Time Series Analysis and Recurrent Neural Network
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Exercise 7

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1 Task 1 - Discrete-time non-linear dynamics

Consider the univariate non-linear map:

$$x_{t+1} = f(x_t, a, b) = a \cdot x_t + b \cdot tanh(x_t)$$

1.0.1 1. Plot the return plot of this map.
Do this for the parameter values: I) {a, b} = {1,3}, II) {a,b} = {0.5,−2}, III) {a,b} = {0.5,3} and IV) {a,b} = {1,0}.

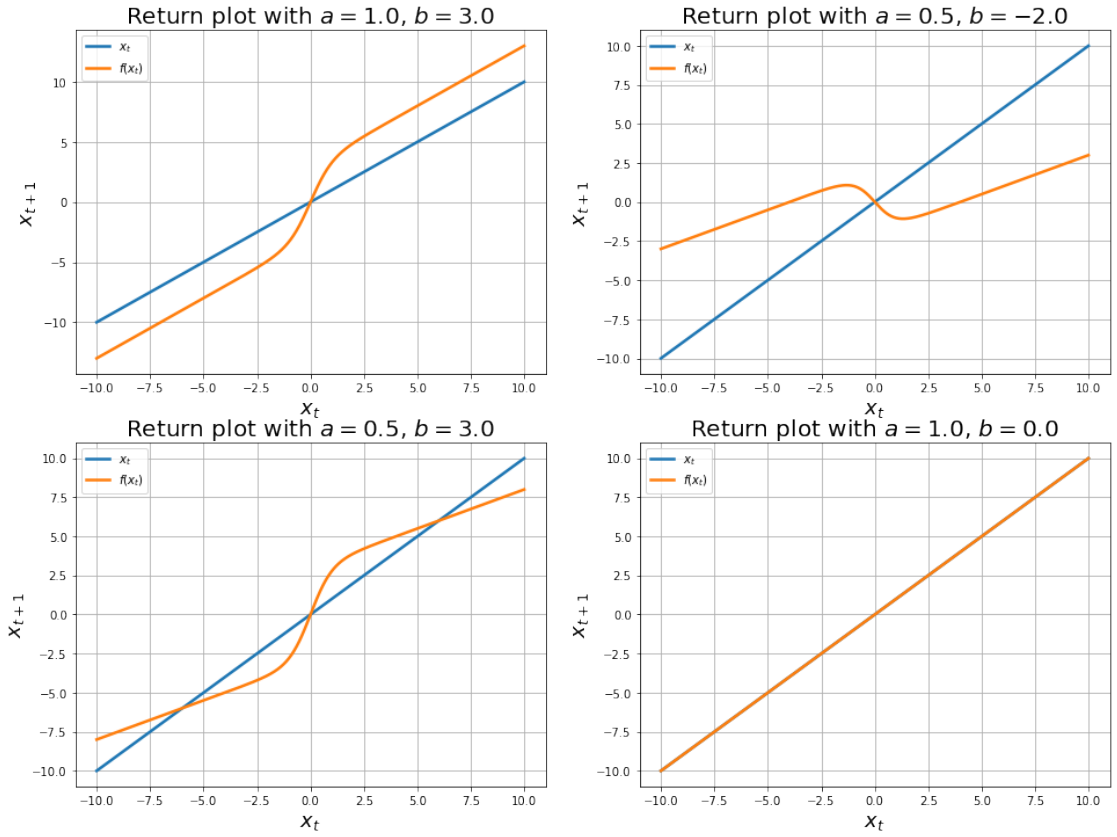
In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import scipy as sp

# hiding warnings
import warnings
warnings.filterwarnings('ignore')
```

In [2]:
```
# define parameters
a = np.array([1.0, 0.5, 0.5, 1.0])
b = np.array([3.0, -2.0, 3.0, 0.0])
```

In [3]:
```
# define non-linear map
def map(x, a, b):
    return a*x + b*np.tanh(x)
```

In [4]:
```
# plot return plot for different parameters
x = np.arange(-10.,10., 1e-2)

# Plot the return plot
fig1 = plt.subplots(figsize=[16,12])

for i in range(a.shape[0]):
    fx = map(x, a[i], b[i])
```

```
        plt.subplot(2,2,i+1)

        plt.plot(x, x, label=r'$x_t$', linewidth=2.5 )
        plt.plot(x, fx, label=r'$f(x_t)$', linewidth=2.5 )

        plt.xlabel(r'$x_{t}$', fontsize = 18)
        plt.ylabel(r'$x_{t+1}$', fontsize = 18)
        plt.grid()
        plt.legend()
        plt.title(r'Return plot with $a={%.1f}$, $b={%.1f}$' %(a[i],b[i]) , fontsize = 20)

    plt.show(fig1)
```



1.0.2 By inspecting the plot say how many fixed points you expect the system to have and comment on their stability.

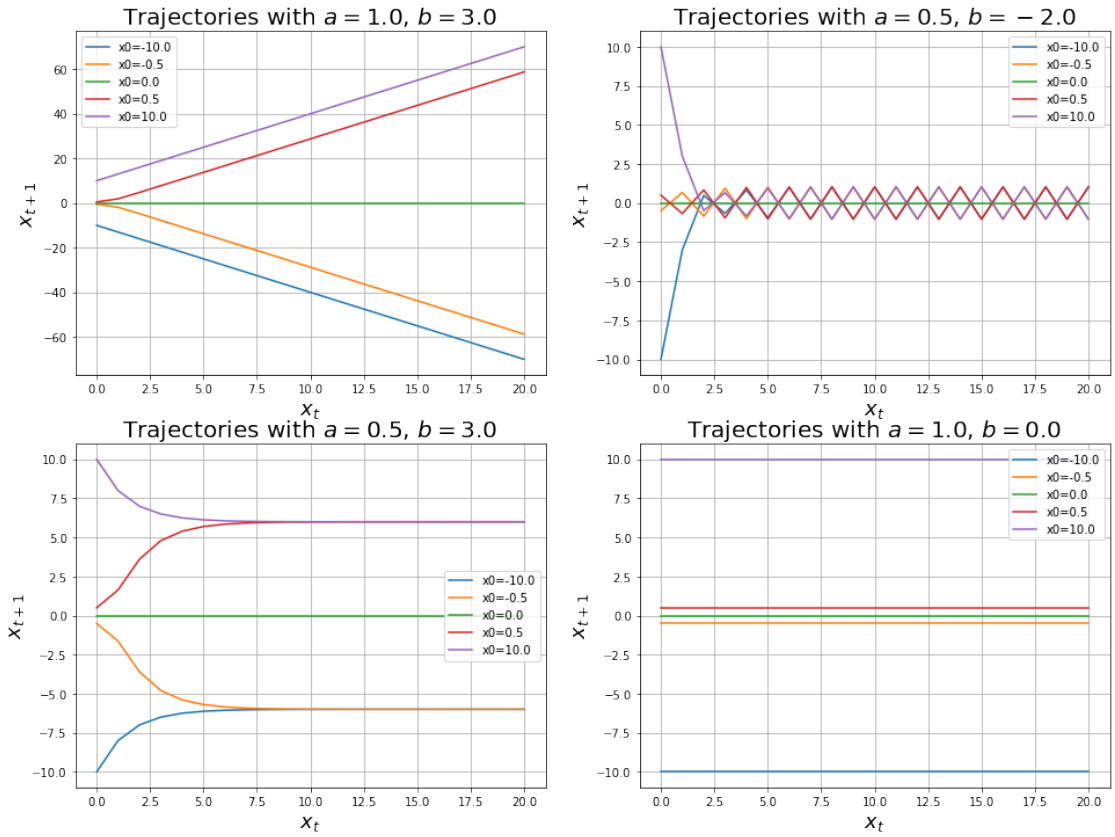
In the two plots on the left, the ones with $b = 3$, for high positive value of the parameter a (i.e. $a = 1$) we can notice that there is only one instable fixed point at x=0, while for low positive value of a (i.e. $a = 0.5$) in addition we have other two stable points at $x = \pm 6$. By keeping fixed $a = 5$ and by decreasing the parameter b to negative values (i.e. $b = -2$) (plot at top right) the two fixed points

disappear. By using the parameters $(a = 1, b = 0)$ (plot at bottom right) we can notice that all the points are neutrally stable.

1.0.3 2. For all parameter sets specified above, plot the trajectory of the system when starting from the initial conditions x0 = −10, x0 = −0.5, x0 = 0, x0 = 0.5, x0 = 10.

In [5]:
```
def trajectory(x0, a, b, steps):

    # define empty array
    x = np.zeros(steps+1)

    # define initial condition
    x[0] = x0

    for i in range(steps):
        x[i+1] = map(x[i], a, b)

    return x
```

In [6]:
```
# plot trajectory for different parameters and different starting points
x0 = np.array( [-10., -0.5, 0., 0.5, 10.] )
steps = 20

fig1 = plt.subplots(figsize=[16,12])

for i in range(a.shape[0]):
    plt.subplot(2,2,i+1)

    for j in x0:
        traj = trajectory(j, a[i], b[i], steps)
        plt.plot(np.arange(steps+1), traj, label='x0=%.1f' %(j))

    plt.xlabel(r'$x_{t}$', fontsize = 18)
    plt.ylabel(r'$x_{t+1}$', fontsize = 18)
    plt.grid()
    plt.legend()
    plt.title(r'Trajectories with $a={%.1f}$, $b=%.1f$' %(a[i],b[i]) , fontsize = 20)

plt.show(fig1)
```



1.0.4 3. Confirm your intuitions by computing the fixed points (numerically) and their stability (analytically) for the parameter set III

fixed point:

$$x^\star = f(x^\star, a, b) = a \cdot x^\star + b \cdot tanh(x^\star)$$

stability condition:

$$|f'(x^\star, a, b)| = |a + b[1 - tanh(x^\star)^2]| < 1$$

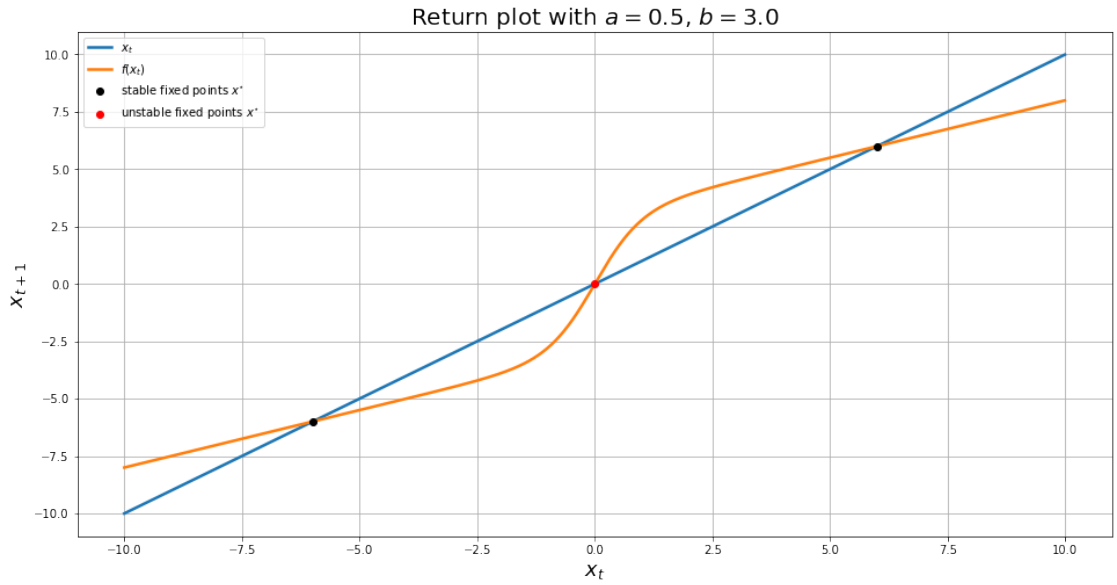
```
In [7]: # find fixed points numerically
        def fixed_points(x, fx, tolerance=1e-2):
            x_star = np.abs(x - fx)

            # find local minima
            index1 = np.r_[True, x_star[1:] < x_star[:-1]] & np.r_[x_star[:-1] < x_star[1:], Tru

            # find similar values (up to tolerance)
            index2 = x_star <= tolerance

            return x[index1 & index2]
```

In [8]:
```
# find stability analitycally
def isStable(x, a, b):
    return np.abs(a + b*(1-np.tanh(x)**2) ) < 1.
```

In [9]:
```
# print fixed points and relative stability
fx = map(x, a[2], b[2])
x_star = fixed_points(x, fx)

stable = isStable(x_star, a[2], b[2])
fp_stable = x_star[stable]
fp_unstable = x_star[ np.logical_not( stable ) ]

np.set_printoptions(formatter={'float_kind':"{:.3f}".format})
print('Fixed points: ', x_star)
print('Stable: ', stable)
#np.set_printoptions()
```

```
Fixed points:  [-6.000 -0.000 6.000]
Stable:  [ True False  True]
```

In [10]:
```
# plot return plot
fig1 = plt.subplots(figsize=[16,8])

plt.plot(x, x, label=r'$x_t$', linewidth=2.5 )
plt.plot(x, fx, label=r'$f(x_t)$', linewidth=2.5 )
plt.plot(fp_stable, fp_stable, 'ok' , label=r'stable fixed points $x^{\star}$', linewid
plt.plot(fp_unstable, fp_unstable, 'or' , label=r'unstable fixed points $x^{\star}$', l

plt.xlabel(r'$x_{t}$', fontsize = 18)
plt.ylabel(r'$x_{t+1}$', fontsize = 18)
plt.grid()
plt.legend()
plt.title('Return plot with $a={%.1f}$, $b=%.1f$' %(a[2],b[2]), fontsize = 20)
plt.show()
```



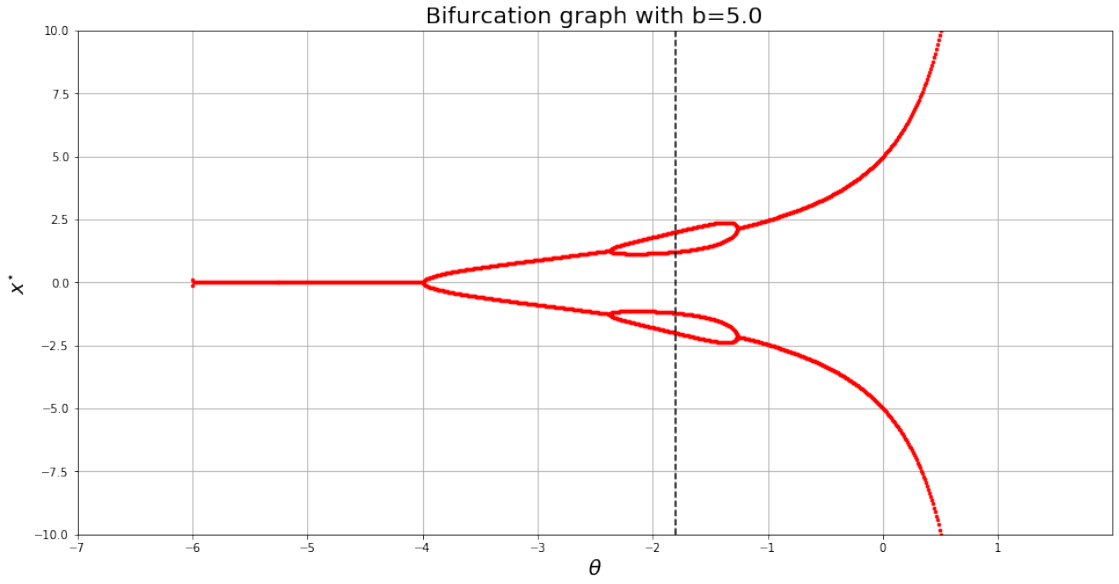
1.0.5 4. Plot the bifurcation graph as a function of $a \in [-7, 2]$ (x-axis) for b = 5 in which you display stable fixed points x^* in the range of $x^* \in [-5, 5]$ (y-axis)

```
In [11]: def trajectory(x0, a, b, steps):
             n = x0.shape[0]

             # define empty array
             x = np.zeros(( steps+1, n ))

             # define initial condition
             x[0] = x0

             for i in range(steps):
                 x[i+1] = map(x[i], a, b)

             return x

In [12]: def findStableObjects(f, a, b, xl=-5., xr=5., N=100, steps=500):
             # generate random initial conditions
             x0 = np.random.uniform(xl,xr,N)

             # create trajectories
             x = trajectory(x0, a, b, steps)

             # take only last element
             stablePoints = np.ravel( x[(steps-5):,:] )

             # discard repetitive elements
```

```
        stablePoints = np.unique(stablePoints)

        # discard similar element up to tolerance
        tol = 0.01
        stablePoints = stablePoints[~(np.triu(np.abs(stablePoints[:,None] - stablePoints) <

        return stablePoints
```

In [13]:
```
a = np.arange(-7., 2., 0.01)
b = 5.

stableObjects = []

for i in a:
    stableObjects.append( findStableObjects(map, i, b ) )
```

In [14]:
```
# plot bifurcation graph
fig1 = plt.subplots(figsize=[16,8])

for i in range(len(a)):
    plt.plot(np.zeros(stableObjects[i].shape)+a[i], stableObjects[i], '.r', markersize=

plt.axvline( x=-1.8, c='k', ls='--' )

plt.xlim([a[0], a[-1]])
plt.ylim([-10.,10.])
plt.xlabel(r'$\theta$', fontsize = 18)
plt.ylabel(r'$x^{\star}$', fontsize = 18)
plt.grid()
plt.title('Bifurcation graph with b=%.1f' % b, fontsize = 20)
plt.show()
```



1.0.6 How would you interpret the bifurcation graph at a = −1.8?

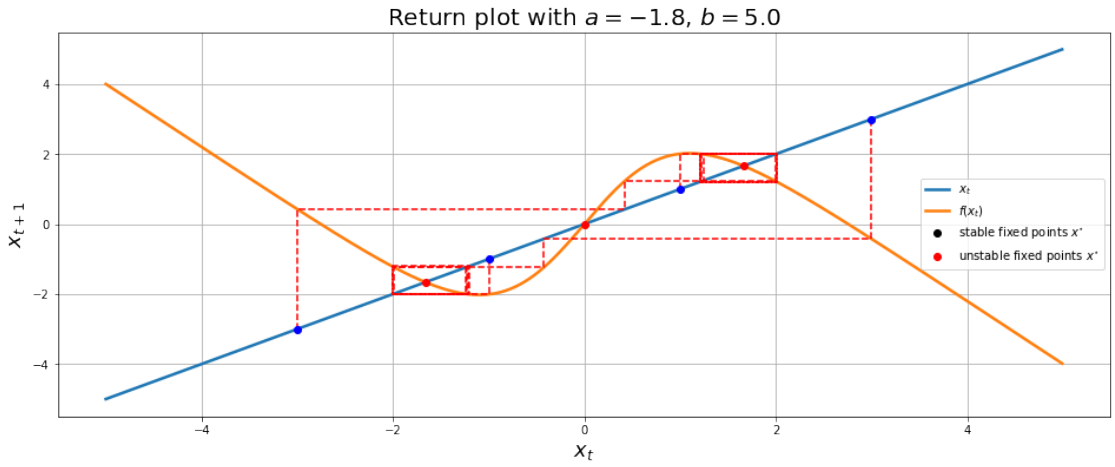
In [15]:
```
# compute cobweb for fixed initial point
def cobweb(x0, a, b, steps = 10):
    p = np.zeros(( 2, steps ))

    p[:,0] = [x0, x0]
    p[:,1] = [x0, map(x0, a, b) ]

    for i in range(2, steps, 2):
        p[:,i] = [ p[1,i-1], p[1,i-1] ]
        p[:,i+1] = [ p[0,i], map(p[0,i], a, b) ]

    return p
```

In [16]:
```
# print fixed points and relative stability
x = np.arange(-5.,5., 1e-2)

a = -1.8

fx = map(x, a, b)
x_star = fixed_points(x, fx)

stable = isStable(x_star, a, b)
fp_stable = x_star[stable]
fp_unstable = x_star[ np.logical_not( stable ) ]

np.set_printoptions(formatter={'float_kind':"{:.3f}".format})
print('Fixed points: ', x_star)
print('Stable: ', stable)
```

```
Fixed points:  [-1.660 -0.000 1.660]
Stable:  [False False False]
```

In [17]:
```
# Plot the return plot
x = np.arange(-5.,5., 1e-2)

x0 = [-3, -1, 1, 3]

a = -1.8

fig1 = plt.figure(figsize=[16,6])

fx = map(x, a, b)
plt.plot(x, x, label=r'$x_t$', linewidth=2.5 )
```

```
plt.plot(x, fx, label=r'$f(x_t)$', linewidth=2.5 )
plt.plot(fp_stable, fp_stable, 'ok' , label=r'stable fixed points $x^{\star}$', linewid
plt.plot(fp_unstable, fp_unstable, 'or' , label=r'unstable fixed points $x^{\star}$', l

for i in x0:
    line = cobweb(i, a, b)
    plt.plot(line[0], line[1], '--r')
    plt.plot(line[0,0], line[1,0], 'ob')

plt.xlabel(r'$x_{t}$', fontsize = 18)
plt.ylabel(r'$x_{t+1}$', fontsize = 18)
plt.grid()
plt.legend()
plt.title(r'Return plot with $a={%.1f}$, $b={%.1f}$' %(a,b) , fontsize = 20)

plt.show(fig1)
```



As we can notice from the plots above, for $a = 1.8$ also the two fixed points different from zero become unstable, but at the same time there is the emerging of four other periodically stable objects.

2 Task 2 - Training an RNN in PyTorch

Consider a recurrent neural network with N neurons:

$$x_t = \Phi(Ax_{t-1} + I_t)$$

$$\Phi(y) = tanh(y)$$

You are given the following mapping from inputs to outputs:

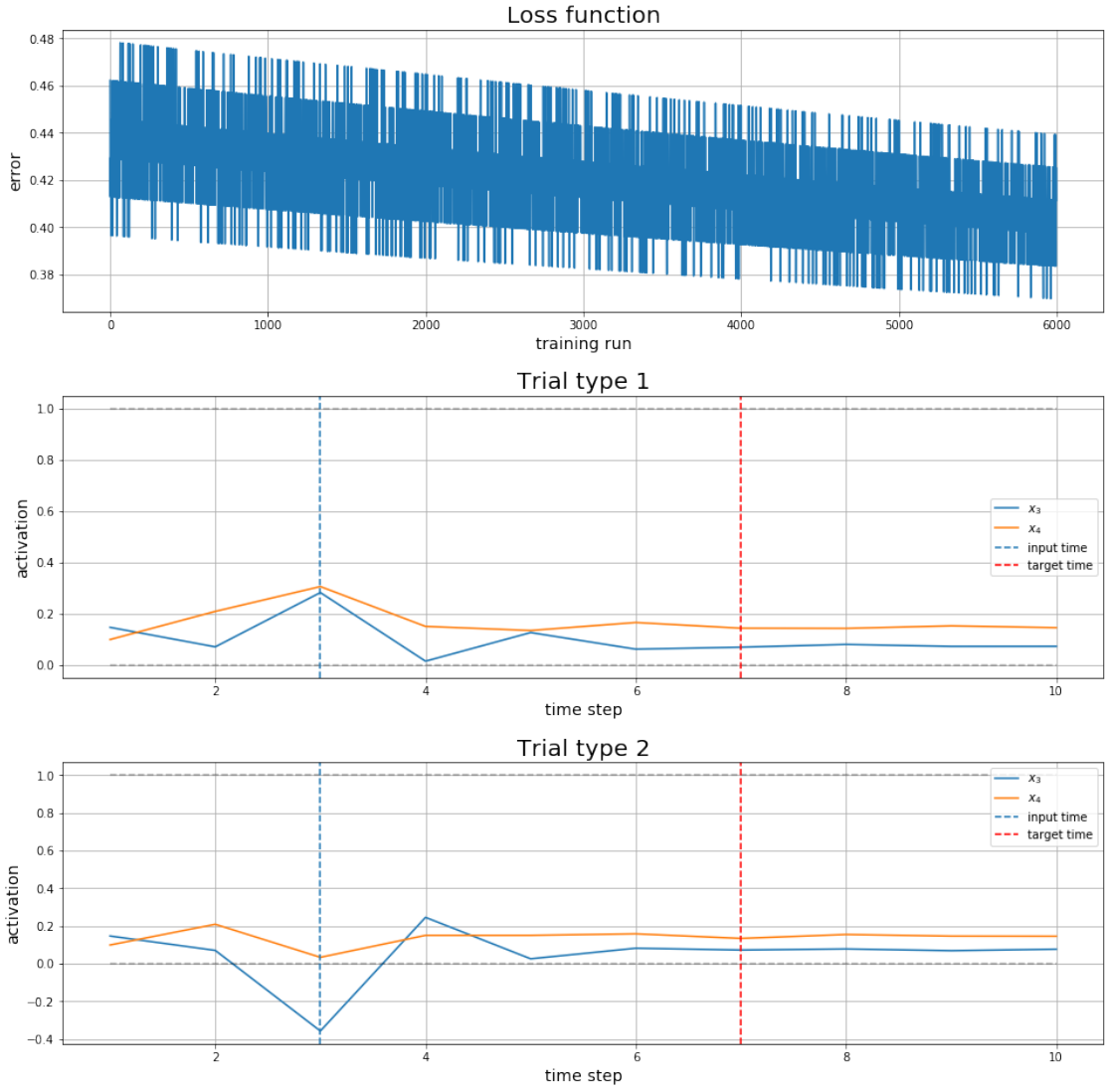
$$I_3 = (1, 0, 0, ...)^T \rightarrow x_7 = (..., 1, 0, ...)^T$$

$$I_3 = (0, 1, 0, ...)^T \rightarrow x_7 = (..., 0, 1, ...)^T$$

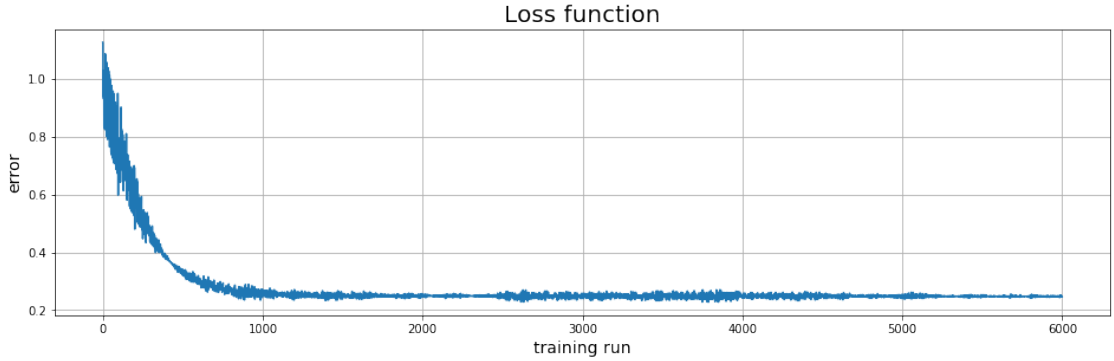
2.0.1 For N = 5, examine the convergence of the error function and the performance of the network after apparent convergence for different learning rates from the range $\alpha \in [10^{-5}, 1]$

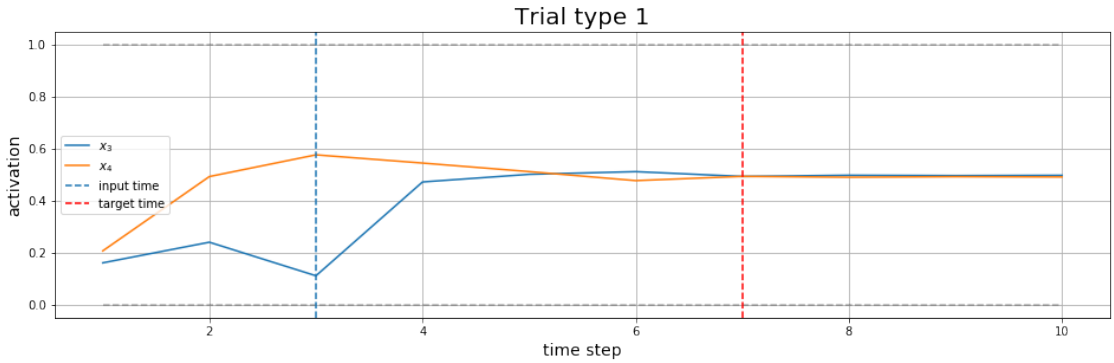
```
In [18]: from working_memory_RNN import RNN

In [19]: alpha = [1e-5, 1e-3, 1e-1, 1.0]

         for i in alpha:
             RNN(alpha = i)
```

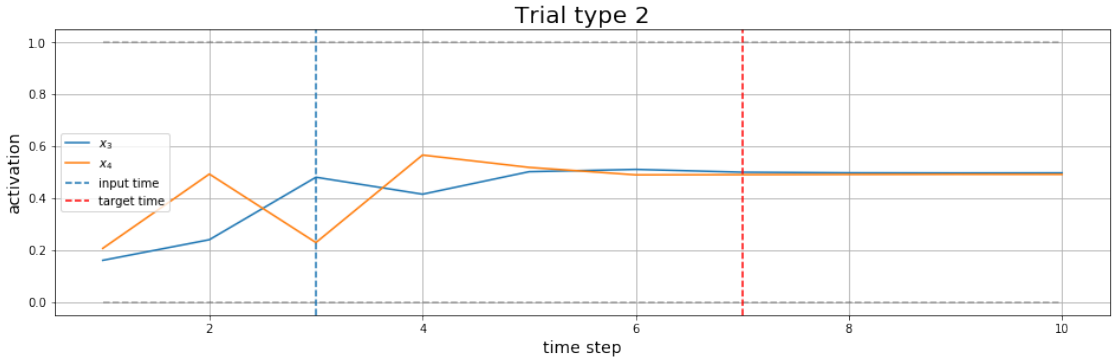
RNN with N=5, alpha=0.000010, act.func.=tanh



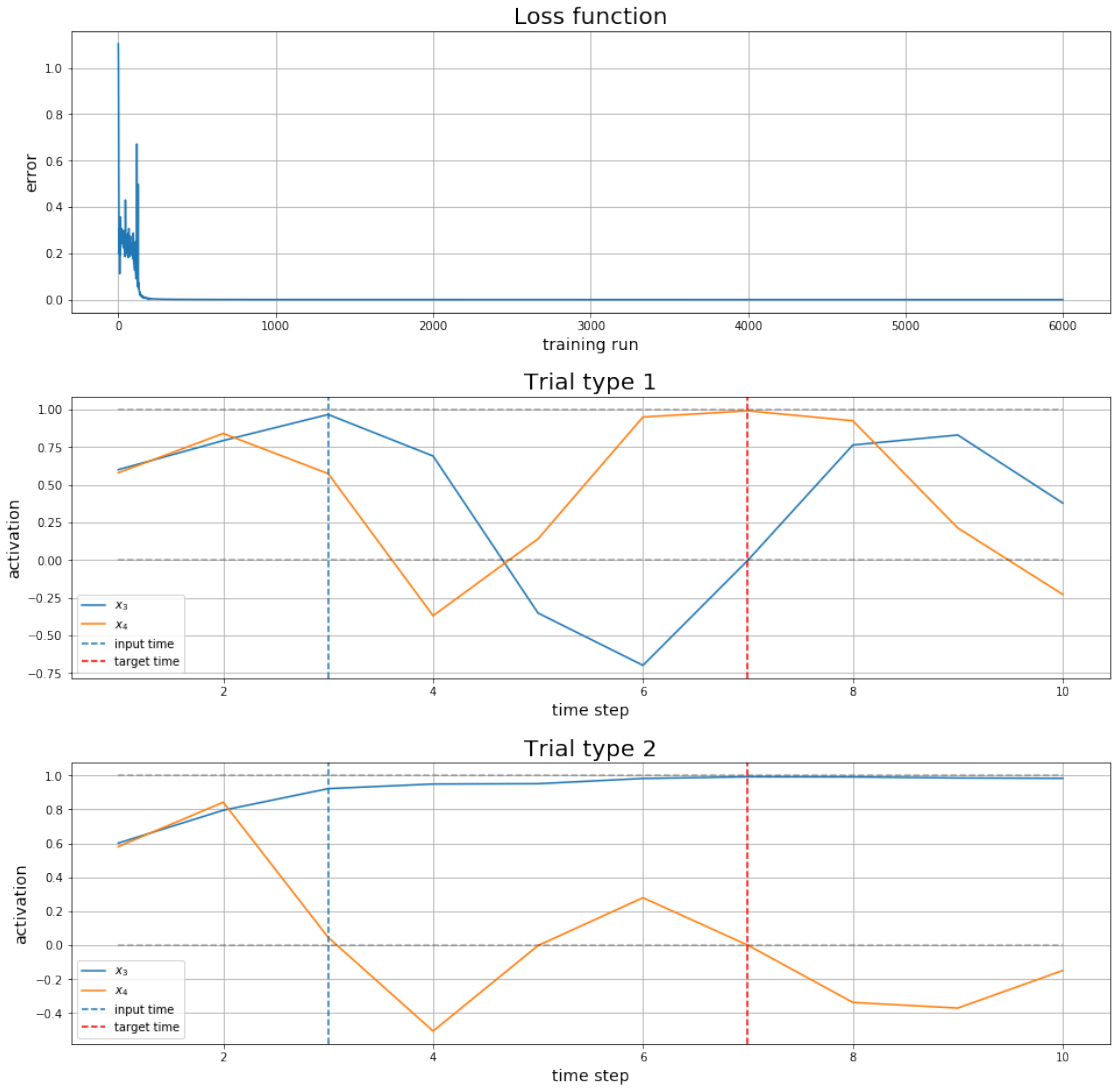
RNN with N=5, alpha=0.001000, act.func.=tanh

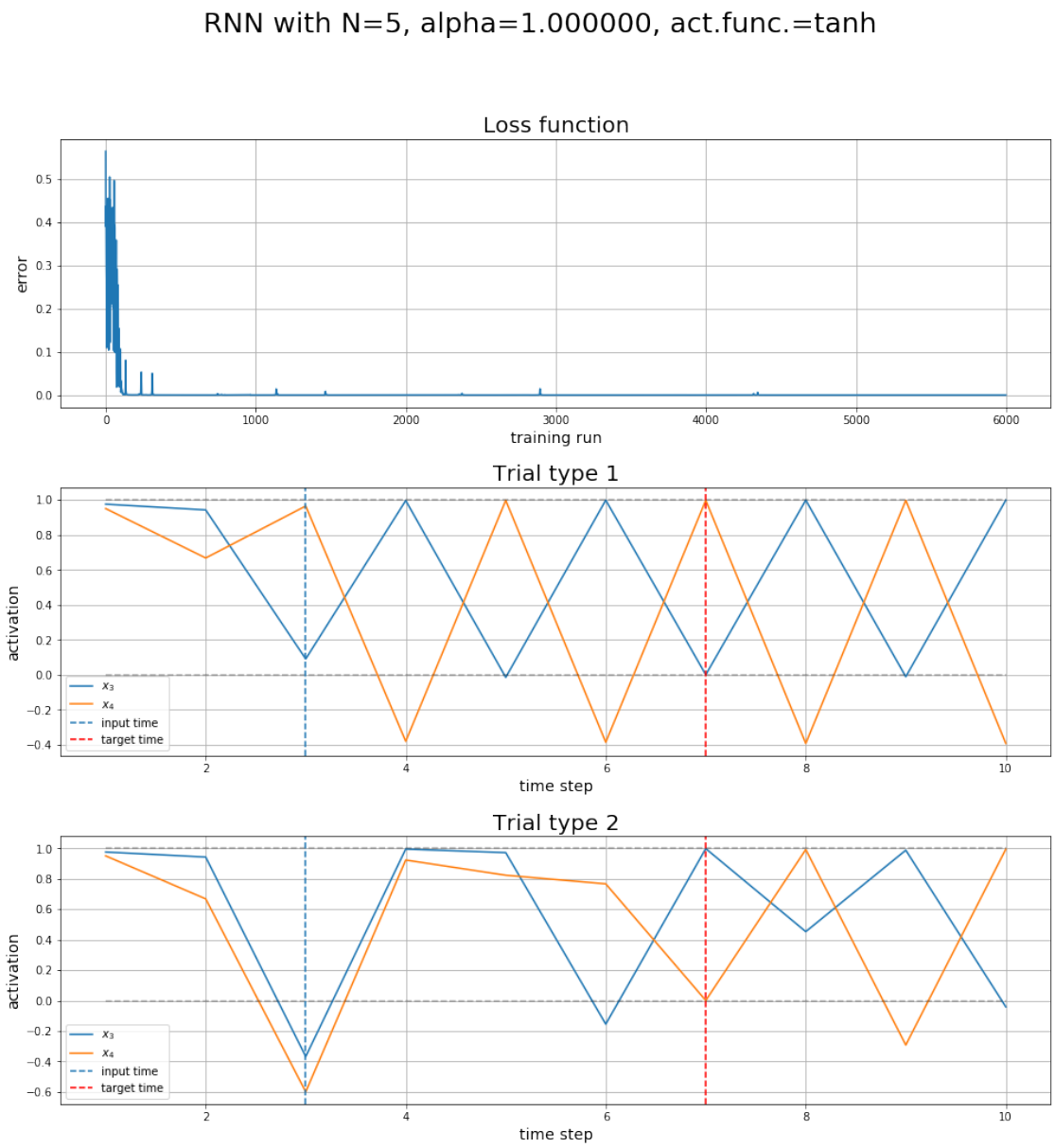






RNN with N=5, alpha=0.100000, act.func.=tanh



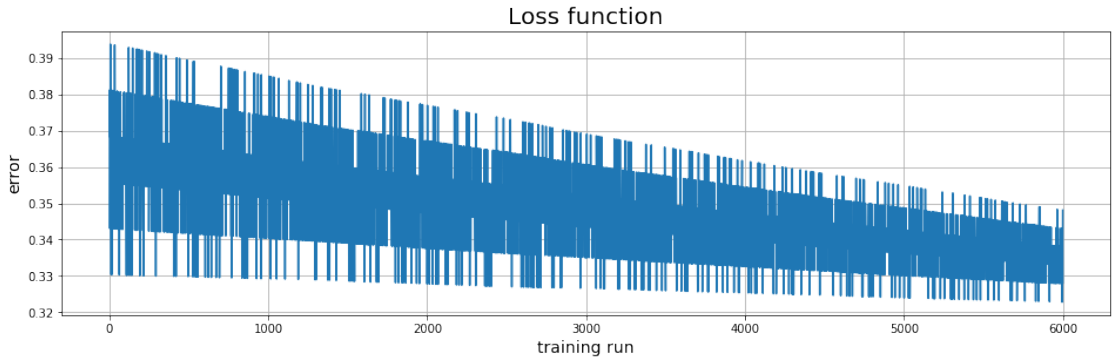


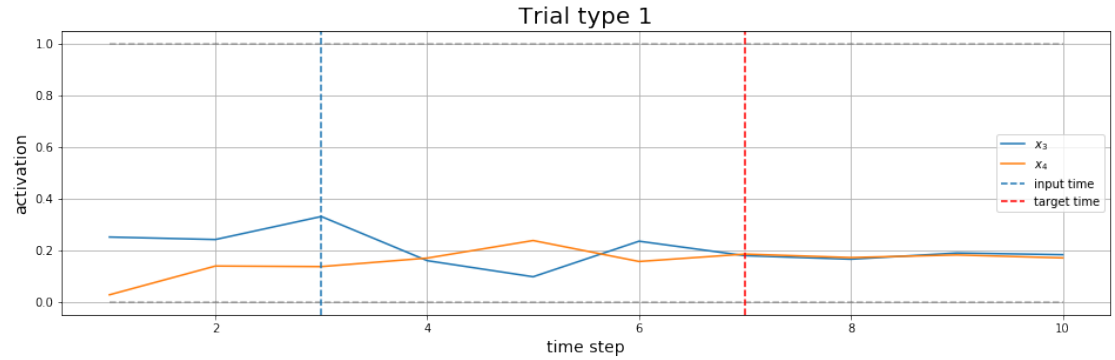
As we can notice, for the two smaller values of α the loss function doesn't converge to zero and the network isn't able to reproduce the exact output values. This is probably because the system falls into a local minima and it's not able to move out from it. On the other hand with the others higher values of α the network correctly reproduces the outputs.

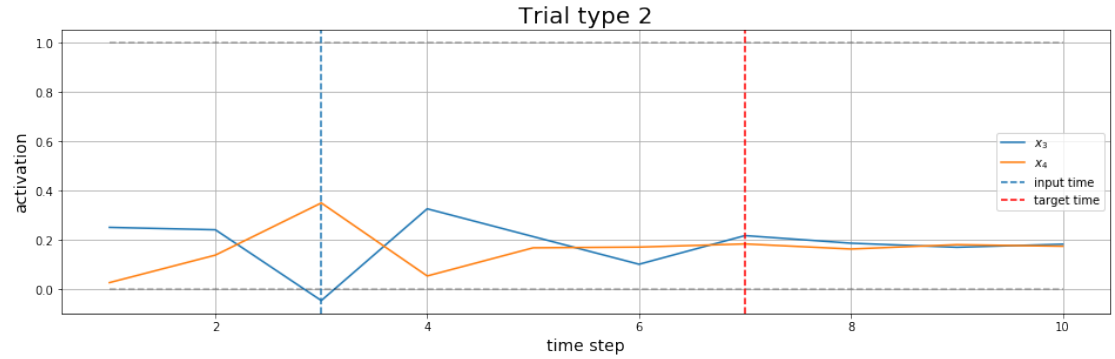
2.0.2 Comparing that to training in networks with N = 10 units.

```
In [20]: alpha = [1e-5, 1e-3, 1e-1, 1.0]

         for i in alpha:
             RNN( N = 10, alpha = i )
```

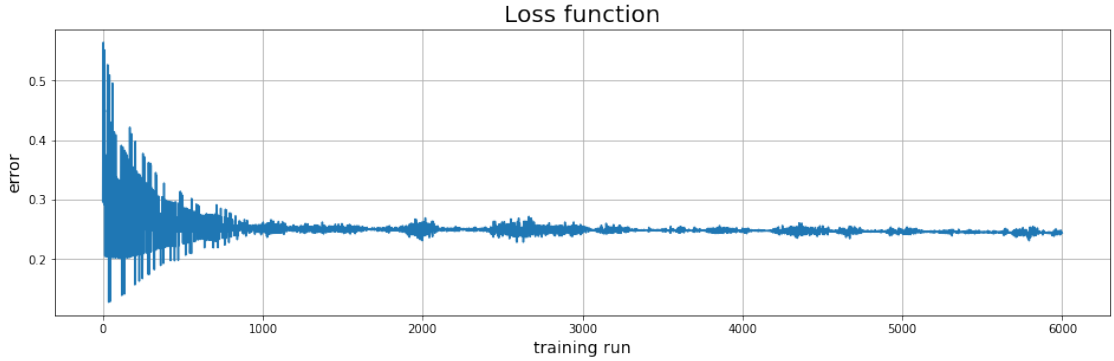
RNN with N=10, alpha=0.000010, act.func.=tanh

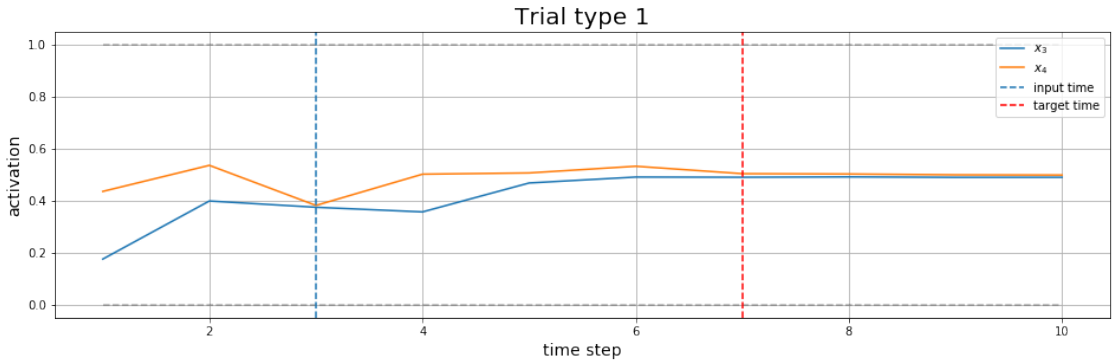


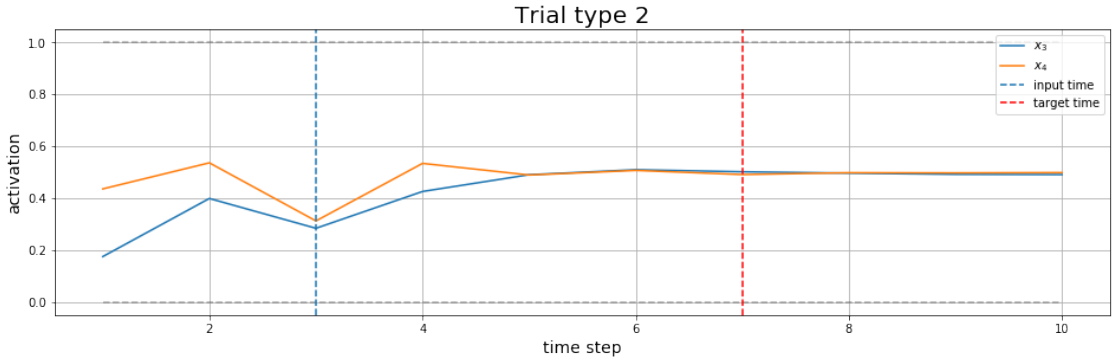




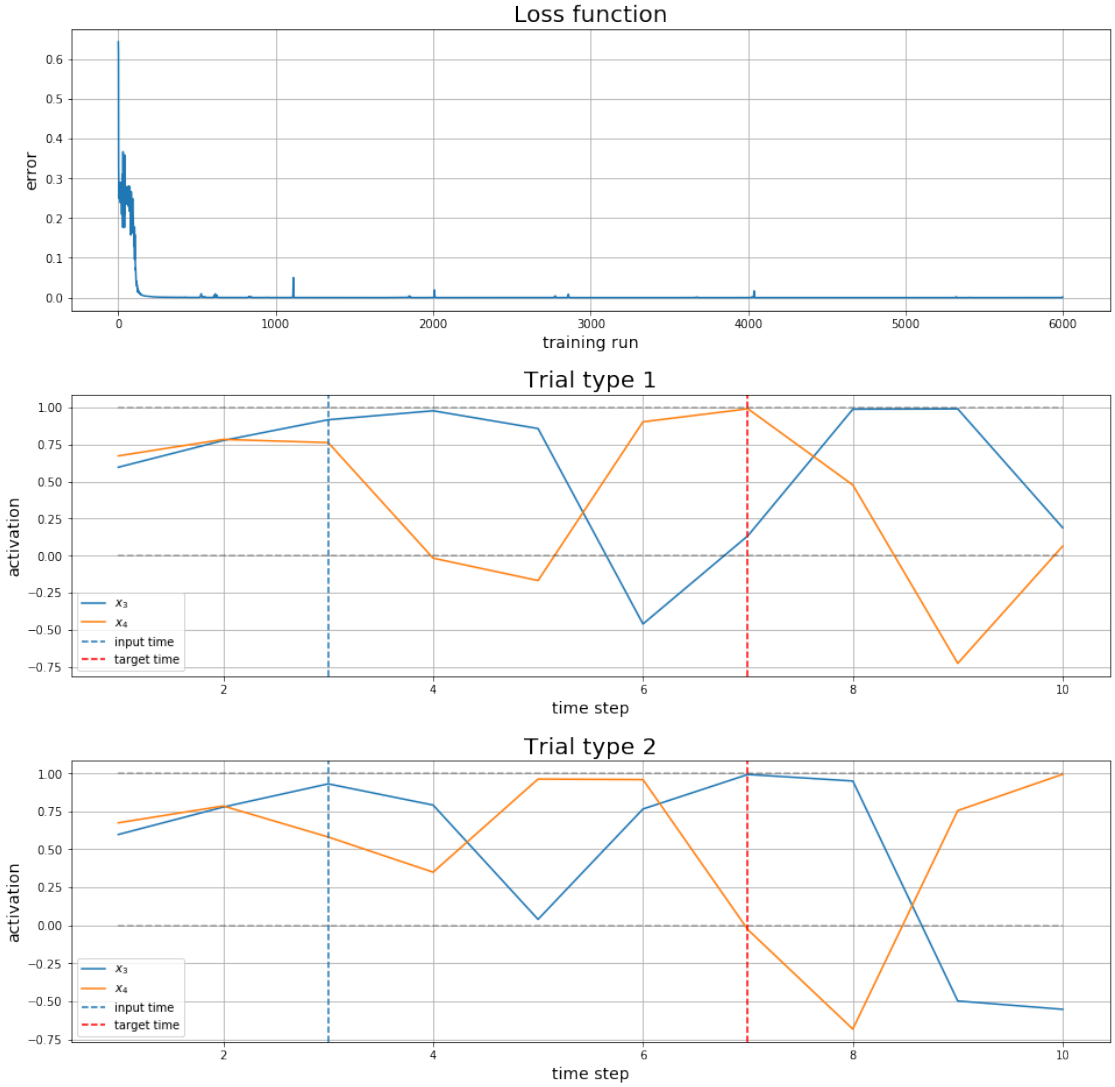
RNN with N=10, alpha=0.001000, act.func.=tanh

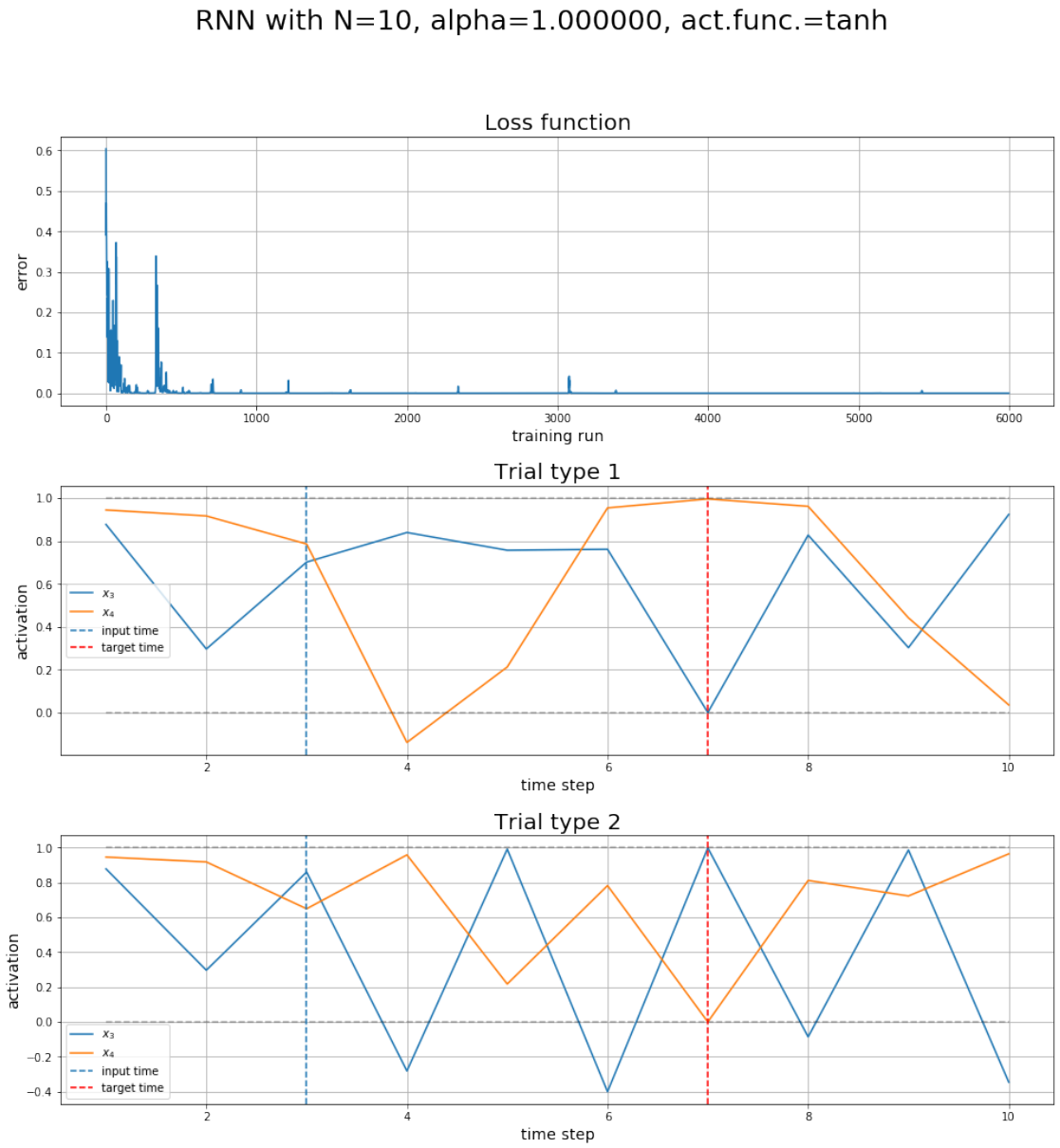






RNN with N=10, alpha=0.100000, act.func.=tanh



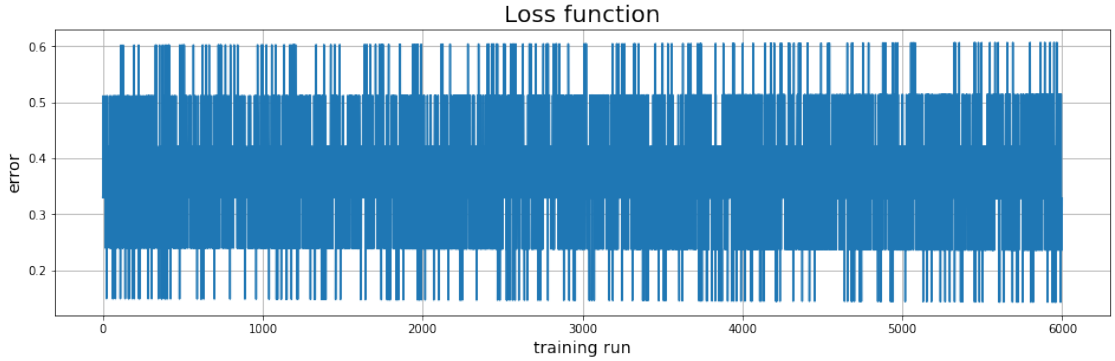


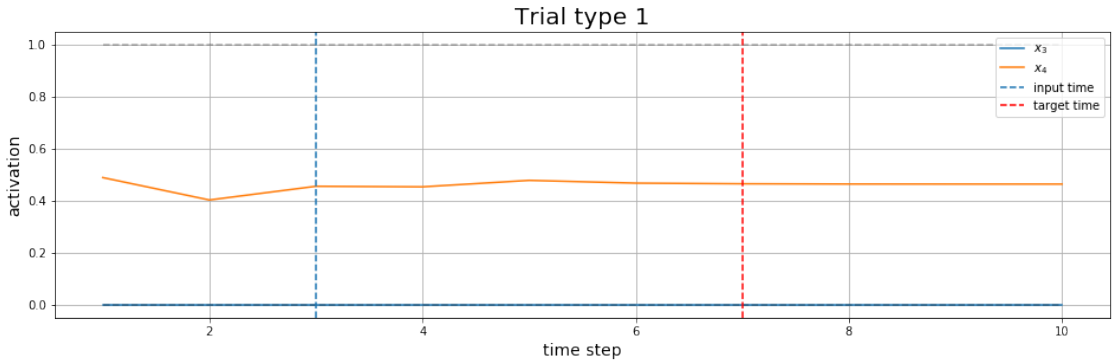
By duplicating the total number of neurons, we observe that the fenomenology is the same.

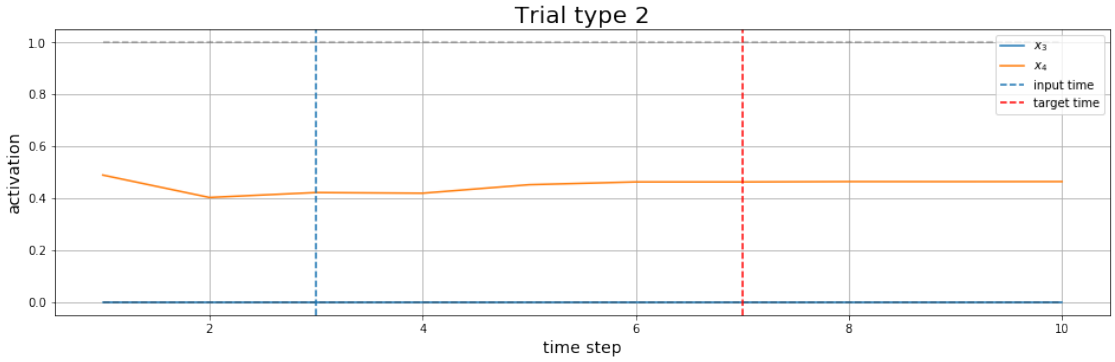
2.0.3 Change the activation function to the rectified linear function (ReLU) and do a comparison.

```
In [22]: alpha = [1e-5, 1e-3, 1e-1, 1.0]

         for i in alpha:
             RNN(alpha = i, nonlinearity = 'relu')

         for i in alpha:
             RNN( N = 10, alpha = i, nonlinearity = 'relu')
```

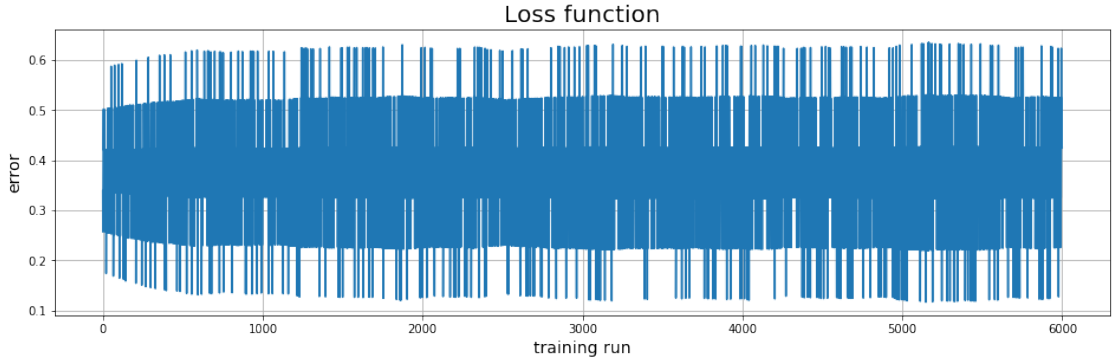
RNN with N=5, alpha=0.000010, act.func.=relu

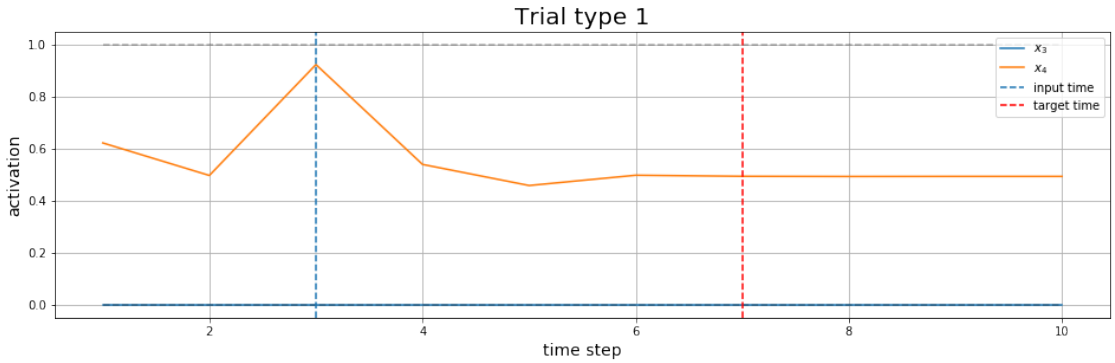


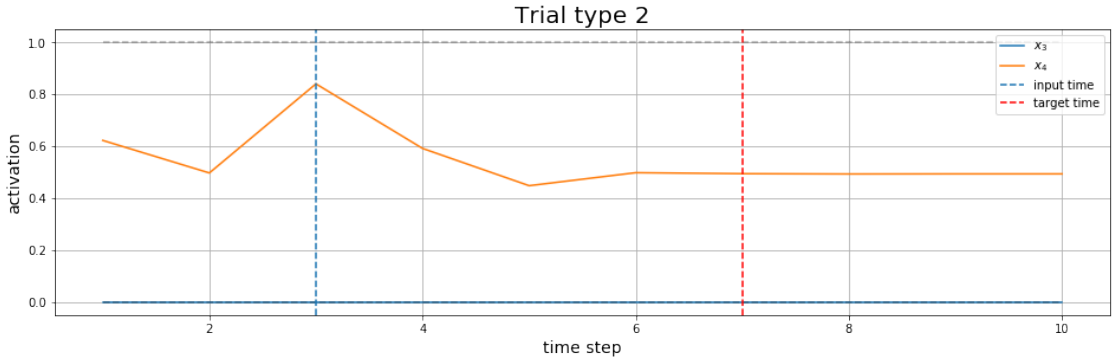




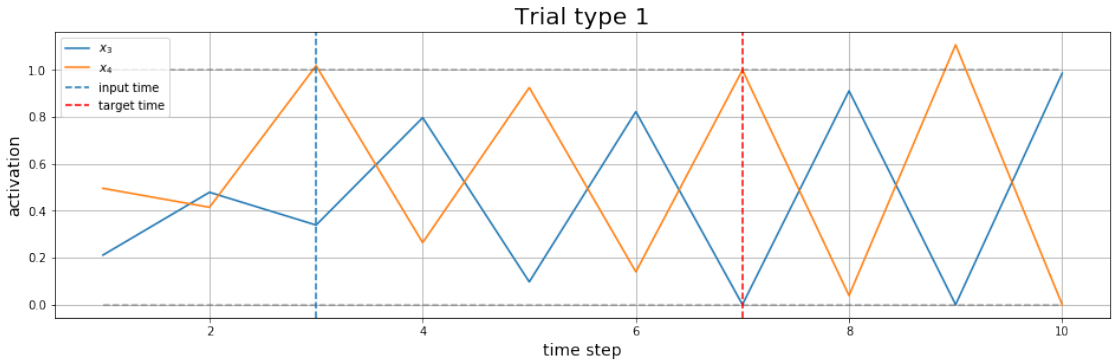
RNN with N=5, alpha=0.001000, act.func.=relu

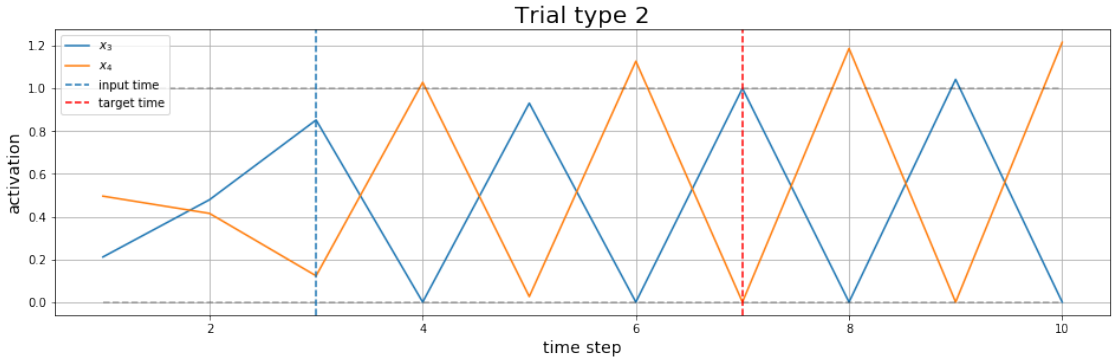




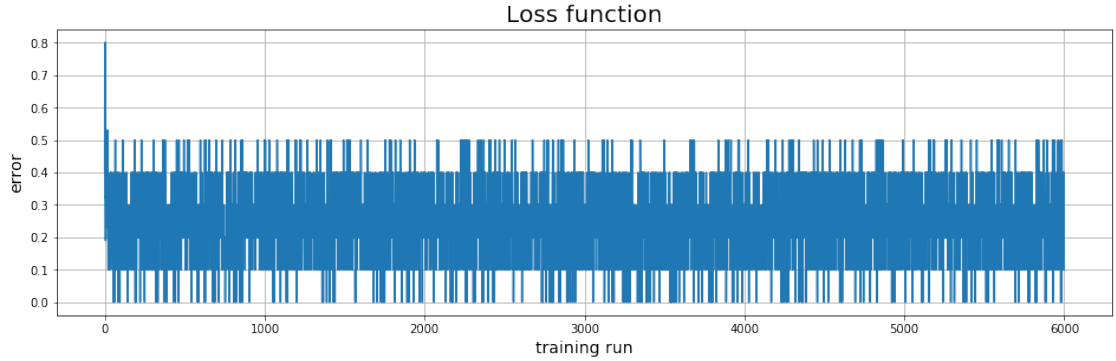


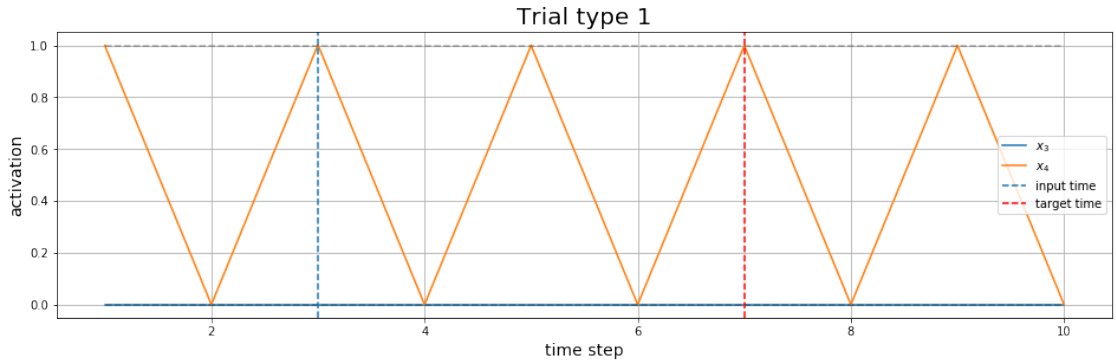
RNN with N=5, alpha=0.100000, act.func.=relu

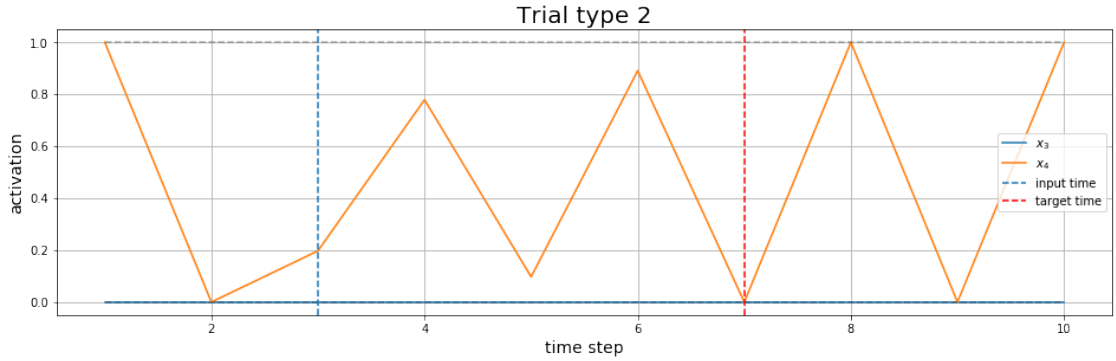




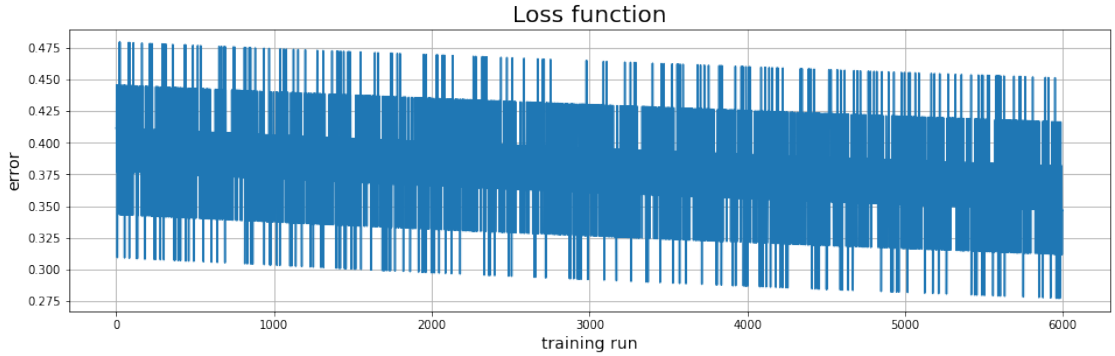
RNN with N=5, alpha=1.000000, act.func.=relu

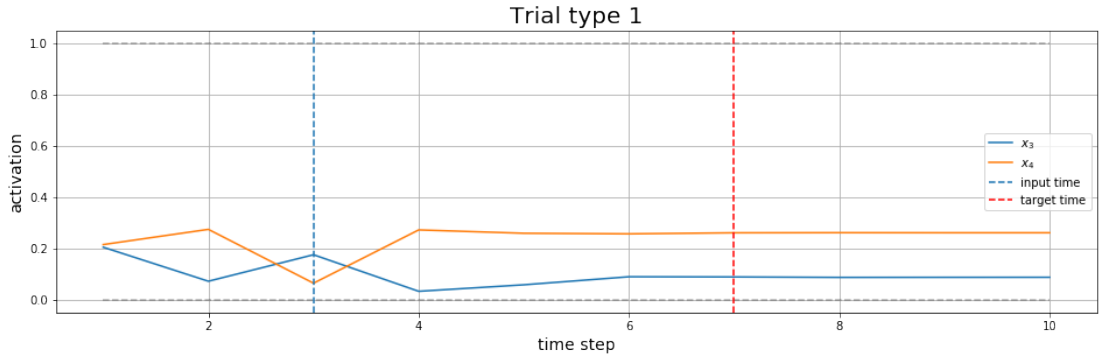


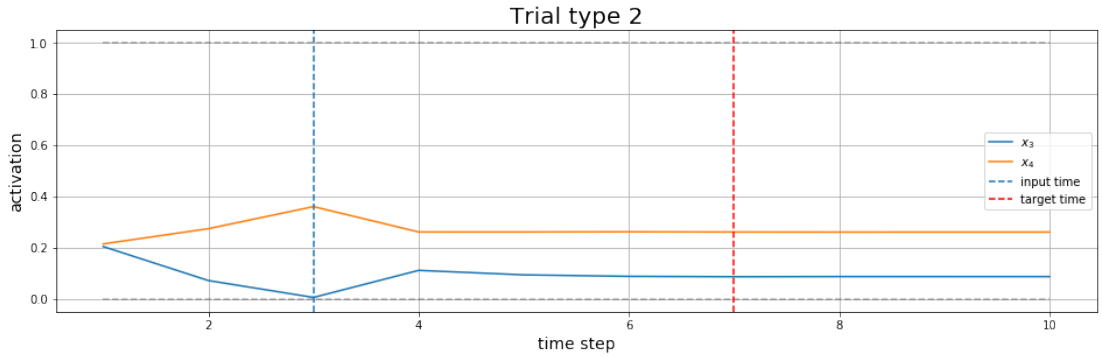




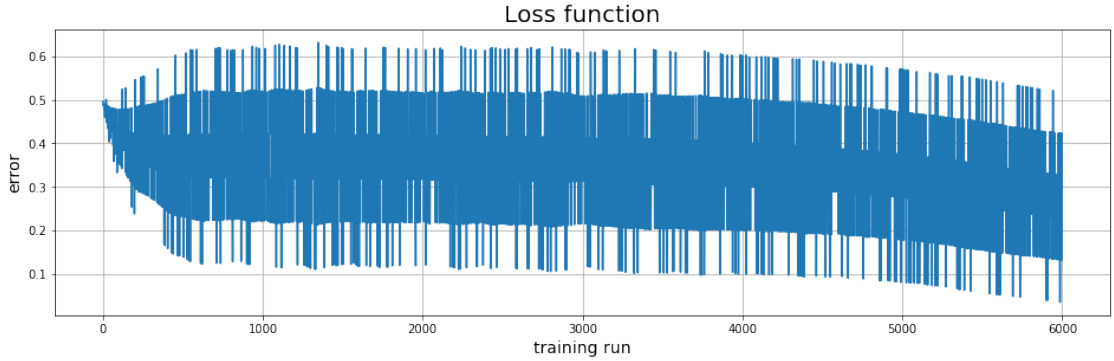
RNN with N=10, alpha=0.000010, act.func.=relu

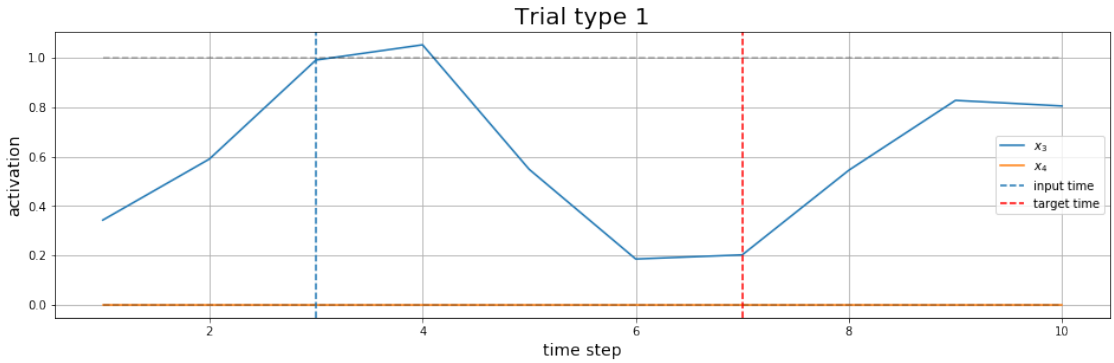


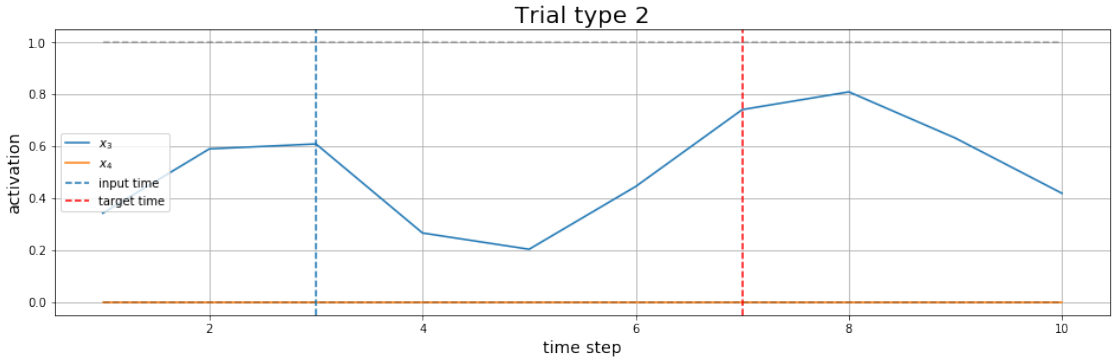




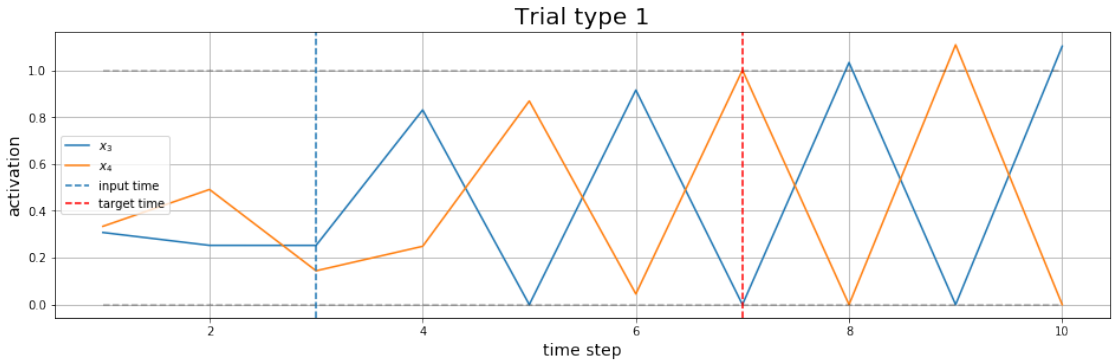
RNN with N=10, alpha=0.001000, act.func.=relu

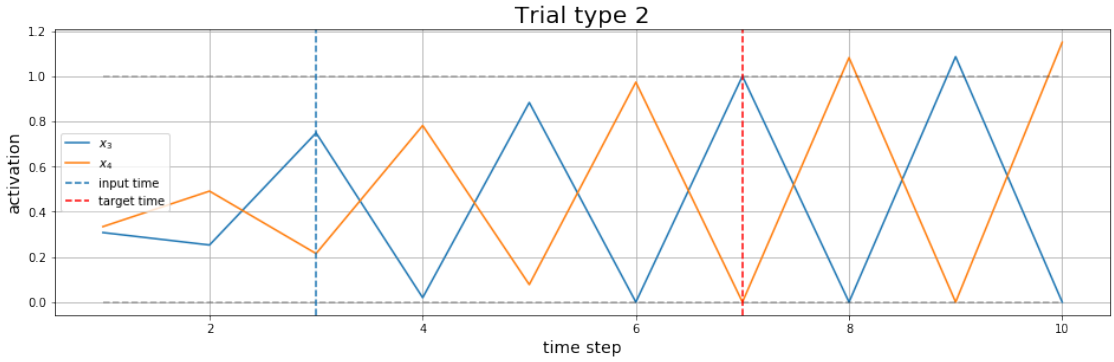


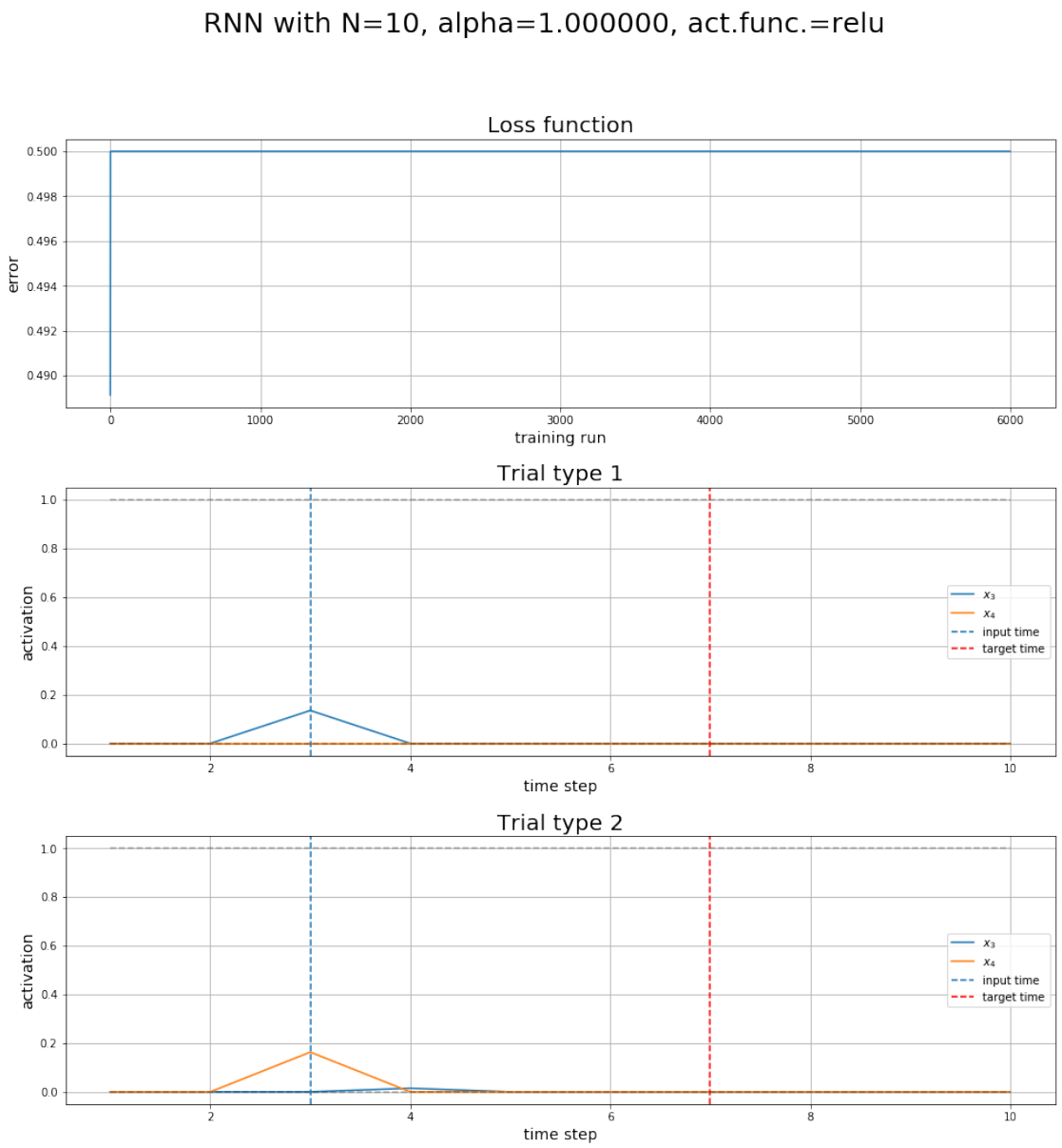




RNN with N=10, alpha=0.100000, act.func.=relu







By changing the activation function to ReLU (Rectified Linear Unit), we can observe that the network is able to reach convergence and correctly reproduce the output only with $\alpha = 10^{-1}$ with both N=5,10 total number of neurons: so we can notice that the performance of a RNN are strictly dependent on the choice of the activation function.

working_memory_RNN.py

```
import torch
import torch.nn as nn
import numpy as np
import matplotlib.pyplot as plt
import torch.optim as optim

def RNN(N=5, alpha=0.1, nonlinearity='tanh'):
    # N:              # Number of neurons
    # alpha:          # learning rate

    #%%
    T = 10            # Length of time series
    Tinp = 2          # Time at which the input is presented
    Tout = [6]        # Time points at which the output is required
    Ntrial = 6000     # Number of trials
    batchsize = 5     # how many trials per training step?

    # Create 1 layer RNN with N neurons
    # Inputs to RNN:
    # No. of input units, No. of total units, No. of layers, nonlinearity
    net = nn.RNN(N, N, 1, nonlinearity=nonlinearity)
    #net = nn.RNN(N, N, 1, nonlinearity='relu')

    #%%
    # Define the inputs for one trial:
    # Inputs to the RNN need to have 3 dimensions, where the first dimension is
    # the length of the time series, the second is the trial number and the third
    # is the number of the unit that receives the input. Here, we define the input
    # for one single trial, so we set the 2nd dimension to 1
    input1 = np.zeros([T,1,N])
    input2 = np.zeros([T,1,N])
    input1[Tinp,0,0] = 1   # Provide input to neuron 0
    input2[Tinp,0,1] = 1   # Provide input to neuron 1

    # Define the targets for a single trial
    target1 = np.zeros([T,1,N])
    target2 = np.zeros([T,1,N])
    for t in Tout:
        target1[t,0,3] = 1
        target2[t,0,2] = 1
    inputs  = {0: input1, 1: input2}
    targets = {0: target1, 1: target2}


    # We want to record losses for plotting
```

```
losses = np.zeros(Ntrial)
# Define the loss function. We want to use mean square error function
criterion = nn.MSELoss()
for i in range(Ntrial):
    # First, stitch multiple input-target pairs together to one batch
    # Notice that the trial number within one batch is given by the second
    # dimension of the input/target tensor!
    inpt    = torch.zeros(T,batchsize,N, dtype=torch.float)
    target  = torch.zeros(T,batchsize,N, dtype=torch.float)
    for n in range(batchsize):
        # Draw randomly from either trial type 1 or trial type 2
        trial_type      = np.random.randint(0,2)
        inpt[:,n,:]     = torch.tensor(inputs[trial_type], dtype=torch.float).s
        target[:,n,:]   = torch.tensor(targets[trial_type], dtype=torch.float).

    # To propagate input through the network, we simply call the network with
    # the input as argument. Output is the whole time series for all trials in
    # the batch
    [outp, _] = net(inpt)

    # Calculate MSE between output and target. Here we specifically select
    # those units (2 and 3) and time points (Tout) that have target outputs.
    loss = criterion(outp[Tout,:,2:4], target[Tout,:,2:4])
    #loss = criterion(outp[:,:,2:4], target[:,:,2:4])
    losses[i] = loss

    # We need to reset the gradients from previous step to zero:
    net.zero_grad()

    # This function backpropagates the loss through the network. PyTorch takes
    # care of calculating the gradients that are created by the backpropagation
    loss.backward()

    # Use the optimizer on the parameters, with learning rate alpha
    # We use stochastic gradient descent, but you can change it if you want:
    optimizer = optim.SGD(net.parameters(), lr=alpha)
    # Finally, do one gradient descent step:
    optimizer.step()


# Now we plot the results. For that, we propagate both input types through the
# network and plot the resulting time series
fig1 = plt.subplots(figsize=[16,16])

plt.subplot(3,1,1)
plt.plot(losses)
plt.title('Loss function', fontsize = 20)
plt.xlabel('training run', fontsize = 14)
```

```
plt.ylabel('error', fontsize = 14)
plt.grid()

inputs_dict  = {0: input1, 1: input2}
targets_dict = {0: target1, 1: target2}
for i in range(2):
    inpt = torch.tensor(inputs_dict[i], dtype=torch.float)

    [outp, _] = net(inpt)
    outp = outp.detach().numpy()
    x3 = np.squeeze(outp[:,0,2])
    x4 = np.squeeze(outp[:,0,3])

    plt.subplot(3,1,i+2)

    plt.plot(np.arange(T)+1, x3, label=r'$x_3$')
    plt.plot(np.arange(T)+1, x4, label=r'$x_4$')

    plt.plot(np.arange(T)+1, np.zeros(T), 'k—', alpha=0.3)
    plt.plot(np.arange(T)+1, np.ones(T), 'k—', alpha=0.3)
    plt.axvline(x = Tinp+1, ls='—', label='input␣time')
    plt.axvline(x = np.array(Tout)+1, ls='—', label='target␣time', c='r')
    plt.title('Trial␣type␣%i' %(i+1), fontsize = 20)
    plt.xlabel('time␣step', fontsize = 14)
    plt.ylabel('activation', fontsize = 14)
    plt.legend()
    plt.grid()

plt.suptitle('RNN␣with␣N=%i,␣alpha=%f,␣act.func.=%s' %(N,alpha,nonlinearity), f
plt.subplots_adjust(hspace = 0.3)
plt.show(fig1)
```