Homework3

April 6, 2024

[ ]:
```python
import matplotlib.pyplot as plt
from scipy.integrate import solve_ivp
import numpy as np
```

0.1 Finding the central force given a particular orbit

The equation describing the curve is $r(\theta) = R + a\cos(k\theta + \phi)$, with $R > 0$ radius of the perturbed circumference, a the amplitude of the perturbation ($|a| < R$ to avoid reaching the origin) and k number of oscillations done in an orbit. If $k \in \mathbb{Z}$, the orbits are periodic. ϕ is the initial phase.

Using the Binet's equation, $u = \frac{1}{r} = \frac{1}{R+a\cos(k\theta+\phi)}$ and deriving twice gives

$$\frac{d^2u}{d\theta^2} = \frac{k^2 a\cos(k\theta + \phi)}{(R + a\cos(k\theta + \phi))^2} + \frac{2k^2a^2\sin^2(k\theta + \phi)}{(R + a\cos(k\theta + \phi))^3}$$

Now to have a central force this has to be rewritten as a function of r by using $a\cos(k\theta+\phi) = r-R$. Substituting this gives the result

$$\frac{d^2u}{d\theta^2} = \frac{k^2(r - R)}{r^2} + \frac{2k^2a^2 - k^2(r - R)^2}{r^3}$$

Now Binet's method states that for a central force the equation of motion $\ddot{r} - \frac{h^2}{r^3} + f(r) = 0$ gives $\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2u^2}f(\frac{1}{u})$. By using the previous solution and grouping the terms together, the result is

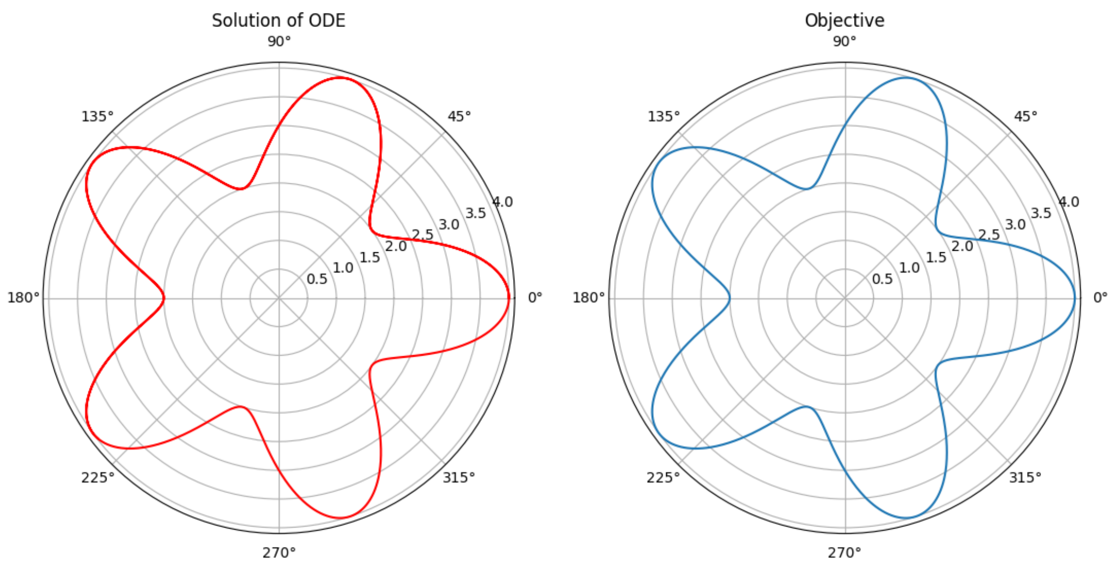
$$f(r) = \frac{h^2 - h^2k^2}{r^3} + 3\frac{h^2k^2R}{r^4} + 2\frac{k^2h^2(a^2 - R^2)}{r^5}$$

Any force of the type $f(r) = \frac{A}{r^3} + \frac{B}{r^4} + \frac{C}{r^5}$ can give rise to this kind of orbits if, after choosing h, a tuple of values R, k, a can be found to solve the system and specific intial condotions are chosen.

In the following cell is reported a solution of the ODE after fixing R,a,k

[ ]:
```python
def f(r, h, k, a, R):
    ''' Defining the central force '''
    # I'm not considering the term h^2/r^3 that cancels out in the equation of␣
↪motion
    return -h**2*k**2/r**3 + 3*h**2*k**2*R/r**4 + 2*h**2*k**2*(a**2-R**2)/r**5
```

```python
R, a, k = 3, 1, 5
r, dr, theta, dtheta = R+a, 0.0, 0.0, 0.5

h = r**2*dtheta

y0 = np.array([r,dr,theta])

def ode(t, y, h, k, a, R, f):
  ''' Defining the ODE associated to the system'''
  r, dr, theta = y[0], y[1], y[2]
  return np.array([dr, -f(r,h,k,a,R), h*r**-2])



Y = solve_ivp(ode, t_span=(0,12), y0=y0, t_eval=np.linspace(0,12,1000),␣
 ↪args=(h,k,a,R, f), rtol=1e-12)

r, dr, theta = Y.y[0], Y.y[1], Y.y[2]

fig, AX = plt.subplots(1,2, figsize = (13,6), subplot_kw={'projection':␣
 ↪'polar'})
AX[0].set_title('Solution of ODE')
AX[0].plot(theta, r, 'r')

AX[1].set_title('Objective')
theta = np.linspace(0,2*np.pi, 1000)
AX[1].plot(theta, R+a*np.cos(k*theta))

plt.show()
```



0.2 Investigating the orbit of a projectile

The projectile is launched east with a velocity of 5 $km\ s^{-1}$ at an angle of 45° in the rotating Earth RF, so the velocity in an inertial RF is $\mathbf{v_I} = \mathbf{v} + \ \times \mathbf{R}$, so as $\omega = \frac{2\pi}{23h56min4.1s} = 7.292 \cdot 10^{-5}$ rad s^{-1}, and the equatorial radius is $R_E = 6378.137$ km, the velocity in the inertial RF is $v_I = 5.339$ $km\ s^{-1}$ with an angle of 41.468° with the horizon.

Now the specific angular momentum is $h = R_E v_I \sin(\alpha + \frac{\pi}{2}) = 2.5516 \cdot 10^4$ $km^2\ s^{-1}$, the gravitational parameter is $\mu = 3.986 \cdot$, the energy constant is $C = \frac{1}{2}v^2 - \frac{\mu}{R_E} = -48.24$ $km^2\ s^{-2}$, so the orbit has $e = 0.78$, $a = 4131$ km. The initial true anomaly is found by reversing $r = \frac{a(1-e^2)}{1+e\cos f}$, thus the angular distance covered in the inertial RF is $\phi = 2(\pi - f) = 33.85°$.

Finally, the period of the orbit is given by $\frac{T^2}{a^3} = \frac{4\pi^2}{\mu}$ and the time from pericenter as $n(t-\tau) = \cos^{-1}[\frac{1}{e}(1-\frac{r}{a})] - e\sqrt{1 - \frac{1}{e^2}(1-\frac{r}{a})^2}$. This gives the time of flight as $T - 2t = 1137$ s and the angular distance covered in the Earth RF as $\phi - \omega\Delta t = 29.09°$

```
[ ]: v = 5 #Initail velocity
     R = 6378.137  #Equatorial radius
     T = 23*3600+56*60+4.100  #Sidereal rotation period
     mu = 398600.4415  #Gravitational parameter
     alpha = np.deg2rad(45)

     omega = 2*np.pi/T
     v = np.array([v*np.cos(alpha)+R*omega, v*np.sin(alpha)])  #velocity vector␣
      ↪(horizontal and vertical components)
     h = R*v[0]
     C = 0.5*np.linalg.norm(v)**2-mu/R
     e = (1+2*C*h**2/mu**2)**0.5
     a = -mu/(2*C)
     f = np.arccos(1/e*(a*(1-e**2)/R-1))

     phi = 2*(np.pi-f)
     print(f'The angular distance covered in the inertial RF is {np.rad2deg(phi)}␣
      ↪deg')

     T = (4*np.pi**2*a**3/mu)**0.5 #Period of the orbit
     n = 2*np.pi/T
     t = (np.arccos((1-R/a)/e) - e*(1-(1-R/a)**2/e**2)**0.5)/n
     delta_t = T-2*t

     print(f'The time of flight is {delta_t} s')
     print(f'The angle covered in the Earth RF is {np.rad2deg(phi-omega*delta_t)}␣
      ↪deg')
```

```
The angular distance covered in the inertial RF is 33.84449835570999 deg
```

```
The time of flight is 1137.1251066749837 s
The angle covered in the Earth RF is 29.09350532725621 deg
```

[ ]:
```python
theta = np.linspace(0, 2*np.pi, 1000)

r =lambda theta: a*(1-e**2)/(1+e*np.cos(theta))

ax = plt.subplot(projection='polar')
ax.plot(theta, r(theta), color='r', label='trajectory')
ax.plot(theta, R*np.ones(shape=1000), color='lightblue', label='Earth')
ax.fill_between(theta, R, 0, color='lightblue', alpha=0.8)
ax.set_yticks(np.arange(2000,8000, 2000))
plt.legend(loc='best')
plt.show()
```