

## Homework I

- Solutions have to be uploaded on the course website by h 23:59 of Sunday November 13, 2022 under the name Homework1.
- The exercises may be solved by hand or by using a numerical code. You may implement your codes in any language you see fit, but the teacher assistant can only guarantee you support with MATLAB and Python. Your code should be written in a quite general manner, i.e., if a question is slightly modified, it should only require slight modifications in your code as well. Upload a PDF lab-report together with your code.
- The PDF should read like a standard lab-report, including a description of what you are doing, a summary of the theory used, proper presentation of results (including readable figures with axis labels, if any). You must report all the results in the report. Writing the report in Latex is strongly encouraged. Clarity of the presentation (especially if the report is hand-written) and ability to synthesize are part of the evaluation of the homework.
- Comment your code well. Clarity is more important than efficiency.
- Collaboration such as exchange of ideas among students is encouraged. However, every student has to submit her/his copy of the final manuscript (in PDF format) and code, and specify whom she/he has collaborated with and on what particular part of the work.

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**Exercise 1.** Consider the network in Figure 1 with link capacities

$$c_2 = c_4 = c_6 = 1, \quad c_1 = c_3 = c_5 = 2.$$

- What is the minimum aggregate capacity that needs to be removed for no feasible flow from  $o$  to  $d$  to exist?
- What is the maximum aggregate capacity that can be removed from the links without affecting the maximum throughput from  $o$  to  $d$ ?
- You are given  $x > 0$  extra units of capacity. How should you distribute them in order to maximize the throughput that can be sent from  $o$  to  $d$ ? Plot the maximum throughput from  $o$  to  $d$  as a function of  $x \geq 0$ .

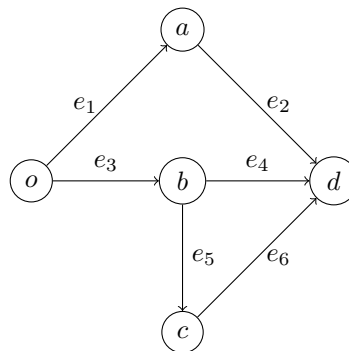


Figure 1

**Exercise 2.** There are a set of people  $\{p_1, p_2, p_3, p_4\}$  and a set of books  $\{b_1, b_2, b_3, b_4\}$ . Each person is interested in a subset of books, specifically

$$p_1 \rightarrow \{b_1, b_2\}, \quad p_2 \rightarrow \{b_2, b_3\}, \quad p_3 \rightarrow \{b_1, b_4\}, \quad p_4 \rightarrow \{b_1, b_2, b_4\}.$$

- (a) Exploit max-flow problems to find a perfect matching (if any).
- (b) Assume now that there are multiple copies books, and the distribution of the number of copies is  $(2, 3, 2, 2)$ . Each person can take an arbitrary number of different books. Exploit the analogy with max-flow problems to establish how many books of interest can be assigned in total.
- (c) Suppose that the library can sell a copy of a book and buy a copy of another book. Which books should be sold and bought to maximize the number of assigned books?

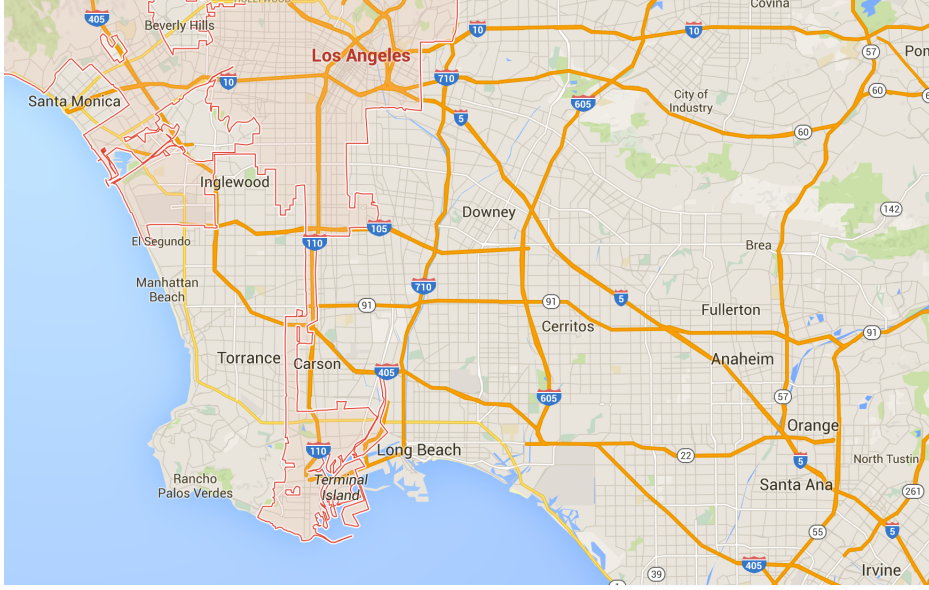


Figure 2: The highway network in Los Angeles.

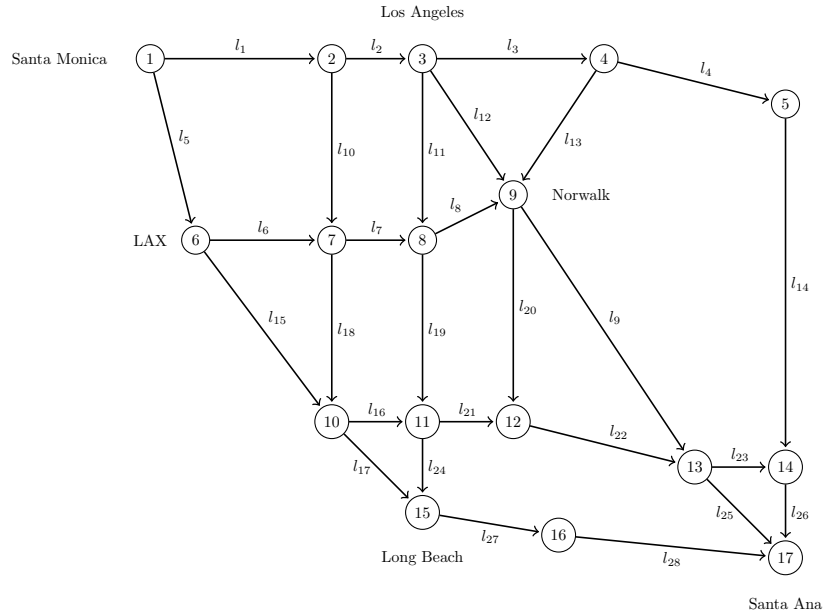


Figure 3: Some possible paths from Santa Monica (node 1) to Santa Ana (node 17).

**Exercise 3.** We are given the highway network in Los Angeles, see Figure 2. To simplify the problem, an approximate highway map is given in Figure 3, covering part of the real highway network. The node-link incidence matrix  $B$ , for this traffic network is given in the file *traffic.mat*. The rows of  $B$  are associated with the nodes of the network and the columns of  $B$  with the links. The  $i$ -th column of  $B$  has 1 in the row corresponding to the tail node of link  $e_i$  and  $(-1)$  in the row corresponding to the head node of link  $e_i$ . Each node represents an intersection between highways (and some of the area around).

Each link  $e_i \in \{e_1, \dots, e_{28}\}$ , has a maximum flow capacity  $c_{e_i}$ . The capacities are given as a vector  $c_e$  in the file *capacities.mat*. Furthermore, each link has a minimum travelling time  $l_{e_i}$ , which the drivers experience when the road is empty. In the same manner as for the capacities, the minimum travelling times are given as a vector  $l_e$  in the file *traveltime.mat*. These values are

simply retrieved by dividing the length of the highway segment with the assumed speed limit 60 miles/hour. For each link, we introduce the delay function

$$\tau_e(f_e) = \frac{l_e}{1 - f_e/c_e}, \quad 0 \leq f_e < c_e.$$

For  $f_e \geq c_e$ , the value of  $\tau_e(f_e)$  is considered as  $+\infty$ .

If you use Python to solve the Exercise, the files `.mat` be loaded by using the following code:

```
f = scipy.io.loadmat('flow.mat')['flow'].reshape(28,)
C = scipy.io.loadmat('capacities.mat')['capacities'].reshape(28,)
B = scipy.io.loadmat('traffic.mat')['traffic']
l = scipy.io.loadmat('traveltime.mat')['traveltime'].reshape(28,)
```

- (a) Find the shortest path between node 1 and 17. This is equivalent to the fastest path (path with shortest traveling time) in an empty network.
- (b) Find the maximum flow between node 1 and 17.
- (c) Given the flow vector in *flow.mat*, compute the external inflow  $\nu$  satisfying  $Bf = \nu$ .

In the following, we assume that the exogenous inflow is zero in all the nodes except for node 1, for which  $\nu_1$  has the same value computed in the point (c), and node 17, for which  $\nu_{17} = -\nu_1$ .

- (d) Find the social optimum  $f^*$  with respect to the delays on the different links  $\tau_e(f_e)$ . For this, minimize the cost function

$$\sum_{e \in \mathcal{E}} f_e \tau_e(f_e) = \sum_{e \in \mathcal{E}} \frac{f_e l_e}{1 - f_e/c_e} = \sum_{e \in \mathcal{E}} \left( \frac{l_e c_e}{1 - f_e/c_e} - l_e c_e \right)$$

subject to the flow constraints.

**Hint:** use the Tutorial (<https://www.cvxpy.org/tutorial/functions/index.html>) to learn how to code functions in cvxpy.

- (e) Find the Wardrop equilibrium  $f^{(0)}$ . For this, use the cost function

$$\sum_{e \in \mathcal{E}} \int_0^{f_e} \tau_e(s) ds.$$

- (e) Introduce tolls, such that the toll on link  $e$  is  $\omega_e = f_e^* \tau_e'(f_e^*)$ , where  $f_e^*$  is the flow at the system optimum. Now the delay on link  $e$  is given by  $\tau_e(f_e) + \omega_e$ . compute the new Wardrop equilibrium  $f^{(\omega)}$ . What do you observe?
- (f) Instead of the total travel time, let the cost for the system be the total additional delay compared to the total delay in free flow, given by

$$\psi_e(f_e) = f_e(\tau_e(f_e) - l_e)$$

subject to the flow constraints. Compute the system optimum  $f^*$  for the costs above. Construct tolls  $\omega^*$  such that the Wardrop equilibrium  $f^{(\omega^*)}$  coincides with  $f^*$ . Compute the new Wardrop equilibrium with the constructed tolls  $f^{(\omega^*)}$  to verify your result.