Code for KTI and KTL quadrature formulas

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We want to present a brief pseudo-code for the calculation of the KTL and KTI quadrature weights using the Kosloff Tal-Ezer map summarized in Algorithm 1 and Algorithm 2.

Algorithm 1: KTL quadrature formula

Input:
- $\mathcal{X} = \{x_i,\, i = 0, ..., m\} \subseteq \mathbb{R}$: quadrature nodes;
- $n \leq m$: polynomial degree for the approximation space;
- $\boldsymbol{\mu} = (\sqrt{\mu_0}, ..., \sqrt{\mu_m})^\top$: weights for least squares problem;
- $\mathbf{f} = (f(x_0), ..., f(x_m))^\top$: sample vector of f on $\mathcal{X}$;
- $M_\alpha : [-1, 1] \longrightarrow [-1, 1]$: Kosloff-Tal-Ezer map with parameter α.

```
begin
    Compute moments τ^α ∈ ℝ^{n+1} through discrete cosine transform of g_α;
    Build diagonal matrix with the weights for the least-squares problem:
      W = diag(√μ_0, ..., √μ_m) ∈ ℝ^{m+1} × ℝ^{m+1};
    Construct matrix: A^α ∈ ℝ^{m+1} × ℝ^{n+1}, A^α_{ij} = T_{j-1}(M_α(x_{i-1})), for i = 1, ..., m+1, and
      j = 1, ..., n+1;
    Find coefficient vector γ as the solution of the least-squares problem W A^α γ = W f;
    Compute the quadrature value I^α_{n,X}(f, I) = γ^⊤ τ^α.
end
```

Output: The value of the KTL quadrature $\mathcal{I}^\alpha_{n,\mathcal{X}}(f, I)$.

Algorithm 2: KTI quadrature formula

Input:
- $\mathcal{X} = \{x_i,\, i = 0, ..., m\} \subseteq \mathbb{R}$: quadrature nodes, $m = n = \dim(\mathbb{P}_n^\alpha) - 1$;
- $\mathbf{f} = (f(x_0), ..., f(x_m))^\top$: vector of function samples on $\mathcal{X}$;
- $M_\alpha : [-1, 1] \longrightarrow [-1, 1]$: Kosloff-Tal-Ezer map with parameter α.

```
begin
    Compute moments τ^α ∈ ℝ^{m+1} through discrete cosine transform of g_α;
    Construct interpolation matrix: A^α ∈ ℝ^{m+1} × ℝ^{m+1}, A^α_{ij} = T_{j-1}(M_α(x_{i-1}));
    Find the quadrature weights w^α as the solution of the linear system (A^α)^⊤ w^α = τ^α.;
    Compute the quadrature value I^α_{m,X}(f, I) = (w^α)^⊤ f.
end
```

Output: The value of the KTI quadrature $\mathcal{I}^\alpha_{m,\mathcal{X}}(f, I)$.

For clarity we introduce the following definitions:

- Auxiliary function for the computation of the moments:

$$g_\alpha(t) = \frac{\sin(t)}{\sqrt{\frac{1}{\sin^2(\alpha\pi/2)} - \cos^2(t)}} \frac{1}{\alpha}, \quad t \in [0, \pi];$$

- Chebyshev polynomials: $\{T_i(x) = \cos(i \arccos(x)),\, i = 0, \ldots, n\}$.