Riccardo Cusinato - 3627460
Giacomo Barzon - 3626438
Damiano Sgarbossa - 3623971

Assignment 7

Exercise 1.1: Simple advection problem

A simple 1D advection problem is defined by the PDE

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$$

with u(x,t) function of time and the position. Show that if we are given any function q(x), then

$$u(x,t) = q(xvt)$$

is a solution of the advection equation.

We call $z \equiv x - vt$. Then we only need to compute both terms:

- $\dfrac{\partial u}{\partial t} = \dfrac{\partial q(z)}{\partial z} \cdot \dfrac{\partial z}{\partial t} = \dfrac{\partial q}{\partial z} \cdot (-v)$
- $\dfrac{\partial u}{\partial x} = \dfrac{\partial q(z)}{\partial z} \cdot \dfrac{\partial z}{\partial x} = \dfrac{\partial q}{\partial z} \cdot (1)$

So

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = -v\frac{\partial q}{\partial z} + v\frac{\partial q}{\partial z} = 0$$

which satifies the advection equation.

Exercise 1.2: Derivation of the wave equation

Now we turn to the Euler equations as defined in the lecture. Take a simple isothermal equation of state, $P = c_s^2\rho$, with c_s being the isothermal sound speed. Show that they lead to a simple wave equation for density perturbations.

To obtain the wave equation for the density perturbations we must look at the behavior of small perturbations around the equilibrium solution of the Euler equations (we can just consider the first 2 of them). We'll use the following notation for the equilibrium state (which is homogeneous and stationary) and then only consider terms up to linear order in the perturbation:

$$\rho(\vec{x},t) = \rho_0 + \delta\rho(\vec{x},t) \qquad \text{and} \qquad \vec{v}(\vec{x},t) = \vec{v_0} + \delta\vec{v}(\vec{x},t) = \delta\vec{v}(\vec{x},t)$$

For what regards the expression used for velocity we made the assumption $v_0 = 0$ because we are in the stationary conditions so we can always consider the system in the reference system of the fluid in which $v_0 = 0$. Now we can start to manipulate the Euler equations (starting from the total derivative formulation) by inserting the new expressions of ρ and v:

$$\begin{cases} \frac{d}{dt}\rho = -\rho\vec{\nabla}\cdot\vec{v} \\ \frac{d}{dt}\vec{v} = -\frac{\vec{\nabla}P}{\rho} \\ P = c_s^2\rho \end{cases} \implies \begin{cases} \partial_t(\rho_0+\delta\rho) + \delta\vec{v}\cdot\vec{\nabla}(\rho_0+\delta\rho) = -(\rho_0+\delta\rho)\vec{\nabla}\cdot\delta\vec{v} \\ \partial_t\delta\vec{v} + \delta\vec{v}\cdot\vec{\nabla}\delta\vec{v} = -\frac{c_s^2}{\rho_0+\delta\rho}\vec{\nabla}(\rho_0+\delta\rho) \\ P = c_s^2\rho \end{cases}$$

Now, considering only the first order terms and then deriving the first eq. w.r.t. the time and the second w.r.t. the position, we obtain:

$$\begin{cases} \partial_t\delta\rho + \rho_0\vec{\nabla}\cdot\delta\vec{v} = 0 \\ \rho_0\partial_t\delta\vec{v} + c_s^2\vec{\nabla}\delta\rho = 0 \end{cases} \implies \begin{cases} \partial_t^2\delta\rho + \rho_0\vec{\nabla}\cdot\partial_t\delta\vec{v} = 0 \\ \rho_0\vec{\nabla}\cdot\partial_t\delta\vec{v} + c_s^2\nabla^2\delta\rho = 0 \end{cases}$$

In the previous equations we used the fact that time and spatial derivatives are interchangeable, now we can finally merge the 2 equations obtaining the wave equation for density perturbations:

$$\left(\partial_t^2 - c_s^2\nabla^2\right)\delta\rho(\vec{x},t) = 0$$

Exercise 1.3: Sound waves

Assume that seismic waves can be approximated as simple sound waves in the Earth's mantle.

- How long will it take for the earthquake from a massive volcano eruption on the other side of the Earth to be measured by a seismic station in Europe?

We assume that the seismic waves propagated in the superior mantle, so the path travelled by the waves is a semicirconference with radius the average distance of the superior mantle from the Earth's center ($R_m \approx 6000km$).
The velocity of sound waves can be calculated as $c_s = \sqrt{\frac{P}{\rho}}$. A typical values of pressure and density in the mantle are $P_m \approx 20\ GPa; \rho_m \approx 4\cdot 10^3 kg\cdot m^{-3}$, so the typical waves velocity is $v_m \approx 2.2km/s$. So seismic wave in the mantle can be measured in the other side of the world after $t_m \approx \frac{\pi R_m}{v_m} \approx 160\ min \approx 2.5\ hours$.

- What is the travel time to this station for the corresponding sound wave through the atmosphere? Will it be noticed in Europe? Justify your answer.

For a seismic wave that travel trough the atmosphere, we can assume as typical values of radius, pressure and density $R_{atm} \approx 6400\ km$; $P_{atm} \approx 1\ atm \approx 10^5\ Pa$; $\rho_{atm}\ \approx 1.2\ kg\cdot m^{-3}$, so the typical velocity is $v_{atm} \approx 290\ m/s$. Then the travel time is $t_{atm} \approx \frac{\pi R_{atm}}{v_{atm}} \approx 20\ hours$.

We can notice that $p_{atm} \approx 10 \cdot p_m$.
For what concerns if the waves will be noticed, we have to look at the intensity of the wave. This varies with the distance, because there is an attenuation due to the viscosity of the medium. This can be written as a factor $10^{-\alpha x}$, with α being the attenuation coefficient. For a typical frequency of $\sim$ 100 Hz, this coefficient in air is about $10^{-4} dB/cm$, while the traveled distance is $D = \pi R \sim 10^9$ cm, thus the total attenuation coefficient is 10^5 dB; then the total attenuation will be of a factor 10^{-10^4}, resulting in the acoustic wave in air being completely negligible for that distance.