Assignment 8

Renormalization Group

Quantum Information and Computing

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Theory

We consider the 1D Quantum Ising Model Hamiltonian:

$$H = g\sum_{i=1}^{N}\sigma_i^x + \sum_{i=1}^{N-1}\sigma_i^z\sigma_{i+1}^z$$

We want to study the ground state energy of this Hamiltonian using two iterative methods for many-bodies:

1- Real space renormalization group (RSRG):

- Double the system size at every step combining two identical blocks
- Diagonalize H and project onto the lowest energy eigenvectors
- Stop after a convergence criterion

2 - Infinite space renormalization group (iDMRG):

- Add one site at a time
- Diagonalize the reduced density matrix and keep states with largest eigenvalues
- Stop after a convergence criterion

Algorithm

RSRG

- **Starting configuration**:
  - $H^{(0)} = \Sigma_i^{N-1} \sigma_i^x \sigma_{i+1}^x + g\Sigma_i^N \sigma_i^z$
  - $A^{(0)} = \mathbf{1}_{N-1} \otimes \sigma^x$
  - $B^{(0)} = \sigma^x \otimes \mathbf{1}_{N-1}$
- **Steps**:
  1. Build the enlarged system H_{2N}
  2. Compute the 2^N lowest eigenvectors and build the projector P
  3. Compute the energy density and check its convergence up to a threshold $\Delta E_n < \tau$
  4. If not converged: project the Hamiltonian with P, otherwise repeat 1-3

Algorithm
iDMRG

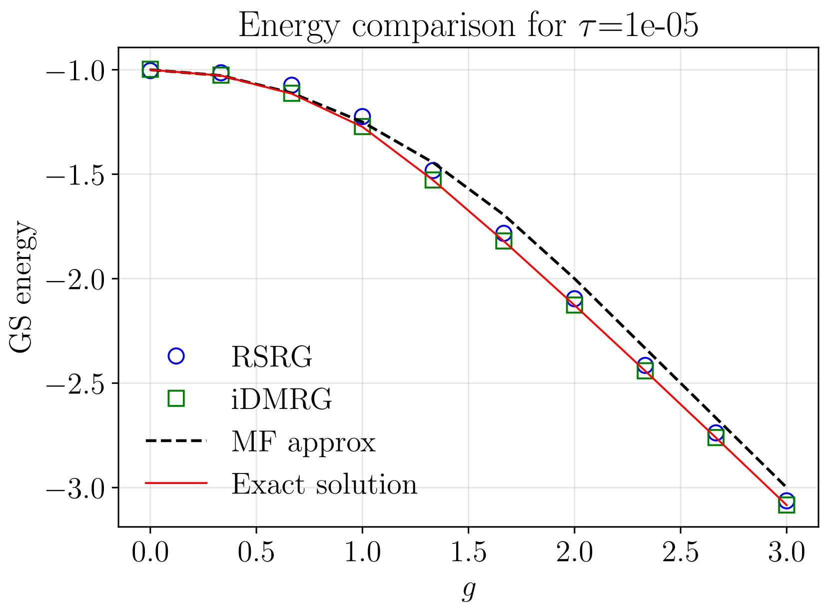
- **Starting configuration** (m=1):
  - $H^{(0)} = g\sigma^x$
  - $A^{(0)} = \sigma^z$
  - $B^{(0)} = \mathbf{1}_m$
- **Steps**:
  1. Enlarge the left block by adding one site
  2. Build the superblock Hamiltonian of size $2m + 2$, find the ground state and build the reduced density matrix ρ tracing out the right block
  3. Build P selecting the k eigenvectors of ρ associated with the largest eigenvalues
  4. Project H, compute the energy density and check its convergence, if not converged repeat 1-3

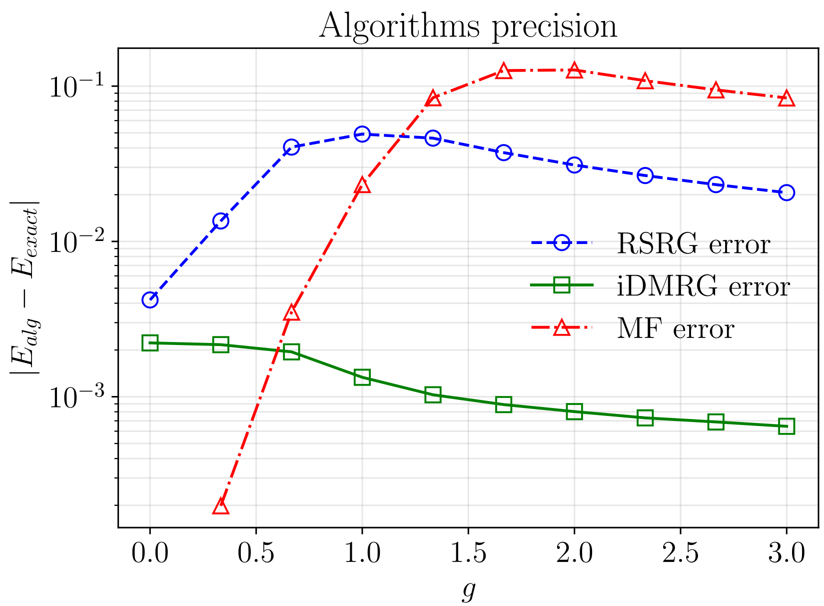
Algorithms analysis

Energy comparison

Compute the Ground State energies for $g \in [0,3]$:

- RSRG and iDMRG converge to the same solution
- More accurate with respect to the exact solution than the Mean-field approximation
- As g increases, the absolute error is lower for iDMRG compared to RSRG and MF





n.b. Exact solution computed following the reference paper: *The One-Dimensional Ising Model with a Transverse Field* (https://www.math.ucdavis.edu/~bxn/pfeuty1970.pdf)

Algorithms analysis

Convergence speed

Convergence speed benchmark confirm the nature of the two algorithms:

- **RSRG**: double the system size at each iteration, it reaches the thermodynamic limit faster in much lower iterations for each value of g
- **iDMRG**: increase the system size by 1 at each iteration, so it needs a higher number of iterations to converge, even if it decreases as g increases