5C Exercises

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1 Prove or give a counterexample: If $T \in \mathcal{L}(V)$ and T^2 has an upper-triangular matrix with respect to some basis of V, then T has an upper-triangular matrix with respect to some basis of V.

let $V = \mathbb{R}^2$. let $T \in \mathcal{L}(V)$ s.t. $\mathcal{M}(T, (e_1, e_2)) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$

$\mathcal{M}(T^2) = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is a upper triangular matrix.

$c_0 I = -T$ has no solution

$c_0 I + c_1 T = -T^2 \Rightarrow \begin{pmatrix} c_0 & c \\ 0 & c_0 \end{pmatrix} + \begin{pmatrix} c_1 & -2c_1 \\ c_1 & -c_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow c_0 = 1, \; c_1 = 0$

$\Rightarrow$ T's minimal polynomial is $1 + z^2$.

$1+z^2$ cant be represented as a product $(z-\lambda_1)(z-\lambda_2)$ in $\mathbb{R}$ hence by 5.44 there is no basis of $\mathbb{R}^2$ s.t. T has an upper-triangular matrix, despite $T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ being upper-triangular.

2 Suppose A and B are upper-triangular matrices of the same size, with $\alpha_1, \dots, \alpha_n$ on the diagonal of A and $\beta_1, \dots, \beta_n$ on the diagonal of B.

(a) Show that $A + B$ is an upper-triangular matrix with $\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n$ on the diagonal.

(b) Show that AB is an upper-triangular matrix with $\alpha_1\beta_1, \dots, \alpha_n\beta_n$ on the diagonal.

The results in this exercise are used in the proof of 5.81.

a) $A + B = \begin{pmatrix} \alpha_1 & & * \\ & \ddots & \\ (0) & & \alpha_m \end{pmatrix} + \begin{pmatrix} \beta_1 & & * \\ & \ddots & \\ (0) & & \beta_m \end{pmatrix} = \begin{pmatrix} \alpha_1+\beta_1 & & * \\ & \ddots & \\ (0) & & \alpha_m+\beta_m \end{pmatrix}$

b) $(AB)_{ii} = \sum_{k=1}^{m} A_{ik} B_{ki}$ and $A_{ik} = 0 \; \forall k < i$, $B_{ki} = 0 \; \forall k > i$ $\Rightarrow (AB)_{ii} = A_{ii} B_{ii} = \alpha_i \beta_i$

let $i, j \in \{1..m\}$ s.t $i > j$.

$(AB)_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj} = \sum_{k=1}^{i-1} \underbrace{A_{ik}}_{=0} B_{kj} + \sum_{k=i}^{m} A_{ik} \underbrace{B_{kj}}_{=0\ (k \geq i > j)} = 0$

$\Rightarrow AB$ upper-triangular matrix with $\alpha_i \beta_i$ on the diagonal.

4 Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.

This exercise and the exercise below show that 5.41 fails without the hypothesis that an upper-triangular matrix is under consideration.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow T$ s.t $\mathcal{M}(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is invertible.

$\Rightarrow T$ s.t $M(T) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is invertible.

5 Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x+y \\ x+y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix} \Rightarrow \begin{cases} x+y = \lambda x \\ x+y = \lambda y \end{cases}$$

Setting $\lambda = 0$ yields $x+y=0 \Rightarrow x = -y \Rightarrow \lambda$ is an e.va. of the operator whose matrix is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ wrt to the standard basis of $\mathbb{F}^2$, meaning the operator is not invertible.

6 Suppose $\mathbf{F} = \mathbf{C}$, V is finite-dimensional, and $T \in \mathcal{L}(V)$. Prove that if $k \in \{1, \ldots, \dim V\}$, then V has a k-dimensional subspace invariant under T.

$\mathbb{F} = \mathbb{C} \overset{(5.47)}{\Rightarrow} T$ has an upper triangular matrix wrt some basis $v_1 \ldots v_n$ of V ($n = \dim V$)

$\overset{(5.39)}{\Rightarrow}$ span$(v_1 \ldots v_k)$ is invariant under T $\forall k = 1 \ldots n$

7 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$.

(a) Prove that there exists a unique monic polynomial p_v of smallest degree such that $p_v(T)v = 0$.

(b) Prove that the minimal polynomial of T is a polynomial multiple of p_v.

a) Existence: The minimal polynomial of T is such that $p(T)w = 0 \ \forall w \in V$, so in particular $p(T)v = 0$, and p is monic. This implies p_v necessarily exists, and its degree is lower or equal to that of p.

Unicity: let $p^1_v, p^2_v \in \mathcal{P}(\mathbb{F})$ of smallest degree s.t $p^1_v \neq p^2_v$

$p^1_v(T)v = p^2_v(T)v = 0$ and p^1_v, p^2_v are monic.

$$p^1_v(T) = \sum_{i=0}^{m-1} a_i T^i + T^m, \quad p^2_v(T) = \sum_{i=0}^{m-1} b_i T^i + T^m$$

$$(p^1_v - p^2_v)(T) = \sum_{i=0}^{m-1} (a_i - b_i) T^i$$

Let k the maximum degree s.t. $a_i - b_i \neq 0$ (exists since $p^1_v \neq p^2_v$).

Then $(a_k - b_k)^{-1}(p^1_v - p^2_v)(T) = \sum_{i=0}^{m-2} \frac{a_i - b_i}{a_k - b_k} T^i + T^{m-1}$ ($\Rightarrow$ monic)

Also, $(a_k - b_k)^{-1}(p^1_v - p^2_v)(T)v = 0$, as $p^1_v(T)v = p^2_v(T)v = 0$.

This contradicts the assumption that p^1_v and p^2_v are polynomials of the smallest degree s.t. $p^1_v \neq p^2_v$, monic and 0 at Tv.

Thus $p^1 v = p^2 v$.

b) We showed in a) in the existence part that if p is the minimal polynomial of T, then $\deg p_v \leq \deg p$. This implies $\exists q, r \in \mathcal{P}(\mathbb{F})$ s.t.:

$p = p_v q + r$, with $\deg r < \deg p_v$ (since $p_v \neq 0$ as it is monic).

$$p(T)v = 0 \Rightarrow \underbrace{(p_v q)(T)v}_{= 0 \text{ as } p_v(T)v = 0} + r(T)v = 0 \Rightarrow r(T)v = 0$$

$\Rightarrow r = 0$, otherwise by dividing it by its highest degree's coefficient, we construct a monic polynomial of smaller degree than p_v s.t. it is 0 at T v.

8 Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and there exists a nonzero vector $v \in V$ such that $T^2v + 2Tv = -2v$.

(a) Prove that if $\mathbf{F} = \mathbf{R}$, then there does not exist a basis of V with respect to which T has an upper-triangular matrix.

(b) Prove that if $\mathbf{F} = \mathbf{C}$ and A is an upper-triangular matrix that equals the matrix of T with respect to some basis of V, then $-1 + i$ or $-1 - i$ appears on the diagonal of A.

a) Let $q \in \mathcal{P}(\mathbb{F})$ s.t. $q(z) = z^2 + 2z + 2$

$b^2 - 4ac = -4 \Rightarrow q$ has no roots in $\mathbb{R}$

q is a polynomial multiple of the minimal polynomial p of T, as $q(T) = 0$.

Since q has no roots in $\mathbb{R}$, p has no roots in $\mathbb{R}$, and thus cannot be factorized in the form $(z - \lambda_1)\dots(z - \lambda_m)$, $\lambda_1 \dots \lambda_m \in \mathbb{R}$.

By 5.44, there is no basis of V s.t. T has an upper triangular matrix.

b) $q(-1+i) = (-1+i)^2 + 2(-1+i) + 2 = 1 - 2i - 1 - 2 + 2i + 2 = 0$

$q(-1-i) = (-1-i)^2 + 2(-1-i) + 2 = 1 + 2i - 1 - 2 - 2i + 2 = 0$

$\Rightarrow$ The two roots of q are $-1+i$ and $-1-i$.

$\Rightarrow$ $-1+i$ or $-1-i$ is/are root/s of p $\Rightarrow$ $-1+i$ or $-1-i$ e.va. of T

$\Rightarrow$ $-1+i$ or $-1-i$ on the diagonal of upper triangular matrix of T.

9 Suppose B is a square matrix with complex entries. Prove that there exists an invertible square matrix A with complex entries such that $A^{-1}BA$ is an upper-triangular matrix.

Let $T \in \mathcal{L}(V)$ s.t. $\mathcal{M}(T, (v_1 \dots v_n)) = B$, $v_1 \dots v_n$ basis of V.

Since $\mathbb{F} = \mathbb{C}$, there exists some basis $w_1 \dots w_n$ of V s.t.

$C = \mathcal{M}(T, (w_1 \dots w_n))$ is upper triangular.

Let $A = \mathcal{M}(I, (w_1 \dots w_n), (v_1 \dots v_n))$. Then by 3.84 (change of basis):

$$C = A^{-1} B A .$$