Assignment 7

Quantum Ising Model

Quantum Information and Computing

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Theory

Quantum Ising Hamiltonian: chain of $1/2$-spin bodies

Two-bodies interaction

$$\hat{H} = \boxed{-g\sum_{i}^{N}\sigma_i^x}\boxed{-\sum_{i}^{N-1}\sigma_i^z\sigma_{i+1}^x}\boxed{-h\sum_{i}^{N}\sigma_i^z}$$

Transverse field interaction **Longitudinal field interaction**

Pauli matrices are defined as: $\sigma_i^z = \mathbf{1}_1 \otimes \ldots \otimes \mathbf{1}_{i-1} \otimes \sigma^z \otimes \mathbf{1}_{i+1} \otimes \ldots \otimes \mathbf{1}_N$

Workflow:

- Diagonalize the Hamiltonian
- Study its behaviour for $g \in [0,3]$
- Compute the average magnetization $m = \langle \frac{1}{N}\sum_{i}^{N}\sigma_i^z \rangle$

Theory

Code efficiency

Setting $h = 0$ and $g = 0.5$ we benchmark the performance:

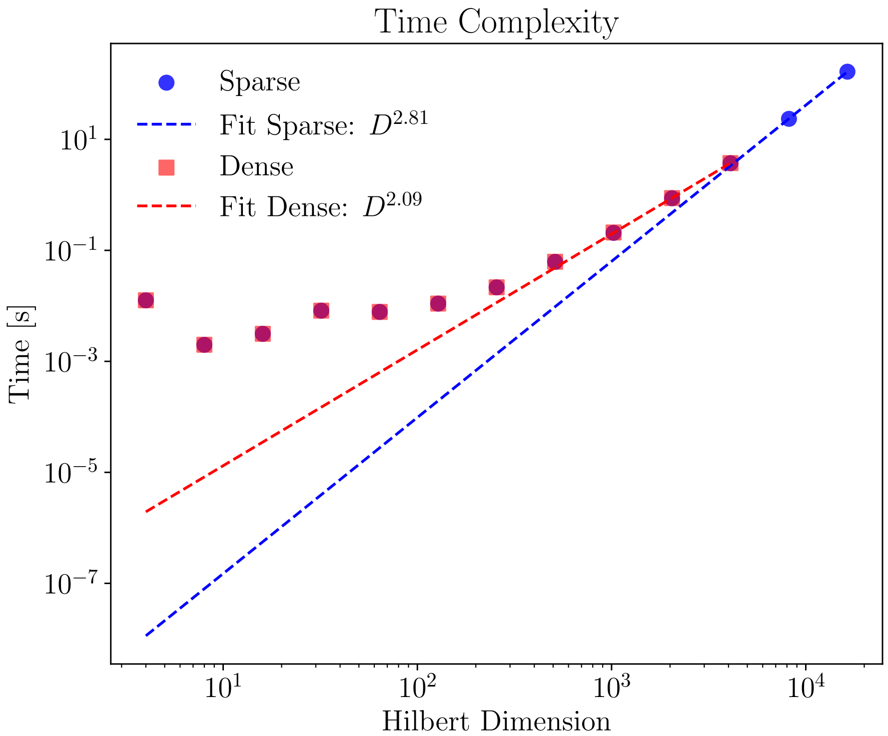
- **Diagonalize** Hamiltonians up to a max dimension $N_{max} = 14$
- **Time complexity** for both Dense and Sparse representations
- **Memory complexity** for both Dense and Sparse representations

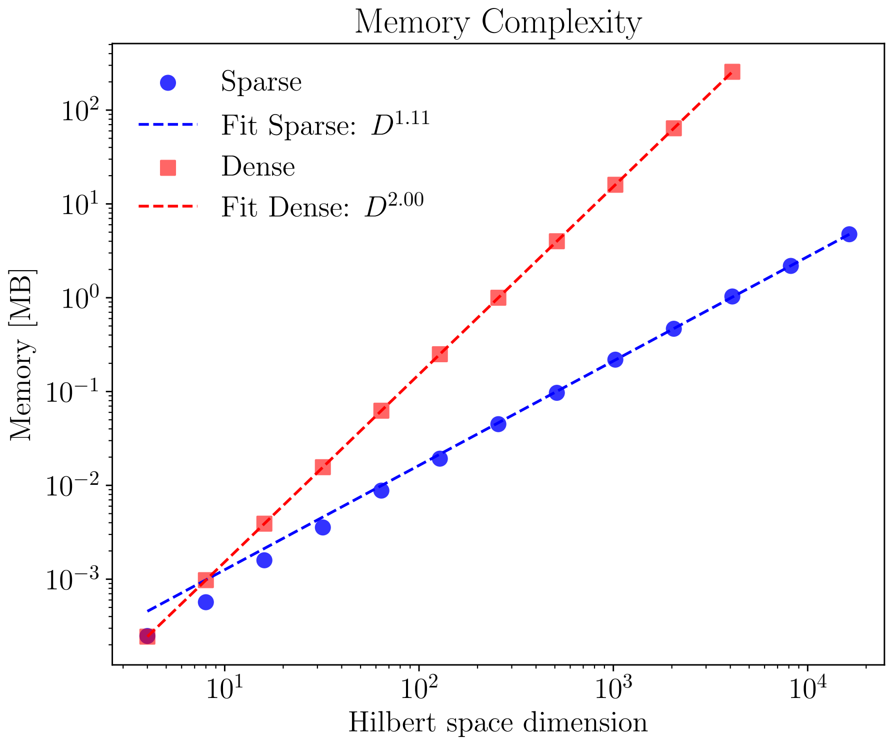
Hilbert space dimension: $D = 2^N$. The matrix dimension increases exponentially with N.

Since the **interactions** between particles are **local** we expect a matrix with many entries equal to zero. For that reason it could be advantageous to store it in memory as a **sparse matrix.**

Advantage of Sparse matrix representation

Time and Memory benchmark





Hilbert space dimension: $D = 2^N$

Memory complexity:

Dense: $D \times D$ matrix to store, complexity $\propto D^2$

Sparse: store only non-zero elements, complexity $\propto D$

No advantage in time but great advantage in memory

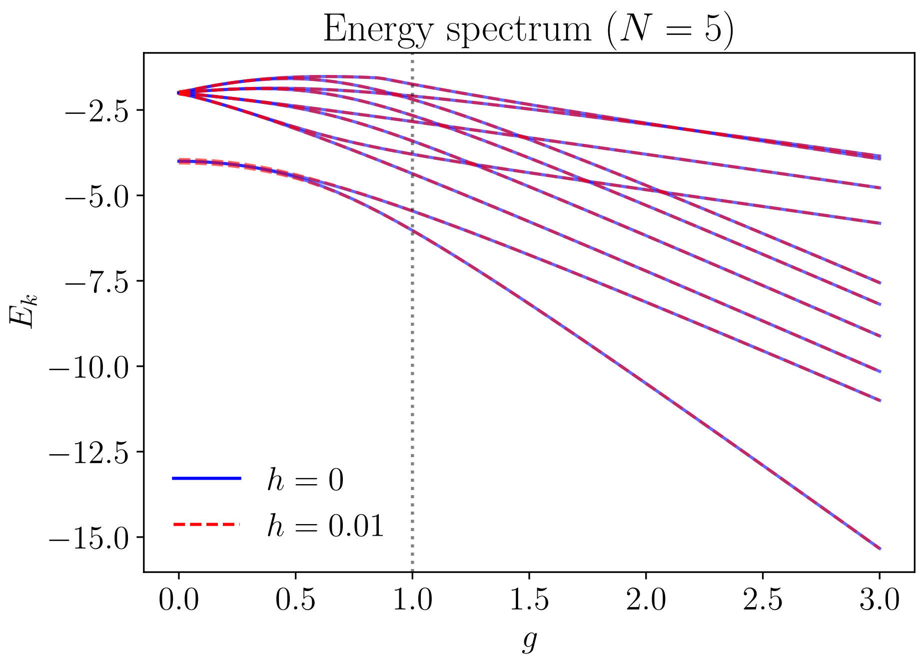
Energy levels

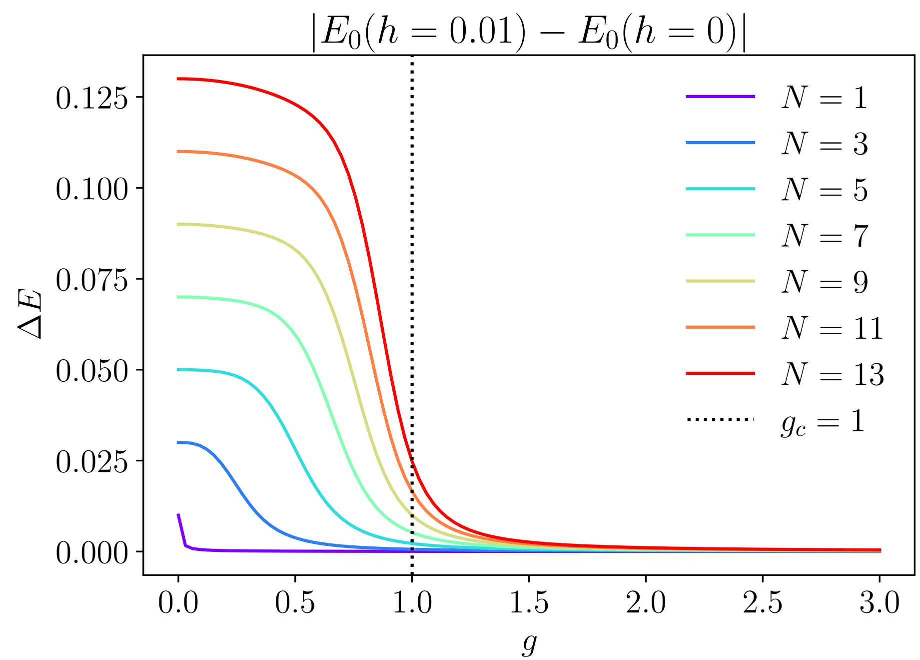
Compute the first $k = 10$ energy levels.

We can distinguish the three phases of the Ising model:

1. $g < 1$: **Ordered phase**, degenerate ground state
2. $g \approx 1$: **Critical point**, quantum phase transition
3. $g > 1$: **Disordered phase**, unique ground state

Lines seem to overlap for both $h = 0$ and $h = 0.01$ but if we look at the energy difference, for low values of g there are some discrepancies





Average magnetization

Quantum phase transition

- $g < 1$: **Ordered phase**, interaction term ($\sum_i^N \sigma_i^z \sigma_{i+1}^z$) dominates. Spins align with the longitudinal field ($h \sum_i^N \sigma_i^z$)

- $g > 1$: **Disordered phase**, transverse field term ($g \sum_i^N \sigma_i^x$) dominates. Spins align along the x-axis

- $h = 0$**: Symmetry is preserved**. The degenerate states are mixed ($|\uparrow\rangle$, $|\downarrow\rangle$), leading to zero net magnetization.
- $h = 0.01$**: Symmetry breaking.** An infinitesimal field breaks the symmetry, selecting the ferromagnetic ground state (all $|\uparrow\rangle$)