Time Series Analysis and Recurrent Neural Network
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Exercise 4

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1 Task 1

1.0.1 1) Create a 2-variate AR(2) time series

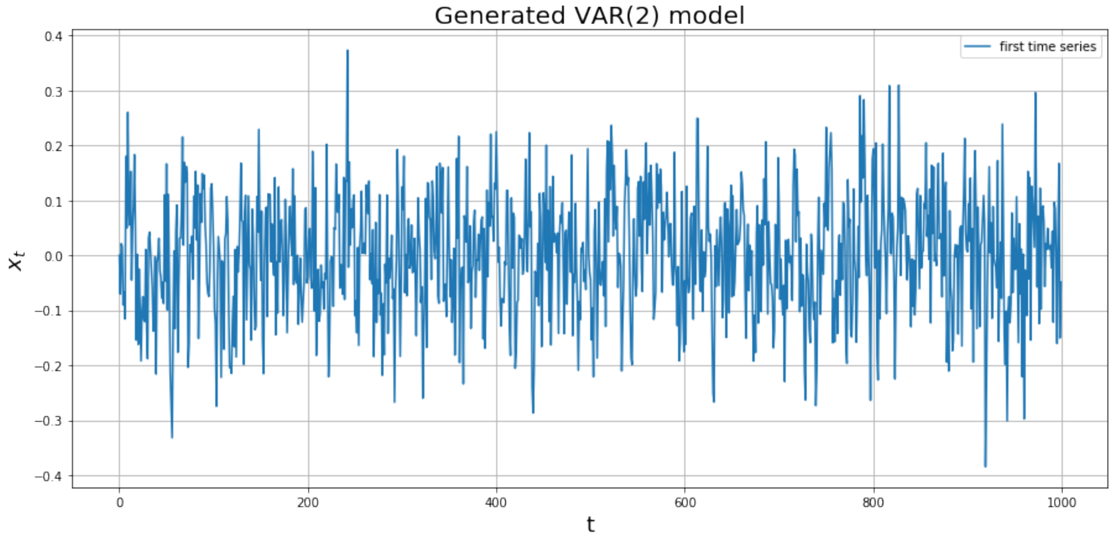
In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import chi2
from scipy.stats import f
```

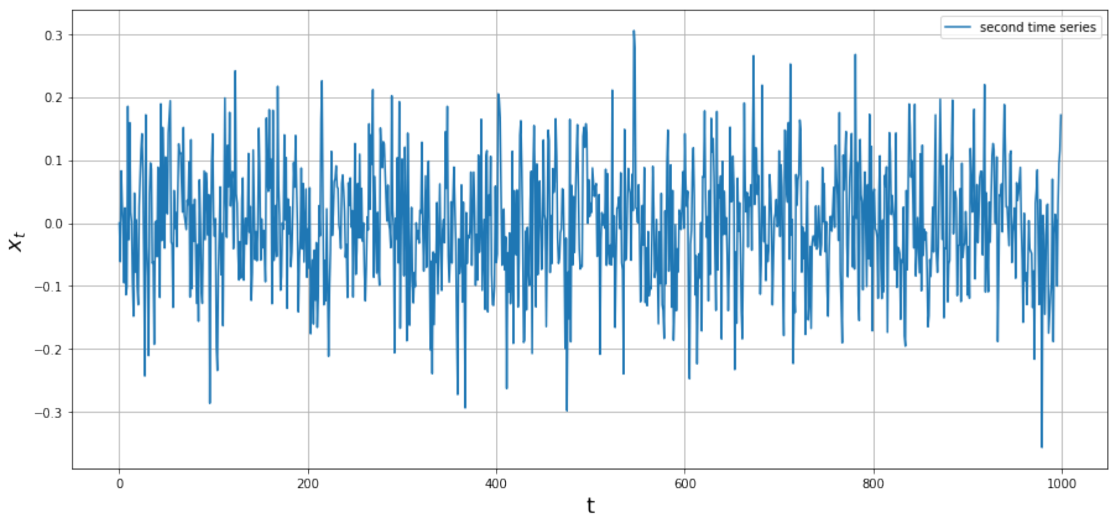
In [2]:
```
# INPUT PARAMETERS
# total time steps
T = 1000

# set random seed
np.random.seed(19062019)

# define empty ts matrix
data = np.zeros((2,T))

# setting initial conditions
data[:,0] = [0., 0.]

# model parameters (a0 = [0,0])
A1 = np.array(( [0.2, -0.2], [0., 0.1] ))
A2 = np.array(( [0.1, -0.1], [0., 0.1] ))

A_model = np.concatenate( ([0.0,0.0], A1, A2), axis=None)

# noise covariance matrix
cov = np.array(( [0.01, 0], [0, 0.01] ))
```

In [3]:
```
# CREATE THE TIME SERIES
# generate values from the gaussian noise
epsilon = np.random.multivariate_normal([0.,0.], cov, T).T
```

```
    # loop over time steps
    data[:, 1] = np.dot(A1, data[:,0]) + epsilon[:,0]

    for i in range(2,T):
        data[:, i] = np.dot(A1, data[:,i-1]) + np.dot(A2, data[:,i-2]) + epsilon[:,i-1]
```

```
In [4]: # PLOT GENERATED TIME SERIES
        fig1 = plt.subplots(figsize=[15,15])

        plt.subplot(2,1,1)
        plt.plot(np.arange(T), data[0,:], label='first time series')
        plt.xlabel('t', fontsize = 18)
        plt.ylabel(r'$x_t$', fontsize = 18)
        plt.grid()
        plt.legend()
        plt.title('Generated VAR(2) model', fontsize = 20)

        plt.subplot(2,1,2)
        plt.plot(np.arange(T), data[1,:], label='second time series')
        plt.xlabel('t', fontsize = 18)
        plt.ylabel(r'$x_t$', fontsize = 18)
        plt.grid()
        plt.legend()

        plt.show(fig1)
```





1.0.2 2) Given your knowledge of model parameters above, does x1 Granger-cause x2, or does x2 Granger-cause x1 according to Granger's definition? Why?

As we can notice by looking at the matrix of the generating model, the lower-left (2,1) coefficients are both null: this means that the generated values of x2 don't depend on the previous value of x1 and for this reason we can deduce that x1 does not Granger-cause x2. On the other hand the upper-left (1,2) are different from zero, so we can deduce that x2 Granger-cause x1.

1.0.3 3) Assuming that the order of the model (p = 2) is known, use the log-likelihood-ratio test statistic to confirm your results from (2)

In [5]:
```
# USEFUL FUNCTIONS
# Coefficient estimation for AR(p) model
def coeff_AR(data, p):
```

```python
    xT = data[p:]
    Xp = np.zeros((len(data)-p,p+1))

    # add 1 as first element
    Xp[:,0] = 1

    # fill matrix Xp with data
    for i in range(1,p+1):
        Xp[:,i] = data[p-i:-i]

    # compute coefficients
    temp = np.matmul(np.transpose(Xp),Xp)
    temp = np.linalg.inv(temp)
    a = np.matmul(np.transpose(Xp),xT)
    a = np.matmul(temp,a)

    return a

# define p as a global variable
p = 2

# coefficient estimation for VAR(p)
def coeff_VAR(data, p):
    data = data.T
    T = data.shape[0]
    n = data.shape[1]

    # define empty matrix
    A = np.zeros(( n, n*p+1 ))
    xT = np.zeros(T-p)
    Xp = np.zeros(( (T-p), n*p+1 ))

    # add column of 1s
    Xp[:,0] = np.zeros(T-p) + 1

    # fill matrix Xp with data
    for i in range(n):
        for k in range(1,p+1):
            Xp[ :, i*p +k ] = data[p-k:-k, i]

    # compute coefficients
    for i in range(n):
        xT = data[p:, i]

        temp = np.matmul(np.transpose(Xp),Xp)
        temp = np.linalg.inv(temp)
        a = np.matmul(np.transpose(Xp),xT)
        a = np.matmul(temp,a)
```

```
            A[i] = a

        return A

    # inferred VAR time series, given coefficient matrices
    def pred_VAR(data, a0, A1, A2):
        T = data.shape[1]
        n = data.shape[0]

        # define empty array
        x_new = np.zeros((n,T))
        # set first p values
        x_new[:,:p] = data[:,:p]

        for i in range(p,T):
            x_new[:, i] = a0 + np.dot(A1, data[:,i-1]) + np.dot(A2, data[:,i-2])

        return x_new

    # Residuals of the VAR(p) model
    def res(data, a0, A1, A2):
        return pred_VAR(data, a0, A1, A2)[:,p:] - data[:,p:]

    # Covariance matrix of the residuals
    def sigma(data, a0, A1, A2):
        return np.cov(res(data, a0, A1, A2))

    # Log-likelihood for VAR(p) model
    def log_likelihood(data, a0, A1, A2):
        T = data.shape[1]
        n = data.shape[0]

        res1 = res(data, a0, A1, A2)
        sigma1 = sigma(data, a0, A1, A2)
        det = np.linalg.det(sigma1)

        t1 = - 0.5 * (T-p)*n * np.log(2*np.pi)
        t2 = - 0.5 * (T-p) * np.log(det)
        t3 = 0.
        for i in range(T-p):
            temp = np.dot(np.linalg.inv(sigma1), res1[:,i])
            t3 += np.dot(res1[:,i], temp)
        t3 = - 0.5 * t3

        return t1 + t2 + t3
```

In [6]:
```
# NULL HYPOTHESIS: no Granger Causalities
# diagonal A1 and A2 matrices / two sigle AR model
```

```
a1 = coeff_AR(data[0,:], p=2)
a2 = coeff_AR(data[1,:], p=2)

A0_null = np.array(([a1[0],a2[0]]))
A1_null = np.array(([a1[1], 0.0], [0.0, a2[1]]))
A2_null = np.array(([a1[2], 0.0], [0.0, a2[2]]))

A_null = np.concatenate( (A0_null,A1_null,A2_null), axis=None)

print('Null hypothesis parameters')
print('A0:\n', A0_null)
print('A1:\n', A1_null)
print('A2:\n', A2_null)
```

```
Null hypothesis parameters
A0:
 [ 9.90742761e-05 -1.68220466e-03]
A1:
 [[0.24370888 0.        ]
 [0.         0.13194647]]
A2:
 [[0.09095963 0.        ]
 [0.         0.10616485]]
```

In [7]:
```
# COMPLETE HYPOTHESIS: VAR(2) model
A = coeff_VAR(data, p=2)

A0_var = A[:,0]
A1_var = np.array((A[:,1], A[:,3])).T
A2_var = np.array((A[:,2], A[:,4])).T

A_var = np.concatenate( (A0_var,A1_var,A2_var), axis=None)

print('Complete model parameters')
print('A0:\n', A0_var)
print('A1:\n', A1_var)
print('A2:\n', A2_var)
```

```
Complete model parameters
A0:
 [-0.00066633 -0.00167826]
A1:
 [[ 0.22035789 -0.17975054]
 [ 0.00785193  0.13189426]]
A2:
 [[ 0.09273513 -0.11843226]
 [-0.00411085  0.10766649]]
```

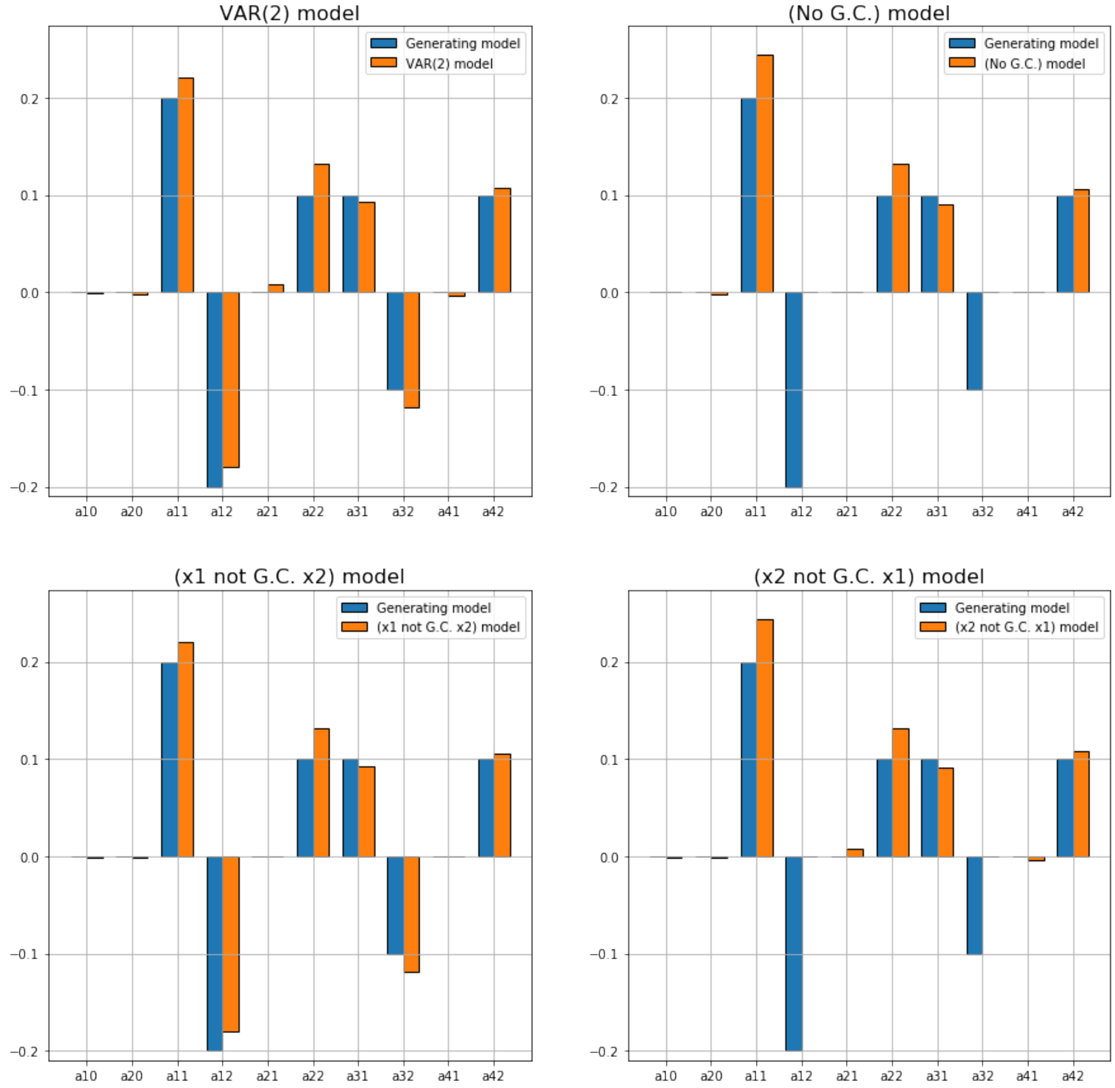
```
In [8]: # FIRST ALTERNATIVE HYPOTHESIS: x1 doesn't Granger causes x2
        A0_x2 = np.array((A[0,0], a2[0]))
        A1_x2 = np.array((A1_var))
        A1_x2[1] = np.array(([0.0, a2[1]]))
        A2_x2 = np.array((A2_var))
        A2_x2[1] = np.array(([0.0, a2[2]]))

        A_x2 = np.concatenate( (A0_x2,A1_x2,A2_x2), axis=None)

        print('(x1 not G.C. x2) model parameters')
        print('A0:\n', A0_x2)
        print('A1:\n', A1_x2)
        print('A2:\n', A2_x2)
```

```
(x1 not G.C. x2) model parameters
A0:
 [-0.00066633 -0.0016822 ]
A1:
 [[ 0.22035789 -0.17975054]
 [ 0.          0.13194647]]
A2:
 [[ 0.09273513 -0.11843226]
 [ 0.          0.10616485]]
```

```
In [9]: # SECOND ALTERNATIVE HYPOTHESIS: x2 doesn't Granger causes x1
        A0_x1 = np.array((A[0,0], a2[0]))
        A1_x1 = np.array((A1_var))
        A1_x1[0] = np.array(([a1[1], 0.0]))
        A2_x1 = np.array((A2_var))
        A2_x1[0] = np.array(([a1[2], 0.0]))

        A_x1 = np.concatenate( (A0_x1,A1_x1,A2_x1), axis=None)

        print('(x2 not G.C. x1) model parameters')
        print('A0:\n', A0_x1)
        print('A1:\n', A1_x1)
        print('A2:\n', A2_x1)
```

```
(x2 not G.C. x1) model parameters
A0:
 [-0.00066633 -0.0016822 ]
A1:
 [[0.24370888 0.        ]
 [0.00785193 0.13189426]]
A2:
 [[ 0.09095963  0.        ]
 [-0.00411085  0.10766649]]
```

In [10]:
```
# PLOTS OF THE COEFFICIENTS FOR DIFFERENT MODELS
A = np.array(([A_var, A_null, A_x2, A_x1]))

fig1 = plt.subplots(figsize=[15,15])
width = 0.35
xs = np.arange(A.shape[1])

x_ticks = ['a10','a20','a11','a12','a21','a22', 'a31', 'a32', 'a41', 'a42']
titles = ['VAR(2) model', '(No G.C.) model', '(x1 not G.C. x2) model', '(x2 not G.C. x1

for i in range(len(A)):
    plt.subplot(2,2,i+1)

    plt.bar(xs - width/2, A_model[:], width, label='Generating model', edgecolor = 'bla
    plt.bar(xs + width/2, A[i,:], width, label=titles[i], edgecolor = 'black')
    plt.xticks( xs, x_ticks )
    plt.ylim( [A.min()-0.03, A.max()+0.03] )
    plt.grid()
    plt.legend()
    plt.title(titles[i], fontsize = 16)

plt.show(fig1)
```




```
In [11]: # LOG-LIKELIHOOD COMPUTATION FOR DIFFERENT MODELS
         k = 2

         ll = log_likelihood(data, A0_var, A1_var, A2_var)
         print('log-likelihood for complete model:, ', ll)

         llnull = log_likelihood(data, A0_null, A1_null, A2_null)
         print('log-likelihood for restricted model: ', llnull)

         llx1 = log_likelihood(data, A0_x1, A1_x1, A2_x1)
         print('log-likelihood for (x2 not G.C. x1) model: ', llx1)

         llx2 = log_likelihood(data, A0_x2, A1_x2, A2_x2)
         print('log-likelihood for (x1 not G.C. x2) model: ', llx2)
```

```
log-likelihood for complete model:,  1786.56817554403
log-likelihood for restricted model:  1764.08138802638
log-likelihood for (x2 not G.C. x1) model:  1764.096829255153
log-likelihood for (x1 not G.C. x2) model:  1786.528964484111
```

```
In [12]: # LLR-TEST - respect to the Complete VAR model
         from scipy.stats import f

         D1 = -2. * (llx1 - ll)
         D2 = -2. * (llx2 - ll)

         print('D for (x2 not G.C. x1) model:', D1)
         print('cdf(D) for (x2 not G.C. x1) model:', chi2.cdf(D1, k) )
         print()
         print('D for (x1 not G.C. x2) model:', D2)
         print('cdf(D) for (x1 not G.C. x2) model:', chi2.cdf(D2, k) )
```

```
D for (x2 not G.C. x1) model: 44.942692577754315
cdf(D) for (x2 not G.C. x1) model: 0.9999999998258922

D for (x1 not G.C. x2) model: 0.07842211983825109
cdf(D) for (x1 not G.C. x2) model: 0.03845225646081886
```

```
In [13]: # F-TEST - respect to the Complete VAR model
         from scipy.stats import f

         sigma_var = sigma(data, A0_var, A1_var, A2_var)
         sigma_x1 = sigma(data,A0_x1, A1_x1, A2_x1)
         sigma_x2 = sigma(data,A0_x2, A1_x2, A2_x2)

         Fx1 = (T-p-1) / k * ( np.log(np.linalg.det(sigma_x1)) - np.log(np.linalg.det(sigma_var)
         Fx2 = (T-p-1) / k * ( np.log(np.linalg.det(sigma_x2)) - np.log(np.linalg.det(sigma_var)

         print('F for (x2 not G.C. x1) model:', Fx1)
         print('cdf(F) for (x2 not G.C. x1) model:', f.cdf(Fx1, k, (T-p-1)-k*(p+1)-1 ) )
         print()
         print('F for (x1 not G.C. x2) model:', Fx2)
         print('cdf(F) for (x1 not G.C. x2) model:', f.cdf(Fx2, k, (T-p-1)-k*(p+1)-1 ) )
```

```
F for (x2 not G.C. x1) model: 22.42235108827798
cdf(F) for (x2 not G.C. x1) model: 0.9999999997006309

F for (x1 not G.C. x2) model: 0.03917177028048702
cdf(F) for (x1 not G.C. x2) model: 0.03841298654551087
```

```
In [14]: # LLR-TEST - respect to the Complete VAR model
         D1 = -2. * (llnull - llx1)
```

```
D2 = -2. * (llnull - llx2)

print('D for (x2 not G.C. x1) model:', D1)
print('cdf(D) for (x2 not G.C. x1) model:', chi2.cdf(D1, k) )
print()
print('D for (x1 not G.C. x2) model:', D2)
print('cdf(D) for (x1 not G.C. x2) model: ', chi2.cdf(D2, k) )
```

```
D for (x2 not G.C. x1) model: 0.0308824575458857
cdf(D) for (x2 not G.C. x1) model: 0.015322624251171488

D for (x1 not G.C. x2) model: 44.89515291546195
cdf(D) for (x1 not G.C. x2) model:  0.9999999998217041
```

```
In [15]: # F-TEST - respect to the 2-AR model
         sigma_null = sigma(data, A0_null, A1_null, A2_null)

         Fx1 = (T-p-1) / k * ( np.log(np.linalg.det(sigma_null)) - np.log(np.linalg.det(sigma_x1
         Fx2 = (T-p-1) / k * ( np.log(np.linalg.det(sigma_null)) - np.log(np.linalg.det(sigma_x2

         print('F for (x2 not G.C. x1) model:', Fx1)
         print('cdf(F) for (x2 not G.C. x1) model:', f.cdf(Fx1, k, (T-p-1)-k*(p+1)-1 ) )
         print()
         print('F for (x1 not G.C. x2) model:', Fx2)
         print('cdf(F) for (x1 not G.C. x2) model:', f.cdf(Fx2, k, (T-p-1)-k*(p+1)-1 ) )
```

```
F for (x2 not G.C. x1) model: 0.04190457815246251
cdf(F) for (x2 not G.C. x1) model: 0.0410370170910969

F for (x1 not G.C. x2) model: 22.425083896149957
cdf(F) for (x1 not G.C. x2) model: 0.9999999997014126
```

By looking at the value of the F test respect to the estimated VAR(2) model, we can reject the assumption that x2 Granger-cause x1 at 1% significance level, while we can accept the assumption that x1 Granger-cause x2 at 1% significance level.

2 Task 2

2.0.1 Consider a linear Gaussian state space model:

$$z_t = Az_{t-1} + \epsilon,\ \epsilon \sim N(0, \Sigma)$$

$$x_t = Bz_t + \epsilon,\ \eta \sim N(0, \Gamma)$$

2.0.2 Derive the M-step for the latent state noise Σ and the observation matrix B by maximizing the expected log-likelihood E[logp(X,Z)], with respect to Σ and B, where X={xt | t∈1...T} and Z={zt | t∈1...T} are the sets of all latent states and observations from time 1 to T

$$ELBO[q,\theta] = E_q[log\ p_\theta(x,z)] + H[q(z)]$$

Since in the M-step we maximize respect to the different parameters θ, the proposal density distribution q is considered fixed, so we take into account only the first term from above. From here on out we ignore that the expectation value is respect to the proposal density distribution $q(z)$ and that the probability depends on the different parameter θ.

Then, we can expand the first term as:

$$E[log\ p(x,z)] = E[log\ p(z_1)] + E\Big[\sum_{t=2}^{T} log\ p(z_t|z_{t-1})\Big] + E\Big[\sum_{t=1}^{T} log\ p(x_t|z_t)\Big]$$

$$= (1) + (2) + (3)$$

where:

$$(1) = E\Big[-\frac{1}{2}log|\Sigma| - \frac{1}{2}(z_1-\mu_0)^T\Sigma^{-1}(z_1-\mu_0)\Big]$$

$$(2) = E\Big[-\frac{T-1}{2}log|\Sigma| - \frac{1}{2}\sum_{t=2}^{T}(z_t - Az_{t-1})^T\Sigma^{-1}(z_t - Az_{t-1})\Big]$$

$$(3) = E\Big[-\frac{T}{2}log|\Gamma| - \frac{1}{2}\sum_{t=1}^{T}(x_t - Bz_t)^T\Gamma^{-1}(x_t - Bz_t)\Big]$$

We can rearrenge the various terms by using the identity:

$$x^TAy = tr(Ayx^T)$$

- Maximization respect to B:

We consider only the terms containing B, which are:

$$-\frac{1}{2}\sum_{t=1}^{T}\Big[-tr(B^T\Gamma^{-1}x_tE[z_t^T]) - tr(\Gamma^{-1}BE[z_t]x_t^T) + tr(B^T\Gamma^{-1}BE[z_tz_t^T])\Big]$$

Maximizing respect to B we obtain:

$$\frac{\partial ELBO(q^*,\theta)}{\partial B} = -\sum_{t=1}^{T}\Big[-\Gamma^{-1}E[z_t]x_t^T + \Gamma^{-1}BE[z_tz_t^T]\Big] = 0$$

$$B = \Big(\sum_{t=1}^{T}x_tE[z_t]^T\Big)\Big(\sum_{t=1}^{T}E[z_tz_t^T]\Big)^{-1}$$

- Maximization respect to Σ:

We consider only the terms containing Σ, which are:

$$-\frac{T}{2}log|\Sigma| - \frac{1}{2}E\Big[z_1^T\Sigma^{-1}z_1 - \mu_0^T\Sigma^{-1}z_1 - z_1^T\Sigma^{-1}\mu_0 + \mu_0^T\Sigma^{-1}\mu_0\Big] -$$

$$-\frac{1}{2}\sum_{t=2}^{T} E\Big[z_t^T\Sigma^{-1}z_t - z_t^T\Sigma^{-1}Az_{t-1} - z_{t-1}^T A^T\Sigma^{-1}z_t + z_{t-1}^T A^T\Sigma^{-1}Az_{t-1}\Big] =$$

$$= -\frac{T}{2}log|\Sigma| - \frac{1}{2}\Sigma^{-1}\Big[-\mu_0^T E[z_1] - E[z_1^T]\mu_0 + \mu_0^T\mu_0\Big] -$$

$$-\frac{1}{2}\sum_{t=1}^{T} tr\Big(\Sigma^{-1}E[z_t z_t^T]\Big) - \frac{1}{2}\sum_{t=2}^{T}\Big[-tr\Big(\Sigma^{-1}AE[z_{t-1}z_t^T]\Big) -$$

$$-tr\Big(A^T\Sigma^{-1}E[z_t z_{t-1}^T]\Big) + tr\Big(A^T\Sigma^{-1}AE[z_{t-1}z_{t-1}^T]\Big)\Big]$$

Maximizing respect to Σ we obtain:

$$\frac{\partial ELBO(q^*,\theta)}{\partial \Sigma} = -\frac{T}{2}\Sigma^{-1} + \frac{1}{2}\Sigma^{-1}\Big[-\mu_0 E[z_1^T] - E[z_1]\mu_0^T + \mu_0\mu_0^T\Big]\Sigma^{-1} +$$

$$+\Sigma^{-1}\Big(\frac{1}{2}\sum_{t=1}^{T} E[z_t z_t^T]\Big)\Sigma^{-1} +$$

$$+\Sigma^{-1}\Big[\frac{1}{2}\sum_{t=2}^{T}\Big(-E[z_t z_{t-1}^T]A^T - AE[z_{t-1}z_t^T] + AE[z_{t-1}z_{t-1}^T]A^T\Big)\Big]\Sigma^{-1} = 0$$

where we have used the identity:

$$\frac{\partial Tr(AX^{-1}B)}{\partial X} = -X^{-1}A^T B^T X^{-1}$$

if the matrix X is symmetric.
Now we can multiply both to left and right by Σ^{-1}, so we obtain:

$$\Sigma = \frac{1}{T}\Big[-\mu_0 E[z_1^T] - E[z_1]\mu_0^T + \mu_0\mu_0^T + \sum_{t=1}^{T} E[z_t z_t^T] +$$

$$+\sum_{t=2}^{T}\Big(-E[z_t z_{t-1}^T]A^T - AE[z_{t-1}z_t^T] + AE[z_{t-1}z_{t-1}^T]A^T\Big)\Big]$$

Also, we know that the parameter A that maximizes $ELBO[q,\theta]$ is:

$$A = \Big(\sum_{t=2}^{T} E[z_t z_{t-1}^T]\Big)\Big(\sum_{t=2}^{T} E[z_{t-1}z_{t-1}^T]\Big)^{-1}$$

and we can use the definition $\mu_0 = E[z_1]$. Plug all into the equation above, finally we obtain:

$$\Sigma = \frac{1}{T}\Big[E[z_1 z_1^T] - E[z_1]E[z_1^T] + \sum_{t=2}^{T}\Big(E[z_t z_t^T] - AE[z_{t-1}z_t^T]\Big)\Big]$$