Introduction to Curved Spaces (Manifolds)

Lecture 1: Smooth manifolds, curves and tangent spaces

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Numerical Calculus class (2018)

Outline

History

Bernhard Riemann

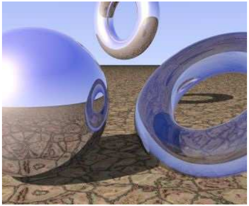
Carl Friedrich Gauss

Albert Einstein

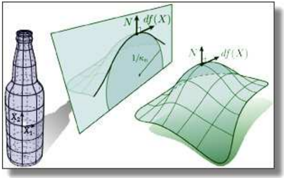
And Ricci-Curbastro, Levi-Civita, Bianchi, Codazzi, Calabi, Chen, Yau, Nash, Perelman, ...

Smooth manifolds
Examples

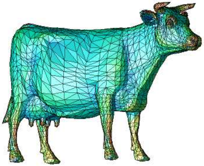
The basic ingredients are *smooth manifolds*



Sphere, Torus



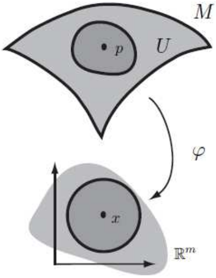
Abstract manifold



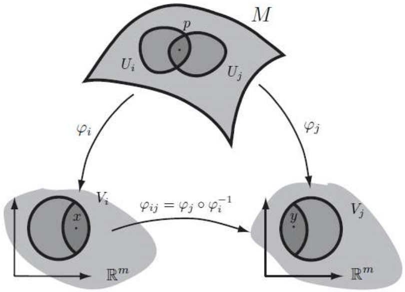
Data manifold

Smooth manifolds
Intuitive definition

The prominent feature of *smooth manifolds* is that it is a dense collection of *abstract objects* to which we can attach labels (real coordinates).



Coordinate chart



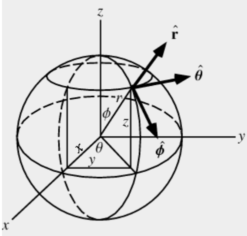
Coordinate patches

Smooth manifolds

Example: 3D sphere

A point on the 3-dimensional (ordinary) sphere may be parameterized as follows:

$$x = \begin{bmatrix} \cos\theta \sin\varphi \\ \sin\theta \sin\varphi \\ \cos\varphi \end{bmatrix}$$



for $0 \leq \theta < 2\pi$ and $0 \leq \varphi \leq \pi$.

Note: This parameterization breaks down at the *north pole* and *south pole* where the azimuth angle θ is not determined uniquely.

Smooth manifolds
Extrinsic coordinates

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Extrinsic coordinates seem natural and are useful to accomplish general/theoretical calculations, to prove theorems and to illustrate general features of manifolds. The minimum number of extrinsic coordinates needed to individuate points on a manifold is called dimension of the manifold

Smooth manifolds

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Extrinsic coordinates are certainly useful to perform practical calculations on 2- or 3- or 4-dimensional manifolds (e.g., ordinary curves and surfaces, space-time theory)

Smooth manifolds

Extrinsic coordinates

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Smooth manifolds

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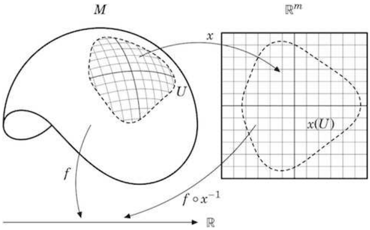
$p \in M$ and $f : M \to \mathbb{R}^m$

How do we define $f(p)$?

$f(p) \stackrel{\text{def}}{=} f(x^{-1}(y)) = (f \circ x^{-1})(y)$

$y \in \mathbb{R}^m$

$f \circ x^{-1}$ is f expressed in local coordinates



Smooth manifolds

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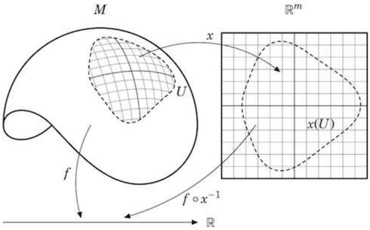

Message: There follows the extrinsic calculus on manifolds

Smooth manifolds

Intrinsic coordinates

Representing a point on a 3-sphere in extrinsic coordinates is fairly easy... But how do we represent a point on a 25-sphere ? We would need 24 angles and a set of complicated trigonometric-function-based maps

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For example, the sphere S^{24} embeds easily into $\mathbb{R}^{25}$ and each point $x \in S^{25}$ may be represented as a 25-components vector $x \in \mathbb{R}^{25}$ such that $\sum_{i=1}^{25} x_i^2 = x^T x = 1$

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Clearly, one of the 25 coordinates is redundant. The x_i's are termed intrinsic coordinates induced by the embedding

Smooth manifolds

Intrinsic coordinates

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Smooth manifolds

Intrinsic coordinates

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- Hyper-rotation group:
  $SO(n) = \{X \in \mathbb{R}^{n \times n} \mid X^T X = I,\ \det(X) = 1\}$

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- Symmetric, positive-definite matrices:
  $S^+(n) = \{X \in \mathbb{R}^{n\times n} \mid X = X^T,\ X > 0\}$

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The size n ranges from some units to some hundreds, depending on the application

Smooth manifolds
Intrinsic coordinates

Intrinsic coordinates are redundant compared with extrinsic coordinates

For example, a $n \times n$ symmetric, positive-definite matrix has n^2 entries (intrinsic coordinates) but only $\frac{n(n+1)}{2}$ independent entries (extrinsic coordinates)

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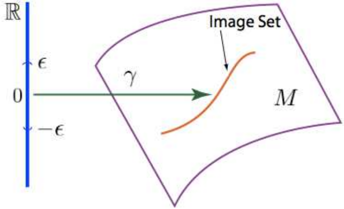
Message: As long as computer-based representation/implementation is concerned, redundancy is not a problem

Smooth manifolds

Smooth curves on manifolds

What is a *smooth curve* on a smooth manifold ?

A smooth curve on a manifold M is a regular map $\gamma : [-\epsilon,\ \epsilon] \to M$



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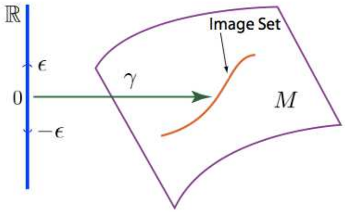

Message: Curves are fairly simple objects, nevertheless, they are overly important! Curves even allow measuring distances on curved spaces!

Smooth manifolds

Examples of smooth curves on manifolds

Examples of curves on the real hypersphere S^{n-1}:

Smooth manifolds

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Examples of curves on the real hypersphere S^{n-1}:

$\gamma(t) = \frac{x+tv}{\sqrt{(x+tv)^T(x+tv)}}$ is a curve on S^{n-1} for $t \in [-\epsilon,\ \epsilon]$, $x, v \in \mathbb{R}^n$ arbitrary. It holds $\gamma(0) = x/\|x\|$.

Hint: To verify it, try computing $\gamma^T(t)\gamma(t)$ and show that it equals 1 for every t.

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$\gamma(t) = x\cos(\sqrt{v^Tv}t) + \frac{v}{\sqrt{v^Tv}}\sin(\sqrt{v^Tv}t)$ is a curve on S^{n-1} for $t \in [-\epsilon,\ \epsilon]$ as long as we choose $x \in S^{n-1}$ and $v^Tx = 0$. It holds $\gamma(0) = x$.

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Did we use extrinsic or intrinsic coordinates ?

Smooth manifolds

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Smooth manifolds

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$\gamma(t) = \begin{bmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{bmatrix}$ is a curve on $SO(2)$ for $t \in [-\epsilon, \ \epsilon]$, $b \in \mathbb{R}$ arbitrary. It holds $\gamma(0) = I_2$.

Hint: To verify it, try computing $\gamma^T(t)\gamma(t)$ and show that it equals the identity matrix I_2 for every t and that $\det(\gamma(t)) = 1$.

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$\gamma(t) = X(I + tH)(I - tH)^{-1}$ is a curve on $SO(n)$ for $t \in [-\epsilon,\ \epsilon]$ as long as we choose $X \in SO(n)$ and $H^T = -H$.

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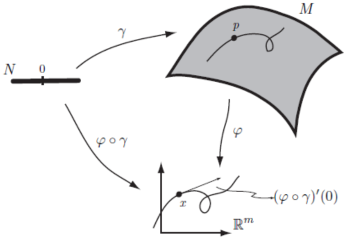
$\gamma(t) = X + t^2 H$ is a curve on $S^+(n)$ for $t \in [-\epsilon,\ \epsilon]$ for every $X, H \in S^+(n)$ arbitrary. It holds $\gamma(0) = X$.

Smooth manifolds
Tangent vectors

Tangent vectors: Definition by extrinsic coordinates

Tangent to a curve γ on a manifold M at a point p (in extrinsic coordinates)



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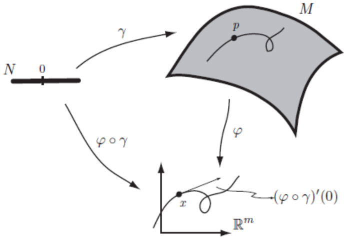
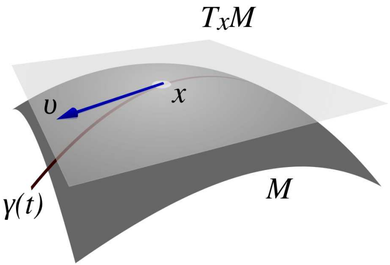


Let $\gamma : [-\varepsilon,\ \varepsilon] \to M$ and $\varphi : U \to \mathbb{R}^m$.
Note that $U \supset \gamma([-\varepsilon,\ \varepsilon])$ and that $\gamma(0) = p$ and $\varphi(p) = x$.
Tangent vector at p: $v = \left.\frac{d\varphi(\gamma(t))}{dt}\right|_{t=0}$

Smooth manifolds
Tangent vectors

Tangent vectors in intrinsic coordinates

Tangent to a curve γ on a manifold M at a point x (in intrinsic coordinates)



Smooth manifolds

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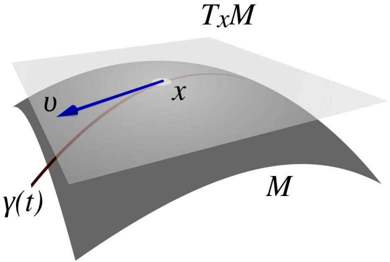

Let $\gamma : [-\varepsilon,\ \varepsilon] \to M$ embedded. Note that $\gamma(0) = x$

Tangent vector at x: $v = \left.\frac{d\gamma(t)}{dt}\right|_{t=0}$

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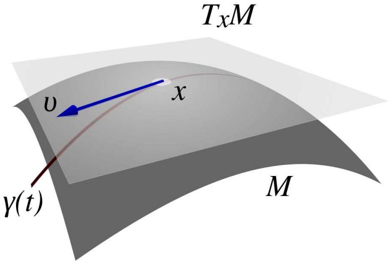


Let $\gamma : [-\varepsilon,\ \varepsilon] \to M$ embedded. Note that $\gamma(0) = x$

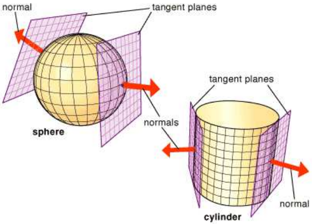
Tangent vector at x: $v = \left.\frac{d\gamma(t)}{dt}\right|_{t=0}$

The set of all tangent vectors at $x \in M$ forms a linear space T_xM termed tangent space

Smooth manifolds

Tangent spaces

To each point of a nonlinear manifold may be attached a linear structure termed tangent space



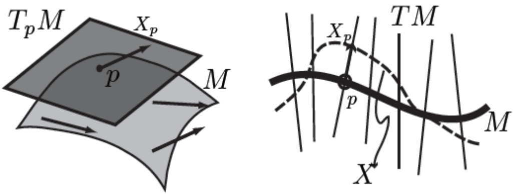
The disjoint union of all tangent spaces forms the *tangent bundle* $TM = \{(x, v) \mid x \in M, v \in T_x M\}$

Smooth manifolds

Generalization: Vector fields

Example: To a manifold may be attached a smooth vector field

A smooth vector field is denoted as $X : M \to TM$



Smooth manifolds

Examples of tangent spaces

Let us calculate the structure of tangent spaces to hyper-spheres S^{n-1} in intrinsic coordinates

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Basic ingredients:
A point $x \in S^{n-1}$
Any smooth curve $\gamma : [-\varepsilon,\ \varepsilon] \to S^{n-1}$ such that $\gamma(0) = x$

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Main idea:
No need to choose any specific curve γ, just recall that for γ to belong entirely to the hyper-sphere, it is necessary that $\gamma^T(t)\gamma(t) = 1$ for any $t \in [-\varepsilon,\ \varepsilon]$
Derive hand-by-hand with respect to the parameter t:
$\dot{\gamma}^T(t)\gamma(t) + \gamma^T(t)\dot{\gamma}(t) = 0$

Smooth manifolds

Examples of tangent spaces

Let us calculate the structure of tangent spaces to hyper-spheres S^{n-1} in intrinsic coordinates

Calculations continued:
$2\dot{\gamma}^T(t)\gamma(t) = 0$
Now set $t = 0$ (we want to know the tangent space at x)
Call $v = \dot{\gamma}(0)$
Gives $v^T x = 0$

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Conclusion:

The sought tangent space has structure:

$T_x S^{n-1} = \{v \in \mathbb{R}^n \mid v^T x = 0\}$

x is a radial vector, normal to the surface of the sphere

v is any orthogonal vector to x

Smooth manifolds

Examples of tangent spaces

Let us calculate the structure of tangent spaces to the manifold of hyper-rotations $SO(n)$ in intrinsic coordinates

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Now set $t = 0$ (we want to know the tangent space at X)
$\dot{\gamma}^T(0)\gamma(0) + \gamma^T(0)\dot{\gamma}(0) = 0$ Call $V = \dot{\gamma}(0)$
Gives $V^T X + X^T V = 0$

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Conclusion:

The sought tangent space has structure:
$T_X SO(n) = \{V \in \mathbb{R}^{n\times n} \mid V^T X + X^T V = 0\}$
Example: $T_I SO(n) = \{H \in \mathbb{R}^{n\times n} \mid H^T + H = 0\}$= set of skew-symmetric matrices

Smooth manifolds

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Let us calculate the structure of tangent spaces to the manifold of SPD matrices $S^+(n)$ in intrinsic coordinates

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Derive hand-by-hand with respect to the parameter t:
$\dot{\gamma}^T(t) = \dot{\gamma}(t)$

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Calculations continued:
Now set $t = 0$ (we want to know the tangent space at X)
Gives $\dot{\gamma}^T(0) = \dot{\gamma}(0)$
Call $V = \dot{\gamma}(0)$
Gives $V^T = V$
The positive-definiteness constraint does not reflect in any condition. Take the case $\gamma(t) = \begin{bmatrix} x_{11}(t) & 0 \\ 0 & x_{22}(t) \end{bmatrix}$: Although $x_{11}(t), x_{22}(t) > 0$, the signs of $\dot{x}_{11}(t), \dot{x}_{22}(t)$ are undetermined

Smooth manifolds

Examples of tangent spaces

Let us calculate the structure of tangent spaces to manifold of SPD matrices $S^+(n)$ in intrinsic coordinates

Calculations continued:

Now set $t = 0$ (we want to know the tangent space at X)
Gives $\dot{\gamma}^T(0) = \dot{\gamma}(0)$
Call $V = \dot{\gamma}(0)$
Gives $V^T = V$
The positive-definiteness constraint does not reflect in any condition. Take the case $\gamma(t) = \begin{bmatrix} x_{11}(t) & 0 \\ 0 & x_{22}(t) \end{bmatrix}$: Although $x_{11}(t), x_{22}(t) > 0$, the signs of $\dot{x}_{11}(t), \dot{x}_{22}(t)$ are undetermined

Conclusion:

The sought tangent space has structure:
$T_X S^+(n) = \{V \in \mathbb{R}^{n\times n} \mid V^T - V = 0\}$=set of symmetric matrices

Thanks

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