1B Exercises

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1 Prove that $-(-v) = v$ for every $v \in V$.

$-v - (-v) = -1v - 1(-v)$ (1.32)

$= -1(v - v)$ (distributivity of $\cdot$ over $+$)

$= -1.0$ (additive inverse)

$= 0$ (1.31)

$\Rightarrow -(-v)$ is the additive inverse of $-v$, so it is v (by unicity, 1.27)

2 Suppose $a \in \mathbf{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Suppose $a \neq 0 \wedge v \neq 0$

$\Rightarrow a^{-1} a v = a^{-1} 0$ ($a \neq 0$)

$\Rightarrow 1v = 0$ (prop of field + 1.31)

$\Rightarrow v = 0$

But we supposed $v \neq 0$.

This implies $a = 0$ or $v = 0$

3 Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

let $x, y \in V$ s.t $v + 3x = w \wedge v + 3y = w$.

let $x, y \in V$ s.t $v + 3x = v + 3y$

$\Rightarrow v + 3x = v + 3y$

$\Rightarrow -v + v + 3x = -v + v + 3y$

$\Rightarrow 3x = 3y$

$\Rightarrow \frac{1}{3} 3x = \frac{1}{3} 3y \Rightarrow \boxed{x = y}$

4 The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in the definition of a vector space (1.20). Which one?

The additive identity requirement is not satisfied,

since: $\nexists!\, x$ s.t $x \in \emptyset$.

5 Show that in the definition of a vector space (1.20), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V.$$

Here the 0 on the left side is the number 0, and the 0 on the right side is the additive identity of V.

(1): $0v = 0 \quad \forall v \in V$

(2): $v + 0 = v \quad \forall v \in V$

(3): conditions of (1.20) that are not (2)

We can show that $(1) \wedge (3) \Leftrightarrow (2) \wedge (3)$

"$\Leftarrow$": already done in (1.30)

"$\Rightarrow$": let $v \in V$.

$v + 0 \overset{(1)}{=} v + 0v \overset{\text{dist.}}{=} (1+0)v = 1v = v$

So both conditions are interchangeable.

6 Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty, -\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \qquad t(-\infty) = \begin{cases} \infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ -\infty & \text{if } t > 0, \end{cases}$$

and

$$\begin{aligned} t + \infty = \infty + t = \infty + \infty &= \infty, \\ t + (-\infty) = (-\infty) + t = (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) = (-\infty) + \infty &= 0. \end{aligned}$$

With these operations of addition and scalar multiplication, is $\mathbf{R} \cup \{\infty, -\infty\}$ a vector space over $\mathbf{R}$? Explain.

No.

$(\infty + \infty) - \infty = \infty - \infty = 0$

$\infty + (\infty - \infty) = \infty + 0 = \infty$

$\Rightarrow$ associativity of $+$ does not hold.

7 Suppose S is a nonempty set. Let V^S denote the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Let $f, g \in V^S$. (We suppose V is a field $+_V$, $\cdot_V$)

$\forall s \in S: (f+g)(s) = f(s) +_V g(s) \in V$

$\forall s \in S, \forall \lambda \in V: (\lambda f)(s) = \lambda \cdot_V f(s) \in V$

We write $+_V, \cdot_V$ as $+, \cdot$.

- Commutativity, associativity hold from

- $0 \in V^S$ defined as $0(s) = 0_V \ \forall s \in S$

$\forall f \in V^S$: $(f+0)(s) = f(s) + 0(s)$
$= f(s) + 0_V = f(s)$
$\implies f + 0 = f$

- let $f \in V^S$. We define $-f \in V^S$ in the following way: $(-f)(s) = -f(s) \ \forall s \in S$

$(f-f)(s) = f(s) - f(s) = 0_V$
$\implies f - f = 0$

- $(1_V f)(s) = 1_V f(s) = f(s) \ \forall s \in S$

- let $v \in V$, $s \in S$

$(v(f+g))(s) = v\,(f+g)(s)$
$= v\,(f(s)+g(s))$
$= vf(s) + vg(s)$ (distrib in field V)
$= (vf)(s) + (vg)(s)$

$\Rightarrow v(f+g) = vf + vg$

$\Rightarrow V^S$ vector field over V

8 Suppose V is a real vector space.

- The *complexification* of V, denoted by V_C, equals $V \times V$. An element of V_C is an ordered pair (u, v), where $u, v \in V$, but we write this as $u + iv$.
- Addition on V_C is defined by
$$(u_1 + iv_1) + (u_2 + iv_2) = (u_1 + u_2) + i(v_1 + v_2)$$
for all $u_1, v_1, u_2, v_2 \in V$.
- Complex scalar multiplication on V_C is defined by
$$(a + bi)(u + iv) = (au - bv) + i(av + bu)$$
for all $a, b \in \mathbf{R}$ and all $u, v \in V$.

Prove that with the definitions of addition and scalar multiplication as above, V_C is a complex vector space.

Think of V as a subset of V_C by identifying $u \in V$ with $u + i0$. The construction of V_C from V can then be thought of as generalizing the construction of $\mathbf{C}^n$ from $\mathbf{R}^n$.

let $x \in V_c$, $x = u_1 + v_1 i$, $u_1, v_1 \in V$
$y \in V_c$, $y = u_2 + v_2 i$, $u_2, v_2 \in V$
$\lambda \in \mathbb{C}$, $\lambda = a + bi$, $a, b \in \mathbb{R}$

$x + y = (\underbrace{u_1 + u_2}_{\in V}) + i(\underbrace{v_1 + v_2}_{\in V}) \Rightarrow x + y \in V_C$

$\lambda x = (\underbrace{au - bv}_{\in V}) + i(\underbrace{av + bu}_{\in V}) \Rightarrow \lambda x \in V_C$

$\Rightarrow$ addition and scalar multiplication closed

• $x+y=(u_1+u_2)+i(v_1+v_2)$
$=(u_2+u_1)+i(v_2+v_1)$ (V vector space)

$=y+x$

• let $z=u_3+iv_3$, $u_3,v_3\in V$

$(x+y)+z=((u_1+u_2)+u_3)+i((v_1+v_2)+v_3)$
$=(u_1+(u_2+u_3))+i(v_1+(v_2+v_3))$
$=x+(y+z)$

• We define $0\in V_c$ to be $0+0i$
(where 0 denotes identity element in V)

$x+0=(u_1+0)+i(v_1+0)$
$=v_1+iv_1=x$

• We define $-x\in V_c$ to be $-u_1+(-v_1)i$

$x-x=(u_1-u_1)+(v_1-v_1)i$
$=0+0i=0$

• $1x=(1+0i)x$
$=(1u_1-0v_1)+(1v_1+0u_1)i$

$=v_1+iv_1=x$

$\cdot\ \lambda(x+y)$

$= \lambda((v_1+v_2)+i(v_1+v_2))$

$= (a(v_1+v_2)-b(v_1+v_2)) + (a(v_1+v_2)+b(v_1+v_2))i$

$= (av_1+av_2-bv_1-bv_2)+(av_1+av_2+bv_1+bv_2)i$

$= \lambda x + \lambda y$