CS271: DATA STRUCTURES

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Project #7: Proofs +5 proof points

Proofs

+3 1. **Prove** your answer from 2b.

Proof. Let $G = (V, E)$ be a clique. Assume for contradiction that G's chromatic number is not equal to $|V|$. Therefore, G's chromatic number is either more or less than $|V|$.

Case 1: Assume G's chromatic number is larger than $|V|$. If we color each vertex a different color, the number of chromatic colors needed is at most $|V|$. Therefore, it is impossible for G's chromatic number is larger than $|V|$ since each $v \in G.V$ ~~$|V|$~~ can only be colored with one color. Therefore, if G's chromatic number is larger than $|V|$, the definition of chromatic number, which states that the chromatic number of a graph is the minimum number of colors in a proper coloring of that graph, is violated. ✓

what does the Pigeon-hole Priciple state?

Case 2: Assume G's chromatic number is smaller than $|V|$. Therefore, according to the Pigeonhole Principle, at least two vertices $\in V$ must have the same color. The definition of a clique states that $\forall u \in G.V$, $G.adj[u] = G.V - u$. If two vertices have the same colors, one vertex in $G.adj[u]$ will have the same color as u. This contradicts the definition of proper coloring, which states that no two adjacent vertices can have the same color. -

Therefore, it is proven by contradiction that G's chromatic number is equal to $|V|$.

□

+2 2. **Prove** the following:

An undirected graph is bipartite if and only if it contains no cycles of odd length.

We must prove that if an undirected graph has no cycles of odd length, the graph is bipartite. We must also prove that no graphs with odd-length cycles are bipartite.

appears to be bipartite ⇒ no odd cycles

Lemma 1. *An undirected graph is bipartite if it contains no cycles of odd length*

Proof. Suppose a graph is bipartite. By definition, a graph $G = (V, E)$'s vertices can be separated into two sets V_1 and V_2 such that $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \varnothing$.

Then, we can see that every step along a traversal from a vertex in V_1 can get us to a connected vertex in V_2, and vice versa since they are adjacent. Assume that there is a cycle from a vertex in V_1, to go through this cycle starting from this arbitrary vertex, we will always have to traverse back and forth between V_1

and V_2 one vertex at a time, making it an even number of vertices. Hence, there are no odd-length cycles.
yes - for clarity actually specify your cycle: $v_1, v_2, \ldots, v_1$ to ground your logic

Therefore, an undirected graph is bipartite if it contains no cycles of odd length

□

Lemma 2. *No graphs with odd-length cycles are bipartite.* need: no odd cycles ⇒ bipartite
rephrase of Lemma 1?

Proof. For the sake of contradiction, suppose there exists a bipartite graph $G = (V, E)$ with at least one odd-length cycle. The definition of a bipartite graph states that it is a graph whose vertices $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, such that there are no edges between vertices of the same set.

We separate set V into subsets V_1 and V_2. If there exists an odd length cycle, the number of vertices and edges within the cycle would be equal to $2n+1$, $n \in R$, $n \geq 1$, $n < \lceil\frac{|V|}{2}\rceil - 1$. Therefore, when partitioning the cycle into 2, $|V_1| \geq |V_2| + 1$, dividing an odd number by 2 will yield a remainder of 1. Because there exists the same number of edges as vertices within the cycle, there do not exist enough vertices in V_2 for vertices in V_1 to connect to. Therefore, there must be at least one edge between vertices in V_1. This contradicts the definition of a bipartite graph. similar logic to Lemma 1?

Hence, no graphs with odd-length cycles are bipartite. □

According to **Lemma 1**, all undirected graphs with no odd-length cycles are bipartite. According to **Lemma 2**, no undirected graphs with odd-length cycles are bipartite. Therefore, it is proven that an undirected graph is bipartite if and only if it contains no cycles of odd length.