 4I)v = 0 \Rightarrow (T - 2I)v = 0 \text{ or } (T - 3I)v = 0 \text{ or } (T - 4I)v = 0 \\ \Rightarrow T_2v = 2v \text{ or } T_3v = 3v \text{ or } T_4v = 4v \\ \Rightarrow \lambda = 2 \text{ or } \lambda = 3 \text{ or } \lambda = 4 \text{ (as } v \neq 0)$$

- 31 Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^4 = -I$.

We can define T s.t. it "applies" a rotation of $\frac{\pi}{4}$ degrees to vectors.

$$T(v_1, v_2) = (v_1 \cos \frac{\pi}{4} - v_2 \sin \frac{\pi}{4}, v_1 \sin \frac{\pi}{4} + v_2 \cos \frac{\pi}{4}) \\ = \frac{\sqrt{2}}{2}(-v_2, v_1, v_1 + v_2)$$

$\therefore T^4 = -I$.

$$\begin{aligned}
T(z_1, z_2) &= z_1 - z_2 + z_1 \cdot z_2 = z_1 - z_2 + \frac{\sqrt{2}}{2}(z_1 - z_2, z_1 + z_2) \\
T^2(z_1, z_2) &= \frac{1}{2}(z_1 - z_2 - (z_1 + z_2), z_1 - z_2 + z_1 + z_2) = \frac{1}{2}(-2z_2, 2z_1) = (-z_2, z_1) \quad (\text{rotation by } \frac{\pi}{2}) \\
T^3(z_1, z_2) &= \frac{\sqrt{2}}{2}(-z_2 - z_1, -z_2 + z_1) \\
T^4(z_1, z_2) &= \frac{1}{2}(-z_2 - z_1 - (-z_2 + z_1), -z_2 - z_1 + (-z_2 + z_1)) = \frac{1}{2}(-2z_1, -2z_2) = (-z_1, -z_2) \\
&= -(z_1, z_2)
\end{aligned}$$

32 Suppose $T \in \mathcal{L}(V)$ has no eigenvalues and $T^4 = I$. Prove that $T^2 = -I$.

T has no e.v.a. $\Rightarrow 0$ is not an e.v.a. of $T \Rightarrow T$ is invertible.

$$T^4 = I \Rightarrow T^{-2} \cdot T^4 = T^{-2} \cdot I \Rightarrow T^2 = T^{-2} \Rightarrow T = I \text{ or } T = -I$$

I has 1 as an e.v.a., therefore $T = -I$.

33 Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

- (a) Prove that T is injective if and only if T^m is injective.
- (b) Prove that T is surjective if and only if T^m is surjective.

a) " Suppose T is injective. Let $v \in \ker T^m$.

$$T^m v = T(T^{m-1} v) = 0 \Rightarrow T^{m-1} v = 0, \text{ as } T \text{ is injective.}$$

Repeating this procedure $m-1$ times yield $Tv = 0 \Rightarrow v = 0$.

" Suppose T^m is injective. Let $v \in \ker T$.

$$Tv = 0 \Rightarrow T^2 v = 0 \Rightarrow \dots \Rightarrow T^m v = 0 \Rightarrow v = 0, \text{ as } T^m \text{ is injective.}$$

b) T injective $\Leftrightarrow T$ injective $\Leftrightarrow T^m$ injective $\Leftrightarrow T^m$ injective

34 Suppose V is finite-dimensional and $v_1, \dots, v_m \in V$. Prove that the list v_1, \dots, v_m is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that v_1, \dots, v_m are eigenvectors of T corresponding to distinct eigenvalues.

\Leftarrow : It is immediate using result 5.11.

\Rightarrow : Suppose v_1, \dots, v_m are linearly independent. This list can be completed to form v_1, \dots, v_n a basis of V .

Define $T \in \mathcal{L}(V)$ by: $Tv_i = \lambda_i v_i$, with $\lambda_1, \dots, \lambda_m = 1, \dots, m$

$\Rightarrow v_1, \dots, v_n$ are eigenvectors of T corresponding to distinct e.v.a..

35 Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on \mathbb{R} .

Hint: Let $V = \text{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$, and define an operator $D \in \mathcal{L}(V)$ by $Df = f'$. Find eigenvalues and eigenvectors of D .

Let $V = \text{span}(e^{\lambda_1 x}, \dots, e^{\lambda_n x})$. Let $D \in \mathcal{L}(V)$, $Df = f'$. $D e^{\lambda_i} = \lambda_i e^{\lambda_i}$ ($D(e^{\lambda_i}) = e^{\lambda_i x}$)

Eigenvalues of D are any $\lambda \in \mathbb{R}$, with associated e.v.e. e^λ . There exists an operator D

s.t. $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ are e.v.e. of D corresponding to distinct e.v.a. (as $\lambda_1, \dots, \lambda_n$ are distinct).

Therefore using previous exercise's result, $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent.

36 Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct positive numbers. Prove that the list $\cos(\lambda_1 x), \dots, \cos(\lambda_n x)$ is linearly independent in the vector space of real-valued functions on \mathbb{R} .

Let $V = \text{span}(\cos(\lambda_1 x), \dots, \cos(\lambda_n x))$. Let $D^2 \in \mathcal{L}(V)$, $D^2 f = f''$. $D^2 \cos(\lambda_i) = -\lambda_i^2 \cos(\lambda_i)$

Eigenvalues of D are any $-\lambda^2 \in \mathbb{R}$ with associated e.v.e. $\cos \lambda$. $\lambda_1, \dots, \lambda_n$ is a

list of distinct positive numbers, meaning $-\lambda_i^2$'s are distinct as well. There

exists an operator s.t. $\cos \lambda_1, \dots, \cos \lambda_n$ are e.v.e. of D^2 corresponding to distinct e.v.a.. Therefore using ex 34's result, $\cos \lambda_1, \dots, \cos \lambda_n$ is linearly independent.

37 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(S) = TS$$

for each $S \in \mathcal{L}(V)$. Prove that the set of eigenvalues of T equals the set of eigenvalues of \mathcal{A} .

\Leftarrow : Let λ e.v.a. of T , with e.v.e. v . Let s, v_1, \dots, v_{n-1} basis of V . (as $V \neq 0$ and V finite dim.)

Let $S \in \mathcal{L}(V)$ s.t. $sv = v$ and $sv_i = 0 \forall i = 1, \dots, n-1$.

$$\stackrel{\sim}{\rightarrow}$$

\subseteq : Let λ e.v.a. of T , with e.v.e. v . Then $Tv = \lambda v$ and $T^{m-1}v = \lambda^{m-1}v$.

Let $S \in L(V)$ s.t. $Sv = v$ and $Sv_i = 0 \forall i=1 \dots m-1$.

Let $w \in V$, $w = av + \sum_{i=1}^{m-1} a_i v_i$.

$$(T(S))(w) = (TS)(w) = Tav = aTv = a\lambda v = \lambda S(av) = \lambda Sw$$

$$\Rightarrow \forall w \in V, T(S) = \lambda S \Rightarrow \lambda \text{ e.v.a. of } T$$

\supseteq : Let λ e.v.a. of T , with e.v.e. $S \in L(V)$: $T(S) = \lambda S, S \neq 0$

$$\forall v \in V, (T(S))(v) = (TS)v = \lambda Sv$$

$$\Rightarrow T(Sv) = \lambda(Sv) \text{ (1)}$$

Since $S \neq 0, \exists v \in V$ s.t. $Sv \neq 0$. Using this v in (1) yields that

λ is an eigenvalue of T .

39 Suppose V is finite-dimensional and $T \in L(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T .

\Leftarrow : Let U of $\dim V - 1$ invariant under T . Let $u_1 \dots u_{m-1}$ basis of U , and $u_1 \dots u_{m-1}, v$ basis of V .

- If $Tv \notin U$, then $Tv = \lambda v$ ($\lambda \neq 0$), and $v \neq 0$ since it is a basis of V .

Therefore T has an e.v.a..

- If $Tv \in U$:
 - If T is not injective, then T is not injective, i.e. $\exists w \in V$ s.t. $Tw = 0, w \neq 0$, meaning T has an e.v.a. ($\lambda = 0$).

- If T is injective: suppose $Tv = u + \lambda v, u \in U, v \neq 0$. Since T is injective, $\exists z \in U$ s.t. $Tz = -u + \lambda v$, since $-u + \lambda v \in U$.

$$\Rightarrow T(v+z) = u + \lambda v - u + \lambda v = \lambda(v+z), \text{ with } v+z \neq 0, (\text{as } v = -z \Rightarrow v+z=0)$$

\Rightarrow : Let λ e.v.a. of T , with e.v.e. v .

Let $p(T) = T - \lambda I$. $p(T) \in L(V) \Rightarrow \dim V = \dim \text{null } p(T) + \dim \text{range } p(T)$

If $\dim \text{null } p(T) = 1$, then $\dim \text{range } p(T) = \dim V - 1$, and $\text{range } p(T)$ is invariant under T (5.18). If $\dim \text{null } p(T) = m > 1$, we can add $m-1$ linearly independent elements $w_1 \dots w_{m-1}$ of $\text{null } p(T)$ to a basis of $\text{range } p(T)$

to form a subspace of dimension $\dim V - 1$ that will still be invariant under T as added vectors satisfy: $Tw = \lambda w$

40 Suppose $S, T \in L(V)$ and S is invertible. Suppose $p \in \mathcal{P}(F)$ is a polynomial.

Prove that

$$p(STS^{-1}) = Sp(TS^{-1}).$$

$$p(STS^{-1}) = \sum_{i=0}^n a_i (STS^{-1})^i$$

$$(STS^{-1})^0 = I \quad (STS^{-1})^1 = STS^{-1}$$

$$(STS^{-1})^2 = \underbrace{(STS^{-1})(STS^{-1})}_{I} = ST^2S^{-1}$$

$$(STS^{-1})^i = ST^i S^{-1}$$

$$= \sum_{i=0}^n a_i ST^i S^{-1}$$

$$= S \left(\sum_{i=0}^n a_i T^i \right) S^{-1}$$

$$= S p(T) S^{-1}$$

41 Suppose $T \in L(V)$ and U is a subspace of V invariant under T . Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(F)$.

$$\forall v \in U, p(T)(v) = \left(\sum_{i=0}^n a_i T^i \right)(v) = \sum_{i=0}^n a_i T^i(v). T^i(v) \in U \text{ for any } i.$$

U is invariant under $p(T)$ for every polynomial $p \in \mathcal{P}(\mathbb{F})$.

$$\text{Let } v \in U. \quad p(T)(v) = \left(\sum_{i=0}^m a_i T^i \right) (v) = \sum_{i=0}^m a_i T^i(v). \quad T^i(v) \in U \text{ for any } i.$$

Then $p(T)(v)$ is a linear combination of elements of U , therefore $p(T)(v) \in U$

$\Rightarrow U$ is invariant under $p(T)$.

42 Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$.

- (a) Find all eigenvalues and eigenvectors of T .
- (b) Find all subspaces of \mathbb{F}^n that are invariant under T .

$$a) (x_1, 2x_2, 3x_3, \dots, nx_n) = \lambda(x_1, \dots, x_n) \Rightarrow kx_k = \lambda x_k \text{ for each } k = 1 \dots n$$

$$\lambda = 1 \text{ e.v.a. of } T \text{ with e.v.e. } (x_1, 0, \dots, 0), \lambda = 2 \text{ e.v.a. with e.v.e. } (0, x_2, 0, \dots, 0)$$

For each $k = 1 \dots n$, $\lambda = k$ e.v.a. of T with e.v.e. with $x_k \in \mathbb{F} \setminus \{0\}$, and $x_i = 0$ for $i \neq k$.

b) $\{0\}, V$ and the spans of any combination of the vectors in the standard basis of \mathbb{F}^n .

$$\text{Let } x \in U, x = \sum_{i=1}^n a_i x_i, \quad x_i = (0 \dots 0, \overset{i}{1}, 0 \dots 0)$$

$$Tx = \sum_{i=1}^n a_i T x_i$$

$$= \sum_{i=1}^n a_i \lambda_i x_i$$