1) Consider the constrained optimization problem (P)

$$\begin{cases} \min\ x_1^2 - \log(x_1 + x_2) \\ 2x_1 + x_2 \le 4 \\ -x_1 - x_2 \le -1 \\ -x_1 \le 0 \\ x \in \mathbb{R}^2 \end{cases}$$

(a) Prove that (P) admits a global optimal solution;

(b) Apply the penalty method with starting point $x^0 = (4, 0)$, $\varepsilon^0 = 5$, $\tau = 0.1$ and using $\max_{i=1,2,3} g_i(x) < 10^{-6}$ as stopping criterion, where g_i denotes the i-th constraint. How many iterations are needed by the algorithm? Write the vector x and the value $\max_{i=1,2,3} g_i(x)$ found at the last three iterations.

(c) Is the obtained solution a global minimum of the given problem? Justify the answer.

SOLUTION

(a) The objective function $f(x)$ is continuous on the feasible set X which is compact, so that (P) admits a global optimal solution.

(b) **Matlab solution**

```
global  A b eps;

A = [2 1 ; -1 -1 ; -1 0 ];         %% data
b = [ 4 ; -1 ; 0 ];
tau = 0.1 ;                  %% parameters
eps0 = 5 ;
tolerance = 1e-6 ;
eps = eps0;
x = [4;0];
iter = 1;

SOL=[];

while true
    [x,pval] = fminunc(@p_eps,x);
    infeas = max(A*x-b);

    SOL=[SOL;iter,eps,x',infeas,pval];

    if infeas < tolerance
            break
    else
          eps = tau*eps;
          iter = iter + 1 ;
    end
end

fprintf('\t iter \t eps \t x(1) \t x(2) \t max(Ax-b) \t pval \n');

SOL

function v= p_eps(x)                %% penalized function
     global  A b eps;
     v = x(1)^2 -log(x(1)+x(2)) ;
     for i = 1 : size(A,1)
          v = v + (1/eps)*(max(0,A(i,:)*x-b(i)))^2 ;
    end
end
```

We obtain the following solution:

```
 iter    eps                   x(1)                x(2)                max(Ax-b)             pval

SOL =


    6.    0.0005         -0.000061     4.000018          0.0006205         -1.386295
    7.    0.000005       -0.000007     4.0000168        0.0000070         -1.3862854
    8.    0.0000005      0.000004      3.9999919        0.00000011         -1.3862933
```

The iterations of the algorithm are 8 and the found solution is $x^* \approx (0, 4)$ with optimal value $val(P) \approx -1.3863$.

(c) The point $x^* = (0, 4)$ is a global minimum. In fact the objective function is convex being the sum of two convex functions $f_1(x_1, x_2) = x_1^2$ and $f_2(x_1, x_2) = -\log(x_1 + x_2)$. In particular f_2 is convex being the composition of the convex function $-\log(y)$ with the linear function $y = x_1 + x_2$. The algorithm converges to a solution of the KKT conditions associated with (P), that in the present case are sufficient for optimality. The KKT conditions for (P) are given by

$$\begin{cases} 2x_1 - \frac{1}{x_1+x_2} + 2\lambda_1 - \lambda_2 - \lambda_3 = 0 \\ -\frac{1}{x_1+x_2} + \lambda_1 - \lambda_2 = 0 \\ \lambda_1(2x_1 + x_2 - 4) = \lambda_2(-x_1 - x_2 + 1) = \lambda_3(-x_1) = 0 \\ 2x_1 + x_2 - 4 \leq 0,\ -x_1 - x_2 + 1 \leq 0,\ -x_1 \leq 0, \\ \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{cases}$$

We obtain the solution $x^* = (0, 4)$ with $\lambda = (1/4, 0, 1/4)$.

2) Consider a clustering problem for the following set of 20 patterns given by the columns of the matrix:

$$D = \begin{pmatrix} 1.2 & 2.7 & 1.1 & 4.8 & 2.7 & 4.8 & 4.0 & 3.7 & 3.5 & 3.2 & 1.8 & 2.4 & 2.6 & 6 & 5.2 & 7.7 & 7.2 & 8 & 6 & 5.5 \\ 6.2 & 7.4 & 7.6 & 7.3 & 6.3 & 8 & 6.7 & 5.1 & 7.8 & 5.7 & 6.5 & 6.2 & 7.2 & 8 & 7.5 & 7 & 6.3 & 4 & 5 & 2.5 \end{pmatrix}$$

(a) Write the optimization model with 2-norm and $k = 3$ clusters;

(b) Solve the problem using the k-means algorithm with $k = 3$ and starting from centroids $x^1 = (1, 1), x^2 = (2, 2), x^3 = (3, 3)$. Write explicitly the vector of the obtained centroids and the value of the objective function;

(c) Solve the problem using the k-means algorithm with $k = 3$ and starting from centroids $x^1 = (1, 6), x^2 = (2, 5), x^3 = (3, 3)$. Write explicitly the vector of the obtained centroids, the value of the objective function and the clusters;

(d) Is any of the solutions found at points (b) and (c) a global optimum? Justify the answer.

SOLUTION

(a) Let $\ell = 20$, $(p^i)^T$ be the i-th row of the matrix D^T, $i = 1, .., \ell$, $k = 3$ the problem can be formulated by

$$\begin{cases} \min\limits_{x} \sum\limits_{i=1}^{\ell} \min\limits_{j=1,\dots,k} \|p^i - x^j\|_2^2 \\ x^j \in \mathbb{R}^2 \qquad \forall\, j = 1, \dots, k \end{cases} \tag{1}$$

which is equivalent to

$$\begin{cases} \min\limits_{x,\alpha} \; f(x,\alpha) := \sum\limits_{i=1}^{\ell} \sum\limits_{j=1}^{k} \alpha_{ij} \|p^i - x^j\|_2^2 \\ \sum\limits_{j=1}^{k} \alpha_{ij} = 1 \quad \forall\, i = 1, \dots, \ell \\ \alpha_{ij} \geq 0 \qquad \forall\, i = 1, \dots, \ell,\; j = 1, \dots, k \\ x^j \in \mathbb{R}^2 \qquad \forall\, j = 1, \dots, k. \end{cases} \tag{2}$$

provided that we look for an optimal solution with $\alpha \in \{0, 1\}^{\ell \times k}$, which can be done by the k-means algorithm.

(b)-(d) **Matlab solution**

```
data = [ 1.2    6.2
         2.7    7.4
         1.1    7.6
         4.8    7.3
         2.7    6.3
         4.8    8
         4.0    6.7
         3.7    5.1
         3.5    7.8
         3.2    5.7
         1.8    6.5
         2.4    6.2
         2.6    7.2
         6.0    8.0
         5.2    7.5
         7.7    7.0
         7.2    6.3
         8      4
         6      5
         5.5    2.5];

l = size(data,1); % number of patterns

k=3;

InitialCentroids=[1,1;2,2;3,,3];
 % InitialCentroids=[1,6;2,5;3,,3];

[x,cluster,v] = kmeans1(data,k,InitialCentroids)
```

```
function [x,cluster,v] = kmeans1(data,k,InitialCentroids)

l = size(data,1); % number of patterns

x = InitialCentroids;

cluster = zeros(l,1);                                    % initialize clusters
for i = 1 : l
    d = inf;
    for j = 1 : k
        if norm(data(i,:)-x(j,:)) < d
            d = norm(data(i,:)-x(j,:));
            cluster(i) = j;
        end
    end
end

vold = 0;                                   % compute the objective function value
for i = 1 : l
    vold = vold + norm(data(i,:)-x(cluster(i),:))^2 ;
end

while true
          for j = 1 : k                                      % update centroids
        ind = find(cluster == j);
        if isempty(ind)==0
            x(j,:) = mean(data(ind,:),1);
        end
    end


    for i = 1 : l                               % update clusters
        d = inf;
        for j = 1 : k
            if norm(data(i,:)-x(j,:)) < d
                d = norm(data(i,:)-x(j,:));
                cluster(i) = j;
            end
        end
    end

    v = 0;                                % update objective function
    for i = 1 : l
        v = v + norm(data(i,:)-x(cluster(i),:))^2 ;
    end

    if vold - v < 1e-5                          % stopping criterion
        break
    else
        vold = v;
    end
end
end
```

In case (b) we obtain the solution:

```
x =

    1.0000    1.0000     % centroids
    6.8800    4.9600
    3.3133    6.9000
```

```
v =    58.0573        %value of the objective function
```

In case (c) we obtain the solution:

```
x =

    2.3778    6.4667
    4.7167    7.5500
    6.8800    4.9600


cluster = 1 1 1 2 1 2 2 1 2 1 1 1 1 2  2 3 3 3 3 3  % The j-th component is the cluster assigned to pattern j

v =    34.1389
```

(d) It is possible to improve the previous solutions, with a multistart approach. For example, starting from centroids $x^1 = (9,8), x^2 = (9,1), x^3 = (4,4)$, we obtain the solution

```
x =

    5.9500    7.3500
    6.5000    3.8333
    2.6273    6.6091

v=  32.7676

cluster=   3 3 3 1 3 1 3 3 3 3 3 3 3 1 1 1 1 2 2 2
```

3) Consider the following multiobjective optimization problem:

$$\begin{cases} \min\ (x_1 - x_2 - x_3,\ x_1 - 2x_2) \\ x_1 + x_2 + x_3 - 4 \leq 0 \\ -x_1 \leq 0 \\ x_2 \leq 0 \end{cases}$$

(a) Prove that the problem admits a Pareto minimum point.

(b) Find the set of all weak Pareto minima.

(c) Find the set of Pareto minima.

(d) Does the problem admit an ideal solution?

SOLUTION

(a) We preliminarily observe that the problem is linear, since the objective and the constraint functions are linear. Therefore the set of weak minima coincides with the set of solutions of the scalarized problems (P_{α_1}), where $0 \leq \alpha_1 \leq 1$ and the set of minima coincides with the set of solutions of the scalarized problems (P_{α_1}), where $0 < \alpha_1 < 1$. (P_{α_1}) is defined by

$$\begin{cases} \min\ \alpha_1(x_1 - x_2 - x_3) + (1 - \alpha_1)(x_1 - 2x_2) =: \psi_{\alpha_1}(x) \\ x_1 + x_2 + x_3 - 4 \leq 0 \\ -x_1 \leq 0 \\ x_2 \leq 0 \end{cases}$$

For $\alpha_1 = \frac{1}{2}$, $\psi_{\alpha_1}(x) = x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3$. By the first inequality constraint

$$-\frac{1}{2}x_3 \geq \frac{x_1 + x_2 - 4}{2}$$

Therefore

$$x_1 - \frac{3}{2}x_2 - \frac{1}{2}x_3 \geq \frac{3}{2}x_1 - \frac{1}{2}x_2 - 2 \geq -2$$

where in the last inequality we have used the second and the third inequality constraint. Then, the objective function $\psi_{\alpha_1}(x)$ is bounded from below on the feasible set $X \neq 0$ which implies that the linear problem $P_{\frac{1}{2}}$ admits an optimal solution, which is a Pareto minimum.

(b) - (d) P_{α_1} can be solved by Matlab for $0 \leq \alpha_1 \leq 1$:

```
% The  Problem: min Cx, s.t. Ax <= b

C = [ 1 -1 -1;
       1  -2 0] ;

 A =[ 1 1  1;
     -1 0 0;
    0 1 0;
     ];
 b = [4,0,0]' ;

% % solve the scalarized problem with 0 =< alfa =< 1

MINIMA=[ ]; LAMBDA=[ ];  DEG=[ ];   % First column: value of alfa1

 for alfa1 = 0 : 0.01 : 1

     [x,fval,exitflag,output,lambda] = linprog(alfa1*C(1,:)+(1-alfa1)*C(2,:),A,b) ;

     MINIMA=[MINIMA;alfa1, x'];
     LAMBDA=[LAMBDA;alfa1,lambda.ineqlin'];

     S=find(lambda.ineqlin <0.01);        % find dual degenerate solutions
     if size(S,1)> 0.1
     DEG=[DEG;alfa1,x',lambda.ineqlin'];
     end
 end
```

By Matlab, we find that $\bar{x} = (0, 0, 4)$ is an optimal solution of (P_{α_1}) for every $0 \le \alpha_1 \le 1$. The dual solution λ is degenerate for $\alpha_1 = 0$ or $\alpha_1 = 1$ and non degenerate for $0 < \alpha_1 < 1$.

Therefore $\bar{x} = (0, 0, 4)$ is the unique Pareto minimum and it is also an ideal solution since it minimize both the objective functions.

For $\alpha_1 = 0$, the dual solution $\lambda = (0, 1, 2)$ is degenerate which implies that the solutions of the system

$$\begin{cases} x_1 + x_2 + x_3 - 4 \le 0 \\ -x_1 = 0 \\ x_2 = 0 \end{cases}$$

are all optimal solutions of P_0 and therefore are weak minima of (P).

For $\alpha_1 = 1$, the dual solution $\lambda = (1, 2, 0)$ is degenerate which implies that the solutions of the system

$$\begin{cases} x_1 + x_2 + x_3 - 4 = 0 \\ -x_1 = 0 \\ x_2 \le 0 \end{cases}$$

are all optimal solutions of P_1 and therefore are weak minima of (P).

We note that the previous solutions can also be obtained by solving the KKT conditions for (P_{α_1}) which is convex,differentiable and fulfils the Abadie constraints qualifications. The following KKT system provides a necessary and sufficient condition for an optimal solution of (P_{α_1}):

$$\begin{cases} 1 + \lambda_1 - \lambda_2 = 0 \\ -2 + \alpha_1 + \lambda_1 + \lambda_3 = 0 \\ -\alpha_1 + \lambda_1 = 0 \\ \lambda_1(x_1 + x_2 + x_3 - 4) = -\lambda_2 x_1 = \lambda_3 x_2 = 0 \\ x_1 + x_2 + x_3 - 4 \le 0 \\ -x_1 \le 0 \\ x_2 \le 0 \\ \lambda \ge 0 \\ 0 \le \alpha_1 \le 1, \end{cases}$$

In particular, solving the system given by the first 3 equation we obtain that

$$\begin{cases} \lambda_1 = \alpha_1 \\ \lambda_2 = \alpha_1 + 1 \\ \lambda_3 = 2 - 2\alpha_1 \\ \lambda_1(x_1 + x_2 + x_3 - 4) = -\lambda_2 x_1 = \lambda_3 x_2 = 0 \\ x_1 + x_2 + x_3 - 4 \le 0 \\ -x_1 \le 0 \\ x_2 \le 0 \\ \lambda \ge 0 \\ 0 \le \alpha_1 \le 1 \end{cases}$$

which leads to obtain the above mentioned solutions.

4) Consider the following bimatrix game:

$$C_1 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 4 & -1 \end{pmatrix} \qquad C_2 = \begin{pmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}$$

(a) Find the set of pure strategies Nash equilibria, if any. Alternatively, show that no pure strategies Nash equilibrium exists.

(b) Find a mixed strategies Nash equilibrium.

SOLUTION

(a) Strategy 2 of Player 1 is dominated by Strategy 1, so that row 2 in the two matrices can be deleted. The reduced game is given by the matrices

$$C_1^R = \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix} \qquad C_2^R = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}$$

Now, it is simple to show that (1,1) and (3,2) are pure strategies Nash equilibria. Indeed, the minima on the columns of C_1^R, (i.e., 1 and -1), are obtained in correspondence of the minima on the rows of C_2^R, (i.e., 2 and 0) and are related to the components $(1,1)$ and $(3,2)$ of the given matrices C_1 and C_2.

This will also be shown in part (b) in the wider context of mixed strategies Nash equilibria.

(b) Consider the reduced game obtained in (a). The optimization problem associated with Player 1 is

$$\left\{ \begin{array}{l} \min\ x^T C_1^R y = (x_1 + 4x_3)y_1 + -x_3 y_2 \\ x_1 + x_3 = 1 \\ x_1, x_3 \geq 0 \end{array} \right. \equiv \left\{ \begin{array}{l} \min\ (1 - 4y_1)x_1 + 5y_1 - 1 \\ 0 \leq x_1 \leq 1 \end{array} \right. \qquad (P_1(y_1))$$

Then, the best response mapping associated with $P_1(y_1)$ is:

$$B_1(y_1) = \left\{ \begin{array}{cl} 0 & \text{if } y_1 \in [0, 1/4) \\ [0,1] & \text{if } y_1 = 1/4 \\ 1 & \text{if } y_1 \in (1/4, 1] \end{array} \right.$$

Similarly, the optimization problem associated with Player 2 is

$$\left\{ \begin{array}{l} \min\ x^T C_2^R y = (2x_1 + x_3)y_1 + 3x_1 y_2 \\ y_1 + y_2 = 1 \\ y_1, y_2 \geq 0 \end{array} \right. \equiv \left\{ \begin{array}{l} \min\ (1 - 2x_1)y_1 + -3x_1 \\ 0 \leq y_1 \leq 1 \end{array} \right. \qquad (P_2(x_1))$$

Then, the best response mapping associated with $P_2(x_1)$ is:

$$B_2(x_1) = \left\{ \begin{array}{cl} 0 & \text{if } x_2 \in [0, 1/2) \\ [0,1] & \text{if } x_1 = 1/2 \\ 1 & \text{if } x_1 \in (1/2, 1] \end{array} \right.$$

The couples (x_1, y_1) such that $x_1 \in B_1(y_1)$ and $y_1 \in B_2(x_1)$ are

1. $x_1 = 0, y_1 = 0,$
2. $x_1 = \frac{1}{2}, y_1 = \frac{1}{4},$
3. $x_1 = 1, y_1 = 1,$

so that, recalling that $x_2 = 0$,

- $(x_1, x_2, x_3) = (0, 0, 1), \quad (y_1, y_2) = (0, 1)$, is a pure strategies Nash equilibrium,
- $(x_1, x_2, x_3) = (\frac{1}{2}, 0, \frac{1}{2}), \quad (y_1, y_2) = (\frac{1}{4}, \frac{3}{4})$, is a mixed strategies Nash equilibrium,
- $(x_1, x_2, x_3) = (1, 0, 0), \quad (y_1, y_2) = (1, 0)$, is a pure strategies Nash equilibrium.