6A Exercises

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1 Prove or give a counterexample: If $v_1, \dots, v_m \in V$, then

$$\sum_{j=1}^{m} \sum_{k=1}^{m} \langle v_j, v_k \rangle \geq 0.$$

$$\sum_{i=1}^{m} \sum_{j=1}^{m} < v_i, v_j > = \sum_{i=1}^{m} < v_i, \sum_{j=1}^{m} v_j > = < \sum_{i=1}^{m} v_i, \sum_{j=1}^{m} v_j > = \| \sum_{i=1}^{m} v_i \|^2 \geq 0$$

2 Suppose $S \in \mathcal{L}(V)$. Define $\langle \cdot, \cdot \rangle_1$ by

$$\langle u, v \rangle_1 = \langle Su, Sv \rangle$$

for all $u, v \in V$. Show that $\langle \cdot, \cdot \rangle_1$ is an inner product on V if and only if S is injective.

"$\Rightarrow$": Suppose $<\cdot,\cdot>_1$ inner product.

Let $v \in \operatorname{null} S$.

$<v, v>_1 = < Sv, Sv > = <0, 0> = 0 \overset{\text{(definiteness)}}{\Longrightarrow} v = 0 \Rightarrow \operatorname{null} S = \{0\} \Rightarrow S$ invertible

"$\Leftarrow$": Suppose S injective.

Positivity, additivity, homogeneity, conjugate symmetry hold because $<\cdot,\cdot>$ is an inner product and S a linear map.

Definiteness: $<v, v>_1 = 0 \Leftrightarrow < Sv, Sv > = 0 \Leftrightarrow Sv = 0 \overset{\text{(S injective)}}{\Longleftrightarrow} v = 0$

3 (a) Show that the function taking an ordered pair $\big((x_1, x_2), (y_1, y_2)\big)$ of elements of $\mathbf{R}^2$ to $|x_1 y_1| + |x_2 y_2|$ is not an inner product on $\mathbf{R}^2$.

(b) Show that the function taking an ordered pair $\big((x_1, x_2, x_3), (y_1, y_2, y_3)\big)$ of elements of $\mathbf{R}^3$ to $x_1 y_1 + x_3 y_3$ is not an inner product on $\mathbf{R}^3$.

a) Homogeneity does not hold for $\lambda < 0$:

$$f(\lambda(x_1, x_2), (y_1, y_2)) = |\lambda x_1 y_1| + |\lambda x_2 y_2| = |\lambda| (|x_1 y_1| + |x_2 y_2|)$$
$$= |\lambda| f((x_1, x_2), (y_1, y_2))$$
$$\neq \lambda f((x_1, x_2), (y_1, y_2))$$

b) Definiteness does not hold. Indeed, $f((0,1,0), (0,1,0)) = 0$, though $(0,1,0) \neq 0_{\mathbb{R}^3}$.

4 Suppose $T \in \mathcal{L}(V)$ is such that $\|Tv\| \leq \|v\|$ for every $v \in V$. Prove that $T - \sqrt{2}I$ is injective.

Let $v \in \operatorname{null}(T - \sqrt{2} I)$.

$(T - \sqrt{2} I) v = 0 \Rightarrow Tv = \sqrt{2} v \Rightarrow \|Tv\| = \|\sqrt{2} v\| = \sqrt{2} \|v\|$

$\Rightarrow \sqrt{2} \|v\| \leq \|v\|$ (using $\|Tv\| \leq \|v\| \; \forall v \in V$)

$\Rightarrow \|v\| = 0 \Rightarrow v = 0 \Rightarrow \operatorname{null}(T - \sqrt{2} I) = \{0\}$

$\Rightarrow T - \sqrt{2} I$ injective

5 Suppose V is a real inner product space.

(a) Show that $\langle u + v, u - v \rangle = \|u\|^2 - \|v\|^2$ for every $u, v \in V$.

(b) Show that if $u, v \in V$ have the same norm, then $u + v$ is orthogonal to $u - v$.

(c) Use (b) to show that the diagonals of a rhombus are perpendicular to each other.

a) $< u + v, u - v > = < u, u - v > + < v, u - v > = < u - v, u > + < u - v, v >$

(c) Use (b) to show that the diagonals of a rhombus are perpendicular to each other.

a) $\langle u+v, u-v\rangle = \langle u, u-v\rangle + \langle v, u-v\rangle = \langle u-v, u\rangle + \langle u-v, v\rangle$
$= \langle u,u\rangle + \langle -v, u\rangle + \langle u, v\rangle + \langle -v, v\rangle$
$= \|u\|^2 - \langle v,u\rangle + \langle v, u\rangle - \|v\|^2 = \|u\|^2 - \|v\|^2$

b) let $u, v \in V$ s.t. $\|u\| = \|v\|$.
$\langle u+v, u-v\rangle \overset{a)}{=} \|u\|^2 - \|v\|^2 = \|u\|^2 - \|u\|^2 = 0 \Rightarrow u+v$ orth. to $u-v$

c) A rhombus is a parallelogram where sides are of equal length/norm, $\|u\| = \|v\|$. The diagonal of a parallelogram are $u+v$ and $u-v$. By b), those vectors are orthogonal.

6 Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0 \iff \|u\| \leq \|u + av\|$ for all $a \in \mathbf{F}$.

$u = cv + w$
$\langle v, w\rangle = 0$
$c = \frac{\langle u,v\rangle}{\|v\|^2}$

"$\Rightarrow$": Suppose $\langle u, v\rangle = 0$.
let $a \in \mathbb{F}$.
$\|u+av\|^2 = \langle u+av, u+av\rangle = \langle u, u+av\rangle + \langle av, u+av\rangle$
$= \langle u,u\rangle + \langle u, av\rangle + \langle av, u\rangle + \langle av, av\rangle$
$= \|u\|^2 + |a|^2\|v\|^2 + \bar{a}\underbrace{\langle u,v\rangle}_{} + a\underbrace{\langle v,u\rangle}_{=0}$
$= \|u\|^2 + |a|^2\|v\|^2 \geq \|u\|^2$ (as $|a|^2\|v\|^2 \geq 0$)
$\Rightarrow \|u+av\| \geq \|u\|$ ($x \mapsto x^2$ increasing fct)

"$\Leftarrow$": Suppose $\|u\| \leq \|u+av\|$ $\forall a \in \mathbb{F}$ (1)

If $v = 0$, then $\langle u, v\rangle = \langle 0, v\rangle = 0$
Else, we can decompose v: $u = cv + w$, $c = \frac{\langle u,v\rangle}{\|v\|^2}$

$\|w\|^2 = \|u - cv\|^2 = \langle u - \frac{\langle u,v\rangle}{\|v\|^2}v, u - \frac{\langle u,v\rangle}{\|v\|^2}v\rangle$

$= \|u\|^2 + \left|\frac{\langle u,v\rangle}{\|v\|^2}\right|^2 \|v\|^2 + \langle u, -\frac{\langle u,v\rangle}{\|v\|^2}v\rangle + \langle -\frac{\langle u,v\rangle}{\|v\|^2}v, u\rangle$

$= \|u\|^2 + \frac{|\langle u,v\rangle|^2}{\|v\|^2} - \frac{\overline{\langle u,v\rangle}}{\|v\|^2}\langle u,v\rangle - \frac{\langle u,v\rangle}{\|v\|^2}\langle v,u\rangle$

$= \|u\|^2 - \frac{|\langle u,v\rangle|^2}{\|v\|^2} \overset{(1)}{\geq} \|u\|^2 \Rightarrow \frac{|\langle u,v\rangle|^2}{\|v\|^2} = 0$

$\overset{v \neq 0}{\Longrightarrow} \langle u,v\rangle = 0$

7 Suppose $u, v \in V$. Prove that $\|au + bv\| = \|bu + av\|$ for all $a, b \in \mathbf{R}$ if and only if $\|u\| = \|v\|$.

"$\Rightarrow$": Suppose $\|au+bv\| = \|bu+av\|$ $\forall a,b \in \mathbb{R}$.
For $a=1, b=0$, we have $\|u\| = \|v\|$.

"$\Leftarrow$": Suppose $\|u\| = \|v\|$. let $a, b \in \mathbb{R}$ (1)
$\|au+bv\|^2 = \langle au+bv, au+bv\rangle = \|au\|^2 + \|bv\|^2 + \langle au, bv\rangle + \langle bv, au\rangle$
$= |a|^2\|u\|^2 + |b|^2\|v\|^2 + 2ab\,\mathrm{Re}\langle u,v\rangle$

$$\|au+bv\|^2 = \langle au+bv, au+bv\rangle = \|au\|^2 + \|bv\|^2 + \langle au, bv\rangle + \langle bv, au\rangle$$
$$= |a|^2\|u\|^2 + |b|^2\|v\|^2 + 2ab\,\mathrm{Re}\langle u,v\rangle$$
$$\overset{(1)}{=} |a|^2\|v\|^2 + |b|^2\|u\|^2 + 2ab\,\mathrm{Re}\langle u,v\rangle$$
$$= \|av+bu\|^2 \Rightarrow \|au+bv\| = \|av+bu\|$$

8 Suppose $a, b, c, x, y \in \mathbf{R}$ and $a^2 + b^2 + c^2 + x^2 + y^2 \leq 1$. Prove that $a + b + c + 4x + 9y \leq 10$.

We have $\|u\|^2 \leq 1 \Rightarrow \|u\| \leq 1$, with $u = (a,b,c,x,y)$. let $v = (1,1,1,4,9)$.

$\langle u,v\rangle = a+b+c+4x+9y$.

Cauchy-Schwarz: $\langle u,v\rangle \leq \|u\|\|v\|$

$\Rightarrow a+b+c+4x+9y \leq \|v\| = 10$

9 Suppose $u, v \in V$ and $\|u\| = \|v\| = 1$ and $\langle u, v\rangle = 1$. Prove that $u = v$.

$u = cv + w$. $c = \dfrac{|\langle u,v\rangle|}{\|v\|^2} = \dfrac{1}{1} = 1$, $\langle v,w\rangle = 0$

$\Rightarrow u = v + w$

$\Rightarrow \|u\|^2 = \|v+w\|^2 = \|v\|^2 + \|w\|^2 + 2\mathrm{Re}\langle v,w\rangle$

$\Rightarrow 1 = 1 + \|w\|^2 \Rightarrow \|w\|^2 = 0 \Rightarrow w = 0$

$\Rightarrow u = v$

$y \leq 1 \Rightarrow -y \geq -1$
$\Rightarrow -1-y \geq -2$
$x \leq 1$
$\Rightarrow -x \geq -1 \Rightarrow -x-y \geq -1-y \geq -2$

$A - x - y \geq A - 2$

10 Suppose $u, v \in V$ and $\|u\| \leq 1$ and $\|v\| \leq 1$. Prove that

$$\sqrt{1-\|u\|^2}\sqrt{1-\|v\|^2} \leq 1 - |\langle u, v\rangle|.$$

$|\langle u,v\rangle| \leq \|u\|\|v\| \leq 1$

Using C.S. inequality: $|\langle u,v\rangle| \leq \|u\|\|v\|$

$\Rightarrow 1 - |\langle u,v\rangle| \geq 1 - \|u\|\|v\|$

11 Find vectors $u, v \in \mathbf{R}^2$ such that u is a scalar multiple of $(1, 3)$, v is orthogonal to $(1, 3)$, and $(1, 2) = u + v$.

$u = k(1,3)$, $\langle u,v\rangle = kv_1 + 3kv_2$, $(1,2) = k(1,3) + (v_1,v_2)$

$\Rightarrow \begin{cases} kv_1 + 3kv_2 = 0 \\ 1 = k + v_1 \\ 2 = 3k + v_2 \end{cases} \Rightarrow \begin{cases} v_1 = 1-k \\ v_2 = 2-3k \\ k(1-k) + 3k(2-3k) = 0 \end{cases} \overset{k\neq 0}{\Rightarrow} \begin{cases} 1-k+6-9k = 0 \\ v_1 = 1-k \\ v_2 = 2-3k \end{cases}$

$\Rightarrow \begin{cases} k = 7/10 \\ v_1 = 3/10 \\ v_2 = -1/10 \end{cases}$

$\|u+v\|^2 \geq \|u\|^2 + \|v\|^2$

$|\langle u,v\rangle| \leq \|u\|\|v\|$

12 Suppose a, b, c, d are positive numbers.

(a) Prove that $(a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16$.

(b) For which positive numbers a, b, c, d is the inequality above an equality?

$\sqrt{\langle u,v\rangle}$

a) $\|u\|^2 = a+b+c+d \Rightarrow u = (\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d})$

$\|v\|^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \Rightarrow v = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{d}}\right)$

$|\langle u,v\rangle| = 4$

C.S.: $\|u\|\|v\| \geq |\langle u,v\rangle|$

$\Rightarrow (a+b+c+d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right) \geq 16$

$$\Longrightarrow (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 16$$

b) By C.S., it is an equality iff $\exists k \in \mathbb{R}$ s.t. $k^2(a,b,c,d) = \left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}\right)$

$$\Longrightarrow \begin{cases} k^2 a = a^{-1} \\ k^2 b = b^{-1} \\ k^2 c = c^{-1} \\ k^2 d = d^{-1} \end{cases} \Longrightarrow \begin{cases} k^2 a^2 = 1 \\ k^2 b^2 = 1 \\ k^2 c^2 = 1 \\ k^2 d^2 = 1 \end{cases} \Longrightarrow a = b = c = d$$

13 Show that the square of an average is less than or equal to the average of the squares. More precisely, show that if $a_1, \dots, a_n \in \mathbf{R}$, then the square of the average of $a_1, \dots, a_n$ is less than or equal to the average of $a_1^2, \dots, a_n^2$.

Let $a = (a_1 \dots a_m) \in \mathbb{R}^m$, and $1 = (1 \dots 1)$.

$$\langle 1, a \rangle^2 = \left(\sum_{i=1}^m a_i\right)^2 \leq \|1\|^2 \|a\|^2 \quad \text{(C.S. inequality)}$$
$$= m \sum_{i=1}^m a_i^2$$

$\times 1/m^2$

$$\Longrightarrow \left(\frac{1}{m}\sum_{i=1}^m a_i\right)^2 \leq \frac{1}{m}\sum_{i=1}^m a_i^2$$

14 Suppose $v \in V$ and $v \neq 0$. Prove that $v/\|v\|$ is the unique closest element on the unit sphere of V to v. More precisely, prove that if $u \in V$ and $\|u\| = 1$, then

$$\left\| v - \frac{v}{\|v\|} \right\| \leq \|v - u\|,$$

with equality only if $u = v/\|v\|$.

$v \neq 0$

$u = cv + w$, $c = \frac{\langle u, v \rangle}{\|v\|^2}$, $\langle v, w \rangle = 0$

$$\left\| v - \frac{v}{\|v\|} \right\|^2 = \|v\|^2 + \left\|\frac{v}{\|v\|}\right\|^2 - 2\,\mathrm{Re}\overbrace{\left\langle v, \frac{v}{\|v\|} \right\rangle}^{= \frac{1}{\|v\|}\langle v, v\rangle = \|v\| \in \mathbb{R}}$$
$$= \|v\|^2 + 1 - 2\|v\|$$

$$\|v - u\|^2 = \|v\|^2 + \|u\|^2 - 2\,\mathrm{Re}\langle v, u \rangle$$
$$= \|v\|^2 + 1 - 2\,\mathrm{Re}\langle v, u \rangle \geq \|v\|^2 + 1 - |\langle v, u \rangle| \overset{\text{(C.S.)}}{\geq} \|v\|^2 + 1 - \|u\|\|v\|$$

$$\Longrightarrow \|v - u\|^2 \geq \|v\|^2 + 1 - \|v\| \geq \|v\|^2 + 1 - 2\|v\| \quad (\text{since } \|v\| \geq 0)$$

15 Suppose u, v are nonzero vectors in $\mathbf{R}^2$. Prove that

$$\langle u, v \rangle = \|u\|\,\|v\| \cos\theta,$$

where θ is the angle between u and v (thinking of u and v as arrows with initial point at the origin).

Hint: Use the law of cosines on the triangle formed by u, v, and $u - v$.

Law of cosines on the triangle formed by $u, v, u - v$.

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\| \cos\theta$$

$$\Longrightarrow \|u\|\|v\| \cos\theta = \frac{1}{2}\left(\|u\|^2 + \|v\|^2 - \left(\|u\|^2 + \|v\|^2 - 2\langle u, v \rangle\right)\right)$$

$$\Longrightarrow \|u\|\|v\| \cos\theta = \langle u, v \rangle$$

16 The angle between two vectors (thought of as arrows with initial point at the origin) in $\mathbf{R}^2$ or $\mathbf{R}^3$ can be defined geometrically. However, geometry is not as clear in $\mathbf{R}^n$ for $n > 3$. Thus the angle between two nonzero vectors $x, y \in \mathbf{R}^n$ is defined to be

$$\arccos \frac{\langle x, y \rangle}{\|x\|\,\|y\|},$$

where the motivation for this definition comes from Exercise 15. Explain why the Cauchy–Schwarz inequality is needed to show that this definition makes sense.

Arccos is a function that takes value in $[-1, 1]$, meaning $\frac{\langle x, y \rangle}{\|x\|\,\|y\|}$ must be in $[-1, 1]$ for all $x, y \in \mathbb{R}^m$.

This is guaranteed by the C.S. inequality $\langle x, y \rangle \leq \|x\|\,\|y\|$.

This is guaranteed by the C.S. inequality $\langle x,y\rangle \le \|x\|\,\|y\|$.

17 Prove that

$$\Big(\sum_{k=1}^{n} a_k b_k\Big)^2 \le \Big(\sum_{k=1}^{n} k a_k^2\Big)\Big(\sum_{k=1}^{n} \frac{b_k^2}{k}\Big)$$

for all real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$.

Let $a = (a_1, \sqrt{2}a_2, \dots, \sqrt{n}a_n)$, $b = \big(b_1, \frac{b_2}{\sqrt{2}}, \dots, \frac{b_n}{\sqrt{n}}\big) \in \mathbb{R}^n$

$$\langle a,b\rangle^2 = \Big(\sum_{k=1}^{n} \sqrt{k}a_k \frac{b_k}{\sqrt{k}}\Big)^2 = \Big(\sum_{k=1}^{n} a_k b_k\Big)^2 \overset{C.S.}{\le} \|a\|^2\|b\|^2 = \sum_{k=1}^{n}(\sqrt{k}a_k)^2 \sum_{k=1}^{n}\Big(\frac{b_k}{\sqrt{k}}\Big)^2$$

$$= \Big(\sum_{k=1}^{n} k a_k^2\Big)\Big(\sum_{k=1}^{n} \frac{b_k^2}{k}\Big)$$

20 Prove that if $u, v \in V$, then $\big|\|u\| - \|v\|\big| \le \|u - v\|$.

The inequality above is called the ***reverse triangle inequality****. For the reverse triangle inequality when $V = \mathbf{C}$, see Exercise 2 in Chapter 4.*

$\big|\,\|u\| - \|v\|\,\big|$

$$\begin{aligned}\|u - v\|^2 &= \|u\|^2 + \|v\|^2 - 2\,\mathrm{Re}\langle u,v\rangle \\ &\ge \|u\|^2 + \|v\|^2 - 2|\langle u,v\rangle| \\ &\ge \|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\| \quad (C.S.) \\ &= (\|u\| - \|v\|)^2\end{aligned}$$

$\Rightarrow \|u-v\|^2 \ge (\|u\| - \|v\|)^2 \Rightarrow \|u-v\| \ge \big|\,\|u\| - \|v\|\,\big|$

21 Suppose $u, v \in V$ are such that

$$\|u\| = 3, \quad \|u + v\| = 4, \quad \|u - v\| = 6.$$

What number does $\|v\|$ equal?

Parallelogram equality:

$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$

$\Rightarrow 52 = 18 + 2\|v\|^2 \Rightarrow \|v\|^2 = 17 \Rightarrow \|v\| = \sqrt{17}$

22 Show that if $u, v \in V$, then

$$\|u + v\|\,\|u - v\| \le \|u\|^2 + \|v\|^2.$$

$$\begin{aligned}\|u+v\|^2\|u-v\|^2 &= (\|u\|^2 + \|v\|^2 + 2\,\mathrm{Re}\langle u,v\rangle)(\|u\|^2 + \|v\|^2 - 2\,\mathrm{Re}\langle u,v\rangle) \\ &= \|u\|^4 + \|u\|^2\|v\|^2 - \cancel{2\|u\|^2\,\mathrm{Re}\langle u,v\rangle} \\ &\quad + \|v\|^2\|u\|^2 + \|v\|^4 - \cancel{2\|v\|^2\,\mathrm{Re}\langle u,v\rangle} \\ &\quad + \cancel{2\|u\|^2\,\mathrm{Re}\langle u,v\rangle} + \cancel{2\|v\|^2\,\mathrm{Re}\langle u,v\rangle} - 4\,\mathrm{Re}\langle u,v\rangle^2 \\ &= \|u\|^4 + 2\|u\|^2\|v\|^2 + \|v\|^4 - 4\,\mathrm{Re}\langle u,v\rangle^2 \\ &= (\|u\|^2 + \|v\|^2)^2 - 4\,\mathrm{Re}\langle u,v\rangle^2\end{aligned}$$

$$\geq (\|u\|^2 + \|v\|^2)^2$$

$$\implies \|u+v\|\|u-v\| \geq \|u\|^2 + \|v\|^2$$

23 Suppose $v_1, \dots, v_m \in V$ are such that $\|v_k\| \leq 1$ for each $k = 1, \dots, m$. Show that there exist $a_1, \dots, a_m \in \{1, -1\}$ such that

$$\|a_1 v_1 + \cdots + a_m v_m\| \leq \sqrt{m}.$$

Suppose $v_1 \dots v_m$ linearly independent.

$$\left\| \sum_{i=1}^{m} a_i v_i \right\|^2 = \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j \langle v_i, v_j \rangle$$

$$= \sum_{i=1}^{m} \|v_i\|^2 + \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j \neq i}}^{m} a_i a_j \langle v_i, v_j \rangle \quad (a_i^2 = 1 \ \forall\, i = 1 \dots m)$$

$$\leq m + \sum_{i=1}^{m} \sum_{j=i}^{m} 2 a_i a_j \operatorname{Re} \langle v_i, v_j \rangle$$

Choose a_i, a_j s.t. $\sum_{i=1}^{m} \sum_{j=1}^{m} 2 a_i a_j \operatorname{Re} \langle v_i, v_j \rangle$ (show it always exists?)

$$\implies \left\| \sum_{i=1}^{m} a_i v_i \right\| \leq \sqrt{m}$$

24 Prove or give a counterexample: If $\|\cdot\|$ is the norm associated with an inner product on $\mathbf{R}^2$, then there exists $(x, y) \in \mathbf{R}^2$ such that $\|(x, y)\| \neq \max\{|x|, |y|\}$.

Let $\|.\|$ be the Euclidean norm.

$$\|(1,2)\| = \sqrt{1+4} = \sqrt{5} \neq 2 = \max\{|1|, |2|\}$$

25 Suppose $p > 0$. Prove that there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by

$$\|(x, y)\| = \left(|x|^p + |y|^p\right)^{1/p}$$

for all $(x, y) \in \mathbf{R}^2$ if and only if $p = 2$.

"$\Leftarrow$": $\|(x,y)\| = \sqrt{x^2+y^2}$ is the Euclidean norm.

"$\Rightarrow$": Let $x = (0,1)$, $y = (1,0)$. Using the parallelogram equality:

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

$$\implies (|x_1+y_1|^p + |x_2+y_2|^p)^{2/p} + (|x_1-y_1|^p + |x_2-y_2|^p)^{2/p} = 2((|x_1|^p + |x_2|^p)^{2/p} + (|y_1|^p + |y_2|^p)^{2/p})$$

$$\implies 2^{2/p} + 2^{2/p} = 4 \implies 2^{2/p} = 2 \implies 2/p = 1 \implies p = 2$$

26 Suppose V is a real inner product space. Prove that

$$\langle u, v \rangle = \frac{\|u+v\|^2 - \|u-v\|^2}{4}$$

for all $u, v \in V$.

$$\frac{\|u+v\|^2 - \|u-v\|^2}{4} = \frac{\|u\|^2 + \|v\|^2 + 2\langle u,v \rangle - \|u\|^2 - \|v\|^2 + 2\langle u,v \rangle}{4} = \langle u, v \rangle$$

($\operatorname{Re}\langle u,v \rangle = \langle u,v \rangle$ as V real)

29 Suppose $V_1, \dots, V_m$ are inner product spaces. Show that the equation

$$\langle (u_1, \dots, u_m), (v_1, \dots, v_m) \rangle = \langle u_1, v_1 \rangle + \cdots + \langle u_m, v_m \rangle$$

defines an inner product on $V_1 \times \cdots \times V_m$.

In the expression above on the right, for each $k = 1, \dots, m$, the inner product $\langle u_k, v_k \rangle$ denotes the inner product on V_k. Each of the spaces $V_1, \dots, V_m$ may have a different inner product, even though the same notation is used here.

- Positivity: $\langle (v_1 \dots v_m), (v_1 \dots v_m) \rangle = \sum_{i=1}^{m} \|v_i\|^2 \geq 0$ (sum of ≥ 0 terms)
- Definiteness: $\langle (v_1 \dots v_m), (v_1 \dots v_m) \rangle = 0 \iff \sum_{i=1}^{m} \|v_i\|^2 = 0 \iff (v_1 \dots v_m) = 0$ (sum of positive terms)
- Additivity: $\langle u+v, w \rangle = \sum_{i=1}^{m} \langle u_i + v_i, w_i \rangle = \sum_{i=1}^{m} \langle u_i, w_i \rangle + \langle v_i, w_i \rangle$ (property of each inner space $V_1 \dots V_m$)

  $= \langle u, w \rangle + \langle v, w \rangle$
- Homogeneity: $\langle \lambda u, v \rangle = \sum_{i=1}^{m} \langle \lambda u_i, v_i \rangle = \lambda \sum_{i=1}^{m} \langle u_i, v_i \rangle = \lambda \langle u, v \rangle$
- Symmetry: $\langle u, v \rangle = \sum_{i=1}^{m} \langle u_i, v_i \rangle = \sum_{i=1}^{m} \overline{\langle v_i, u_i \rangle} = \overline{\sum_{i=1}^{m} \langle v_i, u_i \rangle} = \overline{\langle v, u \rangle}$

31 Suppose $u, v, w \in V$. Prove that

$$\left\|w - \tfrac{1}{2}(u+v)\right\|^2 = \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4}.$$

$\|w - \frac{1}{2}(u+v)\|^2 = \|w\|^2 + \frac{1}{4}\|u+v\|^2 - \operatorname{Re}\langle w, u+v \rangle = \|w\|^2 + \frac{1}{4}(\|u\|^2 + \|v\|^2) + \frac{1}{2}\operatorname{Re}\langle u, v \rangle - \operatorname{Re}(\langle w, u \rangle + \langle w, v \rangle)$

$$\frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4} = \frac{\|w\|^2 + \|u\|^2 - 2\operatorname{Re}\langle w, u \rangle + \|w\|^2 + \|v\|^2 - 2\operatorname{Re}\langle w, v \rangle}{2} - \frac{\|u\|^2 + \|v\|^2 - 2\operatorname{Re}\langle u, v \rangle}{4}$$

$$= \|w\|^2 + \frac{1}{4}(\|v\|^2 + \|u\|^2) - \operatorname{Re}\langle w, u \rangle - \operatorname{Re}\langle w, v \rangle - \frac{\|u\|^2 + \|v\|^2 - 2\operatorname{Re}\langle u, v \rangle}{4} + \frac{1}{4}(\|v\|^2 + \|u\|^2)$$

$$= \|w - \tfrac{1}{2}(u+v)\|^2$$

32 Suppose that E is a subset of V with the property that $u, v \in E$ implies $\frac{1}{2}(u+v) \in E$. Let $w \in V$. Show that there is at most one point in E that is closest to w. In other words, show that there is at most one $u \in E$ such that

$$\|w - u\| \leq \|w - x\|$$

for all $x \in E$.

Existence: Ensured by $\mathbb{R}^+$ being a total order.

Unicity: Suppose $\exists u, v \in E$ s.t $\|w-u\| \leq \|w-x\|$ and $\|w-v\| \leq \|w-x\|$ $\forall x \in E$, and $u \neq v$.

Then $\frac{1}{2}(u+v) \in E$

$$\Rightarrow \|w - \tfrac{1}{2}(u+v)\|^2 = \frac{\|w-u\|^2 + \|w-v\|^2}{2} - \frac{\|u-v\|^2}{4}$$

$$\leq \|w-x\|^2 - \frac{\|u-v\|^2}{4}$$

$\Rightarrow \|w - \frac{1}{2}(u+v)\| < \|w-x\|$ since $\|u-v\| \neq 0$ as $u \neq v$ by hypothesis.

Thus we don't have $\|w-u\| \leq \|w - \frac{1}{2}(u+v)\|$, which contradicts our hypothesis of there being several elements u s.t. $\|w-u\| < \|w-x\|$ $\forall x \in E$.

33 Suppose f, g are differentiable functions from $\mathbf{R}$ to $\mathbf{R}^n$.

(a) Show that

$$\langle f(t), g(t) \rangle' = \langle f'(t), g(t) \rangle + \langle f(t), g'(t) \rangle.$$

(b) Suppose c is a positive number and $\|f(t)\| = c$ for every $t \in \mathbf{R}$. Show that $\langle f'(t), f(t) \rangle = 0$ for every $t \in \mathbf{R}$.

(c) Interpret the result in (b) geometrically in terms of the tangent vector to a curve lying on a sphere in $\mathbf{R}^n$ centered at the origin.

A function $f: \mathbf{R} \to \mathbf{R}^n$ is called differentiable if there exist differentiable functions $f_1, \dots, f_n$ from $\mathbf{R}$ to $\mathbf{R}$ such that $f(t) = (f_1(t), \dots, f_n(t))$ for each $t \in \mathbf{R}$. Furthermore, for each $t \in \mathbf{R}$, the derivative $f'(t) \in \mathbf{R}^n$ is defined by $f'(t) = (f_1'(t), \dots, f_n'(t))$.

a) $\langle f(t), g(t) \rangle' = \left(\sum_{i=1}^{n} f_i(t) g_i(t)\right)' = \sum_{i=1}^{n} (f_i(t) g_i(t))' = \sum_{i=1}^{n} f_i'(t) g_i(t) + f_i(t) g_i'(t)$

$= \sum_{i=1}^{n} f_i'(t) g_i(t) + \sum_{i=1}^{n} f_i(t) g_i'(t) = \langle f'(t), g(t) \rangle + \langle f(t), g'(t) \rangle$

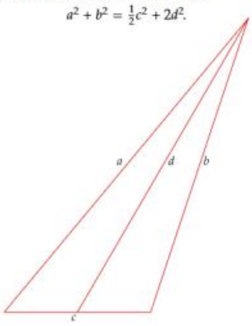
b) $\langle f(t), f(t) \rangle' = \langle f'(t), f(t) \rangle + \langle f(t), f'(t) \rangle$

$\Rightarrow \| f(t) \|' = 2 \langle f'(t), f(t) \rangle$ ($\langle f(t), g(t) \rangle = \langle g(t), f(t) \rangle$ as $f(t), g(t) \in \mathbb{R}^n$)

$\Rightarrow 0 = \langle f'(t), f(t) \rangle$

34 Use inner products to prove Apollonius's identity: In a triangle with sides of length a, b, and c, let d be the length of the line segment from the midpoint of the side of length c to the opposite vertex. Then

$$a^2 + b^2 = \tfrac{1}{2}c^2 + 2d^2.$$



$a^2 = \|u\|^2$

$b^2 = \|u - v\|^2$

$c^2 = \|v\|^2$

$d^2 = \|u - \frac{1}{2}v\|^2$

$$\|u\|^2 + \|u-v\|^2 = 2\|u\|^2 + \|v\|^2 - 2\,\mathrm{Re}\langle u, v\rangle$$
$$= \frac{1}{2}\|v\|^2 + 2\left(\|u\|^2 + \frac{1}{4}\|v\|^2 - \mathrm{Re}\langle u, v\rangle\right)$$
$$= \frac{1}{2}\|v\|^2 + 2\|u - \tfrac{1}{2}v\|^2$$

35 Fix a positive integer n. The *Laplacian* Δp of a twice differentiable real-valued function p on $\mathbf{R}^n$ is the function on $\mathbf{R}^n$ defined by

$$\Delta p = \frac{\partial^2 p}{\partial x_1^2} + \cdots + \frac{\partial^2 p}{\partial x_n^2}.$$

The function p is called *harmonic* if $\Delta p = 0$.

A *polynomial* on $\mathbf{R}^n$ is a linear combination (with coefficients in $\mathbf{R}$) of functions of the form $x_1^{m_1} \cdots x_n^{m_n}$, where $m_1, \dots, m_n$ are nonnegative integers.

Suppose q is a polynomial on $\mathbf{R}^n$. Prove that there exists a harmonic polynomial p on $\mathbf{R}^n$ such that $p(x) = q(x)$ for every $x \in \mathbf{R}^n$ with $\|x\| = 1$.

The only fact about harmonic functions that you need for this exercise is that if p is a harmonic function on $\mathbf{R}^n$ and $p(x) = 0$ for all $x \in \mathbf{R}^n$ with $\|x\| = 1$, then $p = 0$.

Hint: A reasonable guess is that the desired harmonic polynomial p is of the form $q + (1 - \|x\|^2)r$ for some polynomial r. Prove that there is a polynomial r on $\mathbf{R}^n$ such that $q + (1 - \|x\|^2)r$ is harmonic by defining an operator T on a suitable vector space by

$$Tr = \Delta\big((1 - \|x\|^2)r\big)$$

and then showing that T is injective and hence surjective.

Let q a polynomial on $\mathbb{R}^n$, $q(X) = \sum_{i=1}^{M} a_i \prod_{j=1}^{n} X_j^{m_{ij}}$ $\forall X \in \mathbb{R}^n$, with $a_i \in \mathbb{R}$, $m_{ij} \geq 0$.

Let $P_M(\mathbb{R}^n)$ the vector space of polynomials on $\mathbb{R}^n$, over field $\mathbb{R}$.

Let $p(X) = \sum_{i=1}^{M} a_i \prod_{j=1}^{n} X_j^{m_{ij}}$, $q(X) = \sum_{i=1}^{M} b_i \prod_{j=1}^{n} X_j^{k_{ij}}$, $\lambda \in \mathbb{R}$.

- $P_M(\mathbb{R}^n)$ closed under addition:

$$p(X) + q(X) = \sum_{i=1}^{M} a_i \prod_{j=1}^{n} X_j^{m_{ij}} + b_i \prod_{j=1}^{n} X_j^{k_{ij}} = \sum_{i=1}^{M} (a_i + b_i) \prod_{j=1}^{n} X_j^{m_{ij} + k_{ij}} \in P_M(\mathbb{R}^n)$$

- $P_M(\mathbb{R}^n)$ closed under scalar mult.:

$$\lambda p(X) = \sum_{i=1}^{M} \lambda a_i \prod_{j=1}^{n} X_j^{m_{ij}} \in P_M(\mathbb{R}^n)$$

- 0 identity element of +, with $0(X) = 0$
- 1 identity element of ·
- + commutative and associative

- $P_M(\mathbb{R}^m)$ closed under addition:

$$p(X)+q(X) = \sum_{i=1}^{M} a_i \prod_{j=1}^{m} X_j^{m_{ij}} + b_i \prod_{j=1}^{m} X_j^{k_{ij}} = \sum_{i=1}^{M} (a_i+b_i) \prod_{j=1}^{m} X_j^{m_{ij}+k_{ij}} \in P_M(\mathbb{R}^m)$$

- $P_M(\mathbb{R}^m)$ closed under scalar mult.:

$$\lambda p(X) = \sum_{i=1}^{M} \lambda a_i \prod_{j=1}^{m} X_j^{m_{ij}} \in P_M(\mathbb{R}^m)$$

- 0 identity element of $+$, with $0(X) = 0$
- 1 identity element of $\cdot$
- $+$ commutative and associative
- $\cdot$ distributive over $+$
- Every element has its inverse: $-p(X) = \sum_{i=1}^{M} (-a_i) \prod_{j=1}^{m} X_j^{m_{ij}}$

$\Rightarrow P_M(\mathbb{R}^m)$ vector space

Now define $T \in \mathcal{L}(P_M(\mathbb{R}^m))$ by: $Tr = \Delta((1-\|X\|^2)r)$

$$T(p+q) = \Delta((1-\|X\|^2(p+q)) = \Delta((1-\|X\|^2)p + (1-\|X\|^2)q)$$
$$= \Delta((1-\|X\|^2)p) + \Delta((1-\|X\|^2)q) \quad \text{(properties of derivation)}$$
$$= Tp + Tq$$

$$T(\lambda p) = \Delta((1-\|X\|^2)(\lambda p)) \overset{(\text{prop. der.})}{=} \lambda \Delta((1-\|X\|^2)p) = \lambda Tp$$

$\Rightarrow T$ linear map

We can show T is injective:

Let $r \in \operatorname{null} T \Rightarrow Tr = 0 \Rightarrow \Delta((1-\|X\|^2)r) = 0$

$\Rightarrow (1-\|X\|^2)r$ is harmonic.

We also have $(1-\|X\|^2)r(X) = 0 \ \forall X \in \mathbb{R}^m$ s.t. $\|X\| = 1$.

Using the property explained in the exercise: $(1-\|X\|^2)r = 0$ (1)

$$\frac{\partial}{\partial X_i}\left[(1-\|X\|^2)r(X)\right] = -2X_i r(X) + \frac{\partial r(X)}{\partial X_i}(1-\|X\|^2) = 0$$

Let $X \in \mathbb{R}^m$ s.t. $\|X\| = 1$.

Then we have $-2X_i r(X) = 0$. $\|X\| = 1 \Rightarrow \exists i$ s.t. $X_i \neq 0$.

$\Rightarrow r(X) = 0 \ \forall X \in \mathbb{R}^m$ s.t. $\|X\| = 1$.

$r(X) = 0 \ \forall X \in \mathbb{R}^m$ s.t. $\|X\| \neq 1$.

$\implies r(X) = 0 \ \forall X \in \mathbb{R}^n$ s.t. $\|X\| = 1$.

From (1) we also have $r(X) = 0 \ \forall X \in \mathbb{R}^n$ s.t $\|X\| \neq 1$.

$\implies r = 0 \implies \text{null}\, T = \{0\} \implies T$ injective

$\implies T$ injective.

$\implies \exists r \in P_m(\mathbb{R}^n)$ s.t. $\Delta\left((1-\|X\|^2) r\right) = -\underbrace{\Delta q}_{\in P_n(\mathbb{R}^n)}$

$\Delta\left(q + (1-\|X\|^2) r\right) = \Delta q + \Delta(1-\|X\|^2) r = 0 \implies q + (1-\|X\|^2) r$ harmonic.

We also have $q(X) + (1-\|X\|^2) r(X) = q(X) \ \forall X \in \mathbb{R}^n$ s.t. $\|X\| = 1$.

This concludes the proof of existence.