Nonlinear control and aerospace applications

Orbital dynamics

Carlo Novara

Politecnico di Torino

Outline

Introduction

- A spacecraft can be (approximately) described as a rigid body, which moves with respect to some inertial frame.
- The body motion is a combination of
  - a translation of the body center of mass (CoM);
  - a rotation of the body about an axis passing through the CoM.
- The **objective** here is to study the *orbital dynamics*, i.e., the translational motion of a mass in a gravitational field.
  - This study is fundamental for spacecraft control.
- Orbital dynamics is based on celestial mechanics:
  - Kepler's laws: empirical laws describing the motion of a body in unperturbed planetary orbits;
  - Newton's laws: general physical laws that imply the Kepler laws.

Introduction

- Kepler's laws:
  1. The orbit of a planet is an ellipse with the sun located at one focus.
  2. The radius vector drawn from the sun to a planet sweeps out equal areas in equal time intervals.
     - ★ In other words, the areal velocity is constant.
  3. Planetary periods of revolution are proportional to $r_{\mathrm{m}}^{3/2}$, where r_{m} is the mean distance from the sun.
- The Kepler's laws can be derived from the more general Newton's laws (3 laws of motion + 1 law of gravitation).

Introduction

- Newton's laws of motion:
  1. A particle remains at rest or continues to move at a constant velocity, unless acted upon by an external force.
  2. The rate of chance of linear momentum $m\mathbf{v}$ of a particle is given by

     $$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$$

     where

     $$\begin{aligned} m &: \text{ particle mass} \\ \mathbf{v} &: \text{ particle velocity} \\ \mathbf{F} &: \text{ force acting on the particle.} \end{aligned}$$

  3. For any force $\mathbf{F}_{12}$ exerted by a particle 1 on a particle 2, there exists a force

     $$\mathbf{F}_{21} = -\mathbf{F}_{12}$$

     exerted by particle 2 on particle 1.

Introduction

- Newton's law of gravitation:

  Any two particles attract each other with a force

  r vector, ha una direzione

$$\mathbf{F} = \frac{G m_1 m_2 \, \mathbf{r}}{r^3}$$

  r cube=modulo, senza direzione

  where

$$\begin{aligned} &m_1, m_2 : \ \text{particle masses} \\ &\underline{\mathbf{r} : \ \text{vector}} \text{ of magnitude } r = |\mathbf{r}| \text{ connecting the two particles} \\ &G = 6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2} : \ \text{universal constant of gravitation.} \end{aligned}$$

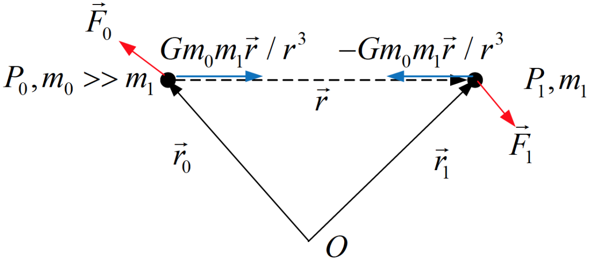
Introduction

Notation

- Scalars: $a, b, A, B \in \mathbb{R}$.
- Column vectors:
$$\mathbf{r} = (r_1, \ldots, r_n) = [r_1 \ \ldots \ r_n]^T = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \in \mathbb{R}^{n\times 1}.$$
- Row vectors: $\mathbf{r}^T = [r_1 \ \ldots \ r_n] \in \mathbb{R}^{1\times n}$.
- Matrices: $\mathbf{M} \in \mathbb{R}^{n\times m}$.
- Products:
$$\mathbf{r}\cdot\mathbf{p} = \mathbf{r}^T\mathbf{p} = \textstyle\sum_{i=1}^n r_i p_i \qquad \text{dot product}$$
$$\mathbf{r}\times\mathbf{p} = \begin{bmatrix} r_2p_3 - r_3p_2 \\ r_3p_1 - r_1p_3 \\ r_1p_2 - r_2p_1 \end{bmatrix} \qquad \text{cross product.}$$
- Vector ℓ_2 (Euclidean) norm:
$$|\mathbf{r}| = \|\mathbf{r}\| = \|\mathbf{r}\|_2 = \sqrt{\mathbf{r}\cdot\mathbf{r}} = \sqrt{\mathbf{r}^T\mathbf{r}} = \sqrt{\textstyle\sum_{i=1}^n r_i^2} = r.$$

The two-body problem

General setting



- Consider two point masses m_0 and m_1 (located at P_0 and P_1):
  - $\mathbf{r}_0$ and $\mathbf{r}_1$: positions of the masses in an inertial frame
  - $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$: relative position of the masses
  - $\mathbf{v}_0$ and $\mathbf{v}_1$: velocities of the masses in an inertial frame
  - $\mathbf{F}_0$ and $\mathbf{F}_1$: external forces (non gravitational) acting on the masses.

The two-body problem

Inertial and relative motion equations

- The Newton's II law and gravity law yield the following equations:

$$\begin{aligned}\dot{\mathbf{v}}_0 &= \frac{Gm_1}{r^3}\mathbf{r} + \frac{1}{m_0}\mathbf{F}_0 \\ \dot{\mathbf{v}}_1 &= -\frac{Gm_0}{r^3}\mathbf{r} + \frac{1}{m_1}\mathbf{F}_1\end{aligned}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0$ is the *r*elative position, $r = |\mathbf{r}|$ and constant masses have been assumed.

- Consider the following transformation:

$$\begin{aligned}&\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_0 \quad \text{(relative position)} \\ &\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_0 \quad \text{(relative velocity)} \\ &\mathbf{r}_c = \tfrac{m_0}{m_0+m_1}\mathbf{r}_0 + \tfrac{m_1}{m_0+m_1}\mathbf{r}_1 \quad \text{(CoM position)} \\ &\mathbf{v}_c = \tfrac{m_0}{m_0+m_1}\mathbf{v}_0 + \tfrac{m_1}{m_0+m_1}\mathbf{v}_1 \quad \text{(CoM velocity)}.\end{aligned}$$

The two-body problem

Restricted two-body equation

- From the above equations, we obtain

$$\text{v1\_dot-v0\_dot} = \dot{\mathbf{v}} = -\frac{G(m_0+m_1)}{r^3}\mathbf{r} + \frac{1}{m_1}\left(\mathbf{F}_1 - \frac{m_1}{m_0}\mathbf{F}_0\right) \qquad \leftarrow \left\{ \text{relative motion} \right.$$

$$\dot{\mathbf{v}}_c = \frac{m_1}{m_0}\frac{\mathbf{F}_1+\mathbf{F}_0}{1+m_1/m_0} \qquad \leftarrow \left\{ \text{CoM motion.} \right.$$

- If $m_1 \ll m_0$, the relative motion equation is

abbiamo una semplificazione, infatti possiamo trascurare alcuni termini, e otteniamo queste due equazioni

$$\dot{\mathbf{v}} + \mu\frac{\mathbf{r}}{r^3} = \frac{1}{m_1}\mathbf{F}_1. \qquad \text{(R2B)}$$

where $\mu = Gm_0$ is the *gravitational parameter*. (R2B) is called the *restricted two-body equation*.

The CoM motion equation is $\dot{\mathbf{v}}_c = 0$, showing that the CoM can be chosen as the origin of an inertial frame.

The two-body problem

Table 1. Gravitational parameters and accelerations					
No.	Planet/star and symbol	Mass [kg]	Equatorial radius [m]	Gravitational parameter [m^3/s^2]	Acceleration at the equatorial radius [m/s^2]
0	Sun ☉	1.99×10^{30}	0.696×10^{9}	0.133×10^{21}	274.0
1	Earth ⊕	5.97×10^{24}	6.38×10^{6}	0.3986×10^{15}	9.78
2	Mars ♂	0.642×10^{24}	3.40×10^{6}	0.0428×10^{15}	3.70
3	Moon ☾	0.0735×10^{24}	1.74×10^{6}	4.90×10^{12}	1.62

Free motion of the restricted two-body problem

- The free R2B equation is

$$\dot{\mathbf{v}} + \mu \frac{\mathbf{r}}{r^3} = 0. \tag{FR2B}$$

- In the following, we will see that
  - the total mechanical energy of the FR2B system is conserved;
  - the angular momentum of the FR2B system is conserved;
  - the free response of the FR2B equation occurs on a plane;

- We will also derive a geometric description of the FR2B system trajectories (orbits).

Free motion of the restricted two-body problem

Energy conservation

- Take the dot product of equation (FR2B) with $\mathbf{v}$:

$$\dot{\mathbf{v}} \cdot \mathbf{v} + \frac{\mu}{r^3}\mathbf{r} \cdot \mathbf{v} = \frac{1}{2}\frac{d}{dt}(\mathbf{v} \cdot \mathbf{v}) + \frac{\mu}{2r^3}\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r})$$

$$= \frac{d}{dt}\frac{v^2}{2} + \frac{\mu}{2r^3}\frac{d}{dt}r^2 = \frac{d}{dt}\frac{v^2}{2} + \frac{\mu\dot{r}}{r^2} = \frac{d}{dt}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) = 0.$$

questo zero è giustifcato dall equazione di sopra

- This proves the principle of energy conservation:

$$\boxed{\dot{\mathcal{E}} = 0, \qquad \mathcal{E} = \text{const}}$$

  - $\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}$: *total (mechanical) energy per unit mass*
  - $\frac{v^2}{2}$: *kinetic energy per unit mass*
  - $-\frac{\mu}{r}$: *potential energy per unit mass.*

total energy of the system

- For a given (constant) total energy $\mathcal{E}$, the corresponding orbital velocity is $v = \sqrt{2\mu/r + 2\mathcal{E}}$.

Free motion of the restricted two-body problem

Angular momentum conservation and planar motion

- Take the cross product of $\mathbf{r}$ with equation (FR2B):

derivata di v per v +
r per la derivata di v

$$\mathbf{r}\times\dot{\mathbf{v}}+\frac{\mu}{r^3}\mathbf{r}\times\mathbf{r}=\mathbf{r}\times\dot{\mathbf{v}}=\mathbf{v}\times\mathbf{v}+\mathbf{r}\times\dot{\mathbf{v}}=\frac{d}{dt}(\mathbf{r}\times\mathbf{v})=\mathbf{0}.$$

quello di prima
cross product con r

- This proves the principle of angular momentum conservation:

$$\boxed{\dot{\mathbf{h}}=\mathbf{0},\qquad \mathbf{h}=\mathrm{const}}$$

  - $\mathbf{h}=\mathbf{r}\times\mathbf{v}$: *angular momentum per unit mass.*

- An important implication of $\mathbf{h}$ being constant is that $\mathbf{r}$ and $\mathbf{v}$ always remain in the same plane, called the *orbital plane*.

- Example: Earth-Sun orbital plane.

Free motion of the restricted two-body problem

Orbit equation

- Take the cross product of equation (FR2B) with $\mathbf{h}$:

$$\left(\dot{\mathbf{v}} + \frac{\mu}{r^3}\mathbf{r}\right) \times \mathbf{h} = \frac{d}{dt}\left(\mathbf{v} \times \mathbf{h} - \frac{\mu}{r}\mathbf{r}\right) = 0.$$

The proof of the first equality is the following:

$$\frac{d}{dt}(\mathbf{v} \times \mathbf{h}) = \dot{\mathbf{v}} \times \mathbf{h} + \mathbf{v} \times \dot{\mathbf{h}} = \dot{\mathbf{v}} \times \mathbf{h} + \mathbf{v} \times \underbrace{(\mathbf{v} \times \mathbf{v} + \mathbf{r} \times \dot{\mathbf{v}})}_{=\mathbf{0}};$$

$$\frac{d}{dt}\left(-\frac{\mathbf{r}}{r}\right) = \frac{\dot{r}}{r^2}\mathbf{r} - \frac{1}{r}\mathbf{v} = \frac{1}{2r^3}\left(\frac{d}{dt}r^2\right)\mathbf{r} - \frac{1}{r}\mathbf{v} = \frac{1}{2r^3}\left(\frac{d}{dt}(\mathbf{r} \cdot \mathbf{r})\right)\mathbf{r} - \frac{1}{r}\mathbf{v}$$
$$= \frac{1}{r^3}\left((\mathbf{r} \cdot \mathbf{v})\,\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\,\mathbf{v}\right) = \frac{1}{r^3}\mathbf{r} \times (\mathbf{r} \times \mathbf{v}) \quad \text{(vector triple product)}.$$

- The above equation shows that

poiche la derivata di questa quantità è uguale a zero allora questa quantità è costante

$$\boxed{\mathbf{v} \times \mathbf{h} - \frac{\mu}{r}\mathbf{r} = \text{const} = \mu\mathbf{e}}$$

where $\mathbf{e}$ is the *eccentricity vector* and $e = |\mathbf{e}|$ is the *eccentricity*.

Free motion of the restricted two-body problem

Orbit equation

- Take the dot product of $\mathbf{r}$ with this equation:

$$\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) - \frac{\mu}{r}\mathbf{r} \cdot \mathbf{r} = \mu \mathbf{r} \cdot \mathbf{e}.$$

- Scalar triple product: $\mathbf{r} \cdot (\mathbf{v} \times \mathbf{h}) = (\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h}$.
  Moreover, $(\mathbf{r} \times \mathbf{v}) \cdot \mathbf{h} = \mathbf{h} \cdot \mathbf{h} = h^2$. Thus,

$$h^2 - \mu\, r = \mu\, r\, e \cos\theta$$

  where θ (the angle between $\mathbf{r}$ and $\mathbf{e}$) is called the *true anomaly*.

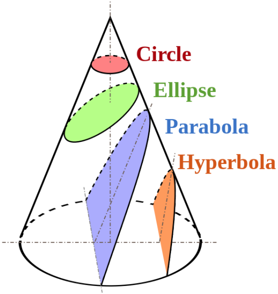
- Expliciting wrt to r and defining $p = h^2/\mu$ (p is called the *parameter* or *semilatus rectum*), we obtain the *orbit equation*:

$$\boxed{r = \frac{p}{1 + e\cos\theta}.} \qquad \text{(ORE)}$$

Orbit geometry

Conic sections

- The ORE
$$r = \frac{p}{1 + e\cos\theta}$$
is the equation of a *conic section*, written in terms of the polar coordinates r and θ: for $\theta \in [0, 2\pi]$, r describes a conic.
- A conic section (or simply a conic) is a curve obtained as the intersection of a cone with a plane.



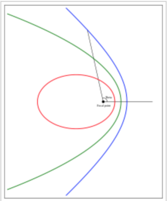
Orbit geometry

Conic sections

- The following conic sections exist:

  - ellipse, $0 \leq e < 1$ (circle, $e = 0$)
  - parabola, $e = 1$
  - hyperbola, $e > 1$.

in fuction of eccentricity we have different shape

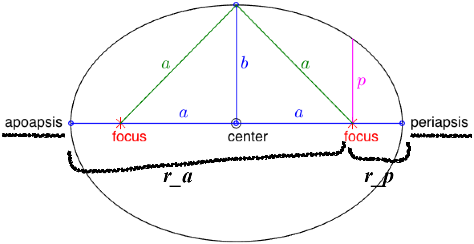


An elliptic, parabolic and hyperbolic Kepler orbit:
elliptic (eccentricity = 0.7)
parabolic (eccentricity = 1)
hyperbolic orbit (eccentricity = 1.3)

- Properties:
  - the origin is located at one focus
  - θ is measured from the point on the conic nearest to the focus ***the angular position of object in the orbit***
  - p determines the size
  - e determines the shape.

Orbit geometry
Ellipse

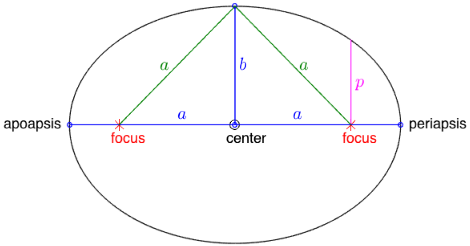
An ellipse is the locus of points the sum of whose distances from two fixed points (*foci*) is constant (and $= 2a$, with $a =$ *semi-major axis*).



- *Apsides*: extreme points of the orbit:
  - the *periapsis* is the point corresponding to $\theta = 0$; its distance from the main focus is $r_p = p/(1+e)$; ***il punto piu vicino al fuoco (principale)***
  - the *apoapsis* is the point corresponding to $\theta = \pi$; its distance from the main focus is $r_a = p/(1-e)$.

Orbit geometry

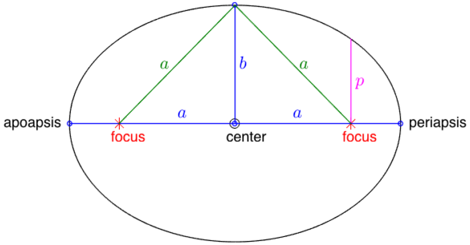
Ellipse



- The periapsis and apoapsis are also said
  - *perihelion* and *aphelion* for a body orbiting around the Sun;
  - *perigee* and *apogee* for a body orbiting around the Earth.

Orbit geometry

Ellipse



- Ellipse parameters:
  - basic: *eccentricity*: e; *semilatus rectum*: p
  - *semi-major axis*: $a = p/(1 - e^2)$
  - *semi-minor axis*: $b = a\sqrt{1 - e^2}$
  - distance center-focus: $c = ae$.
- Other relevant quantities:
  - total energy: $\mathcal{E} = -\mu/(2a) < 0$
  - velocity: $v = \sqrt{2\mu/r - \mu/a}$ $\quad\leftarrow\quad$ *vis-viva equation*.

Orbit geometry

Ellipse: parameters of the solar system planets

Type of body	Body	Distance from Sun at perihelion	Distance from Sun at aphelion
Planet	Mercury	46,001,009 km (28,583,702 mi)	69,817,445 km (43,382,549 mi)
	Venus	107,476,170 km (66,782,600 mi)	108,942,780 km (67,693,910 mi)
	Earth	147,098,291 km (91,402,640 mi)	152,098,233 km (94,509,460 mi)
	Mars	206,655,215 km (128,409,597 mi)	249,232,432 km (154,865,853 mi)
	Jupiter	740,679,835 km (460,237,112 mi)	816,001,807 km (507,040,016 mi)
	Saturn	1,349,823,615 km (838,741,509 mi)	1,503,509,229 km (934,237,322 mi)
	Uranus	2,734,998,229 km (1.699449110×10^9 mi)	3,006,318,143 km (1.868039489×10^9 mi)
	Neptune	4,459,753,056 km (2.771162073×10^9 mi)	4,537,039,826 km (2.819185846×10^9 mi)
Dwarf planet	Ceres	380,951,528 km (236,712,305 mi)	446,428,973 km (277,398,103 mi)
	Pluto	4,436,756,954 km (2.756872958×10^9 mi)	7,376,124,302 km (4.583311152×10^9 mi)
	Makemake	5,671,928,586 km (3.524373028×10^9 mi)	7,894,762,625 km (4.905578065×10^9 mi)
	Haumea	5,157,623,774 km (3.204798834×10^9 mi)	7,706,399,149 km (4.788534427×10^9 mi)
	Eris	5,765,732,799 km (3.582660263×10^9 mi)	14,594,512,904 km (9.068609883×10^9 mi)

Orbit geometry

Ellipse: parameters of the Earth

epoch	J2000.0[nb 3]
aphelion	152.10 million kilometres (94.51×10^6 mi) 1.0167 AU[nb 4]
perihelion	147.10 million kilometres (91.40×10^6 mi) 0.98329 AU[nb 4]
semimajor axis	149.60 million kilometres (92.96×10^6 mi) 1.000001018 AU[11]
eccentricity	0.0167086[11]
inclination	7.155° to Sun's equator 1.578690°[12] to invariable plane
longitude of the ascending node	174.9°[11]
longitude of perihelion	102.9°[11]
argument of periapsis	288.1°[11][nb 5]
period	365.256 363 004 days[13]
average speed	29.78 kilometres per second (18.50 mi/s)[3] 107,200 kilometres per hour (66,600 mph)

- The first law is implied by the ORE, for $0 \leq e < 1$.
- The orbit area swept by $\mathbf{r}$ as it moves through an angle $\Delta\theta$ in a short time interval Δt is $\Delta A \cong \frac{1}{2} r\,(r\,\Delta\theta)$. The areal velocity is

$$\dot{A} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{r\,(r\,\Delta\theta)}{2\Delta t} = \frac{r^2\dot{\theta}}{2} = \frac{h}{2} = \mathrm{const}$$

  which is the second law.
- Let A_p be the total orbital area. Then, the period P of an elliptical orbit is given by

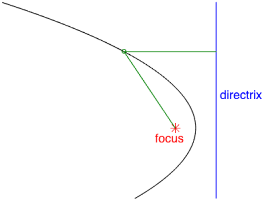
$$P = \frac{A_p}{A_p/P} = \frac{A_p}{\dot{A}} = \frac{2\pi ab}{h} = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}} = 2\pi\sqrt{\frac{a^3}{\mu}}$$

  which is the third law.

Orbit geometry

Parabola

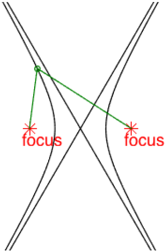
A parabola is the locus of points whose distance from a fixed point (*focus*) is equal to the distance from a fixed line (*directrix*).



- For a parabolic orbit, $e = 1$, implying that
  - $r_a \to \infty$, $a \to \infty$
  - the total energy is null: $\mathcal{E} \to 0$.
- From the vis-viva equation, for any orbital position with radius r, we obtain the corresponding velocity $v_e = \sqrt{2\mu/r}$;
  - this is called the *escape velocity*, and allows leaving a closed orbit.

Orbit geometry

Hyperbola



An hyperbola is the locus of points the difference of whose distances from two fixed points (*foci*) is constant (and $= -2a$).

- For a hyperbolic orbit, $e > 1$, implying that
  - the total energy is positive: $\mathcal{E} = v_\infty^2/2 > 0$.
- Asymptotic quantities ($r \to \infty$):
  - angle: $\theta_\infty = \arccos\left(-1/e\right)$
  - velocity: $v_\infty = \sqrt{\mu/\left|a\right|}$.
- *Hyperbolic passage*: A body passing close to a moving planet is subject to a velocity increase, without being captured by the planet gravity.
  - Adopted first by the US probe Mariner 10 (1973) to fly-by Venus (once) and Mercury (three times).

Orbit geometry

Table 2. Energy and orbital velocity

No.	Orbit	Eccentricity e	Semi-major axis a [m]	Energy per unit mass E [m^2/s^2]	Orbital velocity v [m]
0	Circular	0	$r(t)=a$	$-0.5\mu/a<0$	$v_c=\sqrt{\mu/r}$
1	Elliptic	<1	>0	$-0.5\mu/a<0$	$\sqrt{\mu(2/r-1/a)}$
2	Parabola	1	∞	0	$v_e=\sqrt{2\mu/r}=\sqrt{2}v_c$ (escape velocity)
2 bis	Idem, Earth				$v_e=11.2$ km/s
3	Hyperbola	>1	<0	$-0.5\mu/a>0$	$\sqrt{\mu(2/r-1/a)}$

Orbit geometry

Discussion

- Our study about orbits is based on the equation

$$\dot{\mathbf{v}} + \mu\frac{\mathbf{r}}{r^3} = 0. \tag{FR2B}$$

- The position trajectory generated by (FR2B) is described by the equation

$$r = \frac{p}{1 + e\cos\theta}. \tag{ORE}$$

- Integrating (FR2B) with $z(0) = 0$, $\dot{z}(0) = 0$, we obtain the position $\mathbf{r}(t) = (x(t), y(t), z(t) = 0)$, where $x(t)$ and $y(t)$ satisfy (ORE), when transformed as

$$r(t) = \sqrt{x^2(t) + y^2(t)},$$
$$\cos\theta(t) = x(t)/\sqrt{x^2(t) + y^2(t)}.$$

- Inverse transformation (assuming orbital plane = xy-plane):

$$x(t) = r\cos\theta(t)$$
$$y(t) = r\sin\theta(t)$$
$$z(t) = 0.$$

State equations

- State: $\mathbf{x} = (\mathbf{r}, \mathbf{v}) = (x, y, z, \dot{x}, \dot{y}, \dot{z}) = (x_1, x_2, x_3, x_4, x_5, x_6)$.
- State equations:

$$\boxed{\begin{aligned} \dot{\mathbf{r}} &= \mathbf{v} \\ \dot{\mathbf{v}} &= -\mu \frac{\mathbf{r}}{r^3}. \end{aligned}}$$

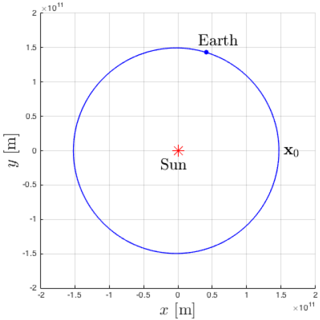
Alternative:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{\mu}{r^3} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{\mu}{r^3} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{\mu}{r^3} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}.$$

State equations

Example: Earth orbit about the Sun

- Parameters (values expressed in the International System of Units): $\mu = 133e20$, $a = 1.496e11$, $r_p = 1.471e11$, $v_p = \sqrt{\mu(2/r_p - 1/a)}$.
- Initial state: $\mathtt{x}_0 = [r_p; 0; 0; 0; v_p; 0]$ (perihelion).
- Simulation duration: $\mathtt{P} = 365 * 24 * 60 * 60$ s.
- Numerical integration: $[\mathtt{t}, \mathtt{x}] =$ode23tb$(@f_{FRTB}, [0, \mathtt{P}], \mathtt{x}_0)$.



Note that, from integration, $z(t) = 0$, $\dot{z}(t) = 0$, $\forall t$.

This confirms that the orbit occurs on a plane.

State equations

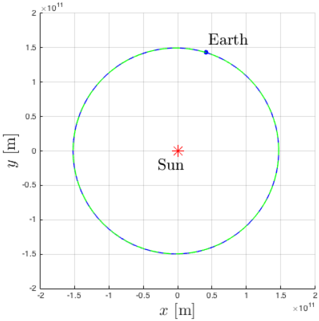
Example: Earth orbit about the Sun

- Parameters $\mathtt{p} = (r_p v_p)^2/\mu$, $\mathtt{e} = 0.0167$. ORE computation:

$$\begin{aligned} &\mathtt{th} = \mathtt{linspace}(0, 2\pi, 1000) \\ &\mathtt{r} = \mathtt{p}./(1 + \mathtt{e} * \mathtt{cos}(\mathtt{th})) \\ &\mathtt{x} = \mathtt{r}. * \mathtt{cos}(\mathtt{th}) \\ &\mathtt{y} = \mathtt{r}. * \mathtt{sin}(\mathtt{th}). \end{aligned}$$

orbit equation

coordinare di r



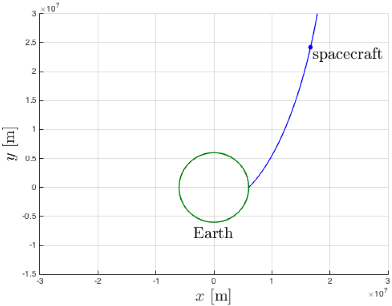
blue: orbit computed by integration;

dashed green: orbit computed with the ORE.

State equations

Example: escape velocity

- Consider a spacecraft taking off from the surface of the Earth with the escape velocity (take off angle 45^o).
- Parameters (values expressed in the International System of Units): $\mu = 0.4e15$, $v_e = 11.2e3$.
- Initial state: $\mathtt{x}_0 = [6e6; 0; 0; v_e/\sqrt{2}; v_e/\sqrt{2}; 0]$ (perigee).
- Numerical integration: $[\mathtt{t}, \mathtt{x}]$ =ode23tb$(@f_{FRTB}, [0, 43200], \mathtt{x}_0)$.



green: Earth surface;

blue: trajectory computed by integration.

Reference frames

- Several reference frames may be associated to an elliptic orbit:
  - **LVLH - local vertical local horizontal frame** (non inertial):

$$R_l = \{P_1, \mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3\};$$

  - **LORF - local orbital frame** (non inertial)**:**

$$R_o = \{P_1, \mathbf{o}_1, \mathbf{o}_2, \mathbf{o}_3\};$$

  - **PF - perifocal (perigee) frame** (inertial)**:**

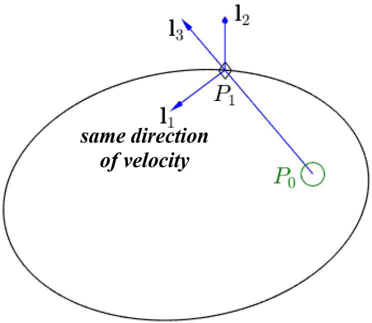
$$R_p = \{P_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}.$$

- Another important frame associated with the Earth, is the **geocentric equatorial (GE) frame** (inertial):

$$R_{ge} = \{P_0, \hat{\mathbf{I}}, \hat{\mathbf{J}}, \hat{\mathbf{K}}\}.$$

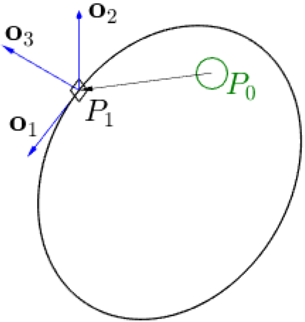
Reference frames

- **LVLH** frame: origin in P_1; the following unit vectors:
  - $\mathbf{l}_3$ (local vertical): defined along the direction $P_0 \rightarrow P_1$, on the orbit plane;
  - $\mathbf{l}_1$ (local horizontal): perpendicular to $\mathbf{l}_3$, on the orbit plane, sign concordant with the orbital velocity;
  - $\mathbf{l}_2$ (orbit pole): perpendicular to the orbit plane.



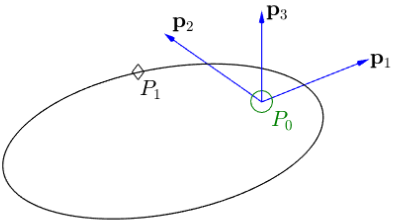
Reference frames

- **LORF** frame: origin in P_1; the following unit vectors:
  - $\mathbf{o}_1$: instantaneous normalized velocity, on the orbit plane, tangent to the orbit; ***and normalized***
  - $\mathbf{o}_2 = \mathbf{l}_2$ (orbit pole): perpendicular to the orbit plane;
  - $\mathbf{o}_3 = \mathbf{o}_1 \times \mathbf{o}_2$: on the orbit plane.



Reference frames

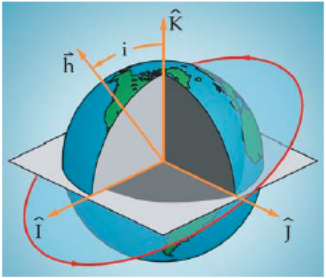
- **PF** frame: origin in P_0; the following unit vectors:
  - $\mathbf{p}_1$: eccentricity unit vector passing through the periapsis, on the orbit plane;
  - $\mathbf{p}_3 = \mathbf{o}_2 = \mathbf{l}_2$ (orbit pole): perpendicular to the orbit plane;
  - $\mathbf{p}_2 = \mathbf{p}_3 \times \mathbf{p}_1$: on the orbit plane.

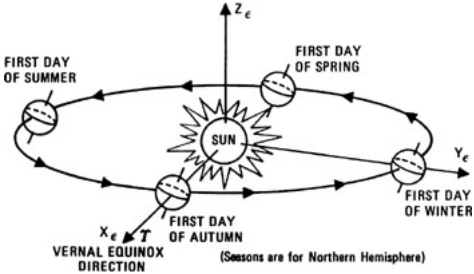


Reference frames

- **GE** frame:
  - $\hat{\mathbf{I}}$ and $\hat{\mathbf{J}}$ are on the equatorial plane;
  - $\hat{\mathbf{I}} \leftrightarrow \mathrm{X}_\epsilon$: vernal equinox: Earth-Sun direction, 1^{st} day of spring;
  - $\hat{\mathbf{K}}$: polar rotation axis;
  - $\hat{\mathbf{j}} = \hat{\mathbf{K}} \times \hat{\mathbf{I}}$: on the equatorial plane.
  - not rotating with the Earth;
  - independent on the orbit.

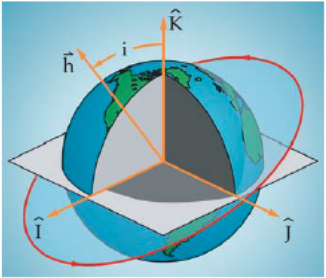
- the origin is in te CoM



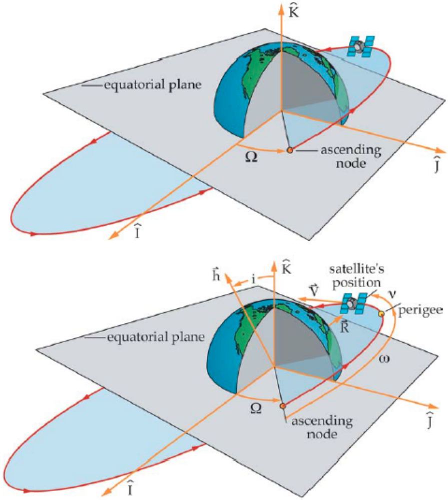


Orbital elements

- We focus on elliptic orbits around the Earth.
- Two planes can be distinguished:
  - the orbital plane
  - the equatorial plane.
- The intersection between these two planes is called the *lines of nodes*.
- The angle i between these two planes is called the *inclination*.



Orbital elements



ascending node: intersection between the orbit and the equatorial plane;

Ω: angle from $\hat{\mathbf{I}}$ to ascending node;

ω: angle from ascending node to perigee;

$\nu = \theta$ (true anomaly): angle from perigee to spacecraft position.

Orbital elements

- The 6 *classical orbital elements* are 5 independent constant quantities $(a, e, \Omega, i, \omega)$, which completely describe the orbit, and one quantity (ν), which gives the spacecraft position on the orbit.

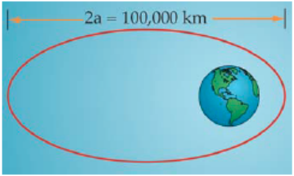
Element	Name	Description	Range of Values	Undefined
a	Semimajor axis	Size	Depends on the conic section	Never
e	Eccentricity	Shape	e = 0: circle 0 < e < 1: ellipse	Never
i	Inclination	Tilt, angle from $\hat{K}$ unit vector to specific angular momentum vector $\hat{h}$	$0 \leq i \leq 180°$	Never
Ω	Right ascension of the ascending node	Swivel, angle from vernal equinox to ascending node	$0 \leq \Omega < 360°$	When i = 0 or 180° (equatorial orbit)
ω	Argument of perigee	Angle from ascending node to perigee	$0 \leq \omega < 360°$	When i = 0 or 180° (equatorial orbit) or e = 0 (circular orbit)
ν	True anomaly	Angle from perigee to the spacecraft's position	$0 \leq \nu < 360°$	When e = 0 (circular orbit)

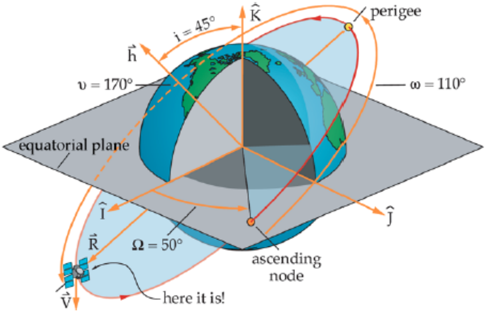
The true anomaly is usually denoted with ν or θ. The *time of perigee passage* t_p is sometimes used instead of the true anomaly.

Orbital elements

Example: a communication satellite orbit

- Semimajor axis, a = 50,000 km
- Eccentricity, e = 0.4
- Inclination, i = 45°
- Right ascension of the ascending node, Ω = 50°
- Argument of perigee, ω = 110°
- True anomaly, ν = 170°





Orbital elements

Position, velocity $\to$ orbital elements

- Suppose the satellite position $\mathbf{r}$ and velocity $\mathbf{v}$ are known and expressed in the GE frame.
- From $\mathbf{r}$ and $\mathbf{v}$, the following quantities can be computed:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}, \quad \mathbf{e} = \tfrac{1}{\mu}\mathbf{v} \times \mathbf{h} - \tfrac{\mathbf{r}}{r}, \quad \hat{\mathbf{I}}' = \hat{\mathbf{K}} \times (\mathbf{h}/h).$$

- The 6 orbital elements can be obtained as follows:

$$\begin{aligned} a &= h^2/(\mu(1-e^2)), & e &= \left|\mathbf{e}\right|, & \cos i &= \hat{\mathbf{K}} \cdot \mathbf{h}/h, \\ \cos\omega &= \hat{\mathbf{I}}' \cdot \mathbf{e}/e, & \cos\Omega &= \hat{\mathbf{I}} \cdot \hat{\mathbf{I}}', & \cos\theta &= \mathbf{r} \cdot \mathbf{e}/(re). \end{aligned}$$

- The eccentric anomaly E and the period can also be computed:

$$\tan\tfrac{E}{2} = \sqrt{\tfrac{1-e}{1+e}}\tan\tfrac{\theta}{2}, \qquad P = 2\pi\sqrt{\tfrac{a^3}{\mu}}.$$

Orbital elements

Orbital elements → position, velocity

- Suppose the 6 orbital elements are known. The semilatus rectum and the radial position can be computed as

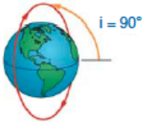
$$p = a(1-e^2), \qquad r = \tfrac{p}{1+e\cos\theta}.$$

- The satellite position and velocity, expressed in PF, are given by

$$\mathbf{r} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} -\sqrt{\mu/p}\sin\theta \\ \sqrt{\mu/p}(e+\cos\theta) \\ 0 \end{bmatrix}.$$

- Transf. PF frame → GE frame: $\mathbf{T}_{313}(\Omega, i, \omega)$.
  Transf. GE frame → PF frame: $\mathbf{T}_{313}(-\omega, -i, -\Omega)$.

Types of orbit
Classification I

Inclination	Orbital Type	Diagram
0° or 180°	Equatorial	
90°	Polar	
$0° \leq i < 90°$	Direct or Prograde (moves in the direction of Earth's rotation)	
$90° < i \leq 180°$	Indirect or Retrograde (moves against the direction of Earth's rotation)	

Remark: any orbit plane passes through the Earth CoM.

Types of orbit

Classification II

Mission	Orbital Type	Semimajor Axis (Altitude)	Period	Inclination	Other
• Communication • Early warning • Nuclear detection	Geostationary	42,158 km (35,780 km)	~24 hr	~0°	$e \cong 0$
• Remote sensing	Sun-synchronous	~6500 – 7300 km (~150 – 900 km)	~90 min	~95°	$e \cong 0$
• Navigation – GPS	Semi-synchronous	26,610 km (20,232 km)	12 hr	55°	$e \cong 0$
• Space Shuttle	Low-Earth orbit	~6700 km (~300 km)	~90 min	28.5°, 39°, 51°, or 57°	$e \cong 0$
• Communication/ intelligence	Molniya	26,571 km (R_p = 7971 km; R_a = 45,170 km)	12 hr	63.4°	$\omega = 270°$ $e = 0.7$

For more details, see https://en.wikipedia.org/wiki/List_of_orbits.

Orbit perturbations

- Orbital dynamics is based on celestial mechanics:
  - Kepler's laws: empirical laws describing the motion of a body in unperturbed planetary orbits;
  - Newton's laws: general physical laws that imply the Kepler laws.
- We have studied non-perturbed orbit (*Keplerian orbits*).
- Real orbits are subject to perturbations (*non-Keplerian orbits*):
  - gravity potential harmonics perturbing the central force, due to an irregular mass distribution of planets (e.g., Earth polar flattening);
  - third-body forces like those due to the Sun or Moon gravity;
  - aerodynamic forces due to the residual atmosphere and wind at low-Earth orbits;
  - solar/cosmic radiation;
  - others, such as Earth radiation and tides, and spacecraft thermal radiation.

Atmospheric drag

- For low Earth orbits (LEO), drag is a significant disturbing force.
- The drag force is given by

$$\mathbf{F}_d = -\frac{1}{2}\,\rho\,C_D\,S\,|\mathbf{v}_{rel}|\,\mathbf{v}_{rel}$$

  - ρ: local atmospheric density
  - C_D: drag coefficient
  - S: spacecraft area projected along the direction of motion
  - $\mathbf{v}_{rel}$: relative velocity of the spacecraft wrt the atmosphere. Assuming a negligible atmospheric velocity, $\mathbf{v}_{rel} \cong \mathbf{v}$.

Atmospheric drag

- A model for the atmospheric density is the following:

$$\rho(r) = \rho_0 \exp\left(-\frac{r - r_0}{H}\right)$$

  - ρ_0, r_0: reference density and height
  - H: scale height coefficient
  - r: distance from the planet CoM.

- Though simple, this model is fine for preliminary simulations. For more accurate simulations, a more refined model may be needed.