1 - Preliminary notions of convex analysis

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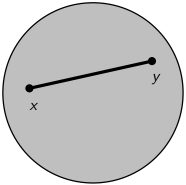
Convex sets

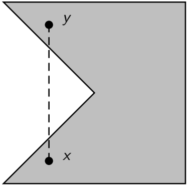
Definition (Convex set)

A set $C \subseteq \mathbb{R}^n$ is convex if, for every $x, y \in C$ and for every $\alpha \in [0, 1]$,

$$\alpha x + (1 - \alpha) y \in C.$$



convex set



non-convex set

Definition (Affine set)

A set $C \subseteq \mathbb{R}^n$ is affine if, for every $x, y \in C$ and every $\alpha \in \mathbb{R}$,

$$\alpha x + (1 - \alpha) y \in C.$$

Examples of affine sets:

- any single point $\{x\}$
- any line
- the solution set of a system of linear equations

$$C = \{x \in \mathbb{R}^n : \; Ax = b\},$$

  where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$
- any subspace

Examples of convex sets: subspaces

Note that a subspace is a particular affine set.

In fact, a set $S \subseteq \mathbb{R}^n$ is a subspace if, for every $x, y \in S$ and every $\alpha, \beta \in \mathbb{R}$,

$$\alpha x + \beta y \in S$$

Examples of subspaces:

- $\{0\}$
- any line which passes through zero
- the solution set of a homogeneous system of linear equations

$$S = \{x \in \mathbb{R}^n : \; Ax = 0\},$$

  where A is a $m \times n$ matrix.

Definition

A convex combination of the points $x^1, x^2, ..., x^k$ is a point

$$y = \sum_{i=1}^{k} \alpha_i x^i \text{ where } \alpha_1, \ldots, \alpha_k \in [0,1] \text{ and } \sum_{i=1}^{k} \alpha_i = 1.$$

Remark

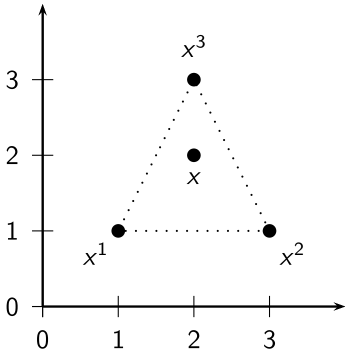
By definition, a set $C \subseteq \mathbb{R}^n$ is convex if it contains all the convex combinations of any two points in C.

Example. Consider the following 3 points in the plane:

$$x^1 = (1,1), \qquad x^2 = (3,1), \qquad x^3 = (2,3).$$

$x = (2,2)$ is a convex combination of x^1, x^2 e x^3, in fact:

$$x = \frac{1}{4}\,x^1 + \frac{1}{4}\,x^2 + \frac{1}{2}\,x^3.$$



A convex set contains any convex combination of its points.

Lemma 1

If C is convex, then for any $x^1, \ldots, x^k \in C$ and $\alpha_1, \ldots, \alpha_k \in [0,1]$ s.t. $\sum_{i=1}^{k} \alpha_i = 1$,

$$\sum_{i=1}^{k} \alpha_i x^i \in C.$$

Proof. By induction on k. For $k = 2$, the thesis holds, by definition of convexity. Assume that the thesis holds for a given k and let us prove it holds for $k + 1$.

Let $x^1, \dots, x^{k+1} \in C$ and $\alpha_1, \dots, \alpha_{k+1} \in [0, 1]$ s.t. $\sum_{i=1}^{k+1} \alpha_i = 1$. With no loss of generality, we assume that $\alpha_1 \neq 0$.

$$\sum_{i=1}^{k+1} \alpha_i x^i = \alpha_1 x_1 + \sum_{i=2}^{k+1} \alpha_i x^i = \alpha_1 x_1 + (1 - \sum_{i=2}^{k+1} \alpha_i) \sum_{i=2}^{k+1} \frac{\alpha_i}{1 - \sum_{i=2}^{k+1} \alpha_i} x^i$$

Since $\sum_{i=2}^{k+1} \frac{\alpha_i}{1-\sum_{i=2}^{k+1} \alpha_i} = 1$, by inductive assumption we have:

$$\bar{x} := \sum_{i=2}^{k+1} \frac{\alpha_i}{1 - \sum_{i=2}^{k+1} \alpha_i} x^i \in C$$

and finally, since C is convex,

$$\alpha_1 x_1 + (1 - \sum_{i=2}^{k+1} \alpha_i) \bar{x} \in C.$$

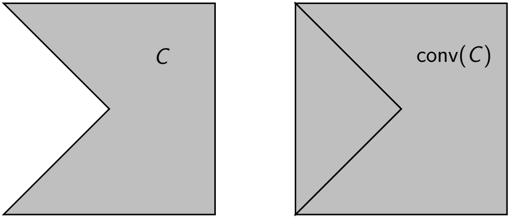
Proposition

If $\{C_i\}_{i \in I}$ is any (possibly infinite) family of convex sets, then $\bigcap_{i \in I} C_i$ is convex.

Definition (Convex hull)

The convex hull $\mathrm{conv}(C)$ of a set C is the intersection of all the convex sets containing C.

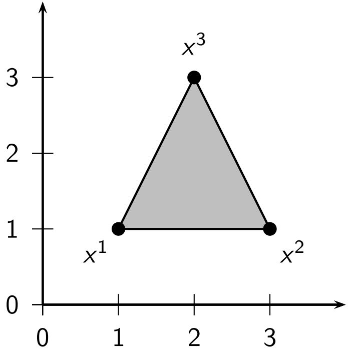
In other words, it is the smallest convex set containing C.



The convex hull of the points

$$x^1 = (1,1), \qquad x^2 = (3,1), \qquad x^3 = (2,3).$$

is the grey triangle with vertexes the three points:



Proposition

$$\text{conv}(C) = \{\text{all convex combinations of points in } C\}$$

Proof. It can be proved that the set of convex combinations of points in C is a convex set containing C, so that

$$\text{conv}(C) \subseteq \{\text{all convex combinations of points in } C\}.$$

Since $C \subseteq \text{conv}(C)$ and $\text{conv}(C)$ is convex, by Lemma 1 it contains any convex combination of its points, and therefore

$$\text{conv}(C) \supseteq \{\text{all convex combinations of points in } C\}.$$

Remark

Observe that C is convex if and only if $C = \text{conv}(C)$.

Examples of convex sets: Polyhedra

Definition (Polyhedron)

A polyhedron P is the intersection of a finite number of closed halspaces in $\mathbb{R}^n$.

A closed halfspace is the set of solutions of a linear inequality:

$$a^\mathsf{T} x \leq \beta, \qquad \text{where } a \in \mathbb{R}^n \text{ e } \beta \in \mathbb{R}.$$

Consequently, a polyhedron

$$P = \{x \in \mathbb{R}^n : \; Ax \leq b\}$$

is the solution set of a system of linear inequalities where A is a $m \times n$ matrix and $b \in \mathbb{R}^m$.

A polyhedron P is convex since any closed halfspace is a convex set and the intersection of convex sets is convex.

Examples of convex sets: Balls

- A ball is defined by $B(\bar{x}, r) := \{z \in \mathbb{R}^n : \|z - \bar{x}\| \leq r\}$, where $\|\cdot\|$ is any norm, e.g.

  $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ (Euclidean norm)

  $\|x\|_1 = \sum_{i=1}^{n} |x_i|$ (Manhattan distance)

  $\|x\|_\infty = \max_{i=1,\dots,n} |x_i|$ (Chebyshev norm)

  $\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$, with $1 \leq p < +\infty$

  $\|x\|_A = \sqrt{x^\mathsf{T} A x}$, where A is a symmetric and positive definite matrix, i.e.,

$$x^\mathsf{T} A x > 0 \qquad \forall\, x \neq 0.$$

Norms

Recall that a norm on a real vector space X is a function $p : X \to \mathbb{R}$ such that:

1. $p(x+y) \leq p(x) + p(y), \quad \forall x, y \in X;$
2. $p(\alpha x) = |\alpha| p(x), \quad \forall x \in X,\ \forall \alpha \in \mathbb{R};$
3. $p(x) = 0 \quad \Leftrightarrow \quad x = 0.$

By the previous conditions it follows that $p(x) \geq 0$, $\forall x \in X$.

Exercise 1.1 Find the unit ball $B(0,1)$ w.r.t. $\| \cdot \|_1$, $\| \cdot \|_\infty$ and $\| \cdot \|_A$, where $A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

Operations that preserve convexity

Algebraic operations

Sum and product by a constant

If C_1 and C_2 are convex, then $C_1 + C_2 := \{x + y : \; x \in C_1, \; y \in C_2\}$ is convex.

If C is convex and $\alpha \in \mathbb{R}$, then $\alpha C := \{\alpha x : \; x \in C, \}$ is convex.

Consequently, if C_1 and C_2 are convex, then $C_1 - C_2 := \{x - y : \; x \in C_1, \; y \in C_2\}$ is convex.

Topological operations

Closure and interior

If C is convex, then $\mathrm{cl}(C)$ is convex.
If C is convex, then $\mathrm{int}(C)$ is convex, provided that $\mathrm{int}(C) \neq \emptyset$.

Relative interior

Given a set $C \subseteq \mathbb{R}^n$ we denote by $aff(C)$ the smallest affine set containing C.

Definition (relative interior)

Let $C \subseteq \mathbb{R}^n$ be a convex set.
The relative interior of C is defined by

$$ri(C) = \{x \in C : \exists \epsilon > 0 \; s.t. \; aff(C) \cap B(x, \epsilon) \subseteq C\}$$

.

Examples

- Let $C := \{(x_1, x_2) \in \mathbb{R}^2 : 1 \leq x_1 \leq 3, x_2 = 0\}$. Then
$$ri(C) := \{(x_1, x_2) \in \mathbb{R}^2 : 1 < x_1 < 3, x_2 = 0\}.$$
- Let $C = \{\bar{x}\}$, then $ri(C) = C$.

Theorem

Let C be a nonempty convex set in $\mathbb{R}^n$. Then the relative interior of C is a nonempty convex set.

Separation of convex sets

The sets A and B in $\mathbb{R}^n$ are said to be linearly separable if there exists $a \in \mathbb{R}^n$, $a \neq 0$, $\beta \in \mathbb{R}$, such that

$$a^T x \geq \beta \quad \forall x \in A, \qquad a^T x \leq \beta \quad \forall x \in B,$$

The separation is said to be proper if strict inequality holds for at least one $x \in A \cup B$.

Theorem

Let A, B be nonempty convex sets in $\mathbb{R}^n$. Then A and B are properly linearly separable if and only if

$$ri(A) \cap ri(B) = \emptyset.$$

In particular two disjoint convex sets are always properly linearly separable.

Example Let $A := \{(x_1, x_2) \in \mathbb{R}^2 : 1 \leq x_1 \leq 2, x_2 = 0\}$,
$B := \{(x_1, x_2) \in \mathbb{R}^2 : 2 \leq x_1 \leq 4, x_2 = 0\}$.
Then $ri(A) \cap ri(B) = \emptyset$ and the sets are properly separable by the hyperplane of equation $x_1 = 2$.

Operations that preserve convexity

Affine functions

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be affine, i.e. $f(x) = Ax + b$, with $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$.

- If $C \subseteq \mathbb{R}^n$ is convex, then $f(C) = \{f(x) : \ x \in C\}$ is convex
- If $C \subseteq \mathbb{R}^m$ is convex, then $f^{-1}(C) = \{x \in \mathbb{R}^n : \ f(x) \in C\}$ is convex

Examples:

- $f(x) = \alpha\, x$, with $\alpha \in \mathbb{R}$
- $f(x) = x + b$, with $b \in \mathbb{R}^n$
- $f(x) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} x$, with $\theta \in (0, 2\pi)$ (rotation)

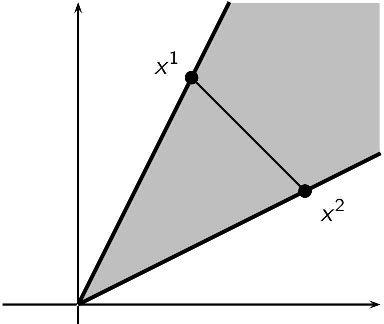
Definition (Cone)

A set $C \subseteq \mathbb{R}^n$ is a cone if, for every $x \in C$ and for every $\lambda \geq 0$, it results

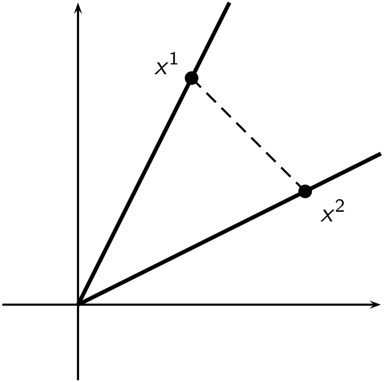
$$\lambda x \in C.$$

In other words, if C contains a point x different from 0 , then it contains the whole halfline starting from 0 and passing through x.

Example. A cone may be convex



or non convex:



Examples of cones

- $\mathbb{R}^n_+$ is a convex cone.
- $\{x \in \mathbb{R}^2 : \; x_1 \, x_2 = 0\}$ is a non-convex cone.
- Given a polyhedron $P = \{x : \; Ax \leq b\}$, the recession cone of P is defined as

$$\text{rec}(P) := \{d : \; x + \alpha \, d \in P \quad \text{for any } x \in P, \, \alpha \geq 0\}.$$

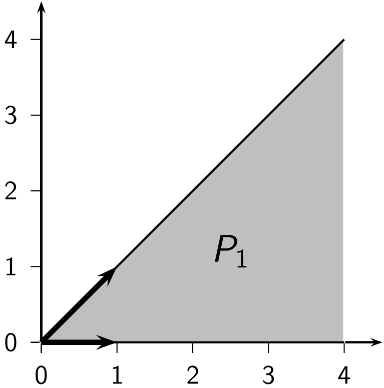
  It can be proved that $\text{rec}(P) = \{d : \; Ad \leq 0\}$, thus it is a polyhedral cone.
- $\left\{x \in \mathbb{R}^3 : \; x_3 \geq \sqrt{x_1^2 + x_2^2}\right\}$ is a non-polyhedral cone.
- Given $\bar{x} \in cl(C) \subseteq \mathbb{R}^n$, the set

$$T_C(\bar{x}) = \left\{ d \in \mathbb{R}^n : \exists \; \{z_k\} \subset C, \exists \; \{t_k\} > 0, \; z_k \to \bar{x}, \; t_k \to 0, \lim_{k \to \infty} \frac{z_k - \bar{x}}{t_k} = d \right\}$$

  is called the *tangent cone* to C at $\bar{x}$.

Example

$P_1 = \{x \in \mathbb{R}^2 : \; x_2 \leq x_1, \quad x_2 \geq 0\}$

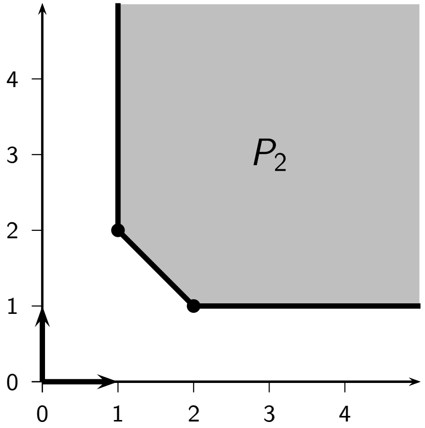


is a polyhedral cone.

$$rec(P_1) = P_1, \quad T_{P_1}((0,0)) = P_1.$$

Example

$P_2 = \{x \in \mathbb{R}^2 : \; x_1 \geq 1, \quad x_2 \geq 1, \quad x_1 + x_2 \geq 3\}$



$$rec(P_2) = \{d \in \mathbb{R}^2 : \; d_1 \geq 0, \quad d_2 \geq 0\}$$

$$T_{P_2}((1,2)) = \{d \in \mathbb{R}^2 : \; d_1 \geq 0, \quad d_1 + d_2 \geq 0\}$$

Exercises

1.2 Let $P = \{x : \ Ax \leq b\}$ where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$. Prove that $rec(P) = \{d \in \mathbb{R}^n : Ad \leq 0\}$.

1.3 If C_1 and C_2 are convex, then is $C_1 \cup C_2$ convex?

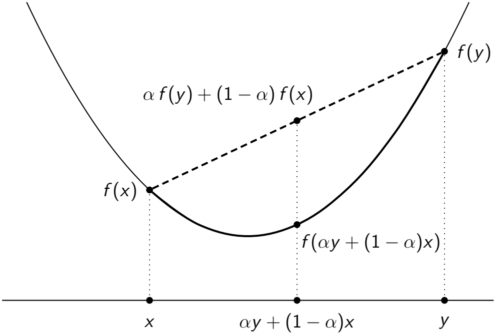
1.4 Prove that $B(\bar{x}, r) := \{z \in \mathbb{R}^n : \|z - \bar{x}\| \leq r\}$, is a convex set, whatever the norm $\| \cdot \|$ may be.

1.5 Write the vector $(1, 1)$ as a convex combination of the vectors $(0, 0), (3, 0), (0, 2), (3, 2)$.

Definition (Convex function)

Let $C \subseteq \mathbb{R}^n$ be convex. A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex on C if

$$f(\alpha y + (1-\alpha)x) \leq \alpha f(y) + (1-\alpha) f(x) \qquad \forall\, x, y \in C, \forall\, \alpha \in [0,1]$$



Remark

When $C = \mathbb{R}^n$ we will simply say that f is convex.

Theorem

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex on $\mathbb{R}^n$ if and only if the set

$$epi\, f_C := \{(x, y) \in C \times \mathbb{R} : y \geq f(x)\}$$

is convex.

Definition (Concave function)

Given a convex set $C \subseteq \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$ is concave on C if $-f$ is convex, i.e.,

$$f(\alpha y + (1 - \alpha)x) \geq \alpha f(y) + (1 - \alpha)f(x) \qquad \forall\, x, y \in C, \forall\, \alpha \in [0, 1]$$

Examples.

- A linear (affine) function $f(x) = c^{\mathsf{T}} x + b$ is both convex and concave.
- Let $\| \cdot \|$ be any norm, then $f(x) = \|x\|$ is convex.

Theorem (continuity of convex functions)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex on the convex set $C \subseteq \mathbb{R}^n$. Then f is continuous on $ri(C)$.

Strictly convex and strongly convex functions

Definition (strictly convex function)

Given a convex set $C \subseteq \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex on C if

$$f(\alpha y + (1-\alpha)x) < \alpha f(y) + (1-\alpha) f(x) \qquad \forall\, x, y \in C, x \neq y, \forall\, \alpha \in (0,1)$$

Definition (strongly convex function)

Given a convex set $C \subseteq \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex on C if there exists $\tau > 0$ s.t.

$$f(\alpha y + (1-\alpha)x) \leq \alpha f(y) + (1-\alpha) f(x) - \frac{\tau}{2}\alpha(1-\alpha)\|y-x\|_2^2$$
$$\forall\, x, y \in C, \forall\, \alpha \in [0,1]$$

Remark

Similarly to convex functions, we say that f is strictly (strongly) concave on C if $-f$ is strictly (strongly) convex on C.

Theorem

f is strongly convex if and only if $\exists\ \tau > 0$ such that $f(x) - \frac{\tau}{2}\|x\|_2^2$ is convex

Remark

By the previous theorem it follows that f is strongly convex if and only if there exists a convex function ψ and $\tau > 0$ such that $f(x) = \psi(x) + \frac{\tau}{2}\|x\|_2^2$.

Exercise 1.6

- Prove that: strongly convex $\Longrightarrow$ strictly convex $\Longrightarrow$ convex
- convex $\Longrightarrow$ strictly convex ?
- strictly convex $\Longrightarrow$ strongly convex ?

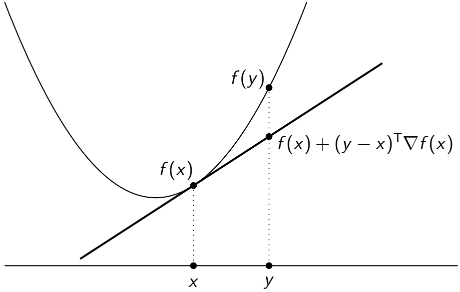
First order conditions

Assume that $C \subseteq \mathbb{R}^n$ is open and convex, $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable on C.

Theorem

f is convex on C if and only if

$$f(y) \geq f(x) + (y-x)^{\mathsf{T}} \nabla f(x) \qquad \forall\, x, y \in C.$$



First-order approximation of f is a global understimator

Theorem

- f is strictly convex on C if and only if

$$f(y) > f(x) + (y-x)^{\mathsf{T}} \nabla f(x) \qquad \forall\, x, y \in C, \text{with } x \neq y.$$

- f is strongly convex on C if and only if there exists $\tau > 0$ such that

$$f(y) \geq f(x) + (y-x)^{\mathsf{T}} \nabla f(x) + \frac{\tau}{2}\|y-x\|_2^2 \qquad \forall\, x, y \in C.$$

Second order conditions

Assume that $C \subseteq \mathbb{R}^n$ is open and convex, $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable on C.

Theorem

- f is convex on C if and only if for all $x \in C$ the Hessian matrix $\nabla^2 f(x)$ is positive semidefinite, i.e.

$$v^{\mathsf{T}} \nabla^2 f(x) v \geq 0 \qquad \forall\ v \in \mathbb{R}^n,\ \forall x \in C,$$

  or, equivalently, the eigenvalues of $\nabla^2 f(x)$ are ≥ 0, $\forall x \in C$.

- If $\nabla^2 f(x)$ is positive definite for all $x \in C$, then f is strictly convex on C.

- f is strongly convex on C if and only if there exists $\tau > 0$ such that $\nabla^2 f(x) - \tau I$ is positive semidefinite for all $x \in C$, i.e.

$$v^{\mathsf{T}} \nabla^2 f(x) v \geq \tau \|v\|_2^2 \qquad \forall\ v \in \mathbb{R}^n, \quad \forall\ x \in C,$$

  or, equivalently, the eigenvalues of $\nabla^2 f(x)$ are $\geq \tau$, $\forall\ x \in C$.

Convexity of quadratic functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$f(x) = \frac{1}{2} x^{\mathsf{T}} Q x + c^{\mathsf{T}} x$$

where Q is a $n \times n$ symmetric matrix, $c \in \mathbb{R}^n$. It is easy to check that

- $\nabla f = \frac{1}{2}(Qx + (x^{\mathsf{T}} Q)^{\mathsf{T}}) + c = Qx + c$
- Q is the Hessian of f.

Then f is:

- convex iff Q is positive semidefinite
- strongly convex iff Q is positive definite
- concave iff Q is negative semidefinite
- strongly concave iff Q is negative definite

Examples

Let $f : \mathbb{R} \to \mathbb{R}$ and $C := \mathbb{R}_+ \setminus \{0\}$.

- $f(x) = e^{px}$ for any $p \in \mathbb{R} \setminus \{0\}$ is strictly convex (on $\mathbb{R}$), but not strongly convex
- $f(x) = x^p$ is strictly convex on C if $p > 1$ or $p < 0$.
  Is it strongly convex?
- $f(x) = x^p$ is strictly concave on C if $0 < p < 1$
- $f(x) = \log(x)$ is strictly concave, but not strongly concave on C

Let $f : \mathbb{R}^n \to \mathbb{R}$.

- $f(x) = \|x\|$ is convex, but not strictly convex
- $f(x) = \max\{x_1, \ldots, x_n\}$ is convex, but not strictly convex

1.7 Prove that $f(x) = \|x\|$ is convex, whatever the norm $\|\cdot\|$ may be.

1.8 Prove that if f is convex, then for any $x^1, \ldots, x^k \in C$ and $\alpha_1, \ldots, \alpha_k \in (0,1)$ s.t. $\sum_{i=1}^{k} \alpha_i = 1$, one has $f\left(\sum_{i=1}^{k} \alpha_i x^i\right) \leq \sum_{i=1}^{k} \alpha_i f(x^i)$.

Hint. Follow the proof given in Lemma 1.

1.9 Prove that $f(x_1, x_2) = \dfrac{1}{x_1\, x_2}$ is convex on the set $\{x \in \mathbb{R}^2 : \ x_1, x_2 > 0\}$.

1.10 Analyse the convexity properties of the function

$$f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + x_3^2 + 3x_1x_2 + x_2x_3 - 6x_1 - 4x_2 - 3x_3$$

1.11 Let f_1 and f_2 be convex, then is the product $f_1\, f_2$ convex?

Operations that preserve convexity

Theorem

- If f is convex and $\alpha > 0$, then αf is convex
- If f_1 and f_2 are convex, then $f_1 + f_2$ are convex
- If f is convex, then $f(Ax + b)$ is convex

Examples

- Log barrier for linear inequalities:

$$f(x) = -\sum_{i=1}^{m} log(b_i - a_i^\top x) \qquad C = \{x \in \mathbb{R}^n : \; b_i - a_i^\top x > 0 \quad \forall\, i = 1, \dots, m\}$$

- Norm of affine function: $f(x) = \|Ax + b\|$

Theorem

- If $f_1, \ldots, f_m$ are convex, then $f(x) = \max\{f_1(x), \ldots, f_m(x)\}$ is convex.
- If $\{f_i\}_{i \in I}$ is a family of convex functions, then $f(x) = \sup_{i \in I} f_i(x)$ is convex.

Example. If $\psi(x, \lambda) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is convex in x and concave in λ, then

$$\begin{aligned} p(x) &= \sup_{\lambda} \psi(x, \lambda) \quad \text{is convex} \\ d(\lambda) &= \inf_{x} \psi(x, \lambda) \quad \text{is concave} \end{aligned}$$

Composition

$f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$.

Theorem

- If f is convex and g is convex and nondecreasing, then $g \circ f$ is convex.
- If f is concave and g is convex and nonincreasing, then $g \circ f$ is convex.

- If f is concave and g is concave and nondecreasing, then $g \circ f$ is concave.
- If f is convex and g is concave and nonincreasing, then $g \circ f$ is concave.

Examples Let $f : \mathbb{R} \to \mathbb{R}$.

- If f is convex, then $e^{f(x)}$ is convex
- If f is concave and positive, then $\log f(x)$ is concave
- If f is convex, then $-\log(-f(x))$ is convex on $\{x : \ f(x) < 0\}$
- If f is concave and positive, then $\dfrac{1}{f(x)}$ is convex
- If f is convex and nonnegative, then $f(x)^p$ is convex for all $p \geq 1$

Sublevel sets

Given $f : \mathbb{R}^n \to \mathbb{R}$ and $k \in \mathbb{R}$, the set

$$S_k(f) = \{x \in \mathbb{R}^n : \; f(x) \le k\}$$

is said the *k-sublevel set* of f.

Exercise 1.12 Prove that if f is convex, then $S_k(f)$ is a convex set for any $k \in \mathbb{R}$.

Is the converse true?

Quasiconvex functions

Definition (Quasiconvex convex function)

Given a convex set $C \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$ is said quasiconvex on C if the sets

$$S_k(f) \cap C = \{x \in C : \ f(x) \leq k\}$$

are convex for all $k \in \mathbb{R}$.

f is said quasiconcave on C if $-f$ is quasiconvex on C.

Examples

- $f(x) = \sqrt{|x|}$ is quasiconvex on $\mathbb{R}$
- $f(x_1, x_2) = x_1 \, x_2$ is quasiconcave on $\{x \in \mathbb{R}^2 : \ x_1 > 0, x_2 > 0\}$
- $f(x) = \log x$ is quasiconvex and quasiconcave