Project 3

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Multigrid in 1d

2.

This part concerns the function $gs - step - 1d$ that implements the Gauss-Seidel numerical method (GS) in order to solve the one dimensional boundary value problem. In Table 1 are reported the CPU times and iteration needed for matching the desired tolerance (10^{-8}) in the pseudo residual size (considered as infinity norm). We choose to report times with millisecond accurancy. Figure 1 shows the correlation between the number of intervals of the grid N and the needed iterations. Please note that the two axis have a different scale.

3.

This part concerns the function $two - grid - step$ that implements the two grid correction scheme numerical method replacing the exact solve in the coarser grid with five iterations of the Gauss Seidel method. In Table 2 are reported the CPU times and iteration needed in order to match the desired tolerance (10^{-8}) in the pseudo residual size. The number of iterations and CPU times needen are notably smaller with the two grid method. Figure 2 shows the correlation between the number of intervals of the grid N and the needed iterations. Please note that the two axis have a different scale and that the scale is different from the one used for the plot in figure 1. From graphical intuition it seems that there is a linear correlation between the two graphs for exercise .2 and .3. Indeed plotting the ratio $\frac{\text{iterations needed with GS-step}}{\text{iterations needed with two grid correction scheme}}$ we obtain Figure 3, that definitively suggest a linear correlation with proportional constant ~ 22.

5.

This part concerns the function $v - cycle - step - 1d$ that implements the V-cycle method in which the relaxing step applies the GS method a certain number of times depending on the variables α_1 and α_2. In particular α_1 is the number of iterations of GS in the pre-smoothing step while α_2 in the post-smoothing step. The In Table 3 are reported the CPU times and iteration needed for

N	Number of iterations	CPU time
2^3	47	0.0
2^4	168	0.004
2^5	616	0.031
2^6	2245	0.231
2^7	8114	1.702

Table 1: Number of iterations and CPU times (in secods) needed with different values of N for the GS method.

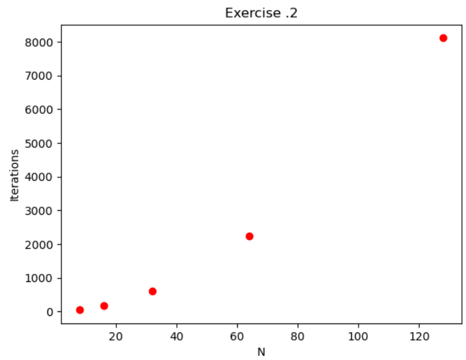

Figure 1: Plot of the needed iterations depending on the numbers N of intervals of the grid.

N	**Number of iterations**	**CPU time**
2^3	7	0.001
2^4	9	0.003
2^5	29	0.014
2^6	103	0.112
2^7	369	1.242

Table 2: Number of iterations and CPU times (in secods) needed with different values of N for the two grid correction scheme method.

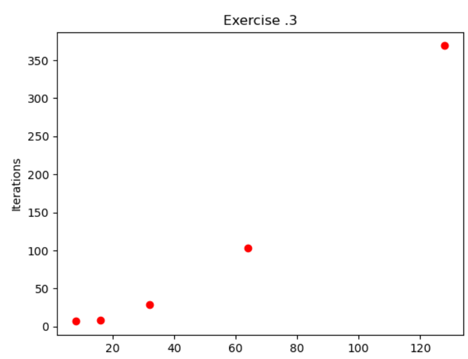

Figure 2: Plot of the needed iterations depending on the numbers N of intervals of the grid.

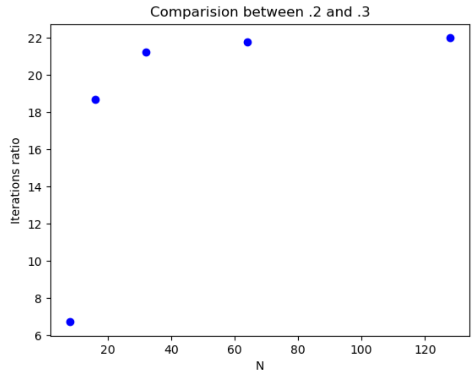

Figure 3: Plot of the ratio of the needed iterations with the GS method and the iteration needed with the two grid correction scheme method.

N	**Iterations and CPU tiwe with** $\alpha_1 = \alpha_2 = 1$		**Iterations and CPU time with** $\alpha_1 = \alpha_2 = 2$	
2^3	7	0.013	4	0.0
2^4	8	0.003	5	0.0
2^5	8	0.004	6	0.016
2^6	8	0.008	6	0.0
2^7	8	0.017	6	0.032
2^8	8	0.037	5	0.043
2^9	8	0.096	5	0.127
2^{10}	7	0.368	5	0.322
2^{11}	7	1.488	5	1.124
2^{12}	7	5.859	5	3.838
2^{13}	7	22.888	5	15.600
2^{14}	7	107.346	5	76.743

Table 3: Number of iterations and CPU times (in secods) needed with different values of N depending on the choiche of the parameters α_1 and α_2 for the V-cycle method.

matching the desired tolerance (10^{-8}) in the pseudo residual size with two different configurations for the parameters. Figure 4 shows the correlation between the number of intervals of the grid N and the required CPU times with the two choiches for the parameters. In this picture we see that the second choice should be preferred for increasing values of N.

7.

This part concerns the function $full - mg - 1d$ that implements the full multigrid method. In Table 4 are reported the CPU times, residual and pseudo residual size with four different configurations for the parameters α_1, α_2 and ν.

8.

In order to plot an approximation of the solution we apply the function V-cycle with parameters(2,2) until we reach the pseudo residual with magnitude less than 10^{-8} in a grid of $N = 2^{14}$ intervals. The plot of the function is reported in figure 6. We also extract the minimum value of the approximation found for u that turns out to be -3.6938796006802006.

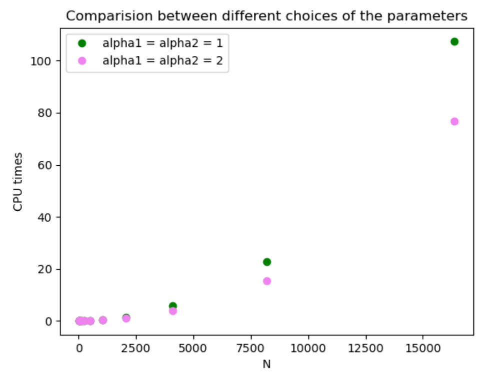

Figure 4: Required CPU times with different chioiches of the parameters: $\alpha_1 = \alpha_2 = 1$ (green) and $\alpha_1 = \alpha_2 = 2$ (violet).

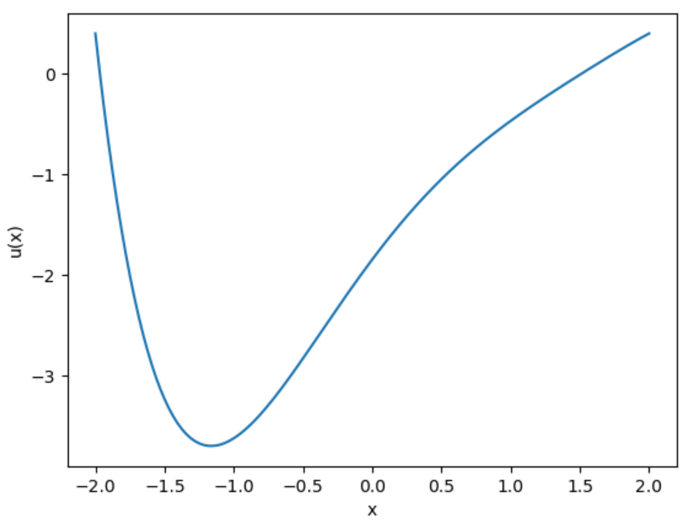

Figure 5: Approximation of the function u solution of the problem.

N	**(1,1,1)**	**(1,1,2)**	**(2,2,1)**	**(2,2,2)**
2^3	0.005	0.0	0.0	0.003
2^4	0.0	0.0	0.0	0.002
2^5	0.0	0.016	0.0	0.004
2^6	0.0	0.079	0.010	0.009
2^7	0.0	0.016	0.0	0.013
2^8	0.010	0.031	0.020	0.026
2^9	0.025	0.054	0.030	0.062
2^10	0.068	0.141	0.120	0.168
2^{11}	0.260	0.502	0.331	0.578
2^{12}	1.311	1.906	1.090	2.362
2^{13}	3.735	7.704	4.365	7.823
2^{14}	18.734	33.270	20.570	33.397

Table 4: CPU times (in seconds) depending on the choiche of the parameters $(\alpha_1, \alpha_2, \nu)$.

N	**(1,1,1)**	**(1,1,2)**	**(2,2,1)**	**(2,2,2)**
2^3	0.1263162217298497	0.0024924665632575227	0.008032889668047716	4.5239444290245956e-05
2^4	0.24400927755755575	0.014194388146876236	0.05209366916431435	0.00046446337134398163
2^5	1.3118733991555445	0.09610353337684074	0.16869440739962727	0.002863003993439861
2^6	2.4721163847927983	0.21469430510524035	0.27804069675789833	0.0060138109009244545
2^7	3.362213308768464	0.2988958050495647	0.3492919518292297	0.008294429114016566
2^8	3.9371920841283554	0.348026914481693	0.3891616577749204	0.009614006339830894
2^9	4.274821168944072	0.3744430195711175	0.4100790053826131	0.010312865148989658
2^10	4.461994757344655	0.3881399201200111	0.4207604021903535	0.010670337665942498
2^{11}	4.5620353257108945	0.395119510780205	0.4261524158064276	0.010850739607121795
2^{12}	4.614246030221693	0.39864472951740026	0.42886053636902943	0.010941300657577813
2^{13}	4.641073377570137	0.4004168894607574	0.4302175146294758	0.010986663051880896
2^{14}	4.6547195428283885	0.40130553126800805	0.4308967161778128	0.011009366236976348

Table 5: Value of the residual size depending on the choiche of the parameters $(\alpha_1, \alpha_2, \nu)$.

N	**(1,1,1)**	**(1,1,2)**	**(2,2,1)**	**(2,2,2)**
2^3	0.0542812107901578	0.005277309556834808	0.002008222417012373	1.1309861073005578e-05
2^4	0.04955466114862506	0.0023980750483254543	0.00325585432276964	2.902896070944294e-05
2^5	0.027150271926458913	0.0031856773633573976	0.002635850115619176	4.473443739749783e-05
2^6	0.009656704628096868	0.0013840588284772526	0.0010860964717105404	2.349144883173615e-05
2^7	0.0032834114343441856	0.0004410967915454249	0.0003411054217082321	8.100028431656803e-06
2^8	0.0009612285361641493	0.00012363247540470024	9.501017035522946e-05	2.3471695165741546e-06
2^9	0.0002609143779873091	3.267164829368063e-05	2.5029236168400137e-05	6.294473357260078e-07
2^10	6.808463680024746e-05	8.394210785134248e-06	6.420294222875267e-06	1.6281643167026516e-07
2^{11}	1.7402783682729517e-05	2.1272040505171397e-06	1.6256424553162674e-06	4.13922867092964e-08
2^{12}	4.400487928601926e-06	5.354048153716207e-07	4.08993278910863e-07	1.0434437425210774e-08
2^{13}	1.1065181201863616e-06	1.3430302930883542e-07	1.0257184851480972e-07	2.6194246771638063e-09
2^{14}	2.7744290487241585e-07	3.363231476649631e-08	2.5683445703528207e-08	6.562093091133647e-10

Table 6: Value of the pseudo residual size depending on the choiche of the parameters $(\alpha_1, \alpha_2, \nu)$.

Multigrid in 2d

2.

Here we report the results of the investigations concerning the two dimensional generalization of the previous functions. In particular:

- In table 7 are reported number of iterations and CPU times needed to reach the tolerance $< 10^{-8}$ in the pseudo residual size using the function $gs - step - 2d$ that implements the Gauss Seidel method.
- In table 8 are reported number of iterations and CPU times needed to reach the tolerance $< 10^{-8}$ in the pseudo residual size using the function $v - cycle - 2d$ that implements the V-cycle method in its two dimensional variant with parameters $\alpha_1 = \alpha_2 = 1$.
- In table 8 are reported number of iterations and CPU times needed to reach the tolerance $< 10^{-8}$ in the pseudo residual size using the function $v - cycle - 2d$ that implements the V-cycle method in its two dimensional variant with parameters $\alpha_1 = \alpha_2 = 2$.
- In table 9, 10 and 11 are reported the CPU times, size of the residual and pseudo residual obtained after a full multigrid step $(full - mg - 2d)$ with different choiches for the parameters $(\alpha_1, \alpha_2, \nu)$.
- In figure 6 we report the plot of an approximation of the solution of the two dimensional problem. In order to obtain this approximation we perform V-cycle steps in a grid of $N = 2^8$ intervals until the pseudo residual has size less than 10^{-8}. We used this approximation in order to extimate the minimum value that the function attains. Our result is -0.04150939164706837.

N	**Number of iterations**	**CPU time**
2^2	17	0.0
2^3	60	0.008
2^4	216	0.114
2^5	777	1.582
2^6	2783	23.120

Table 7: Number of iterations and CPU times (in secods) needed with different values of N for the Gauss Seidel method in the two dimensional version.

N	**Number of iterations**	**CPU time**
2^3	6	0.001
2^4	9	0.009
2^5	10	0.028
2^6	10	0.104
2^6	10	0.411
2^6	10	1.673
2^7	10	7.821

Table 8: Number of iterations and CPU times (in secods) needed with different values of N for the V-cycle method in the two dimensional version with $\alpha_1 = \alpha_2 = 1$.

N	**Number of iterations**	**CPU time**
2^3	4	0.001
2^4	6	0.004
2^5	6	0.020
2^6	7	0.115
2^6	7	0.457
2^6	7	1.974
2^7	7	7.955

Table 9: Number of iterations and CPU times (in secods) needed with different values of N for the V-cycle method in the two dimensional version with $\alpha_1 = \alpha_2 = 2$.

N	(1,1,1)	(1,1,2)	(2,2,1)	(2,2,2)
2^2	0.0	0.0	0.0	0.0
2^3	0.0	0.0	0.0	0.003
2^4	0.010	0.009	0.010	0.017
2^5	0.020	0.031	0.021	0.040
2^6	0.070	0.115	0.095	0.163
2^7	0.301	0.471	0.365	0.723
2^8	1.210	1.930	1.475	2.833

Table 10: CPU times (in seconds) depending on the choiche of the parameters $(\alpha_1, \alpha_2, \nu)$ for the multigrid two dimensional method.

N	(1,1,1)	(1,1,2)	(2,2,1)	(2,2,2)
2^2	0.04210074800000002	0.002415128974989663	0.0031054631999999915	1.833426348563094e-05
2^3	0.0505054238230333	0.003984797383393697	0.005397942942254881	0.00020204994266539988
2^4	0.01688358432910858	0.0015479941228090932	0.0018412711542701576	5.8250971009754515e-05
2^5	0.004597435071587691	0.000454725348762397	0.0005225580170527266	1.2610492462566958e-05
2^6	0.0012351950188426164	0.00012428081129323143	0.00013785561959483994	3.39576408116965e-06
2^7	0.00031892438329239603	3.2176538362516105e-05	3.532833495362153e-05	8.893123182585061e-07
2^8	8.087030033004927e-05	8.164844469804589e-06	8.936593163533235e-06	2.2638169233646366e-07

Table 11: Pseudo residual size depending on the choiche of the parameters $(\alpha_1, \alpha_2, \nu)$ for the multigrid two dimensional method.

N	(1,1,1)	(1,1,2)	(2,2,1)	(2,2,2)
2^2	0.040263256000000025	0.003652657949979765	0.005033326399999982	1.6510644937572927e-05
2^3	0.2047090398465179	0.017716402288310062	0.0334118888057543	0.0012783437481559412
2^4	0.24436057302955572	0.025082120678389774	0.041836558668982615	0.0014528637306912007
2^5	0.29423584458160823	0.02910242232078808	0.04356056353757598	0.0011176681439120095
2^6	0.31620992482376487	0.03181588769110277	0.04476054472437285	0.001042430366325231
2^7	0.32657856849139266	0.032948775283202725	0.045514603018068645	0.0010169582913843733
2^8	0.3312447501511233	0.03344320294897862	0.045925385351983206	0.0010285213621004097

Table 12: Residual size depending on the choiche of the parameters $(\alpha_1, \alpha_2, \nu)$ for the multigrid two dimensional method.

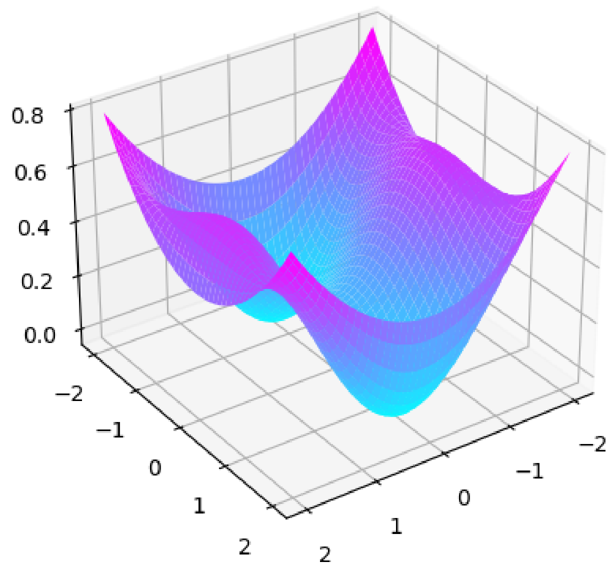

Figure 6: Approximation of the solution of the two dimensional problem.