Università di Pisa

Facoltà di Matematica

Finiteness of Multiplicatively Dependent n-tuples of Singular Moduli

Master Thesis
in Mathematics

Candidate
Giacomo Hermes Ferraro

Advisor
Tamas Szamuely
Università di Pisa

Academic Year 2020 - 2021

Contents

Introduction

In this thesis, our aim is to present a recent result proved by Yves André in [And18], building on some results from [Bha14]: Hochster's direct summand conjecture in commutative algebra; we will focus on the approach given by Bhargav Bhatt in his article [Bha18], in which he streamlines André's original proof. In particular, we will devote two chapters respectively to Gerd Faltings' almost mathematics and Peter Scholze's perfectoid theory, as developed in [Sch12].

Hochster's original conjecture asks if, given a module-finite extension of noetherian rings $i : A_0 \hookrightarrow B_0$ with A_0 regular, the inclusion splits as a map of A_0-modules. Many partial solutions to Hochster's conjecture have been known for quite a long time. For example, Hochster himself reduced the problem to the case of A_0 being local and complete in [Hoc73]. In that same article, Hochster solved the conjecture in the case of equal characteristic, i.e. when A_0 contains a copy of a field. The first chapter will focus on these simplifications, and will contain an easy proof of the equal case in characteristic 0. In their articles, André and Bhatt proved the remaining case of characteristic $(0, p)$, which will be the focus of the rest of the this thesis.

The first approach to simplify the problem is as follows. The short exact sequence of A_0-modules $0 \to A_0 \to B_0 \to Q_0 \to 0$ corresponds to an element $\alpha_0 \in \mathrm{Ext}^1_{A_0}(A_0, Q_0)$, and the sequence splits iff $\alpha_0 = 0$. If we consider a faithfully flat extension $A_0 \to A$ and apply the tensor product we obtain a new short exact sequence $0 \to A \to B \to Q \to 0$, with corresponding element $\alpha \in \mathrm{Ext}^1_A(A, Q)$. The induced map $\mathrm{Ext}^1_{A_0}(A_0, Q_0) \to \mathrm{Ext}^1_A(A, Q)$ sends α_0 to α, and by faithful flatness $\alpha = 0$ iff $\alpha_0 = 0$, so it is sufficient to prove that the tensorized sequence splits (in particular, this happens if $A \hookrightarrow B$ is étale). Similar arguments help reduce the problem from the general to the local case, and from the local to the complete case.

It turns out that only working with A_0-modules is too stringent: Faltings' almost mathematics allows - after the extension of A_0 to a suitably large ring A - to consider the more flexible category of almost-A-modules. This theory will be explored in chapter 2, with the transposition of many definitions and properties in this new language (starting with almost zero modules and concluding with almost finite étale extensions). For a suitable A, Faltings' almost purity theorem states that if the map $A \to B$ is étale after inverting p, it is almost étale.

For a full appreciation of Faltings' almost purity theorem, we will need to introduce Scholze's perfectoid theory, which will be done in chapter 3. This new language we will allow us to give a nice formulation of the theorem, and with some fundamental results of perfectoid theory we will be able to construct a suitable extension A for its application, by pushing all the ramification of the map $A_0 \hookrightarrow B_0$ in p.

Finally, chapter 4 will contain a brief description of pro-modules and almost-pro-modules, while chapter 5 will tackle the main theorem, after presenting some additional useful lemmas.

Chapter 1

Hochster's original work

In this chapter we will follow Hochster's first steps towards the solution of the conjecture, which naturally lead to the solution of the case $\mathrm{char} A = 0$.

1.1 Strengthening of the hypothesis

The first and foremost result from Hochster is the following, which establishes that his conjecture is a local problem.

Proposition 1.1.1. *Let $A \hookrightarrow B$ be a module-finite extension of noetherian rings. Then:*

1. *$A \hookrightarrow B$ splits (as a map of A-modules) if and only if $A_{\mathfrak{p}} \hookrightarrow B_{\mathfrak{p}}$ splits (as a map of $A_{\mathfrak{p}}$-modules) for every $\mathfrak{p} \subseteq A$ prime ideal;*

2. *similarly, if $A \hookrightarrow A'$ is a faithfully flat extension, $A \hookrightarrow B$ splits if and only if $A' \hookrightarrow B \otimes_A A'$ splits.*

1.1.1.i tells us that Hochster's problem is local in nature, while 1.1.1.ii will allow us to use the completion of A instead of A, since it is a faithfully flat extension (see [Sta21, Lemma 10.97.3]).

First, let's remind what we mean by *faithfully flat* extension with the following definition and lemma.

Definition 1.1.2. Let R be a commutative ring, M an R-module. M is faithfully flat if for every sequence $N_1 \to N_2 \to N_3$, it is exact at N_2 if and only if $M \otimes N_1 \to M \otimes N_2 \to M \otimes N_3$ is exact at $M \otimes N_2$. An extension $R \to R'$ is faithfully flat if R' is faithfully flat as an R-module.

Lemma 1.1.3. *Let R be a commutative ring, M a flat R-module. The following are equivalent:*

1. *M is faithfully flat;*

2. *For every pair of R-modules P and Q the natural map $\mathrm{Hom}_R(P, Q) \to \mathrm{Hom}_R(M \otimes P, M \otimes Q)$ is injective;*

3. *For every R-module N, $M \otimes N = 0$ if and only if $N = 0$.*

Proof. First, let's prove (2) $\Leftrightarrow$ (3). Assume (2) is true, and take an R-module N such that $N \otimes M = 0$. We have an injective map: $N \cong \mathrm{Hom}_R(R, N) \hookrightarrow \mathrm{Hom}_R(R \otimes M, N \otimes M) \cong \mathrm{Hom}_R(M, 0) \cong 0$, therefore $N = 0$.

Assume (3) is true. To prove (2), we need to show that for any two R-modules P, Q and any map $f : P \to Q$, if the induced map $f \otimes id_M : P \otimes M \to Q \otimes M$ is zero, then $f = 0$. We can write $f = g \circ h$, with $h : P \to \mathrm{im}(f)$ surjective and $g : \mathrm{im}(f) \to Q$ injective. Since M is flat, $g \otimes id_M$ is injective and $h \otimes id_M$ is surjective, so $f \otimes id_M = 0$ iff $\mathrm{im}(f) \otimes M = 0$, which by hypothesis implies $\mathrm{im}(f) = 0$, i.e. $f = 0$.

Now, let's prove (1) $\Leftrightarrow$ (3). Assume (3) is true and take $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$, a sequence of R-modules such that $M \otimes N_1 \to M \otimes N_2 \to M \otimes N_3$ is exact. Since M is flat, we have that: $\frac{\ker(g)}{\mathrm{im}(f)} \otimes M \cong \frac{\ker(g) \otimes M}{\mathrm{im}(f) \otimes M} \cong \frac{\ker(g \otimes M)}{\mathrm{im}(f \otimes M)} \cong 0$, therefore $\frac{\ker(g)}{\mathrm{im}(f)} = 0$ and $N_1 \to N_2 \to N_3$ is exact.

Vice versa, assume (1) is true and consider the sequence $0 \to N \to 0$. If $M \otimes N = 0$, the tensorized sequence is $0 \to 0 \to 0$, which is exact: this implies that $0 \to N \to 0$ is exact, so $N = 0$. □

We need the following lemma.

Lemma 1.1.4. *Let R be a commutative ring and M a finitely presented R-module. If R' is a flat ring over R, then for every R-module N the following natural map is an isomorphism:*

$$\alpha_{M,N} : \mathrm{Hom}_R(M, N) \otimes_R R' \to \mathrm{Hom}_R(M, N \otimes_R R').$$

Proof. If $M \cong R^n$ for some finite n, both arguments are naturally isomorphic to $(N \otimes_R R')^n$ and the isomorphisms carry $\alpha_{M,N}$ to the identity map.

In general, since M is finitely presented, there is an exact sequence $R^k \to R^n \to M \to 0$. The functors $\mathrm{Hom}_R(\cdot, N) \otimes_R R'$ and $\mathrm{Hom}_R(\cdot, N \otimes_R R')$ are both left exact, thus we get the following diagram:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{Hom}_R(M, N) \otimes_R R' \arrow[r] \arrow[d, "\alpha_{M,N}"] & \mathrm{Hom}_R(R^n, N) \otimes_R R' \arrow[r] \arrow[d, "\alpha_{R^n,N}"] & \mathrm{Hom}_R(R^k, N) \otimes_R R' \arrow[d, "\alpha_{R^k,N}"] \\
0 \arrow[r] & \mathrm{Hom}_R(M, N \otimes_R R') \arrow[r] & \mathrm{Hom}_R(R^n, N \otimes_R R') \arrow[r] & \mathrm{Hom}_R(R^k, N \otimes_R R')
\end{tikzcd}$$

The diagram commutes because α is a natural transformation, and since $\alpha_{R^n,N}$ and $\alpha_{R^k,N}$ are both isomorphism, $\alpha_{M,N}$ is also an isomorphism by the five lemma. □

Corollary 1.1.5. *Using the properties of the tensor product, we can also deduce a natural isomorphism:*

$$\mathrm{Hom}_R(M, N) \otimes_R R' \cong \mathrm{Hom}_{R'}(M \otimes_R R', N \otimes_R R')$$

Now onto the proof of Proposition 1.1.1.

Proof. First of all, let's call $C := B/A$ and observe that $A \hookrightarrow B$ splits if and only if $B \twoheadrightarrow C$ splits, i.e. if the natural map $\mathrm{Hom}_A(C, B) \to \mathrm{Hom}_A(C, C)$ is surjective. This happens if and only if the localization of this map in $\mathfrak{p}$ is surjective for every prime ideal $\mathfrak{p} \subset A$.

B is module-finite over A, so C is a finitely generated A-module, and since A is noetherian C is also finitely presented: by the previous lemma, the functors $\mathrm{Hom}_A(C, \cdot) \otimes_A A_\mathfrak{p}$ and $\mathrm{Hom}_{A_\mathfrak{p}}(C \otimes_A A_\mathfrak{p}, \cdot \otimes_A A_\mathfrak{p})$ are naturally isomorphic. In particular, we have the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_A(C, B) \otimes_A A_\mathfrak{p} \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_A(C, C) \otimes_A A_\mathfrak{p} \arrow[d, "\cong"] \\
\mathrm{Hom}_{A_\mathfrak{p}}(C_\mathfrak{p}, B_\mathfrak{p}) \arrow[r] & \mathrm{Hom}_{A_\mathfrak{p}}(C_\mathfrak{p}, C_\mathfrak{p})
\end{tikzcd}$$

The arrow above is surjective if and only if the arrow below is surjective, which happens if and only if the inclusion $A_p \hookrightarrow B_p$ splits.

For the second point, let's first observe that $\mathrm{Hom}_A(C,B) \to \mathrm{Hom}_A(C,C)$ is surjective if and only if $\mathrm{Hom}_A(C,B) \otimes_A A' \to \mathrm{Hom}_A(C,C) \otimes_A A'$ is surjective, because $A \hookrightarrow A'$ is faithfully flat. With the same reasoning as before we get to the commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_A(C,B) \otimes_A A' \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}_A(C,C) \otimes_A A' \arrow[d, "\sim"] \\
\mathrm{Hom}_{A'}(C \otimes_A A', B \otimes_A A') \arrow[r] & \mathrm{Hom}_{A'}(C \otimes_A A', C \otimes_A A')
\end{tikzcd}$$

and since the arrow below is surjective if and only if $A' \hookrightarrow B \otimes_A A'$ splits, the proposition is proven. $\square$

In the following section, we will include a short proof of the theorem in the case of equal characteristic 0. The study of the finite characteristic will not be included, but a concise proof can be found in [Hoc73, Theorem 2].

1.2 Characteristic 0

Let's prove that we can assume the extensions to be domains with the following proposition.

Proposition 1.2.1. *Let A be a domain and $i : A \to B$ be a module-finite extension of commutative rings. There is a prime $\mathfrak{q} \subset B$ (with projection $\pi_{\mathfrak{q}} : B \to B/\mathfrak{q}$) such that:*

1. *the composite map $\pi_{\mathfrak{q}} \circ i : A \to B/\mathfrak{q}$ is injective;*
2. *if $A \to B/\mathfrak{q}$ splits, then $A \to B$ splits.*

Vice versa, if $i : A \to B$ splits we can choose $\mathfrak{q} \subset B$ prime such that $\pi_{\mathfrak{q}} \circ i : A \to B/\mathfrak{q}$ is injective and splits.

Proof. Since A is a domain, $S := i(A \setminus \{0\})$ is a multiplicatively closed set in B. Take a maximal ideal in $S^{-1}B$: it corresponds to a prime ideal $\mathfrak{q} \subset B$ with empty intersection with S. In particular, the composite map $\pi_{\mathfrak{q}} \circ i : A \to B/\mathfrak{q}$ is injective.

Moreover, if $r_{\mathfrak{q}} : B/\mathfrak{q} \to A$ is a retraction for $\pi_{\mathfrak{q}} \circ i$, we have $r_{\mathfrak{q}} \circ \pi_{\mathfrak{q}} \circ i = id_A$, thus $r_{\mathfrak{q}} \circ \pi_{\mathfrak{q}}$ is a retraction for $i : A \to B$.

Vice versa, if $r : B \to A$ is a retraction for i, set $\mathfrak{q} := \ker(r)$ (which is prime because A is a domain): r factorizes as $r_{\mathfrak{q}} \circ \pi_{\mathfrak{q}}$, where $r_{\mathfrak{q}} : B/\mathfrak{q} \to A$ is the map induced on the quotient, and since $id_A = r \circ i = r_{\mathfrak{q}} \circ \pi_{\mathfrak{q}} \circ i$, the map $\pi_{\mathfrak{q}} \circ i : A \to B/\mathfrak{q}$ is injective with retraction $r_{\mathfrak{q}}$. $\square$

With this result, it's easy to tackle the first case, i.e. when A has characteristic 0. First, two preliminary lemmas.

Lemma 1.2.2. *A ring A has characteristic 0 iff it contains a copy of $\mathbb{Q}$.*

Proof. Let $i : \mathbb{Q} \hookrightarrow A$. Since every element of $\mathbb{Q} \setminus \{0\}$ has an inverse, $i(\mathbb{Q}) \cap \mathfrak{p} = \{0\}$ for every prime ideal $\mathfrak{p} \subset A$, therefore the composite map $\pi_{\mathfrak{p}} \circ i : \mathbb{Q} \to A/\mathfrak{p}$ is injective, and $A/\mathfrak{p}$ has characteristic 0.

On the other hand, if A has characteristic 0, there is an immersion $i : \mathbb{Z} \hookrightarrow A$, and since $A/\mathfrak{p}$ has characteristic 0, it follows that $\mathfrak{p} \cap i(\mathbb{Z}) = \{0\}$ for every prime ideal $\mathfrak{p}$. This means that every element in $i(\mathbb{Z}) \setminus \{0\}$ is invertible, therefore the inclusion i can be extended to $\mathbb{Q}$. $\square$

Lemma 1.2.3. *Let $A \hookrightarrow B$ be a module-finite extension of domains. If A is a field, then B is a field.*

Proof. Fix any $b \in B \backslash A$. Since b is not a zero-divisor, it admits an irreducible minimal polynomial with coefficients in A: $\mu(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, for some $n > 1$. Since μ is irreducible $a_0 \neq 0$; if we call $f(x) := \frac{\mu(x)-a_0}{x}$, we get that $bf(b) = a_0$, and since a_0 is invertible, so is b. □

Corollary 1.2.4. *Let $A \hookrightarrow B$ is a module-finite extension of domains, with $S := A \setminus \{0\}$. $S^{-1}B$ is the fraction field L of B.*

Proof. $S^{-1}A \hookrightarrow S^{-1}B$ is a module-finite extension and $S^{-1}A$ is a field, so by the previous lemma $S^{-1}B$ is also a field; since L is the minimal field that admits an injection $B \hookrightarrow L$, we get an injection $L \hookrightarrow S^{-1}B$. On the other hand, obviously $S^{-1}B$ is contained in L, therefore they are equal. □

Proposition 1.2.5. *Let $A \hookrightarrow B$ be a module-finite extension of rings, where A is a domain with equal characteristic* 0*. Then there is an A-linear retraction $r : B \to A$.*

Proof. As shown in Proposition 1.2.1, it suffices to show that any module-finite extension $A \hookrightarrow B$ with B domain splits. Let $S := A \setminus \{0\}$; since $A \hookrightarrow B$ is a module-finite extension, $S^{-1}B$ is the fraction field of B, and $S^{-1}A \hookrightarrow S^{-1}B$ is a finite extension of fields. Let $tr : S^{-1}B \to S^{-1}A$ be the trace map: for all $a \in A$ $tr(a) = na$, where $n := [S^{-1}B : S^{-1}A]$, and since $A \hookrightarrow B$ is an integral extension, $tr(B) \subseteq A$. Finally, n is invertible is A, thus $\frac{1}{n}tr : B \to A$ is a retraction of $A \hookrightarrow B$. □

Remark 1.2.6. This line of reasoning fails if A has mixed characteristic because n in general will not be invertible; on the other hand, we can still deduce that the localized map $A\left[\frac{1}{n}\right] \hookrightarrow B\left[\frac{1}{n}\right]$ splits. On its own, this result does not help because this localization is not faithful, but it can redirect us towards the solution: one may look for some theorems that from the splitting of the localization deduce the splitting of the original map. Such a result would be too strong to hope for, but there is a weaker version in a particular environment: Faltings' almost purity theorem. It will concern étale morphism (which is a slightly stronger condition than the splitting property), and its setting of almost mathematics will be explored in the next chapter.

Proof. Let $\pi_i : M \hookrightarrow A^n \to A^{(i)}$ be the composite maps given by the projection from A^n to its i-th coordinate: one of these maps takes m to a nonzero element. □

Chapter 2

Almost mathematics

In this chapter we will give an introduction to almost mathematics, a theory first elaborated by Gerd Faltings to facilitate the study of some particular non-noetherian rings. All the concepts and proofs that are contained here can be found, explored in much greater detail and generality, in a comprehensive work from Gabber and Ramero: [GR03].

In the rest of the chapter, V will be a ring with a set element $t \in V$ which is not a zero divisor and admits a system of p-power roots for some prime number p; the ideal generated by all these roots will be called $I := (t^{\frac{1}{p^\infty}})$.

Example 1. Consider the ring $\mathbb{Z}_p$ of p-adic integers (that is the completion of $\mathbb{Z}$ with respect to the ideal p), with quotient field $\mathbb{Q}_p$. Recursively, take $x_0 := p$ and choose $\{x_n\}_n$ in the algebraic closure of $\mathbb{Q}_p$ such that $x_{n+1}^p = x_n$. If we add all the x_i to $\mathbb{Z}_p$ we get the ring $\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$, with a system of p-power roots of p. This ring will be the building block for the construction of perfectoid theory, in Chapter 3.

2.1 The category of almost-modules

Starting from the category $V-\mathbf{Mod}$ of V-modules we want to work with some sort of "quotient" category, where we collapse to zero every "little" module, by which we mean every module which is annihilated after multiplication by any element in I. We will call it the category of *almost*-V-modules: $V^a - \mathbf{Mod}$. To describe this category, we first need to prove the following lemma:

Lemma 2.1.1. *Let V, I be as already defined. Then:*

- *I is a flat V-module;*
- *$I^2 = I$;*
- *The multiplication map $m : I \otimes I \to I$ is an isomorphism.*

Proof.

- The ideal I is the increasing union of the ideals $I_k := (t^{\frac{1}{p^k}})$. Since I_k is principal for all k, and t is not a zero divisor, they are all isomorphic to V as V-modules, hence flat. Since filtered colimits of flat modules are flat, this implies that I is flat.
- Obviously $I^2 \subseteq I$. Viceversa, $t^{\frac{1}{p^k}}$ is a multiple of $t^{\frac{2}{p^{k+1}}} \in I^2$, therefore $I \subseteq I^2$.

- Consider the inclusion $I \hookrightarrow V$: tensoring by I we get the multiplication map $m : I \otimes I \to I$. It is injective because I is flat, but it is surjective by the previous point, therefore it is an isomorphism.

$\square$

Now we can explicitly describe the category $\mathbf{Mod} - V^a$.

- for every object $M \in V - \mathbf{Mod}$ there is an object $M^a \in V^a - \mathbf{Mod}$;
- the morphisms $\mathrm{Hom}_{V^a}(M^a, N^a)$ are identified with $\mathrm{Hom}_V(I \otimes M, N)$;
- the identity in $\mathrm{Hom}_{V^a}(M^a, M^a) \cong \mathrm{Hom}_V(I \otimes M, M)$ is the natural map induced by the inclusion $I \subseteq V$;
- given $f \in \mathrm{Hom}_{V^a}(M^a, N^a)$ and $g \in \mathrm{Hom}_{V^a}(N^a, P^a)$ the composition $g \circ^a f$ is defined as:

$$g \circ (id_{\mathfrak{m}} \otimes f) \circ (m^{-1} \otimes id_M) : I \otimes M \to I \otimes I \otimes M \to I \otimes N \to P.$$

Remark 2.1.2. The almost mathematics depends on the choice of the element t, but when this choice is clear from the context we will omit the dependence for clarity of exposition.

Remark 2.1.3. It is worth noting that we can take $t = 1$: in this case, the almost category is the same as the original category. This means that "usual" mathematics is properly a particular case of almost mathematics.

There is a natural functor $(-)^a : V - \mathbf{Mod} \to V^a - \mathbf{Mod}$, which sends M to M^a and $f : M \to N$ to the induced morphism $f^a : \mathfrak{m} \otimes M \to N$. $V^a - \mathbf{Mod}$ is abelian and inherits via this natural functor a notion of tensor product from the original category.

If we start with a V-algebra R with no t-torsion, we can similarly define $R^a - \mathbf{Mod}$; the morphisms will be:

$$\mathrm{Hom}_{R^a}(M^a, N^a) := \mathrm{Hom}_R(IR \otimes_R M, N).$$

Remark 2.1.4. This is the same as $\mathrm{Hom}_R(I \otimes_V M, N) \cong \mathrm{Hom}_R((I \otimes_V R) \otimes_R M, N)$. The reason is that for all k, since R has no t-torsion, we have the isomorphisms $I_k R \cong R \cong V \otimes_V R \cong I_k \otimes R$, where $I_k = (t^{\frac{1}{p^k}}) \subseteq V$; therefore, by passing to the colimit, we get $IR \cong I \otimes_V R$.

In particular, the different definitions of almost mathematics are compatible.

Remark 2.1.5. Since $V^a - \mathbf{Mod}$ has inherited a notion of tensor product, it is possible to define a V^a-algebra A as an algebra in the category $V^a - \mathbf{Mod}$; similarly we can also define a "module" over A: their category will be simply written as $A - \mathbf{Mod}$. It can be verified that, if $A = R^a$, the categories $A - \mathbf{Mod}$ and $R^a - \mathbf{Mod}$ are equivalent, so there is no ambiguity of definitions.

The main definition in the context of almost mathematics is that of (t-)almost zero module. Let's explore it with a lemma.

Lemma 2.1.6. *Take $M \in R - \mathbf{Mod}$. The following are equivalent:*

- *$IM = 0$;*
- *$I \otimes_V M = 0$;*
- *M^a is the zero object in $R^a - \mathbf{Mod}$.*

In this case, M is said to be t-almost zero, and we will write $M \approx_t 0$.

Proof. Let's prove (2) $\Leftrightarrow$ (3):

$$\begin{aligned} M^a \in R^a - \mathbf{Mod} \text{ is the zero object } &\Leftrightarrow \forall N \in R - \mathbf{Mod},\ \mathrm{Hom}_{R^a}(M^a, N^a) = 0 \\ &\Leftrightarrow \forall N \in R - \mathbf{Mod},\ \mathrm{Hom}_R(IR \otimes_R M, N) = 0 \\ &\Leftrightarrow IR \otimes_R M = 0 \\ &\Leftrightarrow I \otimes_V M = 0. \end{aligned}$$

Since IM is the image of the natural map $I \otimes_V M \to M$, (2) $\Rightarrow$ (1) is obvious. Vice versa, consider a generic element in $I \otimes_V M$: we jist have to prove that it is zero. Since $I \subseteq V$ is an ideal, we can always write such an element as $1 \otimes x$, with $x \in M$. The map $I \otimes_V M \to IM$ is identically zero, so we get $x = m(1 \otimes x) = 0$, which means that $1 \otimes x = 0$. $\square$

Having clarified this basic notion, we can study the relationship between the categories $R - \mathbf{Mod}$ and $R^a - \mathbf{Mod}$.

Proposition 2.1.7. *Let R be a V-algebra.*

- *The functor $(-)^a : R - \mathbf{Mod} \to R^a - \mathbf{Mod}$ admits a right adjoint $(-)_* : R^a - \mathbf{Mod} \to R - \mathbf{Mod}$. For $M \in R^a - \mathbf{Mod}$, with underlying R-module M_0, M_* is defined as follows:*

$$M_* := \mathrm{Hom}_{R^a - \mathbf{Mod}}(R^a, M) \cong \mathrm{Hom}_{R - \mathbf{Mod}}(IR, M_0),$$

while on the maps it is defined in the obvious way.

- *The counit $\varepsilon_M : (M_*)^a \to M$ (i.e. the map that corresponds to the identity $id_{M_*} : M_* \to M_*$ via the adjunction) is an isomorphism.*

- *The functor $(-)^a : R - \mathbf{Mod} \to R^a - \mathbf{Mod}$ admits a left adjoint $(-)_! : R^a - \mathbf{Mod} \to R - \mathbf{Mod}$. It is defined as the composition of the functor $(-)_*$ and the functor $- \otimes_V I$.*

  Proof. In this proof, tensor products will all be considered with respect to R.

  Let $M \in R^a - \mathbf{Mod}$, $N \in R - \mathbf{Mod}$, with M_0 as above:

$$\begin{aligned} \mathrm{Hom}_{R-\mathbf{Mod}}(N, M_*) &= \mathrm{Hom}_{R-\mathbf{Mod}}(N, \mathrm{Hom}_{R-\mathbf{Mod}}(IR, M_0)) \\ &\cong \mathrm{Hom}_{R-\mathbf{Mod}}(IR \otimes N, M_0) \\ &\cong \mathrm{Hom}_{R^a-\mathbf{Mod}}(N^a, (M_0)^a) = \mathrm{Hom}_{R^a-\mathbf{Mod}}(N^a, M), \end{aligned}$$

  so the functors are adjoint.

- If we take $N := M_*$, this adjunction sends the identity map id_{M_*} to:

$$\varepsilon_M \in \mathrm{Hom}_{R^a-\mathbf{Mod}}((M_*)^a, M) \cong \mathrm{Hom}_{R-\mathbf{Mod}}(IR \otimes \mathrm{Hom}_{R-\mathbf{Mod}}(IR, M_0), M_0),$$

  with $\varepsilon_M(a \otimes f) = f(a)$. Let's consider:

$$\phi_M \in \mathrm{Hom}_{R^a-\mathbf{Mod}}(M, (M_*)^a) \cong \mathrm{Hom}_{R-\mathbf{Mod}}(IR \otimes M_0, \mathrm{Hom}_{R-\mathbf{Mod}}(IR, M_0)),$$

  such that $\phi_M(a \otimes x) = (b \to bax)$. Let's prove that it's the inverse of ε_M. In one direction:

$$IR \otimes IR \otimes \mathrm{Hom}_{R-\mathbf{Mod}}(IR, M_0) \xrightarrow{id_I \otimes \varepsilon_M} IR \otimes M_0 \xrightarrow{\phi_M} \mathrm{Hom}_{R-\mathbf{Mod}}(IR, M_0)$$

$$a \otimes b \otimes f \longrightarrow a \otimes f(b) \longrightarrow (c \mapsto caf(b) = abf(c)),$$

therefore $\phi_M \circ^a \varepsilon_M$, viewed as an element of $\mathrm{Hom}_{R-\mathbf{Mod}}(IR \otimes M_*, M_*)$, sends $a \otimes f$ to af, which is the identity map in $R^a - \mathbf{Mod}$.

Vice versa:

$$IR \otimes IR \otimes M_0 \xrightarrow{id_I \otimes \phi_M} IR \otimes \mathrm{Hom}_{R-\mathbf{Mod}}(IR, M_0) \xrightarrow{\varepsilon_M} M_0$$

$$a \otimes b \otimes x \longrightarrow a \otimes (c \mapsto cbx) \longrightarrow abx,$$

therefore $\varepsilon_M \circ^a \phi_M$, viewed as an element of $\mathrm{Hom}_{R-\mathbf{Mod}}(IR \otimes M_0, M_0)$, sends $a \otimes x$ to ax, which is the identity map in $R^a - \mathbf{Mod}$.

- Let $M \in R^a - \mathbf{Mod}$, $N \in R - \mathbf{Mod}$, with M_0 as above:

$$\begin{aligned}\mathrm{Hom}_{R-\mathbf{Mod}}(M_!, N) &= \mathrm{Hom}_{R-\mathbf{Mod}}(IR \otimes M_*, N) \\ &\cong \mathrm{Hom}_{R^a-\mathbf{Mod}}((M_*)^a, N^a) \\ &\cong \mathrm{Hom}_{R^a-\mathbf{Mod}}(M, N^a),\end{aligned}$$

where in the last isomorphism we used that M and M_*^a are isomorphic in $R^a - \mathbf{Mod}$ via ε_M.

□

2.2 First definitions

In this new context we can give many definitions analogous to classical ones, and many properties are preserved - with appropriate alterations - in the new setting. The following examples are the ones we will need in this thesis. Let M, N be R-modules:

- M is almost zero ($M \approx 0$) if $M^a \in R^a - \mathbf{Mod}$ is the zero object, which happens iff M is I-torsion;
- $f : M \to N$ is almost injective (resp. almost surjective) if $\ker(f) \approx 0$ (resp. $\mathrm{coker}(f) \approx 0$);
- $f : M \to N$ is an almost isomorphism if it is almost injective and almost surjective;
- M is almost projective (resp. almost flat) if $\forall N \in R - \mathbf{Mod}$, $\mathrm{Ext}^1_R(M, N) \approx 0$ (resp. if $\mathrm{Tor}^R_1(M, N) \approx 0$);
- M is almost faithfully flat if it is almost flat and $\forall N_1, N_2 \in R - \mathbf{Mod}$ the induced map $\mathrm{Hom}_R(N_1, N_2) \to \mathrm{Hom}_R(N_1 \otimes M, N_2 \otimes M)$ is almost injective;
- the sequence of R-modules $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$ is said to be almost exact at N_2 if $\ker(g)/\operatorname{im}(f) \approx 0$.

Remark 2.2.1. It can be checked that the definition of almost isomorphism is compatible with the intrinsic notion of isomorphism in $R^a - \mathbf{Mod}$.

All the definitions we gave are invariant under almost isomorphism. We will just give an example, and check the following property.

Proposition 2.2.2. *Let M, N, P be R-modules, and $f : M \to N$ an almost isomorphism. The natural map $\mathrm{Ext}^k_R(P, M) \to \mathrm{Ext}^k_R(P, N)$ is an almost isomorphism for all $k \geq 0$.*

First, let's prove another lemma which highlights the good behaviour of almost zero modules.

Lemma 2.2.3. *Let $F : R-\mathbf{Mod} \to R-\mathbf{Mod}$ be an R-linear functor, K an almost zero R-module. $F(K)$ is an almost zero R-module.*

Proof. Since F is R-linear, it preserves the endomorphism $-\cdot t^\alpha$ for all $\alpha > 0$. On the module K this endomorphism is zero, therefore the same happens for $F(K)$; every element of $F(K)$ is t^α-torsion for all $\alpha > 0$, therefore $F(K) \approx 0$. □

Now we can prove the proposition.

Proof. We can write f as the composition $M \xrightarrow{p} \mathrm{im}(f) \xrightarrow{i} N$, where p is surjective and almost injective, while i is injective and almost surjective.

- Consider the exact sequence $0 \to K \to M \to \mathrm{im}(f) \to 0$, where $K \approx 0$. Applying the functor $\mathrm{Hom}_R(P, -)$ we get the following piece of a long exact sequence for all $k \geq 0$:

$$\mathrm{Ext}^k_R(P, K) \to \mathrm{Ext}^k_R(P, M) \to \mathrm{Ext}^k_R(P, \mathrm{im}(f)) \to \mathrm{Ext}^{k+1}_R(P, K).$$

  By the previous lemma, since $K \approx 0$ and $\mathrm{Ext}^k_R(P, -)$ is an R-linear functor for all k, the first and last term of the exact sequence are almost zero, therefore the central map is an almost isomorphism

- Consider the exact sequence $0 \to \mathrm{im}(f) \to N \to C \to 0$, where $C \approx 0$. Applying the functor $\mathrm{Hom}_R(P, -)$ we get the following piece of a long exact sequence for all $k \geq 0$:

$$\mathrm{Ext}^{k-1}_R(P, C) \to \mathrm{Ext}^k_R(P, \mathrm{im}(f)) \to \mathrm{Ext}^k_R(P, N) \to \mathrm{Ext}^k_R(P, C),$$

  where if $k = 0$ the functor $\mathrm{Ext}^{k-1}_R(P, -)$ is identically zero. By the previous lemma, since $C \approx 0$ and $\mathrm{Ext}^k_R(P, -)$ is an R-linear functor for all k, the first and last term of the exact sequence are almost zero, therefore the central map is an almost isomorphism.

Finally, we get that the composite map $\mathrm{Ext}^k_R(P, M) \to \mathrm{Ext}^k_R(P, \mathrm{im}(f)) \to \mathrm{Ext}^k_R(P, N)$ is an almost isomorphism for all $k \geq 0$. □

Remark 2.2.4. It's not obvious a priori that the definitions we gave are the most appropriate generalizations.

For example, we could define an almost projective module M as a module such that M^a is a projective object in the category $R^a - \mathbf{Mod}$. The problem of this definition would be that $R^a - \mathbf{Mod}$ does not have enough projectives (in particular, one could prove that R^a is not a projective object in $R^a - \mathbf{Mod}$).

Some reasons to think of those definitions as the "right" ones will be given in the next section.

2.3 Projectiveness, flatness and faithful flatness

The most convincing argument for the use of the previous definitions is to show that many "classical" properties are translated nicely into the language of the almost mathematics, for example the following:

Proposition 2.3.1. *Let M be an R-module.*

- *The following are equivalent:*

1. *M is almost projective;*
2. *for every surjective map $g : N_2 \to N_3$, for every map $\phi : M \to N_3$, for every $\alpha > 0$, the map $t^\alpha \phi$ factors through g;*
3. *for every exact sequence of R-modules $0 \to N_1 \to N_2 \to N_3 \to 0$, the sequence $0 \to \mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, N_3) \to 0$ is almost exact.*

- *For every sequence of R-modules* $0 \longrightarrow N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$ *which is almost exact, the induced sequence* $0 \longrightarrow \mathrm{Hom}_R(M, N_1) \xrightarrow{\tilde{f}} \mathrm{Hom}_R(M, N_2) \xrightarrow{\tilde{g}} \mathrm{Hom}_R(M, N_3)$ *is almost exact. Moreover, if M is almost projective and g is almost surjective, $\tilde{g}$ is almost surjective.*

Proof. (1) implies (3) because we have an exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M, N_1) \longrightarrow \mathrm{Hom}_R(M, N_2) \longrightarrow \mathrm{Hom}_R(M, N_3) \longrightarrow \mathrm{Ext}^1_R(M, N_1) \ ,$$

where $\mathrm{Ext}^1_R(M, N_1) \approx 0$.

(3) implies (2) because since the map $\mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, N_3)$ is almost surjective, $t^\alpha \phi$ is in its image for every $\phi \in \mathrm{Hom}_R(M, N_3)$ and for every $\alpha > 0$.

Finally, assume (2) is true. For any R-module N there is an injective module I and an injective map $f : N \hookrightarrow I$. Take the short exact sequence induced by f:

$$0 \longrightarrow N \xrightarrow{f} I \xrightarrow{g} C \longrightarrow 0 \ .$$

Since I is injective, $\mathrm{Ext}^1_R(M, I) = 0$ and we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \xrightarrow{\tilde{f}} \mathrm{Hom}_R(M, I) \xrightarrow{\tilde{g}} \mathrm{Hom}_R(M, C) \longrightarrow \mathrm{Ext}^1_R(M, N) \longrightarrow 0 \ .$$

For every map $\phi \in \mathrm{Hom}_R(M, C)$, for every $\alpha > 0$, the map $t^\alpha \phi$ factors through g, therefore $I \, \mathrm{Hom}_R(M, C) \subseteq \mathrm{im}(\tilde{g})$, and $\mathrm{Ext}^1_R(M, N) \cong \mathrm{coker}(\tilde{g}) \approx_0$.

For the second part, since $\ker(f)$ is almost zero, so is $\mathrm{Hom}_R(M, \ker(f)) \cong \ker(\tilde{f})$. For the exactness at N_2, we know that the inclusion $\mathrm{im}(f) \subseteq \ker(g)$ is almost surjective; for every $\phi \in \mathrm{Hom}_R(M, \ker(g))$, for every $\alpha > 0$, $t^\alpha \phi$ has image in $\mathrm{im}(f)$, therefore the inclusion:

$$\mathrm{Hom}_R(M, \mathrm{im}(f)) \hookrightarrow \mathrm{Hom}_R(M, \ker(g)) \cong \ker(\tilde{g})$$

is almost surjective. In particular, if $N_3 = 0$ (i.e. if f is an almost isomorphism), $\tilde{f}$ is an almost isomorphism.

Now, suppose M is almost projective and g is almost surjective. Since $g : N_2 \to \mathrm{im}(g)$ is surjective, the induced map $\mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, \mathrm{im}(g))$ is almost surjective. Moreover, since $0 \to \mathrm{im}(g) \to N_3 \to 0$ is almost exact, $\mathrm{Hom}_R(M, \mathrm{im}(g)) \to \mathrm{Hom}_R(M, N_3)$ is an almost isomorphism, therefore the composition $\mathrm{Hom}_R(M, N_2) \to \mathrm{Hom}_R(M, N_3)$ is almost surjective. $\square$

Lemma 2.3.2. *Let P and Q be almost projective R-modules. $P \otimes Q$ is an almost projective R-module.*

Proof. Take an exact sequence of R-modules $0 \to N_1 \to N_2 \to N_3 \to 0$. Since P and Q are almost projective, by 2.3.1.ii we get the following almost exact sequences:

$$0 \to \mathrm{Hom}_R(Q, N_1) \to \mathrm{Hom}_R(Q, N_2) \to \mathrm{Hom}_R(Q, N_3) \to 0;$$

$$0 \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, N_1)) \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, N_2)) \to \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, N_3)) \to 0.$$

Since the composite functor $\mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, -))$ is naturally isomorphic to $\mathrm{Hom}_R(P \otimes Q, -)$, by Lemma 2.3.1.i $P \otimes Q$ is an almost projective R-module. $\square$

Lemma 2.3.3. *Let P_1 be a direct summand of an almost projective R-module P. P_1 is an almost projective R-module.*

Proof. Let $P \cong P_1 \oplus P_2$. Given a surjective map $f : M \twoheadrightarrow N$, let's call $\tilde{f} : \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$ the induced map. Since P is almost projective, the following modules are almost zero:

$$\frac{\mathrm{Hom}_R(P, N)}{\tilde{f}(\mathrm{Hom}_R(P, M))} \cong \frac{\mathrm{Hom}_R(P_1, N) \oplus \mathrm{Hom}_R(P_2, N)}{\tilde{f}(\mathrm{Hom}_R(P_1, M) \oplus \mathrm{Hom}_R(P_2, N))} \cong \frac{\mathrm{Hom}_R(P_1, N)}{\tilde{f}(\mathrm{Hom}_R(P_1, M))} \oplus \frac{\mathrm{Hom}_R(P_2, N)}{\tilde{f}(\mathrm{Hom}_R(P_2, M))}.$$

In particular $\mathrm{Hom}_R(P_1, N) \approx \tilde{f}(\mathrm{Hom}_R(P_1, M))$, so the morphism $\mathrm{Hom}_R(P_1, M)) \to \mathrm{Hom}_R(P_1, N)$ is almost surjective and P_1 is almost projective. $\square$

Remark 2.3.4. Similar properties are true for almost flat modules, and the proofs are analogous.

Now we will show that almost-projectiveness and almost-flatness are related in a similar way as classical projectiveness and flatness.

Lemma 2.3.5. *Let P be an almost projective R-module. P is almost flat.*

Proof. Let C be any R-module and take an exact sequence $0 \to K \to F \to C \to 0$, where F is a free R-module. Tensoring by P we get the following exact sequence:

$$0 \longrightarrow \mathrm{Tor}^1_R(C, P) \longrightarrow K \otimes P \longrightarrow F \otimes P \longrightarrow C \otimes P \longrightarrow 0,$$

where we used that $\mathrm{Tor}^1_R(F, P) = 0$ because F is free, and hence flat.

Let Q be an arbitrary injective module and apply the exact functor $\mathrm{Hom}_R(\cdot, Q)$:

$$0 \to \mathrm{Hom}_R(\mathrm{Tor}^1_R(C, P), Q) \to \mathrm{Hom}_R(K \otimes P, Q) \xrightarrow{f} \mathrm{Hom}_R(F \otimes P, Q) \xrightarrow{g} \mathrm{Hom}_R(C \otimes P, Q) \to 0.$$

From the properties of the tensor product, the functor $\mathrm{Hom}_R(\cdot \otimes P, Q)$ is naturally isomorphic to $\mathrm{Hom}_R(P, \mathrm{Hom}_R(\cdot, Q))$, composition of the functor $\mathrm{Hom}_R(P, \cdot)$, which is "almost" exact, and $\mathrm{Hom}_R(\cdot, Q)$, which is exact. This means that applying it to $0 \to K \to F \to C \to 0$ we get an exact sequence:

$$\mathrm{Hom}_R(P, \mathrm{Hom}_R(K, Q)) \xrightarrow{f'} \mathrm{Hom}_R(P, \mathrm{Hom}_R(F, Q)) \xrightarrow{g'} \mathrm{Hom}_R(P, \mathrm{Hom}_R(C, Q)) \longrightarrow 0,$$

with the added property that f' is almost injective.

The natural isomorphism of the two functors sends f to f' and g to g', therefore f is almost injective and $\mathrm{Hom}_R(\mathrm{Tor}^1_R(C, P), Q) \approx 0$.

We can choose Q such that there is an injective map $i : \mathrm{Tor}^1_R(C, P) \hookrightarrow Q$. Since the module $\mathrm{Hom}_R(\mathrm{Tor}^1_R(C, P), Q)$ is almost zero, $t^\alpha i$ is the zero map for every $\alpha > 0$. It follows that $t^\alpha \mathrm{Tor}^1_R(C, P) = 0$ for every $\alpha > 0$, i.e. $\mathrm{Tor}^1_R(C, P) \approx 0$. $\square$

Proposition 2.3.6. *Let P be a finitely presented, almost flat R-module. Then P is almost projective.*

Proof. Fix an injective R-module Q, and call $\widetilde{-} : R-\mathbf{Mod} \to R-\mathbf{Mod}$ the functor $\mathrm{Hom}_R(-,Q)$. For any R-modules M, N, there is a natural morphism $\alpha_{M,N} : \widetilde{N} \otimes M \to \widetilde{\mathrm{Hom}_R(M,N)}$ such that $\alpha_{M,N}(f \otimes x)$ is the map that sends $g \in \mathrm{Hom}_R(M,N)$ to $f(g(x))$. Let's prove that if M is finitely presented $\alpha_{M,N}$ is an isomorphism. If $M \cong R^m$ for some finite m, we have the following chain of isomorphism:

$$\widetilde{N} \otimes_R R^m \cong \mathrm{Hom}_R(N,Q)^m \cong \mathrm{Hom}_R(N^m,Q) \cong \widetilde{N^m} \cong \widetilde{\mathrm{Hom}_R(R^m,N)},$$

and the composite isomorphism is $\alpha_{R^m,N}$.

In general, if M is finitely presented, there is an exact sequence $R^k \to R^n \to M \to 0$. The functors $\widetilde{N} \otimes -$ and $\widetilde{\mathrm{Hom}_R(-,N)}$ are both right exact, so we get the following diagram:

$$\begin{tikzcd}
\widetilde{N} \otimes R^k \arrow[r] \arrow[d, "\alpha_{R^k,N}"] & \widetilde{N} \otimes R^n \arrow[r] \arrow[d, "\alpha_{R^n,N}"] & \widetilde{N} \otimes M \arrow[r] \arrow[d, "\alpha_{M,N}"] & 0 \arrow[d, equal] \\
\widetilde{\mathrm{Hom}_R(R^k,N)} \arrow[r] & \widetilde{\mathrm{Hom}_R(R^n,N)} \arrow[r] & \widetilde{\mathrm{Hom}_R(M,N)} \arrow[r] & 0.
\end{tikzcd}$$

The diagram commutes because α is a natural transformation, and since $\alpha_{R^n,N}$ and $\alpha_{R^k,N}$ are both isomorphism, $\alpha_{M,N}$ is also an isomorphism by the five lemma.

Now onto the main part of the proof. Let M be any R-module and take an injective R-module I so that we have a short exact sequence:

$$0 \longrightarrow M \longrightarrow I \longrightarrow C \longrightarrow 0 \; .$$

Let P be a finitely presented, almost flat R-module. If we apply the functor $\mathrm{Hom}_R(P,-)$, and then the exact functor $\mathrm{Hom}_R(-,Q)$, since I is injective we get the following exact sequence:

$$0 \longrightarrow \widetilde{\mathrm{Ext}^1_R(P,M)} \longrightarrow \widetilde{\mathrm{Hom}_R(P,C)} \longrightarrow \widetilde{\mathrm{Hom}_R(P,I)} \longrightarrow \widetilde{\mathrm{Hom}_R(P,M)} \longrightarrow 0 \qquad .$$

If instead we apply the exact functor $\mathrm{Hom}_R(-,Q)$ and then the functor $-\otimes P$, we get an exact sequence:

$$\mathrm{Tor}^1_R(\widetilde{M},P) \longrightarrow \widetilde{C} \otimes P \longrightarrow \widetilde{I} \otimes P \longrightarrow \widetilde{M} \otimes P \longrightarrow 0 \qquad ,$$

where the leftmost module is almost zero because P is almost flat. Since α is a natural isomorphism, the rightmost parts of these sequences are isomorphic. This implies that $\widetilde{\mathrm{Ext}^1_R(P,M)}$ is an almost zero module. We can choose Q such that there is an injective map $i : \mathrm{Ext}^1_R(P,M) \to Q$: this map must be zero when multiplied by t^α for all α; in particular, its image, which is isomorphic to $\mathrm{Ext}^1_R(P,M)$, must be I-torsion, therefore almost zero.

By varying M, we get that P is an almost projective R-module. □

Let's now reexamine the concept of faithful flatness in the almost sense.

Proposition 2.3.7. *Let P be an almost flat R-module. The following are equivalent:*

- *P is almost faithfully flat;*
- *for every R-module M, $M \approx 0$ if and only if $M \otimes P \approx 0$;*
- *for every ideal $I \subseteq R$, if $R/I \not\approx 0$, then $P/IP \not\approx 0$.*

Proof. Assume (2) and take a map $f : M \to N$ such that the induced morphism $\tilde{f} : M \otimes P \to N \otimes P$ is zero. By the right exactness of tensor product we get $0 = \mathrm{im}(\tilde{f}) = \mathrm{im}(f) \otimes P$, therefore $\mathrm{im}(f) \approx 0$, i.e. the map f is almost zero.

Assume (1) and take M such that $M \otimes P \approx 0$. By almost faithful flatness, we have an almost injective map $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(P, M \otimes P) \approx 0$, therefore $M \cong \mathrm{Hom}_R(R, M) \approx 0$.

Finally, (2) obviously implies (3) by taking $M := A/I$. Vice versa, if $M \otimes P \approx 0$, choose $x \in M$ and consider the submodule generated by x, which is isomorphic to $R/\mathrm{Ann}(x)$. Since P is almost flat, the inclusion $R/\mathrm{Ann}(x) \hookrightarrow M$ induces an almost injective map $R/\mathrm{Ann}(x) \otimes P \to M \otimes P \approx 0$, therefore $R/\mathrm{Ann}(x) \approx 0$ by our hypothesis. Since x is arbitrary, this means every element of M is killed by multiplication with t^α for all α, i.e. $M \approx 0$. □

Another way to work with almost faithfully flat modules is to use the evaluation map, as shown in the following lemma.

Lemma 2.3.8. *Let R be an A-algebra, P a $(IR$-$)$almost projective R-module. Consider the natural map $ev_{P/R} : P \otimes \mathrm{Hom}_R(P, R) \to R$ given by the evaluation, and call $\mathcal{E}_{P/R}$ the image of this map. The following statements are true:*

- *for every morphism of algebras $R \to R'$, if $P' := P \otimes_R R'$, we have $\mathcal{E}_{P'/R'} \approx \mathcal{E}_{P/R}R'$;*
- *$\mathcal{E}_{P/R} \approx 0$ if and only if $P \approx 0$;*
- *$\mathcal{E}_{P/R} \approx R$ if and only if P is almost faithfully flat.*

Proof.
- Consider a free module $F := R^{(J)}$ with a surjective map $\phi : F \to P$. For any $j \in J$, denote by $p_j : F \to R$ the projection on the j-th component, and with $e_j : R \to F$ the inclusion on the j-th component, so that $p_i \circ e_j = \delta_{ij} id_R$ and $\sum_j e_j \circ p_j = id_F$ (this sum is finite because for every $x \in F$ there is only a finite number of j such that $p_j(x) \neq 0$). Since P is almost projective, we make use of its lifting property, and take for every $\alpha > 0$ $\psi_\alpha \in \mathrm{Hom}_R(F, P)$ such that $\phi \circ \psi_\alpha = t^\alpha id_P$. Let's prove the following:

  $$I\mathcal{E}_{P/R} \subseteq \langle p_i \circ \psi_\alpha \circ \phi \circ e_j(1) | i, j, \alpha \rangle_R \subseteq \mathcal{E}_{P/R},$$

  Every generator of the center ideal can be obtained as $ev_{P/R}(\phi \circ e_j(1), p_i \circ \psi_\alpha)$.

  On the other hand, take $f : P \to A$ and $x \in P$. Since ϕ is surjective, we can write $x = \phi(\sum_j a_j e_j(1))$, where $a_j \in A$ are all zero except for a finite number of indices. For every $\alpha > 0$ we have:

  $$t^\alpha f = (f \circ \phi) \circ \psi_\alpha = f \circ \phi \circ \left(\sum_i e_i \circ p_i \right) \circ \psi_\alpha = \sum_i (f \circ \phi \circ e_i) \circ p_i \circ \psi_\alpha,$$

  where $f \circ \phi \circ e_i : R \to R$ can be thought of as an element $b_i \in R$. Putting the formulas together we get that $t^\alpha f(x) = \sum_{i,j} a_j b_i p_i \circ \psi_\alpha \circ \phi \circ e_j(1)$, and since α, f and x are arbitrary, we get:

  $$I\mathcal{E}_{P/R} \subseteq \langle p_i \circ \psi_\alpha \circ \phi \circ e_j(1) | i, j, \alpha \rangle_R \subseteq \mathcal{E}_{P/R},$$

  which proves our claim.

  Given a morphism of algebras $R \to R'$, with $P' := P \otimes_R R'$, we have:

  $$\mathcal{E}_{P'/R'} \approx \langle p_i \circ \psi_\alpha \circ \phi \circ e_j(1) | i, j, \alpha \rangle_{R'} \approx \langle p_i \circ \psi_\alpha \circ \phi \circ e_j(1) | i, j, \alpha \rangle_R R' \approx \mathcal{E}_{P/R}R'.$$

- For the second point, the left implication is obvious. Vice versa, if $\mathcal{E}_{P/R} \approx 0$, for every i, j, α the element $p_i \circ \psi_\alpha \circ \phi \circ e_j(1)$ is almost zero. Like before, take any $x \in P$ and write it as $\sum_j a_j \phi \circ e_j(1)$. We have:

$$0 \approx \sum_{i,j} a_j \phi \circ e_i \circ (p_i \circ \psi_\alpha \circ \phi \circ e_j(1)) = \sum_j a_j \phi \circ \psi_\alpha \circ \phi \circ e_j(1) = t^\alpha \sum_j b_j \phi \circ e_j(1) = t^\alpha x,$$

therefore $t^\alpha x$ is I-torsion. Since x and α were arbitrary, every element in P is I-torsion, i.e. $P \approx 0$.

- Finally, for every ideal $J \subseteq R$ such that $R/J \not\approx 0$ (which happens if and only if $IR \not\subseteq J$) we have:

$$P/JP (\cong P \otimes R/J) \not\approx 0 \iff \mathcal{E}_{\frac{P}{JP}/\frac{R}{J}} (\approx \mathcal{E}_{P/R} R/J) \not\approx 0 \iff I\mathcal{E}_{P/R} \not\subseteq J$$

Remembering that by Lemma 2.3.5 P is almost flat, Lemma 2.3.7 tells us that the LHS is true (for all J such that $IR \not\subseteq J$) if and only if P is almost faithfully flat. If the RHS is true for all J, J can't be equal to $I\mathcal{E}_{P/R}$, therefore $IR \subseteq I\mathcal{E}_{P/R}$; vice versa, if $IR \subseteq I\mathcal{E}_{P/R}$, obviously the RHS is true for all J which don't contain IR.

To wrap up, we have proven that P is almost faithfully flat if and only if $IR \subseteq I\mathcal{E}_{P/R}$, i.e. if and only if $\mathcal{E}_{P/R} \approx R$

□

2.4 Étale extensions

In this section we will talk about étale extensions and find the proper way to translate this concept into the language of almost mathematics, using the notions we already explored in the previous sections.

2.4.1 The Kähler module of differentials

We will remind the construction of the Kähler module of differentials and some of its basic properties. All the proofs can be found in [Sta21, Section 10.131].

For the rest of this subsection, A and B will be commutative rings.

Definition 2.4.1. Let $f : A \to B$ be a ring map and M a B-module. An A-*derivation* of B into M is an A-linear map $d : B \to M$ that satisfies the Leibniz rule: for every $b_1, b_2 \in B$, $d(b_1 b_2) = b_1 d(b_2) + b_2 d(b_1)$. The set of A-derivations of B into M is called $\mathrm{Der}_A(B, M)$

Remark 2.4.2. For all $a \in A$, for any derivation $d \in \mathrm{Der}_A(B, M)$, we have $d(f(a)) = d(1 \cdot f(a)) = d(1)f(a) + 1 \cdot d(f(a)) = d(f(a)) + d(f(a))$, so $d(a) = 0$. In particular, if f is surjective, there is only one derivation into any B-module M, given by the zero map.

Remark 2.4.3. The B-module structure on M induces a B-module structure on $\mathrm{Der}_A(B, M)$. Moreover, any map of B-modules $M \to N$ induces, via the composition, a B-linear map $\mathrm{Der}_A(B, M) \to \mathrm{Der}_A(B, N)$: this means that the derivations determine a functor $\mathrm{Der}_A(B, -) : B - \mathbf{Mod} \to B - \mathbf{Mod}$.

Proposition 2.4.4. *Let $f : A \to B$ be a ring map. The functor $Der_A(B, -) : B - \mathbf{Mod} \to B - \mathbf{Mod}$ is representable. In particular, there is a B-module $\Omega_{B/A}$ and a derivation $d : B \to \Omega_{B/A}$ such that the natural transformation $\mathrm{Hom}_B(\Omega_{B/A}, -) \Rightarrow Der_A(B, -)$ which sends f to $f \circ d$ is an isomorphism.*

Definition 2.4.5. The module $\Omega_{B/A}$ is called the *module of Kähler differentials* of B over A.

Remark 2.4.6. If $A \to B$ is surjective, the functor $\mathrm{Der}_A(B, -)$ sends all B-modules to zero, which means that $\Omega_{B/A} = 0$.

Lemma 2.4.7. *Let $A \to B$ and $A \to A'$ be ring maps, and let $B' := A' \otimes_A B$. $\Omega_{B,A}$ and $\Omega_{B'/A'}$ are naturally isomorphic as B-modules.*

Concretely, the construction of $\Omega_{B/A}$ is as follows. Consider the free module $\bigoplus_{b \in B} B[b]$, where $\{[b] | b \in B\}$ are thought as variables, and quotient it by the following relations:

$$\langle [f(a)b_1] - f(a)[b_1], [b_1 b_2] - b_1[b_2] - b_2[b_1] | a \in A, b_1, b_2 \in B \rangle.$$

The resulting module is $\Omega_{B/A}$, and the associated derivation is the A-linear map $d : B \to \Omega_{B/A}$ that sends b to $[b]$.

To conclude this subsection, let's include a statement (without proof) which allows us to give an alternative description of the module of differentials.

Proposition 2.4.8. *Let R be a ring and suppose that $\pi : C \to B$ is a surjective map of R-algebras, with kernel I. Then there is a canonical exact sequence of B-modules:*

$$I/I^2 \longrightarrow \Omega_{C/R} \otimes_C B \longrightarrow \Omega_{B/R} \longrightarrow 0,$$

where the leftmost map is induced by sending $f \in I$ to $df \otimes 1$.

Moreover, if there is a map of R-algebras $r : B \to C$ such that $\pi \circ r : B \to B$ is the identity, the leftmost map is injective.

Corollary 2.4.9. *Let $A \to B$ be a ring map, and $\Omega_{B/A}$ the module of differentials of B over A. Consider the multiplication map $m : B \otimes_A B \to B$ and call I its kernel. There is an isomorphism of B-modules $\Omega_{B/A} \cong I/I^2$.*

Proof. The map $r : B \to B \otimes_A B$ which sends b to $b \otimes 1$ is a ring map such that $m \circ r : B \to B$ is the identity. We can apply Proposition 2.4.8 with $R := B$, $C := B \otimes_A B$, and $\pi := m$ to get the following exact sequence:

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{B \otimes_A B/B} \otimes_{B \otimes_A B} B \longrightarrow \Omega_{B/B} \longrightarrow 0.$$

By Corollary 2.4.6 $\Omega_{B/B} = 0$, and by Lemma 2.4.7 we have that $\Omega_{B \otimes_A B/B} \cong \Omega_{B/A} \otimes_A B$. Therefore we have a chain of isomorphisms:

$$\begin{aligned} I/I^2 \cong \Omega_{B \otimes_A B/B} \otimes_{B \otimes_A B} B &\cong (\Omega_{B/A} \otimes_A B) \otimes_{B \otimes_A B} B \\ &\cong ((\Omega_{B/A} \otimes_B B) \otimes_A B) \otimes_{B \otimes_A B} B \\ &\cong (\Omega_{B/A} \otimes_B (B \otimes_A B)) \otimes_{B \otimes_A B} B \\ &\cong \Omega_{B/A} \otimes_B B \cong \Omega_{B/A}. \end{aligned}$$

□

2.4.2 Unramified morphisms

We will now give two alternative definitions of unramified morphism: the first will use the definition of the module of differentials, while the second will allow us to generalize this concept to almost mathematics.

Definition 2.4.10 (Definition 1)**.** A module-finite ring map $A \to B$ is said to be *unramified* if $\Omega_{B/A} = 0$.

Definition 2.4.11 (Definition 2)**.** A module-finite ring map $A \to B$ is said to be *unramified* if B is projective as a $B \otimes_A B$-module, via the multiplication map $m : B \otimes_A B \to B$.

Proposition 2.4.12. *The previous definitions are equivalent.*

Proof. By Corollary 2.4.9, if $\Omega_{B/A} = 0$, $I := \ker(m) = \ker(m)^2$. To describe I more explicitly, let's prove the following:

$$I = \mathrm{Span}_{B \otimes_A B}(\{b \otimes 1 - 1 \otimes b | b \in B\}).$$

One inclusion is obvious. Vice versa take an element $x \in I$ with $x = \sum_j b'_j \otimes b''_j$. For every j, we can add the element $b'_j \otimes 1(b''_j \otimes 1 - 1 \otimes b''_j)$, which is contained on the right hand side; we get:

$$y := \sum_j \left(b'_j \otimes b''_j + (b'_j b''_j \otimes 1 - b'_j \otimes b''_j)\right) = \sum_j b'_j b''_j \otimes 1 = \left(\sum_j b'_j b''_j\right) \otimes 1.$$

Since $y \in I$, it's in the kernel of the multiplication map, so $\sum_j b'_j b''_j = 0$, therefore $y = 0$ and x is contained in the LHS.

Fixed a set $\{b_1, \cdots, b_k\}$ of A-generators for B, we have that

$$\mathrm{Span}_{B \otimes_A B}(\{b \otimes 1 - 1 \otimes b | b \in B\}) = \mathrm{Span}\ _{B \otimes_A B}(\{b_i \otimes 1 - 1 \otimes b_i\}_i),$$

which means that I is finitely generated as a $B \otimes_A B$-module. Since $I^2 = I$, by Nakayama's lemma we deduce that there is an element $e \in 1 + I$ such that $eI = 0$. If we write $e = 1 + i$, we get that $e^2 = e(1+i) = e + ei = e$, i.e. e is an idempotent. Obviously $B \otimes_A B = (e)_{B \otimes_A B} + (1-e)_{B \otimes_A B}$; moreover, the two submodules are direct summands because, if $x = ey = (1-e)z$ is in their intersection, $x = ex + (1-e)x = e(1-e)z + (1-e)ey = (e-e^2)z + (e-e^2)y = 0$. Since $eI = 0$, this decomposition implies $I \subseteq (1-e)B \otimes_A B$ so $I \cap eB \otimes_A B = 0$. Moreover, $m(e) = m(1+i) = 1$, so m induces a surjective map of $B \otimes_A B$-modules $eB \otimes_A B \to B$: since the kernel of this map is contained in I, it must be 0, therefore $B \cong eB \otimes_A B$ is a direct summand of $B \otimes_A B$, hence projective as a $B \otimes_A B$-module.

Viceversa, if B is projective as a $B \otimes_A B$-module, the multiplication map m induces a decomposition $B \otimes_A B = B \oplus I$, where $I = \ker(m)$. If we identify B with this ideal of $B \otimes_A B$, the multiplication map acts as a projection: on one hand $B = m(1) B \otimes_A B$, while on the other hand, since m is a projection and is $B \otimes B$-linear, we get that $m(1) = m(m(1)) = m(1 \cdot m(1)) = m(1) \cdot m(1)$. Moreover, $m(1)I = m(I) = 0$: with the same reasoning as the previous point we get:

$$I = (1 - m(1))B \otimes_A B \Longrightarrow I^2 = (1 - m(1))^2 B \otimes_A B = (1 - m(1))B \otimes_A B = I,$$

therefore, by Lemma 2.4.9 $\Omega_{B/A} \cong I/I^2 = 0$. $\square$

Now we can properly generalize the definition of ramification:

Definition 2.4.13. Let R, t as in the previous section, and consider almost mathematics with respect to t. Let $R \to S$ be a module finite ring map. It is said to be almost unramified if S is almost projective as an $S \otimes_R S$-module.

2.4.3 Étale coverings

As always, the following definition, albeit formulated in the language of almost mathematics, is a proper generalization of the classical definition of étaleness. Let R, t, I as in the previous section, and consider almost mathematics with respect to t.

Definition 2.4.14. Let $R \to S$ be a module finite ring map. It is said to be an almost étale covering if the following conditions are verified:

- S is almost faithfully flat as an R-module;
- $R \to S$ is almost unramified.

In the rest of the thesis, when we talk about (almost) finite étale maps, we will be referring to the previous definition.

Let's note with the following proposition that S is more than almost faithfully flat.

Proposition 2.4.15. *Let $R \to S$ be a module finite ring map, which is almost étale. Then S is an almost projective R-module.*

Proof. Since S is finitely generated as an R-module, so is $S \otimes_R S$. Moreover, as seen in the proof of Proposition 2.4.12, the kernel I of the multiplication map is finitely generated as an $S \otimes_R S$-module, therefore it is also finitely generated as an R-module. This means that the exact sequence $I \to S \otimes_R S \to S \to 0$ makes S a finitely presented R-module: by Proposition 2.3.6, being almost flat, it is also almost projective as an R-module. $\square$

Proposition 2.4.16. *Let $f : R \hookrightarrow S$ a module finite ring map which is an almost étale covering. Then $C := \operatorname{coker}(f)$ is an almost projective R-module.*

Proof. Tensoring by S the exact sequence $0 \to R \to S \to C \to 0$ we get:

$$S \xrightarrow{\tilde{f}} S \otimes S \longrightarrow C \otimes S \longrightarrow 0 \ ,$$

where $\tilde{f}(b) = b \otimes 1$. The multiplication map $m : S \otimes S \to S$ sends $b_1 \otimes b_2$ to $b_1 b_2$, therefore $m \circ \tilde{f} = id_S$: not only $\tilde{f}$ is injective, but the sequence splits and we can write $S \otimes S \cong S \oplus C \otimes S$ as S-modules.

$S \otimes S$ is an almost projective S-module by Lemma 2.3.2, and $C \otimes S$ is almost projective by Lemma 2.3.3 because it is a direct summand of $S \otimes S$.

Let M be any R-module, and take a short exact sequence $0 \to M \to Q \to K \to 0$, where Q is an injective module. Since Q is injective, $\operatorname{Ext}^1_R(C, Q) = 0$, so if we apply the functor $\operatorname{Hom}_R(C, \cdot)$ we get the following exact sequence:

$$0 \longrightarrow \operatorname{Hom}_R(C, M) \longrightarrow \operatorname{Hom}_R(C, I) \longrightarrow \operatorname{Hom}_R(C, K) \longrightarrow \operatorname{Ext}^1_R(C, M) \longrightarrow 0 \ .$$

Since S is almost projective, applying the functor $\operatorname{Hom}_R(S, \cdot)$ we get an almost exact sequence:

$$0 \to \operatorname{Hom}_R(S \otimes C, M) \to \operatorname{Hom}_R(S \otimes C, Q) \to \operatorname{Hom}_R(S \otimes C, K) \to \operatorname{Hom}_R(S, \operatorname{Ext}^1_R(C, M)) \to 0,$$

where we used that the functors $\operatorname{Hom}_R(S, \operatorname{Hom}_R(C, \cdot))$ and $\operatorname{Hom}_R(S \otimes C, \cdot)$ are naturally isomorphic. Since $S \otimes C$ is almost projective, the following sequence is almost exact:

$$0 \longrightarrow \operatorname{Hom}_R(S \otimes C, M) \longrightarrow \operatorname{Hom}_R(S \otimes C, Q) \longrightarrow \operatorname{Hom}_R(S \otimes C, K) \longrightarrow 0,$$

so we have $\mathrm{Hom}_R(S, \mathrm{Ext}^1_R(C, M)) \approx 0$.

Assume by contradiction that $\mathrm{Ext}^1_R(C, M) \not\approx 0$: there is an element $x \in \mathrm{Ext}^1_R(C, M)$ such that $IR \not\subseteq \mathrm{Ann}(x)$. By Lemma 2.3.8, since S is almost faithfully flat, $IR \subseteq \mathcal{E}_{S/R}$: there is an element $a \in \mathcal{E}_{S/R} \setminus \mathrm{Ann}(x)$. Take any $f : S \to R$ such that for some $b \in S$ $f(b) = a$: f induces a map $\tilde{f} : S \to R/\mathrm{Ann}(x) \cong \langle x \rangle_R \subseteq \mathrm{Ext}^1_R(C, M)$ which is not killed by all the elements in I: it follows that $\mathrm{Hom}_R(S, \mathrm{Ext}^1_R(C, M)) \not\approx 0$, which is a contradiction. □

Remark 2.4.17. The hypothesis of being unramified was not used in this proof, since we only needed that S is almost projective and almost faithfully flat as an R-module

Remark 2.4.18. Since this is the transposition in the language of almost mathematics of the property that finite étale coverings split, we can say that finite almost étale coverings almost split.

Finally, let's prove that any suitable module-finite extension of rings admits a localization which is a finite étale covering.

Proposition 2.4.19. *Let $A \to B$ be a module-finite separable extension of noetherian domains (with quotient fields K and L respectively). There is some $g \in A$, which we will call the discriminant, such that $A[g^{-1}] \to B[g^{-1}]$ is étale.*

Proof. Remember that $A[g^{-1}] \to B[g^{-1}]$ is étale if B is a finitely generated, projective, and almost flat A-module, and it is projective as a $B[g^{-1}] \otimes_{A[g^{-1}]} B[g^{-1}] (\cong B \otimes_A B[g^{-1}])$-module. Let's divide the proof in two parts:

- There is some $f \in A$ such that $A[f^{-1}] \to B[f^{-1}]$ is free.

  Since the extension $K \to L$ is finite and separable, there is some $\alpha \in L$ such that:

$$L = K(\alpha) = \bigoplus_{i=0}^{n-1} K\alpha^i,$$

  where $n = [L : K]$; up to multiplying α by a factor in K, we can assume $\alpha \in B$, whith monic polynomial μ of degree n. We have an inclusion of A-modules $i : A \oplus A\alpha \oplus \cdots \oplus A\alpha^{n-1} \to B$, which becomes an isomorphism after tensorization by the flat A-module K; in particular, if $C := \mathrm{coker}(i)$, $C \otimes_A K = 0$. Since B is a finitely generated A-module, so is C, and we can take a finite set of generators $\{c_1, \cdots, c_k\}$; for every c_i, the corresponding element $c_i \otimes 1 \in C \otimes_A K$ is zero, which means that there is some $a_i \in A \setminus \{0\}$ such that $a_i c_i = 0$. Call f the product of the a_i's: since it kills all the generators of C, it belongs to $Ann_A(C)$, therefore if we invert f the map i becomes an isomorphism, i.e.:

$$B[f^{-1}] = \bigoplus_{i=0}^{n-1} A[f^{-1}]\alpha^i.$$

- There is some $h \in A$ such that $B \otimes_A B[h^{-1}] \to B[h^{-1}]$ is projective.

  Call $I \subseteq B \otimes_A B$ the kernel of the multiplication map $m : B \otimes_A B \to B$. Fixed a set $\{b_1, \cdots, b_k\}$ of A-generators for B, in the proof of 2.4.12 we showed that:

$$I = \mathrm{Span}_{B \otimes_A B}(\{b \otimes 1 - 1 \otimes b | b \in B\}) = \mathrm{Span}_{\ B \otimes_A B}(\{b_i \otimes 1 - 1 \otimes b_i\}_i).$$

  Consider the natural map $\tilde{m} : \mathrm{Hom}_{B \otimes_A B}(B, B \otimes_A B) \to \mathrm{Hom}_{B \otimes_A B}(B, B)$: we just need to prove that, this map becomes surjective after inverting some $h \in A$. The second module is simply B: on one hand there is an inclusion (of B-modules) $\mathrm{Hom}_{B \otimes_A B}(B, B) \subseteq$

$\mathrm{Hom}_B(B,B) \cong B$, while on the other hand every B-linear morphism from B to B is automatically $B\otimes_A B$-linear. Now, take $\phi \in \mathrm{Hom}_{B\otimes_A B}(B, B\otimes_A B)$; for all $b \in B$ we have:

$$\phi(b) = (b\otimes 1)\phi(1) = (1\otimes b)\phi(1) \Longrightarrow \phi(1)(b\otimes 1 - 1\otimes b) = 0,$$

so $\phi(1) \in \mathrm{Ann}_{B\otimes_A B}(I)$. Vice versa, for the same reason, for all $x \in \mathrm{Ann}_{B\otimes_A B}(I)$ the map $\phi(b) := (b\otimes 1)x$ is $B\otimes_A B$-linear, so we have that $\mathrm{Hom}_{B\otimes_A B}(B, B\otimes_A B) \cong \mathrm{Ann}_{B\otimes_A B}(I)$. Via this identifications, we can think of $\tilde{m}$ as the multiplication map from $\mathrm{Ann}_{B\otimes_A B}(I)$ to B.

This map's image is an ideal of B: let's prove that it is not zero. For every b_i, for all $k \geq 0$, define:

$$b_i^{(k)} := \sum_{j=0}^{k-1} b_i^j \otimes b_i^{k-1-j}$$

(so that $b_i^{(0)} = 0$), and note that

$$b_i^{(k)}(b_i\otimes 1 - 1\otimes b_i) = b_i^k \otimes 1 - 1\otimes b_i^k.$$

If $\mu_i(x) := \sum_k a_k x^k$ is the minimal polynomial of b_i with coefficients in A, we can consider the element $\tilde{b}_i := \sum_k a_k b_i^{(k)}$; we have that:

$$\tilde{b}_i(b_i\otimes 1 - 1\otimes b_i) = \sum a_k b_i^{(k)}(b_i\otimes 1 - 1\otimes b_i) = \sum a_k(b_i^k\otimes 1 - 1\otimes b_i^k) = \mu_i(b_i)\otimes 1 - 1\otimes\mu_i(b_i) = 0.$$

This means that the product $\tilde{b} := \prod_i \tilde{b}_i$ is in $\mathrm{Ann}_{B\otimes_A B}(I)$. Moreover, since $m(b_i^{(k)}) = kb_i^{k-1}$, and m is a ring homomorphism, we get that:

$$m(\tilde{b}) = \prod_i m(\tilde{b}_i) = \prod_i m\left(\sum_k a_k b_i^{(k)}\right) = \prod_i \left(\sum_k a_k k b_i^{k-1}\right) = \prod \mu_i'(b_i),$$

and since B is a separable extension of A, for all i $\mu_i'(b_i) \neq 0$, and so is their product.

Finally, the inverse of $m(\tilde{b})$ is in L, therefore there is some $h \in A\setminus\{0\}$ such that $\frac{h}{m(\tilde{b})} \in B$, which means that $m(\tilde{b})$ divides h. In particular, if we invert h, the localization of the map $\tilde{m}$, whose image contains $m(\tilde{b})$, becomes surjective, and therefore $B[h^{-1}]$ is projective as a $B\otimes_A B[h^{-1}]$-module.

To conclude, we just have to take $g := hf$ so that by inverting g both of the previous conditions are satisfied. □

Remark 2.4.20. The hypothesis of separability is satisfied in particular when the quotient fields of A and B are of characteristic 0.

In the following chapter we will introduce some basic concepts of perfectoid theory from Scholze's article [Sch12]. In this context we will finally formulate Faltings' almost purity theorem.

Chapter 3

Perfectoid spaces

3.1 Perfectoid algebras and almost mathematics

Here we will present the basic definitions and results regarding perfectoid fields and algebras - details can be found in [Sch12]. Many of the following propositions will not be directly used in the proof of the main theorem, but they are collected here in an orderly fashion as to provide context for the mathematics used in the next subsection.

Definition 3.1.1. Let's give some definitions for the language that will be used in this chapter.

- Given a field K, we will call *valuation* a function $|\cdot| : K \to \mathbb{R}_{\geq 0}$ such that:
  - $|x| = 0$ if and only if $x = 0$;
  - for all $x, y \in K$, $|xy| = |x| \cdot |y|$;
  - for all $x, y \in K$, $|x + y| \leq \max\{|x|, |y|\}$.

  In this case, the couple $(K, |\cdot|)$ is called a *normed field*, and the image of $K \setminus \{0\}$ via $|\cdot|$ is called the *group of valuation*.

- The normed field $(K, |\cdot|)$ is said to be complete if it is complete with respect the topology induced by $|\cdot|$.

- A Banach K-algebra is a couple $(R, ||\cdot||)$, where R is a K algebra and $||\cdot|| : R \to \mathbb{R}_{\geq 0}$ is a function such that:
  - $||x|| = 0$ if and only if $x = 0$;
  - for all $x, y \in K$, $||xy|| \leq ||x|| \cdot ||y||$;
  - for all $x, y \in K$, $||x + y|| \leq ||x|| + ||y||$.
  - for all $x \in R$, for all $\lambda \in K$, we have $||\lambda x|| = |\lambda| \cdot ||x||$.
  - R is complete with respect to the topology induced by $||\cdot||$.

- The ring R° of *powerbounded* elements consists of all the elements $x \in R$ such that the set of positive real numbers $\{||x^n||\}_n$ is bounded. In particular, if $R = K$, $K^\circ = \{x \in K | |x| \leq 1\}$ is a local ring.

- The set $R^{\circ\circ}$ of *topologically nilpotent* elements consists of all the elements $x \in R$ such that $\text{liminf}_n ||x^n|| = 0$. In particular, if $R = K$, $K^{\circ\circ} = \{x \in K | |x| < 1\}$ is the maximal ideal of K° and it will be called $\mathfrak{m}$.

Example 2. The ring of p-adic integers $\mathbb{Z}_p$ admits a valuation $|\cdot|_p$ that maps any non zero element x to $p^{-e(x)}$, where $e(x)$ is the maximal exponent such that $p^{e(x)}$ divides x. This valuation can be extended to its fraction field $\mathbb{Q}_p$, the field of p-adic numbers, making $(\mathbb{Q}_p, |\cdot|_p)$ a complete normed field.

Let's define perfectoid fields.

Definition 3.1.2. A perfectoid field is a complete normed field $(K, |\cdot|)$ such that:

- its group of valuation $\Gamma \subseteq \mathbb{R}_{\geq 0}$ is nondiscrete;
- the local ring $(K^\circ, \mathfrak{m})$ has residue characteristic p;
- the Frobenius endomorphism on K°/p (i.e. the map that sends x to x^p) is surjective.

For every perfectoid field we will choose $\omega \in K^\circ$ a fixed nonzero element with $|p| \leq \omega < 1$ (if K has characteristic 0, we can take $\omega = p$).

Remark 3.1.3. If K has characteristic p, $K^\circ/p = K^\circ$, and the Frobenius endomorphism is surjective if and only if K is a perfect field.

Example 3. Consider the ring $\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]$ as constructed in Example 1. We can extend (in a unique way) the p-adic valuation $|\cdot|_p$ to this ring, and complete it with respect to the induced topology. We will denote the resulting ring, its quotient field, and its maximal ideal respectively in the following way:

$$\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}, \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}, (p^{\frac{1}{p^\infty}}).$$

It's easy to check that:

- $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$ is a perfectoid field;
- $\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$ is its local ring of powerbounded elements;
- $(p^{\frac{1}{p^\infty}})$ is the ideal of topologically nilpotent elements.

From now on, K will always denote a perfectoid field.

Lemma 3.1.4. *The group of valuations Γ is (multiplicatively) p-divisible.*

Proof. For every $\bar{x} \in K^\circ/p \setminus \{0\}$, any lifting $x \in K^\circ$ has norm greater than $|p|$, therefore they all have the same norm. Since the Frobenius is surjective, for every $x \in K^\circ$ with $|p| < |x| \leq 1$, $\bar{x} \neq 0$ and there is $y \in K^\circ$ such that $\bar{y}^p = \bar{x}$, therefore $|y|^p = |x|$. Since the norm is nondiscrete, there is at least an x with $|p| < |x| < 1$, and every element of K° can be recursively written as a product of elements with this property: thus for every $x \in K^\circ$ there is $y \in K^\circ$ such that $|y|^p = |x|$. ☐

The main concept around which much of the perfectoid theory is built is the *tilting operation*, a construction that we will explain with the next proposition.

Proposition 3.1.5. *Consider the ring K°/ω, of characteristic p, and call $\Phi : x \mapsto x^p$ its Frobenius endomorphism. We have the following.*

- *The projection $\pi : K^\circ \to K^\circ/\omega$ induces an isomorphism of topological multiplicative monoids: $\tilde{\pi} : \varprojlim_{x \to x^p} K^\circ \to \varprojlim_\Phi K^\circ/\omega$. In particular, since Φ is a ring homomorphism, the RHS is a ring (of characteristic p) because it's a projective limit of rings, and the LHS acquires the same ring structure.*

  There is a natural multiplicative map $\varprojlim_{x \to x^p} K^\circ \to K^\circ$, given by the projection on the first coordinate; it sends f to $f^\#$.

- *The norm on K induces on $\varprojlim_\Phi K^\circ/\omega$ a multiplicative map $f \mapsto |f^\#|$: it is a valuation.*

- *There is an element $\omega^\flat \in \varprojlim_\Phi K^\circ/\omega$ such that $|(\omega^\flat)^\#| = |\omega|$. We define K's tilt as $K^\flat := \varprojlim_\Phi K^\circ/\omega[(\omega^\flat)^{-1}]$. The previous multiplicative map can be extended to $K^\flat \to K$.*

  (Without losses of generality, from here on out we will take $\omega := (\omega^\flat)^\#$; in particular, ω will have a system of p power roots, given by $\left((\omega^\flat)^{\frac{1}{p^n}}\right)^\#$).

- *The natural norm $K^\flat \ni f^\flat \mapsto |(f^\flat)^\#|$ induces the limit topology on $\varprojlim_\Phi K^\circ/\omega$. Moreover, $K^{\flat\circ}$ is exactly $\varprojlim_\Phi K^\circ/\omega$, and $K^\flat$ is a perfectoid field.*

- *$K^\flat$ and $\varprojlim_{x \to x^p} K$ are isomorphic as topological multiplicative monoids. Moreover, if $\mathrm{char} K = p$, $K^\flat = K$.*

Proof.
- Let's construct a map from $\varprojlim_\Phi K^\circ/\omega$ to K. Any element on the LHS can be thought of as a sequence $x := (\bar{x}^{\frac{1}{p^n}})_n$; take two different lifting of each element in K°: $(x_n)_n$ and $(x'_n)_n$. Let's consider the sequences $x_n^{p^n}$ and $x'^{p^n}_n$ and write $x'_n - x_n = a\omega$, with $a \in K^\circ$; then $x_n^{p^n} - x'^{p^n}_n = bp^n + a^{p^n}\omega^{p^n}$, which is divisible by ω^n: if the two sequences have a limit in K°, it does not depend on the lifting.

  For the existence, remember that x_{n+1}^p and x_n are congruent modulo ω: call their difference $c\omega$ with $c \in K^\circ$. Like before, $x_{n+1}^{p^{n+1}} - x_n^{p^n}$ is divisible by ω^n, therefore the sequence converges. Call $x^\#$ this limit: our map will send x to $x^\#$.

  Let's prove the multiplicativity: if x lifts to $(x_n)_n$ and y lifts to $(y_n)_n$, one lifting of xy is $(x_n y_n)_n$, and $(xy)^\# = \lim_n (x_n y_n)^{p^n} = \lim_n x_n^{p^n} \lim_n y_n^{p^n} = x^\# y^\#$. For the continuity, if x and y coincide on the first k coordinates, then we can lift them to sequences $(x_n)_n$ and $(y_n)_n$ that also coincide on the first k coordinates. ω divides $x_{k+n}^{p^n} - y_{k+n}^{p^n}$: like before $x_{k+n}^{p^{k+n}} - y_{k+n}^{p^{k+n}}$ is divisible by ω^k: the same will be true for $x^\# - y^\#$.

  Now, if $x = (\bar{x}^{\frac{1}{p^n}})_n \in \varprojlim_\Phi K^\circ/\omega$, we can call $x^{\frac{1}{p^k}} := (\bar{x}^{\frac{1}{p^{k+n}}})_n$, and define a (continuous and multiplicative) map $\tilde{h} : \varprojlim_\Phi K^\circ/\omega \to \varprojlim_{x \to x^p} K^\circ$ which sends $x = (\bar{x}^{\frac{1}{p^n}})_n$ to $((x^{\frac{1}{p^n}})^\#)_n$. Since $(x^{\frac{1}{p^n}})^\# \equiv \bar{x}^{\frac{1}{p^n}} \mod \omega$, $\tilde{\pi} \circ \tilde{h}$ is the identity. Conversely, every $(x^{\frac{1}{p^n}})_n \in \varprojlim_{x \to x^p} K^\circ$ is a lift of its own projection, therefore $(\tilde{\pi}(x^{\frac{1}{p^n}})_n)^\# = \lim_n (x^{\frac{1}{p^n}})^{p^n} = x$; it follows that $\tilde{h} \circ \tilde{\pi}$ is the identity, therefore both maps were isomorphisms.

- To show that the map $f \mapsto |f^\#|$ is a valuation, it suffices to check that, for all $f, g \in \varprojlim_\Phi K^\circ/\omega$, $|(f+g)^\#| \leq \max\{|f^\#|, |g^\#|\}$. We will write $f = (\bar{f}^{\frac{1}{p^n}})_n$ and $g = (\bar{g}^{\frac{1}{p^n}})_n$. From the proof of the previous point, we know that for some big enough m, any lifting $\tilde{f}$ of $\bar{f}^{\frac{1}{p^m}}$ is such that $|\tilde{f}^{p^m}| = |f^\#|$; moreover, we can take an m such that this is true at the same time for f, g, and $f+g$. If we choose the liftings $\tilde{f}$, $\tilde{g}$, and $\tilde{fg}$ respectively of $\bar{f}^{\frac{1}{p^m}}$, $\bar{g}^{\frac{1}{p^m}}$, and $\bar{fg}^{\frac{1}{p^m}}$, such that $\tilde{f}+\tilde{g}=\tilde{fg}$, we get that:

$$|(f+g)^{\#}| = |(\tilde{f}+\tilde{g})|^{p^m} \leq \max\{|\tilde{f}|, |\tilde{g}|\}^{p^n} = \max\{|\tilde{f}|^{p^n}, |\tilde{g}|^{p^n}\} = \max\{|f^{\#}|, |g^{\#}|\}.$$

- Take $x \in K^\circ$ such that $|x|^p = |\omega|$, and let $\bar{x} \in K^\circ/\omega \setminus \{0\}$ be its projection. Consider a system of p power roots $\tilde{x} := (\bar{x}^{\frac{1}{p^n}})_n$ (which exists because the Frobenius is surjective), and take its p-th power: $\omega^\flat := (0, \bar{x}, \bar{x}^{\frac{1}{p}}, \cdots)$. As observed in the previous points, $(\tilde{x})^{\#} \equiv x \mod \omega$, therefore $|(\tilde{x})^{\#}| = |x|$, which implies that $|(\omega^\flat)^{\#}| = |(\tilde{x}^p)^{\#}| = |(\tilde{x})^{\#}|^p = |x|^p = |\omega|$.

- Let's prove the second part of the statement first.

  Obviously, $\varprojlim_{x\to x^p} K^\circ \subseteq K^{\flat\circ}$ by definition of the norm on $K^\flat$. Vice versa, take $f^\flat \in K^\flat$ with $|(f^\flat)^{\#}| \leq 1$. By definition of $K^\flat$ there is some N such that $(\omega^\flat)^N f^\flat \in \varprojlim_{x\to x^p} K^\circ$; write $(\omega^\flat)^N f^\flat = (f_0, f_1, \cdots)$. Denoting with $\omega^{\frac{1}{p^n}}$ the image in K of the only p^n-th root of $\omega^\flat \in K^\flat$, we have:

$$\left(\frac{f_n}{\omega^{\frac{N}{p^n}}}\right)^{p^n} = \frac{f_0}{\omega^N} = \frac{\left((\omega^\flat)^N f^\flat\right)^{\#}}{\left((\omega^\flat)^N\right)^{\#}} = (f^\flat)^{\#} \in K^\circ.$$

  So we can take the sequence $(f_n\omega^{-\frac{N}{p^n}})_n \in \varprojlim_{x\to x^p} K^\circ$. If multiplied by $(\omega^\flat)^N = (\omega^{-\frac{N}{p^n}})_n$, it yields $(\omega^\flat)^N f^\flat$, therefore $f^\flat = (f_n\omega^{-\frac{N}{p^n}})_n \in \varprojlim_{x\to x^p} K^\circ$.

  Now we want to prove that the Frobenius map on ${K^\flat}^\circ$ is surjective, that the norm on $K^\flat$ is nondiscrete, and that $K^\flat$ is complete with respect to the topology induced by the norm.

  The surjectivity of the Frobenius map follows from the identification of ${K^\flat}^\circ$ we just gave. This also implies that the norm is nondiscrete, since it suffices to find elements with norm arbitrarily close to 1, and to do that we can consider all the p-power roots of $\omega^\flat$.

  To show that $(K^\flat, |\cdot^{\#}|)$ is complete, it suffices to prove it for ${K^\flat}^\circ$, and we can think its elements as successions in $\varprojlim_\Phi K^\circ/\omega$. Consider a Cauchy sequence with respect to the norm $|\cdot^{\#}|$: $\{(x_n^{(0)})_n, (x_n^{(1)})_n, (x_n^{(2)})_n, \cdots\}$. For every k, there is m such that for all $i, j > m$ $(x_n^{(i)} - x_n^{(j)})_n$ has norm less that $|\omega|^{p^k}$; in particular, $x_n^{(i)} - x_n^{(j)} = 0$ for all $n \leq k$. Therefore, we get that these sequences converge pointwise (i.e. in the limit topology of $\varprojlim_\Phi K^\circ/\omega$) to some element. With the completeness, we have proven that $(K^\flat, |\cdot^{\#}|)$ is a perfectoid field.

  For the first part of the statement, note that we have just shown that Cauchy sequences in the topology induced by the norm are the same as converging sequences with respect to the limit topology, and the inverse is obvious by a similar reasoning.

- Take $(\bar{x}^{\frac{1}{p^n}})_n \in \varprojlim_{x\to x^p} K$. $x \in K$ can be written as $\frac{y}{\omega^k}$ for some k, where $\omega := (\omega^\flat)^{\#}$, and $y \in K^\circ$. If $\omega^\flat = (\bar{\omega}^{\frac{1}{p^n}})_n$ (with $n > 0$), we can write:

$$(\bar{x}^{\frac{1}{p^n}})_n = \left((\bar{y}(\bar{\omega})^{-k})^{\frac{1}{p^n}}\right)_n = (\bar{y}^{\frac{1}{p^n}})_n(\omega^\flat)^{-k},$$

  so $\varprojlim_{x\to x^p} K \subseteq K^\flat$. Vice versa, as we already wrote, $(\omega^\flat)^{-1} = \left(\frac{1}{\bar{\omega}}^{\frac{1}{p^n}}\right)_n$, so the other containment is also true.

Finally, if $\mathrm{char}K = p$, $\varprojlim_{x \to x^p} K \cong K$ via the map that sends $(x^{\frac{1}{p^n}})_n$ to x, because every element $x \in K$ admits a system of p-power roots.

$\square$

Example 4. Let's consider again $K := \widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$, with $K^\circ = \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$, and let's compute its tilt. We can take $\omega = p$, therefore:

$$\varprojlim_{\Phi} K^\circ/\omega = \varprojlim_{\Phi} \widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}/p \cong \varprojlim_{t \mapsto t^p} \mathbb{F}_p[t^{\frac{1}{p^\infty}}]/t \cong \mathbb{F}_p[[t]][t^{\frac{1}{p^\infty}}].$$

Thus $K^\flat \cong \mathbb{F}_p((t))(t^{\frac{1}{p^\infty}})$, and in this case we can take $\omega^\flat = t$.

Remark 3.1.6. In the previous proposition we noted that we can choose ω such that it admits a system of p-power roots. Moreover, since $\mathfrak{m} \subseteq K^\circ$ consists of all the elements with norm strictly less than 1, we can think of $\{\omega^{\frac{1}{p^k}}\}_k$ as a set of generators for $\mathfrak{m}$. Since this is the same requirement as Chapter 2, we may talk about almost mathematics (with respect to ω) also in the context of perfectoid theory.

After this preliminary work, we may present the main classes of objects studied in perfectoid theory (although we will mainly need just the first category from the following definition).

Definition 3.1.7. As always, let K be a perfectoid field.

- A perfectoid K-algebra R is a Banach K-algebra such that the set of powerbounded elements $R^\circ \subset R$ is open and bounded, and the Frobenius endomorphism on R°/ω is surjective. A morphism of perfectoid K-algebras is a morphism of Banach K-algebras, and the corresponding category is called $K - perf$.
- A perfectoid $K^{\circ a}$-algebra A is an almost ω-adically complete and almost flat $K^{\circ a}$-algebra A on which the Frobenius endomorphism induces an almost isomorphism $A/\omega \cong A/\omega^{\frac{1}{p}}$. A morphism of perfectoid $K^{\circ a}$-algebras is simply a morphism of $K^{\circ a}$-algebras, and the corresponding category is called $K^{\circ a} - perf$.
- A perfectoid $K^{\circ a}/\omega$-algebra is an almost flat $K^{\circ a}/\omega$-algebra $\bar{A}$ on which the Frobenius endomorphism induces an almost isomorphism $\bar{A} \cong \bar{A}/\omega^{\frac{1}{p}}$. A morphism of perfectoid $K^{\circ a}/\omega$-algebras is simply a morphism of $K^{\circ a}/\omega$-algebras, and the corresponding category is $K^{\circ a}/\omega - perf$.

Example 5. The main example of a perfectoid K-algebra is $R = K\langle T^{\frac{1}{p^\infty}}\rangle$, which is the algebra obtained by adjoining a variable T and all its p-power roots to K, and taking the ω-adic completion (we can think of this object as the analogous in perfectoid theory to the ring $\mathbb{C}[t]$ in complex geometry).

To better understand the meaning of a perfectoid $K^{\circ a}$-algebra, let's explore the definition with the following proposition.

Proposition 3.1.8. *Take $M \in K^{\circ a} - \mathbf{Mod}$.*

1. *M is almost flat if and only if M_* is flat, if and only if M_* has no ω-torsion.*
2. *If $N \in K^\circ - \mathbf{Mod}$ is flat and $M = N^a$, then M is almost flat and $M_* \cong \tilde{N}$, where $\tilde{N} = \{x \in N[\omega^{-1}] | x\mathfrak{m} \subseteq N\}$.*
3. *If M is almost flat, then $(xM)_* = x(M_*)$ for all $x \in K^\circ$. Moreover $M_*/xM_* \subseteq (M/xM)_*$ contains the image of the natural map $(M/x\omega^\alpha M)_* \to (M/xM)_*$ for all $\alpha > 0$.*

4. *If M is almost flat, then M is almost ω-adically complete if and only if M_* is ω-adically complete.*

Proof. 1. By Proposition 2.1.7 we know that $(M_*)^a \approx M$, therefore by definition M is almost flat if $\mathrm{Tor}_1^{K^\circ}(M_*, N) \approx 0$ for every K°-module N, which is implied by M_*'s flatness.

Suppose M to be almost flat. Tensoring by M_* the inclusion $\omega K^\circ \subseteq K^\circ$ we get an almost injective map $M_* \to M_*$, whose kernel N is comprised of the elements of ω-torsion in M_*. Always by Proposition 2.1.7 we know that $M_* \cong (M_*^a)_* \cong \mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*)$, and the isomorphism sends $x \in M_*$ to the map $a \to ax$: in particular, for all $x \in M_*$ there is an $a \in \mathfrak{m}$ such that $ax \neq 0$, which means that there are no almost zero elements in M_*. As a result, N must be 0, so there are no ω-torsion elements in M_*.

Finally, suppose that M_* has no ω-torsion, which means that for every k it has no $\omega^{\frac{1}{p^k}}$-torsion. An equivalent condition for flatness is that $\mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/I) = 0$ for every proper ideal $I \subsetneq K^\circ$, and these can be of only two types: $I_\alpha := (\omega^\alpha)$ and $J_\alpha := \bigcup_{\beta>\alpha} I_\beta$. Tensoring by M_* the inclusion $I_\alpha \subseteq K^\circ$ we get the exact sequence:

$$0 \longrightarrow \mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/I_\alpha) \longrightarrow M_* \xrightarrow{-\cdot\omega^\alpha} M_*,$$

but since M_* has no ω^α-torsion, $\mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/I_\alpha) = 0$. For J_α we get the following commutative diagrams for every $\beta > \alpha$:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & I_\beta \arrow[r] \arrow[d] & K^\circ \arrow[d, equal] \\ 0 \arrow[r] & J_\alpha \arrow[r, "i_\alpha"] & K^\circ \end{tikzcd} \implies \begin{tikzcd} 0 \arrow[r] \arrow[d] & M_* \arrow[r, "-\cdot\pi_\beta"] \arrow[d, "i_{\beta,\alpha}"] & M_* \arrow[d, equal] \\ \mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/J_\alpha) \arrow[r] & M_* \otimes J_\alpha \arrow[r, "M_*\otimes j_\alpha"] & M_* \end{tikzcd}.$$

Tensoring by M commutes with direct limits, therefore $M \otimes J_\alpha = \varinjlim_{\beta>\alpha} M \otimes I_\beta$. In particular, every element $y \in M \otimes J_\alpha$ can be written as $i_{\beta,\alpha}(x)$ for some $\beta > \alpha$ and some $x \in M_*$, so:

$$0 \neq x\pi_\beta = (M_* \otimes j_\alpha)(y), \circ i_{\beta,\alpha}(x) = (M_* \otimes j_\alpha)(y),$$

therefore $M_* \otimes j_\alpha$ is injective and $\mathrm{Tor}^1_{K^\circ}(M_*, K^\circ/J_\alpha) = 0$.

2. Again, N flat implies $M = N^a$ almost flat. Consider the morphism $\phi : \tilde{N} \to M_*$ that sends x to the map $a \mapsto ax$ in $\mathrm{Hom}_{K^\circ}(\mathfrak{m}, N) \cong M_*$. Let's prove ϕ is injective. Take $y = \frac{x}{\omega^n} \in \ker(\phi)$, with $x \in N$. $\phi(y) = 0$ implies $y\omega^n = 0$, therefore $x = 0$. For the surjectivity, take $f \in \mathrm{Hom}_{K^\circ}(\mathfrak{m}, N)$ and call $x := f(\omega)$. For any $a \in \mathfrak{m}$, multiplication by $\frac{x}{\omega}$ yields:

$$\frac{ax}{\omega} = \frac{af(\omega)}{\omega} = \frac{f(a\omega)}{\omega} = \frac{\omega f(a)}{\omega} = f(a) \in N,$$

therefore $\frac{x}{\omega} \in \tilde{N}$, and its image via ϕ is f.

3. Since $(M_*)^a \approx M$, $(x(M_*))^a \approx xM$, but $x(M_*)$ has no ω-torsion, therefore it is flat and by the previous point $x(M_*) = (xM)_*$. Since $(-)_*$ is right adjoint to $(-)^a$, it is left exact: the exact sequence $0 \to xM \to M \to M/xM \to 0$ yields the exact sequence $0 \to (xM)_* \to M_* \to (M/xM)_*$; in particular there is an inclusion $M_*/(xM)_* \hookrightarrow (M/xM)_*$. Moreover, the respective almost modules are the same because the composition of functors $((-)_*)^a$ is isomorphic to the identity on $K^{\circ a} - \mathbf{Mod}$, therefore:

$$(M/xM)_* = (((M/xM)_*)^a)_* = ((M_*/(xM)_*)^a)_* = \mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*/(xM)_*).$$

Consider the natural map $(M/x\omega^\alpha M)_* \to (M/xM)_*$ induced by multiplication by ω^α. If $m \in (M/xM)_*$ can be lifted to $\tilde{m} \in (M/x\omega^\alpha M)_* \cong \mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*/x\omega^\alpha M_*)$, call $n := \tilde{m}(\omega^\alpha)$ and lift it to $\tilde{n} \in M_*$. Finally, $\tilde{n}$ is divisible by ω^α if $\omega^\alpha | a\tilde{n}$ for every $a \in \mathfrak{m}$, and we have:

$$\overline{a\tilde{n}} = an = a\tilde{m}(\omega^\alpha) = \tilde{m}(a\omega^\alpha) = \omega^\alpha \tilde{m}(a).$$

Finally, let's take $n' := \frac{\tilde{n}}{\omega^\alpha} \in M_*$. The natural map $M_* \to (M/xM)_*$ sends n' to the following map in $\mathrm{Hom}_{K^\circ}(\mathfrak{m}, M_*/xM_*)$:

$$a \mapsto \overline{an'} = \overline{an'} = \frac{an}{\omega^\alpha} = \frac{a\tilde{m}(\omega^\alpha)}{\omega^\alpha} = \frac{\tilde{m}(a\omega^\alpha)}{\omega^\alpha} = \frac{\omega^\alpha \tilde{m}(a)}{\omega^\alpha} = \tilde{m}(a).$$

4. As already seen in 2.1.7, the functors $(-)_*$ and $(-)^a$ are both right adjoints, so they commute with limits. If M is almost ω-adically complete, we have:

$$M_* = (\varprojlim_n M/\omega^n M)_* = \varprojlim_n (M/\omega^n M)_* \cong \varprojlim_n M_*/\omega^n M_*,$$

where the last isomorphism is due to the fact that $M_*/\omega^n M_* \subseteq (M/\omega^n M)_*$ and the transition map $(M/\omega^{n+1}M)_* \to (M/\omega^n M)_*$ has image contained in $M_*/\omega^n M_*$ for all n, as proven in the previous point.

Vice versa, if M_* is ω-adically complete:

$$M \approx (M_*)^a = (\varprojlim_n M_*/\omega^n M_*)^a = \varprojlim_n (M_*/\omega^n M_*)^a \cong \varprojlim_n M/\omega^n M.$$

$\square$

Remark 3.1.9. Given a perfectoid K-algebra, one can define its tilt in the same way as for K, to obtain a perfectoid $K^\flat$-algebra. A fundamental result of perfectoid theory states that the tilting functor induces an equivalence of the corresponding categories of perfectoid algebras $K - perf \cong K^\flat - perf$ ([Sch12, Theorem 5.2]); this has a nice philosophical implication: that studying rings of mixed characteristic $(0, p)$ is equivalent to studying rings of characteristic p if one works in the context of perfectoid algebras.

Finally, we may formulate in the language of perfectoid algebras Faltings' almost purity theorem:

Theorem 3.1.10 ([Sch12, Theorem 7.9.iii])**.** *Let R be a perfectoid K-algebra, and $R \to S$ a finite étale extension. Then S is perfectoid and S° is almost finite étale over R°.*

3.2 Perfectoid spaces

The following constructions and propositions outline the start of [Sch12, Chapter 6]. Scholze's aim is to assign a topological space and an associated sheaf to a given perfectoid K-algebra, in the same way as with adic spaces. We will follow his line of reasoning, but we will only include the proofs that we will need in the following chapters.

First, we construct the underlying space.

Definition 3.2.1. A perfectoid affinoid K-algebra is a pair (R, R^+), where R is a perfectoid K-algebra and $R^+ \subseteq R^\circ$ is an open and normal subring.

$$X := Spa(R, R^+) = \{v : R \to \Gamma \cup \{0\} \text{ continuous valuation} | \forall f \in R^+ : v(f) \leq 1\}/\sim,$$

where Γ is the value group of K. For any $x \in X$ we will write the associated valuation as $f \mapsto |f(x)|$. This space will be given the topology generated by the following so-called rational subsets:

$$U\left(\frac{f_1 \cdots f_n}{g}\right) := \{x | \forall i \, |f_i(x)| \leq |g(x)|\},$$

where $g \in R$ and $f_1, \cdots, f_n \in R$ generate R as an R-module.

Remark 3.2.2. Since R^+ is integrally closed in R°, it can be easily shown that $\mathfrak{m}R^\circ \subseteq R^+ \subseteq R^\circ$. In particular, all the possible choices for R^+ are almost isomorphic to R°.

We now define a presheaf on rational subsets and then extend it to all open subsets by taking a limit. The definition is as follows.

Definition 3.2.3. Let $U = U\left(\frac{f_1,\cdots,f_n}{g}\right)$. Choose an open subring $R_0 \subseteq R$ such that $\{aR_0\}_{a\in K^*}$ is a basis of open neighborhoods of 0. Consider the algebra $R[g^{-1}]$ and equip it with the topology making $\{aR_0[\frac{f_1}{g}, \cdots, \frac{f_n}{g}]\}_{a\in K^*}$ a basis of open neighborhoods of 0. Finally let $B \subseteq R[g^{-1}]$ the normalization of $R^+[\frac{f_1}{g}, \cdots, \frac{f_n}{g}]$. We define the pair $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ as the completion of the pair $(R[\frac{f_1}{g}, \cdots, \frac{f_n}{g}], B)$.

If $W \subseteq X$ is an open set we define, with obvious notation:

$$(\mathcal{O}_X(W), \mathcal{O}_X^+(W)) := \varprojlim_{U \subseteq W \text{ rational subset}} (\mathcal{O}_X(U), \mathcal{O}_X^+(U)).$$

For any open set W, the pair $(\mathcal{O}_X(W), \mathcal{O}_X^+(W))$ is itself a perfectoid affinoid K-algebra. We now include the statement of the main result of [Sch12, Chapter 6], which makes explicit the interplay between this construction and the tilting operation.

Theorem 3.2.4 ([Sch12, Theorem 6.3])**.** *Let (R, R^+) be a perfectoid affinoid K-algebra, and let $X = Spa(R, R^+)$ with associated presheaves $(\mathcal{O}_X, \mathcal{O}_X^+)$. Also, let $(R^\flat, {R^\flat}^+)$ be its tilt, and let $X^\flat = Spa(R^\flat, {R^\flat}^+)$. Then:*

- *the map $X \to X^\flat$ that sends the valuation $R \ni f \mapsto |f(x)|$ to the valuation $R^\flat \ni g \mapsto |g^\#(x)|$ is a homeomorphism;*
- *for every rational subset $U \subseteq X$ with image $U^\flat \subseteq X^\flat$ the pair $\big(\mathcal{O}_X(U), \mathcal{O}_X^+(U)\big)$ is a perfectoid affinoid K-algebra, with tilt $\big(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat)\big)$;*
- *$\mathcal{O}_X$ is a sheaf, and $\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) | |f(x)| \leq 1 \forall x \in U\}$.*

We will only highlight some intermediate results which will be needed for the main argument of this thesis. First, we observe that the map $X \to X^\flat$ is continuous because the preimage of $U = U\left(\frac{f_1,\cdots,f_n}{g}\right)$ is $U^\# = U\left(\frac{f_1^\#,\cdots,f_n^\#}{g^\#}\right)$. The next proposition explicitly describes $\mathcal{O}_X(U^\#)$ for any rational set $U \subseteq X^\flat$ at the almost-integral level.

Proposition 3.2.5 ([Sch12, Lemma 6.4])**.** *Let U and $U^\#$ be as above.*

(1) Consider the map:

$$R^\circ\left[T_1^{\frac{1}{p^\infty}}, \cdots, T_n^{\frac{1}{p^\infty}}\right] \to R^\circ\left[\left(\frac{f_1^\#}{g^\#}\right)^{\frac{1}{p^\infty}}, \cdots, \left(\frac{f_n^\#}{g^\#}\right)^{\frac{1}{p^\infty}}\right]$$

$$T_i^{\frac{1}{p^k}} \mapsto f_i^{\frac{1}{p^k}}.$$

It is surjective, and its kernel is almost isomorphic to the ideal:

$$I := (g^{\#\frac{1}{p^k}} T_i^{\frac{1}{p^k}} - f_i^{\#\frac{1}{p^k}})_{i,k}$$

(2) The ring $R^\circ \left\langle \left(\frac{f_1^\#}{g^\#}\right)^{\frac{1}{p^\infty}}, \cdots, \left(\frac{f_n^\#}{g^\#}\right)^{\frac{1}{p^\infty}} \right\rangle$ *is a perfectoid* $K^{\circ a}$*-algebra.*

(3) $\mathcal{O}_X(U^\#)^\circ (\approx \mathcal{O}_X^+(U^\#))$ *is almost isomorphic to* $R^\circ \left\langle \left(\frac{f_1^\#}{g^\#}\right)^{\frac{1}{p^\infty}}, \cdots, \left(\frac{f_n^\#}{g^\#}\right)^{\frac{1}{p^\infty}} \right\rangle$.

(4) The tilt of $\mathcal{O}_X(U^\#)$ *is given by* $\mathcal{O}_{X^\flat}(U)$.

From an operative point of view, we are only interested to the proof of point (1) in the following particular case, which we will use in Proposition 4.3.1.

Lemma 3.2.6. *Let* $g \in R^\circ$ *be an element such that:*

- g *is not a zero divisor;*
- R° *contains a system of* p*-power roots of* g*;*
- *there is no* k *such that* $\omega^{\frac{1}{p^k}} | g$.

Fix some m *and consider the map:*

$$\pi : R^\circ \left[T_1^{\frac{1}{p^\infty}}\right] \to R^\circ \left[\left(\frac{\omega^m}{g}\right)^{\frac{1}{p^\infty}}\right]$$
$$T^{\frac{1}{p^k}} \mapsto \left(\frac{\omega^m}{g}\right)^{\frac{1}{p^k}}.$$

It is surjective, with kernel:

$$I := ((Tg)^{\frac{1}{p^k}} - \omega^{\frac{m}{p^k}})_k.$$

Proof. Surjectivity is obvious. Fix some k and consider the induced map:

$$\pi : R^\circ \left[T_1^{\frac{1}{p^k}}\right] \to R^\circ \left[\left(\frac{\omega^m}{g}\right)^{\frac{1}{p^k}}\right]$$
$$T^{\frac{1}{p^k}} \mapsto \left(\frac{\omega^m}{g}\right)^{\frac{1}{p^k}}.$$

If we prove that this map has kernel generated by $x_k := (Tg)^{\frac{1}{p^k}} - \omega^{\frac{m}{p^k}}$, passing to the colimit we get the statement of the lemma.

Inverting ω the RHS becomes $R^\circ[g^{-1}]$, and the kernel becomes exactly:

$$((Tg)^{\frac{1}{p^k}} - \omega^{\frac{m}{p^k}}) R^\circ \left[\omega^{-1}, T_1^{\frac{1}{p^k}}\right],$$

therefore we just have to prove that the intersection between this module and $R^\circ \left[T_1^{\frac{1}{p^k}}\right]$ is (x_k). By contradiction, if this were not the case we could find some $f \in R^\circ \left[T_1^{\frac{1}{p^k}}\right]$ and $\alpha > 0$ such that $x_k \omega^{-\alpha} f$ is contained in $R^\circ \left[T_1^{\frac{1}{p^k}}\right]$, while $\omega^{-\alpha} f$ is not. This implies that there is some $\beta > 0$ such that $\omega^\beta | x_k$, which means that some power of ω divides $(Tg)^{\frac{1}{p^k}}$, which is contrary to the third hypothesis. $\square$

A priori, Proposition 3.2.5 does not cover all the possible rational subsets in X. This problem can be solved with the following approximation lemma, which we will not prove:

Proposition 3.2.7 ([Sch12, Corollary 6.7.i])**.** *For any $f \in R$ and any $c > 0, \varepsilon > 0$, there is $h(f)_{c,\varepsilon} \in R^\flat$ such that for all $x \in X$ we have $|f(x) - h(f)^\#_{c,\varepsilon}(x)| \leq |\omega|^{1-\varepsilon} \max\{|f(x)|, |\omega|^c\}$.*

Corollary 3.2.8. *Every rational subset in X of the form $U\left(\frac{f_1, \cdots, f_n}{g}\right)$ for some $f_i, g \in R$ can be written as a preimage of a rational subset in $X^\flat$, i.e. as $U\left(\frac{(f_1^\flat)^\#, \cdots, (f_n^\flat)^\#}{(g^\flat)^\#}\right)$ for some $f_i^\flat, g^\flat \in R^\flat$.*

Proof. We have already proven that there is some N such that $|\omega^N| < |g(x)|$ for all $x \in X$. If we take $g^\flat := h(g)_{N,\frac{1}{2}}$ we have:

$$|g(x) - (g^\flat)^\#(x)| \leq |\omega|^{\frac{1}{2}} \max\{|g(x)|, |\omega|^N\} < |g(x)|,$$

which implies $|g(x)| = |(g^\flat)^\#(x)|$ for all $x \in X$, so we can substitute g with $g^\flat$ with no problems. If we take $f^\flat := h(f)_{N,\frac{1}{2}}$, we just need to show that there are no $x \in X$ for which only one between $|f(x)|$ and $|(f^\flat)^\#(x)|$ is greater than $|g(x)|$. If this happened, in particular the two norms would be different, and we would have:

$$\max\{|f(x)|, |(f^\flat)^\#(x)|\} = |f(x) - (f^\flat)^\#(x)| \leq |\omega|^{\frac{1}{2}} \max\{|f(x)|, |\omega|^N\} < |f(x)|,$$

which is a contradiction. $\square$

Chapter 4

Almost-pro-modules

In this chapter we talk about pro-modules and their almost analogue, following [Bha18, Section 3-4]. To keep the exposition clean, we will take for granted some notion of category theory.

4.1 Definition of pro-modules

In this section we will give a very quick presentation of pro-modules. Let's first define what we mean by pro-module:

Definition 4.1.1. Let R be a (commutative) ring and a N the poset of natural numbers, viewed as a category where the arrow goes from the bigger number to the smaller. Take any functor $F : \mathbb{N} \to R-\mathbf{Mod}$, and compose it with the Yoneda embedding $Y : R-\mathbf{Mod} \to [R-\mathbf{Mod}^{\mathrm{op}}, \mathbf{Set}]$. A pro-$R$-module is the limit of $Y \circ F$ in $[R-\mathbf{Mod}^{\mathrm{op}}, \mathbf{Set}]$.

We can identify a pro-R-module with a sequence: $M_0 \xleftarrow[f_0^1]{} M_1 \xleftarrow[f_1^2]{} M_2 \xleftarrow[f_2^3]{} M_3 \xleftarrow[f_3^4]{} \cdots$.

Given two pro-R-modules, $\{M_i\}_i$ and $\{N_j\}_j$, the morphisms between them are given by:

$$\varprojlim_j \varinjlim_i \mathrm{Hom}_R(M_i, N_j).$$

Concretely, we can think of it as a collection of morphisms $\phi_j : M_{k_j} \to N_j$, where $k_j \geq j$, with the obvious commutation conditions between the ϕ_j's and the transition maps (furthermore, one can always tweak the indexes without changing the isomorphism class so that $i_j = j$).

Remark 4.1.2. The morphism $\{\phi_j\}_j$ is zero if and only if for all j ϕ_j induces the zero map on $M_i \to N_j$ for some large enough i. Consequently, a pro-R-module $\{M_i\}_i$ is zero when its identity is the zero map, i.e. when for all j there is $k \geq j$ such that the transition map $f_j^k : M_k \to M_j$ is zero.

Remark 4.1.3. We can think of an isomorphism between $\{M_i\}_i$ and $\{N_i\}_i$ - after reindexing - as a collection $\{f_i : M_i \to N_i\}_i$ such that $\{\ker(f_i)\}_i$ and $\{\mathrm{coker}(f_i)\}_i$ (with the obvious transition maps) are zero as pro-R-modules.

4.2 Definition of almost-pro-modules

We need to define an analogue version of pro-modules to be used in almost mathematics (with respect to a certain system $\{t^{\frac{1}{p^k}}\}_k$ in R). As objects, almost-pro-R-modules will be the same as pro-R-modules. As usual, the crux of the matter is understanding when to define an object as almost zero - all the other definitions can be obtained in a similar way to what we did in Chapter 2.

Definition 4.2.1. A pro-R-module $\{M_n\}_n$ is said to be *almost-pro-zero* when for all i, k there is $j \geq i$ such that the transition map $f_i^j : M_j \to M_i$, multiplied by $t^{\frac{1}{p^k}}$, is zero.

It is said to be *uniformly almost-pro-zero* if for all k there is some $c > 0$ such that for all i the transition map $f_i^{i+c} : M_{i+c} \to M_i$, multiplied by $t^{\frac{1}{p^k}}$, is zero.

Definition 4.2.2. Let $\{M_n\}_n$, $\{N_n\}_n$ be two pro-R-modules and let $\{f_n : M_n \to N_n\}_n$ be a morphism of pro-R-modules. We say that $\{f_n\}_n$ is an *almost-pro-isomorphism* if the pro-R-modules $\{\ker(f_n)\}_n$ and $\{\mathrm{coker}(f_n)\}_n$ are almost-pro-zero.

Remark 4.2.3. If we denote by $M_n[t^{\frac{1}{p^k}}] \subseteq M$ the submodule of $t^{\frac{1}{p^k}}$-torsion, $\{M_n\}_n$ is almost-pro-zero if and only if, for all k, the natural map $\{M_n[t^{\frac{1}{p^k}}]\}_n \to \{M_n\}_n$ is an isomorphism of pro-R-modules.

To explain why this is a "good" definition, let's prove some propositions.

Lemma 4.2.4. *Let the following be a pointwise exact sequence of pro-R-modules:*

$$0 \longrightarrow \{K_n\}_n \xrightarrow{\{f_n\}_n} \{M_n\}_n \xrightarrow{\{g_n\}_n} \{C_n\}_n \longrightarrow 0 \ ,$$

where $\{f_n\}_n$ and $\{g_n\}_n$ are morphism of pro-R-modules. If $\{M_n\}_n$ is almost-pro-zero, so are $\{K_n\}_n$ and $\{C_n\}_n$.

Moreover, if $\{M_n\}_n$ is almost-pro-zero, so are $\{K_n\}_n$ and $\{C_n\}_n$.

Proof. We will prove only the first statement, since the proof for the second is practically the same. Let $\{p_i^j\}_{i,j}, \{q_i^j\}_{i,j}, \{r_i^j\}_{i,j}$ be the transition maps respectively of $\{K_n\}_n, \{M_n\}_n, \{C_n\}_n$. For all i, for all k, there is some $j \geq i$ such that the map $t^{\frac{1}{p^k}} q_i^j$ is identically zero. We have:

$$0 = t^{\frac{1}{p^k}} q_i^j \circ f_j = t^{\frac{1}{p^k}} f_i \circ p_i^j,$$

and since f_i is injective, $t^{\frac{1}{p^k}} p_i^j = 0$. Similarly, we have:

$$0 = t^{\frac{1}{p^k}} g_i \circ q_i^j = t^{\frac{1}{p^k}} r_i^j \circ g_j,$$

and since g_j is surjective, $t^{\frac{1}{p^k}} r_i^j = 0$. $\square$

Proposition 4.2.5. *Let $\{M_n\}_n$ be an almost-pro-zero pro-R-module. Then $\varprojlim_n M_n$ is almost zero.*

Proof. Call $\{p_i^j\}$ the transition maps of $\{M_n\}_n$. Fix a sequence $(x_n)_n$ which identifies an element in $\varprojlim_n M_n$. For all k and for all i, there is some $j \geq i$ such that the map $t^{\frac{1}{p^k}} p_i^j$ is identically zero, therefore $0 = t^{\frac{1}{p^k}} p_i^j(x_j) = t^{\frac{1}{p^k}} x_i$. This means that for all k $t^{\frac{1}{p^k}}(x_n)_n = 0$, which implies that $\varprojlim_n M_n$ is almost zero. $\square$

Proposition 4.2.6. *Let $\{M_n\}_n$ be an almost-pro-zero pro-R-module, and let $F : R - \mathbf{Mod} \to R - \mathbf{Mod}$ be an R-linear functor. Then $\{F(M_n)\}_n$ is an almost-pro-zero pro-R-module.*

Moreover, if $\{M_n\}_n$ is uniformly almost-pro-zero, so is $\{F(M_n)\}_n$.

Proof. Like before, we will prove only the first statement, since the proof for the second is practically the same. Fix two integers $n, k > 0$. Choose m such that the $f_n^m : M_m \to M_n$ is $t^{\frac{1}{p^k}}$-torsion, i.e. $f_n^m(M_m) \subseteq M_n[t^{\frac{1}{p^k}}]$; applying F we get that $Ff_n^m : F(M_m) \to F(M_n)$ factors through $F(M_n[t^{\frac{1}{p^k}}])$. Since F is R-linear, it preserves the endomorphism $- \cdot t^{\frac{1}{p^k}}$; on the module $M_n[t^{\frac{1}{p^k}}]$ this endomorphism is zero, therefore the same happens for $F(M_n[t^{\frac{1}{p^k}}])$, and we get a natural map $F(M_n[t^{\frac{1}{p^k}}]) \to F(M_n)[t^{\frac{1}{p^k}}]$. This means that Ff_n^m factors through $F(M_n)[t^{\frac{1}{p^k}}]$, i.e. Ff_n^m is $t^{\frac{1}{p^k}}$-torsion. $\square$

4.3 Quantitative Hebbarkeitssatz

The following proposition is analogous to Riemann's theorem about removable singularities (*Riemannscher Hebbarkeitssatz*) in the context of perfectoid theory.

Proposition 4.3.1. *Consider a perfectoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra A_∞ with a system of p-roots of some non-zero divisor element $g \in A_\infty$. Furthermore, suppose that no power of p divides g. Consider the trivial pro-module $\{A_\infty/p^m\}_n$ and the pro-module $\{A_\infty\langle\frac{p^n}{g}\rangle/p^m\}_n$ with the obvious transition maps for some fixed $m > 0$. Let $\{f_n\}_n$ be the morphism of pro-modules induced by the natural maps $A_\infty/p^m \to A_\infty\langle\frac{p^n}{g}\rangle/p^m$. Then:*

- $\ker(f_n) = 0$ *for every n;*
- *the pro-module $\{\mathrm{coker}(f_n)\}_n$ is uniformly g-almost-pro-zero.*

Proof. By Lemma 3.2.6, we have the following isomorphism:

$$A_\infty\langle\frac{p^n}{g}\rangle/p^m \cong A_\infty[u^{\frac{1}{p^\infty}}]/(p^m, (ug)^{\frac{1}{p^\infty}} - p^{\frac{n}{p^\infty}}).$$

Call the rightmost module M_n: we get an induced pro-module $\{M_n\}_n$, where the transition map $t_{n+c,n} : M_{n+c} \to M_n$ is given by $u^{\frac{1}{p^l}} \mapsto (up^c)^{\frac{1}{p^l}}$ for all l.

The induced maps $A_\infty/p^m \to M_n$ are injective, so $\ker(f_n) = 0$ for all n. For the second part, we will show that given $k > 0$ there is a c such that the c-fold transition map $t_{n+c,n} : M_{n+c} \to M_n$, multiplied by $g^{\frac{1}{p^k}}$, has image in $A_\infty/p^m \subseteq M_n$. Since $t_{n+c,n}(u^{\frac{1}{p^l}}) = u^{\frac{1}{p^l}}p^{\frac{c}{p^l}}$, we have two cases:

- if $\frac{c}{p^l} \geq m$, we get that $t_{n+c,n}(u^{\frac{1}{p^l}}) \in p^m M_n = 0$;
- if $\frac{c}{p^l} < m$, $g^{\frac{1}{p^k}} t_{n+c,n}(u^{\frac{1}{p^l}}) = g^{\frac{1}{p^k}} u^{\frac{1}{p^l}} p^{\frac{c}{p^l}} = g^{\frac{1}{p^k} - \frac{1}{p^l}} (gu)^{\frac{1}{p^l}} p^{\frac{c}{p^l}} = g^{\frac{1}{p^k} - \frac{1}{p^l}} p^{\frac{n+c}{p^l}}$, which is in A_∞/p^m if $l > k$.

It is sufficient that we choose $c = p^k m$, so that $\frac{c}{p^l} < m$ iff $l > k$. $\square$

Remark 4.3.2. The morphism $\{f_n\}_n$ is a pg-almost-pro-isomorphism.

Let's prove a very useful proposition in the hypotheses given to us by the statement of the previous theorem. We will assume to be working on a ring R, with an non zero divisor $t \in R$ which has a system of p^k-roots, and almost mathematics will be considered with respect to the ideal $(t^{\frac{1}{p^\infty}})$.

Proposition 4.3.3. *$\{f_n : M_n \to N_n\}_n$ be an pro-morphism of pro-R-modules such that the pro-modules $\{\ker(f_n)\}$ and $\{\mathrm{coker}(f_n)\}_n$ are both uniformly almost-pro-zero.*

Then, the natural map $\varprojlim_n M_n \to \varprojlim_n N_n$ is an almost isomorphism.

Proof. Let's call $\{p_i^j\}_{i,j}$ the transition maps for $\{M_n\}_n$ and $\{q_i^j\}_{i,j}$ the transition maps for $\{N_n\}_n$.

The induced pro-module $\{\mathrm{im}(f_n)\}_n$ allows us to split the maps f_n as $g_n \circ h_n$, where h_n is surjective and g_n is injective. Therefore, we can prove the statement in these two simpler cases.

- Suppose that $\{f_n\}_n$ is pointwise surjective.

  If $(x_n)_n \in \ker(\tilde{f})$, we have that for all i $f_i(x_i) = 0$, so $\{x_n\}_n \in \{\ker f_n\}_n$. In particular, this means that $(x_n)_n \in \varprojlim_n \ker(f_n)$, which is almost zero by Lemma 4.2.5; therefore $\tilde{f}$ is almost injective.

  Take $(y_n)_n \in \varprojlim_n N_n$. Since f_n is surjective for all n, we can consider a set $\{x_n\}_n$ such that $f_n(x_n) = y_n$. Fix some $k > 0$ and take the constant c such that for all n $t^{\frac{1}{p^k}} p_n^{n+c}$ is zero when restricted to $\ker(f_{n+c})$. We have that, for all n:

  $$f_{n+c}(p_{n+c}^{n+c+1}(x_{n+c+1})) = q_{n+c}^{n+c+1}(f_{n+c+1}(x_{n+c+1})) = q_{n+c}^{n+c+1}(y_{n+c+1}) = y_{n+c} = f_{n+c}(x_{n+c}).$$

  This means that $p_{n+c}^{n+c+1}(x_{n+c+1}) - x_{n+c}$ is contained in $\ker(f_n)$. From the definition of c we get that, if we apply to this element $t^{\frac{1}{p^k}} p_n^{n+c}(-)$ we get zero. Rewriting this equality, we get that:

  $$t^{\frac{1}{p^k}} p_n^{n+c}(x_{n+c}) = t^{\frac{1}{p^k}} p_n^{n+c}(p_{n+c}^{n+c+1}(x_{n+c+1})) = t^{\frac{1}{p^k}} p_n^{n+c+1}(x_{n+c+1}).$$

  This allows us to define the sequence $(x_n^{(k)}) := (t^{\frac{1}{p^k}} p_n^{n+c}(x_{n+c}))_n$. To verify that it identifies an element in $\varprojlim_n M_n$, we just need to show the following equality:

  $$p_n^{n+1}(x_{n+1}^{(k)}) = p_n^{n+1}(t^{\frac{1}{p^k}} p_{n+1}^{n+c+1}(x_{n+c+1})) = t^{\frac{1}{p^k}} p_n^{n+c+1}(x_{n+c+1}) = t^{\frac{1}{p^k}} p_n^{n+c}(x_{n+c}) = x_n^{(k)},$$

  where the second to last equality follows from the previous chain of equalities. Finally:

  $$f_n(x_n^{(k)}) = f_n(t^{\frac{1}{p^k}} p_n^{n+c}(x_{n+c})) = t^{\frac{1}{p^k}} q_n^{n+c}(f_{n+c}(x_{n+c})) = t^{\frac{1}{p^k}} y_n,$$

  therefore $\tilde{f}((x_n^k)_n) = t^{\frac{1}{p^k}} (y_n)_n$. By repeating the construction for all k, we get that $\tilde{f}$ is almost surjective.

- Suppose that $\{f_n\}_n$ is pointwise injective.

  If $(x_n)_n \in \ker(\tilde{f})$, we have that for all i $f_i(x_i) = 0$, so $x_i = 0$ for all i; therefore $\tilde{f}$ is injective.

  Take $(y_n)_n \in \varprojlim_n N_n$. For all j, call $z_j \in \mathrm{coker}(f_j)$ the projection of y_j. Since $\{\mathrm{coker}(f_n)\}_n$ is almost-pro-zero, for all $k \geq 0$, for all i there is some j such that $t^{\frac{1}{p^k}} q_i^j$ is identically zero, and in particular $0 = t^{\frac{1}{p^k}} q_i^j(z_j) = t^{\frac{1}{p^k}} z_i$. This means that for all i and for all k there is some $x_i^{(k)} \in M_i$ such that $f_i(x_i^{(k)}) = t^{\frac{1}{p^k}} y_i$, so we get:

  $$f_i(p_i^{i+1}(x_{i+1}^{(k)})) = q_i^{i+1}(f_{i+1}(x_{i+1}^{(k)})) = q_i^{i+1}(t^{\frac{1}{p^k}} y_{i+1}) = t^{\frac{1}{p^k}} q_i^{i+1}(y_{i+1}) = t^{\frac{1}{p^k}} y_i = f_i(x_i^{(k)}).$$

Since f_i is injective, we get, for all i and for all k, $p_i^{i+1}(x_{i+1}^{(k)}) = x_i^{(k)}$. In particular, the sequence $(x_i^{(k)})_i$ identifies an element in $\varprojlim_n M_n$; the image of this element is $t^{\frac{1}{p^k}}(y_n)_n$, therefore, by varying k, we get that $\tilde{f}$ is almost surjective.

$\square$

Corollary 4.3.4. *In the setting of Proposition 4.3.1, i.e. with $R := A_\infty$, $t := pg$, $\{M_n\}_n := \{A_\infty/p^m\}_n$ and $\{N_n\}_n := \{A_\infty\langle\frac{p^n}{g}\rangle/p^m\}_n$ for some m, the hypotheses of Proposition 4.3.3 are satisfied, therefore we have:*

$$A_\infty/p^m \cong \varprojlim_n A_\infty/p^m \approx_{pg} \varprojlim_n A_\infty\langle\frac{p^n}{g}\rangle/p^m.$$

Proposition 4.3.5. *In the same setting as Proposition 4.3.3, take some R-module Q and consider the following natural map:*

$$\varprojlim_n \mathrm{Ext}^1_R(Q, M_n) \to \varprojlim_n \mathrm{Ext}^1_R(Q, N_n).$$

It is a t-almost isomorphism.

Proof. Like in the proof of Proposition 4.3.3, we just have to prove the two cases in which $\{f_n\}_n$ is respectively pointwise surjective and pointwise injective. We will only prove the first case, since the hypotheses are symmetric and the other case has a very similar proof.

Consider the pro-module $\{K_n\}_n := \{\ker(f_n)\}_n$. Applying the functor $\mathrm{Hom}_R(Q, -)$, we get the following pointwise exact sequence of pro-R-modules for all i:

$$\{\mathrm{Ext}^i_R(Q, K_n)\}_n \longrightarrow \{\mathrm{Ext}^i_R(Q, M_n)\}_n \xrightarrow{\{\phi_n\}_n} \{\mathrm{Ext}^i_R(Q, N_n)\}_n \longrightarrow \{\mathrm{Ext}^{i+1}_R(Q, K_n)\}_n$$

Since for all i $\mathrm{Ext}^i(Q, -)$ is an R-linear functor, and $\{K_n\}_n$ is uniformly almost-pro-zero, by Proposition 4.2.6 the leftmost and rightmost pro-modules in the exact sequence are uniformly almost-pro-zero. The leftmost module admits a pointwise surjective map towards the pro-module $\{\ker(\phi_n)\}$, while the pro-module $\{\mathrm{coker}(\phi_n)\}_n$ admits a pointwise injective map towards the rightmost module. By Lemma 4.2.4 $\{\ker(\phi_n)\}$ and $\{\mathrm{coker}(\phi_n)\}_n$ are uniformly almost-pro-zero, therefore by Proposition 4.3.3 it induces an isomorphism of the limits. $\square$

Chapter 5

Proof of the main theorem

5.1 Useful propositions

Before proceeding with the content of this section, we cite a result from Hochster in [Hoc83].

Proposition 5.1.1. *The following statements are equivalent.*

- *For all noetherian regular local rings* $(A, \mathfrak{m})$ *of mixed characteristic* $(0, p)$*, every module-finite extension of* A *admits* A *as a direct summand.*

- *For all noetherian regular local rings* $(A, \mathfrak{m})$ *of mixed characteristic* $(0, p)$*, which are complete, unramified (i.e.* $p \in \mathfrak{m} \setminus \mathfrak{m}^2$*), and with* $A/\mathfrak{m}$ *algebraically closed, every module-finite extension of* A *admits* A *as a direct summand.*

Remark 5.1.2. Completion and algebraic closure of the residue field can be achieved via faithfully flat extensions, therefore we already proved that the problem could be reduced to this case in Chapter 1. The proof that we can assume A to be unramified can be found in [Hoc83, Theorem 6.1].

Throughout the rest of the chapter, we will assume A_0 to have all the properties of the ring A in the second point of the previous proposition. The first thing that we want to do is to take the problem into the realm of perfectoid theory. We need a suitably "faithful" extension of the ring A, which will also be a perfectoid algebra. For the construction we will need the following structure theorem for regular rings.

Theorem 5.1.3 (Cohen's structure theorem)**.** *Let* $(A, \mathfrak{m})$ *be a complete regular local ring of mixed characteristic* $(0, p)$ *and dimension* d*, and let* k *be its residue field, which we assume to be perfect. Then:*

- *if* $p \in \mathfrak{m} \setminus \mathfrak{m}^2$*, there is an isomorphism* $W(k)[[x_1, \cdots, x_{d-1}]] \cong A$*;*

- *if* $p \in \mathfrak{m}^2$*, there is a surjective map* $W(k)[[x_1, \cdots, x_d]] \twoheadrightarrow A$*;*

where $W(k)$ *is the ring of Witt vectors of* k*.*

We will also need some results on faithful flatness.

Lemma 5.1.4. *Let* A *be a noetherian ring and* $\{B_i\}_i$ *a filtered diagram of faithfully flat* A*-algebras with colimit* B*. Then* B *is a faithfully flat* A*-algebra.*

Proof. If we drop the faithfulness condition, this lemma is true in much more generality, without the requirements of A being noetherian and B_i being algebras. For the faithfulness, take an A-module M - which we can assume to be finitely generated - and suppose that $M \otimes B = 0$: since tensor product commutes with filtered colimits, $0 = \varinjlim_i M \otimes B_i$. This implies that for any i, for any element $m_i \in M \otimes B_i$, there is $j > i$ such that the map $M \otimes B_i \to M \otimes B_j$ sends m_i to 0. In particular, for any element $m \in M$ there is some k such that the natural map $f_k : M \to M \otimes B_k$ sends m to 0. Considered the filtered system $\{\ker(f_i)\}_i$ of submodules of M: by the previous argument, their union is M, but since M is finitely generated over a noetherian ring, there is some k such that $\ker(f_k) = M$. Since $M \otimes B_k$ is generated over B_k by the elements $\{m \otimes 1 = f_k(m) | m \in M\}$, it follows that $M \otimes B_k = 0$, therefore $M = 0$ by faithful flatness. $\square$

Lemma 5.1.5. *Let $f : A \to B$ be a faithfully flat map of commutative rings. Let C be a B-module which is flat (resp. faithfully flat) as an A-module. Then C is also flat (resp. faithfully flat) as a B-module.*

Proof. First, let's prove the flatness. Consider an exact sequence of B modules $0 \to M \to N \to P \to 0$. Since the functor $\_ \otimes_A C$ is the composition of the functors $\_ \otimes_A B$ and $\_ \otimes_B C$, we have the following:

$$\begin{tikzcd}
 & 0 \arrow[r] & M \arrow[r] & N \arrow[r] \arrow[ddl, "\otimes_A C"'] & P \arrow[r] \arrow[dr, "\otimes_B C"] & 0 & & & \\
 & & & & M \otimes_B C \arrow[r] & N \otimes_B C \arrow[r] \arrow[dll, "\otimes_A B"] & P \otimes_B C \arrow[r] & 0 & \\
0 \arrow[r] & M \otimes_A C \arrow[r] & N \otimes_A C \arrow[r] & P \otimes_A C \arrow[r] & 0 & & & &
\end{tikzcd}$$

where every sequence is exact. Since B is faithfully flat over A, and $M \otimes_A C \to N \otimes_A C$ is an injective map, so is $M \otimes_B C \to N \otimes_B C$, therefore C is a flat B-module.

Moreover, suppose that C is a faithfully flat A-module and take a B-module M such that $M \otimes_B C = 0$. Then $M \otimes_A C = (M \otimes_B C) \otimes_A B = 0$, therefore $M = 0$. $\square$

Lemma 5.1.6 ([Bha18, Proposition 5.1])**.** *Let B be a noetherian ring, and $B \to C$ a ring extension. Suppose that there is $\pi \in B$ such that the extension is (faithfully) flat modulo π, and both rings are π-adically complete and π-torsion free. Then, the extension $B \to C$ is (faithfully) flat.*

Corollary 5.1.7. *Let $\{B_k, \alpha_{i,j}\}_{i,j,k}$ a directed system of A_0-algebras with no p-torsion, and let B_∞ be the p-adic completion of their colimit. If $A_0/p \to B_k/p$ is faithfully flat for all k, so is the map $A_0 \to B_\infty$.*

Proof. Since for all i, j $\alpha_{i,j}(pB_i) \subseteq pB_j$, we can work modulo p: in this context, the p-adic completion becomes a trivial functor, so $B_\infty/p = \varinjlim_k B_k/p$. Since A_0/p is noetherian, filtered colimits preserve faithful flatness, therefore $A_0/p \to B_\infty/p$ is faithfully flat. For all k, the map $B_k \xrightarrow{\cdot p} B_k$ is injective because B_k has no p-torsion; since filtered colimits are exact, multiplication by p is injective on $B_{\infty,0} := \varinjlim_k B_k$. Finally, the p-adic completion of a ring with no p-torsion does not have p-torsion: if $x := \{x_i | x_i \in B_{\infty,0}/p^i\}_i$ is a p-torsion element in B_∞, for all n one must have $px_n = 0$, i.e. $p\tilde{x_n} = p^n b$ for some element $b \in B_{\infty,0}$, where $\tilde{x_n} \in B_{\infty,0}$ is a lifting of x_n; therefore $\tilde{x_n} = p^{n-1}c$ because $B_{\infty,0}$ has no p-torsion, which means that $x_{n-1} = 0$, therefore $x = 0$. Since the hypotheses of Lemma 5.1.6 are satisfied, the map $A_0 \to B_\infty$ is faithfully flat. □

Definition 5.1.8. Consider an extension $A_0 \to A$, where A admits a system of p-power roots of p. A is said to be *faithfully almost flat* over A if the following conditions are satisfied. For all $M \in A - \mathbf{Mod}$:

- $\mathrm{Tor}^1_A(M, A)$ is p-almost zero as an A-module;
- if $M \otimes_{A_0} A \approx_p 0$, then $M = 0$.

Remark 5.1.9. The first condition is a weakened version of flatness, hence the term almost flat. On the other hand, the second condition is a priori stronger that faithfulness - it can be proven that it is not (see [Bha14, Lemma 2.1]), but it won't be needed in this thesis.

Remark 5.1.10. Faithful almost flatness is well defined in the following sense: if $A_0 \to A$ is faithfully almost flat and $A \to A'$ is an almost faithfully flat, then $A_0 \to A'$ is also faithfully almost flat.

Proposition 5.1.11 ([Bha18, Proposition 5.2])**.** *There is an extension $A_0 \to A_{\infty,0}$ such that $(A_{\infty,0}[\frac{1}{p}], A_{\infty,0})$ is a perfectoid affinoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra, and that $A_{\infty,0}$ is faithfully almost flat over A_0. Furthermore, $A_{\infty,0}$ is also flat over A_0.*

Proof. Since we are in the hypotheses of Proposition 5.1.1, Cohen's structure theorem tells us that A_0 is isomorphic to the ring of power series $W[[x_1, \cdots, x_{d-1}]]$, where $d := \dim(A_0)$ and W is the ring of Witt vectors associated to the residue field of A_0 (therefore W is a complete DVR, with maximal ideal generated by p).

Define $A_n := W[[p^{\frac{1}{p^n}}, x_1^{\frac{1}{p^n}}, \cdots, x_{d-1}^{\frac{1}{p^n}}]]$, and call $A_{\infty,0}$ the p-adic completion of their filtered colimit. For all m the natural inclusion $A_0/p \hookrightarrow A_m/p$ is free, therefore faithfully flat; by Corollary 5.1.7, the map $A_0 \to A_{\infty,0}$ is faithfully flat.

Let's take an A_0-module M such that $M \otimes A_{\infty,0} \approx 0$, and prove that $M = 0$. We can assume M to be generated by one element, i.e. $M = A_0/I$ for some ideal $I \subseteq A_0$.

Faithful flatness tells us that $I_{\infty,0} := I \otimes A_{\infty,0}$ is an ideal of $A_{\infty,0}$. $0 \approx M \otimes A_{\infty,0} \cong A_{\infty,0}/I_{\infty,0}$, so for every k we have $p^{\frac{1}{p^k}} \in I_{\infty,0}$, or alternatively $p \in (I_{\infty,0})^{p^k}$.

The inclusion $I^{p^k} \subseteq A_0$ induces an injective map $i_k : I^{p^k} \otimes A_{\infty,0} \to A_{\infty,0}$, whose image is contained in $(I_{\infty,0})^{p^k}$. Using the natural surjection $\bigotimes_1^{p^k} I \twoheadrightarrow I^{p^k}$, we get this commutative diagram:

$$\begin{tikzcd}
(I \otimes_{A_0} \cdots \otimes_{A_0} I) \otimes_{A_0} A_{\infty,0} \arrow[r] \arrow[d, "\cong"] & I^{p^k} \otimes_{A_0} A_{\infty,0} \arrow[r] \arrow[d, "i_k"] & 0 \\
I_{\infty,0} \otimes_{A_{\infty,0}} \cdots \otimes_{A_{\infty,0}} I_{\infty,0} \arrow[r] & (I_{\infty,0})^{p^k} \arrow[r] & 0,
\end{tikzcd}$$

so i_k is surjective, and therefore bijective. With a similar line of reasoning, one gets that the ideal $(I_{\infty,0})^{p^k} \cong (I_{\infty,0})^{p^k} + (p)_{A_{\infty,0}} \subseteq A_{\infty,0}$ is almost isomorphic to $(I^{p^k} + (p)_{A_0}) \otimes_{A_0} A_{\infty,0}$ via the natural map. Since $A_0 \to A_{\infty,0}$ is faithfully flat, we get that $I^{p^k} + (p)_{A_0} = I^{p^k}$, therefore $p \in I^{p^k}$ for all k. By hypothesis, $p \neq \mathfrak{m}^2$, so the only possibility is that $I = A_0$, i.e. $M = 0$.

Finally, to show that $(A_{\infty,0}[\frac{1}{p}], A_{\infty,0})$ is an integral perfectoid affinoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra, it suffices to check the following.

- $A_{\infty,0}[\frac{1}{p}]$ is a $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra, p-adically complete by definition.

- $A_{\infty,0} = A_{\infty,0}[\frac{1}{p}]^\circ$ (essentially by definition) and the Frobenius endomorphism Φ on $A_{\infty,0}/p$ is surjective because $A_{\infty,0}/p = \bigcup_m A_m/p$, and $\Phi(A_{m+1}/p) = A_m/p$ for all m.

- $A_{\infty,0}$ is integral. If $ab = 0$ in $A_{\infty,0}$, modulo p^{2n} $\bar{a}, \bar{b} \in A_m/p^{2n}$ for some m, so they have representatives $\tilde{a}, \tilde{b} \in A_m$. Since $p^{\frac{1}{p^m}} A_m$ is a prime ideal and $\tilde{a}\tilde{b} \in (p^{\frac{1}{p^m}})^{2np^m}$, one of them must be contained in $(p^{\frac{1}{p^m}})^{np^m} = (p^n)$; in particular, either $\bar{a}$ or $\bar{b}$ is divisible by p^n. Repeating the argument for all n we get that either a or b is in (p^n) for all n, i.e. it is zero.

- $A_{\infty,0}$ is normal. Take a monic polynomial $f(x) \in A_{\infty,0}[x]$ and suppose it has a root $\frac{b}{a}$ in the fraction field. For all n, we can approximate any $x \in A_{\infty,0}$ with an element $x_n \in \varinjlim_m A_m$ such that they are equal modulo p^n. If we call $\tilde{f}_n$ a monic polynomial whose coefficients approximate those of f in the same fashion, we have that $c_n := \tilde{f}_n(\frac{b_n}{a_n}) \in \varinjlim_m A_m$ is zero modulo p^n: we now define $f_n := \tilde{f}_n - c_n$, as another monic approximation of f, such that $f_n(\frac{a_n}{b_n}) = 0$. a_n, b_n, and the coefficients of f_n are all contained in A_m for some big enough m; but A_m is a power series ring over a DVR, so it is UFD, and in particular normal: $\frac{a_n}{b_n} \in A_m \subseteq \varinjlim_m A_m$. By varying n, we get an approximating sequence $\left\{\frac{a_n}{b_n}\right\}_n$ of $\frac{a}{b}$: since all the elements in the sequence are in $\varinjlim_m A_m$, their limit is contained in $A_{\infty,0}$.

□

We follow with another result which allows us to enrich the extension $A_{\infty,0}$ with a set system of p-power roots.

Theorem 5.1.12 (André)**.** *Let $(A_{\infty,0}[\frac{1}{p}], A_{\infty,0})$ be the perfectoid affinoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra defined above, and $g \in A_0$ coprime with p. There is an integral perfectoid extension $A_{\infty,0} \to A_\infty$ which is almost faithfully flat and admits a system of p-power roots of g.*

Proof. To prove this, let's take $A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle$ (i.e. the ring obtained adjoining a p-power system of roots of T to $A_{\infty,0}$ and taking the p-adic completion). The pair $(A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle[\frac{1}{p}], A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle)$ is a perfectoid affinoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra: let Y be its associated space. Intuitively, we want to impose $T = g$, which geometrically would correspond to considering the subset $\{y \in Y | |(T-g)(y)| = 0\} \subset Y$. To do this, we write this subset as intersection of the open sets $U_n := U(\frac{T-g}{p^n}) = \{y \in Y | |(T-g)(y)| \leq |(p^n)(y)| = |p|^n\}$, and we take the colimit of $\mathcal{O}_Y^+(U_n)$: its p-adic completion will be the ring A_∞ (we omit the proof that A_∞ is almost isomorphic to an integral perfectoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra, but this result can be found in [Sch13, Chapter II.2]).

To explicitly describe the ring $\mathcal{O}_Y^+(U_n)$, we resort to the approximation lemma: using Proposition 3.2.7 we can find $f \in A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle^\flat$ such that the rational subsets $U(\frac{T-g}{p^n})$ and $U(\frac{f^\#}{p^n})$ are the same, and $f^\# \equiv T - g \mod p^{\frac{1}{p}}$. Crucially, $f^\#$ has a system of p-power roots, and by Proposition 3.2.5 we could describe the ring at the almost level as:

$$\mathcal{O}_Y^+(U_n) \approx A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle \left\langle \left(\frac{f^\#}{p^n}\right)^{\frac{1}{p^\infty}} \right\rangle,$$

which is the p-adic completion of $\varinjlim_k C_{n,k}$, where:

$$C_{n,k} := A_{\infty,0}\left\langle T^{\frac{1}{p^\infty}}\right\rangle\left[u_n^{\frac{1}{p^\infty}}\right]\Big/\left((u_np^n)^{\frac{1}{p^k}} - f^{\#\frac{1}{p^k}}\right).$$

If we work modulo p, we need not worry about completions, and we can describe A_∞/p at the almost level as $\varinjlim_{n,k} C_{n,k}/p$, where the maps $C_{n,k} \to C_{n+1,k}$ send $u_n^{\frac{1}{p^m}}$ to $(u_{n+1}p)^{\frac{1}{p^m}}$.

Let's prove first that $A_{\infty,0} \to C_{n,k}$ is faithfully flat modulo p^ε, with $\varepsilon = \frac{1}{p^{k+1}}$:

$$C_{n,k}/p^\varepsilon = A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle[u_n^{\frac{1}{p^\infty}}]/(p^\varepsilon, f^{\#p\varepsilon}).$$

The k-fold Frobenius identifies $C_{n,k}/p^\varepsilon$, as an $A_{\infty,0}/p^\varepsilon$-algebra, with the following $A_{\infty,0}/p^{\frac{1}{p}}$-algebra:

$$A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle[u_n^{\frac{1}{p^\infty}}]/(p^{\frac{1}{p}}, f^{\#}) \cong A_{\infty,0}\langle T^{\frac{1}{p^\infty}}\rangle[u_n^{\frac{1}{p^\infty}}]/(p^{\frac{1}{p}}, T-g),$$

where we used that $f^{\#} \equiv T - g \mod p^{\frac{1}{p}}$. This algebra is free over $A_{\infty,0}/p^{\frac{1}{p}}$, and in particular faithfully flat. Furthermore, since $C_{n,k}/p^\varepsilon$ is free over $A_{\infty,0}/p^\varepsilon$, it has no p-torsion, which implies that $C_{n,k}$ itself has no p-torsion.

Recall that $A_k \to A_{\infty,0}$ is faithfully flat, where A_k is defined as in 5.1.11. The following maps are faithfully flat for all n, k:

- $A_k/p^\varepsilon \to A_{\infty,0}/p^\varepsilon$ being the quotient of $A_k \to A_{\infty,0}$;
- $A_k/p^\varepsilon \to C_{n,k}/p^\varepsilon$, by composition with $A_{\infty,0}/p^\varepsilon \to C_{n,k}/p^\varepsilon$;
- $A_k \to \widehat{C_{n,k}}$, by Lemma 5.1.6, since A_k is noetherian, p^ε-adically complete and with no p^ε-torsion, while $\widehat{C_{n,k}}$ is p^ε-adically complete by definition and has no p-torsion because $C_{n,k}$ has no p^ε-torsion (as seen in the proof of 5.1.7);
- $A_0 \to \widehat{C_{n,k}}$, by precomposition with $A_0 \to A_k$;
- $A_0/p \to C_{n,k}/p$, obtained by the previous one modulo p;
- $A_0 \to A_\infty$, by Corollary 5.1.7;
- $A_{\infty,0} \to A_\infty$, by Lemma 5.1.5.

Finally, let's remark that if we work with A_∞ only up to almost isomorphism (and we want to, since it is almost isomorphic to an integral perfectoid $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$-algebra), the map $A_{\infty,0} \to A_\infty$ has to be considered just almost faithfully flat. □

Let's include an easy lemma:

Lemma 5.1.13. *Let A_0 be a local noetherian ring of characteristics $(0,p)$ with no p-torsion. Then for all finitely generated A_0-modules M the natural map*

$$\mathrm{Ext}^1_{A_0}(M, A_0) \to \varprojlim_k \mathrm{Ext}^1_{A_0}(M, A_0/p^k)$$

is injective.

Proof. Since A_0 is noetherian, M is finitely presented. It is thus possible to find a free resolution $\cdots \to A_0^n \to A_0^m \to M$ with m, n finite. As a consequence, $\operatorname{Ext}^1_{A_0}(M, A_0)$ is finitely generated, since it is the quotient of a submodule of $\operatorname{Hom}_{A_0}(A_0^n, A_0) \cong A_0^n$. The short exact sequence $0 \longrightarrow A_0 \xrightarrow{\cdot p^k} A_0 \longrightarrow A_0/p^k \longrightarrow 0$ yields the following long exact sequence:

$$\cdots \longrightarrow \operatorname{Ext}^1_{A_0}(Q_0, A_0) \xrightarrow{\cdot p^k} \operatorname{Ext}^1_{A_0}(Q_0, A_0) \xrightarrow{f_m} \operatorname{Ext}^1_{A_0}(Q_0, A_0/p^k) \longrightarrow \cdots.$$

In particular, $\ker(f_m) = p^k \operatorname{Ext}^1_{A_0}(Q_0, A_0)$, therefore the limit of the maps $\{f_m\}_m$ has kernel $K := \bigcap p^k \operatorname{Ext}^1_{A_0}(Q_0, A_0)$. Since $K = pK$, by Nakayama's lemma we get that $K = 0$. □

5.2 Proof

Now we are ready to face the main theorem:

Theorem 5.2.1 (André)**.** *Let A_0 be a local and complete regular noetherian ring of characteristic $(0, p)$, and $i : A_0 \hookrightarrow B_0$ a module-finite extension of rings. The inclusion i splits as a map of A_0-modules.*

First, let's give an idea of the proof. By Proposition 2.4.19 there is some $g \in A_0$ such that $A_0[\frac{1}{g}] \hookrightarrow B_0[\frac{1}{g}]$ is finite étale. Since étaleness is preserved by base change, $A_\infty[\frac{1}{g}] \to B_0 \otimes_{A_0} A_\infty[\frac{1}{g}]$ is still étale. We want to push all the ramification of the extension $A_\infty \to B_0 \otimes_{A_0} A_\infty$ in p by adjoining elements to A_∞ such that g divides p in this new ring: in this way the hypothesis of Faltings'almost purity theorem will be satisfied. On one hand, perfectoid theory allows us to find such an extension; on the other hand, this extension won't be almost faithfully flat: this means that we will not be able to deduce from the almost-splitting of the tensorized sequence the almost-splitting of the starting sequence. This problem will be solved by the Hebbarkeitssatz theorem, which allows us to reduce modulo p^m at the cost of shifting from p-almost mathematics to pg-almost mathematics.

Proof. By Proposition 2.4.19, there is an element $g' \in A_0$ such that $A_0[g'^{-1}] \to B_0[g'^{-1}]$ is an étale covering. Since A_0 is a regular local ring, it is a unique factorization domain, therefore there is some k such that $g' = p^k g$ for some g coprime with p. In particular, g is not a zero divisor modulo p^m for any m.

As already seen in the first chapter, the sequence $0 \to A_0 \to B_0 \to Q_0 \to 0$ splits iff $\alpha_0 \in \operatorname{Ext}^1_{A_0}(Q_0, A_0)$ is 0. Since B_0 is finitely generated as an A_0-module, so is Q_0; we can apply Lemma 5.1.13 to get the following immersion:

$$\operatorname{Ext}^1(Q_0, A_0) \hookrightarrow \varprojlim_m \operatorname{Ext}^1(Q_0, A_0/p^m).$$

We can write α_0 as the limit of $\{\alpha_0/p^m\}_m$, where α_0/p^m is the image of α_0 via the map $\operatorname{Ext}^1(Q_0, A_0) \to \operatorname{Ext}^1(Q_0, A_0/p^m)$ induced by the projection. In particular, we just need to show that $\alpha_0/p^m = 0$ for a large enough m. Let's call α_∞ and α_∞/p^m the images respectively of α_0 and α_0/p^m via the following commutative diagram:

$$\begin{tikzcd}
\operatorname{Ext}^1_{A_0}(Q_0, A_0) \arrow[r] \arrow[d] & \operatorname{Ext}^1_{A_0}(Q_0, A_\infty) \arrow[r, "\approx_p"] \arrow[d] & \operatorname{Ext}^1_{A_\infty}(Q_0 \otimes A_\infty, A_\infty) \arrow[d] \\
\operatorname{Ext}^1_{A_0}(Q_0, A_0/p^m) \arrow[r] & \operatorname{Ext}^1_{A_0}(Q_0, A_\infty/p^m) \arrow[r, "\approx_p"] & \operatorname{Ext}^1_{A_\infty}(Q_0 \otimes A_\infty, A_\infty/p^m),
\end{tikzcd}$$

where the rightmost maps are p-almost isomorphisms because A_∞ is p-almost isomorphic to a flat A_0-module.

Suppose that $\alpha_0/p^m \neq 0$, which is equivalent to saying that $I := \mathrm{Ann}_{A_0}(\alpha_0/p^m) \subsetneq A_0$. Since A_0 is noetherian and I is a proper ideal, $\bigcap_n I^n = 0$ so, for a large enough k, $pg \notin I^{p^k}$, or equivalently $\frac{I^{p^k}+(pg)_{A_0}}{I^{p^k}} \neq 0$. Let $I_\infty := \mathrm{Ann}_{A_\infty}(\alpha_\infty/p^m)$. Since $A_0 \to A_\infty$ is faithfully almost flat (as in Definition 5.1.8), we have:

$$\begin{tikzcd}
 & I \otimes A_\infty \arrow[r, "i"] \arrow[d, "\pi"] & A_\infty \arrow[r] \arrow[d, equal] & \langle \alpha_0/p^m \rangle_{A_0} \otimes A_\infty \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I_\infty \arrow[r] & A_\infty \arrow[r] & \langle \alpha_\infty/p^m \rangle_{A_\infty} \arrow[r] & 0,
\end{tikzcd}$$

where i is p-almost injective, which means that π is not only surjective, but also p-almost injective - in particular, it is a p-almost isomorphism: $I \otimes A_\infty \approx_p I_\infty$. In a similar fashion, we can obtain that $(pg)_{A_0} \otimes_{A_0} A_\infty \approx_p (pg)_{A_\infty}$.

Using the natural surjection $\bigotimes_1^{p^k} I \twoheadrightarrow I^{p^k}$, we get this commutative diagram:

$$\begin{tikzcd}
(I \otimes_{A_0} \cdots \otimes_{A_0} I) \otimes_{A_0} A_\infty \arrow[r] \arrow[d, "\approx_p"] & I^{p^k} \otimes_{A_0} A_\infty \arrow[r] \arrow[d, "\pi_{p^k}"] & 0 \\
I_\infty \otimes_{A_\infty} \cdots \otimes_{A_\infty} I_\infty \arrow[r] & (I_\infty)^{p^k} \arrow[r] & 0,
\end{tikzcd}$$

so π_{p^k} is itself an almost isomorphism (for the sake of accuracy, it is surjective and almost injective). With a similar line of reasoning, one gets that the ideal $(I_\infty)^{p^k} + (pg)_{A_\infty} \subseteq A_\infty$ is almost isomorphic to $(I^{p^k} + (pg)_{A_0}) \otimes_{A_0} A_\infty$ via the natural map. Since $A_0 \to A_\infty$ is faithfully almost flat, and $\frac{I^{p^k}+(pg)_{A_0}}{I^{p^k}} \neq 0$, we get:

$$0 \not\approx_p \frac{I^{p^k} + (pg)_{A_0}}{I^{p^k}} \otimes_{A_0} A_\infty \approx_p \frac{(I^{p^k} + (pg)_{A_0}) \otimes_{A_0} A_\infty}{I^{p^k} \otimes_{A_0} A_\infty} \approx_p \frac{(I_\infty)^{p^k} + (pg)_{A_\infty}}{(I_\infty)^{p^k}}.$$

In particular, the last module is not 0, which means that $pg \notin (I_\infty)^{p^k}$. Since A_∞ has a system of p-power roots for both p and g, it means that $(pg)^{\frac{1}{p^k}} \notin I_\infty = \mathrm{Ann}_{A_\infty}(\alpha_\infty/p^m)$. Recapping, we have proven that if $\alpha/p^m \neq 0$, then α_∞/p^m is not killed by all the p-power roots of pg: to conclude the theorem we must come to a contradiction by showing that indeed α_∞/p^m is pg-almost zero.

Let $A_\infty^{(n)} := A_\infty\langle \frac{p^n}{g} \rangle$, $B_\infty^{(n)} := B_0 \otimes_{A_0} A_\infty^{(n)}$, and call their cokernel $Q_\infty^{(n)}$. Similarly, in the extension $A_\infty \to B_0 \otimes_{A_0} A_\infty$, call the second ring B_∞ and the cokernel of the extension Q_∞.

The natural map $A_\infty^{(n)} \to B_\infty^{(n)}$ becomes an étale extension after inverting p: since g divides p^n in $A_\infty^{(n)}$, inverting p also inverts g and thus kills all algebraic obstructions to étaleness (including injectivity, by virtue of faithful flatness).

The inclusion $A_\infty^{(n)}[\frac{1}{p}] \hookrightarrow B_\infty^{(n)}[\frac{1}{p}]$ is a finite étale covering, and the first ring is a perfectoid $\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}$-algebra, so by Theorem 3.1.10 $A_\infty^{(n)} \to B_\infty^{(n)}$ is a finite étale covering in the category $\widehat{\mathbb{Z}_p[p^{\frac{1}{p^\infty}}]}^a - perf$. Consider the p-almost exact sequence:

$$0 \longrightarrow A_\infty \longrightarrow B_\infty \longrightarrow Q_\infty \longrightarrow 0.$$

If we tensor by $A_\infty^{(n)}$ over A_∞ we get the p-almost exact sequence:

$$A_\infty^{(n)} \longrightarrow B_\infty^{(n)} \longrightarrow Q_\infty^{(n)} \longrightarrow 0,$$

where, being a p-almost étale covering, the leftmost map is also p-almost injective. Applying the functor $\mathrm{Hom}_{A_\infty}(-, A_\infty^{(n)})$ to the first sequence and the functor $\mathrm{Hom}_{A_\infty^{(n)}}(-, A_\infty^{(n)})$ to the second sequence, we can consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{A_\infty}(B_\infty, A_\infty^{(n)}) \arrow[r] & \mathrm{Hom}_{A_\infty}(A_\infty, A_\infty^{(n)}) \arrow[r] & \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)}) \arrow[r] & \cdots \\
\mathrm{Hom}_{A_\infty^{(n)}}(B_\infty^{(n)}, A_\infty^{(n)}) \arrow[r] \arrow[u] & \mathrm{Hom}_{A_\infty^{(n)}}(A_\infty^{(n)}, A_\infty^{(n)}) \arrow[r] \arrow[u, "\cong"'] & \mathrm{Ext}^1_{A_\infty^{(n)}}(Q_\infty^{(n)}, A_\infty^{(n)}) \arrow[r] \arrow[u, dotted, "j"'] & 0,
\end{tikzcd}$$

where the horizontal maps are p-almost exact, and the vertical maps are induced by the natural map $A_\infty \to A_\infty^{(n)}$. For the p-almost surjectivity of the rightmost map on the bottom, we used that $\mathrm{Ext}^1_{A_\infty^{(n)}}(B_\infty^{(n)}, A_\infty^{(n)}) \approx_p 0$ because $B_\infty^{(n)}$ is p-almost projective as an $A_\infty^{(n)}$-module; This induces the p-almost injective map j. Let $\alpha_\infty^{(n)} \in \mathrm{Ext}^1_{A_\infty^{(n)}}(Q_\infty^{(n)}, A_\infty^{(n)})$ be the element which represents the extension $A_\infty^{(n)} \to B_\infty^{(n)}$, in the same way as α_0 represents the extension $A_0 \to B_0$: via the p-almost injective map j, we can identify it with an element of $\mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)})$, which maps to $\alpha_\infty^{(n)}/p^m \in \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)}/p^m)$ via the obvious map. Since $A_\infty^{(n)} \to B_\infty^{(n)}$ is a p-almost étale covering, $Q_\infty^{(n)}$ is p-almost projective as an $A_\infty^{(n)}$-module, which means that $\alpha_\infty^{(n)}$ is p-almost zero, so $\alpha_\infty^{(n)}/p^m$ is p-almost zero for every m.

For every m we have the following canonical maps. First:

$$\mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty/p^m) \to \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)}/p^m),$$

which sends α_∞/p^m to $\alpha_\infty^{(n)}/p^m \approx_p 0$; passing to the limit we get the other map:

$$f : \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty/p^m) \to \varprojlim_n \mathrm{Ext}^1_{A_\infty}(Q_\infty, A_\infty^{(n)}/p^m),$$

which sends α_∞/p^m to a p-almost zero element.

By Theorem 4.3.1, the pro-morphism from $\{A_\infty/p^m\}_n$ to $\{A_\infty^{(n)}/p^m\}_n$ has kernel and cokernel uniformly pg-almost-pro-zero, so by Proposition 4.3.5 the map f is a pg-almost isomorphism: in particular, $\alpha_\infty/p^m \approx_{pg} 0$, which concludes the proof. $\square$

Bibliography

[And18] Yves André. "La conjecture du facteur direct". In: *Publications mathématiques de l'IHÉS* 127 (2018), pp. 71–93. DOI: `https://doi.org/10.1007/s10240-017-0097-9`.

[Bha14] Bhargav Bhatt. "Almost direct summands". In: *Nagoya Mathematical Journal* 214 (2014), pp. 195–204. DOI: `10.1215/00277630-2648180`.

[Bha18] Bhargav Bhatt. "On the direct summand conjecture and its derived variant". In: *Inventiones mathematicae* 212 (2018), pp. 297–317. DOI: `https://doi.org/10.1007/s00222-017-0768-7`.

[GR03] Ofer Gabber and Lorenzo Ramero. *Almost ring theory*. Vol. 1800. Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2003, pp. vi+307. ISBN: 3-540-40594-1. DOI: `10.1007/b10047`. URL: `https://doi.org/10.1007/b10047`.

[Hoc73] M. Hochster. "Contracted ideals from integral extensions of regular rings". In: *Nagoya Mathematical Journal* 51.none (1973), pp. 25–43. DOI: `nmj/1118794784`. URL: `https://doi.org/`.

[Hoc83] M. Hochster. "Canonical elements in local cohomology modules and the direct summand conjecture". In: *Journal of Algebra* 84 (1983), pp. 503–553.

[Sch12] Peter Scholze. "Perfectoid Spaces". In: *Publications mathématiques de l'IHÉS* 116 (2012), pp. 245–313. DOI: `https://doi.org/10.1007/s10240-012-0042-x`.

[Sch13] Peter Scholze. "On torsion in the cohomology of locally symmetric varieties". In: *Annals of Mathematics* 182 (June 2013). DOI: `10.4007/annals.2015.182.3.3`.

[Sta21] The Stacks project authors. *The Stacks project*. `https://stacks.math.columbia.edu`. 2021.