`rising.c`: a basilisk simulation
Possible developments in the code

Giacomo Lorenzon, Davide Repetto, Davide Rinaldoni, Luca Saverio

In regards to the study we have made on the code `rising.c`, we believe that some improvements could be made on the code.
First of all, a possible evolution could be to make the code parallelizable. This could be done by substituting the function `mask` in `rising.c`, which is quite old and only-serial, with a more recent and scalable one. It is crucial to do so in order to get a lower computational time for simulations with high refinement levels. Please also note that `mask` prevents dumping with simulation files. The aforementioned advancement is fundamental for the `CASE 2`, which is way more demanding and computationally expensive than `CASE 1`.
Indeed, it would be interesting to further study the convergence of the code. Our results have shown an order of convergence between 1 and 2, if compared against the most refined simulation with `LEVEL = 10`. This could be explained by the fact that Mixed Youngs-centered method is not fully second order. If an MPI implementation would be available, `LEVEL = 11` may confirm our convergence rate estimate. However, at this time, `LEVEL = 9` and `10` take an order of 10^4s, 10^5s respectively to finish.
Another improvement could be to compute the skirts' width within `rising.c` so that the output can be easily post-processed. In this regard, Legendre (2022)[1] could be a great reference. In particular, our results of the second test case with `LEVEL = 10` are in accordance with the estimate given by Guthrie[2] [3].
We think that it would be also interesting to study the velocity field both within and outside the bubble to observe toroidal recirculation behind the bubble. For instance:

```
event output_velocity_field_bubble(t = 3.0) {
    // Initialize the file-name with time-indexing
    char nameF[80];
    sprintf (nameF, "output_velocity_field_bubble", t);

    // Compute vorticty
    scalar vort[];
    vorticity (u, vort);

    static FILE * fp2;

     if (i == 0) {
       fp2 = fopen (nameF, "w");
       foreach() {
         fprintf (fp2, "%f %f %f %f %f %f\r\n",x,y,u.x[]*(1.-f[]),
                  u.y[]*(1.-f[]), p[]*(1.-f[]),
```

[1] D. Legendre, Free Rising Skirt Bubbles, *Physical Review Fluids*, vol. 7, n. 9, pag. 093601, 2022

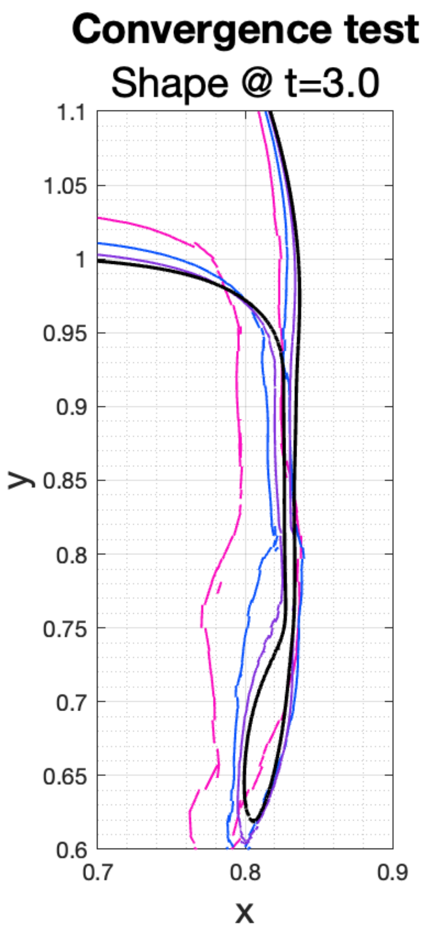
[2] R. I. L. Guthrie, Dynamic and MAS transfer phenomena of spherical capped bubble, Ph.D. thesis, *Imperial College*, 1967

[3] R. I. L. Guthrie and A. V. Bradshaw, The stability of gas envelopes trailed behind large spherical cap bubbles rising through viscous liquids, *Chem. Eng. Sci.24*, 913, 1969

```
                      vort[]*(1.-f[]));
                     }
          fclose(fp2);
        } else {
          fp2 = fopen (nameF, "a");
          foreach() {
            fprintf (fp2, "%f %f %f %f %f %f\r\n",x,y,u.x[]*(1.-f[]),
                      u.y[]*(1.-f[]), p[]*(1.-f[]),
                      vort[]*(1.-f[]));
                     }
          fclose(fp2);
      }
   }

```

Below we include, by way of example, the shape of the bubble w.r.t. different refinement levels.



`LEVEL = 7`: pink; `LEVEL = 8`: blue; `LEVEL = 9`: purple; `LEVEL = 10`: black.