7A Exercises

vendredi 8 novembre 2024 19:50

1 Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by
$$T(z_1, \ldots, z_n) = (0, z_1, \ldots, z_{n-1}).$$
Find a formula for $T^*(z_1, \ldots, z_n)$.

$\langle Tv, w \rangle = \langle v, T^* w \rangle$

Let $x, y \in \mathbb{F}^n$.

$\langle Tx, y \rangle = \langle (0, x_1 \ldots x_{n-1}), (y_1 \ldots y_n) \rangle = \sum_{i=1}^{n-1} x_i y_{i+1} = \langle (x_1 \ldots x_n), (y_2 \ldots y_n, 0) \rangle$

$\Rightarrow T^*(x_1 \ldots x_n) = (x_2 \ldots x_n, 0)$

2 Suppose $T \in \mathcal{L}(V, W)$. Prove that
$$T = 0 \iff T^* = 0 \iff T^*T = 0 \iff TT^* = 0.$$

a → b c d

- Suppose $T = 0$: We have $\operatorname{null} T^* = (\operatorname{range} T)^\perp = \{0\}^\perp = W \Rightarrow T^* = 0$. By a similar argument, $T^* = 0 \Rightarrow T = 0$.
- Suppose $T^* = 0$: Then obviously $T^*T = 0$.
- Suppose $T^*T = 0$: $T^*Tv = 0 \ \forall v \in V \Rightarrow \langle T^*Tv, v \rangle = 0 \Rightarrow \langle Tv, Tv \rangle = 0 \Rightarrow \|Tv\|^2 = 0 \Rightarrow T = 0$
- Suppose $TT^* = 0$: $TT^*w = 0 \ \forall w \in W \Rightarrow \langle TT^*w, w \rangle = 0 \Rightarrow \langle T^*w, T^*w \rangle = 0 \Rightarrow \|T^*w\|^2 = 0 \Rightarrow T^* = 0$

3 Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. Prove that
$$\lambda \text{ is an eigenvalue of } T \iff \bar{\lambda} \text{ is an eigenvalue of } T^*.$$

λ not an e.va. of $T \Rightarrow T - \lambda I$ invertible $\Rightarrow \underbrace{T^* - \bar{\lambda} I}_{= (T - \lambda I)^*}$ invertible $\Rightarrow \bar{\lambda}$ not an e.va. of T^*

$\bar{\lambda}$ not an e.va. of $T^* \Rightarrow T^* - \bar{\lambda} I$ invertible $\Rightarrow T - \lambda I$ invertible $\Rightarrow \lambda$ not an e.va. of T

4 Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that
$$U \text{ is invariant under } T \iff U^\perp \text{ is invariant under } T^*.$$

"$\Rightarrow$": Let $u_\perp \in U^\perp$, $u \in U$.

$\langle Tu, u_\perp \rangle = 0$, since $Tu \in U$.

$\Rightarrow \langle u, T^* u_\perp \rangle = 0 \ \forall u \in U \Rightarrow T^* u_\perp \in U^\perp \ \forall u_\perp \in U^\perp \Rightarrow U^\perp$ invariant under T^*

"$\Leftarrow$": similar reasoning.

5 Suppose $T \in \mathcal{L}(V, W)$. Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Prove that
$$\|Te_1\|^2 + \cdots + \|Te_n\|^2 = \|T^*f_1\|^2 + \cdots + \|T^*f_m\|^2.$$

The numbers $\|Te_1\|^2, \ldots, \|Te_n\|^2$ in the equation above depend on the orthonormal basis $e_1, \ldots, e_n$, but the right side of the equation does not depend on $e_1, \ldots, e_n$. Thus the equation above shows that the sum on the left side does not depend on which orthonormal basis $e_1, \ldots, e_n$ is used.

Since these are orthonormal bases, $\mathcal{M}(T^*) = (\mathcal{M}(T))^*$ wrt to these bases.

If $Te_i = \sum_{j=1}^{m} a_{ji} f_j$ for $i \in \{1 \ldots n\}$, then $T^* f_j = \sum_{i=1}^{n} \overline{a_{ji}} e_i$

$\langle Te_i, Te_i \rangle = \langle e_i, T^*Te_i \rangle = \langle e_i, T^* \sum_j a_{ji} f_j \rangle = \sum_j \overline{a_{ji}} \langle e_i, T^* f_j \rangle$

$\Rightarrow \sum_i^n \|Te_i\|^2 = \sum_i^n \sum_j^m \overline{a_{ji}} \langle e_i, T^* f_j \rangle = \sum_j^m \langle \sum_i^n \overline{a_{ji}} e_i, T^* f_j \rangle = \sum_j^m \|T^* f_j\|^2$

6 Suppose $T \in \mathcal{L}(V, W)$. Prove that
(a) T is injective $\iff$ T^* is surjective;
(b) T is surjective $\iff$ T^* is injective.

a) T injective $\iff \operatorname{null} T = \{0\} \iff (\operatorname{null} T)^\perp = V \overset{(7.6)}{\iff} \operatorname{range} T^* = V \iff T^*$ surjective

a) T injective $\iff \operatorname{null} T = \{0\} \iff (\operatorname{null} T)^{\perp} = V \overset{(7.6)}{\iff} \operatorname{range} T^* = V \iff T^*$ injective

b) T injective $\iff (T^*)^*$ injective $\overset{a)}{\iff} T^*$ injective

7 Prove that if $T \in \mathcal{L}(V, W)$, then
(a) $\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$;
(b) $\dim \operatorname{range} T^* = \dim \operatorname{range} T$.

a) $\dim \operatorname{null} T^* \overset{(\text{rank thm on } T^*)}{=} \dim W - \dim \operatorname{range} T^* = \dim W - \overbrace{\dim (\operatorname{null} T)^{\perp}}^{\dim V - \dim \operatorname{null} T \text{ by } (6.49)} = \dim W - \dim V + \dim \operatorname{null} T$

b) $\dim \operatorname{range} T^* = \dim (\operatorname{null} T)^{\perp} = \dim V - \dim \operatorname{null} T \overset{(\text{rank thm})}{=} \dim \operatorname{range} T$

8 Suppose A is an m-by-n matrix with entries in $\mathbf{F}$. Use (b) in Exercise 7 to prove that the row rank of A equals the column rank of A.

This exercise asks for yet another alternative proof of a result that was previously proved in 3.57 and 3.133.

Let T the linear map and $e_1 \dots e_n$, $f_1 \dots f_m$ orth. basis of $\mathbb{F}^n, \mathbb{F}^m$ s.t. $A = \mathcal{M}(T, (e_1 \dots e_n), (f_1 \dots f_m))$

$\dim \operatorname{range} T = $ col. rank A. Also, $\mathcal{M}(T)^* = \mathcal{M}(T^*)$ ($e_1 \dots e_n$, $f_1 \dots f_m$ orth.)

$\implies A^* = \mathcal{M}(T^*) \implies$ col rank $\mathcal{M}(T^*) =$ col rank $A^* = \dim \operatorname{range} T^* \overset{(7b)}{=} \dim \operatorname{range} T$

$\implies$ col rank $A^* =$ col rank A

col rank $A^* = \dim \operatorname{span}(A^*_{\cdot,1}, \dots, A^*_{\cdot,m})$

$= \dim \operatorname{span}(\overline{A}_{1,\cdot}, \dots, \overline{A}_{m,\cdot})$ (where $A_{i,\cdot}$ is the i-th row of A, holds since $A^*_{ij} = \overline{A_{ji}}$)

$= \dim \operatorname{span}(A_{1,\cdot} \dots A_{m,\cdot})$ (easy to show)

$=$ col rank A

9 Prove that the product of two self-adjoint operators on V is self-adjoint if and only if the two operators commute.

Let $T, S \in \mathcal{L}(V)$ s.t. $T = T^*$, $S = S^*$ (1)

$(ST)^* = ST \iff T^* S^* = ST \overset{(1)}{\iff} TS = ST$

10 Suppose $\mathbf{F} = \mathbf{C}$ and $T \in \mathcal{L}(V)$. Prove that T is self-adjoint if and only if

$$\langle Tv, v \rangle = \langle T^* v, v \rangle$$

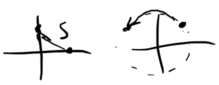
for all $v \in V$.

"$\Rightarrow$": Immediate.

"$\Leftarrow$": Suppose $\langle Tv, v \rangle = \langle T^* v, v \rangle \ \forall v \in V$: $\langle (T - T^*) v, v \rangle = 0 \ \forall v \in V \overset{(7.13,\, \mathbf{F} = \mathbf{C})}{\implies} T - T^* = 0 \implies T = T^*$

11 Define an operator $S : \mathbf{F}^2 \to \mathbf{F}^2$ by $S(w, z) = (-z, w)$.
(a) Find a formula for S^*.
(b) Show that S is normal but not self-adjoint.
(c) Find all eigenvalues of S.

If $\mathbf{F} = \mathbf{R}$, then S is the operator on $\mathbf{R}^2$ of counterclockwise rotation by 90°.



a) $\langle S(w,z), (x,y) \rangle = \langle (-z, w), (x, y) \rangle = -zx + wy = \langle (y, -x), (w, z) \rangle \implies S^*(x,y) = (y, -x)$

b) $\left. \begin{array}{l} SS^*(a,b) = S(b,-a) = (a,b) \\ S^*S(a,b) = S^*(-b,a) = (a,b) \end{array} \right\} \implies SS^* = S^*S \implies S$ normal

$S(0,1) = (-1, 0)$, while $S^*(0,1) = (1, 0) \implies$ not self-adjoint

c) $S(a,b) = \lambda(a,b) \implies (-b, a) = \lambda(a,b) \implies \begin{cases} -b = \lambda a \\ a = \lambda b \end{cases} \implies \begin{cases} -b = \lambda^2 b \\ a = \lambda b \end{cases} \overset{b \neq 0}{\implies} \begin{cases} \lambda^2 = -1 \\ a = \lambda b \end{cases} \implies \begin{cases} \lambda = \pm i \\ a = \pm i b \end{cases}$

$S(0,1) = (-1,0)$, ...

c) $S(a,b) = \lambda(a,b) \Rightarrow (-b,a) = \lambda(a,b) \Rightarrow \begin{cases} -b = \lambda a \\ a = \lambda b \end{cases} \Rightarrow \begin{cases} -b = \lambda^2 b \\ a = \lambda b \end{cases} \overset{b\neq 0}{\Rightarrow} \begin{cases} \lambda^2 = -1 \\ a = \lambda b \end{cases} \Rightarrow \begin{cases} \lambda = \pm i \\ a = \pm i b \end{cases}$

$\Rightarrow$ 2 eva., i and $-i$.

12 An operator $B \in \mathcal{L}(V)$ is called *skew* if

$$B^* = -B.$$

Suppose that $T \in \mathcal{L}(V)$. Prove that T is normal if and only if there exist commuting operators A and B such that A is self-adjoint, B is a skew operator, and $T = A + B$.

"$\Rightarrow$": Suppose T is normal. By 7.23, $\exists\, C, D$ self-adjoint s.t. $T = C + iD$

$(iD)^* = \bar{i} D^* = -iD^* = -iD$, since D self-adjoint. Hence iD is skew.

We thus have $A = C$, $B = iD$ where A is self-adjoint and iD is skew.

"$\Leftarrow$": Suppose $\exists A$ self-adjoint, B skew s.t. $T = A + B$. Let $D = i^{-1}B$. We can show $i^{-1}B$ is self-adjoint.

$(i^{-1}B)^* = \overline{i^{-1}} B^* = i^{-1}B$. Thus $T = A + iD$, with A, D self-adjoint, and by 7.23 T is normal.

13 Suppose $\mathbf{F} = \mathbf{R}$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by $\mathcal{A}T = T^*$ for all $T \in \mathcal{L}(V)$.

(a) Find all eigenvalues of $\mathcal{A}$.

(b) Find the minimal polynomial of $\mathcal{A}$.

a) Let $T \in \mathcal{L}(V)$, $T \neq 0$.

We can notice $\mathcal{A}^2 T = \mathcal{A}(T^*) = T$.

$\mathcal{A}T = \lambda T \Rightarrow T^* = \lambda T$

$\mathcal{A}^2 T = \mathcal{A}(\lambda T) \Rightarrow T = (\lambda T)^* \Rightarrow T = \underbrace{\bar{\lambda}}_{=\lambda \text{ since } \mathbb{F}=\mathbb{R}} T^* = \lambda^2 T \Rightarrow \lambda = 1$ or $\lambda = -1$

b) Let $T \in \mathcal{L}(V)$. $a_0 T + a_1 \mathcal{A} T = -\mathcal{A}^2 T \Rightarrow a_0 T + a_1 T^* = -T$

$\Rightarrow$ minimal polynomial is $p \in \mathcal{P}(\mathbb{R})$ s.t.: $p(z) = -1 + z^2$

14 Define an inner product on $\mathcal{P}_2(\mathbf{R})$ by $\langle p, q \rangle = \int_0^1 pq$. Define an operator $T \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$ by

$$T(ax^2 + bx + c) = bx.$$

(a) Show that with this inner product, the operator T is not self-adjoint.

(b) The matrix of T with respect to the basis $1, x, x^2$ is

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

b) $1, x, x^2$ is not an orthonormal basis of $\mathcal{P}_2(\mathbb{R})$, so $\mathcal{M}(T^*)$ might not be $(\mathcal{M}(T))^*$.

15 Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

(a) T is self-adjoint $\iff$ T^{-1} is self-adjoint;

(b) T is normal $\iff$ T^{-1} is normal.

a) T self-adjoint $\iff T = T^*$

$\iff T^{-1} = (T^*)^{-1}$ (T^* invertible, of inverse $(T^{-1})^*$)

$\iff T^{-1} = (T^{-1})^*$

$\iff T^{-1}$ self-adjoint

b) T normal $\iff TT^* = T^*T$

$\iff (TT^*)^{-1} = (T^*T)^{-1}$

$\iff (T^*)^{-1}T^{-1} = T^{-1}(T^*)^{-1}$

$\iff (T^{-1})^* T^{-1} = T^{-1}(T^{-1})^*$

$\iff (T^*)^{-1} T^{-1} = T^{-1} (T^*)^{-1}$

$\iff (T^{-1})^* T^{-1} = T^{-1} (T^{-1})^*$

$\iff T^{-1}$ normal

16 Suppose $\mathbf{F} = \mathbf{R}$.

(a) Show that the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.
(b) What is the dimension of the subspace of $\mathcal{L}(V)$ in (a) [in terms of $\dim V$]?

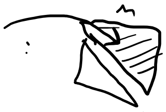
a) let U the set of self-adjoint operators on V.

- $0 \in U$
- let $S, T \in U$: $(S+T)^* = S^* + T^* = S + T \implies S+T \in U$
- let $\lambda \in \mathbb{R}$, $S \in U$: $(\lambda S)^* = \bar{\lambda} S^* = \lambda S \implies \lambda S \in U$

$\implies U$ subspace of $\mathcal{L}(V)$

b) Suppose V is finite-dimensional, with orth. basis $e_1 \dots e_n$.

$M(T^*, (e_1 \dots e_n)) = M(T^*) = M(T) = (M(T))^* \implies a_{ij} = \overline{a_{ji}} \ \forall i,j \in \{1 \dots n\}$

$\implies a_{ij} = a_{ji} \ \forall i,j \in \{1 \dots n\}$. Hence $M(T)$ is fully determined by its upper triangle:  which contains $n + (n-1) + (n-2) + \dots + (n-(n-1))$ elements.

$\dim U = n^2 - \sum_{i=1}^{n-1} i = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ (with $n = \dim V$).

17 Suppose $\mathbf{F} = \mathbf{C}$. Show that the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

let $\lambda \in \mathbb{C}$, $T \in \mathcal{L}(V)$, T self-adjoint.

$(\lambda T)^* = \bar{\lambda} T^* = \bar{\lambda} T \neq \lambda T$ if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, which is possible here since $\mathbb{F} = \mathbb{C}$.

18 Suppose $\dim V \geq 2$. Show that the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

let $T, S \in \mathcal{L}(V)$ s.t. T, S normal. $\quad (T+S)(T+S)^* = TT^* + TS^* + ST^* + SS^*$

$(T+S)^*(T+S) = (S^* + T^*)(T+S) = S^*T + \underbrace{S^*S}_{SS^*} + \underbrace{T^*T}_{TT^*} + T^*S$

$= (T+S)(T+S)^* + \underbrace{S^*T + (S^*T)^* - ((TS^*)^* + TS^*)}_{\text{not necessarily } 0 \text{ (could find example)}}$

20 Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that the following are equivalent.
(a) P is self-adjoint.
(b) P is normal.
(c) There is a subspace U of V such that $P = P_U$.

$a \to b \to c$

- P self-adjoint $\implies P = P^* \implies P^2 = PP^* \implies P = PP^*$
  $\implies P^*P = P^{*2} = P^2 = P$ $\Big| \implies PP^* = P^*P \implies P$ normal

- P normal $\implies V = \operatorname{null} P \oplus \operatorname{range} P$. let $v \in V$, $v = m + n$, $m \in \operatorname{null} P$, $n \in \operatorname{range} P$, $n = Pw$, $w \in V$

• P normal $\Rightarrow V = \text{null } P \oplus \text{range } P$. Let $v \in V$, $v = m + n$, $m \in \text{null } P$, $n \in \text{range } P$

$$\left.\begin{array}{l} Pv = P(m+n) = Pn = P(Pn) = P^2 n = Pn \\ P_{\text{range } P}\, v = P_{\text{range } P}(m+n) = n = Pn \end{array}\right\} \Rightarrow P = P_{\text{range } P} \quad \text{((c) holds with } U = \text{range } P)$$

• c) $\Rightarrow$ a) ???

21 Suppose $D: \mathcal{P}_8(\mathbf{R}) \to \mathcal{P}_8(\mathbf{R})$ is the differentiation operator defined by $Dp = p'$. Prove that there does not exist an inner product on $\mathcal{P}_8(\mathbf{R})$ that makes D a normal operator.

Suppose $\exists$ an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{P}_8(\mathbb{R})$ s.t. D is a normal operator.
Then we should have $\mathcal{P}_8(\mathbb{R}) = \text{null } D \oplus \text{range } D$. However, $\text{null } D = \{c : c \in \mathbb{R}\}$,
and $\text{range } D = \{p \in \mathcal{P}_8(\mathbb{R})$ s.t. $\deg p \in \{0..7\}\}$. Hence polynomials of degree 8
cannot be represented as a linear combination of elements of null D and range D,
s.t. $V \neq \text{null } D + \text{range } D$ and hence $V \neq \text{null } D \oplus \text{range } D \Rightarrow$ contradiction.

23 Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$\|v\| = \|w\| = 2, \quad Tv = 3v, \quad Tw = 4w.$$

Show that $\|T(v+w)\| = 10$.

Since T is normal and 3, 4 distinct e.va. of T, we have $\langle v, w \rangle = 0$.

$$\begin{aligned} \|T(v+w)\|^2 &= \|Tv\|^2 + \|Tw\|^2 + 2\,\text{Re}\langle Tv, Tw\rangle \\ &= \|3v\|^2 + \|4w\|^2 + 2\,\text{Re}\langle 3v, 4w\rangle \\ &= 9 \times 4 + 16 \times 4 \\ &= 100 \Rightarrow \|T(v+w)\| = 10 \end{aligned}$$

26 Fix $u, x \in V$. Define $T \in \mathcal{L}(V)$ by $Tv = \langle v, u \rangle x$ for every $v \in V$.

(a) Prove that if V is a real vector space, then T is self-adjoint if and only if the list u, x is linearly dependent.

(b) Prove that T is normal if and only if the list u, x is linearly dependent.

a) $\langle Tv, w \rangle = \langle \langle v, u \rangle x, w \rangle = \langle v, u \rangle \langle x, w \rangle = \langle v, \langle x, w \rangle u \rangle$ $(\mathbb{F} = \mathbb{R})$
$= \langle v, T^* w \rangle \Rightarrow T^* v = \langle x, v \rangle u$

"$\Rightarrow$" $\forall v \in V$: $Tv = T^* v \Rightarrow \langle v, u \rangle x = \langle x, v \rangle u$.

• If u or $x = 0$, then u and x are linearly dependent.

• Suppose $\begin{cases} x \neq 0 \\ u \neq 0 \end{cases}$. Pick $v = u$: $\|u\|^2 x = \langle x, u \rangle u \Rightarrow x = \dfrac{\langle x, u \rangle}{\|u\|^2} u$ ($\langle x, u \rangle \neq 0$ otherwise $x = 0$)
$\Rightarrow x, u$ linearly dependent

"$\Leftarrow$": Suppose $x = \lambda u$, $\lambda \in \mathbb{R}$. $\forall v \in V$, $Tv = \langle v, u \rangle \lambda u = \langle \lambda u, v \rangle u = T^* v$ (since $\mathbb{F} = \mathbb{R}$)
$\Rightarrow T = T^*$, T self-adjoint

b) "$\Rightarrow$": T normal $\Rightarrow TT^* v = T^* T v$ $\forall v \in V \Rightarrow T(\langle x, v \rangle u) = T^*(\langle v, u \rangle x)$
$\Rightarrow \langle x, v \rangle \|u\|^2 x = \langle v, u \rangle \|x\|^2 u$
$\Rightarrow x = \lambda u \Rightarrow x, u$ linearly dependent

"$\Leftarrow$": Suppose $x = \lambda u$, $\lambda \in \mathbb{F}$.
$\|\ \|^2 = \lambda^2 \langle u, v \rangle \|u\|^2 u$

"⇐": Suppose $u = \lambda v$, $\lambda \in \mathbb{F}$.

$TT^* v = \langle u, v \rangle \|v\|^2 u = \lambda^2 \langle v, v \rangle \|v\|^2 v$

$T^* T v = \langle v, u \rangle \|u\|^2 v = \lambda^2 \langle v, v \rangle \|v\|^2 v$

27 Suppose $T \in \mathcal{L}(V)$ is normal. Prove that

$$\operatorname{null} T^k = \operatorname{null} T \quad \text{and} \quad \operatorname{range} T^k = \operatorname{range} T$$

for every positive integer k.

"⊇": Immediate.

"⊆": Let $v \in \operatorname{null} T^k$.

$(T^*T)^k v = T^k (T^*)^k v = (T^*)^k T^k v = 0 \Rightarrow \operatorname{null} T^k \subseteq \operatorname{null} (T^*T)^k$

$0 = \langle \underbrace{(T^*T)^k v}_{0}, (T^*T)^{k-2} v \rangle = \langle (T^*T)^{k-1} v, (T^*T)^{k-1} v \rangle = 0 \Rightarrow v \in \operatorname{null} (T^*T)^{k-1}$

Recursively, we conclude $v \in \operatorname{null} T^*T$, so $\operatorname{null} T^k \subseteq \operatorname{null} T^*T$

$0 = \langle T^*Tv, v \rangle = \langle Tv, Tv \rangle \Rightarrow v \in \operatorname{null} T \Rightarrow \operatorname{null} T^k \subseteq \operatorname{null} T$

28 Suppose $T \in \mathcal{L}(V)$ is normal. Prove that if $\lambda \in \mathbf{F}$, then the minimal polynomial of T is not a polynomial multiple of $(x - \lambda)^2$.

Suppose $T \in \mathcal{L}(V)$ is normal and $\exists \lambda \in \mathbb{F}$ s.t. $p(z) = (z-\lambda)^2 q(z)$, where $p \in \mathcal{P}(\mathbb{F})$ is the minimal polynomial of T.

$p(T) = (T - \lambda I)^2 q(T) = 0 \Rightarrow \operatorname{range} T \subseteq \operatorname{null} (T-\lambda I)^2 = \operatorname{null} (T - \lambda I)$ (see ex. 27)

$\Rightarrow (T - \lambda I) q(T) = 0 \Rightarrow \exists$ monic poly. of degree $<$ deg p that is 0 evaluated at T.

Thus p is not the minimal polynomial of T: contradiction.

30 Suppose that $T \in \mathcal{L}(\mathbf{F}^3)$ is normal and $T(1,1,1) = (2,2,2)$. Suppose $(z_1, z_2, z_3) \in \operatorname{null} T$. Prove that $z_1 + z_2 + z_3 = 0$.

$\operatorname{null} T = \operatorname{null} T^*$ (T normal)

and $(z_1, z_2, z_3) \in \operatorname{null} T$

$z_1 + z_2 + z_3 = \langle (1,1,1), (z_1, z_2, z_3) \rangle = \langle \frac{1}{2} T(1,1,1), (z_1, z_2, z_3) \rangle = \langle \frac{1}{2}(1,1,1), \underbrace{T^*(z_1, z_2, z_3)}_{=0} \rangle = 0$

32 Suppose $T: V \to W$ is a linear map. Show that under the standard identification of V with V' (see 6.58) and the corresponding identification of W with W', the adjoint map $T^*: W \to V$ corresponds to the dual map $T': W' \to V'$. More precisely, show that

$$T'(\varphi_w) = \varphi_{T^*w}$$

for all $w \in W$, where φ_w and φ_{T^*w} are defined as in 6.58.

Let $w \in W$. Let $v \in V$.

$[T'(\varphi_w)] v = (\varphi_w \circ T) v$

$= \varphi_w (Tv)$

$= \langle Tv, w \rangle$

$= \langle v, T^* w \rangle$

$= \varphi_{T^* w} v$

$\Rightarrow T'(\varphi_w) = \varphi$

$\cdot \; w$

$\Rightarrow \tau(\varphi_w) = \varphi_{\tau^* w}$