Exercises

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2 Prove that if $w, z \in \mathbf{C}$, then $\big||w| - |z|\big| \leq |w - z|$.

The inequality above is called the ***reverse triangle inequality***.

$$
\begin{aligned}
|\omega - z|^2 &= (\omega - z)(\bar{\omega} - \bar{z}) \\
&= \omega\bar{\omega} + z\bar{z} - z\bar{\omega} - \underbrace{\bar{z}\omega}_{\overline{z\bar{\omega}}} \\
&= |\omega|^2 + |z|^2 - 2\operatorname{Re}(\bar{\omega} z) \\
&\geqslant |\omega|^2 + |z|^2 - 2|\bar{\omega} z| \\
&= |\omega|^2 + |z|^2 - 2|\omega||z| \\
&= (|\omega| - |z|)^2 \\
&= \big||\omega| - |z|\big|^2
\end{aligned}
$$

3 Suppose V is a complex vector space and $\varphi \in V'$. Define $\sigma\colon V \to \mathbf{R}$ by $\sigma(v) = \operatorname{Re}\varphi(v)$ for each $v \in V$. Show that

$$\varphi(v) = \sigma(v) - i\sigma(iv)$$

for all $v \in V$.

$\sigma(v) = \operatorname{Re}\varphi(v)$ by definition.

$\sigma(iv) = \operatorname{Re}\varphi(iv) = -\operatorname{Im}\varphi v$, as $i(a+bi) = ai - b$

$\Rightarrow \sigma(v) - i\sigma(iv) = \operatorname{Re}\varphi(v) + i\operatorname{Im}\varphi(v) = \varphi(v)$

4 Suppose m is a positive integer. Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p = m\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

Let $p(x) = a_0 + a_m x^m$ and $q(x) = b_0 + a_m x^m$, with $a_0 \neq b_0$

$p(x) - q(x) = a_0 - b_0 \neq 0$ and $\deg(p-q) = 0 \neq m$

$\Rightarrow$ not closed under addition $\Rightarrow$ not a subspace of $\mathcal{P}(\mathbb{F})$

5 Is the set

$$\{0\} \cup \{p \in \mathcal{P}(\mathbf{F}) : \deg p \text{ is even}\}$$

a subspace of $\mathcal{P}(\mathbf{F})$?

let $p(x) = a_1 x + a_2 x^2$, $q(x) = b_1 x + a_2 x^2$, $b_1 \neq a_1$

$p(x) - q(x) = (a_1 - b_1)x \neq 0$, $\deg(p - q) = 1$ not even

$\Rightarrow$ not closed under addition $\Rightarrow$ not a subspace of $\mathcal{P}(\mathbb{F})$

6 Suppose that m and n are positive integers with $m \leq n$, and suppose $\lambda_1, \dots, \lambda_m \in \mathbf{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros.

let $p \in \mathcal{P}(\mathbb{F})$, $p(x) = (x - \lambda_1) \dots (x - \lambda_m)^{n-m+1}$

$p(\lambda_1) = \dots = p(\lambda_m) = 0$, $\deg p = n$, no other roots

7 Suppose that m is a nonnegative integer, $z_1, \dots, z_{m+1}$ are distinct elements of $\mathbf{F}$, and $w_1, \dots, w_{m+1} \in \mathbf{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbf{F})$ such that

$$p(z_k) = w_k$$

for each $k = 1, \dots, m+1$.

This result can be proved without using linear algebra. However, try to find the clearer, shorter proof that uses some linear algebra.

let $T \in \mathcal{L}(\mathcal{P}_m(\mathbb{F}), \mathbb{F}^{m+1})$ s.t. $Tp = (p(z_1), \dots, p(z_{m+1}))$

T is a linear map:

- let $p, q \in \mathcal{P}_m(\mathbb{F})$: $T(p+q) = (\underbrace{p+q(z_1)}_{= p(z_1) + q(z_1)} \dots \underbrace{p+q(z_{m+1})}_{= p(z_{m+1}) + q(z_{m+1})}) = Tp + Tq$
- let $\lambda \in \mathbb{F}$: $T(\lambda p) = (\lambda p(z_1) \dots \lambda p(z_{m+1})) = \lambda(p(z_1) \dots p(z_{m+1})) = \lambda Tp$

T is injective:

T is injective:

$Tp = 0 \Rightarrow (p(z_1) \dots p(z_{m+1})) = 0 \Rightarrow \begin{cases} p(z_1) = 0 \\ \vdots \\ p(z_{m+1}) = 0 \end{cases} \Rightarrow$ p has $m+1$ roots, so

if $p \neq 0$, $\deg p \geq m+1$. However, $p \in \mathcal{P}_m(\mathbb{F})$. Hence $p = 0$.

Furthermore, $\dim \mathcal{P}_m(\mathbb{F}) = m+1 = \dim \mathbb{F}^{m+1}$, so T is an isomorphism from $\mathcal{P}_m(\mathbb{F})$ to $\mathbb{F}^{m+1}$, meaning $\forall w \in \mathbb{F}^{m+1}$, $\exists! p \in \mathcal{P}_{m+1}(\mathbb{F})$ s.t. $Tp = w$

$\Leftrightarrow (p(z_1) \dots p(z_{m+1})) = (w_1 \dots w_{m+1})$

8 Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m. Prove that p has m distinct zeros if and only if p and its derivative p' have no zeros in common.

"$\Rightarrow$": Suppose $\exists \lambda \in \mathbb{C}$ s.t. $p(\lambda) = p'(\lambda) = 0$

$p(z) = (z-\lambda) q(z)$

$p'(z) = q(z) + (z-\lambda) q'(z) = (z-\lambda) s(z) \Rightarrow q(z) = (z-\lambda)[s(z) - q'(z)]$

$\Rightarrow p(z) = (z-\lambda)^2 (s(z) - q'(z))$

We can notice $\deg q' = \deg s = \deg s - q' = m-2$, meaning $s - q'$ has at most $m-2$ roots.

$\Rightarrow$ By the fundamental theorem of algebra, p has at most $m-1$ roots

$\Rightarrow$ p does not have m distinct roots.

"$\Leftarrow$": Suppose $\exists \lambda \in \mathbb{C}, q \in \mathcal{P}_{m-2}(\mathbb{C})$ s.t. $p(z) = (z-\lambda)^2 q(z)$

$p(z) = (z-\lambda)^2 q(z)$

$p'(z) = 2(z-\lambda) q(z) + (z-\lambda)^2 q'(z) \Rightarrow p'(\lambda) = 0$

$\Rightarrow$ p and p' have a common root.

9 Prove that every polynomial of odd degree with real coefficients has a real zero.

Let $p \in \mathcal{P}_m(\mathbb{C})$ with real coefficients. Then for each root λ of p, $\bar{\lambda}$ is also a root of p. The multiplicity of root λ is the same as $\bar{\lambda}$. It is obvious in the case the root is real ($\lambda = \bar{\lambda}$). Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ root of p s.t. its multiplicity is 2. We can write p as: $p(z) = (z-\lambda)(z-\bar{\lambda}) q(z)$, where q is also a polynomial in $\mathcal{P}_m(\mathbb{C})$ with real coefficients, and λ is a root of q. This implies $\bar{\lambda}$ is also a root of q and hence has multiplicity 2 too. This process can be repeated for any multiplicity. This implies the number of roots that are not real is even, meaning at least one root of an odd degree polynomial has to be real.

11 Suppose $p \in \mathcal{P}(\mathbf{C})$. Define $q : \mathbf{C} \to \mathbf{C}$ by
$$q(z) = p(z)\,\overline{p(\bar{z})}.$$
Prove that q is a polynomial with real coefficients.

$m = \deg p$

$p(z) = c \prod_{i=1}^{m} (z - \lambda_i)$, $\overline{p(\bar{z})} = \bar{c} \prod (z - \lambda_i) \prod (z - \bar{\lambda}_i)$

$$\begin{aligned} p(z)\overline{p(\bar z)} &= \bar{c}c \prod (z-\lambda_i)(z-\bar{\lambda}_i) \\ &= |c|^2 \prod (z^2 - z(\lambda_i + \bar{\lambda}_i) - \lambda_i\bar{\lambda}_i) \\ &= \underbrace{|c|^2}_{\in \mathbb{R}} \prod (z^2 - z \underbrace{2\operatorname{Re}(\lambda_i)}_{\in \mathbb{R}} - \underbrace{|\lambda_i|^2}_{\in \mathbb{R}}) \end{aligned}$$

This is the factorization of a polynomial with real coefficients.

13 Suppose $p \in \mathcal{P}(\mathbf{F})$ with $p \neq 0$. Let $U = \{pq : q \in \mathcal{P}(\mathbf{F})\}$.
(a) Show ~~that~~ dim $\mathcal{P}(\mathbf{F})/U = \deg p$.
(b) Find a basis of $\mathcal{P}(\mathbf{F})/U$.

a) $U = \{pq : q \in P(\mathbb{F})\}$. let $m = \deg p$.

let $u = pq \in U$. $\deg u = 0$ (case where $q = 0$, or p is constant)

or $\deg u = \deg p + \deg q \geq m$.

let $s \in P(\mathbb{F})$, $s \neq 0$, $\exists!\, q\, r \in P(\mathbb{F})$ s.t.:

$s = \underbrace{pq}_{\in U} + r$, $\deg r < \deg p$

$\Rightarrow P(\mathbb{F}) = U \oplus W$, $W = P_{m-1}(\mathbb{F})$

By 3E, Ex 18.b), $\dim W = \dim P(\mathbb{F})/U$ (as $V = A \oplus U = B \oplus U \Rightarrow A = B$)

$\Rightarrow \dim P(\mathbb{F})/U = m$

b) By 3E, Ex 14, $1+U, z+U, \dots, z^{m-1}+U$ is a basis of $P(\mathbb{F})/U$, as $1, z, \dots, z^{m-1}$ is a basis of W.