Time Series Analysis and Recurrent Neural Network
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Exercise 5

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1 Task 1 - Kullback-Leibler divergence of two normal distributions

The Kullback-Leibler divergence of two continuous distributions P and Q is:

$$KL(P||Q) = \int_{-\infty}^{\infty} p(x) \log \frac{p(x)}{q(x)} dx$$

- Compute it analytically in the case where P and Q are two normal distributions P = N(μ_1,σ_1) and Q = N(μ_2,σ_2)

$$KL(P||Q) = \int_{-\infty}^{\infty} p(x) \log \left(\frac{p(x)}{q(x)} \right) dx$$

$$= \int_{-\infty}^{\infty} (2\pi\sigma_1)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2} \log \left(\frac{(2\pi\sigma_1^2)^{-1/2} e^{-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2}}{(2\pi\sigma_2^2)^{-1/2} e^{-\frac{1}{2}(x-\mu_2)^2/\sigma_2^2}} \right) dx$$

By using the logarithm properties we obtain:

$$= log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2} \int_{-\infty}^{\infty} (2\pi\sigma_1)^{-\frac{1}{2}} e^{-\frac{1}{2}(x-\mu_1)^2/\sigma_1^2} \left[\frac{(x-\mu_2)^2}{\sigma_2^2} - \frac{(x-\mu_1)^2}{\sigma_1^2} \right] dx$$

We change the integration variable $x \to y = \frac{x-\mu_1}{\sigma_1}$, so the integral becomes:

$$\frac{1}{2} \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} \left[\frac{(\sigma_1 y + \sigma_1\mu_1 - \mu_2)^2}{\sigma_2^2} - y^2 \right] dy$$

Now we can split the terms that multiply the gaussian:

$$\left[\frac{(\sigma_1 y + \mu_1 - \mu_2)^2}{\sigma_2^2} - y^2 \right] = \left(\frac{\sigma_1^2}{\sigma_2^2} - 1 \right) y^2 + 2\frac{\sigma_1\mu_1 - \mu_2}{\sigma_2^2} y + \frac{(\mu_1 - \mu_2)^2}{\sigma_2^2}$$

So we can apply the definition of the moments for a gaussian distribution:

$$\int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} = 1 \ (normalization)$$

$$\int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} y = 0 \ (mean)$$

$$\int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{y^2}{2}} y^2 = 1 \ (variance)$$

Finally, we obtain:

$$KL(P||Q) = log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2}\left[\frac{\sigma_1^2}{\sigma_2^2} - 1 + \frac{(\mu_1 - \mu_2)^2}{\sigma_2^2}\right]$$

$$= log\left(\frac{\sigma_2}{\sigma_1}\right) + \frac{1}{2}\left[\frac{\sigma_1^2 + (\mu_1 - \mu_2)^2}{\sigma_2^2}\right] - \frac{1}{2}$$

1.0.1 1. What is the specific value of KL(P||Q) when μ_1= μ_2 and σ_1 = σ_2?

$$KL(N(\mu, \sigma)||N(\mu, \sigma)) = 0$$

If the two distibution are equal, the K-L divergence is 0, as we could have expected, since the K-L is a measure of how one probability distribution is different from another distribution.

1.0.2 2. What are the specific values of KL(P||Q) and KL(Q||P) when μ_1 =0, σ_1 =2, μ_2 =1, σ_2 =1?

$$KL(N(\mu = 0, \sigma = 2)||N(\mu = 1, \sigma = 1)) = 2 - ln(2)$$

$$KL(N(\mu = 1, \sigma = 1)||N(\mu = 0, \sigma = 2)) = ln(2) - \frac{1}{4}$$

2 Task 2 - Kalman filter smoother

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt
        from scipy.io import loadmat                    #import mat data
```

```
In [2]: # Load data
        mat_file = loadmat('ex5.mat')
        #print(mat_file)

        # z: latent states -> position and velocity of the drone
        z = mat_file['z']
```

```
# x: measurements
x = mat_file['x']

# u: external forces
u = mat_file['u']

# model matrices
A = mat_file['A']
B = mat_file['B']
C = mat_file['C']
Sigma = mat_file['Sigma']
Gamma = mat_file['Gamma']

# initial conditions
L0 = mat_file['L0']
mu0 = mat_file['mu0']
```

2.0.1 1) Retrieve an estimate for the latent states 'z' purely from the observations 'x' by implementing the Kalman-filter recursions

In [3]:
```
# useful function for Kalman-filter
def Kt(B, Gamma, Lt_1):
    return Lt_1 @ B.T @ np.linalg.inv( B @ Lt_1 @ B.T + Gamma )

def Vt(B, Kt, Lt_1):
    return ( np.identity(Kt.shape[0]) - Kt @ B ) @ Lt_1

def Lt(A, Sigma, V_t):
    return A @ V_t @ A.T + Sigma
```

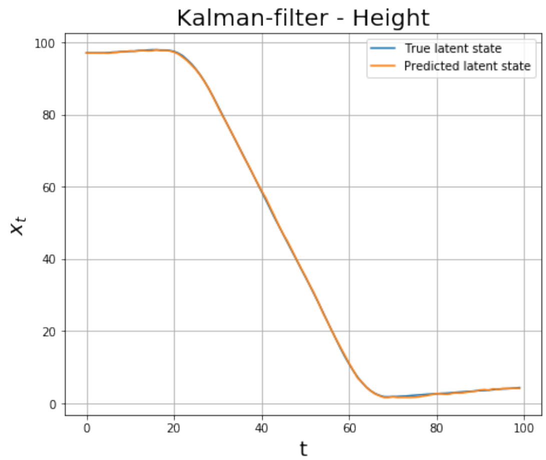
In [4]:
```
# KALMAN FILTER
def KalmanFilter(A, B, C, Sigma, Gamma, L0, mu0, x, u):
    steps = x.shape[1]
    n = x.shape[0]

    # define empty state array
    mu = np.zeros(( n, steps ))
    mu[:,0] = mu0.ravel()

    # define empty matrix
    V = np.zeros(( steps, L0.shape[0], L0.shape[1] ))

    # compute first element
    K = Kt(B, Gamma, L0)
    V[1,:,:] = Vt(B, K, L0)
    mu[:,1] = A @ mu[:,0] + (C * u[:,1]).reshape(n) +
    + K @ (x[:,1]  - B@( A @ mu[:,0] + (C * u[:,1]).reshape(2) ) )
```

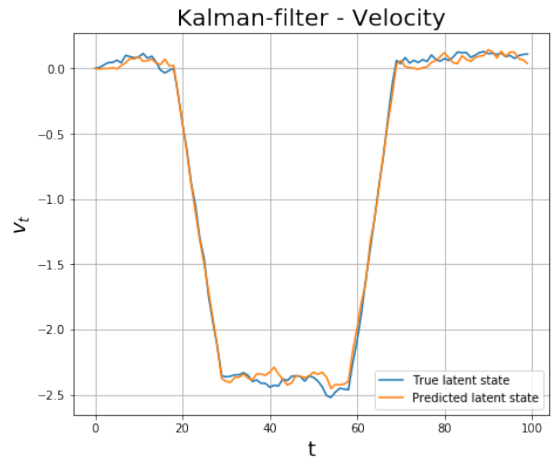
```
    # loop over time steps
    for i in range(2, steps):
        Lt_1 = Lt(A, Sigma, V[i-1])
        K = Kt(B, Gamma, Lt_1)
        V[i] = Vt(B, K, Lt_1)
        mu[:,i] = A @ mu[:,i-1] + (C * u[:,i]).reshape(n) +
        + K @ (x[:,i] - B@( A @ mu[:,i-1] + C @ u[:,i]) )

    return mu, V
```

In [5]:
```
# Estimation of variable z with Kalman-filter
mu, V = KalmanFilter(A, B, C, Sigma, Gamma, L0, mu0, x, u)
```

2.0.2 2) Plot the obtained predicted latent states against the true latent states (in variable 'z')

In [6]:
```
# Plot the time series
fig1 = plt.subplots(figsize=[16,6])

plt.subplot(1,2,1)
plt.plot(np.arange(z.shape[1]), z[0], label='True latent state' )
plt.plot(np.arange(z.shape[1]), mu[0], label='Predicted latent state' )
plt.xlabel('t', fontsize = 18)
plt.ylabel(r'$x_t$', fontsize = 18)
plt.grid()
plt.legend()
plt.title('Kalman-filter - Height', fontsize = 20)

plt.subplot(1,2,2)
plt.plot(np.arange(z.shape[1]), z[1], label='True latent state'  )
plt.plot(np.arange(z.shape[1]), mu[1], label='Predicted latent state')
plt.xlabel('t', fontsize = 18)
plt.ylabel(r'$v_t$', fontsize = 18)
plt.grid()
plt.legend()
plt.title('Kalman-filter - Velocity', fontsize = 20)

plt.show(fig1)
```





2.0.3 How well can you recover the true latent states? What could the drone be doing?

The true heigth is recovered pretty well, while the true velocity oscillates a bit.
The drone starts from a heigth of about 100 and remains still for about 20 time units: this can be noticed also in the velocity plot, where the velocity is null. Then, the height of the drone begins to decline and the corresponding speed grows negatively until it stops again. So the drone can be coming down.

2.0.4 3) Examine the parameter values of A and C? Why are they chosen the way they are?

```
In [7]: print('Parameter A:\n', A)
        print('Parameter C:\n', C)
```

```
Parameter A:
 [[1 1]
 [0 1]]
Parameter C:
 [[0.5]
 [1. ]]
```

The matrices A and C propagates the mean of the latent state through time. We can notice that the new position estimate depends on the previous values of position and velocity, while the new velocity estimate depends only on previous values of velocity. In addition, the new estimates depends on the external forces by a factor 0.5 and 1 respectively: so the system $\mu_t = A\mu_{t-1} + C$

```
In [8]: # KALMAN SMOOTHER
        def KalmanSmoother(A, C, Sigma, Vt, Lt, mu, u):
            T = mu.shape[1]
            n = mu.shape[0]

            mu_tilde = np.zeros(( n, T ))
```

```
    # define last element
    mu_tilde[:,-1] = mu[:,-1]

    for i in range(1,T):
        L = Lt(A, Sigma, Vt[T-i,:,:])
        mu_tilde[:,T-1-i] = mu[:,T-1-i] + Vt[T-1-i,:,:]
        @A.T@np.linalg.inv(L)@(mu_tilde[:,T-i] - A@mu[:,T-1-i] - C@u[:,T-1-i])

    return mu_tilde
```

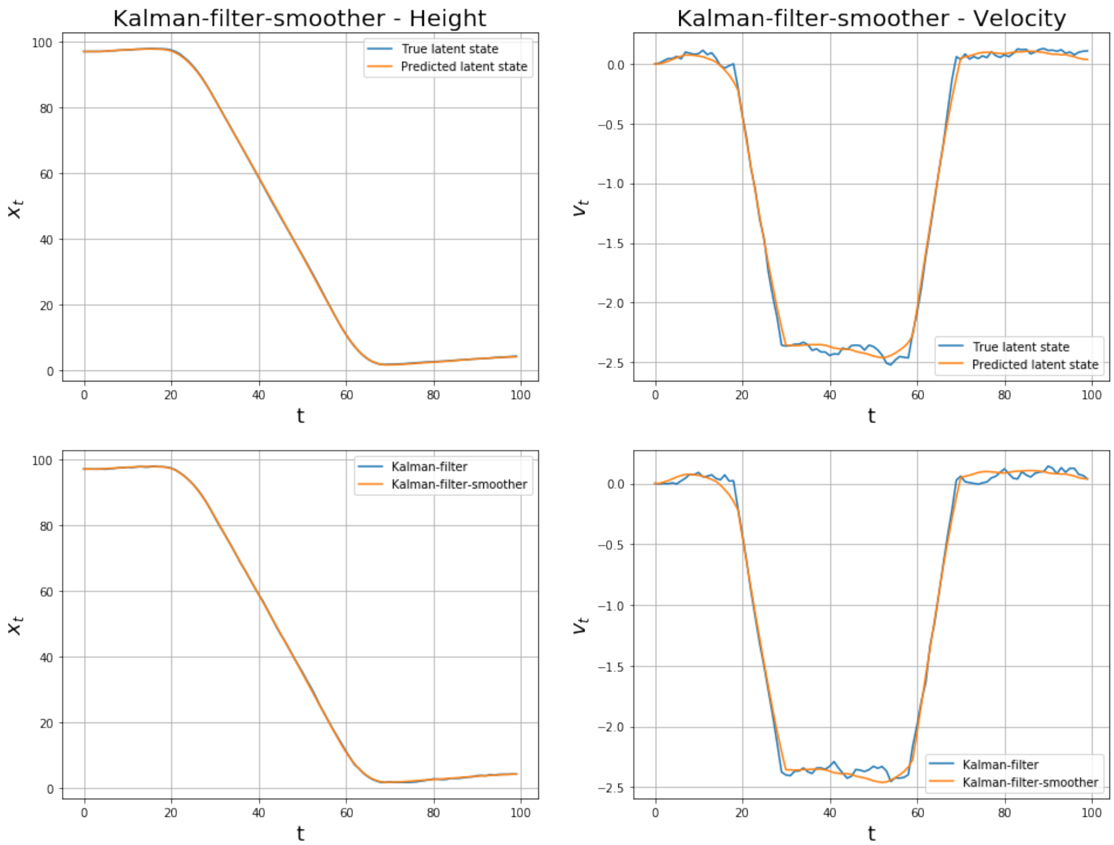
2.0.6 5) Apply the Kalman-filter-smoother to obtain estimates for your latent state path

In [9]:
```
# Estimation of variable z with Kalman-smoother
mu_tilde = KalmanSmoother(A, C, Sigma, V, Lt, mu, u)
```

2.0.7 Plot the obtained predicted latent states against the true latent states (variable 'z'), and against the latent states obtained by only using the filter

In [10]:
```
# Plot the time series
fig1 = plt.subplots(figsize=[16,12])

plt.subplot(2,2,1)
plt.plot(np.arange(z.shape[1]), z[0], label='True latent state' )
plt.plot(np.arange(z.shape[1]), mu_tilde[0], label='Predicted latent state' )
plt.xlabel('t', fontsize = 18)
plt.ylabel(r'$x_t$', fontsize = 18)
plt.grid()
plt.legend()
plt.title('Kalman-filter-smoother - Height', fontsize = 20)

plt.subplot(2,2,2)
plt.plot(np.arange(z.shape[1]), z[1], label='True latent state'  )
plt.plot(np.arange(z.shape[1]), mu_tilde[1], label='Predicted latent state')
plt.xlabel('t', fontsize = 18)
plt.ylabel(r'$v_t$', fontsize = 18)
plt.grid()
plt.legend()
plt.title('Kalman-filter-smoother - Velocity', fontsize = 20)

plt.subplot(2,2,3)
plt.plot(np.arange(z.shape[1]), mu[0], label='Kalman-filter' )
plt.plot(np.arange(z.shape[1]), mu_tilde[0], label='Kalman-filter-smoother' )
plt.xlabel('t', fontsize = 18)
plt.ylabel(r'$x_t$', fontsize = 18)
plt.grid()
plt.legend()
```

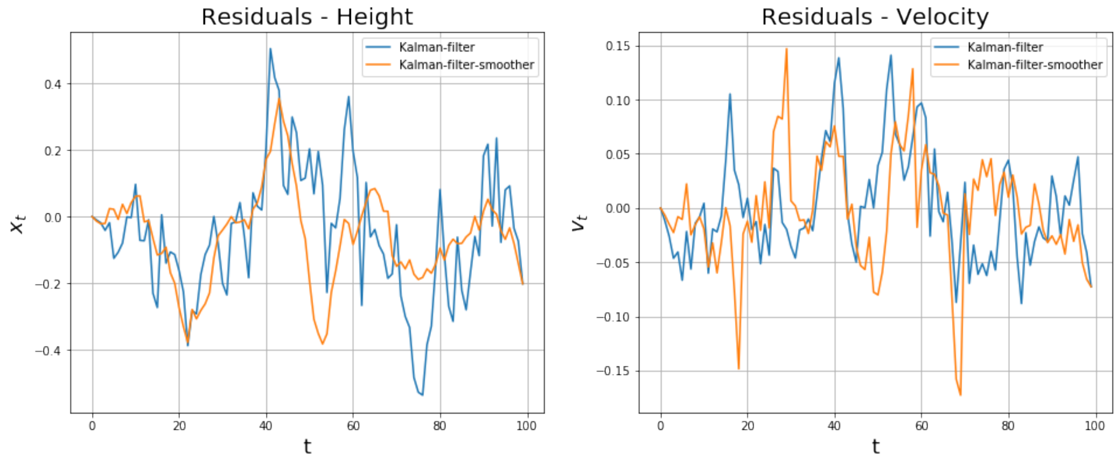
```
plt.subplot(2,2,4)
plt.plot(np.arange(z.shape[1]), mu[1], label='Kalman-filter'  )
plt.plot(np.arange(z.shape[1]), mu_tilde[1], label='Kalman-filter-smoother')
plt.xlabel('t', fontsize = 18)
plt.ylabel(r'$v_t$', fontsize = 18)
plt.grid()
plt.legend()

plt.show(fig1)
```



2.0.8 Quantify these results by computing the mean squared error between true and recovered latent states

```
In [11]: # Plots of the residuals respect to the true latent state
         fig1 = plt.subplots(figsize=[16,6])

         plt.subplot(1,2,1)
         plt.plot(np.arange(z.shape[1]), mu[0] - z[0], label='Kalman-filter' )
         plt.plot(np.arange(z.shape[1]), mu_tilde[0] - z[0], label='Kalman-filter-smoother' )
         plt.xlabel('t', fontsize = 18)
         plt.ylabel(r'$x_t$', fontsize = 18)
```

```
plt.grid()
plt.legend()
plt.title('Residuals - Height', fontsize = 20)

plt.subplot(1,2,2)
plt.plot(np.arange(z.shape[1]), mu[1] - z[1], label='Kalman-filter'  )
plt.plot(np.arange(z.shape[1]), mu_tilde[1] - z[1], label='Kalman-filter-smoother')
plt.xlabel('t', fontsize = 18)
plt.ylabel(r'$v_t$', fontsize = 18)
plt.grid()
plt.legend()
plt.title('Residuals - Velocity', fontsize = 20)

plt.show()
```




```
In [12]: # Compute MSE - position
         KF_pos = np.mean( (mu[0] - z[0])**2 )
         KF_vel = np.mean( (mu[1] - z[1])**2 )
         KFS_pos = np.mean( (mu_tilde[0] - z[0])**2 )
         KFS_vel = np.mean( (mu_tilde[1] - z[1])**2 )

         print('Mean-squared error for Kalman-filter:')
         print('Height: %.4f' % KF_pos)
         print('Velocity: %.4f' % KF_vel)


         print('Mean-squared error for Kalman-filter-smoother:')
         print('Height: %.4f' % KFS_pos)
         print('Velocity: %.4f' % KFS_vel)
```

```
Mean-squared error for Kalman-filter:
Height: 0.0411
```

```
Velocity: 0.0024
Mean-squared error for Kalman-filter-smoother:
Height: 0.0235
Velocity: 0.0027
```

2.0.9 How do they differ?

It can be noticed that the MSE for the height is smaller for the Kalman-filter-smoother method respect to the only Kalman-filter (as it can be expected), while the MSE for the velocity are almost the same for the two methods.