Control of Cyber-Physical Systems Project Report

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Distributed local decision and control algorithms for CPS are often desirable due to their computational simplicity, flexibility, and robustness to the loss of single agents. Hence the objective of cooperative control is to devise control protocols for the individual agents that guarantee synchronized behavior of the states of all the agents in some prescribed sense. This project report, in detail, focuses on the implementation of the optimal State Variable FeedBack (SVGB) cooperative control based on output feedback (OPFB) obtained using distributed and local observers for a distributed multi-agents magnetic levitation system, which consists of a distributed control protocol on the communication graph represented by the network between the nodes, divided in a leader node and N followers, where each one depends only on the local neighbor information as allowed by the topology.

Distributed control of a multi-agents magnetic levitation system

In the case of the magnetic levitation distributed system, given the following matrices

$$A = \begin{bmatrix} 0 & 1 \\ 880.87 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ -9.9453 \end{bmatrix} \quad C = \begin{bmatrix} 708.27 & 0 \end{bmatrix}$$

it is possible to prove the properties of stability, as a particular case of controllability, and detectability, as a subcase of observability.

$$rank\left(\begin{bmatrix} B & AB \end{bmatrix}\right) = 2 \equiv n \implies controllable, \quad rank\left(\begin{bmatrix} C & CA \end{bmatrix}^\top\right) = 2 \equiv n \implies observable$$

1. Effects of the communication network on cooperative control

In cooperative control systems on graphs, there are intriguing interactions between the individual agent dynamics and the topology of the communication graph. The graph topology can affect the possible performance of any control laws used by the agents. A fundamental problem in multi-agent dynamical systems on graph is the design of distributed protocols that guarantee consensus or synchronization in the sense that the states of all the agents reach the same value.
Each graph will be analyzed under three different scenarios: the first without noise, the second with only one node affected by a large noise and the third where all nodes are affected by noise. Moreover, their behaviour will be analyzed with both distributed and local observers. Both the State Variable Feedback controllers and the Output Feedback observers are designed with the following parameters, to compare different topologies.

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \quad R = 20 \quad x_0 = \begin{bmatrix} 0.00141 \\ 0.00125 \end{bmatrix} \quad c = 5$$

The coupling gain c is chosen so that it is greater than the minimum eigenvalue of every Laplacian matrix added to the pinning gains matrix for the relative graph.
Each topology is tested to follow three different kinds of references and this is achieved by creating a feedback loop around the leader node to impose specific eigenmodes:

- Constant reference, $\lambda = \begin{bmatrix} 0 & -1 \end{bmatrix} \implies A - BK = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$
- Ramp reference $\lambda = \begin{bmatrix} 0 & 0 \end{bmatrix} \implies A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
- Sinusoidal reference $\lambda = \begin{bmatrix} -\omega i & +\omega i \end{bmatrix} \implies A - BK = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} \quad \omega = 1, 3$

Two different evaluations are done with respect to the noise-free case and the noisy case:

- Noise-Free case
  In this case it is analyzed how the graph behaves when nodes are not affected by noise. Because of this, every topology will bring the system to the same signal at steady-state so for this part it is necessary to analyze its transient behaviour so that it is possible to make comparisons. The differences between topologies will be carried out by means of convergence time t_c and Root Mean Square Error (RMSE) of the displacemente error calculated from 0 to t_c. In particular, given the output of i-th follower y_i and the output of the leader y_0, the convergence time is here defined as

$$t_c : |y_i(t) - y_0(t)| \leqslant 10^{-3} \quad \forall t > t_c$$

- Noisy case
  In this scenario it is decided to not calculate the RMSE anymore, but to check the value of the output of each node at steady-state to understand if even with noise they are able to bring the displacement error to 0. The convergence time loses importance because with the noise it is impossible to have a tracking error less than 1% on all the nodes.

The choice of graph topology can have a significant impact on the dynamics of distributed control algorithms, which is equal to the global disagreement error, and their effectiveness in achieving desired system objectives. In this analysis, we will explore and compare five different graph topologies:

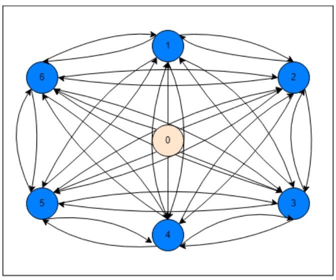


Figure 1. Complete Graph

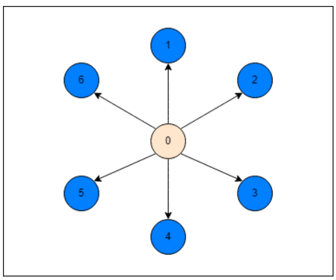


Figure 2. Directed Star

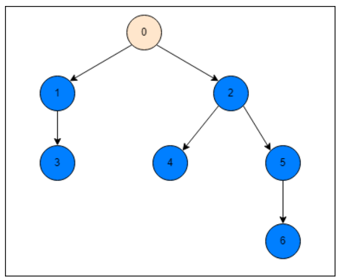


Figure 3. Directed Graph

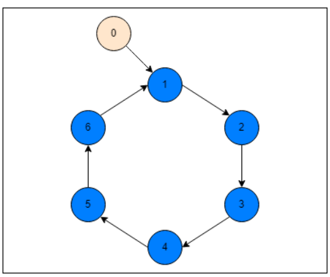


Figure 4. Directed Ring

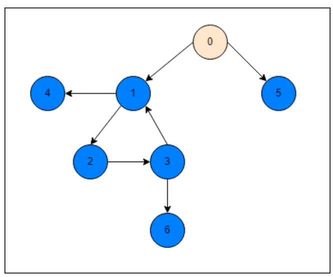


Figure 5. Random Graph

1.1. *Complete Graph*

1.1.1. *Noise-Free case*

	Distributed		Local	
Reference	convergence time (tc)	RMSE	convergence time (tc)	RMSE
Constant	$10.0488s$	0.7745	$11.1830s$	0.6199
Ramp	$11.3511s$	0.8797	$12.3824s$	0.6263
Sinusoidal, $w = 1$	$11.7000s$	0.8697	$12.6504s$	0.6307
Sinusoidal, $w = 3$	$14.9s$	1.0626	$15.5513s$	0.8808

1.2. *Directed Star*

1.2.1. *Noise-Free case*

	Distributed		Local	
Reference	convergence time (tc)	RMSE	convergence time (tc)	RMSE
Constant	$10.0492s$	0.8301	$11.1924s$	0.6262
Ramp	$11.3000s$	0.7777	$12.3555s$	0.6549
Sinusoidal, $w = 1$	$11.6000s$	0.7234	$12.5680s$	0.6595
Sinusoidal, $w = 3$	$14.3000s$	1.1111	$14.5000s$	0.6512

1.3. *Directed Tree Graph*

1.3.1. *Noise-Free case*

	Distributed		Local	
Reference	convergence time (tc)	RMSE	convergence time (tc)	RMSE
Constant	$10.04930s$	0.8944	$11.1927s$	0.6295
Ramp	$11.3513s$	0.8492	$12.3679s$	0.6701
Sinusoidal, $w = 1$	$11.6093s$	0.4083	$12.6000s$	0.6712
Sinusoidal, $w = 3$	$13.8000s$	1.1543	$14.8303s$	0.6673

1.4. *Directed Ring*

1.4.1. *Noise-Free case*

	Distributed		Local	
Reference	convergence time (tc)	RMSE	convergence time (tc)	RMSE
Constant	$10.1000s$	0.9623	$11.2157s$	0.6255
Ramp	$11.3000s$	0.9350	$12.3647s$	0.6374
Sinusoidal, $w = 1$	$11.5481s$	0.8608	$12.5822s$	0.6371
Sinusoidal, $w = 3$	$13.8511s$	1.0145	$14.7300s$	0.6437

1.5. *Random Graph*

1.5.1. *Noise-Free case*

	Distributed		Local	
Reference	convergence time (tc)	RMSE	convergence time (tc)	RMSE
Constant	$10.0498s$	0.9567	$11.2000s$	0.6361
Ramp	$11.3000s$	0.8682	$12.3624s$	0.6405
Sinusoidal, $w = 1$	$11.6000s$	0.9071	$12.5813s$	0.6410
Sinusoidal, $w = 3$	$13.3493s$	1.0399	$13.3493s$	1.0399

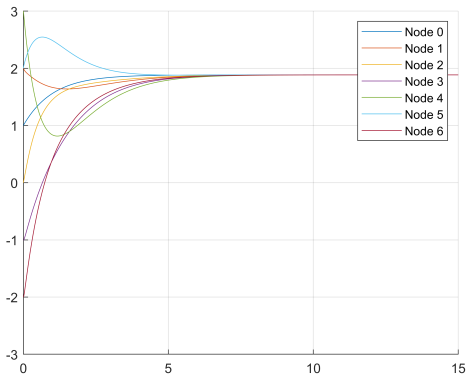


FIGURE 6. Constant Reference

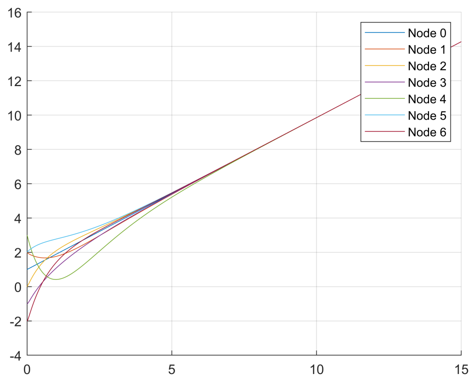


FIGURE 7. Ramp Reference

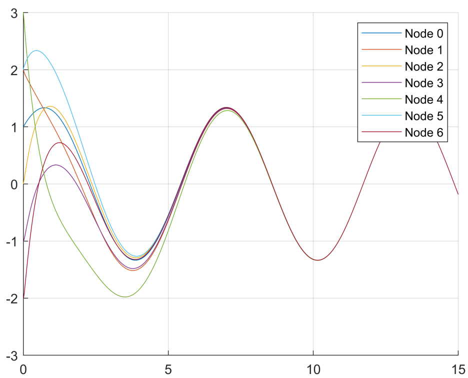


FIGURE 8. Sine Reference $\omega = 1$

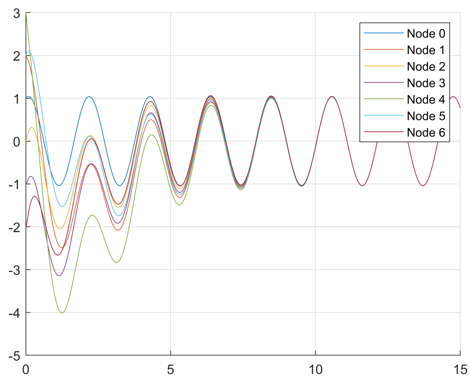


FIGURE 9. Sine Reference $\omega = 3$

2. Data analysis

It is possible to see that the distributed approach has a less or equal convergence time than the local but an higher RMSE with every type of signal (in the sinusoidal case with frequency equal to $\frac{3}{2\pi}$ the difference is neglectible).
It is observable that the Complete Graph and the Directed Star Graph seem to work better in most occasions since all their nodes are directly connected to the leader which provides them its exact position so that they can achieve smaller convergence times and in other cases lower RMSE.
However, those topology will not be considered for the choice of the best topology because in real life situations the leader can only communicate with few nodes.
Assuming that all the topologies converge at the same value at steady-state, the best

topology is then chosen by comparing the convergence times and choosing the one which is faster in most situations. By doing that the best graph results to be the Random Graph.

3. Effect of noise

Once the best topology is chosen, it is now time to analyze how system behaves whether one or more nodes are affected by some noise. We can distinguish two extreme scenarios:

- The noise affects only one follower node, in order to compute how this noise is passed along the noise in both distributed and local enviroments.
- The noise affects every node in the topology.

With the addition of noise a change related to the eigenvalues of the local observer is required. According to the theory, the eigenvalues should be large enough to provide a good tracking response, but choosing lower eigenvalues allows each local observer to act as filters reducing noise. This effect can be seen through the following transfer function, which represents how the local estimate of the state varies with respect to the noise as the frequency changes:

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF(\tilde{y}_i + \eta_i) = A\hat{x}_i + Bu_i - cF(y_i - \hat{y} + \eta_i) =$$
$$A\hat{x}_i + Bu_i - cFy_i + cFC\hat{x}_i - cF\eta_i = (A + cFC)\hat{x}_i + Bu_i - cFy_i - cF\eta_i$$

Using the Laplace transform, with $\hat{x}_i(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$s\hat{X}_i(s) = (A + cFC)\hat{X}_i(s) + BU_i(s) - cFY_i(s) - cF\mathrm{H}_i(s)$$
$$\hat{Y}_i(s) = C[s - (A + cFC)]^{-1}[BU_i(s) - cFY_i(s) - cF\mathrm{H}_i(s)]$$
$$G_\eta(s) = \left.\frac{\hat{Y}_i(s)}{\mathrm{H}_i(s)}\right|_{\substack{U=0\\Y=0}} = -C[s - (A + cFC)]^{-1}cF$$

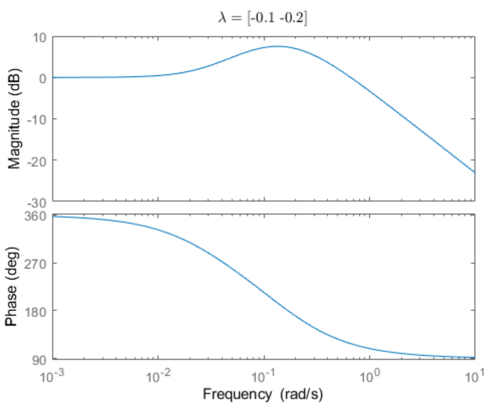


FIGURE 10. Bode plot with $\lambda = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}$

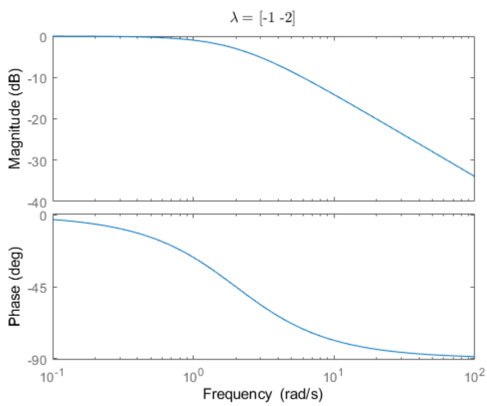
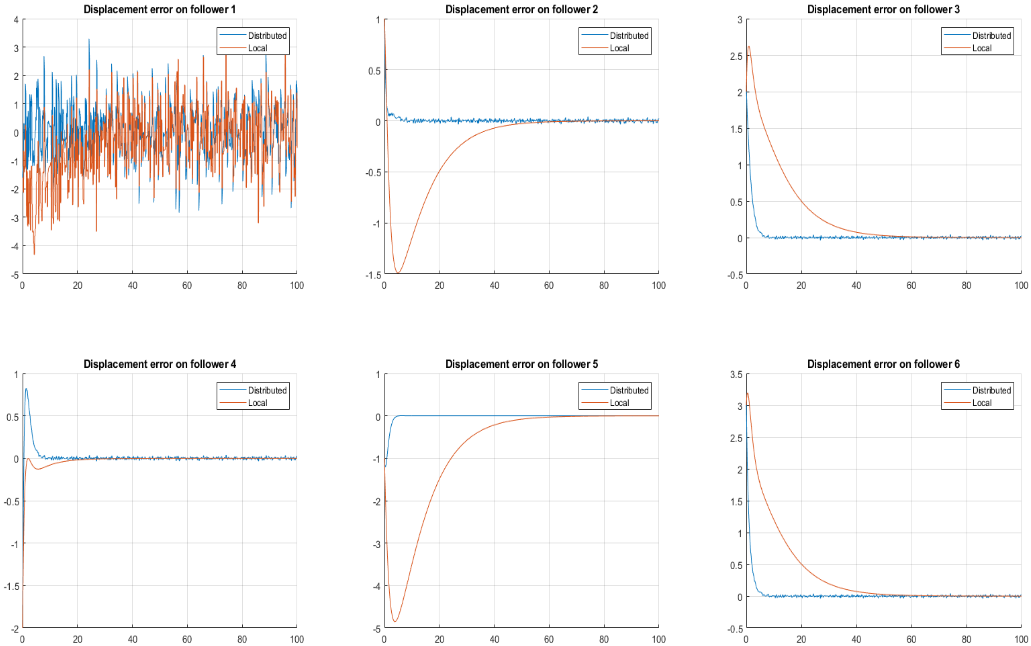


FIGURE 11. Bode plot with $\lambda = \begin{bmatrix} -1 & -2 \end{bmatrix}$

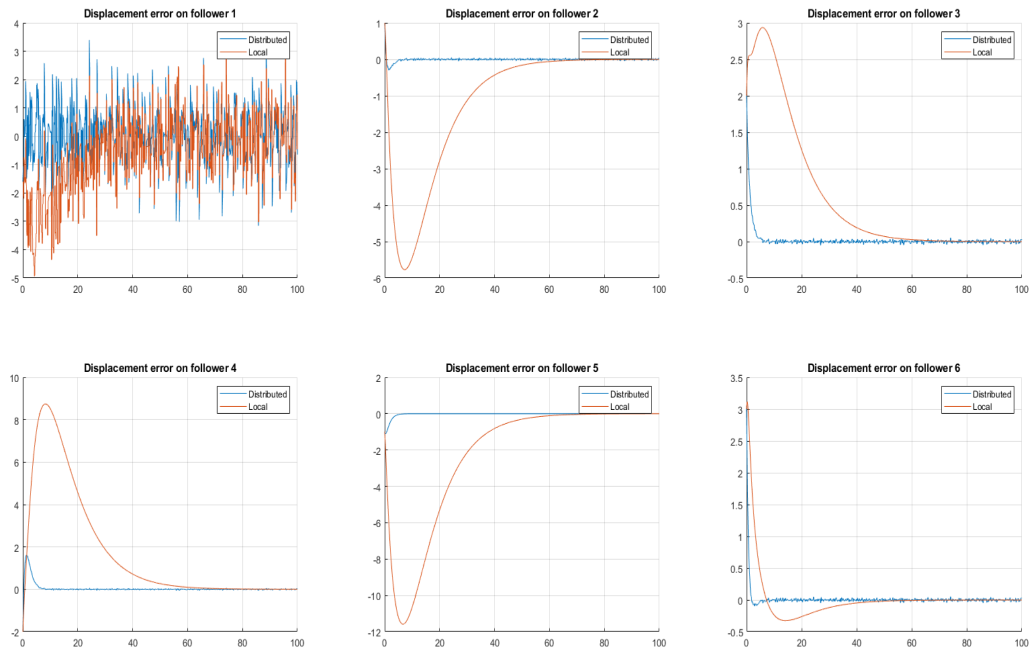
From the bode plots it is observable that by decreasing the eigenvalues, the cutoff frequency decreases so the filter has a tighter frequency band.

So it is then decided to put the noise on the Follower 1 and to set the eigenvalues of the local observer to $\lambda = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}$ in order to analyze the behaviour of the system with the four references.

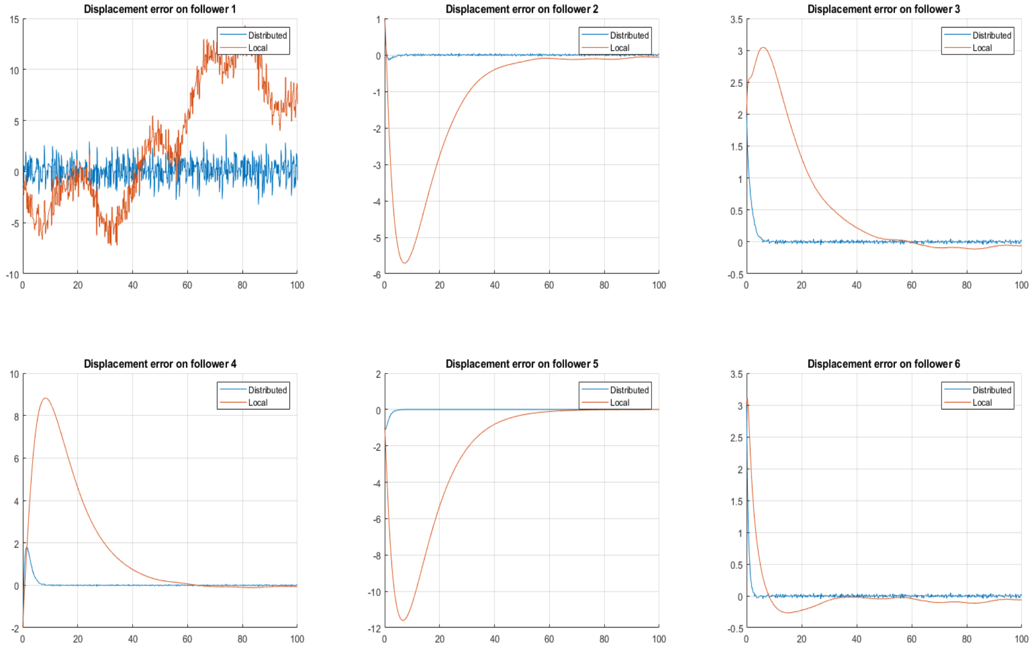
- Constant Reference



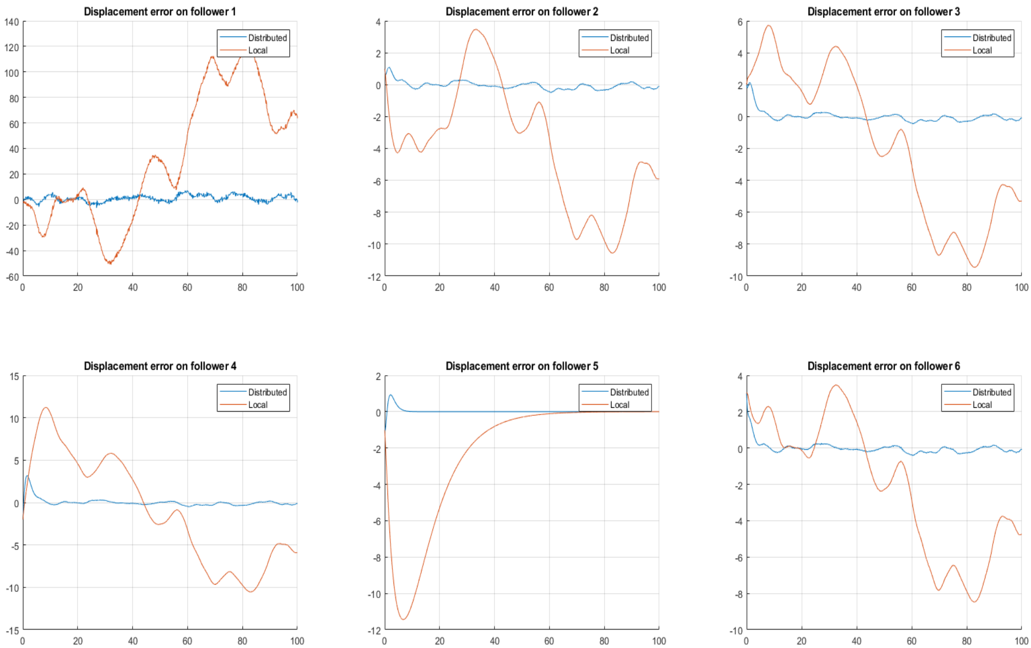
- Ramp Reference



- Sinusoidal Reference with $\omega = 1$



- Sinusoidal Reference with $\omega = 3$



It is observable from the graphs above, that for almost all reference the local observers tend to work better since they filter out most of the noise because the noise frequency is put outside of the low pass filter created by the local observers, whereas the distributed observers tend to create a pass-band filter so the noise is still attenuated but not filtered out given by the following transfer function (see appendix for details):

$$G_\eta(s) = \left.\frac{\hat{Y}(s)}{\mathrm{H}(s)}\right|_{\substack{U=0\\ Y=0\\ \hat{Y}_0=0}} = (I_{Nn} \otimes C)\{sI_{Nn} - [(I_N \otimes A) - c(L+G) \otimes FC]\}^{-1}\{c[(L+G) \otimes F]\}$$

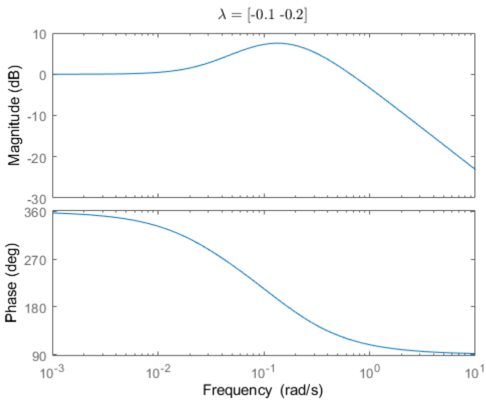


FIGURE 12. Bode plot of local observer

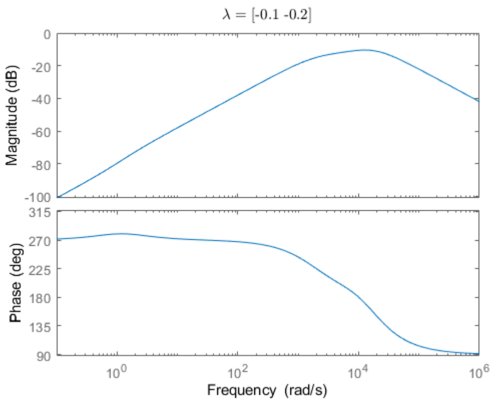


FIGURE 13. Bode plot of distributed observer

This is not the case of the sinusoidal reference with $\omega = 3$ because even if the noise frequency is outside the low pass filter generated by the local observers it is necessary to choose eigenvalues high enough so that the frequency of the sinusoidal is not attenuated by the low pass filter generated by the following transfer function:

$$s\hat{X}_i(s) = (A + cFC)\hat{X}_i(s) + BU_i(s) - cFY_i(s) - cF\mathrm{H}_i(s)$$
$$\hat{X}_i(s) = [s - (A + cFC)]^{-1}[BU_i(s) - cFY_i(s) - cF\mathrm{H}_i(s)]$$
$$\hat{Y}_i(s) = C[s - (A + cFC)]^{-1}[BU_i(s) - cFY_i(s) - cF\mathrm{H}_i(s)]$$
$$G_\eta(s) = \left.\frac{\hat{Y}_i(s)}{\mathrm{U}_i(s)}\right|_{\substack{H=0\\ Y=0}} = -C[s - (A + cFC)]^{-1}B$$

If this does not happen the followers are not able to track the leader because their behaviour is significantly changed.

In the case that all the nodes are corrupted with different values of noise it is not noticeable any big difference between the distributed and local approach, so the graphs of this case have not been shown.

4. Changing the values of the weighting matrices and of the coupling gain

It is now analyzed the effect of the weighting matrices and the coupling gain on the magnetic levitation system. For the study of the weighting matrices Q and R, which are used to calculate the controller and observer gains using a Riccati approach, we suppose to be in a noise-free environment in the "random" topology considering only the ramp signal with eigenvalues $\begin{bmatrix}-1 & -2\end{bmatrix}$ of the local observers, as a representative example. The following table shows the convergence time measured as the maximum among the six agents, using both distributed and local observers, and the mean energy of the inputs of the nodes, keeping a fixed state weighting matrix $Q = diag(q, q) \in \mathbb{R}^{2\times 2}$ and varying the value of the input weighting matrix $R = r \in \mathbb{R}$, noticing that is not so much important the absolute value of the weights themselves but rather the ratio between q and r.

q	r	Distributed [s]	Local [s]	Mean energy of ε
100	1	11.1494	12.1772	0.1046
100	10	11.2479	11.2479	0.0103
100	20	11.3000	12.3624	0.0052
100	100	11.6000	12.6413	0.0011
100	1000	12.6483	13.7000	1.1476×10^{-4}
100	100000	22.7775	24.0000	4.2656×10^{-7}

It is observable that if the ratio between Q and R increases the convergence time decreases so we have better performances, but we also need to provide more energy to the system. However, if the ratio between Q and R decreases we have poorer performances, but in return we save a great amount of energy.
To study the effect of the coupling gain the weighting matrices are set back to

$$Q = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} \qquad R = 20$$

to understand the behaviour of the system under different values of the coupling gain. Not many differences are witnessed under the free-noise environment so we suppose to have noise only on the Follower 1. The table lists the steady state values of average displacement errors on distributed observer of the agents with respect to the average input signal energy.

c	Follower 1	Follower 2	Follower 3	Follower 4	Follower 5	Follower 6	Energy of ε
5	-0.6444	0.0209	0.0215	0.0209	0.0001	0.0222	0.0783
50	-0.6532	0.0019	0.0019	0.0019	0.0001	0.0019	6.8608
500	-0.6520	0.0001	0.0002	0.0001	0.0001	0.0002	656.9474

It is noticeable that even though by increasing the coupling gain the effect of the noise decreases and the value of the displacement error get closer to zero, it also requires more and more energy to be able to filter that noise. In a virtual simulation the coupling gain can be increased without a limit, but in real life scenario it must be chosen a compromise between the noise filtered and the required energy. This behaviour numerically proves how the coupling gain affects the asymptotic stability of the state according to the theoretical relationship with the amplification of the feedback effect on the dynamics for which increasing c the eigenvalues of the global state matrix A_c decrease provided that $A - c\lambda_i BK$ is Hurwitz ($\lambda = eig(L + G)$), as proven in the appendix.

Conclusion

In conclusion, this paper it is shown that the locally optimal design, in terms of a local Riccati equation, guarantees synchronization of multi-agent systems regardless of graph topology, as long as the arbitrary communication graph has a spanning tree. The goal is numerically analyzing these properties using different graph topologies and scenarios with and without noise.

Comparing the distributed and local observers, we prove that the distributed approach has either equal or faster convergence times but higher RMSE compared to the local approach. The Complete Graph and Directed Star Graph show promising performance due to direct connectivity with the leader, resulting in faster convergence times and lower RMSE. However, these topologies are not considered the best choices due to limitations in real-life situations where the leader can only communicate with a few nodes.

Assuming that all topologies converge to the same signal, the best topology is determined by comparing convergence times across various situations. The Random Graph emerges as the best choice in this regard. Additionally, it notes that the local observers tend to perform better in filtering out noise, for the fact that in the distributed version the estimate errors related to noise propagate to the neighbors through the network, except for cases involving a sinusoidal reference signal with certain frequencies where the noise can distort the signal shape and introduce fluctuations due to the transfer function between estimated state and noise from which we notice the system manages to track low frequency oscillations attenuating the effect of higher frequency noise as a pass low filter.

Comparing the effects of the weighting matrices, we find that increasing the ratio between Q and R leads to decreased convergence time and improved performance, at the expense of increased energy consumption. While choosing a larger c allows the system to converge with more accuracy at the cost of an exponential energy consumption of control inputs, so this is particular useful to filter eventual noise.

In summary, the project report presents a comprehensive analysis of distributed local decision and control algorithms for cooperative control of a multi-agent magnetic levitation system. The findings highlight the importance of topology selection, noise considerations, observer type, and the coupling gain in achieving desired performance and energy efficiency in such systems.

Appendix

Cooperative tracking control of multi-agent dynamical systems

In the cooperative tracking problem each agent or node of the Cyber-Physical System is mathematically modeled by identical continuous LTI systems, connected by a communication network described by a directed graph associated to an adjency matrix, where a command generator or leader node provides the desired tracking trajectory to which all N agents should synchronize, keeping in mind that in general only a few nodes are aware of the direct information from the leader node.

$$\begin{cases} \dot{x}_0 = Ax_0 & y_0 = Cx_0 & x_0 \in \mathbb{R}^n,\, y_0 \in \mathbb{R}^p & \textit{leader node} \\ \dot{x}_i = Ax_i + Bu_i & y_i = Cx_i & x_i \in \mathbb{R}^n,\, y_i \in \mathbb{R}^p,\, u_i \in \mathbb{R}^m & \textit{follower nodes} \end{cases}$$

$$\mathcal{G} = \{\mathcal{V}, \mathcal{E}\} \quad s.t.\ \ \mathcal{V} = \{\nu_1, \nu_2, ..., \nu_N\},\, \mathcal{E} \subset \mathcal{V} \times \mathcal{V}$$

To design a cooperative tracking controller it is necessary to comply further assumptions:

- The matrices A,B and C must be stabilizable and detectable;
- The global topology represented by the augmented graph, which includes the leader, $\bar{\mathcal{G}}$ has a spanning tree where there is at least one directed path from the leader node to every follower.

The objective of the cooperative tracking problem is to design a local distributed controller (u_i) for all the follower nodes such that they can track the state trajectory of the leader node (so $\lim_{t\to\infty}(x_i(t) - x_0(t)) = 0 \ \ \forall i \in \mathbb{N}$).
In a first analysis we assume that state variable of each node is directly measurable to design a cooperative state variable feedback (SVFB) control law of distributed form using a Riccati design approach. To obtain a fully distributed controller, the control law of each agent must respect the graph topology so it means each agent can only use restricted scope information to compute the local neighborhood tracking error as the weighted sum of differences between the states of node i and its neighbors, given the adjacency matrix $\mathcal{A}$ where a_{ij} is the weight of the link from i to j of $\mathcal{G}$ and the pinning gain g_i which is the weight of the link between the leader and the agent i.

$$\varepsilon_i \triangleq \sum_{j=1}^{N} a_{ij}(x_i - x_j) + g_i(x_0 - x_i)$$

And each node $i \in \mathcal{N}$ calculates a linear control input with respect to the tracking error, based on a coupling gain $c > 0$ and a so called feedback gain matrix $K \in \mathbb{R}^{m\times n}$. The closed-loop system for each node i becomes:

$$u_i = cK\varepsilon_i \implies \dot{x}_i = Ax_i + Bu_i = Ax_i + cBK\left(\textstyle\sum_{j=1}^{N} a_{ij}(x_i - x_j) + g_i(x_0 - x_i)\right)$$

So it is possible to define a global closed-loop dynamics putting the LTI systems together

$$\begin{aligned} \dot{x} &= (I_N \otimes A)x - c\left[(L+G)\otimes BK\right](x - x_0) = \\ &= \left[I_N \otimes A - c(L+G)\otimes BK\right]x + \left[c(L+G)\otimes BK\right]\bar{x}_0 = A_c x + B_c \bar{x}_0 \end{aligned}$$

where $x = col(x_1, x_2, ..., x_N) \in \mathbb{R}^{nN}$ is the global state, $\bar{x}_0 = col(x_0, x_0, ..., x_0) \in \mathbb{R}^{nN}$ is the control input, $G = diag(g_1, g_2, ..., g_N)$ is the diagonal matrix of pinning gains, and $L = \mathcal{D} - \mathcal{A}$ is the Laplacian matrix given by diagonal of in-degrees and adjacency matrix. Defined the global disagreement error with respect to the leader we found its dynamics as

$$\delta_i = x_i - x_0 \qquad \delta = col(\delta_1, \delta_2, ..., \delta_n) = x - x_0 \in \mathbb{R}^{nN} \qquad \dot{\delta} = \dot{x} - \dot{\bar{x}}_0 = A_c\delta$$

the cooperative tracking problem is solved when $\lim_{t\to\infty}\delta(t) = 0$.
Necessary and sufficient condition for the entire Cyber-Physical System to be asymptoti-

cally stable is that the matrices $A - c\lambda_i BK$ for each agent i are themselves Hurwitz.

$$\lambda = eig(L+G) \qquad \lambda' = eig(A_c) = \bigcup_{i=1}^{N} eig(A - c\lambda_i BK) \;\; \text{if} \;\; \lambda'_j < 0 \;\; \forall\, j \implies A_c \text{ Hurwitz}$$

An arbitrary SVFB control gain K that stabilizes the single-agent dynamics $A - BK$ in general can fail to stabilize the global dynamics A_c given a network topology. Thus the feedback matrix K can be designed to decouple the global control protocol from the graph properties using a local Riccati design approach and a proper choice of the coupling gain c. Given that each agent system is stabilizable and choosing matrices $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$ positive definite and computing P as the solution of the Algebraic Riccati Equation $A^\top P + PA + Q - PBR^{-1}B^\top P = 0$ therefore K is taken as

$$K = R^{-1}B^\top P$$

The global disagreement error dynamics is asymptotically stable if the coupling gain satisfies

$$c \geqslant \frac{1}{2\min_{i\in\mathcal{N}} Re\{\lambda_i\}}$$

Cooperative Observer Design

In the previous controller design we assumed that the full state information of all nodes can be measured, but generally in practice it is not said that they are directly available. In these cases it is possible to design a cooperative state observer for networked multi-agent systems which provides estimates of the full state starting from the detected output. Considering the provided system model, we denote $\hat{x}_i \in \mathbb{R}^n$ as the estimate of the state x_i, $\hat{y}_i = C\hat{x}_i$ as the estimated output, $\tilde{x}_i = x_i - \hat{x}_i$ as the state estimation error and $\tilde{y}_i = y_i - \hat{y}_i$ as the output estimation error for node i. Similarly to the local neighborhood tracking error, we define the local neighborhood output estimation error for each node i

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(\tilde{y}_0 - \tilde{y}_i)$$

The cooperative observer for each node i is designed based on a Luenberger observer distributed version as

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF\zeta_i$$

where $c > 0$ is the coupling gain and $F \in \mathbb{R}^{n\times p}$ is the observer gain, which are selected so that the state estimation errors $\tilde{x}_i$ go to zero for each node to fullfil the cooperative observer purpose. Since the leader node acts like a signal generator, we assume it knows its own state $\hat{x}_0 = x_0$ i.e. $\tilde{y} = 0$ so the global cooperative observer dynamics is

$$\begin{aligned}
\dot{\hat{x}} &= (I_N \otimes A)\hat{x} + (I_N \otimes B)u - \{-c\,[(L+G)\otimes F]\,(\tilde{y} - \tilde{y}_0)\} = \\
&= (I_N \otimes A)\hat{x} + (I_N \otimes B)u + c\,[(L+G)\otimes F]\,(y - \hat{y}) = \\
&= (I_N \otimes A)\hat{x} + (I_N \otimes B)u + c\,[(L+G)\otimes F]\,y - c\,[(L+G)\otimes F]\,\hat{y} = \\
&= (I_N \otimes A)\hat{x} - c\,[(L+G)\otimes F]\,C\hat{x} + (I_N \otimes B)u + c\,[(L+G)\otimes F]\,y = \\
&= [I_N \otimes A - c(L+G)\otimes FC]\,\hat{x} + (I_N \otimes B)u + c\,[(L+G)\otimes F]\,y
\end{aligned}$$

where $\hat{x} = col(\hat{x}_1, \hat{x}_2, ..., \hat{x}_N)$, $y = col(y_1, y_2, ..., y_N)$, $u = col(u_1, u_2, ..., u_N)$. So that the global state estimation error has a dynamics matrix $A_o = I_N \otimes A - c(L+G)\otimes FC$

$$\tilde{x} = x - \hat{x} \quad \implies \quad \dot{\tilde{x}} = A_o\tilde{x}$$

The task of the cooperative observer design is to find a suitable coupling gain c and observer gain F such that the error state matrix A_0 is Hurwitz, guaranteeing the state estimation error goes to zero id est $\lim_{t\to\infty}\hat{x}(t) = x(t)$. Necessary and sufficient condition for asymptotical stability of the global cooperative observer dynamics is that all the matrices $A - c\lambda_i FC$ are Hurwitz, so the global state estimation error converges to zero.

$$\lambda = eig(L+G) \qquad \lambda' = eig(A_0) = \bigcup_{i=1}^{N} eig(A - c\lambda_i FC) \quad \text{if} \quad \lambda'_j < 0 \quad \forall\, j \implies A_0 \text{ Hurwitz}$$

Again, it is possible to note the graph topology interferes with the proper effectiveness of the local agent observers, and since it is something unwanted, it is necessary that they must be designed decoupled from the graph properties. To accomplish this goal, given that each agent is detectable and choosing properly $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{p\times p}$ the observer gain F is obtained using a local Riccati design approach, finding the unique positive definite solution P of the ARE $AP + PA^\top + Q - PC\top R^{-1}CP = 0$, such that

$$F = PC^\top R^{-1}$$

and taking a coupling gain c as

$$c \geqslant \frac{1}{2\min_{i\in\mathcal{N}} Re\{\lambda_i\}}$$

Cooperative Dynamic Output Feedback Tracker

The final aim is integrating the state feedback controller (SVFB) along with the cooperative observer to design dynamic regulators based on reduced state or output feedback (OPFB) that guarantee global synchronization given only the available output measurements. It is possible to devise three possible cooperative dynamic output feedback regulators based on using one of the combinations obtained using a local/neighborhood controller and a local/neighborhood observer.

Neighborhood Controller and Neighborhood Observer

Given the SVFB cooperative control protocol where the local neighborhood output estimation error ζ and the OPFB local neighborhood state estimation tracking error ε are

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(\tilde{y}_0 - \tilde{y}_i) \qquad \hat{\varepsilon}_i = \sum_{j=1}^{N} a_{ij}(\hat{x}_i - \hat{x}_j) + g_i(\hat{x}_0 - \hat{x}_i)$$

Each node i computes these two local errors from the measured information from itself and received from its neighbors in the network. Starting from the previous quantities the cooperative regulator is designed giving as input u_i and having an estimated state $\hat{x}_i$

$$u_i = cK\hat{\varepsilon}_i \qquad \dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF\zeta_i$$

So the closed-loop dynamics of node i is described by the real system and the filter

$$\dot{x}_i = Ax_i + cBK\left[\sum_{j\in N_i} a_{ij}(\hat{x}_j - \hat{x}_i) + g_i(\hat{x}_0 - \hat{x}_i)\right]$$
$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF\left[\sum_{j\in N_i} a_{ij}(\tilde{y}_j - \tilde{y}_i) + g_i(\tilde{y}_0 - \tilde{y}_i)\right]$$

Assuming the leader state is known a-priori without the necessity of an observer, i.e. $\hat{x}_0 = x_0$ and $\tilde{y}_0 = 0$, the global closed loop dynamics can be expressed as

$$\begin{aligned}
\dot{x} &= (I_N \otimes A)x - c\left[(L+G) \otimes BK\right](\hat{x} - \bar{x}_0) \overset{*}{=} \qquad\qquad * \ \hat{x} = x - \tilde{x} \\
&\overset{*}{=} (I_N \otimes A)x - c[(L+G) \otimes BK](x - \tilde{x} - \bar{x}_0) = \\
&= (I_N \otimes A)x - c[(L+G) \otimes BK]x + c[(L+G) \otimes BK](\tilde{x} + \bar{x}_0) = \\
&= \{(I_N \otimes A) - c[(L+G) \otimes BK]\}\, x + c[(L+G) \otimes BK](\tilde{x} + \bar{x}_0) = A_c x + B_c(\tilde{x} + \bar{x}_0) \\
\dot{\hat{x}} &= (I_N \otimes A)\hat{x} + (I_N \otimes B)u - \{-c[(L+G) \otimes F]\}\,(\tilde{y} - y_0) = \\
&= (I_N \otimes A)\hat{x} - c[(L+G) \otimes BK](\hat{x} - x_0) + c[(L+G) \otimes F]\tilde{y} = \\
&= (I_N \otimes A)\hat{x} - c[(L+G) \otimes BK](\hat{x} - \bar{x}_0) + c[(L+G) \otimes F](y - \hat{y}) = \\
&= (I_N \otimes A)\hat{x} - c[(L+G) \otimes F]C\hat{x} + c[(L+G) \otimes F]y - c[(L+G) \otimes BK](\hat{x} - x_0) = \\
&= \{(I_N \otimes A) - c[(L+G) \otimes FC]\}\,\hat{x} - c[(L+G) \otimes BK](\hat{x} - x_0) + c[(L+G) \otimes F]y = \\
&= A_o\hat{x} + B_o(\hat{x} - x_0) + C_o y
\end{aligned}$$

The goal is obtaining a global disagreement error $\delta = x - \bar{x}_0$ and a global state estimation error $\tilde{x} = x - \hat{x}$, described by the following dynamics, such that go to zero asymptotically.

$$\begin{aligned}
\dot{\delta} &= \dot{x} - \dot{\bar{x}}_0 = (I_N \otimes A)x - c[(L+G) \otimes BK](\hat{x} - \bar{x}_0) - (I_N \otimes A)\bar{x}_0 = \\
&= (I_N \otimes A)(x - \bar{x}_0) - c[(L+G) \otimes BK](x - \tilde{x} - \bar{x}_0) = \\
&= [(I_N \otimes A) - c(L+G) \otimes BK](x - \bar{x}_0) + c[(L+G) \otimes BK]\tilde{x} = A_c\delta + B_c\tilde{x} \\
\dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} = \{(I_N \otimes A)x - c[(L+G) \otimes BK](\hat{x} - \bar{x}_0)\} - \{[(I_N \otimes A) - c(L+G) \otimes FC]\hat{x} - \\
&\quad - c[(L+G) \otimes BK](\hat{x} - \bar{x}_0) + c[(L+G) \otimes F]y\} = \\
&= (I_N \otimes A)x - c[(L+G) \otimes BK](\hat{x} - \bar{x}_0) - [(I_N \otimes A) - c(L+G) \otimes FC]\hat{x} + c[(L+G) \otimes \\
&\quad \otimes BK](\hat{x} - \bar{x}_0) - c[(L+G) \otimes F]y = \\
&= (I_N \otimes A)x - c[(L+G) \otimes BK]\hat{x} + c[(L+G) \otimes BK]\bar{x}_0 - (I_N \otimes A)\hat{x} + c(L+G) \otimes FC\hat{x} \\
&\quad + c[(L+G) \otimes BK]\hat{x} - c[(L+G) \otimes BK]\bar{x}_0 - c[(L+G) \otimes F]y = \\
&= (I_N \otimes A)(x - \hat{x}) + c(L+G) \otimes FC\hat{x} - c(L+G) \otimes FCx = \\
&= (I_N \otimes A)(x - \hat{x}) - c(L+G) \otimes FC(x - \hat{x}) = \\
&= [(I_N \otimes A) - c(L+G) \otimes FC](x - \hat{x}) = A_o\tilde{x}
\end{aligned}$$

We can represent the disagreement and state estimation errors with the block LTI system

$$\begin{bmatrix} \dot{\delta} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ 0 & A_o \end{bmatrix} \begin{bmatrix} \delta \\ \tilde{x} \end{bmatrix}$$

Considering that the eigenvalues of a block triangular matrix correspond to the combined eigenvalue of its diagonal blocks and having proved in the previous paragraph A_c and A_o are Hurwitz so, in turn, the errors system is asymptotically stable.

$$eig\left(\begin{bmatrix} A_c & B_c \\ 0 & A_o \end{bmatrix}\right) = \{eig(A_c),\ eig(A_o)\} \quad \begin{matrix} A_c\ Hurwitz \\ A_o\ Hurwitz \end{matrix} \quad \Longrightarrow \quad \begin{matrix} \lim_{t\to\infty} \delta(t) = 0 \\ \lim_{t\to\infty} \tilde{x}(t) = 0 \end{matrix}$$

In conclusion, given a Cyber-Physical System made by a leader and N agents described by LTI systems defined by A, B, C matrices, which are stabilizable and detectable, linked by a network graph which has a spanning tree with the leader as the root, designing a SVFB tracking control law based u_i on a OPFB estimate $\hat{x}_i$, where the coupling gain c, the state feedback gain K and the observer gain F are chosen according to the relative theorems, then all nodes synchronize to the leader node 0 asymptotically and the state estimation error approaches zero asymptotically.

In presence of noise on the measurement it is necessary to study the effect of the noise on the measurements obtained from the estimator. With the following demonstration, we emphasize the link between the estimated output and the noise on the estimate made by the neighborhood observer itself, in terms of the whole global distributed system.

$$\dot{\hat{x}} = (I_N \otimes A)\hat{x} + (I_N \otimes B)u + c[(L+G) \otimes F](\tilde{y} - \tilde{y}_0 + \eta)$$
$$\dot{\hat{x}} = (I_N \otimes A)\hat{x} + (I_N \otimes B)u + c[(L+G) \otimes F]\tilde{y} - c[(L+G) \otimes F]\tilde{y}_0 + c[(L+G) \otimes F]\eta$$
$$\dot{\hat{x}} = (I_N \otimes A)\hat{x} + (I_N \otimes B)u + c[(L+G) \otimes F](y - \hat{y}) - c[(L+G) \otimes F]\tilde{y}_0 + c[(L+G) \otimes F]\eta$$
$$\dot{\hat{x}} = (I_N \otimes A)\hat{x} + (I_N \otimes B)u + c[(L+G) \otimes F]y - c[(L+G) \otimes F]\hat{y} - c[(L+G) \otimes F]\tilde{y}_0 + \\ + c[(L+G) \otimes F]\eta$$
$$\dot{\hat{x}} = (I_N \otimes A)\hat{x} + (I_N \otimes B)u + c[(L+G) \otimes F]y - c[(L+G) \otimes F]C\hat{x} - c[(L+G) \otimes F]\tilde{y}_0 + \\ + c[(L+G) \otimes F]\eta$$
$$\dot{\hat{x}} = [(I_N \otimes A) - c(L+G) \otimes FC]\hat{x} + (I_N \otimes B)u + c[(L+G) \otimes F]y - c[(L+G) \otimes F]\tilde{y}_0 + \\ + c[(L+G) \otimes F]\eta$$
$$\mathcal{L}\{\dot{\hat{x}}\} = \mathcal{L}\{[(I_N \otimes A) - c(L+G) \otimes FC]\hat{x} + (I_N \otimes B)u + c[(L+G) \otimes F]y - c[(L+G) \otimes \\ F]\tilde{y}_0 + c[(L+G) \otimes F]\eta\}$$
$$s\hat{X}(s) - \hat{x}(0) = [(I_N \otimes A) - c(L+G) \otimes FC]\hat{X}(s) + (I_N \otimes B)U(s) + c[(L+G) \otimes F]Y(s) \\ - c[(L+G) \otimes F]\tilde{Y}_0 + c[(L+G) \otimes F]\mathrm{H}(s)$$
$$s\hat{X}(s) = [(I_N \otimes A) - c(L+G) \otimes FC]\hat{X}(s) + (I_N \otimes B)U(s) + c[(L+G) \otimes F]Y(s) - \\ - c[(L+G) \otimes F]\tilde{Y}_0 + c[(L+G) \otimes F]\mathrm{H}(s) + \hat{x}(0)$$
$$s\hat{X}(s) - [(I_N \otimes A) - c(L+G) \otimes FC]\hat{X}(s) = (I_N \otimes B)U(s) + c[(L+G) \otimes F]Y(s) - \\ - c[(L+G) \otimes F]\tilde{Y}_0 + c[(L+G) \otimes F]\mathrm{H}(s) + \hat{x}(0)$$
$$\{sI_N n - [(I_N \otimes A) - c(L+G) \otimes FC]\}\hat{X}(s) = (I_N \otimes B)U(s) + c[(L+G) \otimes F]Y(s) \\ - c[(L+G) \otimes F]\tilde{Y}_0 + c[(L+G) \otimes F]\mathrm{H}(s) + \hat{x}(0)$$
$$\hat{X}(s) = \{sI_{Nn} - [(I_N \otimes A) - c(L+G) \otimes FC]\}^{-1}\{(I_N \otimes B)U(s) + c[(L+G) \otimes F]Y(s) \\ - c[(L+G) \otimes F]\tilde{Y}_0 + c[(L+G) \otimes F]\mathrm{H}(s) + \hat{x}(0)\}$$
$$\hat{Y}(s) = (I_{Nn} \otimes C)\{sI_{Nn} - [(I_N \otimes A) - c(L+G) \otimes FC]\}^{-1}\{(I_N \otimes B)U(s) + c[(L+G) \otimes F]Y(s) \\ - c[(L+G) \otimes F]\tilde{Y}_0 + c[(L+G) \otimes F]\mathrm{H}(s) + \hat{x}(0)\}$$
$$G_\eta(s) = \frac{\hat{Y}(s)}{\mathrm{H}(s)} = \{(I_{Nn} \otimes C)\{sI_{Nn} - [(I_N \otimes A) - c(L+G) \otimes FC]\}^{-1}\{(I_N \otimes B)U(s) + c[(L+ \\ + G) \otimes F]Y(s) - c[(L+G) \otimes F]\tilde{Y}_0 + c[(L+G) \otimes F]\mathrm{H}(s) + \hat{x}(0)\}\}/\mathrm{H}(s)$$
$$G_\eta(s) = \frac{\hat{Y}(s)}{\mathrm{H}(s)}\bigg|_{\substack{U=0\\Y=0\\\tilde{Y}_0=0}} = \frac{(I_{Nn} \otimes C)\{sI_{Nn} - [(I_N \otimes A) - c(L+G) \otimes FC]\}^{-1}\{c[(L+G) \otimes F]\mathrm{H}(s)\}}{\mathrm{H}(s)}$$
$$G_\eta(s) = \frac{\hat{Y}(s)}{\mathrm{H}(s)}\bigg|_{\substack{U=0\\Y=0\\\tilde{Y}_0=0}} = (I_{Nn} \otimes C)\{sI_{Nn} - [(I_N \otimes A) - c(L+G) \otimes FC]\}^{-1}\{c[(L+G) \otimes F]\}$$

Neighborhood Controller and Local Observer

In cooperative multi-agent systems on graphs, achieving synchronization does not require the use of neighbor information in both the control protocol and the observer protocol. It is sufficient for either one of them to contain neighbor information. The OPFB cooperative regulator, developed in this section, employs the neighborhood estimate tracking error information $\hat{\varepsilon}_i$ for controller design, while relying exclusively on the local agent output estimation error information $\tilde{y}_i$ for observer design. In this context, the term "local" specifically refers to the information pertaining to the $i-th$ node itself.
For each node i, the controller uses complete neighbor information and provides an input u_i where the observers use only local agent information $\hat{x}$ so each one is designed as a classical Luenberger observer where the observer gain is $-cF$.

$$u_i = cK\hat{\varepsilon}_i = cK\left(\sum_{j=1}^{N} a_{ij}(\hat{x}_i - \hat{x}_j) + g_i(\hat{x}_0 - \hat{x}_i)\right) \qquad \dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF\tilde{y}_i$$

The closed-loop dynamics of each node is

$$\begin{gathered}\dot{x}_i = Ax_i + Bu_i\\ \dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF(y_i - \hat{y}_i)\end{gathered}$$

The global closed-loop dynamics can be written as

$$\begin{aligned}\dot{x} &= (I_N \otimes A)x + (I_N \otimes B)u =\\ &= (I_N \otimes A)x - c\left[(L+G)\otimes BK\right](\hat{x} - \bar{x}_0) \overset{*}{=} \qquad\qquad * \ \hat{x} = x - \tilde{x}\\ &\overset{*}{=} (I_N \otimes A)x - c[(L+G)\otimes BK](x - \tilde{x} - \bar{x}_0) =\\ &= \{(I_N \otimes A) - c[(L+G)\otimes BK]\}\, x + c[(L+G)\otimes BK](\tilde{x} + \bar{x}_0) = A_c x + B_c(\tilde{x} + \bar{x}_0)\\ \dot{\hat{x}} &= (I_N \otimes A)\hat{x} + (I_N \otimes B)u - (I_N \otimes cF)(y - \hat{y}) =\\ &= (I_N \otimes A)\hat{x} - c[(L+G)\otimes BK](\hat{x} - \bar{x}_0) - (I_N \otimes cF)(Cx - C\hat{x}) =\\ &= (I_N \otimes A)\hat{x} - c[(L+G)\otimes BK](\hat{x} - \bar{x}_0) - (I_N \otimes cFC)x + (I_N \otimes cFC)\hat{x} =\\ &= [I_N \otimes (A + cFC)]\hat{x} - c[(L+G)\otimes BK](\hat{x} - \bar{x}_0) - (I_N \otimes cFC)x\end{aligned}$$

Given the global disagreement error δ and the global state estimation $\tilde{x}$ their dynamics is

$$\begin{aligned}\dot{\delta} &= \dot{x} - \dot{\bar{x}}_0 = (I_N \otimes A)x - c[(L+G)\otimes BK](\hat{x} - \bar{x}_0) - (I_N \otimes A)\bar{x}_0 =\\ &= (I_N \otimes A)(x - \bar{x}_0) - c[(L+G)\otimes BK](x - \tilde{x} - \bar{x}_0) =\\ &= [(I_N \otimes A) - c(L+G)\otimes BK](x - \bar{x}_0) + c[(L+G)\otimes BK]\tilde{x} = A_c\delta + B_c\tilde{x}\\ \dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} = \{[(I_N \otimes A) - c(L+G)\otimes BK]x + c[(L+G)\otimes BK](\tilde{x} + \bar{x}_0)\} - \{[I_N \otimes (A+\\ &\quad + cFC)]\hat{x} - c[(L+G)\otimes BK](\hat{x} - \bar{x}_0) - (I_N \otimes cFC)x\} =\\ &= (I_N \otimes A)x - c[(L+G)\otimes BK]x + c[(L+G)\otimes BK]\tilde{x} + c[(L+G)\otimes BK]\bar{x}_0 - [I_N\otimes\\ &\quad \otimes (A + cFC)]\hat{x} + c[(L+G)\otimes BK]\hat{x} - c[(L+G)\otimes BK]\bar{x}_0 + (I_N \otimes cFC)x =\\ &= [I_N \otimes (A + cFC)]x - c[(L+G)\otimes BK]x + c[(L+G)\otimes BK]x - c[(L+G)\otimes BK]\hat{x}\\ &\quad + c[(L+G)\otimes BK]\bar{x}_0 - [I_N \otimes (A + cFC)]\hat{x} + c[(L+G)\otimes BK]\hat{x} - c[(L+G)\otimes BK]\bar{x}_0\\ &= [I_N \otimes (A + cFC)]x - [I_N \otimes (A + cFC)]\hat{x} =\\ &= [I_N \otimes (A + cFC)](x - \hat{x}) = [I_N \otimes (A + cFC]\tilde{x}\end{aligned}$$

Defined the LTI system which represents the disagreement and estimate error equations

$$\begin{bmatrix}\dot{\delta}\\ \dot{\tilde{x}}\end{bmatrix} = \begin{bmatrix}A_c & B_c\\ 0 & I_N \otimes (A + cFC)\end{bmatrix}\begin{bmatrix}\delta\\ \tilde{x}\end{bmatrix}$$

The system is asymptotically stable only if the state matrix, which is a block triangular matrix, is Hurwitz, so when the eigenvalues of its diagonal blocks are themselves Hurwitz.

$$eig\left(\begin{bmatrix} A_c & B_c \\ 0 & I_N \otimes (A + cFC) \end{bmatrix}\right) = \{eig(A_c),\ eig(A + cFC)\}$$

A_c is Hurwitz because of the SVFB theorem mentioned above therefore the disagreement error δ goes to zero asymptotically, but the state estimation error $\tilde{x}$ is not guaranteed to be driven to zero as the time increases, since $I_N \otimes (A + cFC)$ is not said to be Hurwitz, thus F has to be designed such that $(A + cFC)$ has all eigenvalues with a negative real part. Furthermore to guarantee the stability of the whole distributed system F must be designed using the local Riccati equation but with the opposite sign $F = -PC^\top R^{-1}$.

$$\begin{matrix} A_c & Hurwitz \\ (A + cFC) & Hurwitz \end{matrix} \implies \begin{matrix} \lim_{t\to\infty} \delta(t) = 0 \\ \lim_{t\to\infty} \tilde{x}(t) = 0 \end{matrix}$$

In the end, given a distributed LTI system where the nodes have matrices A, B, C stabilizable and detectable and are connected by a graph which has a spanning tree from the leader node, designed a local agent observer for each node with observer gain F controlled by the OPFB cooperative dynamic tracking law based on the SVFB gain K and coupling gain c, then all nodes syncronyze to the leader asymptotically and the state estimation approaches zero asymptotically.

We prove in the following demostration how the noise affects the estimate of the output made by each single Luenberger observer from a local view, where in this case L refers to the general gain observer.

$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i + L(\tilde{y}_i + \eta_i)$$
$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF(\tilde{y}_i + \eta_i)$$
$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF(y_i - \hat{y} + \eta_i)$$
$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cF(y_i - \hat{y}_i + \eta_i)$$
$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cFy_i + cF\hat{y}_i - cF\eta_i$$
$$\dot{\hat{x}}_i = A\hat{x}_i + Bu_i - cFy_i + cFC\hat{x}_i - cF\eta_i$$
$$\dot{\hat{x}}_i = (A + cFC)\hat{x}_i + Bu_i - cFy_i - cF\eta_i$$
$$\mathcal{L}\{\dot{\hat{x}}_i\} = \mathcal{L}\{(A + cFC)\hat{x}_i + Bu_i - cFy_i - cF\eta_i\}$$
$$s\hat{X}_i(s) - \hat{x}_i(0) = (A + cFC)\hat{X}_i(s) + BU_i(s) - cFY_i(s) - cF\mathrm{H}_i$$
$$s\hat{X}_i(s) - (A + cFC)\hat{X}_i(s) = BU_i(s) - cFY_i(s) - cF\mathrm{H}_i + \hat{x}_i(0)$$
$$[s - (A + cFC)]\hat{X}_i(s) = BU_i(s) - cFY_i(s) - cF\mathrm{H}_i + \hat{x}_i(0)$$
$$\hat{X}_i(s) = [sI_n - (A + cFC)]^{-1}[BU_i(s) - cFY_i(s) - cF\mathrm{H}_i(s) + \hat{x}_i(0)]$$
$$\hat{X}_i(s) = [sI_n - (A + cFC)]^{-1}[BU_i(s) - cFY_i(s) - cF\mathrm{H}_i(s) + \hat{x}_i(0)]$$
$$\hat{Y}_i(s) = C[sI_n - (A + cFC)]^{-1}[BU_i(s) - cFY_i(s) - cF\mathrm{H}_i(s) + \hat{x}_i(0)]$$

$$G_\eta(s) = \frac{\hat{Y}_i(s)}{\mathrm{H}_i(s)} = \frac{C[sI_n - (A + cFC)]^{-1}[BU_i(s) - cFY_i(s) - cF\mathrm{H}_i + \hat{x}_i(0)]}{\mathrm{H}_i(s)}$$

$$G_\eta(s) = \left.\frac{\hat{Y}_i(s)}{\mathrm{H}_i(s)}\right|_{\substack{U=0 \\ Y=0}} = -C[sI_n - (A + cFC)]^{-1}cF$$