2C Exercises

dimanche 14 juillet 2024 21:15

1 Show that the subspaces of $\mathbf{R}^2$ are precisely $\{0\}$, all lines in $\mathbf{R}^2$ containing the origin, and $\mathbf{R}^2$.

$\dim(\mathbb{R}^2) = 2 \Rightarrow$ The only subspace of $\mathbb{R}^2$ of dimension 2 is itself.

The only subspace of dimension 0 is $\{0\}$, as shown in a previous exercise.

This leaves only subspaces of dimension 1 (as $\dim U \leq \dim V$ $\forall$ subspaces U of V)

Let U be a subspace of dim 1 of $\mathbb{R}^2$, with the base (x_b, y_b)

If $y_b = 0$, then the subspace is the constant line $y = 0$. Otw, we can divide the basis vector by y_b $(\in \mathbb{F})$ to obtain:

$\left(\frac{x_b}{y_b}, 1\right)$. It is also a basis of the subspace (length of 1 and in the subspace)

$\operatorname{span}\left(\frac{x_b}{y_b}, 1\right) = \left\{(x, y) \in \mathbb{R}^2 : y = \frac{x_b}{y_b} x\right\}$

which is the definition of a line going through the origin.

$\frac{x_b}{y_b}$ is arbitrary in $\mathbb{F}$ so every line of $\mathbb{R}^2$ going through the origin can be expressed as such.

3 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(6) = 0\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

a) $(X-6), (X-6)^2, (X-6)^3, (X-6)^4 \in U$

$P(X) = 1 \in \mathcal{P}_4(\mathbb{F})$ and $\notin U$, so $U \neq \mathcal{P}_4(\mathbb{F})$

$\Rightarrow \dim(U) < \dim \mathcal{P}_4(\mathbb{F}) = 5$

Furthermore, $(X-6), (X-6)^2, (X-6)^3, (X-6)^4$ is linearly independent, as the polynomials have different degrees. The list contains 4 vectors and is linearly independent, therefore it is a basis of U.

b) We can add 1 to the list so that it becomes a basis of $\mathcal{P}_4(\mathbb{F})$.

c) $\operatorname{span}(1) = \{P(X) \in \mathcal{P}_4(\mathbb{F}) : P(X) = c, c \in \mathbb{F}\}$

$\operatorname{span}(1) \cap U = \{0\}$

$span(1) \cap U = \{0\}$

$\Rightarrow span(1) \oplus U = \mathcal{P}_4(\mathbb{F})$

4 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : p''(6) = 0\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

a) $1, X, (X-6)^3, (X-6)^4$

b) $1, X, (X-6)^3, (X-6)^4, X^2$

c) $W = \{P(X) = cX^2 : c \in \mathbb{F}\}$

5 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5)\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

a) Let $a, b, c, d, e \in \mathbb{F}$

- $a = a \Rightarrow \boxed{1 \in U}$
- $a + 2b = a + 5b \Rightarrow b = 0 \Rightarrow$ No polynom of degree 1 in U
- $a + 2b + 4c = a + 5b + 25c$
$\Rightarrow 3b + 21c = 0 \Rightarrow b = -7c \Rightarrow -c7X + cX^2 \in U \Rightarrow \boxed{-7X + X^2 \in U}$
- $a + 2b + 4c + 8d = a + 5b + 25c + 125d$
$\Rightarrow 3b + 21c + 117d = 0 \Rightarrow b = -c7 - 39d$
$\Rightarrow (-c7 - 39d)X + cX^2 + dX^3 \in U \Rightarrow \boxed{-46X + X^2 + X^3}$
- $a + 2b + 4c + 8d + 16e = a + 5b + 25c + 125d + 625e$
$\Rightarrow b = -c7 - 39d - 203e$
$\Rightarrow (-c7 - 39d - 203e)X + cX^2 + dX^3 + eX^4 \in U \Rightarrow \boxed{-249X + X^2 + X^3 + X^4}$

b) Add X to the previous list

c) $W = \{P(X) = aX : a \in \mathbb{F}\}$

6 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{F}) : p(2) = p(5) = p(6)\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{F})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{F})$ such that $\mathcal{P}_4(\mathbf{F}) = U \oplus W$.

a)
- $1 \in U$
- No polynom of degree 1.
- $\begin{cases} a + 2b + 4c = a + 5b + 25c \\ a + 2b + 4c = a + 6b + 36c \end{cases} \Rightarrow \begin{cases} b = -7c \\ 4b + 32c = 0 \end{cases}$

$\Rightarrow \begin{cases} b = -7c \\ b = -8c \end{cases} \Rightarrow c = 0$ No polynom degree 2

$$\Rightarrow \begin{cases} b = -7c \\ b = -8c \end{cases} \Rightarrow c = 0 \quad \text{No polynom degree } 2$$

$\vdots$

b) Can add X, X^2

c) $W = \{ P(X) = aX + bX^2 : a, b \in \mathbb{F} \}$

7 (a) Let $U = \{p \in \mathcal{P}_4(\mathbf{R}) : \int_{-1}^{1} p = 0\}$. Find a basis of U.
(b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbf{R})$.
(c) Find a subspace W of $\mathcal{P}_4(\mathbf{R})$ such that $\mathcal{P}_4(\mathbf{R}) = U \oplus W$.

a) $\cdot \int_{-1}^{1} a = a[X]_{-1}^{1} = 2a = 0 \iff a = 0$ (constant polynoms are not in U, except 0)

$\cdot \int_{-1}^{1} aX = 0 \iff \left[\frac{a}{2}X^2\right]_{-1}^{1} = \frac{a}{2} - \frac{a}{2} = 0$

$\Rightarrow aX \in U \;\forall a \in \mathbb{F}$

$\Rightarrow X \in U$

$\cdot \int_{-1}^{1} a + bX + cX^2 = \left[aX + \frac{b}{2}X^2 + \frac{c}{3}X^3\right]_{-1}^{1}$

$= 2a + \frac{2}{3}c = 0$

$\iff c = -3a$

$\Rightarrow 1 - 3X^2 \in U$

$\cdot \int_{-1}^{1} a + bX + cX^2 + dX^3 = \left[aX + \frac{b}{2}X^2 + \frac{c}{3}X^3 + \frac{d}{4}X^4\right]_{-1}^{1}$

$= 2a + \frac{2}{3}c$

$\Rightarrow 1 - 3X^2 + X^3 \in U$

$\cdot \int_{-1}^{1} a + bX + cX^2 + dX^3 + eX^4 = \left[aX + \frac{b}{2}X^2 + \frac{c}{3}X^3 + \frac{d}{4}X^4 + \frac{e}{5}X^5\right]$

$= 2a + \frac{2}{3}c + \frac{1}{2}e = 0$

$= -4a - \frac{4}{3}c$

$$\Longleftrightarrow e = -4a - \frac{4}{3}c$$

$$\Rightarrow x^2 - \frac{4}{3}x^4 \in U$$

b) Add 1 to the list

c) $W = \{ p(x) = a \in \mathcal{P}_4(\mathbb{F}) : a \in \mathbb{F} \}$

8 Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that

$$\dim \operatorname{span}(v_1 + w, \dots, v_m + w) \geq m - 1.$$

First we can notice $v_1 \dots v_m$ is a basis of $\operatorname{span}(v_1 \dots v_m)$.

We note $\operatorname{span}(v_1 \dots v_m) = U$, and $\operatorname{span}(v_1 + w, \dots, v_m + w) = U^w$

$\dim(U) = m$

$\dim \operatorname{span}(U^w) \leq \dim U = m$

$$\left| \begin{array}{l} v_1 + w - v_2 - w = v_1 - v_2 \\ \vdots \\ v_{m-1} + w - v_m - w = v_{m-1} - v_m \end{array} \right.$$

We can check if $v_1 - v_2, \dots, v_{m-1} - v_m$ is linearly $\perp\!\!\!\perp$

Let $a_1 \dots a_{m-1} \in \mathbb{F}$:

$$\sum_{i=1}^{m-1} a_i (v_i - v_{i+1}) = a_1 v_1 + (a_2 - a_1) v_2 + \dots + (a_{m-2} - a_{m-1}) v_{m-2} - a_{m-1} v_m$$

$$\Rightarrow \begin{cases} a_1 = 0 \\ a_2 - a_1 = 0 \\ \vdots \\ a_{m-2} - a_{m-1} = 0 \\ a_{m-1} = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ \vdots \\ a_{m-1} = 0 \end{cases} \Rightarrow v_1 - v_2, \dots, v_{m-1} - v_m \text{ linearly } \perp\!\!\!\perp$$

$$\Rightarrow \dim U^w \geq \operatorname{length}(v_1 - v_2, \dots, v_{m-1} - v_m) = m - 1$$

9 Suppose m is a positive integer and $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbf{F})$ are such that each p_k has degree k. Prove that $p_0, p_1, \dots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

We have to check whether $p_0, \dots, p_m$ is linearly independent, as the list contains $m+1$ vectors, and $\dim \mathcal{P}_m(\mathbb{F}) = m+1$.

Let $a_0 \dots a_m \in \mathbb{F}$

$$\sum_{i=0}^{m} a_i p_i = 0 \Rightarrow \begin{cases} a_0 = 0 \\ \vdots \\ a_m = 0 \end{cases}$$, as the polynom on the right side has no x^k term $\forall k$

$\Rightarrow p_0, p_1, \dots, p_m$ basis of $\mathcal{P}_m(\mathbb{F})$

10 Suppose m is a positive integer. For $0 \leq k \leq m$, let

$$p_k(x) = x^k (1 - x)^{m-k}.$$

Show that $p_0, \dots, p_m$ is a basis of $\mathcal{P}_m(\mathbf{F})$.

The basis in this exercise leads to what are called ***Bernstein polynomials****. You can do a web search to learn how Bernstein polynomials are used to approximate continuous functions on* $[0, 1]$.

$\forall k \in \{0 \dots m\}$

Let $a_0 \dots a_m \in \mathbb{F}$

$\forall k \in \{0..m\}$

let $a_0 \dots a_m \in \mathbb{F}$

$$\sum_{i=0}^{m} a_i x^i (1-x)^{m-i} = \sum_{i=0}^{m} a_i x^i \sum_{k=0}^{m-i} \binom{m-i}{k} (-1)^{m-i-k} x^k$$

$$= \sum_{i=0}^{m} \sum_{k=0}^{m-i} a_i \binom{m-i}{k} (-1)^{m-i-k} x^{k+i}$$

We can show $a_i = 0 \, \forall i$ by induction, by considering factors of x^k for $k = 0 \dots m$

Initialization: x^0 appears for $i = k = 0$

Its factor is: $a_0 \binom{m}{0} (-1)^m$

$a_0 \binom{m}{0} = 0 \Rightarrow a_0 = 0$

Hypothesis: $a_i = 0 \, \forall i \in \{0 \dots l\}$ for $l > 0$

Induction step: $a_{l+1} = ?$

x^{l+1} appears when $i = 0, k = l+1$
$i = 1, k = l$
etc

However, $a_i = 0 \, \forall i \leq l$.

Thus only the term where $i = l+1$ is non-zero.

We thus have $a_{l+1} \binom{m-l-1}{0} (-1)^{b \,(\in \mathbb{N})} = 0$ (as $1, x, x^2, \dots, x^m$ linearly $\perp\!\!\!\perp$)

$\Rightarrow a_{l+1} = 0$

$\Rightarrow a_i = 0 \, \forall i \in \{0, \dots, m\}$

11 Suppose U and W are both four-dimensional subspaces of $\mathbf{C}^6$. Prove that there exist two vectors in $U \cap W$ such that neither of these vectors is a scalar multiple of the other.

We essentially have to show $\dim U \cap W \geq 2$.

$\dim \mathbb{C}^6 = 6 \geq \dim U+W \geq 4 = \dim U = \dim W$

$\Rightarrow 6 \geq \dim U + \dim W - \dim U \cap W \geq 4$

$\Rightarrow 6 \geq 8 - \dim U \cap W \geq 4$

$\Rightarrow -2 \geq -\dim U \cap W \geq -4$

$\Rightarrow 2 \leq \dim U \cap W \leq 4$

Since $\dim U \cap W$ is at least 2, it has bases of size 2, meaning there are at least 2 vectors in it that can't be expressed as a scalar multiple of one another.

12 Suppose that U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $\mathbf{R}^8 = U \oplus W$.

We know that:

$\dim U+W = \dim U + \dim W - \dim U \cap W$

$U+W = \mathbb{R}^8$ and $\dim \mathbb{R}^8 = 8$ implies $\dim U+W = 8$, so:

$8 = 3 + 5 - \dim U \cap W \Rightarrow \dim U \cap W = 0$

$\Rightarrow U \oplus W = \mathbb{R}^8$

13 Suppose U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

$\dim \mathbb{R}^9 = 9 \geq \dim U+W \geq 5$

$\Rightarrow 9 \geq \dim U + \dim W - \dim U \cap W \geq 5$

$\Rightarrow 9 \geq 10 - \dim U \cap W \geq 5$

$\Rightarrow -1 \geq -\dim U \cap W \geq -5$

$\Rightarrow 1 \leq \dim U \cap W \leq 5$

$\Rightarrow U \cap W \neq \{0\}$ (as $\dim \{0\} = 0$)

14 Suppose V is a ten-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

$$\dim V = 10 \geq \dim V_1 + V_2 \geq 7$$
$$\Rightarrow 10 \geq \dim V_1 + \dim V_2 - \dim V_1 \cap V_2 \geq 7$$
$$\Rightarrow 10 \geq 14 - \dim V_1 \cap V_2 \geq 7$$
$$\Rightarrow 4 \leq \dim V_1 \cap V_2 \leq 7$$

$$10 \geq \dim (V_1 \cap V_2) + V_3 \geq 7$$
$$\Rightarrow 10 \geq \dim (V_1 \cap V_2) + \dim (V_3) - \dim V_1 \cap V_2 \cap V_3 \geq 7$$
$$\Rightarrow 3 \geq \dim (V_1 \cap V_2) - \dim V_1 \cap V_2 \cap V_3 \geq 0$$
$$\Rightarrow 4 \leq \dim V_1 \cap V_2 \cap V_3 \leq 7$$
$$\Rightarrow \dim V_1 \cap V_2 \cap V_3 \neq 0$$
$$\Rightarrow V_1 \cap V_2 \cap V_3 \neq \{0\}$$

15 Suppose V is finite-dimensional and V_1, V_2, V_3 are subspaces of V with $\dim V_1 + \dim V_2 + \dim V_3 > 2 \dim V$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

$$\dim V = m$$
$$m \geq \dim \left[(V_1 \cap V_2) + V_3 \right]$$
$$\Rightarrow m \geq \dim V_1 \cap V_2 + \dim V_3 - \dim V_1 \cap V_2 \cap V_3$$
$$\Rightarrow m \geq \dim V_1 + \dim V_2 - \dim V_1 + V_2 + \dim V_3 - \dim V_1 \cap V_2 \cap V_3 \geq 1$$
$$\Rightarrow m \geq \underbrace{\dim V_1 + \dim V_2 + \dim V_3}_{> 2m} - \underbrace{\dim V_1 + V_2}_{\leq m} - \dim V_1 \cap V_2 \cap V_3$$
$$\Rightarrow m + \dim V_1 + V_2 \geq \dim V_1 + \dim V_2 + \dim V_3 - \dim V_1 \cap V_2 \cap V_3$$
$$\Rightarrow 2m - (\dim V_1 + \dim V_2 + \dim V_3) \geq - \dim V_1 \cap V_2 \cap V_3$$
$$\Rightarrow 0 > - \dim V_1 \cap V_2 \cap V_3 \Rightarrow \dim V_1 \cap V_2 \cap V_3 > 0$$
$$\Rightarrow V_1 \cap V_2 \cap V_3 \neq \{0\}$$

16 Suppose V is finite-dimensional and U is a subspace of V with $U \neq V$. Let $n = \dim V$ and $m = \dim U$. Prove that there exist $n - m$ subspaces of V, each of dimension $n - 1$, whose intersection equals U.

Let $V_1 \dots V_k$ subspaces of V s.t. $V_1 \cap \dots \cap V_k = U$ and $\dim V_i = p$ $\forall i$

$\dim V_1 \cap \dots \cap V_{k-1} \geq m$

$\exists \bar{U}$ subspace of V s.t. $\bar{U} + U = V$, $\dim \bar{U} = n - m$

Let $\bar{u}_1 \dots \bar{u}_{n-m}$ basis of $\bar{U}$, and $u_1 \dots u_m$ basis of U.

We can construct, $\forall i \in \{1, \dots, n-m\}$, the list of vectors:

$l_i = u_1 \dots u_m, \bar{u}_1 \dots \bar{u}_{i-1}, \bar{u}_{i+1} \dots \bar{u}_{n-m}$, ie the list of vectors in basis of U and the list of vectors of a basis of $\bar{U}$ excluding the ith component. There are $n-m$ such lists ($\dim \bar{U}$).

These lists are all linearly independent, as $\bar{U} \cap U = \{0\}$, meaning each of them is a basis of a subspace of dimension $m + (n - m - 1) = n - 1$. Now we can prove their intersection is U. The intersection of l_i's gives $u_1 \dots u_m$, as for every element $\bar{u}_i$ of $\bar{u}_1 \dots \bar{u}_{n-m}$, there exists l_i which does not have this element. So $\bigcap_{i=1}^{n-m} l_i = u_1 \dots u_m =$ basis of U.

This implies the intersection of these $n-m$ spaces of size $n-1$ is U.

17 Suppose that $V_1, \dots, V_m$ are finite-dimensional subspaces of V. Prove that $V_1 + \dots + V_m$ is finite-dimensional and

$$\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m.$$

The inequality above is an equality if and only if $V_1 + \dots + V_m$ is a direct sum, as will be shown in 3.94.

$$\dim ((V_1 + V_2) + V_3 + \dots + V_m) = \dim V_1 + \dim V_2 - \dim \left[(V_1 + V_2) \cap V_3 + \dots + V_m \right]$$

In general: $\dim \dots = \dim ((V_i + V_j) + \sum V_k) = \dim V_i + \dim V_j - p_{ij}, \; p \geq 0$

$\dim((V_1+V_2)+V_3+\dots+V_m) = \dim V_1 + \dim V_2 - \dim[(V_1+V_2)\cap V_3+\dots+V_m]$

In general: $d_{ij} = \dim((V_i+V_j)+\sum_{k\neq i,j} V_k) = \dim V_i + \dim V_j - p_{ij},\ p \geqslant 0$

$\sum_{i,j} d_{ij} = m\dim V_1 + \dots + m\dim V_m - C\ ,\ C \geqslant 0$

$\Rightarrow 2m\dim(\sum_i V_i) = m\sum \dim V_i - C$

$\underset{m\neq 0}{\Rightarrow} 2\dim(\sum_i V_i) = \sum \dim V_i - C$

$\Rightarrow \sum \dim V_i \geqslant \dim(\sum_i V_i)$

18 Suppose V is finite-dimensional, with $\dim V = n \geq 1$. Prove that there exist one-dimensional subspaces $V_1, \dots, V_n$ of V such that

$$V = V_1 \oplus \dots \oplus V_n.$$

Let $v_1 \dots v_m$ a basis of V ($n \geqslant 1$).

v_i is linearly independent $\forall i$, s.t each v_i is a basis of a subspace of dimension 1 in V, that we call V_i

We can show $V_1 \cap \dots \cap V_m = \{0\}$.

Let $v \in V_1 \cap \dots \cap V_m$.

$\begin{cases} v\in V_1 \Rightarrow \exists a \in \mathbb{F} \text{ s.t } aV_1 = v \\ v \in V_2 \Rightarrow \exists b \in \mathbb{F} \text{ s.t } bv_2 = v \end{cases} \Rightarrow av_1 = bv_2$

Either $a=0$, in which case $v=0$.

Otherwise, we have $v_1 = \frac{b}{a}v_2$, which is not possible as v_1 and v_2 are linearly independent.

Thus $v=0$, and $V_1 \cap \dots \cap V_m = \{0\}$, which implies

$V_1 \oplus \dots \oplus V_m = V$.

19 Explain why you might guess, motivated by analogy with the formula for the number of elements in the union of three finite sets, that if V_1, V_2, V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(V_1+V_2+V_3) &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \dim(V_1\cap V_2) - \dim(V_1\cap V_3) - \dim(V_2 \cap V_3) \\ &\quad + \dim(V_1\cap V_2\cap V_3).\end{aligned}$$

Then either prove the formula above or give a counterexample.

The formula for the number of elements in a union of two sets is

$\#U_1 \cup U_2 = \#U_1 + \#U_2 - \#U_1 \cap U_2$

Similarly, the formula for the dimension of V_1+V_2 is

$\dim(V_1+V_2) = \dim V_1 + \dim V_2 - \dim V_1 \cap V_2$ (same structure)

The formula for the number of elements in the union of three sets is:

$\#U_1\cup U_2 \cup U_3 = \#U_1 + \#U_2 + \#U_3 - \#U_1\cap U_2 - \#U_2 \cap U_3 - \#U_1\cap U_3 + \#U_1\cap U_2 \cap U_3$

Extending the analogy, we might expect a similar structure for the sum of three sets.

However, let $V_1 = \{(x,0)\in\mathbb{R}^2 : x\in\mathbb{R}\}$, $\dim V_1 = 1$

$V_2 = \{(0,x)\in\mathbb{R}^2 : x\in\mathbb{R}\}$, $\dim V_2 = 1$

$V_3 = \{(x,x)\in\mathbb{R}^2 : x\in\mathbb{R}\}$, $\dim V_3 = 1$

We can easily check that $V_1 \cap V_2 = \{0\}$,

$V_1\cap V_3 = \{0\}$, $V_2\cap V_3 = \{0\}$, and of course $V_1\cap V_2\cap V_3 = \{0\}$.

The right side of the formula gives:

$\dim V_1 + \dim V_2 + \dim V_3 - \dots = 3$

all 0

This is not possible, as $V_1+V_2+V_3 \subseteq \mathbb{R}^2$, whose dimension is 2.

Therefore the formula is wrong.

20 Prove that if V_1, V_2, and V_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(V_1+V_2+V_3) &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \frac{\dim(V_1\cap V_2)+\dim(V_1\cap V_3)+\dim(V_2\cap V_3)}{3} \\ &\quad - \frac{\dim((V_1+V_2)\cap V_3)+\dim((V_1+V_3)\cap V_2)+\dim((V_2+V_3)\cap V_1)}{3}.\end{aligned}$$

The formula above may seem strange because the right side does not look like an integer.

$$\dim(V_1+V_2+V_3) = \dim V_1 + \dim V_2+V_3 - \dim V_1\cap(V_2+V_3) = \dim V_1+\dim V_2+\dim V_3 - \dim V_2\cap V_3 - \dim V_1\cap(V_2+V_3)$$

$$\dim(V_1+V_2+V_3) = \dim V_2 + \dim V_1+V_3 - \dim V_2\cap(V_1+V_3) = \dim V_1+\dim V_2+\dim V_3 - \dim V_1\cap V_3 - \dim V_2\cap(V_1+V_3)$$

$$\dim(V_1+V_2+V_3) = \dim V_3 + \dim V_1+V_2 - \dim V_3\cap(V_1+V_2) = \dim V_1+\dim V_2+\dim V_3 - \dim V_1\cap V_2 - \dim V_3\cap(V_1+V_2)$$

$$\Rightarrow 3\dim \sum V_i = 3\sum \dim V_i - \dim V_1\cap(V_2+V_3) - \dots$$

Dividing both sides by 3 gives the expected result.