5B Exercises

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1 Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

"$\Rightarrow$": let p the minimal polynomial of T^2.

9 e.va. of $T^2 \Rightarrow p(T^2) = (T^2 - 9I)\,q(T^2)$ for some $q(T)$ with degree $\deg(p)-1$

$= (T^2 - (3I)^2)\,q(T^2)$

$= (T-3I)(T+3I)\,q(T^2) = 0$

q has a smaller degree than p, thus there must exist $v \in V$ s.t $q(T^2)v \neq 0$, otherwise p is not the minimal polynomial of T^2. Thus, $(T-3I)(T+3I)v = 0$.

$\Rightarrow (T-3I)v = 0$ or $(T+3I)v = 0 \Rightarrow Tv = 3v$ or $Tv = -3v \Rightarrow$ 3 or -3 is an e.va. of T (since $v \neq 0$ as $q(T^2)v \neq 0$).

"$\Leftarrow$": 3 e.va. of T with e.ve. v: $Tv = 3v \Rightarrow T^2 v = 3Tv \Rightarrow T^2 v = 9v \Rightarrow 9$ e.va. of T^2

Same reasoning with -3. Thus if 3 or -3 e.va. of T, then 9 e.va. of T^2.

2 Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T is either $\{0\}$ or infinite-dimensional.

Suppose U is a subspace of V s.t. U is finite dimensional, and $U \neq \{0\}$.

5.19 $\Rightarrow T|_U$ has an e.va. $\Rightarrow T$ has an e.va., which is not the case.

Hence U is either $\{0\}$ or infinite dimensional.

3 Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is defined by

$$T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n).$$

(a) Find all eigenvalues and eigenvectors of T.

(b) Find the minimal polynomial of T.

The matrix of T with respect to the standard basis of $\mathbf{F}^n$ consists of all 1's.

a) $T(x_1 \dots x_n) = (\sum x_i, \dots, \sum x_i) = \lambda(x_1 \dots x_n) \Rightarrow$ $\sum x_i = \lambda x_1 = \lambda x_2 = \dots = \lambda x_n$

If $\lambda = 0$, $\sum x_i = 0$. E.ve.: $\{(x_1 \dots x_n) \in \mathbb{F}^n : \sum_{i=1}^{n} x_i = 0\} \setminus \{0\}$

If $x_1 = x_2 = \dots = x_n$, $\lambda = n$ E.ve.: $\{(x \dots x) \in \mathbb{F}^n\} \setminus \{0\}$

b) $c_0 I = -T$ no solution

$c_0 I + c_1 T = -T^2 \Rightarrow \begin{pmatrix} c_0 & & 0 \\ & \ddots & \\ 0 & & c_0 \end{pmatrix} + \begin{pmatrix} c_1 & \cdots & c_1 \\ \vdots & & \vdots \\ c_1 & \cdots & c_1 \end{pmatrix} = -\begin{pmatrix} n & \cdots & n \\ \vdots & & \vdots \\ n & \cdots & n \end{pmatrix}$

Choosing $c_0 = 0$ and $c_1 = -n$ solves the system.

Thus the minimal polynomial of T is $-nz + z^2$

4 Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $\alpha \in \mathbf{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

"$\Leftarrow$": let λ e.va. of T s.t $\alpha = p(\lambda)$: $Tv = \lambda v$

$p(T)v = \sum_{i=0}^{n} a_i T^i v = \sum_{i=0}^{m} a_i \lambda^i v = (\sum_{i=0}^{m} a_i \lambda^i) v = p(\lambda) v \Rightarrow \alpha$ e.va. of $p(T)$

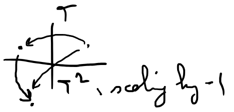
"$\Rightarrow$": let α e.va. of $p(T)$: $p(T)v = \alpha v$.

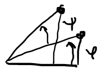
$(p(T) - \alpha I)v = (c \prod_{i=1}^{m} T - \lambda_i) v = 0$ for some $c, \lambda_1 \dots \lambda_m \in \mathbb{C}$

$(p(T) - \alpha I)v = (c\prod_{i=1}^{m} T - \lambda_i)v = 0$ for some $c, \lambda_1 \dots \lambda_m \in \mathbb{C}$

$\Rightarrow \exists i \in \{1 \dots m\}$ s.t $(T - \lambda_i)v = 0 \Rightarrow Tv = \lambda_i v \Rightarrow \lambda_i$ e.va. of T

Also, $p(\lambda_i) - \alpha = 0 \Rightarrow p(\lambda_i) = \alpha$





$\cos(\psi + \varphi)$
$= \cos(\psi)\cos(\varphi) - \sin(\psi)\sin(\varphi)$
$= \cos(\frac{\pi}{2})x_1 - \sin(\frac{\pi}{2})x_2$
$= -x_2$

5 Give an example of an operator on $\mathbf{R}^2$ that shows the result in Exercise 4 does not hold if $\mathbf{C}$ is replaced with $\mathbf{R}$.

Let $p \in \mathcal{P}(\mathbb{R})$ s.t $p(z) = z^2$. Let $T \in \mathcal{L}(\mathbb{R}^2)$: $T(x_1, x_2) = (-x_2, x_1)$

$p(T)(x_1, x_2) = T^2(x_1, x_2) = T(-x_2, x_1) = (-x_1, -x_2) = -1(x_1, x_2) \Rightarrow -1$ e.va. of $p(T)$

Suppose $-1 = p(\lambda)$ for some e.va. of T. Then we would have $\lambda^2 = -1$, but e.va. belong to the field of the vector space, so $\lambda \in \mathbb{R}$ and there is no solution in $\mathbb{R}$ for this equation.

6 Suppose $T \in \mathcal{L}(\mathbf{F}^2)$ is defined by $T(w, z) = (-z, w)$. Find the minimal polynomial of T.

Let $e_1 = (1, 0)$.

$c_0 e_1 + c_1 T e_1 = -T^2 e_1 \Rightarrow (c_0, 0) + (0, c_1) = (1, 0) \Rightarrow \begin{cases} c_0 = 1 \\ c_1 = 0 \end{cases}$

The minimal polynomial of T is $1 + z^2$.

7 (a) Give an example of $S, T \in \mathcal{L}(\mathbf{F}^2)$ such that the minimal polynomial of ST does not equal the minimal polynomial of TS.

(b) Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that if at least one of S, T is invertible, then the minimal polynomial of ST equals the minimal polynomial of TS.

Hint: Show that if S is invertible and $p \in \mathcal{P}(\mathbf{F})$, then $p(TS) = S^{-1}p(ST)S$.

a) $T(x_1, x_2) = (0, x_1)$ $\quad ST(x_1, x_2) = S(0, x_1) = (0, 0)$

$S(x_1, x_2) = (x_1, 0)$ $\quad TS(x_1, x_2) = T(x_1, 0) = (0, x_1)$, matrix with standard base: $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- $ST = 0$, so its minimal polynomial is z.
- $c_0 I = -c_1 TS \Rightarrow \begin{pmatrix} c_0 & 0 \\ 0 & c_0 \end{pmatrix} = -c_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ no solution

$c_0 I + c_1 TS = -TS^2 \Rightarrow c_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + c_1 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} c_0 = 0 \\ c_1 = 0 \end{cases}$

$\Rightarrow$ minimal polynomial of TS is z^2, different from minimal polynomial of ST, z.

b) Let us show that if S is invertible and $\in \mathcal{P}(\mathbb{F})$, $p(TS) = S^{-1}p(ST)S$ $(*)$

$p(TS) = \sum_{i=0}^{m} a_i (TS)^i = \sum_{i=0}^{m} a_i (S^{-1}S)(TS)^i = \sum_{i=0}^{m} a_i S^{-1}(ST)^i S = S^{-1}p(ST)S$

Let q be the minimal polynomial of TS and WLOG S be invertible.

$q(TS) = S^{-1} q(ST) S = 0 \Leftrightarrow q(ST)S = 0$

Let $v \in V \setminus \{0\}$, S^{-1} injective. $\exists w$ s.t $S^{-1}w = v$, $w \neq 0$

$q(ST)Sv = q(ST)SS^{-1}w = q(ST)w = 0 \Leftrightarrow q(ST) = 0$

Suppose $\exists p \in \mathcal{P}(\mathbb{F})$ s.t. $p(ST) = 0$ and p has a lower degree than q.

Then $p(TS) = 0$, meaning q would not be the minimal polynomial of TS.

$\Rightarrow TS$ and ST have the same minimal polynomial.

8 Suppose $T \in \mathcal{L}(\mathbf{R}^2)$ is the operator of counterclockwise rotation by 1°. Find the minimal polynomial of T.

Because $\dim \mathbf{R}^2 = 2$, *the degree of the minimal polynomial of T is at most 2. Thus the minimal polynomial of T is not the tempting polynomial $x^{180} + 1$, even though $T^{180} = -I$.*

$c_0 I = -T$ not possible

$c_0 I + c_1 T = -T^2 \Rightarrow c_0 I + c_1 T = -2T \Rightarrow c_1 = -2$

$\Rightarrow$ minimal polynomial of T is $-2T + T^2$.

9 Suppose $T \in \mathcal{L}(V)$ is such that with respect to some basis of V, all entries of the matrix of T are rational numbers. Explain why all coefficients of the minimal polynomial of T are rational numbers.

Suppose one of the coefficients is irrational. This would imply at least one of the entries of $-T^m$ is irrational, with m the degree of the minimal polynomial, as the sum of an irrational with rationals is irrational.
However, the rationals are closed under addition and multiplication, meaning $\forall m$, $-T^m$ is rational. This would thus lead to a contradiction.

10 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$. Prove that

$$\operatorname{span}(v, Tv, \dots, T^m v) = \operatorname{span}(v, Tv, \dots, T^{\dim V - 1} v)$$

for all integers $m \geq \dim V - 1$.

Since $m \geq \dim V - 1$, the "$\supseteq$" way of the equality is trivial.
"$\subseteq$": let $p \in \mathcal{P}(\mathbb{F})$ the minimal polynomial of T, of degree $m \leq \dim V$.

$p(T)v = 0 \Rightarrow \sum_{i=0}^{m-1} a_i T^i v = -T^m v \Rightarrow T^m v \in \operatorname{span}(v, Tv, \dots, T^{m-1} v) \subseteq \operatorname{span}(v, Tv, \dots, T^{\dim V - 1} v)$

By applying T on both sides of the equation: $\sum_{i=1}^{m} a_{i-1} T^i v = -T^{m+1} v$

$\Rightarrow T^{m+1} v \in \operatorname{span}(Tv, \dots, T^{\dim V} v) \subseteq \operatorname{span}(v, Tv, \dots, T^{\dim V - 1} v)$

By repeating this process, $T^k v \in \operatorname{span}(v, Tv, \dots, T^{\dim V - 1} v)$ $\forall k$.
Therefore $\operatorname{span}(v, Tv, \dots, T^m v) \subseteq \operatorname{span}(v, Tv, \dots, T^{\dim V - 1} v)$ $\forall m$,
in particular for $m \geq \dim V - 1$

12 Define $T \in \mathcal{L}(\mathbf{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$. Find the minimal polynomial of T.

From 5A ex 42., T has n real e.va.: $1, 2, \dots, n$
Therefore its minimal polynomial is: $(T-1)(T-2)\dots(T-n)$

13 Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbf{F})$. Prove that there exists a unique $r \in \mathcal{P}(\mathbf{F})$ such that $p(T) = r(T)$ and $\deg r$ is less than the degree of the minimal polynomial of T.

Let q minimal polynomial: $q(T) = 0$, of degree m: $\sum_{i=0}^{m-1} c_i T^i = -T^m$

$\Rightarrow T^m \in \operatorname{span}(I, T, \dots, T^{m-1})$ (subspace of $\mathcal{L}(V)$)

$\sum_{i=1}^{m} c_{i-1} T^i = -T^{m+1} \Rightarrow 0I + \sum_{i=1}^{m} c_{i-1} T^i = -T^{m+1} \Rightarrow T^{m+1} \in \operatorname{span}(I, T \dots T^m) = \operatorname{span}(I, T \dots T^{m-1})$

$\vdots$

$\Rightarrow p(T) \in \operatorname{span}(I, T \dots T^{m-1})$ $\forall p \in \mathcal{P}(\mathbb{F})$

$\Rightarrow \forall p \in \mathcal{P}(\mathbb{F})$, $\exists r \in \mathcal{P}_{m-1}(\mathbb{F})$ s.t $p(T) = r(T)$

Furthermore, $I, T \dots T^{m-1}$ is a basis of $\operatorname{span}(I, T \dots T^{m-1})$, as otherwise q would not be the

$\in \mathbb{F}$ s.t $\sum^{m-1} a_i T^i = -T^{m-1}$, $m-1 < m$).

Furthermore, $I, T \dots T^{m-1}$ is a basis of span$(I, T \dots)$

minimal polynomial of T (there would be $a_0 \dots a_{m-2} \in \mathbb{F}$ s.t $\sum_{i=0}^{m-1} a_i T^i = -T^{m-1}$, $m-1 < m$).

Thus $\exists! \; r \in P_{m-1}(\mathbb{F})$ s.t $p(T) = r(T)$

14 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has minimal polynomial $4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5$. Find the minimal polynomial of T^{-1}.

let $p(z) = 4 + 5z - 6z^2 - 7z^3 + 2z^4 + z^5$.

$p(T) = 0 \Rightarrow 4I + 5T - 6T^2 - 7T^3 + 2T^4 + T^5 = 0$

Applying T^{-1} 5 times

$\Rightarrow I + 2T^{-1} - 7T^{-2} - 6T^{-3} + 5T^{-4} + 4T^{-5} = 0$

$\Rightarrow I + 2T^{-1} - 7(T^{-1})^2 - 6(T^{-1})^3 + 5(T^{-1})^4 + 4(T^{-1})^5 = 0$

This polynomial is T^{-1}'s minimal polynomial, as if one with degree $n < 5$ existed, we could apply T n times to obtain a polynomial satisfying the properties of the minimal polynomial of T despite having a lower degree.

15 Suppose V is a finite-dimensional complex vector space with $\dim V > 0$ and $T \in \mathcal{L}(V)$. Define $f \colon \mathbf{C} \to \mathbf{R}$ by

$$f(\lambda) = \dim \operatorname{range}(T - \lambda I).$$

Prove that f is not a continuous function.

V finite dim. complex v.s. with $\dim V > 0 \Rightarrow T$ has at least one e.va. λ^*, with e.ve. v.

T has a finite amount of e.va.. let $\rho = \underset{\lambda \in \text{e.va. of } T}{\arg\min} |\lambda^* - \lambda|$.

Then there exists a non-empty neighborhood of λ^* s.t. no other e.va. are in it:

$\{\lambda \in \mathbb{C} \text{ s.t } |\lambda^* - \lambda| < |\lambda^* - \rho|\}$.  (if no other e.va., then it is $\mathbb{C}$)

In this neighborhood of λ^*, $f(\lambda) = \dim V - \dim \operatorname{null}(T - \lambda I) = \dim V$ (no e.va.) $\forall \lambda \neq \lambda^*$

However, $f(\lambda^*) = \dim V - \dim \operatorname{null}(T - \lambda I) < \dim V$, meaning $f(\lambda^*) \neq f(\lambda)$.

This contradicts the continuity condition, implying f is not continuous.

17 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda \in \mathbf{F}$. Show that the minimal polynomial of $T - \lambda I$ is the polynomial q defined by $q(z) = p(z + \lambda)$.

$q(T - \lambda I) = p((T - \lambda I) + \lambda I) = p(T) = 0$. q is monic.

Suppose $\exists r \in P(\mathbb{F})$ s.t. $r(T - \lambda I) = 0$ and $\deg r < \deg q = \deg p$ and r monic.

$r(T - \lambda I) = \sum_{i=0}^{m} a_i (T - \lambda I)^i = \sum_{i=0}^{m-1} a_i (T - \lambda I)^i + (T - \lambda I)^m = 0$

$\Rightarrow \sum_{i=0}^{m-1} a_i (T - \lambda I)^i = -T^m + s(T)$, with s a polynomial of degree at most $m-1$

$\Rightarrow \sum_{i=0}^{m-1} a_i (T - \lambda I)^i - s(T) = -T^m$

$\Rightarrow$ Minimal polynomial of T has degree $m < \deg p$, which is a contradiction

$\Rightarrow$ Minimal polynomial of T has degree $m < \deg p$, which is a contradiction.

There is no polynomial r s.t. $r(T - \lambda I) = 0$ and $\deg r < \deg q$, thus q is the minimal polynomial of $T - \lambda I$.

18 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda \in \mathbf{F}\backslash\{0\}$. Show that the minimal polynomial of λT is the polynomial q defined by $q(z) = \lambda^{\deg p} p\left(\frac{z}{\lambda}\right)$.

- $q(\lambda T) = \lambda^m \; p(T) = 0$, $m = \deg p$
- q is monic; indeed, let p_m, q_m be the coefficients of highest degree of p, q.

$$q_m = \lambda^m \left(\frac{\overset{=1}{p_m}}{\lambda}\right)^m = 1$$

- If $\exists r \in \mathcal{P}(\mathbf{F})$ monic s.t. $\deg r < m$, $r(\lambda T) = 0$ we could define $s \in \mathcal{P}(\mathbf{F})$ s.t. $s(z) = \lambda^{-\deg r} r(\lambda z)$ and s would be the minimal polynomial of T, not p, leading to a contradiction. Therefore q is the minimal polynomial of λT.

19 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\mathcal{E}$ be the subspace of $\mathcal{L}(V)$ defined by

$$\mathcal{E} = \{q(T) : q \in \mathcal{P}(\mathbf{F})\}.$$

Prove that $\dim \mathcal{E}$ equals the degree of the minimal polynomial of T.

In ex. 13 we showed $I, T, \dots, T^{n-1}$ with n the degree of the minimal polynomial of T is a basis of this subspace. Therefore its dimension is n.

20 Suppose $T \in \mathcal{L}(\mathbf{F}^4)$ is such that the eigenvalues of T are 3, 5, 8. Prove that $(T - 3I)^2(T - 5I)^2(T - 8I)^2 = 0$.

Let $p(T) = (T - 3I)^2 (T - 5I)^2 (T - 8I)^2$

E.va. of T are zeros of the minimal polynomial of T and reciprocally.

$\Rightarrow$ The minimal polynomial of T can be factorized in the following manner:

$p(T) = (T - 3I)(T - 5I)(T - 8I)\, q(T)$, with $q(T) = T - \lambda I$, $\lambda = 3, 5$ or 8.

Thus $q(T)$ is a polynomial multiple of $p(T)$, meaning $q(T) = 0$ (by 5.29)

22 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if $I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.

"$\Rightarrow$": T invertible $\Rightarrow$ null $T = \{0\}$. $\quad p(z) = \sum_{i=1}^{m} c_i z^i \quad p(T) = 0$.

$\Rightarrow$

Let $p \in \mathcal{P}(\mathbf{F})$ the minimal polynomial of T: $p(z) = \sum_{i=0}^{m} c_i z^i$. Since T is invertible, $c_0 = 0$.

Let $k^* = \arg\min_k \{c^k \neq 0\}$. k^* cannot be 0, as $c_0 = 0$.

$$p(T) = c_{k^*} T^{k^*} + \sum_{i=k^*+1}^{m} c_i T^i = 0 \xRightarrow[\text{sides}]{\text{apply } T^{-k^*} \text{ on both}} c_{k^*} I + \sum_{i=k^*+1}^{m} c_i T^{i-k^*} = 0$$

$$\xRightarrow{c_{k^*} \neq 0} I = \frac{-1}{c_{k^*}} \sum_{i=k^*+1}^{m} c_i T^{i-k^*} . \quad \forall i \in \{k^*+1 \dots m\},\ i - k^* > 0$$

$$\overset{c_{k^*}\neq 0}{\Longrightarrow} I = -\frac{1}{c_{k^*}}\sum_{i=k^*+1}^{m} c_i T^{i-k^*} \quad \forall i \in \{k^*+1 \dots m\},\ i-k^* > 0$$

$$\Rightarrow I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$$

"$\Leftarrow$": Suppose $I \in \operatorname{span}(T, T^2, \dots, T^{\dim V})$.

$$\Rightarrow \exists a_1 \dots a_{\dim V} \text{ s.t } I = \sum_{i=1}^{\dim V} a_i T^i$$

$$= T\left(\sum_{i=1}^{\dim V} a_i T^{i-1}\right)$$

$\Rightarrow T$ is invertible with inverse $\left(\sum_{i=1}^{\dim V} a_i T^{i-1}\right)$ (using 3.68)

23 Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $n = \dim V$. Prove that if $v \in V$, then $\operatorname{span}(v, Tv, \dots, T^{n-1}v)$ is invariant under T.

We can reduce $v, Tv \dots T^{n-1}v$ to a basis of its span: $v, Tv \dots T^m v$, $m \leq n-1$

Let $i \in \{0 \dots m\}$: $T(T^i v) = T^{i+1} v \in \operatorname{span}(v \dots T^m v)$

We know from previous exercises that $T^n v \in \operatorname{span}(v \dots T^{n-1} v)$.

Therefore $\forall i \in \{0 \dots m\}$, $T(T^i v) \in \operatorname{span}(v \dots T^{n-1} v)$: All vectors of a base are mapped to a vector of the space by T, so this space is invariant under T (easy to prove).

24 Suppose V is a finite-dimensional complex vector space. Suppose $T \in \mathcal{L}(V)$ is such that 5 and 6 are eigenvalues of T and that T has no other eigenvalues. Prove that $(T-5I)^{\dim V - 1}(T-6I)^{\dim V - 1} = 0$.

$(T-5I)^{\dim V-1}(T-6I)^{\dim V-1}$ is a polynomial multiple of the minimal polynomial which is $(T-5I)^m(T-6I)^k$ (5.27b)) hence it is equal to 0 (5.29).

29 Show that every operator on a finite-dimensional vector space of dimension at least two has an invariant subspace of dimension two.

Exercise 6 in Section 5C will give an improvement of this result when $\mathbf{F} = \mathbf{C}$.

- Initialization: Let V vector space s.t $\dim V = 2$
  For each $T \in \mathcal{L}(V)$, V is an invariant subspace of V of dimension 2.

- Suppose $2 \leq i \leq k$, any vector space of dimension i has an invariant subspace of dimension 2 under any operator T. Let V vector space of dimension $k+1$.

If T has an e.va. $\lambda \in \mathbb{F}$ then by 3A.39, $\exists U$ subspace of V with $\dim U = \dim V - 1 = k$. This vector space has an invariant subspace of dimension 2 under any operator $T|_U$, hence V has an invariant subspace of dimension 2 under any operator T.

This leaves real vector spaces with an even number of dimensions.

Let V a vector space with an even number of dimensions, and $T \in \mathcal{L}(V)$.

Let $p \in \mathcal{P}(\mathbb{F})$ the minimal polynomial of T. p has the form:

$p(x) = (x^2 + b_1 x + c_1) \dots (x^2 + b_M x + c_M)$

Also, $p(T) = 0 \Rightarrow \exists\, i = 1..M,\ v \in V$ s.t $(T^2 + b_i T + c_i)v = 0$

$\Rightarrow T^2 v = -b_i T v - c_i v \quad (1)$

Tv, v is linearly independent (as T has no e.va, hence no e.ve., hence v not a e.ve.)

and obviously $Tv \in \text{span}(Tv, v)$ and $T^2 v \in \text{span}(Tv, v)$ from (1).

$\Rightarrow Tv, v$ is a 2 dim invariant subspace of V.