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Notes on the curvature tensor ☆

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ABSTRACT

We present a collection of formulas for computing the curvature tensor on parametrized surfaces, on implicit surfaces, and on surfaces obtained by space deformation.

1. Introduction

The *curvature tensor*, introduced to the field of Applied Geometry by Taubin [9], is commanding increasing interest, presumably due to the following reasons: First, unlike fundamental forms or the shape operator, the curvature tensor is independent of a given parametrization. In particular, this is useful when dealing with piecewise defined surfaces, like spline surfaces, facetted surfaces, or surfaces generated by subdivision. For instance, curvature continuity of a surface is equivalent to continuity of the curvature tensor. This insight was used in [8] to characterize curvature continuity of subdivision schemes. It should be noted that a similar result cannot be formulated in terms of fundamental forms or principal curvatures and -directions. Second, the curvature tensor is defined on the surface itself and thus can be computed equally for surfaces given by a parametrization, in implicit form, or in any other way.

In this work, we present formulas for the computation of the curvature tensor of hypersurfaces in $\mathbb{R}^n$ and discuss also some other aspects like integrability conditions. Most results do not appear in the literature, while a few others are well known, and included here for the sake of completeness.

After introducing the concept in the next section, we elaborate on different setups in Section 3: First, we consider parametrized surfaces. Here, the formulas bear a striking resemblance to the standard shape operator, however, without sharing its dependence on the parametrization. In particular, Eq. (7) suggests an elegant and easy-to-implement scheme for estimating curvature properties of facetted surfaces. Further, we present integrability conditions for reconstructing surfaces from curvature information. These are of substantially lower complexity than when expressed in fundamental forms. Second, we derive formulas for surfaces in implicit form. Amazingly, in the literature, this case is rarely discussed in full detail. For instance, Goldman's nice treatise [4] yields a host of different formulas for the principal (and other) curvatures, but does not address principal directions. Third, we discuss surfaces obtained from a given surface via space deformation. Our results show in detail, how curvature data of the given surface interact with the deformation function to yield the curvature tensor of the new surface.

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2. The generalized curvature tensor

We start with introducing notations and recalling some basic facts from elementary differential geometry. Since our analysis is local, we consider hypersurfaces in $\mathbb{R}^n$ parametrized by a single function $\mathbf{f}$. More precisely, letting $m := n - 1$ throughout, $\mathcal{H}^m$ denotes the space of all m-dimensional hypersurfaces $\mathbf{H} = \mathbf{f}(U) \subset \mathbb{R}^n$, where

- $U \subset \mathbb{R}^m$ is an open domain,
- $\mathbf{f} : U \to \mathbb{R}^n$ is twice differentiable, injective, and regular in the sense that rank $(\mathrm{d}\mathbf{f}) = m$.

Here and below, bold face letters are used for objects or functions with values in the geometry space $\mathbb{R}^n$. Vectors are understood as columns, while the differential operator $\mathrm{d} := [\partial_1, \ldots, \partial_m]$ is generating rows. For example, the Jacobian $\mathrm{d}\mathbf{f} = [\partial_1\mathbf{f}, \ldots, \partial_m\mathbf{f}]$ of $\mathbf{f}$ is a full rank $(n \times m)$-matrix.

The *normal vector field* of $\mathbf{H} \in \mathcal{H}^m$ is a continuous mapping $\mathbf{n} : \mathbf{H} \to \mathbb{S}^m$ from the surface to the m-dimensional unit sphere. The *Gauss map* $\bar{\mathbf{n}} : U \to \mathbb{S}^m$ corresponding to $\mathbf{f}$ is defined by $\bar{\mathbf{n}} := \mathbf{n} \circ \mathbf{f}$ and satisfies

$$\bar{\mathbf{n}}^{\mathrm{t}}\,\mathrm{d}\mathbf{f} = 0.$$

Differentiating $\bar{\mathbf{n}}^{\mathrm{t}}\bar{\mathbf{n}} = 1$ yields $\bar{\mathbf{n}}^{\mathrm{t}}\,\mathrm{d}\bar{\mathbf{n}} = 0$, implying range $\mathrm{d}\bar{\mathbf{n}} \subseteq$ range $\mathrm{d}\mathbf{f}$. Hence, there is a unique linear map $W : \mathbb{R}^m \to \mathbb{R}^m$, called the *Weingarten map* or also the *shape operator*, satisfying

$$-\mathrm{d}\bar{\mathbf{n}} = \mathrm{d}\mathbf{f}\,W. \tag{1}$$

Multiplication by $\mathrm{d}\mathbf{f}$ from the left yields the normal equation

$$B = GW,$$

where $G := \mathrm{d}\mathbf{f}^{\mathrm{t}}\,\mathrm{d}\mathbf{f}$ and $B := -\mathrm{d}\mathbf{f}^{\mathrm{t}}\,\mathrm{d}\bar{\mathbf{n}}$ are the first and second fundamental form of $\mathbf{f}$, respectively. By regularity of $\mathbf{f}$, the matrix G is invertible. Hence, we may solve for W to find

$$W = G^{-1}B = -\mathrm{d}\mathbf{f}^{+}\,\mathrm{d}\bar{\mathbf{n}}, \tag{2}$$

where

$$\mathrm{d}\mathbf{f}^{+} := G^{-1}\,\mathrm{d}\mathbf{f}^{\mathrm{t}} \tag{3}$$

is the pseudo-inverse of $\mathrm{d}\mathbf{f}$. Further analysis reveals that W is self-adjoint with respect to G. This fact guarantees existence of a complete set of real eigenvalues κ_i, called the *principal curvatures* of $\mathbf{H}$, corresponding to eigenspaces V_i,

$$V_i := \{v \in \mathbb{R}^m : Wv = \kappa_i v\}, \quad i = 1, \ldots, m.$$

It is easy to see that $v \in V_i$ implies

$$\mathrm{d}\bar{\mathbf{n}}\,v = -\kappa_i\,\mathrm{d}\mathbf{f}\,v. \tag{4}$$

When mapping vectors $v \in V_i$ from the parameter space $\mathbb{R}^m$ to geometry space $\mathbb{R}^n$, we obtain *principal directions* $\mathbf{v} := \mathrm{d}\mathbf{f}v$ corresponding to κ_i. The according *principal subspaces* are denoted by $\mathbf{V}_i := \{\mathrm{d}\mathbf{f}v\colon v \in V_i\}$.

It is well known that the principal curvatures κ_i and subspaces $\mathbf{V}_i$ are uniquely defined (up to ordering) and independent of the given parametrization of the surface $\mathbf{H}$. By contrast, the matrices G, B, W depend on the given parametrization. When considering piecewise defined surfaces, as generated by spline or subdivision techniques, this fact may result in discontinuities of these objects at patch boundaries, even if the surface is geometrically smooth. Examples can be found in [7]. The concept of curvature tensors resolves this problem. We characterize it as a linear map in $\mathbb{R}^n$ with κ_i and $\mathbf{v} \in \mathbf{V}_i$ as eigenvalues and -vectors, respectively. The normal vector $\mathbf{n}$ is a further eigenvector, where the corresponding eigenvalue λ can be prescribed at will. Of course, the canonical choice is $\lambda = 0$, but for instance, the case that $\lambda = -\kappa_m$ equals the negative mean curvature is considered in [5].

Below,

$$\mathcal{E}^n := \{\mathbf{A} \in \mathbb{R}^{n\times n} : \mathbf{A}^{\mathrm{t}} = \mathbf{A}\}$$

denotes the space of symmetric endomorphisms in $\mathbb{R}^n$.

Definition 1. Let $\mathbf{H} \in \mathcal{H}^m$ be a hypersurface with normal vector field $\mathbf{n}$, principal curvatures $\kappa_1, \ldots, \kappa_n$, and principal subspaces $\mathbf{V}_1, \ldots, \mathbf{V}_n$. Further, let $\lambda : \mathbf{H} \to \mathbb{R}$ be some function defined on $\mathbf{H}$. The mapping $\mathbf{E}_\lambda : \mathbf{H} \to \mathcal{E}(\mathbb{R}^n)$, characterized by

$$\mathbf{E}_\lambda\mathbf{n} = \lambda\mathbf{n} \quad \text{and} \quad \mathbf{E}_\lambda\mathbf{v} = \kappa_i\mathbf{v}, \quad \mathbf{v} \in \mathbf{V}_i, \;\; i = 1, \ldots, n-1$$

is called the *generalized curvature tensor of* $\mathbf{H}$ *corresponding to* λ. For $\lambda = 0$, we obtain the special case $\mathbf{E} := \mathbf{E}_0$, which is simply called the *curvature tensor* or also the *embedded Weingarten map* of $\mathbf{H}$.

It is important to note that, by definition, curvature tensors do not depend on the parametrization, but only on the geometry of $\mathbf{H}$ and the choice of λ. Clearly, the mean curvature κ_m and the Gaussian curvature κ_g are given by

$$\kappa_m = \frac{\operatorname{trace}\mathbf{E}_\lambda - \lambda}{m}, \quad \kappa_g = \frac{\det\mathbf{E}_\lambda}{\lambda},$$

respectively, the latter of course only if $\lambda \neq 0$.

Finally, we note that $\mathbf{E}$ and $\mathbf{E}_\lambda$ are related by

$$\mathbf{E}_\lambda = \mathbf{E} + \lambda\mathbf{n}\mathbf{n}^{\mathrm{t}}.$$

To show this, we consider two cases: First, $\mathbf{E}_\lambda\mathbf{n} = \lambda\mathbf{n} = \mathbf{E}\mathbf{n} + \lambda\mathbf{n}\mathbf{n}^{\mathrm{t}}\mathbf{n}$ since $\mathbf{E}\mathbf{n} = 0$ and $\mathbf{n}^{\mathrm{t}}\mathbf{n} = 1$. Second, if $\mathbf{v} = \mathrm{d}\mathbf{f}\,v$ is a principal direction with associated principal curvature κ, then $\mathbf{n}^{\mathrm{t}}\mathbf{v} = 0$ so that $\mathbf{E}_\lambda\mathbf{v} = \mathbf{E}\mathbf{v} = \kappa\mathbf{v}$, as requested. As a consequence of the latter display, it is sufficient to specify formulas for $\mathbf{E}$. We will elaborate on that issue in the next section.

3. Three different setups

In this section, we consider the curvature tensor of hypersurfaces given in three different forms: parametrized, implicit, and resulting form space deformation.

3.1. Parametrized surfaces

Formulas for the curvature tensor of parametrized surfaces $\mathbf{H} = \mathbf{f}(U)$, given in Theorem 1, strongly resemble those for the standard shape operator. As briefly sketched then, the crucial relation (7) suggests a most simple scheme for estimating curvature properties of facetted surfaces. Further, in Theorem 2, we provide integrability conditions which admit the reconstruction of a surface from given curvature data. The case of parametrized surfaces in $\mathbb{R}^3$ has been discussed in a similar way in [8].

Analogous to the Gauss map $\bar{\mathbf{n}} := \mathbf{n} \circ \mathbf{f} : U \to \mathbb{S}^m$, we introduce the function $\overline{\mathbf{E}} := \mathbf{E} \circ \mathbf{f} : U \to \mathcal{E}^n$, and call it the *parametrized curvature tensor*. By Definition 1, the conditions characterizing $\overline{\mathbf{E}}$ are

$$\overline{\mathbf{E}}\bar{\mathbf{n}} = 0 \quad \text{and} \quad \overline{\mathbf{E}}\mathbf{v} = \kappa_i \mathbf{v}, \quad \mathbf{v} \in \mathbf{V}_i, \; i = 1, \ldots, n.$$

In this setting, anything depends on the argument $u \in U$, which is omitted throughout to improve readability. The following theorem provides explicit formulas for the computation of $\overline{\mathbf{E}}$:

Theorem 1. *Let $\mathbf{H} = \mathbf{f}(U) \in \mathcal{H}^m$ be a hypersurface with Gauss map $\bar{\mathbf{n}}$. The parametrized curvature tensor $\overline{\mathbf{E}}$ is given by*

$$\overline{\mathbf{E}} = -\mathrm{d}\bar{\mathbf{n}}\,\mathrm{d}\mathbf{f}^+, \tag{5}$$

or, equivalently, by

$$\overline{\mathbf{E}} = (\mathrm{d}\mathbf{f}^+)^{\mathrm{t}} B\,\mathrm{d}\mathbf{f}^+, \tag{6}$$

where $\mathrm{d}\mathbf{f}^+ = G^{-1}\,\mathrm{d}\mathbf{f}^{\mathrm{t}}$ is the pseudo-inverse of $\mathrm{d}\mathbf{f}$, as defined in (3), and G, B are the first and second fundamental form of $\mathbf{f}$. Furthermore,

$$-\mathrm{d}\bar{\mathbf{n}} = \overline{\mathbf{E}}\,\mathrm{d}\mathbf{f}. \tag{7}$$

Proof. We denote the matrices given by the right hand sides of (5) and (6) by $\overline{\mathbf{E}}_1$ and $\overline{\mathbf{E}}_2$, respectively. First, $\mathrm{d}\mathbf{f}^{\mathrm{t}}\,\bar{\mathbf{n}} = 0$ shows that $\overline{\mathbf{E}}_1\mathbf{n} = \overline{\mathbf{E}}_2\mathbf{n} = 0$. Now, let $\mathbf{v} = \mathrm{d}\mathbf{f}\,v \in \mathbf{V}_i$ be a principal direction corresponding to κ_i. We obtain using (4)

$$\overline{\mathbf{E}}_1\mathbf{v} = -\mathrm{d}\bar{\mathbf{n}}G^{-1}\,\mathrm{d}\mathbf{f}^{\mathrm{t}}\,\mathrm{d}\mathbf{f}\,v = -\mathrm{d}\bar{\mathbf{n}}\,v = \kappa_i\,\mathrm{d}\mathbf{f}\,v = \kappa_i\mathbf{v},$$

and equally, using $B = -\mathrm{d}\mathbf{f}^{\mathrm{t}}\,\mathrm{d}\bar{\mathbf{n}}$,

$$\overline{\mathbf{E}}_2\mathbf{v} = \mathrm{d}\mathbf{f}G^{-1}\,\mathrm{d}\mathbf{f}^{\mathrm{t}}\,\overline{\mathbf{E}}_1\mathbf{v} = \kappa_i\mathrm{d}\mathbf{f}G^{-1}\,\mathrm{d}\mathbf{f}^{\mathrm{t}}\mathrm{d}\mathbf{f}\,v = \kappa_i\mathrm{d}\mathbf{f}\,v = \kappa_i\mathbf{v}.$$

Hence, $\overline{\mathbf{E}} = \overline{\mathbf{E}}_1 = \overline{\mathbf{E}}_2$. Finally, multiplying (5) by $\mathbf{df}$ from the right yields (7). □

Let us briefly comment on the three equations appearing in the theorem: First, when comparing (5) with (2), we find that W and $\overline{\mathbf{E}}$ differ just by the ordering of their factors,

$$W = -\mathrm{d}\mathbf{f}^+\,\mathrm{d}\mathbf{n}, \quad \overline{\mathbf{E}} = -\mathrm{d}\mathbf{n}\,\mathrm{d}\mathbf{f}^+.$$

Second, since the second fundamental form B is symmetric (6) shows that also $\overline{\mathbf{E}}$ is symmetric, thus reconfirming the well known fact that the principal curvatures are all real and that it is possible to choose an orthonormal set of principal directions $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Third, when comparing (7) with (1), we see that both operators establish a relation between $\mathrm{d}\bar{\mathbf{n}}$ and $\mathrm{d}\mathbf{f}$,

$$-\mathrm{d}\bar{\mathbf{n}} = \mathrm{d}\mathbf{f}\,W, \quad -\mathrm{d}\bar{\mathbf{n}} = \overline{\mathbf{E}}\,\mathrm{d}\mathbf{f}.$$

However, W is acting on the parameter space $\mathbb{R}^m$, while $\overline{\mathbf{E}}$ is acting on the geometry space $\mathbb{R}^n$, in which the surface $\mathbf{H}$ is embedded. This observation accounts for the name *embedded Weingarten map*, as it was proposed in [8].

There exist a few schemes for estimating the curvature tensor on facetted surfaces, see e.g., [10,3,9,2]. Compared with these approaches (7) suggests an extremely simple alternative: Given vertices $\mathbf{p}_j \in \mathbf{H}$ and (possibly estimated) corresponding normal vectors $\mathbf{n}_j \in \mathbb{S}^m, j = 1, \ldots, J$, one can proceed as follows. Without loss of generality, let us consider some triangle $\mathbf{H}_\Delta$ with vertices $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, approximating some part $\mathbf{H}_0$ of the surface $\mathbf{H}$. An approximation $\mathbf{E}_\Delta$ of the curvature tensor $\mathbf{E}_0$ at the center of $\mathbf{H}_0$ is sought. Regarding $\mathbf{H}_\Delta = \mathbf{f}_\Delta(T)$ and $\mathbf{H}_0 = \mathbf{f}_0(T)$ as images of the unit triangle T, we may employ the trivial discretization

$$\mathrm{d}\mathbf{f}_0 \approx \mathrm{d}\mathbf{f}_\Delta := [\mathbf{p}_2 - \mathbf{p}_1, \mathbf{p}_3 - \mathbf{p}_1],$$
$$\mathrm{d}\bar{\mathbf{n}}_0 \approx \mathrm{d}\bar{\mathbf{n}}_\Delta := [\mathbf{n}_2 - \mathbf{n}_1, \mathbf{n}_3 - \mathbf{n}_1],$$

to obtain the matrix

$$\widetilde{\mathbf{E}}_\Delta := -\mathrm{d}\bar{\mathbf{n}}_0\,\mathrm{d}\mathbf{f}_\Delta^+,$$

which is already an approximation of $\mathbf{E}_0$. However, to remedy the typical lack of symmetry of $\widetilde{\mathbf{E}}_\Delta$, we define

$$\mathbf{E}_\Delta := (\mathrm{Id} - \mathbf{n}_\Delta\mathbf{n}_\Delta^{\mathrm{t}})(\widetilde{\mathbf{E}}_\Delta + \widetilde{\mathbf{E}}_\Delta^{\mathrm{t}})(\mathrm{Id} - \mathbf{n}_\Delta\mathbf{n}_\Delta^{\mathrm{t}}),$$
$$\mathbf{n}_\Delta := (\mathbf{n}_1 + \mathbf{n}_2 + \mathbf{n}_3)/3.$$

One can show that this matrix is closest to $\widetilde{\mathbf{E}}_\Delta$ with respect to the Frobenius norm among all symmetric matrices satisfying $\mathbf{E}_\Delta\mathbf{n}_\Delta = 0$ for given $\mathbf{n}_\Delta$. First experiments [6] indicate that this approach combines extremely simple implementation with approximation order $\|\mathbf{E}_0 - \mathbf{E}_\Delta\| = O(h^2)$ provided that sufficiently accurate normal vectors are available. Details and further aspects of this issue are beyond the scope of this paper and will be addressed in a forthcoming report.

Another interesting application of curvature tensors concerns the reconstruction of surfaces from given curvature information. When prescribing fundamental forms, fairly complicated integrability conditions, known as Gauss and Mainardi–Codazzi equations, have to be obeyed. By contrast, the integrability conditions on the parametrized curvature tensor have a *much* simpler form.

Theorem 2. *Let $U \subset \mathbb{R}^m$ be an open, simply connected domain. Consider any C^2-function $\overline{\mathbf{E}} : U \to \mathcal{E}^n$ with the following properties:*

- *$\overline{\mathbf{E}}$ has a simple eigenvalue $\lambda = 0$. The corresponding normalized eigenvector is denoted by $\bar{\mathbf{n}} : U \to \mathbb{S}^m$, where the sign is chosen consistently to obtain a continuous map.*
- *For all $k, \ell = 1, \ldots, m$, it holds*

$$\partial_k\overline{\mathbf{E}}^+\partial_\ell\bar{\mathbf{n}} = \partial_\ell\overline{\mathbf{E}}^+\partial_k\bar{\mathbf{n}}, \tag{8}$$

where $\overline{\mathbf{E}}^+$ denotes the pseudo-inverse of $\overline{\mathbf{E}}$.

Then there exists a parametrization $\mathbf{f} : U \to \mathbb{R}^n$, unique up to translation, such that $\bar{\mathbf{n}}$ and $\overline{\mathbf{E}}$ are the Gauss map and the parametrized curvature tensor of $\mathbf{f}$, respectively. Further, no such parametrization exists if the second property above is violated.

Proof. We recall that, by definition of the pseudo-inverse, $\overline{\mathbf{E}}^+\bar{\mathbf{n}} = 0$, and $\overline{\mathbf{E}}^+\mathbf{v} = \kappa_i^{-1}\mathbf{v}$ for any $\mathbf{v} \in \mathbf{V}_i$. In particular, $\overline{\mathbf{E}}^+ \in \mathcal{E}^n$ is symmetric, too. Let us assume that $\overline{\mathbf{E}}$ and $\bar{\mathbf{n}}$ are given as specified above. Since 0 is a simple eigenvalue of $\bar{\mathbf{E}}$, the vector field $\bar{\mathbf{n}}$ satisfying $\bar{\mathbf{E}}\bar{\mathbf{n}} = 0$ is not only continuous, but as smooth as $\bar{\mathbf{E}}$. Hence, $\partial_k\partial_\ell\bar{\mathbf{n}} = \partial_\ell\partial_k\bar{\mathbf{n}}$. Define the $(n \times m)$-matrix $\mathbf{J} := -\bar{\mathbf{E}}^+ \mathrm{d}\bar{\mathbf{n}}$, and denote its columns by $\mathbf{J}_1, \ldots, \mathbf{J}_m$. Then, by (8),

$$\partial_k \mathbf{J}_\ell = \partial_k(\bar{\mathbf{E}}^+\partial_\ell\bar{\mathbf{n}}) = \partial_k\bar{\mathbf{E}}^+\partial_\ell\bar{\mathbf{n}} + \bar{\mathbf{E}}^+\partial_k\partial_\ell\bar{\mathbf{n}},$$

equals

$$\partial_\ell \mathbf{J}_k = \partial_\ell(\bar{\mathbf{E}}^+\partial_k\bar{\mathbf{n}}) = \partial_\ell\bar{\mathbf{E}}^+\partial_k\bar{\mathbf{n}} + \bar{\mathbf{E}}^+\partial_\ell\partial_k\bar{\mathbf{n}}.$$

Hence, there exists a potential $\mathbf{f} : U \to \mathbb{R}^n$ of $\mathbf{J}$, i.e., $\mathrm{d}\mathbf{f} = \mathbf{J}$. The vector field $\bar{\mathbf{n}}$ is the Gauss map of $\mathbf{f}$ since $\bar{\mathbf{n}}^t\mathrm{d}\mathbf{f} = -\bar{\mathbf{n}}^t\bar{\mathbf{E}}^+\mathrm{d}\bar{\mathbf{n}} = 0$ by the first condition on $\bar{\mathbf{E}}$. Further, $\mathrm{d}\mathbf{f} = -\bar{\mathbf{E}}^+\mathrm{d}\bar{\mathbf{n}}$ implies $\bar{\mathbf{E}}\mathrm{d}\mathbf{f} = -\mathrm{d}\bar{\mathbf{n}}$, showing that $\bar{\mathbf{E}}$ is indeed the parametrized curvature tensor of $\mathbf{f}$. Let us assume that $\mathbf{f}'$ is another parametrization complying with $\bar{\mathbf{n}}$ and $\bar{\mathbf{E}}$. Then $\mathrm{d}\mathbf{f} = \mathrm{d}\mathbf{f}' = -\bar{\mathbf{E}}^+\mathrm{d}\bar{\mathbf{n}}$ yields $\mathrm{d}\,(\mathbf{f} - \mathbf{f}') = 0$, showing that $\mathbf{f}$ and $\mathbf{f}'$ differ only by a constant. To establish necessity of (8), let us assume that there exists $\mathbf{f}$ with $\mathrm{d}\mathbf{f} = -\bar{\mathbf{E}}^+\mathrm{d}\bar{\mathbf{n}}$. Then $\mathbf{f}$ is at least as smooth as $\bar{\mathbf{E}}$ so that $\partial_k\partial_\ell\mathbf{f} = \partial_\ell\partial_k\mathbf{f}$ must hold. Now (8) follows immediately. □

3.2. Implicit surfaces

Curvature formulas are less commonly stated for hypersurfaces that are level sets of real-valued functions. Only recently, Goldman [4] collected old and provided new formulas for the principal curvatures in this context. Our results addressing the curvature tensor are different and reveal a simple underlying structure. Of course, the result (11) on the signed distance function is well known, but to the best of our knowledge (10) is new in this form. The matrix defined in [1] is somehow similar, but it is not symmetric, and the surface normal is a left rather than a right eigenvector.

The C^2-function $F : \Sigma \to \mathbb{R}$, defined on some open neighborhood $\Sigma \subset \mathbb{R}^n$ of $\mathbf{H}$, is called an *implicit representation* of the hypersurface $\mathbf{H} \in \mathcal{H}^m$, if

- $F_{|\mathbf{H}} = 0$, and
- F is regular in the sense that $\|\mathrm{D}F\| \neq 0$.

While d denotes differentiation with respect to parameters in $\mathbb{R}^m$, the operator $\mathrm{D} := [\partial_1, \ldots, \partial_n]$ denotes differentiation with respect to the coordinates in geometry space $\mathbb{R}^n$. It is our goal to establish formulas for the curvature tensor $\mathbf{E}$ in terms of the function F.

Given a parametrization $\mathbf{f} : U \to \mathbb{R}^n$ of $\mathbf{H}$, equality of two functions $\Phi, \Psi : \Sigma \to \mathbb{R}$ on $\mathbf{H}$ can be shown by the following simple argument:

$$\Phi \circ \mathbf{f} = \Psi \circ \mathbf{f} \quad \text{if and only if} \quad \Phi_{|\mathbf{H}} = \Psi_{|\mathbf{H}}.$$

In particular, $F_{|\mathbf{H}} = 0$ is equivalent to $F \circ \mathbf{f} = 0$. Hence, by the chain rule,

$$0 = \mathrm{d}(F \circ \mathbf{f}) = (\mathrm{D}F \circ \mathbf{f})\mathrm{d}\mathbf{f}.$$

Defining the function $\mathbf{N} : \Sigma \to \mathbb{S}^m$ by

$$\mathbf{N} := \frac{\mathrm{D}^t F}{\|\mathrm{D}F\|},$$

this implies that $\mathbf{N} \circ \mathbf{f}$ is perpendicular to $\mathbf{H}$. In other words,

$$\mathbf{n} := -\mathbf{N}_{|\mathbf{H}} \quad \text{and} \quad \bar{\mathbf{n}} = \mathbf{n} \circ \mathbf{f} = -\mathbf{N} \circ \mathbf{f},$$

are the normal vector field of $\mathbf{H}$ and the corresponding Gauss map, respectively. The deliberate choice of the orientation of the normal vector field opposite to $\mathbf{N}$ will yield simpler formulas later on. Below, $\mathrm{D}^2F = (\partial_i\partial_j F)_{i,j}$ denotes the Hessian of F. Further, we use the following convention: If functions defined on Σ and functions defined on the subset $\mathbf{H} \subset \Sigma$ appear in the same formula, then the functions defined on Σ are understood to be restricted to $\mathbf{H}$.

Theorem 3. *Let $\mathbf{H} \in \mathcal{H}^m$ be a hypersurface with implicit representation $F : \Sigma \to \mathbb{R}$. With $\mathbf{N}$ as defined above and $\mathbf{T} := \mathrm{Id} - \mathbf{n}\mathbf{n}^t : \mathbf{H} \to \mathcal{E}^n$ the orthogonal projector onto the tangent space, the curvature tensor is given by*

$$\mathbf{E} = \mathrm{D}\mathbf{N}\mathbf{T}, \tag{9}$$

or, equivalently, by

$$\mathbf{E} = \frac{1}{\|\mathrm{D}F\|}\mathbf{T}\mathrm{D}^2F\mathbf{T}. \tag{10}$$

In particular, if F is a *signed distance function*, i.e., $\|\mathrm{D}F\| = 1$, then

$$\mathbf{E} = \mathrm{D}^2F. \tag{11}$$

Proof. We denote the matrices given by the right hand sides of (9) and (10) by $\mathbf{E}_1$ and $\mathbf{E}_2$, respectively. First, $\mathbf{T}\mathbf{n} = 0$ yields $\mathbf{E}_1\mathbf{n} = \mathbf{E}_2\mathbf{n} = 0$. Now, let $\mathbf{v} = \mathrm{d}\mathbf{f}v \in \mathbf{V}_i$ be a principal direction corresponding to κ_i. We obtain using (4) and the chain rule

$$\begin{aligned}(\mathbf{E}_1 \circ \mathbf{f})\mathbf{v} &= (\mathrm{D}\mathbf{N} \circ \mathbf{f})(\mathrm{Id} - \bar{\mathbf{n}}\bar{\mathbf{n}}^t)\mathrm{d}\mathbf{f}\,v = (\mathrm{D}\mathbf{N} \circ \mathbf{f})\mathrm{d}\mathbf{f}\,v = -\mathrm{d}\bar{\mathbf{n}}v \\ &= \kappa_i\mathrm{d}\mathbf{f}\,v = \kappa_i\mathbf{v}.\end{aligned}$$

Thus, $\mathbf{E} = \mathbf{E}_1$. Regarding (10), we differentiate the identity $\|\mathrm{D}F\|\mathbf{N} = \mathrm{D}^tF$ using $\mathrm{D}\|\mathrm{D}F\| = (\mathrm{D}F\mathrm{D}^2F)/\|\mathrm{D}\ \ F\| = \mathbf{N}^t\mathrm{D}^2F$. This yields $\mathbf{N}\mathbf{N}^t\mathrm{D}^2F + \|\mathrm{D}F\|\,\mathrm{D}\mathbf{N} = \mathrm{D}^2F$. Solving for $\mathrm{D}\mathbf{N}$, we find

$$\mathrm{D}\mathbf{N} = \frac{1}{\|\mathrm{D}F\|}\mathbf{T}\mathrm{D}^2\ F,$$

and

$$\mathbf{E}_2 = \frac{1}{\|\mathrm{D}F\|}\mathbf{T}\mathrm{D}^2\ F\ \mathbf{T} = \mathrm{D}\mathbf{N}\mathbf{T} = \mathbf{E}_1 = \mathbf{E}.$$

Finally, if $\|\mathrm{D}F\| = 1$, we differentiate $\mathrm{D}F\mathrm{D}^tF = 1$ to obtain $\mathrm{D}F\mathrm{D}^2F = 0$. Hence, by symmetry of the Hessian, $\mathbf{N}^t\mathrm{D}^2F = 0$ and $\mathrm{D}^2F\mathbf{N} = 0$, showing that $\mathbf{T}\mathrm{D}^2F\mathbf{T} = \ \mathrm{D}^2F$. □

3.3. Space deformation

Surfaces can be defined from an initial surface by a deformation of the embedding space. This approach is used, for instance, in the context of surface morphing. Here, we want to relate curvature properties of the deformed surface with curvature properties of the initial surface and both the vector field representing the space deformation and its inverse. We are not aware of any references addressing this issue in a systematic way.

Consider a hypersurface $\mathbf{H} \in \mathcal{H}^m$ with normal vector field $\mathbf{n}$ and curvature tensor $\mathbf{E}$. As before, $\Sigma \subset \mathbb{R}^n$ denotes some open neighborhood of $\mathbf{H}$. Given a C^2-diffeomorphism $\Psi : \Sigma \to \Sigma_* \subset \mathbb{R}^n$, we define the new surface $\mathbf{H}_* := \Psi(\mathbf{H})$. That is, $\mathbf{H}$ is transformed into $\mathbf{H}_*$ by deforming the surrounding geometry space. Denoting the inverse of Ψ by Ψ_*: $\Sigma_* \to \Sigma$, we may also write $\mathbf{H} = \Psi_*(\mathbf{H}_*)$. Depending on the application, either Ψ or Ψ_* may be known, and it is our goal to derive formulas for the curvature tensor $\mathbf{E}_*$ of $\mathbf{H}_*$ in terms of $\mathbf{E}$, $\mathbf{n}$, and either Ψ or Ψ_*. The Jacobians of Ψ and Ψ_* are denoted by

$$\mathbf{J} := D\Psi \quad \text{and} \quad \mathbf{J}_* := D\Psi_*,$$

respectively. Clearly, these are inverse matrices,

$$\mathbf{J}\mathbf{J}_* = \mathrm{Id}.$$

Here and below, we use the following convention: If functions defined on Σ and Σ_* appear in the same formula, then they are understood to be evaluated at corresponding points $\mathbf{x} \in \Sigma$ and $\mathbf{x}_* \in \Sigma_*$, related by $\mathbf{x} = \Psi_*(\mathbf{x}_*)$ and $\mathbf{x}_* = \Psi(\mathbf{x})$. For instance, the above formula actually reads $\mathbf{J}(\mathbf{x})\mathbf{J}_*(\mathbf{x}_*) = \mathrm{Id}$.

Referring to Section 3.2, let $F : \Sigma \to \mathbb{R}$ be the signed distance function of $\mathbf{H}$, i.e., $F_{|\mathbf{H}} = 0$ and $\|DF\| = 1$. Then $F_* := F \circ \Psi_*$ is an implicit representation of $\mathbf{H}_*$. By the chain rule, $DF_* = DF\mathbf{J}_*$ and $DF = DF_*\mathbf{J}$. Hence, the unit vector fields $\mathbf{N}$ and $\mathbf{N}_*$ corresponding to F and F_*, are related by

$$\mathbf{N}_* = \frac{\mathbf{J}_*^t \mathbf{N}}{\|\mathbf{J}_*^t \mathbf{N}\|} = \frac{\mathbf{J}^{-t}\mathbf{N}}{\|\mathbf{J}^{-t}\mathbf{N}\|}.$$

When restricting $-\mathbf{N}$ and $-\mathbf{N}_*$ to $\mathbf{H}$ and $\mathbf{H}_*$, we obtain the normal vector fields $\mathbf{n}$ and $\mathbf{n}_*$, respectively. Below, we use the notation

$$D^2_{\mathbf{v}}\Psi := \sum_{i=1}^{n} \mathbf{v}_i D^2\Psi_i,$$

for the linear combination of Hessians of the coordinate functions of Ψ with the components of the vector $\mathbf{v} \in \mathbb{R}^n$.

Theorem 4. *Let $\mathbf{H} \in \mathcal{H}^m$ be a hypersurface with curvature tensor $\mathbf{E}$, and $\mathbf{H}_* := \mathbf{H}(\Psi)$ for some C^2-diffeomorphism, as introduced above. Then, with $\mathbf{T}_* := \mathrm{Id} - \mathbf{n}_*\mathbf{n}_*^t$ and the notation introduced above, the curvature tensor of $\mathbf{H}_*$ is given in terms of Ψ_* by*

$$\mathbf{E}_* = \|\mathbf{J}_*^t \mathbf{n}\|^{-1} \mathbf{T}_* \left(\mathbf{J}_*^t \mathbf{E} \mathbf{J}_* - D^2_{\mathbf{n}} \Psi_* \right) \mathbf{T}_*, \tag{12}$$

and in terms of Ψ by

$$\mathbf{E}_* = \mathbf{T}_* \mathbf{J}^{-t} (\|\mathbf{J}^{-t}\mathbf{n}\|^{-1} \mathbf{E} + D^2_{\mathbf{n}_*} \Psi) \mathbf{J}^{-1} \mathbf{T}_*. \tag{13}$$

Proof. Let F be the signed distance function of $\mathbf{H}$, as above. First, differentiating $F_* = F \circ \Psi_*$ twice yields $D^2 F_* = D\Psi_*^t D^2 F D\Psi_* + D^2_{DF}\Psi_*$. Since F is assumed to be the signed distance function, $D^t F = \mathbf{N}$ and $D^2 F = \mathbf{E}$. Further, $D^t F_* = \mathbf{J}_*^t \mathbf{F}^t = \mathbf{J}_*^t \mathbf{N}$, and comparison with (10) proves (12). Second, differentiating $F = F_* \circ \Psi$ twice yields

$$\begin{aligned}
\mathbf{E} &= D^2 F = D\Psi^t D^2 F_* D\Psi + D^2_{DF_*}\Psi \\
&= \mathbf{J}^t D^2 F_* \mathbf{J} - \|DF_*\| D^2_{\mathbf{n}_*}\Psi.
\end{aligned}$$

Hence,

$$\|\mathbf{J}^{-t}\mathbf{n}\|^{-1} D^2 F_* = \mathbf{J}^{-t}(\|\mathbf{J}^{-t}\mathbf{n}\|^{-1}\mathbf{E} + D^2_{\mathbf{n}_*}\Psi)\mathbf{J}^{-1},$$

and, in view of (10), multiplication with T_* from the left and the right verifies (13). □

If Ψ: $\mathbf{x} \to \mathbf{A}\mathbf{x} + \mathbf{x}_0$ is an invertible affine map, then formulas (12) and (13) attain a particularly simple form,

$$\mathbf{E}_* = \|\mathbf{A}^{-t}\mathbf{n}\|^{-1} \mathbf{T}_* \mathbf{A}^{-t} \mathbf{E} \mathbf{A}^{-1} \mathbf{T}_*.$$

4. Conclusion

This paper summarizes formulas for the computation of the curvature tensor for hypersurfaces in $\mathbb{R}^n$. They all have a relatively simple structure, and they are fairly easy to derive and to implement. Thus, we hope to promote the use of the curvature tensor in theory and applications.

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