Celestial Mechanics - Homework n. 1

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1 Problem 1

The canonical equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with the two foci at $(c, 0)$, $(-c, 0)$. The eccentricity is defined as $c = ea$, $0 \le e < 1$, and $a^2 = b^2 + c^2$ which gives $b = a\sqrt{1 - e^2}$.
Now I apply the transformation in polar coordinates centered on the focus at $(c, 0)$.

$$\begin{cases} x = c + r\cos\theta \\ y = r\sin\theta \end{cases}$$

Substituting x and y in the equation gives

$$\frac{(c + r\cos\theta)^2}{a^2} + \frac{(r\sin\theta)^2}{a^2(1 - e^2)} = 1$$

that after some manipulating becomes

$$\frac{1 - e^2\cos^2\theta}{1 - e^2} + 2rc\cos\theta - a^2(1 - e^2) = 0$$

This quadratic equation gives two solutions

$$r_{1,2} = \frac{-ae\cos\theta \pm a}{(1 - e\cos\theta)(1 + e\cos\theta)}(1 - e^2) = \pm\frac{a(1 - e^2)}{1 \mp e\cos\theta}$$

Now because $r > 0$, the correct solution is

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}$$

2 Problem 2

The Stokes theorem states that the integral of a vectorial field $\boldsymbol{F}$ calculated over a closed path C is equal to the integral of the curl of the field over the surface Σ enclosed by C

$$\int_\Sigma \nabla \times \boldsymbol{F} \cdot d\boldsymbol{\Sigma} = \oint_C \boldsymbol{F} \cdot d\boldsymbol{\Gamma}$$

In this case, I will use $d\boldsymbol{\Sigma} = \hat{\boldsymbol{n}}dxdy$ with $\hat{\boldsymbol{n}} = \boldsymbol{e}_z$, so $\nabla \times \boldsymbol{F} \cdot d\boldsymbol{\Sigma} = (\partial_x \boldsymbol{F}_y - \partial_y \boldsymbol{F}_x)\hat{\boldsymbol{n}}$. To calculate the area inside C I choose $\boldsymbol{F} = (-y, x, 0)$, so

$$\int_\Sigma (\partial_x \boldsymbol{F}_y - \partial_y \boldsymbol{F}_x)dxdy = \int_\Sigma 2dxdy = 2A$$

By doing the change of coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

I get the differential form

$$\begin{cases} dx = dr\cos\theta - r\sin\theta d\theta \\ dy = dr\sin\theta + r\cos\theta d\theta \end{cases}$$

and by substituting in the path integral $\oint_C \boldsymbol{F}_x dx + \boldsymbol{F}_y dy = \oint_C -ydx + xdy$ I obtain

$$\oint_C -rdr\sin\theta\cos\theta + r^2\sin^2\theta d\theta + rdr\cos\theta\sin\theta + r^2\cos^2\theta d\theta = \int_0^{2\pi} r^2 d\theta$$

Now I choose as a coordinate basis $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_z)$. In this basis the radius $\boldsymbol{r}$ has representation $(r, 0, 0)$, $d\boldsymbol{r} = (0, rd\theta, 0)$ and the cross product between the two gives indeed $r^2 d\theta\boldsymbol{e}_z$, which proves that

$$A = \frac{1}{2}\hat{\boldsymbol{n}} \cdot \int_0^{2\pi} \boldsymbol{r} \times d\boldsymbol{r}$$

For the ellipse $A = \frac{1}{2}\int_0^{2\pi} \frac{a^2(1-e^2)^2}{(1-e\cos\theta)^2} d\theta = \frac{a^2(1-e^2)^2}{2}\int_0^{2\pi} \frac{d\theta}{(1-e\cos\theta)^2}$.
The indefinite integral $\int \frac{d\theta}{(1-e\cos\theta)^2}$ has been solved by Wolfram Alpha by substituting $u = \tan\frac{\theta}{2}$ and the solution is

$$F(\theta) = \frac{2\,e\,\tan\left(\frac{\theta}{2}\right)}{(1-e^2)\left(e\,\tan^2\left(\frac{\theta}{2}\right) + \tan^2\left(\frac{\theta}{2}\right) - e + 1\right)} + \frac{2\,\arctan\left(\frac{\sqrt{e+1}\,\tan\left(\frac{\theta}{2}\right)}{\sqrt{1-e^2}}\right)}{\sqrt{1-e^2}\,(1-e^2)} + C$$

This integral function is not continue in $\theta = \pi$, so the value of the definite integral is $F(2\pi) - lim_{\theta\to\pi^+}F(\theta) + lim_{\theta\to\pi^-}F(\theta) - F(0)$. In 0 and 2π $\tan\frac{\theta}{2} = 0$ and also $\arctan(...) = 0$, while $lim_{\theta\to\pi^+}\tan\frac{\theta}{2} = -\infty$: the first part goes to zero while the second one gives $\frac{-\pi}{(1-e^2)^{1.5}}$. Similarly, $lim_{\theta\to\pi^-}\tan\frac{\theta}{2} = +\infty$ and the result is $\frac{\pi}{(1-e^2)^{1.5}}$. Combining all of this together gives

$$A = \frac{a^2(1-e^2)^2}{2}\frac{2\pi}{(1-e^2)^{1.5}} = \pi a \cdot a\sqrt{1-e^2} = \pi ab$$

3 Problem 3

Given the position at a given time $s(t) = \frac{g}{2\omega}(\sinh\omega t - \sin\omega t)$ and the value $s(1) = 1$, to find the value of ω a root finding algorithm is used with the function $f(\omega) = \frac{\sinh\omega - \sin\omega}{\omega} - \frac{2}{g}$. The calculation are done in different ways in the jupyter file attached. The result, with an accuracy of 10^{-8}, is $\omega = 0.782286418$.

4 Problem 4

Given the ODE

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

with $m, c, k > 0$, this is solved by a linear combination of solutions of the type $x = e^{zt}$, where z is a complex number.
The conditions on z are found by substituting the solution inside the ODE, giving

$$mz^2 + cz + k = 0 \implies z_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Using $\gamma = \frac{c}{2m}$ and $\omega_0^2 = \frac{k}{m}$, the solution becomes

$$z_{1,2} = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Now there are three cases:

1. $\gamma > \omega_0$ ($c^2 > 4mk$): the roots are two real numbers, and the general solution is

$$x = Ae^{(-\gamma+\omega)t} + Be^{(-\gamma-\omega)t}$$

   with $\omega = \sqrt{\gamma^2 - \omega_0^2}$ and A, B to be determined from the initial conditions $x(t_0)$ and $\dot{x}(t_0)$. The system is in the overdampened regime and quickly tends to zero.

2. $\gamma = \omega_0$ ($c^2 = 4mk$): the solution has one root of molteplicity two, and the general solution is

$$x = (A + Bt)e^{-\gamma t}$$

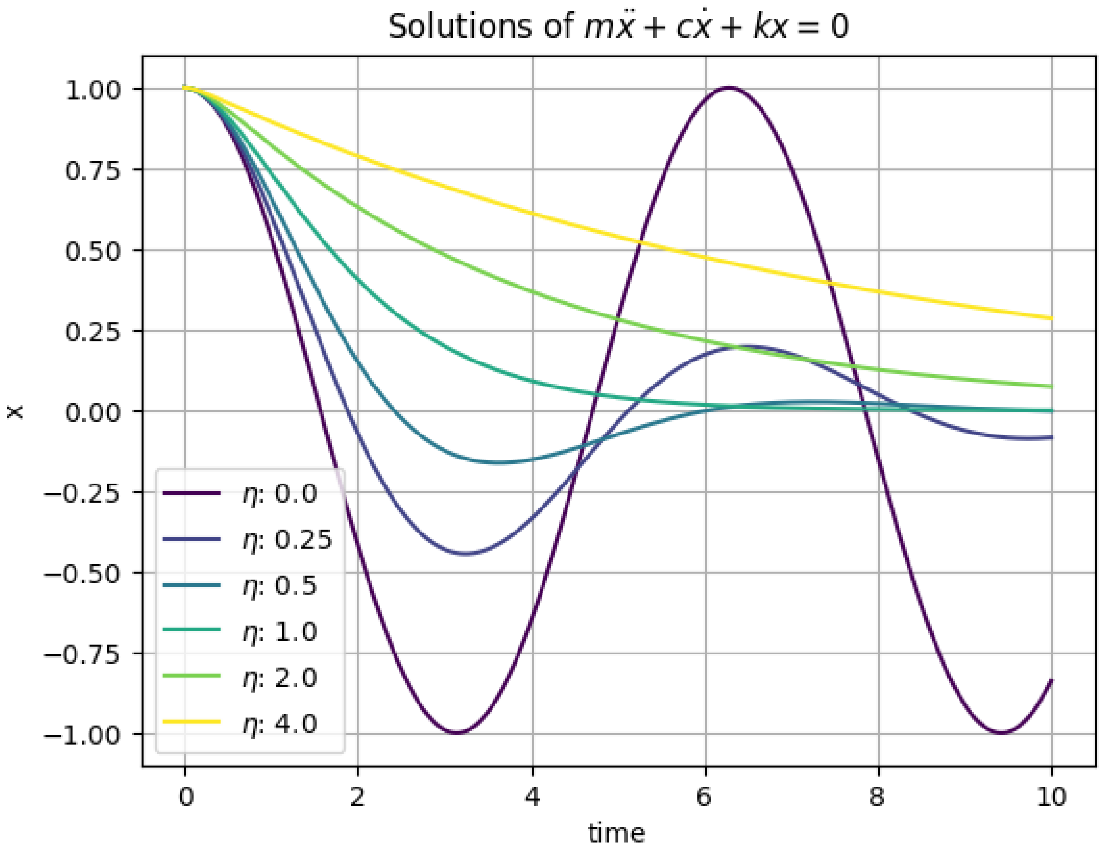
   The system is said to be critically dampened.

3. $\gamma < \omega_0$ ($c^2 < 4mk$): the roots are complex numbers $z_{1,2} = -\gamma \pm i\omega$, with $\omega = \sqrt{\omega_0^2 - \gamma^2}$. Grouping together the imaginary parts in a trigonometric function, the general solution is

$$x = Ae^{-\gamma t} cos(\omega t + \phi)$$

The system is in the underdampened regime and oscillates around zero with increasingly smaller amplitude. If $c = 0$ the system reduces to a lossless harmonic oscillator.

In the graph are plotted several solution with initial conditions $x(0) = 1$, $\dot{x}(0) = 0$ for different values of $\eta = \frac{\gamma}{\omega_0} = \frac{c}{2\sqrt{mk}}$: for $\eta > 1$ the system is overdampened, for $\eta = 1$ is critically dampened, for $\eta < 1$ is underdampened. In this plot I used $m = k = 1$, so $\eta = \frac{c}{2}$.



5 Problem 5

The formula describing the relationship between the angles and the sides of a spherical triangle is known as the **law of cosines**.
Because the sides of the triangle are circle segments of circles with radius 1, the length of the segment is equal to the angle between the vectors $\boldsymbol{e}_i$ (expressed in radians). So $\boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \cos b$, $\boldsymbol{e}_2 \cdot \boldsymbol{e}_3 = \cos a$, $\boldsymbol{e}_1 \cdot \boldsymbol{e}_3 = \cos c$. Then the vectors $\boldsymbol{e}_1 \times \boldsymbol{e}_2$ and $\boldsymbol{e}_1 \times \boldsymbol{e}_3$ are each normal to the plane described by the two vectors and have norms $\sin b$ and $\sin c$. The angle α is the angle between the two planes found before, so

$$\cos\alpha = \frac{(\boldsymbol{e}_1 \times \boldsymbol{e}_2) \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_3)}{\sin b \sin c}$$

and by applying the rule $(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d}) = (\boldsymbol{a}\cdot\boldsymbol{c})\cdot(\boldsymbol{b}\cdot\boldsymbol{d}) - (\boldsymbol{a}\cdot\boldsymbol{d})\cdot(\boldsymbol{b}\cdot\boldsymbol{c})$, this gives $\cos\alpha \sin b \sin c = 1 \cdot \cos a - \cos b \cos c$ which can be arranged in the final form

$$\cos a = \cos b \cos c + \sin b \sin c \cos\alpha$$

6 Problem 6

By putting the origin of the reference frame in the radar ant $t = 0$ at the moment of closest approach, the position of the plane is given by

$$\boldsymbol{r} = (vt)\boldsymbol{e}_x + s\boldsymbol{e}_y + h\boldsymbol{e}_z$$

The angular velocity is defined as $\dot{\hat{\boldsymbol{r}}} = \boldsymbol{\omega} \times \hat{\boldsymbol{r}}$, where $\hat{\boldsymbol{r}}$ is the radial versor, so it can be obtained by deriving $\dot{\boldsymbol{r}} = (r\dot{\hat{\boldsymbol{r}}}) = \dot{r}\hat{\boldsymbol{r}} + \boldsymbol{\omega} \times r\hat{\boldsymbol{r}}$ and by cross-multiplying both members by $\boldsymbol{r}$, $\boldsymbol{r} \times \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{r} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times (\boldsymbol{\omega} \times \boldsymbol{r})$ and using the triple product rule $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$, this gives $\boldsymbol{r} \times \dot{\boldsymbol{r}} = (\boldsymbol{r} \cdot \boldsymbol{r})\boldsymbol{\omega} - (\boldsymbol{r} \cdot \boldsymbol{\omega})\boldsymbol{r}$ with $\boldsymbol{r} \cdot \boldsymbol{\omega} = 0$ because the vectors are perpendicular.
The final result is

$$\boldsymbol{\omega} = \frac{\boldsymbol{r} \times \dot{\boldsymbol{r}}}{r^2}$$

and using $\dot{\boldsymbol{r}} = (v, 0, 0)$, $\boldsymbol{\omega} = \frac{(0, vh, -vs)}{r^2}$. The horizontal angular velocity at the closest approach is $\boldsymbol{\omega} \cdot \boldsymbol{e}_z = -\frac{vs}{s^2+h^2}$, while the vertical one is $\frac{vh}{s^2+h^2}$. The angular acceleration is

$$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \frac{\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}}}{r^2} + \frac{\boldsymbol{r} \times \ddot{\boldsymbol{r}}}{r^2} - 2\dot{r}\frac{\boldsymbol{r} \times \dot{\boldsymbol{r}}}{r^3}$$

and because $\ddot{\boldsymbol{r}} = \boldsymbol{0}$, $\boldsymbol{\alpha} = -2\dot{r}\frac{\boldsymbol{r} \times \dot{\boldsymbol{r}}}{r^3} = -2\frac{\dot{r}}{r}(vh\boldsymbol{e}_y - vs\boldsymbol{e}_z)$. Now $r = \sqrt{v^2t^2 + s^2 + h^2}$, so $\dot{r} = \frac{d}{dt}\sqrt{v^2t^2 + s^2 + h^2} = \frac{v^2 t}{\sqrt{v^2t^2+s^2+h^2}}$ is zero at the time of closest approach, so $\boldsymbol{\alpha} = (0, 0, 0)$: the angular acceleration is zero at the closest approach.
Below is a plot of the angular velocities and accelerations for $v = 10\frac{m}{s}$, $s = 20m$, $h = 15m$.