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EXPLICIT EQUATIONS FOR SINGULAR DEL PEZZO SURFACES

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Abstract

Regarding Manin's conjecture for singular Del Pezzo surfaces, it is useful to have a concrete classification of such surfaces. In this thesis we develop what is necessary to obtain such a classification, specifically the case of rational singular Del Pezzo surfaces of degree 4 over a field of characteristic 0.

Contents

Introduction

Algebraic geometry is the study of geometric shapes described by polynomial equations. This apparently simple premise is the starting point of one of the most complex and fascinating topics in modern mathematics – but let's start simple. One of the fundamental examples in algebraic geometry are the non-singular cubic surfaces in projective space: roughly speaking, three-dimensional space can be described by three variables x, y, z, and a cubic surface is the set of solutions of a polynomial equation of degree 3 in those variables, for instance

$$xy - 2xy^2 - z + 3x^2z - yz + 2y^2z - 2xz^2 = 0. \tag{1}$$

A remarkable fact in algebraic geometry is that such a surface has always exactly 27 lines entirely contained in it – that is, if we examine the equation over an algebraically closed field like $\mathbb{C}$ and consider so called "points at infinity" – but even taking these extra conditions into account, it still is quite a special result.

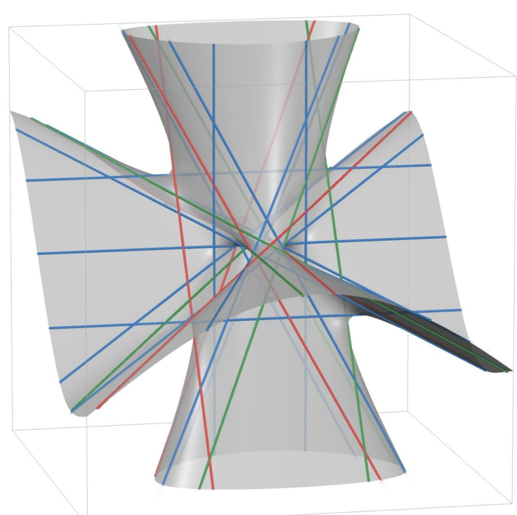


Figure 1: The surface given by equation (1) and its 27 lines. Image generated by Desmos, available at `https://www.desmos.com/3d/ccb1cae5a4`.

One way to understand why this is true is to realize that any cubic surface can be obtained from the ordinary (projective) two-dimensional plane $\mathbb{P}^2$ through six blow-ups, a blow-up being the process of replacing a point with a curve (called the exceptional curve) representing the linear directions to the point, a process which quite interestingly can be carried out in an algebraic manner. In plain terms, this means that by choosing six points on the plane we can "cook up" a cubic surface: for instance, choose points

$$P_1 = (0,0),\ P_2 = (1,1),\ P_3 = (-1,1),\ P_4 = (2,4),\ P_5 = (-2,4),\ P_6 = (0,3).$$

Now find out a basis f_0, f_1, f_2, f_3 of the vector space of cubic polynomials which has these six points among their solutions, like

$$f_0 = y^3-2x^2-5y^2+6y,\ f_1 = xy^2-5xy+4x,\ f_3 = x^2y-3x^2-y^2+3y,\ f_3 = x^3-xy.$$

Then a cubic surface can be obtained as the image X of the function

$$\begin{aligned} f\colon \mathbb{P}^2 &\to \mathbb{P}^3 \\ (x,y) &\mapsto (f_1/f_0,\ f_2/f_0,\ f_3/f_0), \end{aligned}$$

which happens to be the surface described by equation (1). Note that all of the necessary calculations can be dealt with the use of a computer [22], while the reasoning behind these steps is described in Chapter 3.

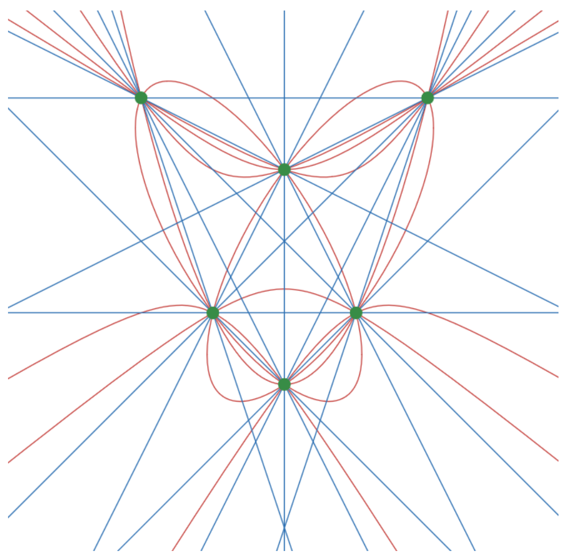


Figure 2: The six points $P_1, \ldots, P_6$, together with the lines passing through two points among them and the non-degenerate conics which pass through five points among them. Image generated by Desmos.

It can be proven that through the morphism f, the lines on X correspond to the lines of $\mathbb{P}^2$ which pass through two points among $P_1, \ldots, P_6$, the non-degenerate conics which pass through five points among $P_1, \ldots, P_6$, and the exceptional curves arising from the six points themselves. If we choose the six points "at random" (that is, in general position, i.e. the points are distinct, no three of them are aligned, and they do not lie on the same conic), then there will be $\binom{6}{2} = 15$ lines passing through pairs of points and $\binom{6}{5} = 6$ conics passing through five of the six points, so adding up we get $15 + 6 + 6 = 27$.

If we choose the six points not quite "at random" then something else happens: if a line of $\mathbb{P}^2$ passes through three points among $P_1, \ldots, P_6$, or a non-degenerate conic passes through all six of them, or we simply blow up a point on an exceptional curve, then through f that curve is contracted to a singular point. For instance, choose $P_6 = (0, 2)$ instead of $(0, 3)$.

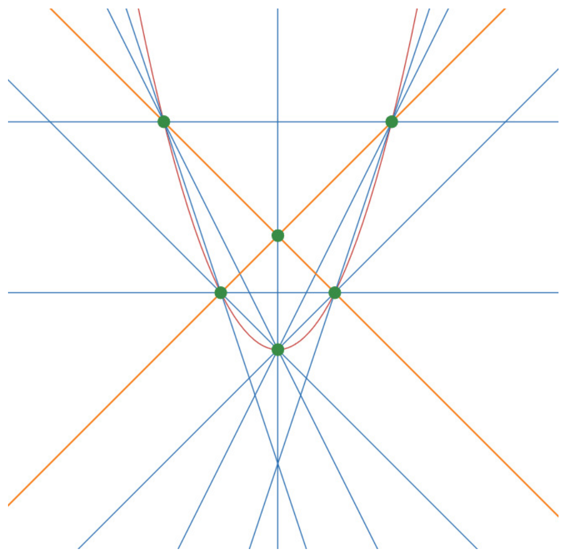


Figure 3: The modified configuration of points, lines and conics.

Then following the same steps as before, the corresponding cubic has equation

$$xy - 3xy^2 - z + 2x^2z + yz + 2y^2z - 2xz^2 = 0, \qquad (2)$$

which does no longer yield a smooth surface.

Consequently, there are also less lines on X. The relationship between surfaces and configurations of points is at the heart of the classification carried out in this thesis – which is, however, not a classification of cubic surfaces.

We instead proceed in Chapter 3 to study surfaces obtained as the intersection of two quadrics (that is, the system of solutions of two equations of degree 2) on four-dimensional space $\mathbb{P}^4$. The great tragedy of high-dimensional space is that it is not possible to visualize it in the manner seen e.g. in Figures 1 and 4. However, we treasure such visualizations and carry them in our minds even when trailing off to new geometric landscapes.

We will see in Chapter 2 that cubic surfaces on $\mathbb{P}^3$ and intersections of two

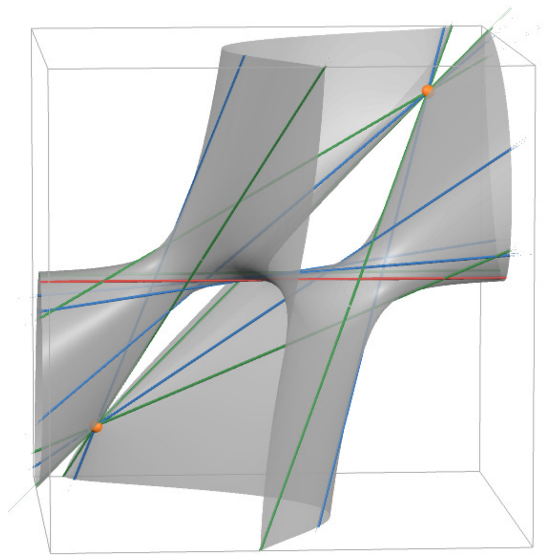


Figure 4: The surface given by equation (2), available at `https://www.desmos.com/3d/5d03bf5d3d`.

quadrics on $\mathbb{P}^4$ have a lot in common, being respectively the rational singular Del Pezzo surfaces of degree 3 and 4.

We also deal with another complication: that of working over (perfect) non-algebraically closed fields. This is introduced in Chapter 1, where the key idea is to examine a geometric object X defined over a field k as a geometric object $\overline{X}$ defined over the algebraic closure $\overline{k}$ together with the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$ on $\overline{X}$.

The chosen language is that of modern algebraic geometry and schemes, and this thesis is not an introduction to it.

Now a more detailed outline of the thesis. The project goal was to investigate so called k-forms of singular Del Pezzo surfaces, i.e. schemes X over a field k such that the algebraic closure $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ is isomorphic to the algebraic closure of a singular Del Pezzo surface over k. Chapter 1 is a general introduction to k-forms, and as previously mentioned, Galois extensions and the action of the Galois group on $\overline{X}$ are a key concept. The chapter also deals with some fundamental examples – namely, k-forms of $\mathbb{P}^1_k, \mathbb{P}^2_k$ and $\mathbb{P}^1_k \times \mathbb{P}^1_k$ – which, together with a section on the Hirzebruch surface $F_{2,k}$ and its k-forms in Chapter 2, become very important in the classification.

Chapter 2 begins by proving that any k-form of a weak (ordinary) Del Pezzo surface is itself a weak (ordinary) Del Pezzo surface. The same is true for singular Del Pezzo surfaces, which we define in a natural and general way as normal geometrically integral projective surfaces X over k such that the canonical divisor K_X is locally principal (that is, X is Gorenstein) and $-K_X$ is ample;

it turns out that if such a surface is (geometrically) rational then its minimal desingularization X' is a weak Del Pezzo surface, and for $d \geq 3$ the anticanonical image of X' is X. We assume from other sources these facts in the case of an algebraically closed field, and prove them for a perfect field.

Chapter 3 is the heart of this thesis. The explicit goal is to work out explicit equations for all rational singular Del Pezzo surfaces of degree 4 over a field of characteristic 0, a goal motivated by Manin's conjecture. The basic plan is illustrated in Section 3.2 and executed in Section 3.7.

Notation

A scheme X over a field k is an *algebraic variety* over k if the structural morphism $X \to \operatorname{Spec} k$ is of finite type. An algebraic variety is a *curve* (resp. a *surface*) if its irreducible components have dimension 1 (resp. 2). A *projective variety* over k is a projective scheme over k.

When blowing up points, we mean reduced closed points.

If K/k is a field extension and X is a scheme over k then

$$X_K = X \otimes_k K := X \times_{\operatorname{Spec} k} \operatorname{Spec} K$$

and $p_K \colon X_K \to X$ is the canonical projection morphism. The algebraic closure of k is $\overline{k}$, and similarly $\overline{X} := X \otimes_k \overline{k}$; the canonical projection morphism is denoted $\overline{p} \colon \overline{X} \to X$. If D is a (Cartier) divisor on X then $D_K := p_K^*(D)$ and $\overline{D} := \overline{p}^*(D)$.

After fixing a field k, we write $\mathbb{P}^n$, $\mathbb{A}^n$ for the varieties $\mathbb{P}^n_{\overline{k}}$, $\mathbb{A}^n_{\overline{k}}$ over the algebraic closure.

Chapter 1

Base change and Galois extensions

Definition 1.0.1. Let k be a field. A *form* or *k-form* of a scheme X over k is a scheme Y over k such that the two are isomorphic over the algebraic closure: $\overline{X} \simeq \overline{Y}$. A *$k$-model* of a scheme X over the algebraic closure $\overline{k}$ is a scheme Y over the base field k such that $\overline{Y} \simeq X$.

For what follows, recall that the set of *rational points* of a scheme X over k is the set of sections $\operatorname{Spec} k \to X$ over k. If X is embedded in $\mathbb{P}^n_k$, the rational points are simply those of the form $(x_0 : \cdots : x_n)$ inside X with $x_0, \ldots, x_n \in k$ [21, Corollary 2.3.44].

Example 1.0.2. Consider the curve C given by $x_0^2 + x_1^2 + x_2^2$ on $\mathbb{P}^2_{\mathbb{R}}$. This curve does not have any rational points, that is, points lying on it of the form $(x_0 : x_1 : x_2)$ with $x_0, x_1, x_2 \in \mathbb{R}$. But if we consider it as a scheme, it is not empty: for example, it has the point P represented by the ideal $(x_0, x_1^2 + x_2^2)$.

When we consider C as a curve in $\mathbb{P}^2_{\mathbb{C}}$, i.e. we take the complexification $\overline{C} = C \otimes_{\mathbb{R}} \mathbb{C}$, since it is a nonsingular plane conic over an algebraically closed field we can conclude that it is isomorphic to the projective line $\mathbb{P}^1_{\mathbb{C}}$. This means that C is an $\mathbb{R}$-form of $\mathbb{P}^1_{\mathbb{R}}$, but the lack of rational points of C prevents it from being isomorphic to $\mathbb{P}^1_{\mathbb{R}}$.

It is interesting to note that over the algebraic closure, P no longer represents a single point. To see this, note that by definition of fibered product there is a canonical morphism $\overline{p}\colon \overline{C} \to C$ (henceforth called the *projection*, see Notation), and we can check the preimage $\overline{p}^{-1}(P)$. These are the points of $\overline{C}$ corresponding to the homogeneous prime ideals of $\mathbb{C}[x_0, x_1, x_2]$ contained inside $(x_0, x_1^2 + x_2^2) = (x_0, (x_1 + ix_2)(x_1 - ix_2))$, which are $(x_0, x_1 + ix_2)$ and $(x_0, x_1 - ix_2)$, written alternatively as

$$P_1 = (0 : -i : 1), \qquad P_2 = (0 : i : 1).$$

Notice that the coordinates of these two points are one the complex conjugate of the other.

A k-form of the projective line is called a *Brauer-Severi curve*. These are discussed more in detail in Section 1.1.1.

We now take a more thorough look at base changes of schemes over Galois extensions (for example, the algebraic closure of a perfect field). In the following, fix a Galois extension K/k and a scheme X over k. Let $G = \mathrm{Gal}(K/k)$. Given $\sigma \in G$, the Spec functor provides a k-automorphism

$$\operatorname{Spec}\sigma\colon \operatorname{Spec}K \to \operatorname{Spec}K$$

and this induces a right action of G on $\operatorname{Spec}K$ through K-automorphisms: it is a right action because

$$(\operatorname{Spec}\sigma)(\operatorname{Spec}\tau) = \operatorname{Spec}(\tau \circ \sigma)$$

for any $\sigma, \tau \in G$. Furthermore, G naturally acts from the right on X_K: given $\sigma \in G$, the morphism

$$f_\sigma := \mathrm{id}_X \times_k \operatorname{Spec}\sigma\colon X_K \to X_K$$

is a k-automorphism of X_K.

While this construction is natural in one way, it is unnatural in another: namely, if X is embedded in $\mathbb{P}^n_k$ then for any closed point $(x_0 : \dots : x_n) \in X_K \subseteq \mathbb{P}^n_K$ the result of applying f_σ is

$$f_\sigma(x_0 : \dots : x_n) = (\sigma^{-1}(x_0) : \dots : \sigma^{-1}(x_n))$$

instead of the more obvious $(\sigma(x_0) : \dots : \sigma(x_n))$; and if $Y = V(g_1, \dots, g_m)$ is a subscheme of X_K then the image of Y through f_σ is $V(\sigma^{-1}(g_1), \dots, \sigma^{-1}(g_m))$, where σ refers to

$$\mathrm{id} \otimes_k \sigma\colon k[x_0, \dots, x_n] \otimes_k K = K[x_0, \dots, x_n] \to K[x_0, \dots, x_n]$$

which fixes the variables $x_0, \dots, x_n$ and acts on the coefficients as $\sigma\colon K \to K$.

Remark 1.0.3. If K/k is a field extension and A is a k-algebra, then the canonical map $A \to A_K = A \otimes_k K$ is injective, as any field extension is (faithfully) flat. We can, and will, identify A the image $A \otimes 1$ of this morphism.

Lemma 1.0.4. *Let K/k be an algebraic field extension and let A be a k-algebra. Suppose that $\mathfrak{p} \subset A$ is a prime ideal. Then a prime ideal $\mathfrak{Q} \subset A_K$ is minimal among those containing $\mathfrak{p}A_K$ if and only if $\mathfrak{Q} \cap A = \mathfrak{p}$.*

Proof.

($\neg \Leftarrow \neg$) Suppose that $\mathfrak{Q} \cap A \neq \mathfrak{p}$. One possibility is that $\mathfrak{Q} \cap A \not\supseteq \mathfrak{p}$, but then $\mathfrak{Q} \not\supseteq \mathfrak{p}A_K$. Another is that $\mathfrak{Q} \cap A \supsetneq \mathfrak{p}$. In this case, by the going-down theorem for flat morphisms [26, Theorem 5.D] there exists a prime $\mathfrak{P} \subset A_K$ such that $\mathfrak{P} \cap A = \mathfrak{p}$ and $\mathfrak{P} \subsetneq \mathfrak{Q}$ i.e. $\mathfrak{Q}$ is not minimal among primes containing $\mathfrak{p}A_K$.

($\Leftarrow$) Suppose that $\mathfrak{Q} \cap A = \mathfrak{p}$ and that $\mathfrak{Q} \supseteq \mathfrak{R} \supseteq \mathfrak{p}A_K$ for some prime ideal $\mathfrak{R} \subset A_K$. Then $\mathfrak{Q} \cap A = \mathfrak{R} \cap A = \mathfrak{p}$. But A_K is integral over A, because K/k is an algebraic extension, hence $\mathfrak{Q} = \mathfrak{R}$ by [2, Corollary 5.9].

□

Theorem 1.0.5. *Let K/k be an algebraic extension and let X be a scheme over k. Then the projection morphism $p_K \colon X_K \to X$ is surjective and closed.*

Proof. The properties of closedness and surjectivity can be checked locally, so we can suppose $X = \operatorname{Spec} A$ affine where A is a k-algebra. Then $p_K = \operatorname{Spec}(A \hookrightarrow A_K)$ and $p_K([\mathfrak{P}]) = [\mathfrak{P} \cap A]$ for all $\mathfrak{P} \subset A_K$ prime ideals.

To prove surjectivity, suppose that $\mathfrak{p} \subset A$ is a prime ideal. The ideal $\mathfrak{p}A_K$ is not the whole of A_K: if it was, there would be $\lambda_i \in K$, $a_i \in A$, $p_i \in \mathfrak{p}$ such that

$$1 = \sum_i \lambda_i \, a_i \, p_i$$

or in other words, 1 would be in the K-linear span of $a_1p_1, \dots, a_np_n$. But by the theory of Galois descent on vector spaces [4, Theorem 3.1], 1 would also be in their k-span, i.e. $\mathfrak{p} = A$, which is absurd. So the ideal $\mathfrak{p}A_K$ is not the whole of A_K and we can find a minimal prime ideal $\mathfrak{Q} \supseteq \mathfrak{p}A_K$, which is the same thing as $\mathfrak{Q} \cap A = \mathfrak{p}$ by Lemma 1.0.4.

For closedness, let $I \subseteq A_K$ be an ideal. We prove that $p_K(V(I)) = V(I \cap A)$: indeed, it is easy to see that

$$\{[\mathfrak{P} \cap A] \mid I \subseteq \mathfrak{P} \text{ prime} \subset A_K\} \subseteq \{[\mathfrak{q}] \mid I \cap A \subseteq \mathfrak{q} \text{ prime} \subset A\}$$

and the other inclusion follows from surjectivity. □

Definition 1.0.6. Let K/k be a field extension, and let X be a scheme over k. A closed subset $C \subseteq X_K$ is said to be *defined over k* if $p_K^{-1}(p_K(C)) = C$.

Theorem 1.0.7. *Let K/k be a Galois field extension, and let X be an algebraic variety over k. Let $P \in X$ be a point (not necessarily closed). Then the fiber $p_K^{-1}(P)$ is non-empty and finite, and the action of the Galois group $G = \operatorname{Gal}(K/k)$ on X restricts to and is transitive on $p_K^{-1}(P)$.*

Proof. We can suppose $X = \operatorname{Spec} A$ affine with A finitely generated k-algebra, and that is because if $U \subseteq X$ is an open affine neighborhood of P, then surely the fiber $p_K^{-1}(P)$ is contained in U_K and the action of G restricts to U_K.

Let $P = [\mathfrak{p}]$, then the fiber $p_K^{-1}([\mathfrak{p}])$ is non-empty by surjectivity 1.0.5, and by 1.0.4 consists of the minimal primes of A_K containing $\mathfrak{p}A_K$. These are in 1-1 correspondence with the irreducible components of $\operatorname{Spec}(A_K/\mathfrak{p}A_K)$, which are finitely many because $\operatorname{Spec}(A_K/\mathfrak{p}A_K)$ is Noetherian [21, Proposition 2.4.9].

If $[\mathfrak{Q}] \in p_K^{-1}([\mathfrak{p}])$ and $\sigma \in G$ then $f_\sigma([\mathfrak{Q}]) = [\sigma^{-1}(\mathfrak{Q})]$ which is still a minimal prime containing $\mathfrak{p}A_K$, so the action of G restricts to the fiber. It is also transitive: by contradiction, suppose that $[\mathfrak{P}], [\mathfrak{Q}] \in p_K^{-1}([\mathfrak{p}])$ but $\sigma(\mathfrak{P}) \neq \mathfrak{Q}$ for all $\sigma \in G$. By the prime avoidance lemma [2, Proposition 1.11]

$$\bigcup_{\sigma \in G} \sigma(\mathfrak{P}) \not\supseteq \mathfrak{Q}$$

so there exist an element $x \in \mathfrak{Q}$ such that $\sigma(x) \notin \mathfrak{P}$ for all $\sigma \in G$. Now, let $K \supseteq F \supseteq k$ be a finite Galois intermediate field with $x \in A_F$, and let $H = \mathrm{Gal}(F/k)$. Then

$$y = \prod_{\sigma \in H} \sigma(x)$$

is invariant under the action of G i.e. $y \in \mathfrak{p}A_K \subseteq \mathfrak{P}$, so $\sigma(x) \in \mathfrak{P}$ for some $\sigma \in H$, absurd. □

Using the well-known correspondence between points of a scheme and closed irreducible subsets on it, we will frequently apply Theorem 1.0.7 to curves: if C is an integral curve on X, then $C_K = p_K^{-1}(C)$ has finitely many irreducible components, on which the Galois group acts transitively.

Corollary 1.0.8. *Let K/k be a Galois extension, let X be an algebraic variety over k, then a closed subset $C \subseteq X_K$ is defined over k if and only if it is invariant under the action of the Galois group, i.e. for all $\sigma \in \mathrm{Gal}(K/k)$*

$$f_\sigma(C) = C.$$

Proof. Let C be invariant under the Galois action, and let $\{\xi\}^-$ be an irreducible component. Suppose that

$$p_K^{-1}(p_K(\xi)) = \{\xi_1, \dots, \xi_n\}.$$

Then

$$p_K^{-1}(p_K(\{\xi\}^-)) = \{\xi_1\}^- \cup \cdots \cup \{\xi_n\}^-$$

are Galois conjugate and $\{\xi\}^-$ is among them, so on top of being a closed subset defined over k, $p_K^{-1}(p_K(\xi))$ is also contained in C by the hypothesis of invariance under the Galois action. Since C has finitely many irreducible components, this means that it can be written as a finite union of closed subsets defined over k, hence it is itself defined over k.

The other direction is obvious. □

Definition 1.0.9. Let X be an algebraic variety over a perfect field k. Let $P \in X$ be a point (not necessarily closed). Then we say that P *splits* over an algebraic extension K/k if the cardinality of $p_K^{-1}(P)$ equals the cardinality of $\overline{p}^{-1}(P)$.

By extension, the irreducible closed subset $\{P\}^-$ splits over K/k if P does.

Theorem 1.0.10. *Suppose that k is a perfect field, X is an algebraic variety over k and that $P \in X$ is a closed point. Then*

$$\#\overline{p}^{-1}(P) = [\kappa(P) : k]$$

and P splits over the normal closure of $\kappa(P)$. In particular, P is k-rational if and only if $\#\overline{p}^{-1}(P) = 1$.

Proof. Let us consider P as the image of the canonical injective morphism $\operatorname{Spec} k(P) \to X$. Taking the algebraic closure we obtain the injective morphism $\operatorname{Spec} \overline{\kappa(P)} \to \overline{X}$ with image $\overline{p}^{-1}(P)$, so we need to find the cardinality of $\operatorname{Spec} \overline{\kappa(P)}$.

First, notice that $d = [k(P) : k] < \infty$ by some version of the Nullstellensatz [2, Corollary 7.10], so it is a finite separable extension and by the primitive element theorem [17, Theorem V.4.6] there exists $\alpha \in \kappa(P)$ such that $\kappa(P) = k(\alpha)$. Let $p(t) = (t - \alpha_1) \dots (t - \alpha_d)$ be the minimal polynomial of α, then

$$\begin{aligned} k(P) \otimes_k \overline{k} = \frac{k[t]}{(t-\alpha_1)\dots(t-\alpha_d)} \otimes \overline{k} &\simeq \frac{\overline{k}[t]}{(t-\alpha_1)\dots(t-\alpha_d)} \\ &\simeq \frac{\overline{k}[t]}{(t-\alpha_1)} \times \dots \times \frac{\overline{k}[t]}{(t-\alpha_d)} \simeq \overline{k}^d \end{aligned}$$

where the isomorphisms are of rings, hence $\operatorname{Spec} \overline{\kappa(P)}$ has cardinality d.

In order to split $p(t)$ we could have taken its splitting field, which is its normal closure. □

Theorem 1.0.11. *On an algebraic variety X over a perfect field k, suppose that a point P (not necessarily closed) is such that $\overline{p}^{-1}(P)$ has cardinality n. Then P splits over an extension K with*

$$n \leq [K : k] \leq n!$$

In particular, if $n = 2$ then P splits over a quadratic extension.

Proof. Let $\overline{p}^{-1}(P) = \{P_1, \dots, P_n\}$. Let $G = \operatorname{Gal}(\overline{k}/k)$. Recall that the stabilizer G_{P_1} of P_1 is a subgroup of G with index the cardinality of the orbit of P_1, which is n. Now, the orbit of P_2 by G_{P_1} has cardinality between 1 and $n-1$, hence the subgroup G_{P_1,P_2} of elements of G which fix both P_1 and P_2 has index between n and $n(n-1)$, and similarly

$$n \leq (G : G_{P_1,\dots,P_n}) \leq n!$$

where $H := G_{P_1,\dots,P_n}$ is the subgroup of G of elements fixing all the P_i.

Now, let $K = \overline{k}^H$. Then $n \leq [K : k] \leq n!$ and P splits over K: if there was an element $P' \in p_K^{-1}(P)$ such that $\overline{p}^{-1}(P') = \{P_1, \dots, P_m\}$ with $m \geq 2$ then by the transitivity of the action of the Galois group on fibers, there should be a $\sigma \in H$ such that for instance $\sigma(P_1) = P_2$, which is absurd as H was supposed to fix all these elements. □

The following propositions say that intersection numbers are invariant under base changes and isomorphisms (not necessarily over the field k), and in particular invariant under the action of the Galois group (recall that $\mathrm{Gal}(K/k)$ acts on X_K through k-automorphisms, not K-automorphisms).

Proposition 1.0.12. *Let X be a regular integral projective surface over a field k, and let K/k be a field extension. Then for any $C, D \in \mathrm{Div}(X)$ we have that*

$$C \cdot D = C_K \cdot D_K.$$

Proof. [21, Proposition 9.2.15]. □

Proposition 1.0.13. *Let $f\colon X \xrightarrow{\sim} Y$ be an isomophism between regular Noetherian connected schemes of dimension 2. Let C and D be two effective divisors on Y, with no common irreducible components. Let $x \in X$ be a closed point, and $y = f(x)$. Then*

$$(f^*(C) \cdot f^*(D))_x = (C \cdot D)_y.$$

Proof. Recall that

$$(C \cdot D)_y = \mathrm{length}_{\mathcal{O}_{Y,y}} \mathcal{O}_{Y,y}/(\mathcal{O}_Y(-C)_y + \mathcal{O}_Y(-D)_y).$$

By [21, Remark 7.1.35], the image of the ideal $\mathcal{O}_Y(-C)_y \subset \mathcal{O}_{Y,y}$ through $f_x^{\#}\colon \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is the ideal $\mathcal{O}_X(-f^*(C))_x \subset \mathcal{O}_{X,x}$, and the same goes for $\mathcal{O}_Y(-D)_y$. Since $f_x^{\#}$ is an isomorphism, this concludes the proof. □

Corollary 1.0.14. *Let $f\colon X \xrightarrow{\sim} Y$ be an isomorphism (not necessarily over k) between regular integral projective surfaces over a field k. Let $C, D \in \mathrm{Div}(Y)$. Then*

$$C \cdot D = f^*(C) \cdot f^*(D).$$

Proof. The intersection product is entirely determined by its restriction to effective divisors with no common irreducible components, and by the properties of symmetry, additivity and the fact that it only depends on linear equivalence classes [21, Theorem 9.1.12]. Since an isomorphism preserves all of these properties, it also preserves the intersection product. □

1.1 Some examples

1.1.1 Forms of $\mathbb{P}^1_k$

Let X be a form of $\mathbb{P}^1_k$, also known as *Brauer-Severi curves*. The anticanonical divisor $-K_X$ is defined over k, very ample and induces an immersion $X \hookrightarrow \mathbb{P}^2_k$. The image is a nonsingular conic C.

If X has a k-rational point, then it is isomorphic to $\mathbb{P}^1_k$. To see this, note that the rational point P lets us defines an invertible sheaf $\mathcal{L} = \mathcal{O}_X(P)$. Over the algebraic closure, $\overline{p}^*\mathcal{L}$ is generated by global sections and induces an isomorphism $\overline{C} \xrightarrow{\sim} \mathbb{P}^1$. Since [21, Corollary 5.2.27]

$$H^0(\overline{X}, \overline{p}^*\mathcal{L}) = H^0(X, \mathcal{L}) \otimes_k \overline{k},$$

$\mathcal{L}$ is also generated by global sections; as we have seen, the induced morphism $C \xrightarrow{\sim} \mathbb{P}^1_k$ becomes an isomorphism over the algebraic closure, so by [12, Proposition 14.51] it is itself an isomorphism. This also means that a k-form X of $\mathbb{P}^1_k$ is isomorphic to $\mathbb{P}^1_k$ if and only if $X(k) \neq \varnothing$ if and only if it is k-rational.

It turns out [15, Proposition 3.3.5] that two forms X, X' of $\mathbb{P}^1_k$ are isomorphic if and only if there is a k-automorphism of $\mathbb{P}^2_k$ such that one of the corresponding nonsingular conics is sent to the other one, and that through a k-automorphism any nonsingular conic can be reduced to the form

$$ax_0^2 + bx_1^2 = x_2^2$$

where $a, b \in k^\times$. Moreover, there is a $1{-}1$ correspondence between isomorphisms classes of nonsingular conics over k and isomorphism classes of quaternion algebras over k:

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{nonsingular conics over } k \end{array} \right\} & \xleftrightarrow{1:1} & \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{quaternion algebras over } k \end{array} \right\} \\ ax_0^2 + bx_1^2 = x_2^2 & \leftrightarrow & \left(\dfrac{a, b}{k} \right) \end{array}$$

where $\left(\frac{a,b}{k}\right)$ is the k-algebra generated by $1, i, j, k$ with relations $i^2 = a,\ j^2 = b,\ k = ij = -ji$.

For instance, let $k = \mathbb{R}$. There are two $\mathbb{R}$-forms of $\mathbb{P}^1_{\mathbb{R}}$: one is $\mathbb{P}^1_{\mathbb{R}}$ itself, the other one is the "imaginary" conic

$$x_0^2 + x_1^2 + x_2^2 = 0$$

and there are no others by Sylvester's law of inertia. Likewise, there are two quaternion algebras over $\mathbb{R}$: one is the matrix algebra $M_2(\mathbb{R})$, which can also be described as $\left(\frac{1,1}{\mathbb{R}}\right)$ through the following isomorphism:

$$i \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad j \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix};$$

in general, when $\left(\frac{a,b}{k}\right) \simeq M_2(k)$ we say that the quaternion algebra and the pair (a, b) *splits*. The other one consists of Hamilton's ring of quaternions, which by definition is $\left(\frac{-1,-1}{\mathbb{R}}\right)$.

A classification of nonsingular conics, equivalently of quaternion algebras, over the field $k = \mathbb{Q}$ is trickier to describe. First of all, it is not hard to see [15, Exercise 3.1.4] that for all $p/q, p'/q \in \mathbb{Q}^\times$

$$\left(\frac{p/q, p'/q'}{\mathbb{Q}} \right) \simeq \left(\frac{pq, p'q'}{\mathbb{Q}} \right)$$

so we can consider $a, b \in \mathbb{Z} \setminus \{0\}$ instead of in $\mathbb{Q}^\times$. Given $a, b, c, d \in \mathbb{Z} \setminus \{0\}$ we would like to be able to say whether $\left(\frac{a,b}{\mathbb{Q}}\right) \simeq \left(\frac{c,d}{\mathbb{Q}}\right)$ or not. To do this, we introduce very briefly the notion of local fields and use a subcase of the Albert-Brauer-Hasse-Noether theorem.

In short, given a number field k, the *places* of k are the equivalence classes of absolute values over $\mathbb{Q}$, where two absolute values $|\cdot|_1,\ |\cdot|_2$ are equivalent if for all $x \in k,\ |x|_1 < 1 \Leftrightarrow |x|_2 < 1$. For instance, if $k = \mathbb{Q}$ then we have the following absolute values:

- $|x|_\infty = \begin{cases} +x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ is the usual archimedean absolute value;
- $|x|_p = p^{-k}$, with $k = \nu_p(x)$ the maximum integer such that $p^k|x$: this is called the p-adic absolute value, and it possesses the *non-archimedean* property;
- these are all distinct as places, and by Ostrowski's theorem there are no others over $\mathbb{Q}$.

There is a construction called the *completion of a field with respect to an absolute value*, and the resulting fields are called *local fields*. For example, the completion of $\mathbb{Q}$ with respect to $|\cdot|_\infty$ is the field of real numbers $\mathbb{R}$; the completion of $\mathbb{Q}$ with rispect to the p-adic absolute value $|\cdot|_p$ is the field of p-adic numbers $\mathbb{Q}_p$.

Local fields k_ν arising from a number field k are of interest to us. The first reason is that it is easy to classify quaternion algebras on local rings. If $\left(\frac{a,b}{k_\nu}\right) \not\cong M_2(k_\nu)$ (that is, (a,b) does not split) we say that $\left(\frac{a,b}{k_\nu}\right)$ and the pair (a,b) *ramifies* over k_ν, and on local fields all ramified quaternion algebras are isomorphic to each other, i.e. there are only two isomorphism classes of quaternion algebras.

The second reason is that there is a strong relationship between quaternion algebras over a number field k and over its completions: given $a, b \in k^\times$, the set

$$\mathrm{Ram}(a,b) = \{\nu \text{ places of } k \mid (a,b) \text{ ramifies over } k_\nu\}$$

is finite, of even cardinality and determines $\left(\frac{a,b}{k}\right)$ up to isomorphism.

This is an instance of a much more general and powerful theorem named the Albert-Brauer-Hasse-Noether theorem. Basically speaking, the isomorphism classes of quaternion algebras over k are in bijection with the elements of order one or two $\mathrm{Br}\,k[2]$ of a group $\mathrm{Br}\,k$ called the *Brauer group* of k, whose elements are classes of central simple algebras over k up to Brauer equivalence. The Albert-Brauer-Hasse-Noether theorem states that there is an exact sequence

$$0 \to \mathrm{Br}\,k \to \bigoplus_\nu \mathrm{Br}\,k_\nu \to \mathbb{Q}/\mathbb{Z} \to 0,$$

where ν ranges over the places of k; $\mathrm{Br}\,k_\nu[2] = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is a group with two elements, the identity $0+\mathbb{Z}$ representing split quaternion algebras and the other $\frac{1}{2} + \mathbb{Z}$ representing quaternion algebras which ramify.

Returning to the case $k = \mathbb{Q}$, for any $a, b \in \mathbb{Z} \setminus \{0\}$ there is a simple way of determining the set $\mathrm{Ram}(a,b)$:

- (a, b) ramifies over $\mathbb{R}$ if and only if $a, b < 0$, like what we saw with Hamilton's quaternions;
- let $p > 2$ be an odd prime: assuming a, b squarefree (otherwise divide by the square) and $\nu_p(a) \leq \nu_p(b)$, (a, b) ramifies over $\mathbb{Q}_p$ if and only if either $p \nmid a,\ p \mid b$ and a is not a square modulo p, or $p \mid a,\ p \mid b$ and $-a^{-1}b$ is not a square modulo p [15, Exercise 3.3.7];
- the previous criteria means that for odd primes, we only need to check for p dividing a or b. At this point, we know $\mathrm{Ram}(a, b) \setminus \{2\}$ i.e. we only need to determine whether (a, b) ramifies over $\mathbb{Q}_2$ or not, but since $\mathrm{Ram}(a, b)$ must be of even cardinality, this is easy to infer.

We can represent isomorphism classes of quaternion algebras over $\mathbb{Q}$ by picturing a grid in which squares of coordinates (a, b) and (c, d) are colored the same if the corresponding quaternion algebras $\left(\frac{a,b}{\mathbb{Q}}\right), \left(\frac{c,d}{\mathbb{Q}}\right)$ are isomorphic.

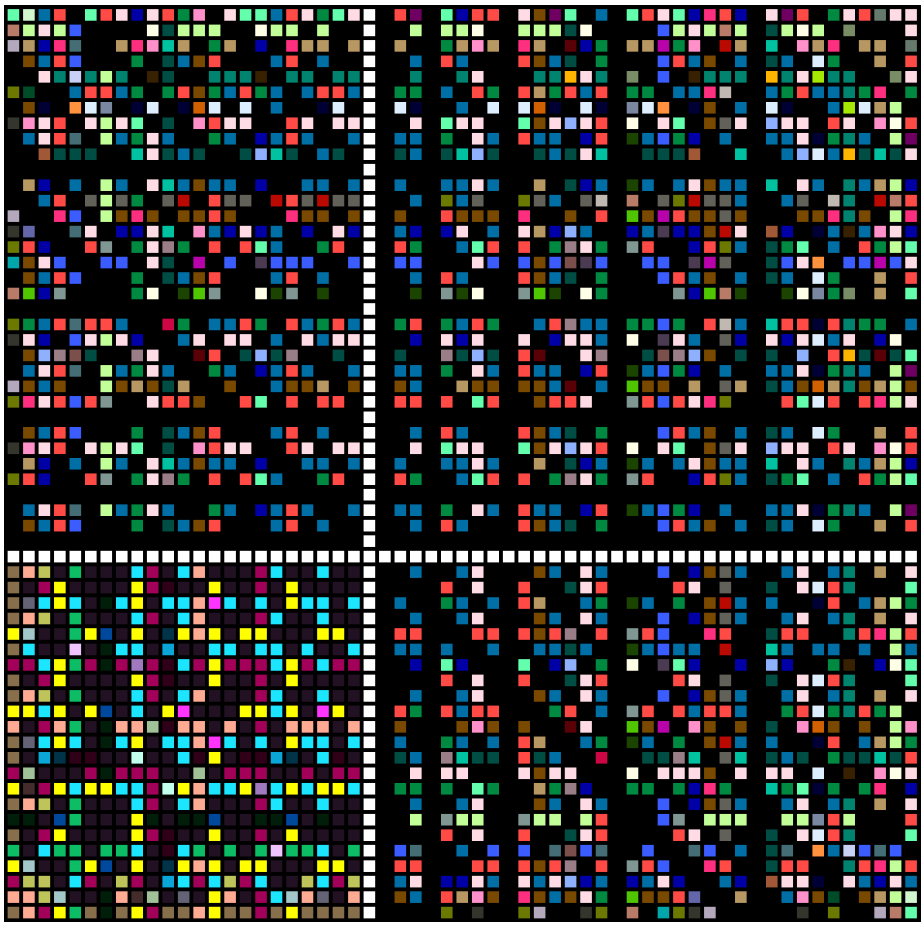


Figure 1.1: Isomorphism classes of quaternion algebras over $\mathbb{Q}$. White cells represent coordinates $(a, 0), (0, b)$ which are not associated to quaternion algebras, while black cells represent quaternion algebras which split. This is a screenshot of an interactive tool, see [23].

1.1.2 Forms of $\mathbb{P}^2_k$

The k-forms of $\mathbb{P}^2_k$ are called *Brauer-Severi surfaces*; more in general, the k-forms of $\mathbb{P}^n$ are called *Brauer-Severi varieties.* As in the case of $\mathbb{P}^1_k$, there is a tight relationship between Brauer-Severi surfaces and certain algebraic structures, called central simple algebras, which we will not talk about here. We only state the following theorem (which we have seen for $n = 1$) which is crucial for the rest of this thesis:

Theorem 1.1.1 (Châtelet). *Let X be a Brauer-Severi variety over k of dimension n. Then X is isomorphic to $\mathbb{P}^n_k$ if and only if $X(k) \neq \varnothing$ if and only if X is k-rational.*

Proof. [12, Proposition 14.91]. □

Of course, $\mathbb{P}^2_k = \operatorname{Proj} k[x_0, x_1, x_2]$ is k-rational. In order to fix notation, we describe an affine open subscheme $U \subset \mathbb{P}^2_k$ such that $U \simeq \mathbb{A}^2_k$: simply let

$$U\colon x_0 \neq 0, \quad x = x_1/x_0, \quad y = x_2/x_0.$$

1.1.3 Forms of $\mathbb{P}^1_k \times \mathbb{P}^1_k$

We immediately see that if P, P' are k-forms of $\mathbb{P}^1_k$ then $X = P \times P'$ is a k-form of $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Moreover, X is isomorphic to $\mathbb{P}^1_k \times \mathbb{P}^1_k$ if and only if $X(k) \neq \varnothing$ if and only if X is k-rational: suppose that $X(k)$ is not empty. Then through the morphisms $X \to P$, $X \to P'$ we see that $P(k), P'(k) \neq \varnothing$ hence $P \simeq P' \simeq \mathbb{P}^1_k$.

In order to fix notation, we specify the following open subscheme $U \subset \mathbb{P}^1_k \times \mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1] \times \operatorname{Proj} k[y_0, y_1]$ isomorphic to $\mathbb{A}^2_k$:

$$U\colon x_0 \neq 0, y_0 \neq 0, \quad x = x_1/x_0, \quad y = y_1/y_0.$$

It might seem reasonable to assume that what we have already found are all the k-forms of $\mathbb{P}^1_k \times \mathbb{P}^1_k$, but in fact there is another family of surfaces. To construct them, let $\alpha^2 = a \in k^\times$ and consider the Segre embedding

$$\begin{gathered} \mathbb{P}^1_k \times \mathbb{P}^1_k \hookrightarrow V(a_0a_3 - a_1a_2) \subset \mathbb{P}^3_k \\ (x_0 : x_1, y_0 : y_1) \mapsto (x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1). \end{gathered}$$

The pencils of curves of degree $(1, 0)$ and $(0, 1)$ given by

$$L_{A,B}\colon Ax_0 + Bx_1, \quad L'_{C,D}\colon Ay_0 + By_1$$

through the embedding become

$$L_{A,B}\colon \begin{cases} Aa_0 + Ba_2 \\ Aa_1 + Ba_3, \end{cases} \quad L'_{C,D}\colon \begin{cases} Ca_0 + Da_1 \\ Ca_2 + Da_3. \end{cases}$$

On $\mathbb{P}^3_{k(\alpha)}$, consider now a $k(\alpha)$-automorphism of projective space which makes the elements of these two pencils no longer defined over k, but $k(\alpha)/k$-conjugate: set

$$\begin{cases} a_0 = b_0 \\ a_1 = b_2 + \alpha b_1 \\ a_2 = b_2 - \alpha b_1 \\ a_3 = b_3 \end{cases} \Rightarrow \quad \begin{cases} b_0 = a_0 \\ b_1 = \frac{1}{2\alpha}(a_1 - a_2) \\ b_2 = \frac{1}{2}(a_1 + a_2) \\ b_3 = a_3. \end{cases}$$

Then $V(a_0a_3 - a_1a_2) = V(b_0b_3 + ab_1^2 - b_2^2)$ and

$$L_{A,B} \colon \begin{cases} Ab_0 + B(b_2 - \alpha b_1) \\ A(b_2 + \alpha b_1) + Bb_3, \end{cases} \quad L'_{C,D} \colon \begin{cases} Cb_0 + D(b_2 + \alpha b_1) \\ C(b_2 - a\alpha b_1) + Db_3. \end{cases}$$

Let $G(k(\alpha)/k) = \{1, \sigma\}$. We notice that

$$f_\sigma^{-1}(L_{A,B}) = \begin{cases} A^\sigma b_0 + B^\sigma(b_2 + \alpha b_1) \\ A^\sigma(b_2 - \alpha b_1) + B^\sigma b_3 \end{cases} = L'_{A^\sigma, B^\sigma}$$

and similarly $f_\sigma^{-1}(L'_{C,D}) = L_{C^\sigma, D^\sigma}$. Thus, as a k-variety

$$Q_\alpha := V(b_0b_1 + ab_1^2 - b_2^2) \subset \mathbb{P}^3_k$$

has Picard group $\mathbb{Z}$, because the curves of degree $(1,0)$ and $(0,1)$ are individually not defined over k, but their sum of degree $(1,1)$ is. This alone makes Q_α distinct from all other k-forms of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ we have defined before. Naturally, $Q_\alpha \otimes_k k(\alpha) \simeq \mathbb{P}^1_{k(\alpha)} \times \mathbb{P}^1_{k(\alpha)}$.

Another property which sets apart Q_α from products of Brauer-Severi curves is that the former is k-rational: consider the morphism

$$\begin{aligned} \mathbb{A}^2_k &\to \mathbb{P}^3_k \\ (x, y) &\mapsto (1 : x : y : y^2 - ax^2). \end{aligned}$$

It is an embedding of $\mathbb{A}^2_k$ in Q_α, with image the open subscheme $Q_\alpha \cap D_+(b_0)$. The anticanonical sheaf of Q_α is given by the (tensor) square of the hyperplane section (modulo the equality $b_0b_3 + ab_1^2 - b_2^2$), so

$$H^0(Q_\alpha, \omega^{-1}) = \langle b_0^2,\ b_0b_1,\ b_0b_2,\ b_1^2,\ b_1b_2,\ b_2^2,\ b_1b_3,\ b_2b_3,\ b_3^2 \rangle$$

The image of the global sections in $H^0(U, \omega^{-1})$ is

$$H^0(Q_\alpha, \omega^{-1})|_U = \langle 1,\ x,\ y,\ x^2,\ xy,\ y^2,\ x(y^2 - ax^2),\ y(y^2 - ax^2),\ (y^2 - ax^2)^2 \rangle.$$

It is useful to examine the surface Q_α from another point of view: consider now $V(b_0b_3 + ab_1^2 - b_2^2)_{k(\alpha)} = Q_\alpha \otimes_k k(\alpha)$. The Galois group $H = \{1, \sigma\}$ of the extension $k(\alpha)/k$ acts naturally on $V(b_0b_3 + ab_1^2 - b_2^2)_{k(\alpha)}$, and its objects are defined over k if and only if they are invariant under this action. Through the $k(\alpha)$-isomorphism

$$\mathbb{P}^1_{k(\alpha)} \times \mathbb{P}^1_{k(\alpha)} \xrightarrow{\sim} V(b_0b_3 + ab_1^2 - b_2^2)_{k(\alpha)}$$
$$(X_0 : X_1, Y_0 : Y_1) \mapsto (X_0Y_0 : \frac{1}{2\alpha}(X_0Y_1 - X_1Y_0) : \frac{1}{2}(X_0Y_1 + X_1Y_0) : X_1Y_1)$$

we can consider the same Galois action on $\mathbb{P}^1_{k(\alpha)} \times \mathbb{P}^1_{k(\alpha)}$, which is given by

$$f_\sigma^{-1}(X_0 : X_1, Y_0 : Y_1) = (Y_0^\sigma : Y_1^\sigma, X_0^\sigma : X_1^\sigma).$$

By composing the maps $\mathbb{P}^1_{k(\alpha)} \times \mathbb{P}^1_{k(\alpha)} \to V(b_0b_3 + ab_1^2 - b_2^2)_{k(\alpha)} \to \mathbb{A}^2_{k(\alpha)}$ we also get the relations

$$X := \frac{X_1}{X_0} = y - \alpha x, \quad Y := \frac{Y_1}{Y_0} = y + \alpha x$$

which are useful e.g. in order to get the equations of the (conjugate) curves of degree $(1,0)$ and $(0,1)$ on $U_{k(\alpha)}$:

$$L_{A,1} \colon y - \alpha x + A, \quad L'_{C,1} \colon y + \alpha x + C.$$

This makes it more obvious that $f_\sigma^{-1}(L_{A,1}) = L'_{A^\sigma,1}$, and also that their union $L_{A,1} \cup L'_{A^\sigma,1}$ is a curve of degree $(1,1)$ defined over k:

$$AA^\sigma + \alpha(A - A^\sigma)x + (A + A^\sigma)y + y^2 - ax^2;$$

in fact, a generic curve of degree $(1,1)$ on $\mathbb{P}^1_{k(\alpha)} \times \mathbb{P}^1_{k(\alpha)}$ given by $AX_0Y_0 + BX_0Y_1 + CX_1Y_0 + DX_1Y_1$ is mapped to

$$A + \alpha(B - C)x + (B + C)y + D(y^2 - ax^2)$$

whence we deduce that in order for it to be defined over k, it ought to be that $A, D \in k,\ B \in k(\alpha),\ C = B^\sigma$.

We do not prove the following fact, which seem to require knowledge of the relative Picard functor and the étale and fppf topologies on schemes.

Proposition 1.1.2. *Let X be a k-form of $\mathbb{P}^1_k \times \mathbb{P}^1_k$. Then either*

- *$X \simeq P \times P'$ where P, P' are Severi-Brauer curves, or*
- *$X \simeq (P \times P)_{k(\alpha)}/H$ where P is a Severi-Brauer curve, $k(\alpha)/k$ is a quadratic extension with Galois group H, acting on $(P \times P)_{k(\alpha)}$ by permuting its factors (similarly to Q_α, which is the case $P = \mathbb{P}^1_k$).*

If X is k-rational, then either $X \simeq \mathbb{P}^1_k \times \mathbb{P}^1_k$ or $X \simeq Q_\alpha$.

Proof. [19, Proposition 5.2 and Corollary 5.5]. □

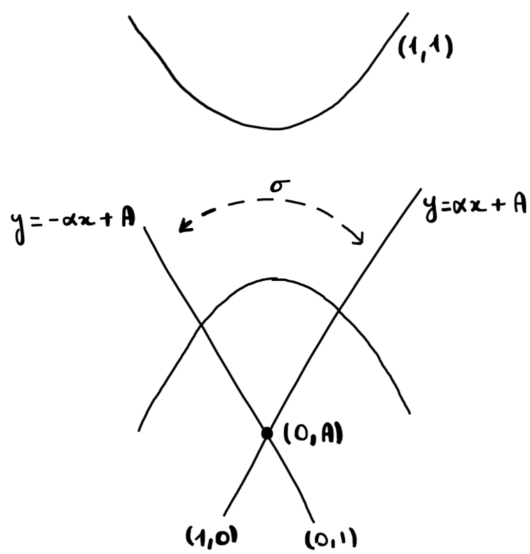


Figure 1.2: Some curves on Q_α.

Chapter 2

Del Pezzo surfaces

In this chapter, k is a perfect field.

2.1 Ordinary and weak Del Pezzo surfaces

Definition 2.1.1. An *ordinary Del Pezzo surface* X over k is a regular geometrically integral projective surface over k such that the anticanonical divisor $-K_X$ is ample. By the Nakai-Moishezon criterion [13, Theorem V.1.10] this is equivalent to asking that $K_X^2 > 0$ and $K_X \cdot C < 0$ for all effective divisors (equivalently, all integral curves) C. The *degree* of X is the number K_X^2.

Definition 2.1.2. A *weak Del Pezzo surface* X over k is a regular integral projective surface over k such that the anticanonical divisor $-K_X$ is big and nef, that is, $K_X^2 > 0$ and $K_X \cdot C \leq 0$ for all effective divisors C (equivalently, all integral curves). Its *degree* is the number K_X^2.

Theorem 2.1.3. *Let D be a divisor on a regular geometrically integral projective surface over k. Then D is big and nef (ample) if and only if $\overline{D}$ is big and nef (ample).*

Proof. It is clear by Proposition 1.0.12 that $D^2 > 0$ if and only if $\overline{D}^2 > 0$.

Suppose that $\overline{D}$ is nef, i.e. $\overline{D} \cdot C' \geq 0$ for all C' effective divisors on $\overline{X}$. Let C be an effective divisor on X. Then $\overline{C}$ is an effective divisor on $\overline{X}$, and

$$D \cdot C = \overline{D} \cdot \overline{C} \geq 0.$$

Conversely, suppose that D is nef, and let C' be an integral curve on $\overline{X}$ with image $C = \overline{p}(C')$ on k. Then by Theorem 1.0.7, C' is among the irreducible components of $\overline{C} = C_1' + \cdots + C_n'$ which are conjugate under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$. Since the Galois action preserves intersection numbers (Corollary 1.0.14) then $\overline{D} \cdot C_i' = \overline{D} \cdot C'$ for all i, so

$$n(\overline{D} \cdot C') = \overline{D} \cdot (nC') = \overline{D} \cdot (C_1' + \cdots + C_n') = \overline{D} \cdot \overline{C} = D \cdot C \geq 0.$$

In the case of ampleness, substitute $\geq$ with $>$. $\square$

Theorem 2.1.4. *Let X be a scheme over a perfect field k. Then X is a weak (ordinary) Del Pezzo surface over k if and only if $\overline{X}$ is a weak (ordinary) Del Pezzo surface over $\overline{k}$. If this is the case, then they have the same degree.*

Proof. The properties of being a regular geometrically integral projective surface are preserved under the base change $\overline{k}/k$, when k is perfect ([12, Proposition 14.51, 14.55, Exercise 6.18] and [21, Remark 3.2.9, Proposition 3.2.7]). The rest is Theorem 2.1.3. □

Definition 2.1.5. Let X be a weak Del Pezzo surface over $\overline{k}$. Consider the graph $\mathcal{G}$ of its negative curves, where black dots represent (-1)-curves, white circles represent (-2)-curves, and an edge joining two vertices represents the fact that the corresponding curves intersect in a point. Then we say that X is of *type* $\mathcal{G}$. By extension, the *type* of a weak Del Pezzo surface X over k is the type of $\overline{X}$.

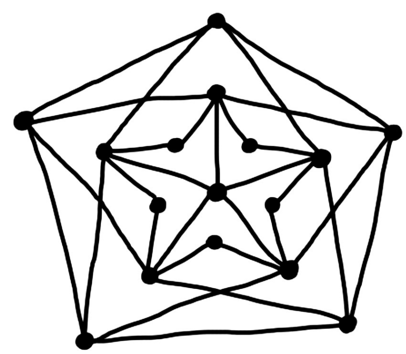


Figure 2.1: The type of an ordinary Del Pezzo surface of degree 4.

Proposition 2.1.6. *Let*

$$X = X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = \mathbb{P}^2$$

be such that each morphism is the blow-up of a point which does not lie on any (-2)-curve, with $0 \leq r \leq 8$ (i.e. the blown-up points are in almost general position). Then X is a weak Del Pezzo surface over $\overline{k}$ of degree $9-r$. Surfaces obtained in this way, together with $\mathbb{P}^1 \times \mathbb{P}^1$ and the Hirzebruch surface F_2 (both of degree 8) comprise all weak Del Pezzo surfaces over $\overline{k}$.

Proof. [5, Proposition 0.4], [8, Proposition III.3]. □

Proposition 2.1.7. *Let X be a weak Del Pezzo surface over k of degree d. The blow-down of an exceptional curve E on X is a weak Del Pezzo surface of degree $d+n$, where n is the number of irreducible components of $\overline{E}$.*

Proof. Over $\overline{k}$, let $\pi\colon X \to X'$ be the blow-down of E. Using known properties of blow-ups [13, Proposition V.3.3 and Proposition V.3.6], we see that

$$K_{X'}^2 = K_X^2 + 1$$

and that for any integral curve $C \subset X'$,

$$K_{X'} \cdot C = \pi^* K_X \cdot \pi^* C = (K_X - E) \cdot \pi^* C = K_X \cdot \pi^* C \leq 0.$$

Over k, use Lemma 3.3.2 and Theorem 2.1.4. □

2.2 Some facts about singularities

Before talking about singular Del Pezzo surfaces, we recall some facts about singularities and desingularizations.

Definition 2.2.1. Let X be a scheme. A morphism of schemes $f\colon X' \to X$ is called a *desingularization* if f is proper and birational and X' is regular. It is a *minimal desingularization* if every desingularization $g\colon Z \to X$ factors uniquely through f, i.e. there is a unique morphism $Z \to X'$ such that g is given by the composition $Z \to X' \to X$.

Proposition 2.2.2. *Let C be a reduced curve over a field k. Then C admits a minimal desingularization.*

Proof. It is enough to consider the *normalization* $C' \to C$ [21, Example 4.2.9]. If C is an integral projective curve, it is possible to obtain the normalization through a series of blow-ups at the singular locus (the set of singular points with the reduced scheme structure)

$$\cdots \to C_{n+1} \to C_n \to \cdots \to C_1 \to C,$$

a process which gives a desingularization in a finite number of steps [21, Proposition 8.1.26]. □

Theorem 2.2.3. *Let X be a reduced surface over a field k. Let $X_1 \to X$ be the normalization of X, and for $i \geq 1$ let $X_{i+1} \to X_i$ be the blow-up of the singular locus of X_i followed by the normalization. Then the sequence*

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \to X$$

gives a desingularization of X in a finite number of steps.

Proof. The result is stated for excellent reduced Noetherian schemes with irreducible components of dimension 2 in [20, Theorem 2.1], and algebraic varieties are excellent schemes [21, Corollary 8.2.40]. □

Definition 2.2.4. Let X be a scheme and let $x \in X$ be a closed point of X such that the local ring $\mathcal{O}_{X,x}$ is normal of dimension 2. Then x is a *rational singularity* if there exists a desingularization of its local scheme $Y \to \operatorname{Spec}(\mathcal{O}_{X,x})$ such that $H^1(Y, \mathcal{O}_Y) = 0$.

Theorem 2.2.5. *Let X be a normal reduced separated surface with only a finite number of singular points, all of which are rational singularities. For $i \geq 0$ let $X_{i+1} \to X$ be the blow-up of X in a singular point. Then the sequence*

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \to X$$

is finite, and gives a minimal desingularization of X.

Proof. Following [20, Theorem 4.1], we make use of the two following facts, which we do not prove.

1. If X is a normal reduced separated surface having a finite number of singularities, all of which are rational, and if $X' \to X$ is a blow-up at a reduced point, then X' is also a normal reduced separated surface with only a finite number of singularities which are all rational.

2. If x is a rational singularity and $f\colon Y \to \operatorname{Spec}(\mathcal{O}_{X,x})$ is a desingularization which is not the identity (i.e. x is not already regular) then f factors through the blow-up of X at the reduced point x.

Using Fact 1 we see that the given sequence of blow-ups is the same as the one of Theorem 2.2.3, so after a finite number of steps we get a regular surface X_n. By iterating Fact 2 it is also evident that this desingularization is minimal. □

Definition 2.2.6. Let X be a scheme, and let $x \in X$ be a closed point. Then x is a *rational double point* if it is a rational singularity and the local ring $\mathcal{O}_{X,x}$ has multiplicity 2.

Definition 2.2.7. Let X' be an integral normal projective surface over a field k, and let $\mathcal{E}$ be a set of integral curves on X'. A *contraction* of $\mathcal{E}$ is an integral normal projective surface X over k together with a projective birational morphism $f : X' \to X$ such that for every integral curve E on X', the set $f(E)$ is a point if and only if $E \in \mathcal{E}$.

Proposition 2.2.8. *Let X' be a normal integral projective surface, and let $\mathcal{E}$ be a set of integral curves on X'. If a contraction $f\colon E' \to X$ of $\mathcal{E}$ exists, then it is unique up to unique isomorphism.*

Proof. [21, Proposition 8.3.28]. □

2.3 Singular Del Pezzo surfaces

The following constructions are explained in more detail in [16, Section 5].

Let X be a normal geometrically integral projective surface over k. Recall that a normal surface is nonsingular in codimension 1, i.e. admits a regular open subscheme $U \subseteq X$. Let $\iota\colon U \to X$ be the inclusion. We define the *canonical sheaf* of X as $\omega_X := \iota_* \omega_U$. This is not necessarily an invertible sheaf - in fact, ω_X is invertible if and only if X is *Gorenstein*, and we will put this as an hypothesis.

Similarly, it is possible to define the canonical divisor $K_X := \iota_* K_U$ as the pushforward of K_U as a Weil divisor (the closure of K_U in X). Note that

$$\mathcal{O}_X(K_X) := U \mapsto \{f \in K(U) \setminus \{0\} \mid K_X + \operatorname{div}(f) \geq 0\}$$

is isomorphic to ω_X, and by saying that we want X to be Gorenstein we are asking for K_X to be locally principal, i.e. also a Cartier divisor.

Definition 2.3.1. Let X be a normal Gorenstein geometrically integral projective surface over k. We say that X is a *singular Del Pezzo surface* if the anticanonical divisor $-K_X$ is ample.

Theorem 2.3.2. *Let X be a scheme over k, then X is a singular Del Pezzo surface over k if and only if $\overline{X}$ is a singular Del Pezzo surface over $\overline{k}$.*

Proof. As in Theorem 2.1.4, the properties of being a normal geometrically integral projective surface are preserved under the base change $\overline{k}/k$, when k is perfect. Furthermore, X is Gorenstein if and only if $\overline{X}$ is [27, Tag 0C03]. Finally, note that $-K_X$ is ample if and only if ω_X^{-1} is an ample invertible sheaf, which happens if and only if $\overline{p}^*\omega_X^{-1} = \omega_{\overline{X}}^{-1}$ is [12, Proposition 14.56]. □

Theorem 2.3.3. *Let X be a singular Del Pezzo surface over an algebraically closed field $\overline{k}$. Let $\pi \colon X' \to X$ be a minimal resolution. Then one of the following cases occurs.*

- *X is rational; in this case X' is a weak Del Pezzo surface, π is the contraction of the (-2)-curves of X', and X has a finite number of singularities, all of which are rational double singularities.*
- *X is a cone over an elliptic curve. More precisely, X' is the projective bundle $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$ over an elliptic curve C, where $\mathcal{L}$ is an invertible sheaf with $\deg \mathcal{L} > 0$. X is obtained by contracting the minimal section of X', that is, the section of minimal self-intersection, which is also an elliptic curve.*

Proof. [14, Theorem 2.2]. □

Theorem 2.3.4. *Let X be a rational singular Del Pezzo surface over an algebraically closed field $\overline{k}$, and let $f \colon X' \to X$ be its minimal desingularization, where X' is a weak Del Pezzo surface of degree d. Let*

$$i = \begin{cases} 1 & \text{if } d \geq 3, \\ 2 & \text{if } d = 2, \\ 3 & \text{if } d = 1. \end{cases}$$

Then $|-iK_{X'}|$ is base-point-free, $|-iK_X|$ is very ample and the diagram

$$\begin{tikzcd} X' \arrow[rr, "g"] \arrow[dr, "f"'] & & Y \\ & X \arrow[ur, "h", "\sim"'] & \end{tikzcd}$$

commutes, where g is the morphism associated to $|-iK_{X'}|$, h is the mophism associated to $|-iK_X|$, and Y is the image of g and h. In particular, X is isomorphic to the anticanonical image of X' for $d \geq 3$.

Conversely, if X' is a weak Del Pezzo surface of degree d, then the image of the morphism associated to $|-iK_{X'}|$ is a singular Del Pezzo surface.

Proof. [14, Theorem 3.4, Corollary 4.5]. □

Theorem 2.3.5. *Let X be a scheme over k such that $\overline{X}$ is a rational singular Del Pezzo surface over $\overline{k}$ (so X is a geometrically rational singular Del Pezzo surface over k, by Theorem 2.3.2). Let $f\colon X' \to X$ be its minimal desingularization. Then X' is a weak Del Pezzo surface over k, and letting $i \in \mathbb{N}$ be as in Theorem 2.3.4, the image of the morphism associated to $|-iK_{X'}|$ is isomorphic to X.*

Proof. We know from Theorem 2.3.3 that $\overline{X}'$ is a normal surface which has a finite number of singularities, all of which are rational. Let

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \to X$$

be such that $X_{i+1} \to X_i$ is the blow-up of X_i in the singular locus S_i, that is, the set of singular points with the reduced subscheme structure. The singular locus of $\overline{X}_i$ is precisely $\overline{p}^{-1}(S)$, because k is perfect [12, Exercise 6.18]. So, Theorem 2.2.5 guarantees that at each step S_i is finite and that the process terminates in a finite number of steps, i.e. we reach

$$\begin{array}{ccccccccc}
\overline{X}' = \overline{X}_n & \longrightarrow & \overline{X}_{n-1} & \longrightarrow & \dots & \longrightarrow & \overline{X}_1 & \longrightarrow & \overline{X} \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
X' = X_n & \longrightarrow & X_{n-1} & \longrightarrow & \dots & \longrightarrow & X_1 & \longrightarrow & X
\end{array}$$

where $\overline{f}\colon \overline{X}' \to \overline{X}$ is a minimal desingularization of X, hence $\overline{X}'$ is a singular Del Pezzo surface. By the universal property of blow-ups [21, Corollary 8.1.16], it is clear that also $f\colon X' \to X$ is a minimal desingularization, and X' is a weak Del Pezzo surface by Theorem 2.1.4. Finally, let g and h be respectively the morphisms associated to $|-iK_{X'}|$ and $|-iK_X|$; then $\overline{g}$ and $\overline{h}$ are the morphisms associated to $|-iK_{\overline{X}'}|$ and $|-iK_{\overline{X}}|$, and the diagram

$$\begin{tikzcd}
\overline{X}' \arrow[rr, "\overline{g}"] \arrow[dr, "\overline{f}"] \arrow[dd] & & \overline{Z} \arrow[dd] \\
& \overline{X} \arrow[ur, "\overline{h}", "\sim"'] \arrow[dd] & \\
X' \arrow[rr, "g"] \arrow[dr, "f"] & & Z \\
& X \arrow[ur, "h", "\sim"'] &
\end{tikzcd}$$

commutes. As $\overline{h}$ is an isomorphism, also h is by [12, Proposition 14.51]. $\square$

Example 2.3.6. Let X be a geometrically rational singular Del Pezzo surface over k such that its minimal desingularization X' is a weak Del Pezzo surface of degree d; identify X with the anticanonical image of X'.

- If $d = 4$, then X is the complete intersection of two quadrics in $\mathbb{P}^4_k$; every such surface can be obtained in this way.
- If $d = 3$, then X is a cubic surface of $\mathbb{P}^3_k$; every such surface can be obtained in this way.

In both cases, the lines on $\mathbb{P}^n_k$ which lie entirely on X are the image of the (-1)-curves C of X'.

2.4 The Hirzebruch surface

For $n \geq 0$, the *Hirzebruch surface* F_n is a ruled surface over the projective line $\mathbb{P}^1_k$ defined as the projective bundle of the locally free sheaf $\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(n)$ [3, Chapter IV]. Note that $F_0 = \mathbb{P}^1_k \times \mathbb{P}^1_k$ and F_1 is the blow-up of $\mathbb{P}^1_k$ at a point, so they are ordinary Del Pezzo surfaces of degree 8.

We are interested in particular on F_2, which from now on we simply call the Hirzebruch surface. We focus on two alternative characterizations of this surface.

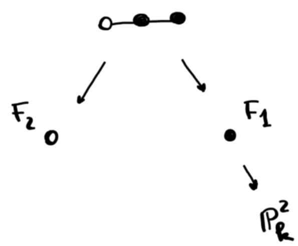


Figure 2.2: F_2 from $\mathbb{P}^2_k$.

The first one is geometric. Starting from $\mathbb{P}^2_k$, blow up a rational point P_0, obtaining the surface F_1 with an exceptional divisor E_0. Proceed by blowing up a rational point P_1 infinitely close to P_0, that is, lying on E_0. The surface thus obtained has the (-2)-curve E_0' which is the strict transform of E_0, the exceptional (-1)-curve E_1 and one additional (-1)-curve L, which is the strict transform of the line on $\mathbb{P}^1_k$ that after the first blow-up intersects E_0 in P_1. Now instead of blowing down E_1 (which would give us F_1 again), blow down L: the result is F_2. From this we can deduce that F_2 has a (-2)-curve and no other negative curves, and that it is a weak Del Pezzo surface of degree 8.

The second characterization comes from the description of F_2 as a toric variety ([6], [1, Exercise 4.2]). Recall that $\mathbb{P}^1_k \times \mathbb{P}^1_k$ can be defined as

$$\{\mathfrak{p} \subset S \mid \mathfrak{p} \text{ is prime and bihomogeneous not containing } (x_0, x_1) \text{ nor } (y_0, y_1)\},$$

where $S = k[x_0, x_1, y_0, y_1]$ and an ideal is bihomogeneous if it is generated by bihomogeneous elements. Specifically, a monomial $x_0^{a_0} x_1^{a_1} y_0^{b_0} y_1^{b_1}$ has by definition bidegree $(a_0 + a_1, b_0 + b_1)$, and an element $f \in S$ is bihomogeneous if it is the sum of monomials of the same bidegree. Moreover, $S_{(a,b)}$ is the k-vector space generated by the monomials of bidegree (a, b), and also immediately gives a concrete description of the Picard group of $\mathbb{P}^1_k \times \mathbb{P}^1_k$: a curve has degree (a, b) if and only if the defining equation belongs to $S_{(a,b)}$. For example, a curve of degree $(1, 1)$ can be written as

$$Ax_0y_0 + Bx_0y_1 + Cx_1y_0 + Dx_1y_1$$

with $A, B, C, D \in k$, because

$$S_{(1,1)} = \langle x_0y_0,\ x_0y_1,\ x_1y_0,\ x_1y_1\rangle.$$

We now give a similar treatment to F_2. We define it as

$$\{\mathfrak{p} \subset S \mid \mathfrak{p} \text{ is prime and bihomogeneous not containing } (x_0, x_1) \text{ nor } (y_0, y_1)\},$$

where S is still $k[x_0, x_1, y_0, y_1]$ but we change the grading: we say that the monomial $x_0^{a_0}x_1^{a_1}y_0^{b_0}y_1^{b_1}$ has bidegree $(a_0 + a_1 + 2b_0, b_0 + b_1)$. Consequently, the vector spaces $S_{(a,b)}$ change: we write explicitly the first few ones.

$$\begin{aligned}
&S_{(0,0)} = \langle 1\rangle && S_{(1,0)} = \langle x_0,\ x_1\rangle\\
&S_{(0,1)} = \langle y_1\rangle && S_{(2,0)} = \langle x_0^2,\ x_0x_1,\ x_1^2\rangle\\
&S_{(1,1)} = \langle x_0y_1,\ x_1y_1\rangle && S_{(0,2)} = \langle y_1^2\rangle\\
&S_{(3,0)} = \langle x_0^3,\ x_0^2x_1,\ x_0x_1^2,\ x_1^3\rangle && S_{(2,1)} = \langle x_0^2y_1,\ x_0x_1y_1,\ x_1^2y_1,\ y_0\rangle\\
&S_{(1,2)} = \langle x_0y_1^2,\ x_1y_1^2\rangle && S_{(0,3)} = \langle y_1^3\rangle
\end{aligned}$$

Since $S_{(a,b)}$ can be seen as representing curves of degree (a, b) in the Picard group, we can calculate geometrically the intersection numbers. We notice that $(1, 0)^2 = 0$, because taking $x_0, x_1 \in S_{(1,0)}$ we see that their intersection (x_0, x_1) contains no points of F_2 (ideals containing (x_0, x_1) are explicitly left out). We have that $(2, 1) \cdot (1, 0) = 1$, because the intersection of $x_0 \in S_{(1,0)},\ y_0 \in S_{(2,1)}$ is (x_0, y_0) which is prime, hence is just one point of multiplicity 1. And we also have that $(2, 1)^2 = 2$, because the intersection of $x_0^2y_1, y_0 \in S_{(2,1)}$ is $(x_0^2y_1, y_0)$ which contains just the prime (x_0, y_0): this is inside the affine open subset $x_1 \neq 0, y_1 \neq 0$, and using affine coordinates $x = x_0,\ y = y_0,\ x_1 = y_1 = 1$ we get

$$\dim_k \left(\frac{k[x, y]}{(x^2, y)}\right)_{(x,y)} = \dim_k \frac{k[x]}{(x^2)} = 2.$$

From this we deduce that

$$(0, 1)^2 = ((2, 1) - 2(1, 0))^2 = (2, 1)^2 - 4(2, 1) \cdot (1, 0) + 4(1, 0)^2 = -2$$

whence

$$(a, b) \cdot (c, d) = ad + bc - 2bd \quad \text{and} \quad (a, b)^2 = 2ab - 2b^2.$$

Now it is clear that F_2 contains no (-1)-curves and just one integral (-2)-curve, which is $V(y_1)$, the unique curve of type $(0, 1)$. In fact, if $(a, b)^2 < 0$ necessarily $a < b$, but in that case all elements of $S_{(a,b)}$ contain y_1, hence the corresponding curves are not integral and have $V(y_1)$ as an irreducible component.

Later on it will be very useful to consider an affine open subscheme $U \subset F_2$ isomorphic to $\mathbb{A}^2_k$: let

$$U\colon x_0 \neq 0, x_1 \neq 0, \quad x = x_1, \quad y = y_1, \quad x_0 = y_0 = 1.$$

We write down the first few vector spaces $S_{(a,b)}|_U$:

$$\begin{aligned} S_{(0,0)}|_U &= \langle 1\rangle & S_{(1,0)}|_U &= \langle 1,\ x\rangle \\ S_{(0,1)}|_U &= \langle y\rangle & S_{(2,0)}|_U &= \langle 1,\ x,\ x^2\rangle \\ S_{(1,1)}|_U &= \langle y,\ xy\rangle & S_{(0,2)}|_U &= \langle y^2\rangle \\ S_{(3,0)}|_U &= \langle 1,\ x,\ x^2,\ x^3\rangle & S_{(2,1)}|_U &= \langle y,\ xy,\ x^2y,\ 1\rangle \\ S_{(1,2)}|_U &= \langle y^2,\ xy^2\rangle & S_{(0,3)}|_U &= \langle y^3\rangle \end{aligned}$$

In general, to find out the generators of the vector space $S_{(a,b)}|_U$, start by $x^a y^b$ and write $x^{a-1}y^b, x^{a-2}y^b, \dots, y^b$, then join with the generators of $S_{(a-2,b-1)}|_U$ if $a \geq 2, b \geq 1$. Notice that writing for instance about "the curve $y = 0$" would be ambiguous (are we referring to $y \in S_{(0,1)}|_U$ or $y \in S_{(2,1)}|_U$ or something else?) but by specifying both the equation of a curve and its degree, we eliminate any ambiguity.

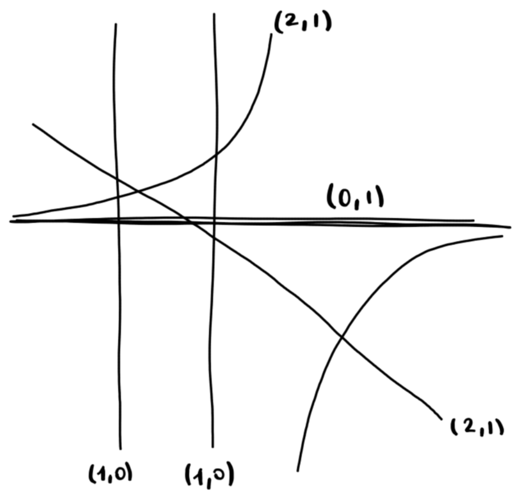


Figure 2.3: Some curves on U.

We will also need to know explicitly the global sections of the anticanonical divisor on U. To find out the degree (a, b) of the anticanonical divisor, recall that F_2 is a weak Del Pezzo surface of degree 8. This implies that $(a, b)^2 = 2a(a - b) = 8$, hence our only candidates are $(5, 1)$, $(4, 2)$, $(5, 4)$. But we also know that $S_{(a,b)}$ should induce a morphism to $\mathbb{P}^8_k$ i.e. should be 9-dimensional as a k-vector space, and we quickly check that the only possibility is $(4, 2)$, so

$$H^0(F_2, \omega^{-1})|_U = S_{(4,2)}|_U = \langle x^4y^2,\ x^3y^2,\ x^2y^2,\ xy^2,\ y^2,\ x^2y,\ xy,\ y,\ 1\rangle.$$

Hereafter, after fixing a field k, the notation F_2 will stand for the Hirzebruch surface F_2 over the algebraic closure $\overline{k}$, while $F_{2,k}$ for the Hirzebruch surface over k; in other words, $\overline{F}_{2,k} = F_2$.

2.4.1 Forms of the Hirzebruch surface

We only focus on k-rational forms of $F_{2,k}$, with the goal of proving that they are isomorphic to $F_{2,k}$. In fact, let X be a k-form of $F_{2,k}$ with a rational point P. The unique curve C of degree $(1,0)$ passing through P (which is defined over k) is a k-form of $\mathbb{P}^1_k$ with a rational point, so it is full of rational points. Choose one not on the (-2)-curve and blow it up: now C is a (-1)-curve. Blow it down to get a Del Pezzo surface of degree 8 with only one (-1)-curve. Since this is necessarily defined over k, blow it down to get a k-form of $\mathbb{P}^2_k$ which has a rational point (for example, the image of the blown-down curve) and which therefore is actually $\mathbb{P}^2_k$. Our description of X in terms of blow-ups and blow-downs of $\mathbb{P}^2_k$ match that of $F_{2,k}$, so they are necessarily isomorphic.

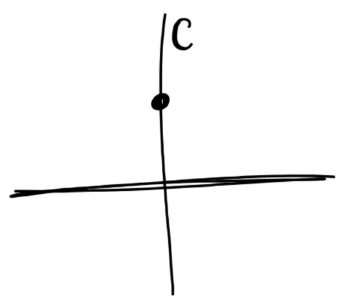


Figure 2.4: A rational point on F_2.

Chapter 3

Singular Del Pezzo surfaces of degree 4

Throughout this chapter, k is a field of characteristic 0. Singular Del Pezzo surfaces are assumed to be geometrically rational, i.e. we leave out the case of cones over elliptic curves. The *degree* of a singular Del Pezzo surface X is the degree of its minimal desingularization X' seen as a weak Del Pezzo surface, and the *type* of X is the type of X'.

3.1 Motivation

We are about to write down explicitly the defining equations of as many singular Del Pezzo surfaces of degree 4 as we can. One might be naturally led to the question of why should we attempt such a thing.

The answer is that, apart from being a large pool of examples from which to draw upon, such a classification could be useful for a conjecture in arithmetic geometry called Manin's conjecture. We state it specifically for singular Del Pezzo surfaces of degree 4 over the field $\mathbb{Q}$; it applies more generally to Fano varieties over number fields.

Conjecture. *Let $X \subset \mathbb{P}^4_{\mathbb{Q}}$ be an anticanonically embedded singular Del Pezzo surface of degree 4 with dense rational points. Consider the exponential height $H\colon \mathbb{P}^4(\mathbb{Q}) \to \mathbb{R}_{>0}$ defined for a vector $(x_0, \dots, x_4) \in \mathbb{Z}^5$ subject to the condition $\gcd(x_0, \dots, x_4) = 1$ by*

$$H(x_0 : \cdots : x_4) = \max\{|x_0|, \dots, |x_4|\}.$$

Let U be the complement of the union of the (-1)-curves of X (that is, the points outside the lines). Let

$$N_{U,H}(B) = \#\{x \in U(\mathbb{Q}) \mid H(x) \le B\}$$

be the number of rational points of X away from the lines and with height bounded by B. Denote by X' the minimal desingularization of X and by ρ the rank of its Picard group. Then it is expected that

$$N_{U,H}(B) = c_{V,H} B \log(B)^{\rho-1}(1 + o(1)),$$

where $c_{V,H}$ is a positive constant [18].

At the moment, there does not seem to be a general method for proving the conjecture for singular Del Pezzo surfaces all at once [10]; instead, it has been proven for some families of surfaces and individual cases, for which it is useful to have concrete knowledge of their equations, lines, singular points and other information.

Manin's conjecture also motivates the fact that we assume our surface X to have dense rational points. For many types, this is equivalent to not having any hypotheses; for others, it is equivalent to $X'(k) \neq \varnothing$, or to having X be k-rational. The only type in which this does not happen is Type III, where we also assume X to be k-rational (equivalently, non-minimal) in order to avoid the case of minimal Iskovskih surfaces (Manin's conjecture for Iskovskih surfaces is discussed in [7]).

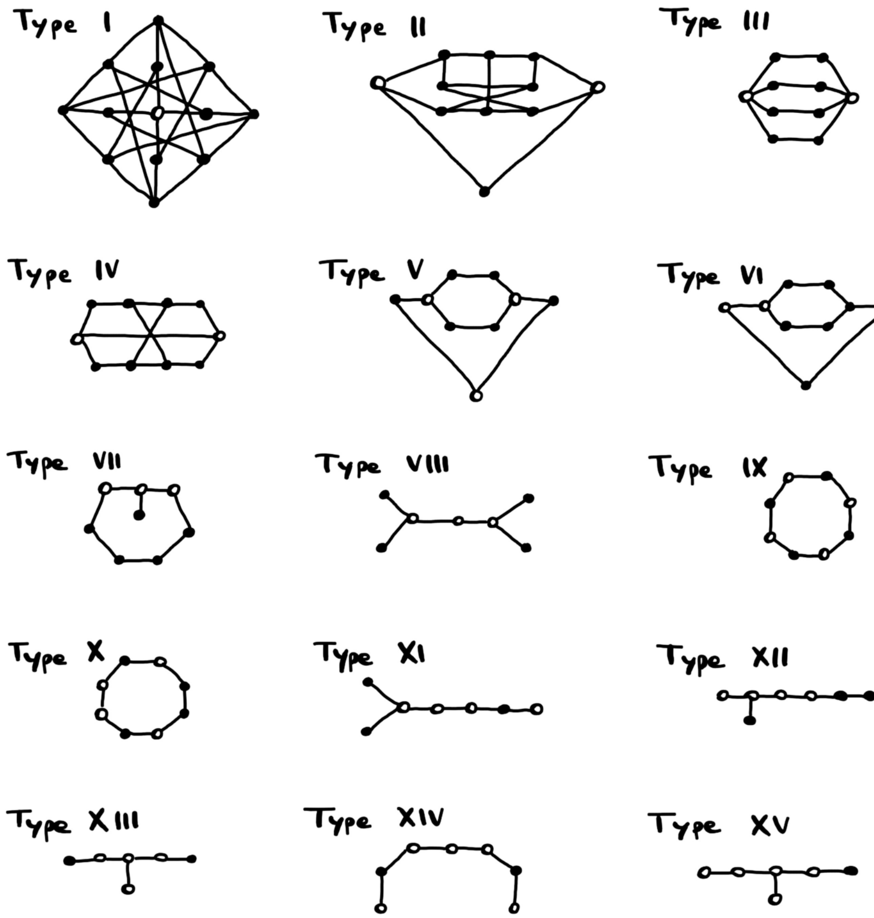
The procedure we are going to follow is an extension of what has been done in [10] for Type XI.

The classification of singular Del Pezzo surfaces of degree 4, which is the same thing as intersections of two quadrics on $\mathbb{P}^4$, goes back a long way. Segre classifies them using the Segre symbol (equivalent to the classification based on types) introduced earlier by A. Weiler [11, page 424]. [28] gives a list of examples for each type along with all singular Del Pezzo surfaces of degree ≥ 3, specifying lines and singularities. [5] discusses arithmetic on singular Del Pezzo surfaces, following the work of Manin on cubic surfaces [25], and provides a description of the various types in terms of their graph of negative curves. [9] gives a concrete description of the geometry and the Cox rings of some types.

3.2 The plan

The foundation of our approach is the following classification, which we take for granted.

Proposition 3.2.1 ([5, Proposition 6.1]). *Let X' be a weak Del Pezzo surface over $\overline{k}$ of degree 4 which is not an ordinary Del Pezzo surface. Then it has one of the following fifteen types.*



The plan is, for each type, to take a generic weak Del Pezzo surface X' of that type and find the image of the morphism $X' \to X \subset \mathbb{P}^4_k$ induced by $H^0(X', \omega_{X'}^{-1})$, that is, by the global sections of the anticanonical sheaf. However, obtaining explicit equations for X' could be as difficult as for X, so instead we apply a series of blow-downs $\pi\colon X' \to Z$ (which we find by just looking at the diagram of the type from Proposition 3.2.1) given by

$$X' = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = Z$$

where Z is a minimal surface (Section 3.3), and calculate the image of the birational map $Z \dashrightarrow X$ induced by $H^0(Z, \pi_* \omega_{X'}^{-1})$. This works because X' and Z are birationally equivalent, and over the open subset where they are isomorphic the two maps coincide, so overall they have the same (closure of the) image.

In order to find out $H^0(Z, \pi_* \omega_{X'}^{-1})$, we first need to describe geometrically the morphism $\pi \colon X' \to Z$ in terms of blow-ups (Section 3.4). Usually we try to simplify as much as possible the arrangement of points and curves through k-automorphisms of Z, so that the equations are shorter and there are less free parameters (Section 3.5).

Finally, through the knowledge of the geometry of blown-up points and curves on them, we are able to calculate $H^0(Z, \pi_* \omega_{X'}^{-1})$ or rather, the image of the injective restriction map $H^0(Z, \pi_* \omega_{X'}) \to H^0(U, \pi_* \omega_{X'})$ which we denote by $H^0(Z, \pi_* \omega_{X'})|_U$, where U is a "preferred" affine open subset on Z isomorphic to $\mathbb{A}^2_k$ (Section 3.6). The image of the associated morphism is determined either by hand or with the help of the software SageMath. The same software is used extensively to confirm the validity of the calculations, and `.sage` files are available for each type at [24]. If there are discrepancies between what is written here and the code, the code is right.

Apart from one instance of Type I, we do not leave out any k-rational singular Del Pezzo surfaces of degree 4.

3.3 Blow-downs to Z

Fix a type. We begin by examining the diagram of its negative curves from Proposition 3.2.1. In these diagrams and over $\overline{k}$ any (-1)-curve is an exceptional curve: all we need to prove is that such a curve E is isomorphic to $\mathbb{P}^1$, which is the same as having arithmetic genus p_a equal to zero. These being weak Del Pezzo surfaces, $K_{\overline{X}} \cdot E \leq 0$, so by the adjunction formula [21, Theorem 9.1.37]

$$2p_a - 2 = E \cdot (E + K_{\overline{X}}) \leq -1 \quad \Rightarrow \quad p_a = 0.$$

We have established that any (-1)-curve can be blown down over $\overline{k}$, but to ensure that the blow-down is also defined over k we need to use Lemma 3.3.2, i.e. make sure that the set of (-1)-curves blown down is defined over k (recall that the diagram of negative curves is drawn over $\overline{k}$).

Lemma 3.3.1. *Let X be a regular integral projective surface over a field k, then an integral curve E on X is exceptional if and only if $K \cdot E < 0$ and $E^2 < 0$, where K is a canonical divisor.*

Proof. [21, Proposition 8.3.10]. □

Lemma 3.3.2. *Let X be a regular integral projective surface over a field k. Let E be an integral curve on X, and let $E_1, \dots, E_n$ be the irreducible components of $\overline{E} = p^{-1}(E)$. Then E is exceptional if and only if the E_i are exceptional and pairwise disjoint.*

Proof. Suppose that the curves $E_1, \dots, E_n$ are exceptional and pairwise disjoint. By Lemma 3.3.1, for each $i = 1, \dots, n$

$$K_{\overline{X}} \cdot E_i < 0, \quad E_i^2 < 0.$$

Through Proposition 1.0.12,

$$K_X \cdot E = K_{\overline{X}} \cdot \overline{E} = K_{\overline{X}} \cdot E_1 + \cdots + K_{\overline{X}} \cdot E_n < 0$$

and using the disjointness hypothesis

$$E^2 = (E_1 + \cdots + E_n)^2 = E_1^2 + \cdots + E_n^2 < 0,$$

therefore E is exceptional by the same Lemma.

Conversely, suppose that E_1 is not exceptional. By Theorem 1.0.7, the curves E_i are conjugate under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$, while $-K_{\overline{X}}$ is fixed since it is defined over k. Galois actions preserve intersection numbers (Corollary 1.0.14), so the other E_i are also not exceptional, meaning $K_{\overline{X}} \cdot E_i \geq 0$ or $E_i^2 \geq 0$, hence E is also not exceptional. □

The way we ensure that a particular set of (-1)-curves is defined over k is by leveraging the symmetries (or lack thereof) in the diagram of negative curves. By Corollary 1.0.8, a set of integral curves on $\overline{X}'$ is defined over k if and only if it is invariant under the action of the absolute Galois group $G = \mathrm{Gal}(\overline{k}/k)$. Any such action is only a k-automorphism of X' (not a $\overline{k}$-automorphism), but by Corollary 1.0.14 this is enough to preserve intersection numbers. In particular, G acts on the diagram of negative curves by symmetries. Thus, any subset of black dots on the diagram (which represent (-1)-curves) which is not preserved by the symmetries of the diagram is not necessarily defined over k. On the other hand, any subset maximal among those invariant under the action of G must be defined over k.

We can leverage this by repeatedly blowing down sets of (-1)-curves defined over k. This sequence of blow-downs is represented by a series of diagrams representing the negative curves over $\overline{k}$, which change according to the following rules: by blowing down a bunch of curves $E_1, \dots, E_n$,

- any (-1)-curve among $E_1, \dots, E_n$ or meeting one of them disappears from the picture, while the others are unaffected;
- any (-2)-curve becomes a (-1)-curve if it meets only one among the E_i, and disappears from the picture if it meets two or more, otherwise it remains unaffected.

Eventually we end up with a minimal surface Z; for its algebraic closure $\overline{Z}$ we get only three possibilities:

- five curves have been blown down, so $\overline{Z}$ is a weak Del Pezzo surface of degree 9, that is, $\overline{Z} = \mathbb{P}^2$;
- four curves have been blown down, and the resulting weak Del Pezzo surface of degree 8 has no negative curves, so $\overline{Z} = \mathbb{P}^1 \times \mathbb{P}^1$;
- four curves have been blown down, but $\overline{Z}$ has a (-2)-curve: in this case, $\overline{Z}$ is the Hirzebruch surface F_2.

Assuming $Z(k) \neq \varnothing$, there are few possibilities for Z itself:

- if $\overline{Z} = \mathbb{P}^2$ then Z is a rational Brauer-Severi surface i.e. $\mathbb{P}^2_k$;
- if $\overline{Z} = \mathbb{P}^1 \times \mathbb{P}^1$ then Z is either $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or the twisted self-product Q_α;
- if $\overline{Z} = F_2$ then $Z = F_{2,k}$.

Each of these has a "preferred" affine open subset $U \simeq \mathbb{A}^2_k$ and coordinates (x, y) described in Sections 1.1.2, 1.1.3 and 2.4.

The hypothesis of existence of k-rational points can be put interchangeably on X' and Z: if $X'(k) \neq \varnothing$ then the rational point moves through the morphism $X' \to Z$. If $Z(k) \neq \varnothing$ then either the rational point is in the open subset where $X' \to Z$ is an isomorphism, or it is blown-up in the process - but in that case, it is replaced by $\mathbb{P}^1_k$ which is full of rational points.

It turns out that these conditions are equivalent to the existence of a regular k-rational point on X: certainly, if such a point exist, then it is in the open subset where $X' \to X$ is an isomorphism. If $X'(k) \neq \varnothing$ but the rational point is on a (-2)-curve (so that it is mapped to a singular point), X cannot be an Iskovskih surface (otherwise the two singularities would not be conjugate, see Type III), and that was the only case in which $X'(k) \neq \varnothing$ did not imply that X' is k-rational (as can be verified by going through all the types as we do). So X' is k-rational and so is X.

The fact that it is not enough to ask that $X(k) \neq \varnothing$ to have $X'(k) \neq \varnothing$ is illustrated by the following surface of type IX over $\mathbb{Q}$, where the only $\mathbb{Q}$-rational points are singular:

$$\begin{cases} x_0^2 - 17x_1^2 = x_2^2 \\ x_0^2 - 17x_1^2 = -x_3^2 - x_4^2. \end{cases}$$

This is "Ellison's example" [5, page 75].

3.4 Recovery of X' from Z

Having obtained Z, we discuss the inverse process of recovering X' from Z through a series of blow-ups. We do not describe it in detail; usually the reasoning goes as follows: from X_{i-1}, through the blow-up $\pi_i \colon X_i \to X_{i-1}$, some new negative curves appear (or become "more negative"). Those which are already negative on X_{i-1} follow the rules which we have just outlined in the previous section but in reverse. Some of the new (-1)-curves are those which we chose to blow down, and so they represent the exceptional locus. The other new negative curves are curves of Z on which we have blown up enough point. The degree of each of these curves can be inferred using the following Lemma:

Lemma 3.4.1. *Let $\overline{Z}$ be either $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or F_2. Let C be an integral curve on $\overline{Z}$; if $\overline{Z} = \mathbb{P}^2$, let d be the degree, otherwise let $d = a + b$ where C is of degree (a, b). Suppose that n distinct points are blown up on C resulting in a weak Del Pezzo surface X', and let C' be the strict transform of C. Then*

$$-K_{\overline{X}'} \cdot C' = -K_{\overline{Z}} \cdot C - \Sigma$$

where Σ can be bounded in the following way:

$$\begin{aligned} d = 1,2 &: \Sigma = n \\ d > 2, n = 1 &: \Sigma \leq d - 1 \\ n = 2 &: \Sigma \leq d \\ n = 3 &: \Sigma \leq 2d - 2 \\ n = 4 &: \Sigma \leq 2d - 1 \\ n = 5 &: \Sigma \leq 2d \end{aligned}$$

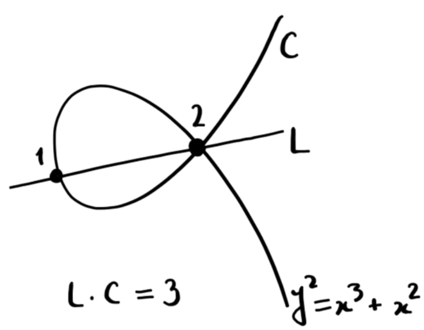


Figure 3.1: Calculating multiplicities of points through intersection numbers.

Proof. Suppose that $\overline{Z} = \mathbb{P}^2$, and let $P_1, \ldots, P_n$ be the blown-up points. Then [21, Proposition 9.2.23]

$$\Sigma = \sum_{i=1}^{n} \mu_{P_i}(C)$$

where $\mu_{P_i}(C)$ is the multiplicity of P_i at C. If $n = 1, 2$ then C is a line or a non-degenerate conic, so each point has multiplicity 1. Suppose now that $d > 2$.

If $n = 2$, consider the line L joining P_1 and P_2. By [13, Exercise I.5.4] and Bézout's Theorem,

$$d = L \cdot C = \sum_{i=1}^{2} (L \cdot C)_{P_i} \geq \sum_{i=1}^{2} \mu_{P_i}(C) = \Sigma.$$

If $n = 5$, let D be the unique conic passing through the five points (they are in general position because $d > 2$). By a similar reasoning $2d = D \cdot C \geq \Sigma$. Now, when blowing up a point, Σ increases at least by one, so we get the inequalities for $n = 1, 3, 4$. Finally, if $\overline{Z} = \mathbb{P}^1 \times \mathbb{P}^1$ or F_2 and C is a curve of degree (a, b), since Σ can be determined locally we can use an affine open subset isomorphic to $\mathbb{A}^2$, on which C is at most of degree $d = a + b$, and then consider the projective closure. □

In order to see how this result can be useful, recall that on a weak Del Pezzo surface X' we have $C^2 = -2$ if and only if $-K_{X'} \cdot C = 0$ [8, Théorème III.1] and if $d > 1$ then $C^2 = -1$ if and only if $-K_{X'} \cdot C = 1$ [8, Lemme III.9].

For example suppose as in Type I that an integral curve C of degree (a, b) on $\mathbb{P}^1 \times \mathbb{P}^1$ becomes a (-2)-curve after blowing up 4 points on it. Then,

- (a, b) cannot be $(1, 0)$ or $(0, 1)$, because $-K_{\overline{Z}} \cdot C = 2 \neq -K_{\overline{X}'} \cdot C' + \Sigma = 0 + 4$;
- (a, b) cannot be $(a, 0)$ or $(0, b)$ for $a, b > 1$, as C wouldn't be integral;
- if $(a, b) = (1, 1)$ there are no problems;
- if $a + b > 2$ and $a, b \neq 0$ then

$$-K_{\overline{X}'} \cdot C' = -K_{\overline{Z}} \cdot C - \Sigma \geq 2(a + b) - (2(a + b) - 1) = 1$$

so C' cannot be a (-2)-curve.

In conclusion, the only possibility for C is to be of degree $(1, 1)$.

3.5 Simplification through k-automorphisms

The following lemmas are used throughout the classification in order to reduce in complexity the most general case.

Lemma 3.5.1. *Let P_1, P_2, P_3 be closed points on $\mathbb{P}^1$ such that P_1 is k-rational and $\{P_2, P_3\}$ is defined over k, splitting over an extension $k(\alpha)/k$ with $\alpha^2 \in k^\times$. Then up to k-automorphisms*

$$P_1 = (1 : 0), \quad P_2 = (1 : \alpha), \quad P_3 = (1 : -\alpha).$$

Proof. After applying an automorphism of $\mathbb{P}^1_k$, we can suppose that $P_1 = (1:0)$ and that $P_2, P_3 \neq (0:1)$ so that there exist $A, B \in k$ such that

$$P_2 = (1 : A + \alpha B), \quad P_3 = (1 : A - \alpha B)$$

with $B \neq 0$ (otherwise P_2, P_3 would not be distinct). It is now sufficient to apply the k-automorphism $\begin{pmatrix} aB^2 - A^2 & A \\ 0 & -aB \end{pmatrix}$:

$$\begin{pmatrix} aB^2 - A^2 & A \\ 0 & -aB \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (aB^2 - A) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\begin{pmatrix} aB^2 - A^2 & A \\ 0 & -aB \end{pmatrix} \begin{pmatrix} 1 \\ A \pm \alpha B \end{pmatrix} = (aB^2 \pm \alpha AB) \begin{pmatrix} 1 \\ \pm\alpha \end{pmatrix}$$

where $aB^2 - A^2, aB^2 \pm \alpha AB$ are both non-zero, otherwise $B = 0$. □

Lemma 3.5.2. *Let L_1, L_2 be lines on $\mathbb{P}^2$. Up to k-automorphisms,*

- $L_1 \colon x = 0, \quad L_2 \colon y = 0$ *if they are both defined over k;*
- $L_1 \colon y = \alpha x, \quad L_2 \colon y = -\alpha x$ *if together they are defined over k and split over an extension $k(\alpha)/k$ with $\alpha^2 \in k^\times$.*

Proof. In both cases, the intersection point of the two lines is a k-rational point, hence we can suppose it to be $Q = (1:0:0)$. Consider now the line $T \colon x_0 = 0$. We can extend any k-automorphism $\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ of $\mathbb{P}^1_k \simeq T$ to a k-automorphism of $\mathbb{P}^1_k$ fixing Q by considering the transformation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & m_{11} & m_{12} \\ 0 & m_{21} & m_{22} \end{pmatrix}.$$

Now, let P_1, P_2 be the intersection points of T with L_1, L_2. If the lines are both defined over k, then P_1, P_2 will both be k-rational, so through an automorphism of T fixing Q we can suppose them to be $P_1 = (0:1:0)$, $P_2 = (0:0:1)$ from which follows that

$$L_1 \colon x_1 = 0, \quad L_2 \colon x_2 = 0.$$

If instead L_1, L_2 are conjugate under a quadratic extension $k(\alpha)/k$ then the points will also be conjugate, and by Lemma 3.5.1 we can suppose that $P_1 = (0:1:\alpha)$, $P_2 = (0:1:-\alpha)$ so that

$$L_1 \colon x_2 = \alpha x_1, \quad L_2 \colon x_2 = -\alpha x.$$

□

Lemma 3.5.3. *The k-automorphisms of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ are generated by the swap*

$$(x_0 : x_1, y_0 : y_1) \mapsto (y_0 : y_1, x_0 : x_1)$$

and by those of the form

$$\left(\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}\right) \mapsto \left(A\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, B\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}\right)$$

where $A, B \in PGL_2(k)$.

Proof. An automorphism φ of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ induces an automorphism φ^* of the Picard group $\mathrm{Pic}(\mathbb{P}^1_k \times \mathbb{P}^1_k) \simeq \mathbb{Z} \times \mathbb{Z}$, which also sends effective divisors to effective divisors. Thus, φ^* is either the identity or the swap $(a, b) \mapsto (b, a)$.

In the first case, consider the projections $p_1, p_2 \colon \mathbb{P}^1_k \times \mathbb{P}^1_k \to \mathbb{P}^1_k$. The curves of degree $(1, 0)$, resp. $(0, 1)$ are the fibers of p_1, resp. p_2, and since φ^* is the identity, φ preserves these families of fibers and induces automorphisms on the two copies of $\mathbb{P}^1_k$. Since the action on these fibers determines the automorphism, in this case all we get is $PGL_2(k) \times PGL_2(k)$.

In the second case, after a swap $(x_0 : x_1, y_0 : y_1) \mapsto (y_0 : y_1, x_0 : x_1)$ we are once again led to the first case, so the previously found automorphisms and the swap generate all the k-automorphisms of $\mathbb{P}^1_k \times \mathbb{P}^1_k$. □

Lemma 3.5.4. *Consider the twisted self-product Q_α with coordinates $(X_0 : X_1, Y_0 : Y_1)$. Its k-automorphisms are generated by the swap*

$$(X_0 : X_1, Y_0 : Y_1) \mapsto (Y_0 : Y_1, X_0 : X_1)$$

and by those of the form

$$\left(\begin{pmatrix} X_0 \\ X_1 \end{pmatrix}, \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix}\right) \mapsto \left(\begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}\begin{pmatrix} X_0 \\ X_1 \end{pmatrix}, \begin{pmatrix} m_{00}^\sigma & m_{01}^\sigma \\ m_{10}^\sigma & m_{11}^\sigma \end{pmatrix}\begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix}\right)$$

where $\begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} \in PGL_2(k(\alpha))$.

Proof. Any k-automorphism of Q_α extends to a $k(\alpha)$-automorphism of $Q_\alpha \otimes_k k(\alpha) \simeq \mathbb{P}^1_{k(\alpha)} \times \mathbb{P}^1_{k(\alpha)}$, and should be compatible with the equivalence $(X_0 : X_1, Y_0 : Y_1) \sim (Y_0^\sigma : Y_1^\sigma, X_0^\sigma : X_1^\sigma)$ where σ is a generator of $\mathrm{Gal}(k(\alpha)/k)$. The $k(\alpha)$-automorphisms of $\mathbb{P}^1_{k(\alpha)} \times \mathbb{P}^1_{k(\alpha)}$ which satisfy this requirement are precisely those of the claimed form. □

Lemma 3.5.5. *Let P_1, P_2 be closed points on $\overline{Q}_\alpha$ not on the same curve of degree $(1, 0)$ or $(0, 1)$, together defined over k and splitting over an extension $k(\beta)/k$ with $\beta^2 \in k^\times$. Then up to k-automorphisms*

- $P_1 = [(X - \beta, Y - \beta)] = [(x, y - \beta)]$, $\quad P_2 = [(X + \beta, Y + \beta)] = [(x, y + \beta)]$ *if $k(\beta) \neq k(\alpha)$;*

- $P_1 = [(X+a, Y-a)] = [(x-\alpha, y)], \quad P_2 = [(X-a, Y+a)] = [(x+\alpha, y)]$ *if* $k(\beta) = k(\alpha)$.

Proof. We use (X, Y) coordinates.

First suppose that $k(\beta) \neq k(\alpha)$, so that $k(\alpha, \beta)/k$ is a Galois extension which admits the automorphism $\sigma \in G(k(\alpha, \beta)/k)$ such that $\sigma(\alpha) = -\alpha,\ \sigma(\beta) = \beta$.

The points P_1, P_2 have generically coordinates

$$P_1 = (A + \beta B,\ A^\sigma + \beta B^\sigma), \quad P_2 = (A - \beta B,\ A^\sigma - \beta B^\sigma)$$

with $A, B \in k(\alpha)$: we want $Y = X^\sigma$ because otherwise the point would not split over $k(\beta)$, but instead over $k(\alpha, \beta)$. Let $M \in PGL_2(k(\alpha))$ be the $k(\alpha)$-automorphism such that

$$M(A + \beta B) = \beta, \quad M(A - \beta B) = -\beta$$

which exists by Lemma 3.5.1. Then this extends to the k-automorphism (M, M^σ) of Q_α by Lemma 3.5.4, and

$$(M, M^\sigma)(P_1) = (\beta, \beta), \quad (M, M^\sigma)(P_2) = (-\beta, -\beta)$$

which is what we wanted.

If instead $k(\alpha) = k(\beta)$, then the points are

$$P_1 = (A + \alpha B, C + \alpha D), \quad P_2 = (A - \alpha B, C - \alpha D)$$

with $A, B, C, D \in k$. In this case, let $M \in PGL_2(k(\alpha))$ be the $k(\alpha)$-automorphism such that

$$M(A + \alpha B) = -a, \quad M(A - \beta B) = a.$$

Then

$$(M, M^\sigma)(P_1) = (-a, a), \quad (M, M^\sigma)(P_2) = (a, -a)$$

which is again what we wanted. □

3.6 Determination of $\pi_*\omega_{X'}^{-1}$

Theorem 3.6.1. *Let X be a regular integral algebraic surface over k, and let $\pi \colon X' \to X$ be the blow-up at a closed point P. Let $\mathcal{I}_P$ be the invertible sheaf associated to $\{P\}_{\text{red}}$. Then*

$$\pi_*\omega_{X'}^{-1} \simeq \omega_X^{-1} \otimes \mathcal{I}_P$$

and through this isomorphism

$$H^0(X, \pi_*\omega_{X'}^{-1}) = \{f \in H^0(X, \omega_X^{-1}) \mid f(P) = 0\}.$$

Proof. Let $Y = V(\mathcal{I}_P) = \{P\}_{\mathrm{red}} \simeq \mathrm{Spec}(\kappa(P))$ be the blown-up point, and let E be the exceptional divisor on X'. Some well-known facts about blow-ups are that $E \simeq \mathbb{P}^1_{\kappa(P)}$ [21, Theorem 8.1.19(b)], that the canonical morphism $\mathcal{O}_X \to \pi_* \mathcal{O}_{X'}$ is an isomorphism [21, Proposition 8.1.12 and Corollary 4.4.3(a)] and that [21, Proposition 9.2.24]

$$\omega_{X'} = \pi^* \omega_X \otimes \mathcal{O}_X(E)$$

which implies, as tensor product and inverse image commute, that

$$\omega_{X'}^{-1} = \pi^* \omega_X^{-1} \otimes \mathcal{O}_X(-E).$$

Now, obviously $H^0(X', \omega_{X'}^{-1}) = H^0(X, \pi_* \omega_{X'}^{-1})$ and through the projection formula [13, II.5.1(d)]

$$\pi_* \omega_{X'}^{-1} = \pi_*(\pi^* \omega_X^{-1} \otimes \mathcal{O}_X(-E)) \simeq \omega_X^{-1} \otimes \pi_* \mathcal{O}_X(-E).$$

We now show that $\pi_* \mathcal{O}_X(-E) \simeq \mathcal{I}_P$. The canonical morphism $\pi_* \mathcal{O}_Y \to \mathcal{O}_E$ is an isomorphism: this is easy to see after noticing that it is induced by the morphism $E \to Y$ which is none other than $\mathbb{P}^1_{\kappa(P)} \to \mathrm{Spec}(\kappa(P))$. Considering the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_P \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \pi_* \mathcal{O}_X(-E) \arrow[r] & \pi_* \mathcal{O}_{X'} \arrow[r] & \pi_* \mathcal{O}_E &
\end{tikzcd}$$

where the rows are exact and the second and third vertical arrows are isomorphisms, it is now clear that the canonical morphism $\mathcal{I}_P \to \pi_* \mathcal{O}_X(-E)$ is also an isomorphism.

The only thing left to show is that

$$H^0(X, \omega_X^{-1} \otimes \mathcal{I}_P) = \{f \in H^0(X, \omega_X^{-1}) \mid f(P) = 0\}.$$

On an open affine $U \subseteq X$, the section of the tensor product $(\omega_X^{-1} \otimes \mathcal{I}_P)(U)$ is equal to the tensor product of the sections $\omega_X^{-1}(U) \otimes \mathcal{I}_P(U)$ ([21, Proposition 5.1.12(b)]) and ω_X^{-1} being an invertible sheaf, we can suppose that $\omega_X^{-1}|_U \simeq \mathcal{O}_U$ so that certainly $\omega_X^{-1}(U)$ is a flat $O_X(U)$-module and the canonical map $\omega_X^{-1}(U) \otimes \mathcal{I}_P(U) \to \omega_X^{-1}(U)$ is an injection with image

$$\{f \in \omega_X^{-1}(U) \mid f(P) = 0\}.$$

We conclude by noting that the map $U \mapsto \{f \in \omega_X^{-1}(U) \mid f(P) = 0\}$ forms a sheaf which coincide with $\omega_X^{-1} \otimes \mathcal{I}_P$ on a base of open affine subschemes of X, hence they are the same sheaf. $\square$

Theorem 3.6.2. *Let X_0 be a regular integral algebraic surface over k, and let $\pi\colon X_n \to X_0$ be a sequence of blow-ups at closed points $P_i \in X_i$*

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0$$

where $\pi_i^{-1}(P_{i-1}) = E_i$ for $i = 1, \dots, n$. Define

$$\begin{aligned} \mathcal{F}_n &= \mathcal{O}_{X_n}, \\ \mathcal{F}_{i-1} &= (\pi_i)_*(\mathcal{O}_{X_i}(-E_i) \otimes \mathcal{F}_i) \quad i = 1, \dots, n. \end{aligned}$$

Then

$$\pi_*\omega_{X_n}^{-1} \simeq \omega_{X_0}^{-1} \otimes \mathcal{F}_0.$$

Proof. Proceed by induction on n. The case $n = 1$ is covered in Theorem 3.6.1. For $n > 1$, let $\pi' = \pi_2 \circ \dots \circ \pi_n$ and suppose that $\pi'_*\omega_{X_n}^{-1} \simeq \omega_{X_1}^{-1} \otimes \mathcal{F}_1$. Then through the projection formula we see that

$$\pi_*\omega_{X_n}^{-1} \simeq (\pi_1)_*(\omega_{X_1}^{-1} \otimes \mathcal{F}_1) \simeq \omega_{X_0}^{-1} \otimes (\pi_1)_*(\mathcal{O}_{X_1}(-E_1) \otimes \mathcal{F}_1) = \omega_{X_0}^{-1} \otimes \mathcal{F}_0.$$

□

Remark 3.6.3. In the following, the isomorphisms $\mathcal{O}_{X_{i-1}} \xrightarrow{\sim} (\pi_i)_*\mathcal{O}_{X_i}$ will be considered as identities, i.e. $\mathcal{O}_{X_{i-1}} = (\pi_i)_*\mathcal{O}_{X_i}$. In this fashion, $\mathcal{F}_n, \dots, \mathcal{F}_0$ are sheaves of ideals.

Lemma 3.6.4. *Let X be an integral scheme. Let $\mathcal{F}$ be a sheaf on X locally isomorphic to ideal sheaves (for example, the tensor product of an invertible sheaf and an ideal sheaf). Let $U \subseteq X$ be a non-empty open subscheme. Then the restriction map*

$$H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F})$$

is injective.

Proof. Let $\{U_i\}_i$ be an open cover of X such that $\mathcal{F}|_{U_i} \simeq \mathcal{I}_i$ ideal sheaf of $\mathcal{O}_{U_i}$. Since X is integral, the restriction map

$$\mathrm{res}^{U_i}_{U_i \cap U}\colon H^0(U_i, \mathcal{O}_X) \to H^0(U_i \cap U, \mathcal{O}_X)$$

is injective. On the other hand, the restriction map

$$H^0(U_i, \mathcal{I}_i) \to H^0(U_i \cap U, \mathcal{I}_i)$$

is none other than $\mathrm{res}^{U_i}_{U_i \cap U}|_{H^0(U_i, \mathcal{I}_i)}$, so it is also injective. Thus, if $f \in H^0(X, \mathcal{F})$ is such that $f|_U = 0$, then $f|_{U \cap U_i} = 0$ for all i, whence $f|_{U_i} = 0$ and thus $f = 0$. □

Theorem 3.6.5. *Let $X_0 \subseteq \mathbb{A}^2_k = \mathrm{Spec}(k[x, y])$ be an open subscheme of the affine plane over k, and let $\pi\colon X_n \to X_0$ be a sequence of blow-ups at rational closed points $P_i \in X_i(k)$*

$$X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \dots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0$$

where $\pi_i^{-1}(P_{i-1}) = E_i$ for $i = 1, \dots, n$; let $P_0 \in E_0 = V(x)$ and suppose that $P_i \in E_i \setminus E'_{i-1}$, where E'_{i-1} is the strict transform of E_{i-1} along π_i. Then there exist $a_0, \dots, a_{n-1} \in k$ such that, using notation from Theorem 3.6.2*,*

$$H^0(X_0, \mathcal{F}_0) = (x^n, y - a_0 - a_1 x - \cdots - a_{n-1}x^{n-1}).$$

Moreover, let

$$C\colon y = b_0 + b_1 x + \cdots + b_m x^m$$

be a curve on X_0, with strict transforms $C_i \subset X_i$. Let $k \in \mathbb{N}$ be the maximum such that $a_0 = b_0, \dots, a_{k-1} = b_{k-1}$. Then P_i lie on C_i if and only if $i < k$, in which case C_i has multiplicity 1 *on P_i.*

Proof. It is the same if we just pick $X_0 = \mathbb{A}^2_k = \operatorname{Spec}(k[x, y_0])$. Set $U_0 = X_0$. The blow-up of U_0 at $P_0 = (0, a_0)$ is $\pi_1^{-1}(U_0) \subset \mathbb{A}^2_k \times \mathbb{P}^1_k$:

$$\begin{aligned} X_1 = \pi_1^{-1}(U_0)\colon & s_1 x - t_1(y_0 - a_0) = 0 \\ E_1\colon & x = y_0 - a_0 = 0 \\ E'_0\colon & t_1 = 0 \end{aligned}$$

Let us consider the open subset $U_1\colon t_1 \neq 0$. Here we set $y_1 := s_1/t_1$ so that the defining equation becomes $y_1 x = y_0 - a_0$. In fact, this means that $U_1 = \operatorname{Spec}(k[x, y_1])$ and we can repeat the same process: the blow-up of U_1 at $P_1 = (0, a_1)$ is

$$\begin{aligned} X_2 \supseteq \pi_2^{-1}(U_1)\colon & s_2 x - t_2(y_1 - a_1) = 0 \\ E_2\colon & x = y_1 - a_1 = 0 \\ E'_2\colon & t_2 = 0 \\ U_2\colon & t_2 \neq 0, \quad y_2 := s_2/t_2 \\ U_2\colon & y_2 x = y_1 - a_1 \quad \rightsquigarrow \quad U_2 = \operatorname{Spec}(k[x, y_2]). \end{aligned}$$

Repeating as much as needed, we arrive at

$$\begin{aligned} X_n \supseteq \pi_n^{-1}(U_{n-1})\colon & s_n x - t_n(y_{n-1} - a_{n-1}) = 0 \\ E_n\colon & x = y_{n-1} - a_{n-1} = 0 \\ E'_n\colon & t_n = 0 \\ U_n\colon & t_n \neq 0, \quad y_n := s_n/t_n \\ U_n\colon & y_n x = y_{n-1} - a_{n-1} \quad \rightsquigarrow \quad U_n = \operatorname{Spec}(k[x, y_n]). \end{aligned}$$

We now proceed backwards. We have

$$\mathcal{O}_{X_n}(-E_n)(\pi^{-1}(U_{n-1})) = (x, y_{n-1} - a_{n-1}) = \mathcal{F}_{n-1}(U_{n-1})$$

and $\mathcal{F}_{n-1}$ is just the structure sheaf away of E_{n-1}. On U_{n-1}

$$\mathcal{O}_{X_{n-1}}(-E_{n-1})(U_{n-1}) = (x),$$

$$(\mathcal{O}_{X_{n-1}}(-E_{n-1}) \otimes \mathcal{F}_{n-1})(U_{n-1}) = (x)(x, y_{n-1} - a_{n-1}) = (x^2, x(y_{n-1} - a_{n-1}))$$

which extends on X_{n-1} to

$$(\mathcal{O}_{X_{n-1}}(-E_{n-1}) \otimes \mathcal{F}_{n-1})(U_{n-1}) = (x^2, y_{n-2} - a_{n-2} - a_{n-1}x) = \mathcal{F}_{n-2}(U_{n-2}).$$

In the same fashion we eventually get

$$\mathcal{F}_0(U_0) = (x^n, y_0 - a_0 - a_1 x - \cdots - a_{n-1}x^{n-1})$$

as claimed.

Now we consider the strict transforms of C. On U_1, assuming $b_0 = a_0$ we have $y_0 - b_0 = y_1 x$, so

$$y_0 - b_0 - b_1 x - \cdots - b_m x^m = x(y_1 - b_1 - b_2 x - \cdots - b_m x^{m-1})$$

which means that $C_1\colon y_1 = b_1 + b_2 x + \cdots + b_m x^{m-1}$. If $k \in \mathbb{N}$ is maximum such that $a_i = b_i$ for $i < k$, proceeding in the same way

$$C_k\colon y_k = b_k + b_{k+1}x + \cdots + b_m x^{m-k}$$

but $a_k \neq b_k$ and so $P_k = (0, a_k)$ does not lie on C_k, and neither do the points $P_{k+1}, \dots, P_{n-1}$ which are infinitely close to P_k. Looking back, k points were blown-up on C, and each had multiplicity 1 because C is nonsingular. □

Theorem 3.6.6. *Let X be a regular integral algebraic surface over k which admits an affine open subscheme $U \subseteq X$ isomorphic to $\mathbb{A}^2_k$. Suppose that $\{P_0^{(i)}\}_{i=1}^n$ are distinct points on U, and set*

$$U^{(i)} = U \setminus \{P_0^{(j)} \mid j \neq i\} \quad \textit{and} \quad V = X \setminus \{P_0^{(1)} \dots, P_0^{(n)}\}.$$

Let $\pi\colon X' \to X$ be a sequence of blow-ups which is an isomorphism on V and of the form described in Theorem 3.6.5 *on $U^{(i)}$, with $H^0(U^{(i)}, \mathcal{F}_0^{(i)}) = J^{(i)}$. Then*

$$H^0(X, \pi_*\omega_{X'}^{-1}) \to H^0(U, \pi_*\omega_{X'}^{-1})$$

is injective with image

$$H^0(X, \pi_*\omega_{X'}^{-1})|_U = H^0(X, \omega_X^{-1})|_U \cap J^{(1)} \cap \cdots \cap J^{(n)}.$$

Proof. On U, the invertible sheaf ω_X^{-1} is isomorphic as an $\mathcal{O}_U$-module to $\mathcal{O}_U$ (recall that $\operatorname{Pic} \mathbb{A}^2_k$ is trivial); the isomorphism $\varphi\colon \omega_X^{-1}|_U \xrightarrow{\sim} \mathcal{O}_U$ is usually left implicit, but here we write it explicitly. What we really want to prove is that

$$\varphi\big(H^0(X, \pi_*\omega_{X'}^{-1})|_U\big) = \varphi\big(H^0(X, \omega_X^{-1})|_U\big) \cap J^{(1)} \cap \cdots \cap J^{(n)}.$$

Since the points $P_0^{(i)}$ are distinct, with reasoning similar to Theorem 3.6.2 we see that $\pi_*\omega_{X'}^{-1} = \omega_X^{-1} \otimes \mathcal{F}_0^{(1)} \otimes \cdots \otimes \mathcal{F}_0^{(n)}$. This proves the injectivity of the restriction morphism, through Lemma 3.6.4. We now show that

$$\varphi\big(H^0(X, \omega_X^{-1} \otimes \mathcal{F}_0^{(1)} \otimes \cdots \otimes \mathcal{F}_0^{(n)})|_U\big) = \varphi\big(H^0(X, \omega_X^{-1})|_U\big) \cap (J^{(1)} \dots J^{(n)})$$

by double inclusion.

($\subseteq$) By Remark 3.6.3 we regard $\mathcal{F}_0^{(1)} \otimes \cdots \otimes \mathcal{F}_0^{(n)}$ as an ideal sheaf, so obviously

$$H^0(X, \omega_X^{-1} \otimes \mathcal{F}_0^{(1)} \otimes \cdots \otimes \mathcal{F}_0^{(n)}) \subseteq H^0(X, \omega_X^{-1}).$$

Since U is affine and $\omega_X^{-1}(U)$ is a flat $\mathcal{O}_U(U)$-module (because it is isomorphic to $\mathcal{O}_U(U)$), by [21, Theorem 1.2.4] and [21, Proposition 5.1.14(b)]

$$\begin{aligned} H^0(U, \omega_X^{-1} \otimes \mathcal{F}_0^{(1)} \otimes \cdots \otimes \mathcal{F}_0^{(n)}) &= \omega_X^{-1}(U) \otimes \mathcal{F}_0^{(1)}(U) \otimes \cdots \otimes \mathcal{F}_0^{(n)}(U) \\ &= J^{(1)} \dots J^{(n)} \, \omega_X^{-1}(U) \end{aligned}$$

whence

$$\varphi\big(H^0(X, \omega_X^{-1} \otimes \mathcal{F}_0^{(1)} \otimes \cdots \otimes \mathcal{F}_0^{(n)})|_U\big) \subseteq J^{(1)} \dots J^{(n)}.$$

($\supseteq$) Let $f \in \varphi\big(H^0(X, \omega_X^{-1})|_U\big) \cap (J^{(1)} \dots J^{(n)})$. Using the equality we just found,

$$\varphi^{-1}(f) \in J^{(1)} \dots J^{(n)} \, \omega_X^{-1}(U) = H^0(U, \omega_X^{-1} \otimes \mathcal{F}_0^{(1)} \otimes \cdots \otimes \mathcal{F}_0^{(n)}).$$

Now, since $\varphi^{-1}(f) \in H^0(X, \omega_X^{-1})|_U$, by definition there exists $g \in H^0(X, \omega_X^{-1})$ such that $g|_U = \varphi^{-1}(f)$. Then the condition

$$g|_V \in H^0(V, \omega_X^{-1} \otimes \mathcal{F}_0^{(1)} \otimes \cdots \otimes \mathcal{F}_0^{(n)})$$

is trivially verified, so also globally

$$g \in H^0(X, \omega_X^{-1} \otimes \mathcal{F}_0^{(1)} \otimes \cdots \otimes \mathcal{F}_0^{(n)}).$$

Finally, note that $\sqrt{J^{(i)}}$ is maximal for each i, hence

$$J^{(1)} \dots J^{(n)} = J^{(1)} \cap \cdots \cap J^{(n)}.$$

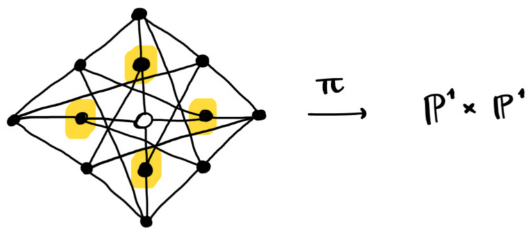
□

3.7 The classification

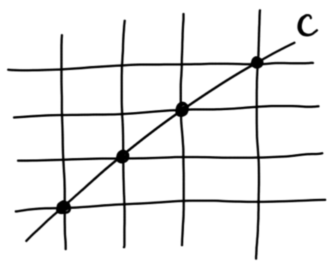
A few words on the conventions in use: when talking about points or curves, we imagine them "living" over the algebraic closure and being defined over k: for example, "two points on $\mathbb{P}^1_k$" could mean two rational points belonging to $\mathbb{P}^1(k)$ or two points on $\mathbb{P}^1$ conjugate under a quadratic extension. We use the same name for a curve and its strict transforms.

3.7.1 Type I

Blow down the four (-1)-curves which intersect the unique (-2)-curve. The result is a Del Pezzo surface of degree 8 with no negative curves, that is, $\mathbb{P}^1 \times \mathbb{P}^1$. This operation corresponds to the blow-up of four points lying on a curve C of degree $(1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. This curve is defined over k. Supposing dense rational points on X, Z is either $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or Q_α.



$\boxed{Z = \mathbb{P}^1_k \times \mathbb{P}^1_k}$



The automorphisms of $\mathbb{P}^1_k \times \mathbb{P}^1_k$ (Lemma 3.5.3) are "enough" to guarantee that all integral curves of degree $(1,1)$ defined over k on $\mathbb{P}^1_k \times \mathbb{P}^1_k$ are the same up to k-automorphisms: starting from $x_0y_0 + x_1y_1$, to get any other curve $ax_0y_0 + bx_0y_1 + cx_1y_0 + dx_1y_1$ just apply the transformation

$$(x_0 : x_1, y_0 : y_1) \mapsto (x_0 : x_1, ay_0 + by_1 : cy_0 + dy_1).$$

Thus, using (x, y) coordinates we may assume $C\colon x = y$, and that the four blown-up points have coordinates (A_1, A_1), (A_2, A_2), (A_3, A_3), (A_4, A_4) with $A_1, A_2, A_3, A_4 \in \overline{k}$ distinct such that the polynomial

$$(t - A_1)(t - A_2)(t - A_3)(t - A_4) =: s_0 + s_1t + s_2t^2 + s_3t^3 + t^4$$

has coefficients $s_0, s_1, s_2, s_3 \in k$.

$$\begin{aligned}
H^0(Z, \mathcal{F}_0)|_U &= (x - A_1, y - A_1) \cap (x - A_2, y - A_2) \\
&\qquad \cap (x - A_3, y - A_3) \cap (x - A_4, y - A_4) \\
&= (x - y,\ s_0 + s_1x + s_2x^2 + s_3x^3 + x^4) \\
H^0(Z, \pi_*\omega_{X'}^{-1})|_U &= \langle x - y,\ x(x - y),\ y(x - y),\ xy(x - y), \\
&\qquad s_0 + s_1x + s_2xy + s_3xy^2 + x^2y^2\rangle
\end{aligned}$$

$$X\colon \begin{cases} z_0z_3 - z_1z_2 \\ s_0z_0^2 + s_1z_0z_1 + s_2z_1z_2 + s_3z_2z_3 + z_3^2 - (z_1 - z_2)z_4 \end{cases}$$

$$X \dashrightarrow U\colon x = z_1/z_0,\ y = z_2/z_0$$

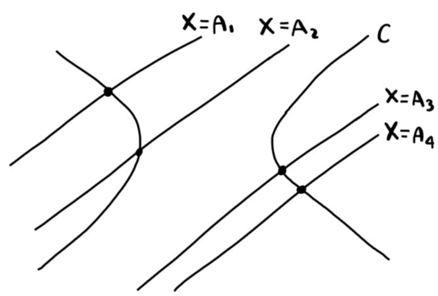
singularities: $(0:0:0:0:1)$

$$\text{lines:}\ \begin{cases} z_1 - A_i z_0 \\ z_2 - A_i z_0 \\ z_3 - A_i^2 z_0 \end{cases} \qquad i = 1,\dots,4$$

$$\begin{cases} z_1 - A_i z_0 \\ z_3 - A_i z_2 \\ z_4 + (A_i + s_3) z_3 + (s_2 + A_i s_3 + A_i^2) z_1 \end{cases} \qquad i = 1,\dots,4$$

$$\begin{cases} z_2 - A_i z_0 \\ z_3 - A_i z_1 \\ z_4 - A_i z_3 + A_i^{-1} s_0 z_0 \end{cases} \qquad i = 1,\dots,4$$

$\boxed{Z = Q_\alpha}$



The description of $Z = Q_\alpha$ as a quotient of $\mathbb{P}^1_{k(\alpha)} \times \mathbb{P}^1_{k(\alpha)}$ by the equivalence relation $(X_0 : X_1, Y_0 : Y_1) \sim (Y_0^\sigma : Y_1^\sigma, X_0^\sigma : X_1^\sigma)$ suggests that the curves of degree $(1,1)$ defined over k have the form

$$AX_0Y_0 + BX_0Y_1 + B^\sigma X_1Y_0 + CX_1Y_1,$$

also expressible in matrix form

$$\begin{pmatrix} X_0 & X_1 \end{pmatrix} \begin{pmatrix} A & B \\ B^\sigma & C \end{pmatrix} \begin{pmatrix} Y_0 \\ Y_1 \end{pmatrix}$$

where $A, C \in k$ and $B \in k(\alpha)$. The k-automorphisms of Q_α (Lemma 3.5.4) are such that not all integral curves of degree $(1,1)$ defined over k are the same up to k-automorphisms: for instance, the orbit of the curve $X_0X_0 + X_1Y_1$ is given by curves with matrix form

$$\begin{pmatrix} m_{00} & m_{10} \\ m_{01} & m_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m_{00}^\sigma & m_{01}^\sigma \\ m_{10}^\sigma & m_{11}^\sigma \end{pmatrix} = \begin{pmatrix} m_{00}m_{00}^\sigma + m_{01}m_{01}^\sigma & m_{00}m_{10}^\sigma + m_{01}m_{11}^\sigma \\ m_{10}m_{00}^\sigma + m_{11}m_{01}^\sigma & m_{01}m_{01}^\sigma + m_{11}m_{11}^\sigma \end{pmatrix}$$

and supposing e.g. that $k = \mathbb{Q}$, $\alpha = \sqrt{-1}$, necessarily $m_{00}m_{00}^\sigma + m_{01}m_{01}^\sigma$ and $m_{01}m_{01}^\sigma + m_{11}m_{11}^\sigma$ are both positive integers.

I have found the equations pertaining the most general case too complicate to be of any use; instead, we will suppose that the curve C of degree $(1,1)$ on which the blown-up points lie has a k- or $k(\alpha)$-rational point $(m_{00} : m_{10}, m_{01} : m_{11})$ i.e. is isomorphic to $\mathbb{P}^1_k$ or $\mathbb{P}^1_{k(\alpha)}$. In this case, C can be represented by a diagonal matrix: it is enough to consider the transformation

$$\begin{pmatrix} m_{00} & m_{10} \\ m_{01} & m_{11} \end{pmatrix} \begin{pmatrix} A & B \\ B^\sigma & C \end{pmatrix} \begin{pmatrix} m_{00}^\sigma & m_{01}^\sigma \\ m_{10}^\sigma & m_{11}^\sigma \end{pmatrix}.$$

Hence, we can assume C to be given by equation $-bX_0Y_0 + X_1Y_1$, which in (x, y) coordinates becomes

$$y^2 - ax^2 - b.$$

We blow up C in the four points

$$\left(\frac{1}{2\alpha}(b/A_i - A_i), \frac{1}{2}(b/A_i + A_i)\right), \qquad i = 1, \dots, 4$$

which is the intersection of C with the degree $(1,0)$ curves $X = y - \alpha x = A_i$, where we require the polynomial

$$(t - A_1)(t - A_2)(t - A_3)(t - A_4) =: s_0 + s_1t + s_2t^2 + s_3t^3 + t^4 = p(t)$$

to be such that $p(y - \alpha x)$ has coefficients in k, and that the $A_i \in \overline{k}$ are distinct and non-zero.

$$\begin{aligned} H^0(Z, \mathcal{F}_0)|_U &= (y^2 - ax^2 - b, y - \alpha x - A_1) \cap (y^2 - ax^2 - b, y - \alpha x - A_2) \\ &\quad \cap (y^2 - ax^2 - b, y - \alpha x - A_3) \cap (y^2 - ax^2 - b, y - \alpha x - A_4) \\ &= (y^2 - ax^2 - b,\ p(y - \alpha x)) \\ H^0(Z, \pi_*\omega_{X'}^{-1})|_U &= \langle y^2 - ax^2 - b,\ x(y^2 - ax^2 - b),\ y(y^2 - ax^2 - b),\ (y^2 - ax^2)^2 - b^2, \\ &\quad b^2(y - \alpha x)^2 + s_3b^2(y - \alpha x) + s_2b^2 + s_1b(y + \alpha x) + s_0(y + \alpha x)^2\rangle \end{aligned}$$

$$X\colon \begin{cases} z_0z_3 - z_2^2 + az_1^2 - bz_0^2 \\ b^2(z_2 - \alpha z_1)^2 + s_3b^2(z_2 - \alpha z_1)z_0 + s_2b^2z_0^2 + s_1b(z_2 + \alpha z_1)z_0 + s_0(z_2 + \alpha z_1)^2 - (z_3 - 2bz_0)z_4 \end{cases}$$

$$X \dashrightarrow U\colon x = z_1/z_0,\ y = z_2/z_0$$

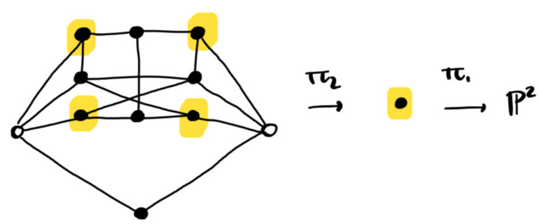
$$\text{singularities}\colon (0:0:0:0:1)$$

$$\text{lines}\colon \begin{cases} 2\alpha z_1 - (b/A_i - A_i)z_0 \\ 2z_2 - (b/A_i + A_i)z_0 \\ z_3 - 2bz_0 \end{cases} \qquad i = 1, \dots, 4$$

$$\begin{cases} z_2 - \alpha z_1 - A_i z_0 \\ z_3 - 2A_i\alpha z_1 - (A_i^2 + b)z_0 \\ A_i^2 z_4 - s_0 z_3 - bs_1 A_i z_0 \end{cases} \qquad i = 1, \dots, 4$$

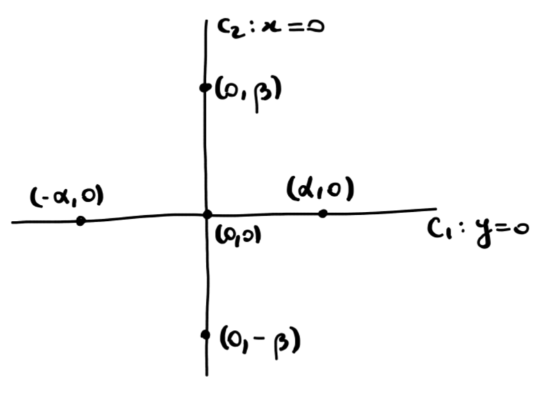
$$\begin{cases} z_2 + \alpha z_1 - (b/A_i)z_0 \\ z_3 + 2(b/A_i)\alpha z_1 - ((b/A_i)^2 + b)z_0 \\ z_4 - A_i^2 z_3 - bs_3 A_i z_0 \end{cases} \qquad i = 1, \dots, 4$$

3.7.2 Type II



The last blow-down contracts a curve defined over k to a rational point P, so $Z = \mathbb{P}^2_k$. The curves C_1, C_2 on $\mathbb{P}^2$ which become the (-2)-curves of X' are lines passing through P and on which lie the four points blown up in π_2, two on each line. Over k, C_1 and C_2 are either defined over k or conjugate under a quadratic extension $k(\alpha)/k$.

$\boxed{C_1, C_2 \textbf{ are defined over } k}$



By Lemma 3.5.2 we may suppose

$$C_1 : y = 0, \quad C_2 : x = 0$$

so that $P = (0,0)$, and by Lemma 3.5.1 that the other four blown-up points are $(\pm\alpha, 0)$ and $(0, \pm\beta)$ where $\alpha^2 = a \in k^\times$, $\beta^2 = b \in k^\times$.

$$H^0(Z, \mathcal{F}_0)|_U = (x, y) \cap (x^2 - a, y) \cap (x, y^2 - b) = (xy, x^3 - ax, y^3 - ay)$$

$$H^0(Z, \pi_* \omega_{X'}^{-1})|_U = \langle xy,\ x^2y,\ xy^2,\ x^3 - ax,\ y^3 - by\rangle$$

$$X\colon \begin{cases} z_2z_3 + az_0^2 - z_1^2 \\ z_1z_4 + bz_0^2 - z_2^2 \end{cases}$$

$$X \dashrightarrow U\colon x = z_1/z_0,\ y = z_2/z_0$$

$$\text{singularities: } (0:0:0:0:1), (0:0:0:1:0)$$

$$\text{lines: } \begin{array}{ll}
\begin{cases} z_1 - \alpha z_0 \\ z_2 \\ \alpha z_4 + bz_0 \end{cases} & \begin{cases} z_1 + \alpha z_0 \\ z_2 \\ -\alpha z_4 + bz_0 \end{cases} \\
\begin{cases} z_1 \\ z_2 - \beta z_0 \\ \beta z_3 + az_0 \end{cases} & \begin{cases} z_1 \\ z_2 + \beta z_0 \\ -\beta z_3 + az_0 \end{cases} \\
\begin{cases} \alpha z_2 - \alpha\beta z_0 + \beta z_1 \\ \beta z_3 + az_0 + \alpha z_1 \\ az_4 + 2\alpha b z_0 - bz_1 \end{cases} & \begin{cases} -\alpha z_2 + \alpha\beta z_0 + \beta z_1 \\ \beta z_3 + az_0 - \alpha z_1 \\ az_4 - 2\alpha b z_0 - bz_1 \end{cases} \\
\begin{cases} \alpha z_2 + \alpha\beta z_0 - \beta z_1 \\ -\beta z_3 + az_0 + \alpha z_1 \\ az_4 + 2\alpha b z_0 - bz_1 \end{cases} & \begin{cases} -\alpha z_2 - \alpha\beta z_0 - \beta z_1 \\ -\beta z_3 + az_0 - \alpha z_1 \\ az_4 - 2\alpha b z_0 - bz_1 \end{cases} \\
\begin{cases} z_0 \\ z_1 \\ z_2 \end{cases} &
\end{array}$$

$\boxed{C_1, C_2 \textbf{ are conjugate under } k(\alpha)/k}$

By Lemma 3.5.2 we may suppose

$$C_1\colon y = \alpha x, \quad C_2\colon y = -\alpha x$$

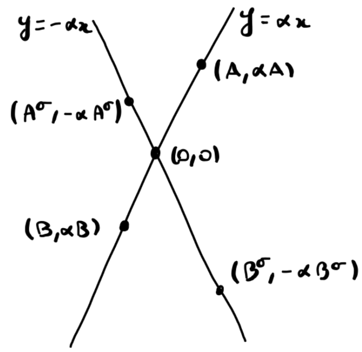
so that $P = (0,0)$. The other four blown-up points are

$$(A, \alpha A), \quad (B, \alpha B), \quad (A^\sigma, -\alpha A^\sigma), \quad (B^\sigma, -\alpha B^\sigma)$$

where $A, B \in \overline{k}$ are non-zero with $A \neq B$ such that

$$(t - A)(t - B) =: t^2 + s_1 t + s_0$$

has coefficients in $k(\alpha)$, and σ is an extension of the conjugation on $k(\alpha)/k$. This is because C_1 is defined over $k(\alpha)$, so $\{P_1, P_2\}$ is defined over $k(\alpha)$, while $\{P_3, P_4\}$ must be their $k(\alpha)/k$-conjugate.

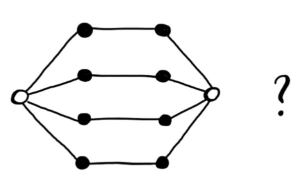


Let

$$s_0 = AB + A^\sigma B^\sigma,\ r_0 = \alpha(AB - A^\sigma B^\sigma),$$
$$s_1 = -(A + B + A^\sigma + B^\sigma),\ r_1 = -\alpha(A + B - A^\sigma - B^\sigma).$$

$$\begin{aligned}
H^0(Z, \mathcal{F}_0)|_U &= (y - \alpha x, x - A) \cap (y - \alpha x, x - B) \\
&\quad \cap (y + \alpha x, x - A^\sigma) \cap (y + \alpha x, x - B^\sigma) \cap (x, y) \\
&= (y^2 - ax^2,\ 2x^2 y + s_1 xy + s_0 y + r_1 x^2 + r_0 x, \\
&\quad r_1 xy + r_0 y + 2ax^3 + as_1 x^2 + as_0 x, \\
&\quad x(x - A)(x - B)(x - A^\sigma)(x - B^\sigma)) \\
H^0(Z, \pi_* \omega_{X'}^{-1})|_U &= \langle y^2 - ax^2,\ x(y^2 - ax^2),\ y(y^2 - ax^2), \\
&\quad 2x^2 y + s_1 xy + s_0 y + r_1 x^2 + r_0 x, \\
&\quad r_1 xy + r_0 y + 2ax^3 + as_1 x^2 + as_0 x\rangle \\
X &\colon \begin{cases} (r_0 s_1 - r_1 s_0) z_0 z_1 + 2r_0 z_1^2 - as_0 z_1 z_3 - r_0 z_2 z_3 + r_0 z_1 z_4 + s_0 z_2 z_4 \\ (r_1 s_0 - r_0 s_1) z_0^2 + 2r_1 z_1^2 - as_1 z_1 z_3 - r_1 z_2 z_3 + r_1 z_1 z_4 + s_1 z_2 z_4 \end{cases} \\
X \dashrightarrow U &\colon x = z_1/z_0,\ y = z_2/z_0 \\
\text{singularities} &\colon (0:0:0:1:\alpha), (0:0:0:1:-\alpha)
\end{aligned}$$

3.7.3 Type III



This type represents a notable exception to our list of surfaces: a priori, there is no way to choose a Galois-invariant set of (-1)-curves to blow down, because each (-1)-curve could be conjugate to the other (-1)-curve one intersecting it. In this case, the two singularities of X are conjugate: when this happens, we say that X is an *Iskovskih surface.* The work [5] presents a number of theorems in arithmetic geometry which hold for all singular Del Pezzo surfaces of degree 4 apart from the Iskovskih surfaces. A result which is of interest to us is the following:

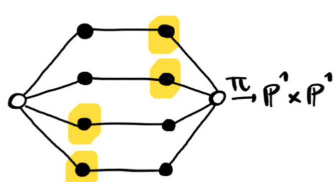
Proposition 3.7.1. *Let X be an Iskovskih surface. Suppose that the associated weak Del Pezzo surface X' is a minimal surface. Then X is not k-rational.*

Since our entire approach is based on the non-minimality of X' and the rationality of X, we suppose that X be k-rational. In all other 14 cases of singular Del Pezzo surfaces of degree 4, this is either always true, or is equivalent to $X'(k) \neq \varnothing$, but here it is a stronger assumption.

X is an Iskovskih surface

Assuming rationality, X' is non-minimal, so there is a way to blow down some of its (-1)-curves in a Galois-invariant way; but there are multiple ways in which this might be true, depending on how the (-1)-curves are related to each other through conjugation, and these different ways lead to different outcomes. More precisely, we distinguish the case in which it is possible to immediately blow down four curves on X', and the case in which it is not.

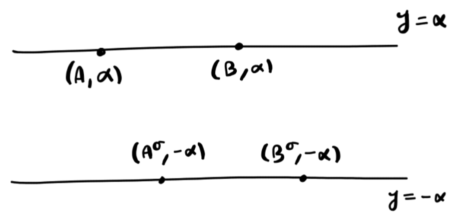
Four (-1)-curves can be immediately blown down



We may describe π as the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in four points, two on a curve of degree $(0, 1)$ and two on another (an equivalent situation would arise if these were of degree $(1, 0)$). Since in this way the number of degree $(0, 1)$ curves which become negative (two) is different from the number of degree $(1, 0)$ curves which become negative (four), Z cannot be Q_α.

Now, using coordinates (x, y) we suppose that the two curves of degree $(0, 1)$ are $y = \alpha$ and $y = -\alpha$, where $\alpha^2 = a \in k^\times$ and $k(\alpha) \neq k$ (otherwise it would not be an Iskovskih surface). The four blown-up points are

$$(A, \alpha), \quad (B, \alpha), \quad (A^\sigma, -\alpha), \quad (B^\sigma, -\alpha)$$



where $A, B \in \overline{k}$ are distinct such that

$$(t-A)(t-B) =: t^2 + s_1 t + s_0$$

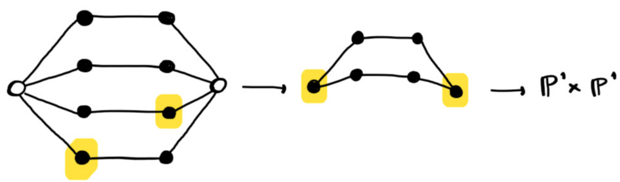
has coefficients in $k(\alpha)$ and σ is an extension of the conjugation on $k(\alpha)/k$ with $A \neq A^\sigma,\ B \neq B^\sigma$. Let

$$s_0 = AB + A^\sigma B^\sigma,\ r_0 = \alpha(AB - A^\sigma B^\sigma),$$
$$s_1 = -(A + B + A^\sigma + B^\sigma),\ r_1 = -\alpha(A + B - A^\sigma - B^\sigma).$$

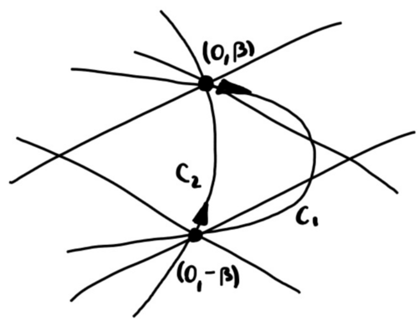
$$\begin{aligned}
H^0(Z, \mathcal{F}_0)|_U &= (x-A, y-\alpha) \cap (x-B, y-\alpha) \\
&\quad \cap (x-A^\sigma, y-\alpha) \cap (x-A^\sigma, y-\alpha) \\
&= (y^2 - a,\ 2x^2y + s_1xy + s_0y + r_1x + r_0, \\
&\quad r_1xy + r_0y + 2ax^2 + as_1x + as_0, \\
&\quad (x-A)(x-B)(x-A^\sigma)(x-B^\sigma)) \\
H^0(Z, \pi_*\omega_{X'}^{-1})|_U &= \langle y^2 - a,\ x(y^2-a),\ x^2(y^2-a), \\
&\quad 2x^2y + s_1xy + s_0y + r_1x + r_0, \\
&\quad r_1xy + r_0y + 2ax^2 + as_1x + as_0 \rangle \\
X\colon & \begin{cases} z_1^2 - z_0z_2 \\ r_0z_0z_3 + r_1z_1z_3 + az_3^2 - s_0z_0z_4 - s_1z_1z_4 - 2z_2z_4 - z_4^2 \end{cases} \\
X \dashrightarrow U\colon & x = z_1/z_0,\ y = (z_4 + 2z_2 + s_1z_1 + s_0z_0)/z_3 \\
\text{singularities}\colon & (0:0:0:1:\alpha), (0:0:0:1:-\alpha)
\end{aligned}$$

Two (-1)-curves can be immediately blown down

Blow up Z in two points P_1, P_2 conjugate under a quadratic extension $k(\beta)/k$. Let E_1, E_2 be resp. the exceptional curve arising from resp. P_1, P_2. Now two $(1,0)$ and two degree $(0,1)$ curves are negative. We are assuming that we are not in the previous case, so each of the degree $(1,0)$ curves should be conjugate to a degree $(0,1)$ curve under a quadratic extension. Since they have different types over $\overline{k}$, this is only possible if $Z = Q_\alpha$ and the extension they are conjugate under is $k(\alpha)/k$. For now, assume that $k(\beta) \neq k(\alpha)$. After this, choose two



degree $(1,1)$ curves on Z passing through P_1 and P_2 which are $k(\beta)$-conjugate, say C_1, C_2, and blow up $E_1 \cap C_1$ and $E_2 \cap C_2$: the resulting surface is X'.



Up to a k-automorphism of Z, by Lemma 3.5.5 we can assume that P_1, P_2 have (x,y) coordinates $(0, \pm\beta)$. Suppose that

$$C_1\colon A' + B'x + C'y + D'(y^2 - ax^2)$$

with $A', B', C', D' \in k(\beta)$; the condition that C_1 passes through both P_1 and P_2 means that $C = 0$ and $A = -Bb$, so the equation becomes

$$B'x + D'(y^2 - ax^2 - b).$$

Notice that neither B' nor D' can be zero, otherwise C_1 would be defined over k. Thus, we can assume that

$$C_1\colon y^2 - ax^2 - b + Ax + \beta Bx$$

where $A, B \in k$, and similarly

$$C_2\colon y^2 - ax^2 - b + Ax - \beta Bx.$$

We want these two curves to be distinct (otherwise we would blow four points on it and end up with Type V) so $B \neq 0$. To calculate $H^0(Z, \mathcal{F}_0)|_U$ we need the equations of the lines of U passing through P_i and tangent to C_i for $i = 1, 2$, which are

$$y + \frac{1}{2}Bx \pm \beta(\frac{1}{2b}Ax - 1).$$

$$\begin{aligned}
H^0(Z,\mathcal{F}_0)|_U &= \Big(y+\frac{1}{2}Bx+\beta(\frac{1}{2b}Ax-1),x^2\Big)\cap\Big(y+\frac{1}{2}Bx-\beta(\frac{1}{2b}Ax-1),x^2\Big)\\
&= (y^2+Ax+Bxy-b,x^2)\\
H^0(Z,\pi_*\omega_{X'}^{-1})|_U &= \langle x^2,\ y^2-ax^2-b+Ax+Bxy,\ x(y^2-ax^2-b),\\
&\quad y(y^2-ax^2-b)+Axy+Bbx,\ (y^2-ax^2-b)^2\rangle
\end{aligned}$$

$$X\colon\begin{cases} z_2^2-z_0z_4\\ a(bB^2-A^2)z_0^2-bz_1^2-2aAz_0z_2-Az_1z_2+Bz_2z_3+z_3^2-az_0z_4-z_1z_4\end{cases}$$

$$\begin{aligned}
X \dashrightarrow Z\colon X &= y-\alpha x=(z_3-\alpha Az_0+(B-\alpha)z_2)/(z_1+\alpha Bz_0),\\
Y &= y+\alpha x=(z_3+\alpha Az_0+(B+\alpha)z_2)/(z_1-\alpha Bz_0)
\end{aligned}$$

$$\text{singularities}\colon (0:1:0:\beta:0),(0:1:0:-\beta:0)$$

Until now we have assumed $k(\beta)\neq k(\alpha)$. If instead P_1,P_2 are conjugate under $k(\alpha)/k$, the geometric picture changes but the calculations are similar. Now we assume that P_1,P_2 have coordinates $(\pm\alpha,0)$ and that the two $(1,1)$-curves passing through both of those points are

$$C_1\colon y^2-ax^2+a^2+Ay+\alpha By,\quad C_2\colon y^2-ax^2+a^2+Ay-\alpha By$$

where $A,B\in k$ and $B\neq 0$.

$$\begin{aligned}
H^0(Z,\mathcal{F}_0)|_U &= \Big(\alpha x-a-\frac{A}{2a}y-\frac{\alpha}{2a}By,y^2\Big)\cap\Big(\alpha x-a-\frac{A}{2a}y+\frac{\alpha}{2a}By,y^2\Big)\\
&= (y^2-ax^2+a^2+Bxy+Ay,y^2)\\
H^0(Z,\pi_*\omega_{X'}^{-1})|_U &= \langle y^2,\ y^2-ax^2+a^2+Ay+Bxy,\ x(y^2-ax^2+a^2)+aBy+Axy,\\
&\quad y(y^2-ax^2+a^2),\ (y^2-ax^2+a^2)^2\rangle
\end{aligned}$$

$$X\colon\begin{cases} z_3^2-z_0z_4\\ (aB^2-A^2)z_0^2-a^2z_1^2-2Az_0z_3+Az_1z_3-Bz_2z_3+az_2^2-z_0z_4+z_1z_4\end{cases}$$

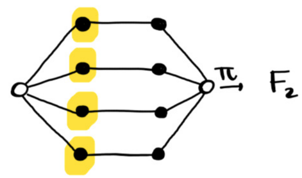
$$\begin{aligned}
X \dashrightarrow Z\colon X &= y-\alpha x=(Az_0-\alpha z_2+(1+B/\alpha)z_3)/(z_1+B/\alpha z_0),\\
Y &= y+\alpha x=(Az_0+\alpha z_2+(1-B/\alpha)z_3)/(z_1-B/\alpha z_0)
\end{aligned}$$

$$\text{singularities}\colon (0:1:\alpha:0:0),(0:1:-\alpha:0:0)$$

X is not an Iskovskih surface

After a series of blow-downs, the resulting curve is a weak Del Pezzo surface of degree 8 with a (-2)-curves, so the Hirzebruch surface F_2. Supposing rationality (or just $X'(k)\neq\varnothing$), we have $Z=F_{2,k}$, and π can be described as the blow-up of Z in four points on an integral curve C of degree $(2,1)$. Such a curve has equation

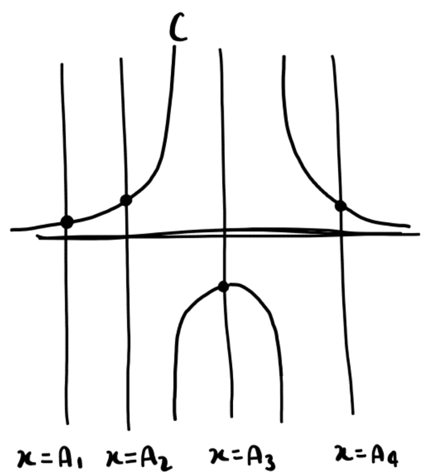
$$1+ay+bxy+cx^2y$$



where $a, b, c \in k$; we blow up the points lying on C with $x = A_1, A_2, A_3, A_4$ where $A_1, A_2, A_3, A_4 \in \overline{k}$ are distinct and such that

$$(t - A_1)(t - A_2)(t - A_3)(t - A_4) =: t^4 + s_3 t^3 + s_2 t^2 + s_1 t + s_0 = p(t)$$

is a polynomial with coefficients in k.



$$\begin{aligned}
H^0(Z, \mathcal{F}_0)|_U &= (1 + ay + bxy + cx^2y, x - A_1) \cap (1 + ay + bxy + cx^2y, x - A_2) \\
&\quad \cap (1 + ay + bxy + cx^2y, x - A_3) \cap (1 + ay + bxy + cx^2y, x - A_4) \\
&= (1 + ay + bxy + cx^2y, p(x)) \\
H^0(Z, \pi_* \omega_{X'}^{-1})|_U &= \langle 1 + ay + bxy + cx^2y, \\
&\quad y(1 + ay + bxy + cx^2y), \\
&\quad xy(1 + ay + bxy + cx^2y), \\
&\quad x^2y(1 + ay + bxy + cx^2y), \\
&\quad y^2(x^4 + s_3x^3 + s_2x^2 + s_1x + s_0)\rangle
\end{aligned}$$

$$X\colon \begin{cases} z_2^2 - z_1 z_3 \\ z_3^2 + s_3 z_2 z_3 + s_2 z_2^2 + s_1 z_1 z_2 + s_0 z_1^2 - z_4(z_0 + a z_1 + b z_2 + c z_3) \end{cases}$$

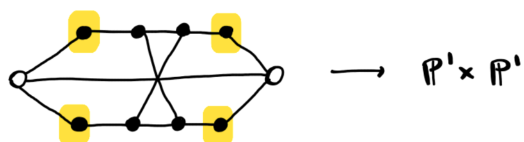
$$X \dashrightarrow Z\colon x = z_2/z_1,\ y = z_1/z_0$$

$$\text{singularities}\colon (1:0:0:0:0), (0:0:0:0:1)$$

$$\text{lines}\colon \begin{cases} z_2 - A_i z_1 \\ z_3 - A_i^2 z_1 \\ z_4 \end{cases}, \qquad i = 1, \dots, 4$$

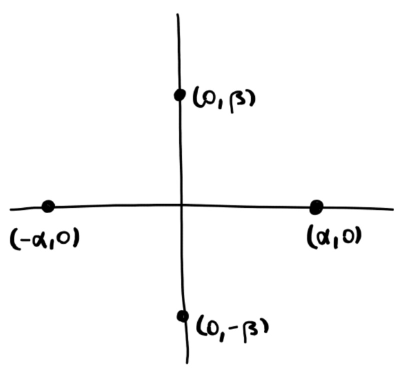
$$\begin{cases} z_2 - A_i z_1 \\ z_3 - A_i^2 z_1 \\ z_0 + a z_1 + bA z_1 + cA_i^2 z_1 \end{cases}, \qquad i = 1, \dots, 4$$

3.7.4 Type IV



The intersection of the two (-2)-curves is a rational point on X', hence $Z = \mathbb{P}^1_k \times \mathbb{P}^1_k$ or Q_α. To recover X' from Z, blow up four points, two on a degree $(1,0)$ curve C_1 and two on a degree $(0,1)$ curve C_2.

$\boxed{Z = \mathbb{P}^1_k \times \mathbb{P}^1_k}$



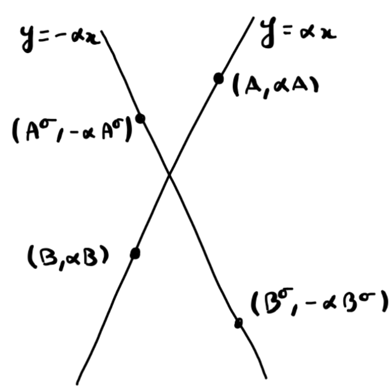
The (-2)-curves are defined over k, since C_1, C_2 can't possibly be conjugate since they have different types over $\overline{k}$. We can assume that

$$C_1\colon x = 0, \quad C_2\colon y = 0$$

and that the points have (x,y) coordinates $(\pm\alpha,0),(0,\pm\beta)$ where $\alpha^2=a\in k^\times$, $\beta^2=b\in k^\times$.

$$\begin{aligned}H^0(Z,\mathcal{F}_0)|_U &= (x^2-a,y)\cap(x,y^2-b)\\ &= (xy,y^3-by,bx^2+ay^2-ab)\\ H^0(Z,\pi_*\omega_{X'}^{-1})|_U &= \langle xy,x^2y,xy^2,x^2y^2,bx^2+ay^2-ab\rangle\\ X\colon &\begin{cases} z_0z_3-z_1z_2\\ z_3z_4-bz_1^2-az_2^2+abz_0^2\end{cases}\\ X \dashrightarrow Z\colon & x=z_1/z_0,\ y=z_2/z_0\\ \text{singularities}\colon &(0:0:0:0:1)\\ \text{lines}\colon &\begin{matrix}\begin{cases}z_1-\alpha z_0\\ z_2\\ z_3\end{cases} & \begin{cases}z_1+\alpha z_0\\ z_2\\ z_3\end{cases} & \begin{cases}z_1\\ z_2-\beta z_0\\ z_3\end{cases} & \begin{cases}z_1\\ z_2+\beta z_0\\ z_3\end{cases}\\ \begin{cases}z_1-\alpha z_0\\ z_3-\alpha z_2\\ z_4-\alpha z_2\end{cases} & \begin{cases}z_1+\alpha z_0\\ z_3+\alpha z_2\\ z_4+\alpha z_2\end{cases} & \begin{cases}z_2-\beta z_0\\ z_3-\beta z_1\\ z_4-\beta z_1\end{cases} & \begin{cases}z_2+\beta z_0\\ z_3+\beta z_1\\ z_4+\beta z_1\end{cases}\end{matrix}\end{aligned}$$

$\boxed{Z=Q_\alpha}$



The (-2)-curves are $k(\alpha)/k$-conjugate; we may assume that

$$C_1\colon y=\alpha x,\quad C_2\colon y=-\alpha x$$

with intersection $(0,0)$, and that the four blown-up points are

$$(A,\alpha A),\quad (B,\alpha B),\quad (A^\sigma,-\alpha A^\sigma),\quad (B^\sigma,-\alpha B^\sigma)$$

where $A, B \in \overline{k}$ are distinct and non-zero such that

$$(t - A)(t - B) =: t^2 + s_1 t + s_0$$

has coefficients in $k(\alpha)$, and σ is an extension of the conjugation on $k(\alpha)/k$. Let

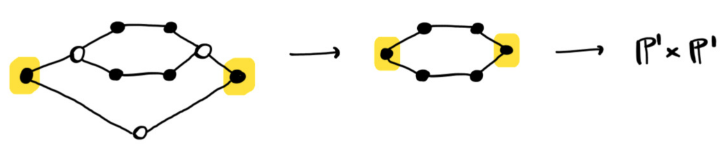
$$s_0 = AB + A^\sigma B^\sigma,\ r_0 = \alpha(AB - A^\sigma B^\sigma),$$
$$s_1 = -(A + B + A^\sigma + B^\sigma),\ r_1 = -\alpha(A + B - A^\sigma - B^\sigma).$$

$$\begin{aligned}
H^0(Z, \mathcal{F}_0)|_U &= (y - \alpha x, x - A) \cap (y - \alpha x, x - B) \\
&\quad \cap (y + \alpha x, x - A^\sigma) \cap (y + \alpha x, x - B^\sigma) \\
&= (y^2 - ax^2,\ 2x^2 y + s_1 xy + s_0 y + r_1 x^2 + r_0 x, \\
&\quad r_1 xy + r_0 y + 2ax^3 + as_1 x^2 + as_0 x, \\
&\quad (x - A)(x - B)(x - A^\sigma)(x - B^\sigma)) \\
H^0(Z, \pi_* \omega_{X'}^{-1})|_U &= \langle y^2 - ax^2,\ x(y^2 - ax^2), \\
&\quad y(y^2 - ax^2),\ (y^2 - ax^2)^2, \\
&\quad (as_0 s_1 - r_0 r_1)x - 2r_0 xy + 2s_0 y^2 + (r_1 s_0 - r_0 s_1)y + as_0^2 - r_0^2 \rangle
\end{aligned}$$

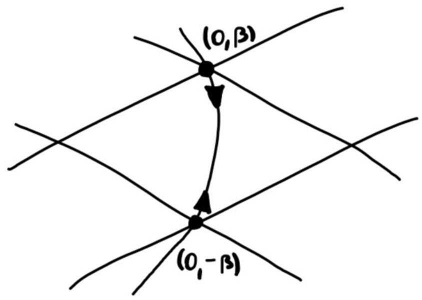
$$X\colon \begin{cases} z_0 z_3 + az_1^2 - z_2^2 \\ (as_0^2 - r_0^2)z_0^2 + (as_0 s_1 - r_0 r_1)z_0 z_1 + (r_1 s_0 - r_0 s_1)z_0 z_2 - 2r_0 z_1 z_2 + 2s_0 z_2^2 - z_3 z_4 \end{cases}$$

$$X \dashrightarrow U\colon x = z_1/z_0,\ y = z_2/z_0$$
$$\text{singularities: } (0:0:0:0:1)$$

3.7.5 Type V



To get X' from Z, blow up two points P_1, P_2 conjugate over a quadratic extension $k(\beta)/k$ or both k-rational (in this case set $\beta = 1$), and then the intersection of the exceptional locus with a degree $(1,1)$ curve C defined over k passing through both points. Geometrically, this is similar to the case of an Iskovskih surface with two (-1)-curves which can be immediately blown down, the difference being that instead of having two conjugate degree $(1,1)$ curves, this time it is the same. Thus, we proceed in the same way. As usual, assuming rationality Z is either $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or Q_α; the former case is covered by setting $\alpha = 1$ in the subsequent calculations.



For now, assume that $k(\beta) \neq k(\alpha)$. As in the case of an Iskovskih surface, up to a k-automorphism of Z, we can assume that P_1, P_2 have (x, y) coordinates $(0, \pm\beta)$ and that

$$C\colon y^2 - ax^2 - b + Ax$$

where $A \in k$.

$$\begin{aligned} H^0(Z, \mathcal{F}_0)|_U &= \Big(y + \beta(\frac{1}{2b}Ax - 1), x^2\Big) \cap \Big(y - \beta(\frac{1}{2b}Ax - 1), x^2\Big) \\ &= (y^2 + Ax - b, x^2) \\ H^0(Z, \pi_*\omega_{X'}^{-1})|_U &= \langle x^2,\ y^2 - ax^2 - b + Ax,\ x(y^2 - ax^2 - b + Ax) \\ &\quad y(y^2 - ax^2 - b + Ax),\ (y^2 - ax^2 - b + Ax)^2,\ \rangle \\ X\colon &\begin{cases} z_2^2 - z_0 z_4 \\ bz_1^2 - Az_1 z_2 - z_3^2 + az_0 z_4 + z_1 z_4 \end{cases} \\ X \dashrightarrow Z\colon & x = z_2/z_1,\ y = z_3/z_1 \\ \text{singularities}\colon & (0:1:0:\beta:0), (0:1:0:-\beta:0), (1:0:0:0:0) \end{aligned}$$

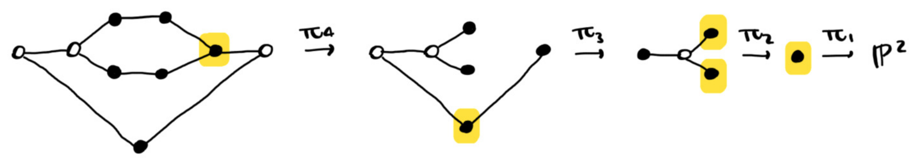
Suppose now instead that P_1, P_2 are conjugate under $k(\alpha)/k$. Like before, let $P_1, P_2 = (\pm\alpha, 0)$ and

$$C\colon y^2 - ax^2 + a^2 + Ay.$$

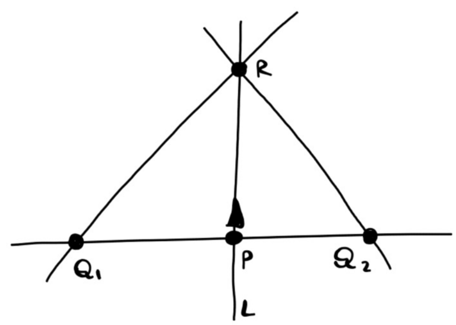
$$\begin{aligned} H^0(Z, \mathcal{F}_0)|_U &= \Big(\alpha x - a - \frac{A}{2a}y - \frac{\alpha}{2a}By, y^2\Big) \cap \Big(\alpha x - a - \frac{A}{2a}y + \frac{\alpha}{2a}By, y^2\Big) \\ &= (y^2 - ax^2 + a^2 + Bxy + Ay, y^2) \\ H^0(Z, \pi_*\omega_{X'}^{-1})|_U &= \langle y^2,\ y^2 - ax^2 + a^2 + Ay,\ x(y^2 - ax^2 + a^2 + Ay), \\ &\quad y(y^2 - ax^2 + a^2 + Ay),\ (y^2 - ax^2 + a^2 + Ay)^2\rangle \\ X\colon &\begin{cases} z_3^2 - z_0 z_4 \\ a^2 z_1^2 - az_2^2 + Az_1 z_3 + z_0 z_4 - z_1 z_4 \end{cases} \end{aligned}$$

$$X \dashrightarrow Z\colon x = z_2/z_1,\ y = z_3/z_1$$
$$\text{singularities}\colon (0:1:0:\alpha:0), (0:1:0:-\alpha:0), (1:0:0:0:0)$$

3.7.6 Type VI



The last blow-down contracts a curve defined over k to a rational point P, hence $Z = \mathbb{P}^2_k$. To obtain X' from Z, blow up P, then two points $Q_1, Q_2 \in Z$ such that P, Q_1, Q_2 are aligned, and then a point on the exceptional curve arising from P; now there is a new exceptional curve E, and a unique line $L \subset Z$ has become a (-1)-curve: to conclude, blow up a rational point R on $L \setminus E$.



We may suppose that, in (x, y) coordinates,

$$P = (0,0), \quad Q_1, Q_2 = (\pm\alpha, 0), \quad L\colon x = 0, \quad R = (0,1)$$

where $\alpha^2 = a \in k$.

$$\begin{aligned} H^0(Z, \mathcal{F}_0)|_U &= (x^2 - a, y) \cap (x, y^2) \cap (x, y-1) \\ &= (xy, y^3 - y^2, x^3 - ax) \\ H^0(Z, \pi_*\omega_{X'}^{-1})|_U &= \langle xy,\ x^2y,\ xy^2,\ y^3 - y^2,\ x^3 - ax \rangle \end{aligned}$$

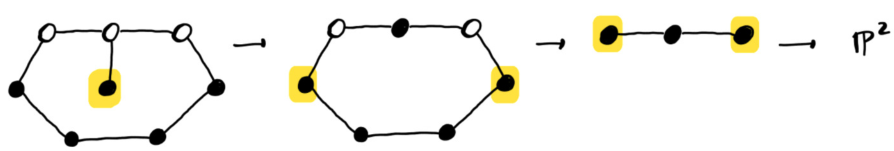
$$X\colon \begin{cases} z_1 z_3 - z_2^2 + z_0 z_2 \\ z_2 z_4 - z_1^2 + a z_0^2 \end{cases}$$

$$X \dashrightarrow Z\colon x = z_1/z_0,\ y = z_2/z_0$$

$$\text{singularities}\colon (0:0:0:1:0), (0:0:0:0:1)$$

$$\text{lines}\colon \begin{cases} z_1 - \alpha z_0 \\ z_2 \\ z_3 \end{cases} \quad \begin{cases} z_1 + \alpha z_0 \\ z_2 \\ z_3 \end{cases}$$
$$\begin{cases} \alpha z_2 - \alpha z_0 + z_1 \\ a z_3 + \alpha z_0 - z_1 \\ z_4 + a z_0 + \alpha z_1 \end{cases} \quad \begin{cases} \alpha z_2 - \alpha z_0 - z_1 \\ a z_3 - \alpha z_0 - z_1 \\ z_4 + a z_0 - \alpha z_1 \end{cases}$$
$$\begin{cases} z_1 \\ z_2 - z_0 \\ z_4 + a z_0 \end{cases} \quad \begin{cases} z_0 \\ z_1 \\ z_2 \end{cases}$$

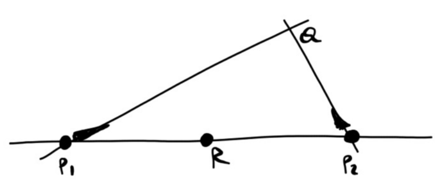
3.7.7 Type VII



Notice that a curve defined over k is blown down; this means that $X'(k) \neq \varnothing$ and that $Z = \mathbb{P}^2_k$. To get X' from Z, blow up two points P_1, P_2; the line L passing through both points is now a (-1)-curve. Blow up two points on the exceptional locus, one for each point, not on L: now, two lines $L_1, L_2 \subset Z$ has become (-1)-curves. These two lines meet at a rational point $Q \in Z$. Finally, blow up a rational point $R \in Z$ such that P_1, P_2, R are aligned. By Lemma 3.5.1, we can suppose that $P_1, P_2 = (\pm\alpha, 0)$ with $\alpha^2 = a \in k^\times$, that Q is at infinity such that

$$L_1, L_2\colon y = \pm\alpha,$$

and that $R = (0,0)$.



$$H^0(Z, \mathcal{F}_0)|_U = (x^2 - a, y^2) \cap (x,y) = (y^2, x(x^2-a), y(x^2-a))$$
$$H^0(Z, \pi_*\omega_{X'}^{-1})|_U = \langle y^2,\ xy^2,\ y^3,\ (x^2-a)x,\ (x^2-a)y \rangle$$

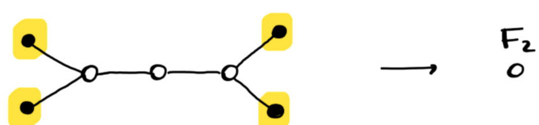
$$X\colon \begin{cases} z_1z_4 - z_2z_3 \\ z_2z_4 - z_1^2 + az_0^2 \end{cases}$$

$$X \dashrightarrow Z\colon x = z_1/z_0,\ y = z_2/z_0$$

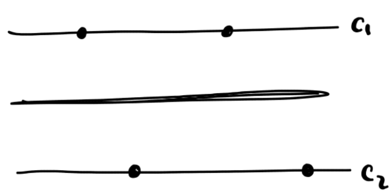
$$\text{singularities}\colon (0:0:0:1:0)$$

$$\text{lines}\colon \begin{matrix} \begin{cases} z_1 - \alpha z_0 \\ z_2 \\ z_4 \end{cases} & \begin{cases} z_1 - \alpha z_0 \\ z_3 \\ z_4 \end{cases} & \\ \begin{cases} z_1 + \alpha z_0 \\ z_2 \\ z_4 \end{cases} & \begin{cases} z_1 + \alpha z_0 \\ z_3 \\ z_4 \end{cases} & \begin{cases} z_0 \\ z_1 \\ z_2 \end{cases} \end{matrix}$$

3.7.8 Type VIII



After the blow-down π we get the Hirzebruch surface F_2, and assuming $X'(k) \neq \varnothing$, $Z = F_{2,k}$. The morphism π is the blow-up of Z in four points, two lying on a curve C_1 of degree $(1,0)$ and two on another C_2 of the same degree. The two curves are either individually defined over k or not.



$\boxed{C_1, C_2 \textbf{ are defined over } k}$

Up to k-automorphisms, the two curves are $x = \pm 1$, and the four points are (in (x, y) coordinates)

$$(1, A),\ (1, B),\ (-1, C),\ (-1, D)$$

where $A, B, C, D \in \overline{k}$ with $A \neq B$, $C \neq D$, such that

$$(t - A)(t - B) =: t^2 + s_1t + s_0, \quad (t - C)(t - D) =: t^2 + r_1t + r_0$$

have coefficients in k.

$$\begin{aligned} H^0(Z,\mathcal{F}_0)|_U &= (x-1,y-A)\cap(x-1,y-B)\cap(x+1,y-C)\cap(x+1,y-D)\\ &= (x^2-1,(x-1)(y^2+r_1y+r_0),(x+1)(y^2+s_1y+s_0))\\ H^0(Z,\pi_*\omega_{X'}^{-1})|_U &= \langle y(x^2-1),\ y^2(x^2-1),\ xy^2(x^2-1),\ x^2y^2(x^2-1),\\ &\quad r_0(x+1)(y^2+s_1y+s_0)-s_0(x-1)(y^2+r_1y+r_0)\rangle \end{aligned}$$

$$X\colon \begin{cases} z_2^2-z_1z_3\\ (r_0-s_0)z_1z_2+(r_0s_1-s_0r_1)z_0z_2+(r_0+s_0)z_1^2+(r_0s_1+s_0r_1)z_0z_1+2r_0s_0z_0^2-z_4(z_3-z_1) \end{cases}$$

$$\begin{aligned} &X \dashrightarrow Z\colon x=z_2/z_1,\ y=z_1/z_0\\ &\text{singularities: } (0:0:0:0:1)\\ &\text{lines: } \begin{cases} z_1-Az_0\\ z_2-z_1\\ z_3-z_2 \end{cases} \begin{cases} z_1-Bz_0\\ z_2-z_1\\ z_3-z_2 \end{cases}\\ &\phantom{\text{lines: }} \begin{cases} z_1+Cz_0\\ z_2+z_1\\ z_3+z_2 \end{cases} \begin{cases} z_1+Dz_0\\ z_2+z_1\\ z_3+z_2 \end{cases} \end{aligned}$$

$\boxed{C_1, C_2 \textbf{ are conjugate over } k(\alpha)/k}$

We may assume that

$$C_1\colon y=\alpha x,\quad C_2\colon y=-\alpha x$$

with intersection $(0,0)$, and that the four blown-up points are

$$(\alpha,A),\quad (\alpha,B),\quad (-\alpha,A^\sigma),\quad (-\alpha,B^\sigma)$$

where $A,B\in\overline{k}$ are distinct and non-zero such that

$$(t-A)(t-B)=:t^2+s_1t+s_0$$

has coefficients in $k(\alpha)$, and σ is an extension of the conjugation on $k(\alpha)/k$. Let

$$\begin{aligned} s_0 &= AB+A^\sigma B^\sigma,\ r_0=\alpha(AB-A^\sigma B^\sigma),\\ s_1 &= -(A+B+A^\sigma+B^\sigma),\ r_1=-\alpha(A+B-A^\sigma-B^\sigma). \end{aligned}$$

$$\begin{aligned} H^0(Z,\mathcal{F}_0)|_U &= (x-\alpha,y-A)\cap(x-\alpha,y-B)\cap(x+\alpha,y-A^\sigma)\cap(x+\alpha,y-B^\sigma)\\ &= (x^2-a,2xy^2+s_1xy+s_0x+r_1y+r_0,\\ &\quad r_1xy+r_0x+2ay^2+as_1y+as_0,\\ &\quad (y-A)(y-B)(y-A^\sigma)(y-B^\sigma)) \end{aligned}$$

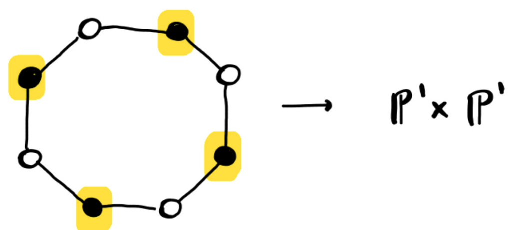
$$\begin{aligned}H^0(Z, \pi_*\omega_{X'}^{-1})|_U = \langle & y(x^2-a),\ y^2(x^2-a),\ xy^2(x^2-a),\ x^2y^2(x^2-a),\\ & 2r_0xy^2 - 2as_0y^2 + (s_1r_0 - s_0r_1)xy + (r_0r_1 - as_0s_1)y + (r_0^2 - as_0^2)\rangle\end{aligned}$$

$$X\colon \begin{cases} z_2^2 - z_1z_3 \\ (r_0^2 - as_0^2)z_0^2 + (r_0r_1 - as_0s_1)z_0z_1 - 2as_0z_1^2 + (s_1r_0 - s_0r_1)z_0z_2 + 2r_0z_1z_2 + az_1z_4 - z_3z_4 \end{cases}$$

$$X \dashrightarrow Z\colon x = z_2/z_1,\ y = z_1/z_0$$

singularities: $(0:0:0:0:1)$

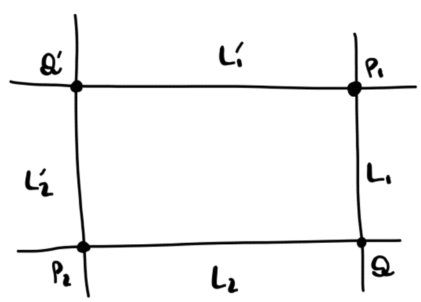
3.7.9 Type IX



Assuming rationality, Z is either $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or Q_α. To recover X' from Z, blow up the four points which form the intersection between two degree $(1,0)$ curves L_1, L_2 and two degree $(0,1)$ curves L'_1, L'_2. Let

$$P_1 = L_1 \cap L'_1,\ P_2 = L_2 \cap L'_2,\ Q = L_1 \cap L_2,\ Q' = L'_1 \cap L'_2.$$

$\boxed{Z = \mathbb{P}^1_k \times \mathbb{P}^1_k}$

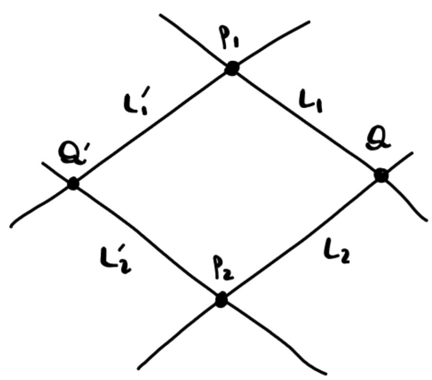


We can suppose that L_1, L_2, L'_1, L'_2 are $x = \pm\alpha,\ y = \pm\beta$ where $\alpha^2 = a \in k^\times,\ \beta^2 = b \in k^\times$.

$$H^0(Z, \mathcal{F}_0)|_U = (x-\alpha, y-\beta) \cap (x+\alpha, y-\beta) \cap (x-\alpha, y+\beta) \cap (x+\alpha, y+\beta)$$

$$
\begin{aligned}
&= (x^2-a, y^2-b)\\
H^0(Z,\pi_*\omega_{X'}^{-1})|_U &= \langle x^2-a,\ y(x^2-a),\ y^2(x^2-a),\ y^2-b,\ x(y^2-b)\rangle\\
X\colon &\begin{cases} z_1^2 - z_0z_2 \\ bz_0z_3 - z_2z_3 - az_3^2 + z_4^2 \end{cases}\\
X \dashrightarrow Z\colon & x = z_4/z_3,\ y = z_1/z_0\\
\text{singularities}\colon & (0:0:0:1:\alpha), (0:0:0:1:-\alpha),\\
& (1:\beta:b:0:0), (1:-\beta:b:0:0)\\
\text{lines}\colon & \begin{matrix} \begin{cases} z_1-\beta z_0 \\ z_2 - bz_0 \\ z_4 - \alpha z_3 \end{cases} & \begin{cases} z_1-\beta z_0 \\ z_2 - bz_0 \\ z_4 + \alpha z_3 \end{cases} \\ \begin{cases} z_1+\beta z_0 \\ z_2 - bz_0 \\ z_4 - \alpha z_3 \end{cases} & \begin{cases} z_1+\beta z_0 \\ z_2 - bz_0 \\ z_4 + \alpha z_3 \end{cases} \end{matrix}
\end{aligned}
$$

$\boxed{Z = Q_\alpha}$



In this setting, every degree $(1,0)$ curve is $k(\alpha)/k$-conjugate with a degree $(0,1)$ curve; suppose that L_1, L_1' are conjugate, and similarly for L_2, L_2'. Then $\{P_1, P_2\}$ is defined over k, so there exists $\beta^2 = b \in k^\times$ such that P_1, P_2 are $k(\beta)$-rational. Notice that $\alpha \notin k(\beta)$, as P_1, P_2 cannot be related by the $k(\alpha)/k$-conjugation (as each of them is the intersection of two $k(\alpha)/k$-conjugate curves) so $Z_{k(\beta)} = \mathbb{P}^1_{k(\beta)} \times \mathbb{P}^1_{k(\beta)} \wedge k(\alpha,\beta)$. Thus, by Lemma 3.5.5 we can suppose $P_1, P_2 = (0, \pm\beta)$. The points Q, Q' are thus necessarily $(\pm\beta\alpha^{-1}, 0)$ $\alpha^2 = a \in k^\times$, $\beta^2 = b \in k^\times$.

$$H^0(Z,\mathcal{F}_0)|_U = (x, y-\beta) \cap (x, y+\beta) \cap (x - \beta\alpha^{-1}, y) \cap (x + \beta\alpha^{-1}, y)$$

$$\begin{aligned} &= (xy, y^2 + ax^2 - b, y^3 - by) \\ H^0(Z, \pi_* \omega_{X'}^{-1})|_U &= \langle xy,\ y^2 + ax^2 - b,\ x(y^2 - ax^2 + b), \\ &\quad y(y^2 - ax^2 - b),\ (y^2 - ax^2)^2 - b^2 \rangle \end{aligned}$$

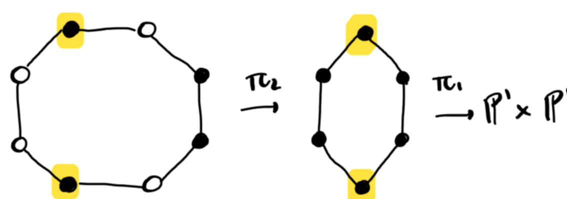
$$X\colon \begin{cases} z_2 z_3 - z_0 z_4 \\ 4ab z_0^2 - b z_1^2 - a z_2^2 - z_3^2 + z_1 z_4 \end{cases}$$

$$\begin{aligned} X \dashrightarrow Z\colon X &= y - \alpha x = (\alpha z_2 + z_3)/(2\alpha z_0 + z_1), \\ Y &= y + \alpha x = (\alpha z_2 - z_3)/(2\alpha z_0 - z_1) \end{aligned}$$

$$\begin{aligned} \text{singularities}\colon &(1 : 2\alpha : 2\beta : 2\alpha\beta : 4\alpha b), (1 : -2\alpha : 2\beta : -2\alpha\beta : -4\alpha b), \\ &(1 : 2\alpha : -2\beta : -2\alpha\beta : 4\alpha b), (1 : -2\alpha : -2\beta : 2\alpha\beta : -4\alpha b) \end{aligned}$$

$$\text{lines}\colon \begin{matrix} \begin{cases} z_2 - 2\beta z_0 \\ z_3 - \beta z_1 \\ z_4 - 2\beta z_3 \end{cases} & \begin{cases} z_2 + 2\beta z_0 \\ z_3 + \beta z_1 \\ z_4 + 2\beta z_3 \end{cases} \\ \begin{cases} z_2 + \beta\alpha^{-1} z_1 \\ z_3 + 2\alpha\beta z_0 \\ z_4 - 2b z_1 \end{cases} & \begin{cases} z_2 - \beta\alpha^{-1} z_1 \\ z_3 - 2\alpha\beta z_0 \\ z_4 - 2b z_1 \end{cases} \end{matrix}$$

3.7.10 Type X

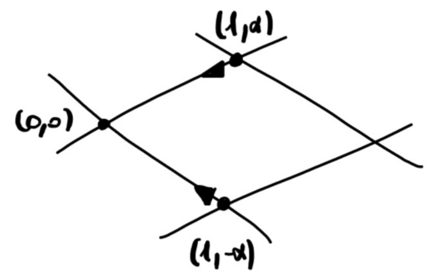


Note that the set of blown-down curves through π_1 is defined over k, because it is the image through π_2 of the two isolated (-2)-curves. To recover X' from Z, blow up two points P_1, P_2 not on the same degree $(1,0)$ or $(0,1)$ curve. Two degree $(1,0)$, resp. $(0,1)$ curves have now self-intersection number -1; name them L_1, L_2', resp. L_1', L_2. Proceed by blowing up the intersection of the exceptional locus with $L_1 \cup L_2$; notice that in this way the intersection point $Q = L_1 \cap L_2$ is k-rational, so $X'(k) \neq \varnothing$ and Z is either $\mathbb{P}^1_k \times \mathbb{P}^1_k$ or Q_α.

$\boxed{Z = \mathbb{P}^1_k \times \mathbb{P}^1_k}$

The two points P_1, P_2 are k-rational: if they were conjugate over a quadratic extension, then so would Q and another point Q', which is absurd. Thus, we can suppose that $P_1 = (1,1),\ P_2 = (-1,-1)$ on (x,y) coordinates. This is just what follows with $\alpha = 1$.

$\boxed{Z = Q_\alpha}$



The rational point Q can be assumed to be $(0,0)$ in (x,y) coordinates. The points P_1, P_2 are necessarily $k(\alpha)/k$-conjugate (as each $(1,0)$ and $(0,1)$ have only one rational point, and in the case of L_1, L_2 it is Q), lying on the curves $y = \alpha x,\ y = -\alpha x$, and up to a k-automorphism fixing Q are $(1, \pm\alpha)$.

$$\begin{aligned}
H^0(Z, \mathcal{F}_0)|_U &= (y - \alpha x, (x-1)^2) \cap (y + \alpha x, (x-1)^2) \\
&= (y^2 - ax^2, (x-1)^2)
\end{aligned}$$

$$H^0(Z, \pi_* \omega_{X'}^{-1})|_U = \langle y^2 - ax^2,\ x(y^2 - ax^2),\ y(y^2 - ax^2),\ (y^2 - ax^2)^2,\ (x-1)^2 \rangle$$

$$X \colon \begin{cases} z_0 z_3 + a z_1^2 - z_2^2 \\ a z_0^2 - 2a z_0 z_1 + z_2^2 - z_0 z_3 - a z_3 z_4 \end{cases}$$

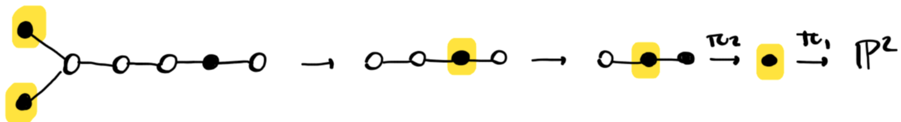
$$X \dashrightarrow Z \colon x = z_1/z_0,\ y = z_2/z_0$$

$$\text{singularities:}\ (1:1:\alpha:0:0), (1:1:-\alpha:0:0), (0:0:0:0:1)$$

$$\text{lines:}\ \begin{matrix} \begin{cases} z_2 + \alpha z_1 - 2\alpha z_0 \\ z_3 - 4a z_0 + 4a z_1 \\ 4a z_4 - z_0 + z_1 \end{cases} & \begin{cases} z_2 - \alpha z_1 + 2\alpha z_0 \\ z_3 - 4a z_0 + 4a z_1 \\ 4a z_4 - z_0 + z_1 \end{cases} \\ \begin{cases} z_1 - z_0 \\ z_2 - \alpha z_0 \\ z_3 \end{cases} & \begin{cases} z_1 - z_0 \\ z_2 + \alpha z_0 \\ z_3 \end{cases} \end{matrix}$$

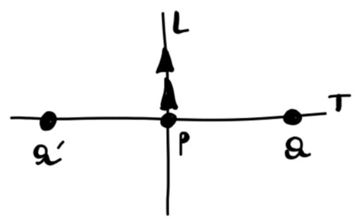
3.7.11 Type XI

Notice that π_1 is the contraction of a curve defined over k, hence $X'(k) \neq \varnothing$ and so $Z = \mathbb{P}^2_k$. To recover X' from Z, blow up a rational point P, the a rational point on the exceptional line; the strict transform of a line $L \subset Z$ is now a (-1)-curve. Proceed by blowing up the intersection of L with the exceptional



locus of π_2, and finally blow up two points $Q, Q' \in Z$ such that P, Q, Q' lie on a line T distinct from L. Up to k-automorphisms, we may suppose that in (x, y) coordinates

$$T\colon x = 0,\ L\colon y = 0,\ P = (1, 0),\ Q = (\alpha, 0),\ Q' = (-\alpha, 0).$$

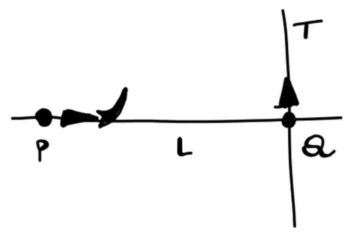


$$\begin{aligned} H^0(Z, \mathcal{F}_0)|_U &= (y^3, x) \cap (x^2 - a, y) \\ &= (xy, y^3, x^3 - ax) \\ H^0(Z, \pi_* \omega_{X'}^{-1})|_U &= \langle xy, x^2y, xy^2, y^3, x^3 - ax \rangle \\ X\colon &\begin{cases} z_2^2 - z_1 z_3 \\ a z_0^2 - z_1^2 + z_2 z_4 \end{cases} \\ X \dashrightarrow Z\colon & x = z_1/z_0,\ y = z_2/z_0 \\ \text{singularities}\colon & (0:0:0:0:1), (0:0:0:1:0) \\ \text{lines}\colon & \begin{cases} z_1 - \alpha z_0 \\ z_2 \\ z_3 \end{cases} \begin{cases} z_1 + \alpha z_0 \\ z_2 \\ z_3 \end{cases} \begin{cases} z_0 \\ z_1 \\ z_2 \end{cases} \end{aligned}$$

3.7.12 Type XII

The last blow-down contracts a curve defined over k to a k-rational point, hence $Z = \mathbb{P}^2_k$. To recover X' from Z, blow up a point P, obtaining an exceptional curve E_1. Blow up a point on E_1, and name E_2 the new exceptional curve. There is a unique line $L \subset Z$ which is now of self-intersection -1. Blow up a point on E_2 distinct from $E_2 \cap L$. Continue by blowing up a point Q on L away from P, and name E_3 the exceptional locus. Finally, blow up a point on $E_3 \setminus L$, turning a line $L' \subset Z$ passing through Q into a (-1)-curve. Up to k-automorphisms and using (x, y) coordinates,

$$P = (0,0),\ L\colon y = 0,\ Q = (1,0),\ T\colon x = 1.$$



By Theorem 3.6.5, there is a unique $a \in k$ such that the curve

$$C\colon y = ax^2$$

has strict transform of self-intersection number $C^2 - 3$. However, note that $a \neq 0$, otherwise $C = L$ becomes a (-3)-curve; and by a change of variables $y = ay'$, we can suppose $a = 1$.

$$\begin{aligned} H^0(Z, \mathcal{F}_0)|_U &= (y - x^2, x^3) \cap (x-1, y^2) \\ &= (y^2, x^2y - xy, x^3 - x^2 - xy + y) \\ H^0(Z, \pi_*\omega_{X'}^{-1})|_U &= \langle y^2, xy^2, y^3, x^2y - xy, x^3 - x^2 - xy + y\rangle \end{aligned}$$

$$X\colon \begin{cases} z_0z_1 - z_1^2 + z_2z_3 \\ z_0^2 - z_0z_1 + z_1z_3 - z_2z_4 \end{cases}$$

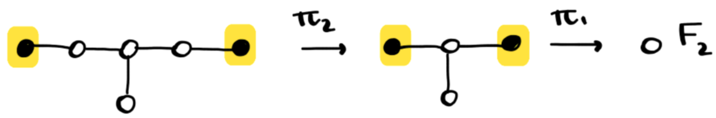
$$X \dashrightarrow Z\colon x = z_1/z_0,\ y = z_2/z_0$$

$$\text{singularities}\colon (0:0:0:0:1)$$

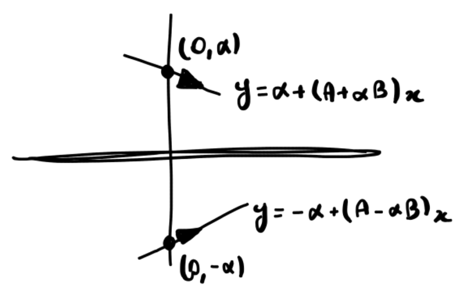
$$\text{lines}\colon \begin{cases} z_0 \\ z_1 \\ z_2 \end{cases} \begin{cases} z_0 - z_1 \\ z_3 \\ z_4 \end{cases} \begin{cases} z_0 - z_2 \\ z_1 \\ z_2 \end{cases}$$

3.7.13 Type XIII

Since the resulting surface is a singular Del Pezzo surface of degree 8 with a (-2)-curve, $\overline{Z} = F_2$. To recover X' from Z, blow up two points P_1, P_2 on the



same degree $(1,0)$ curve L, but not on the (-2)-curve C. Proceed by blowing up two points, one on each exceptional curve, and not on L. Notice that $C \cap L$ is a rational point, hence $Z = F_{2,k}$.



The automorphisms of Z include projective transformations involving only the y coordinate which fix the origin, and projective transformations of the x coordinate. We can use them to suppose that $P_1, P_2 = (0, \pm\alpha)$, where $\alpha^2 = a \in k^\times$. By Theorem 3.6.5, there exist $A, B \in k$ (so that $A + \alpha B$ is a generic element of $k(\alpha)$) such that the blow-up of P_1 and an infinitely close point is associated to the ideal

$$(y - \alpha - (A + \alpha B)x),$$

and the $k(\alpha)/k$-conjugate to P_2 and an infinitely close point.

$$\begin{aligned} H^0(Z, \mathcal{F}_0)|_U &= (y - \alpha - (A+\alpha B)x, x^2) \cap (y + \alpha - (A - \alpha B)x, x^2) \\ &= (y^2 - 2Axy - a - 2Bxy^2, x^2) \end{aligned}$$

$$H^0(Z, \pi_* \omega_{X'}^{-1})|_U = \langle x^2y,\ x^2y^2,\ x^3y^2,\ x^4y^2,\ y^2 - 2Axy - a - 2Bxy^2 \rangle$$

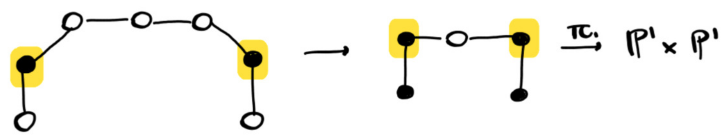
$$X\colon \begin{cases} z_2^2 - z_1z_3 \\ z_1^2 - 2Az_0z_2 - az_0^2 - 2Bz_1z_2 - z_3z_4 \end{cases}$$

$$X \dashrightarrow Z\colon x = z_2/z_1,\ y = z_1/z_0$$

$$\text{singularities}\colon (0:0:0:0:1)$$

$$\text{lines}\colon \begin{cases} z_1 - \alpha z_0 \\ z_2 \\ z_3 \end{cases} \qquad \begin{cases} z_1 + \alpha z_0 \\ z_2 \\ z_3 \end{cases}$$

3.7.14 Type XIV



To get X' from Z, blow up two points P_1, P_2 on the same degree $(1,0)$ curve L; now two curves L_1, L_2 of degree $(1,0)$ are negative. Blow up the intersection of the exceptional locus of π_1 with $L_1 \cup L_2$. Since by this construction a different number of degree $(1,0)$ and $(0,1)$ curves become negative, Z cannot be a twisted product, hence $Z = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Up to k-automorphisms, using (x,y) coordinates $P_1, P_2 = (\pm\alpha, 0)$ and $L_1, L_2\colon x = \pm\alpha$.

$$H^0(Z, \mathcal{F}_0)|_U = (x-\alpha, y^2) \cap (x+\alpha, y^2) = (x^2 - a, y^2)$$

$$H^0(Z, \pi_*\omega_{X'}^{-1})|_U = \langle y^2,\ xy^2,\ x^2y^2,\ x^2 - a,\ y(x^2 - a)\rangle$$

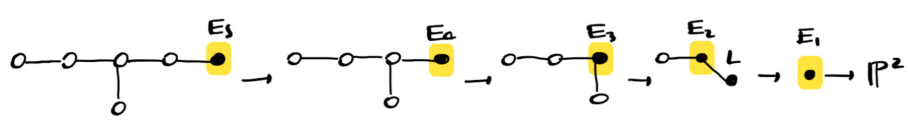
$$X\colon \begin{cases} z_1^2 - z_0 z_2 \\ z_4^2 - z_2 z_3 + a z_0 z_3 \end{cases}$$

$$X \dashrightarrow Z\colon x = z_1/z_0,\ y = z_4/z_3$$

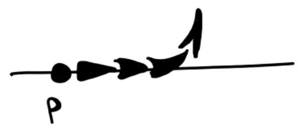
$$\text{singularities}\colon (0:0:0:1:0), (1:\alpha:a:0:0), (1:-\alpha:a:0:0)$$

$$\text{lines}\colon \begin{cases} z_1 - \alpha z_0 \\ z_2 - a z_0 \\ z_4 \end{cases} \quad \begin{cases} z_1 + \alpha z_0 \\ z_2 - a z_0 \\ z_4 \end{cases}$$

3.7.15 Type XV



Notice that every blow-down is a contraction of a curve defined over k, so $X'(k) \neq \varnothing$ and $Z = \mathbb{P}^2_k$. To get X' from Z, start by blowing up a point P and an infinitely close point, resulting in a new exceptional curve E. Now a line $L \subset Z$ is negative: blow up the point $E \cap L$, and keep blowing up points infinitely close to the previous blown-up point and not on the previous exceptional curve or on L.



We can assume $P = (0,0)$ and $L\colon y = 0$. By Theorem 3.6.5, there are unique $a_3, a_4 \in k$ (with $a_3 \neq 0$, as it should not lie on L) such that

$$C\colon y - a_3x^3 - a_4x^4$$

has strict transform with self-intersection number $C^2 - 5$. By setting $y = a_3y'$, the equation becomes

$$y' - x^3 - ax^4$$

with $a = a_4/a_3 \in k$, and if $a \neq 0$, by setting $x = a^{-1}x'$, $y' = a'^{-3}y''$ the equation is now

$$y'' - x'^3 - x'^4.$$

Thus, the only potential cases we have to deal with are

$$H^0(Z, \mathcal{F}_0)|_U = (y - x^3 - x^4, x^5), \quad H^0(Z, \mathcal{F}_0)|_U = (y - x^3, x^5)$$

depending on whether a is zero or not. However, we perform the calculations using a generic $a \in k$, and then show that actually $X_{a=0}$ is isomorphic to $X_{a=-2}$, hence X does not depend on a up to isomorphisms.

$$H^0(Z, \mathcal{F}_0)|_U = (y - x^3 - ax^4, x^5) = (x^3 + axy - y, x^2y, y^2)$$

$$H^0(Z, \pi_*\omega_{X'}^{-1})|_U = \langle y^2,\ xy^2,\ y^3,\ x^2y,\ x^3 + axy - y\rangle$$

$$X\colon \begin{cases} z_1^2 - z_2z_3 \\ z_0^2 - az_0z_1 - z_1z_3 + z_2z_4 \end{cases}$$

$$X \dashrightarrow Z\colon x = z_1/z_0,\ y = z_2/z_0$$

$$\text{singularities}\colon (0:0:0:0:1)$$

$$\text{lines}\colon \begin{cases} z_0 \\ z_1 \\ z_2 \end{cases}$$

Starting from

$$X_{a=0}\colon \begin{cases} z_1^2 - z_2 z_3 \\ z_0^2 - z_1 z_3 + z_2 z_4 \end{cases}$$

set $z_0 = z_0' + z_1,\ z_4 = z_4' - z_3$ to get

$$\begin{cases} z_1^2 - z_2 z_3 \\ z_0'^2 + 2z_0' z_1 - z_1 z_3 + z_2 z_4' \end{cases}$$

which is $X_{a=-2}$.

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