Laurea MAGISTRALE in COMPUTER SCIENCE

ARTIFICIAL INTELLIGENCE

Logics / 2

Modal Logics – Non-Monotonic Reasoning, Reasoning under Uncertainty

Non-classic Logics

- *Classic* logic
  - Propositional logic
  - First-Order Predicate logic
- *Non-classic* logics adopt different or more additional rules
  - To solve different problems than deductive calculus
  - To represent other forms of reasoning
    - Weaker
    - Linked to contextual factors

Extension/Modification Directions

- Using classical logic in a different way
- Abandoning truth-functionality principle
  - Truth value of a proposition not required anymore to be only a function of its components
  - Dismission of truth tables
- Abandoning bivalence principle
  - A proposition is not assumed anymore to be only true or false

Assumptions in Predicate Logic Reasoning

- 1. descriptions in the form of predicates must be “sufficient” to represent the application domain
  - Each fact is known to be true or false
  - What if information is incomplete or uncertain?
    - Logic programming (CWA): if a fact cannot be proved true, it is assumed to be false
    - Human reasoning: if a fact cannot be proven false, it is assumed to be true
- 2. knowledge base must be consistent
  - In human reasoning, alternate hypotheses are considered, even conflicting ones, and are removed as soon as new evidence is available

Assumptions in Predicate Logic Reasoning

- 3. knowledge monotonically grows through the use of inference rules
  - Need for mechanisms to
    - Add knowledge based on assumptions
      - Non-monotonic reasoning
    - Delete inferences based on such assumptions, in case subsequent evidence shows the hypothesized assumptions are wrong
      - Truth maintenance

Minsky’s Criticisms to Logic

- “logical” reasoning not flexible for thinking
- Cannot handle inconsistent data
- Feasibility of representing knowledge by “true” propositions is doubtful
- Separation of knowledge and rules too radical
- Logic is monotonic
- Procedural descriptions over declarative descriptions

Classical Reasoning

- Systems based on predicate logic are conceptually elegant and rigorous
- Albeit logic is often unsuitable to represent common knowledge, it is used for its ease in automatizing the deduction process
- The *truth* of an assumed or derived assertion is such *indefinitely* and assumes the existence of a static world of objects
- Propositions in classical logic are declarative and atemporal
  - E.g.: Property *P* holds for object *x*
- In real problems less strict claims may happen

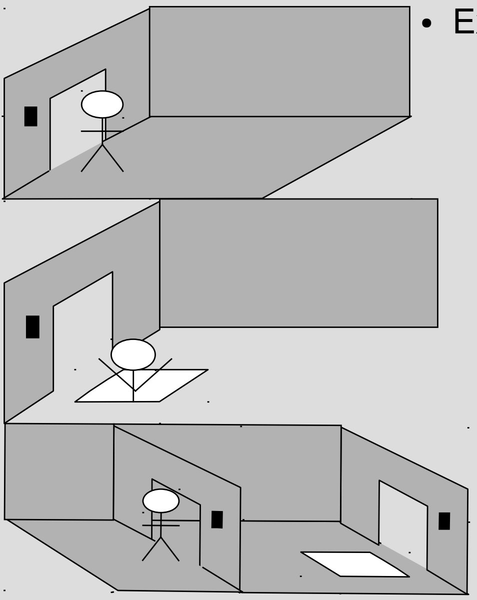
- Moreover, one does not want / cannot express
  - that some property is necessarily true, or
  - the difference between believing, supposing and knowing something
    - E.g., when formalizing juridic norms one must express
      - That taking an action is *prohibited* or *permitted*
      - And that if someone takes a prohibited action he *must* be punished
  - To represent common-sense reasoning, *modal formalism* were defined that allow to express in what way a proposition is true
    - Propositions may be characterized by a *mode* (necessary, possible, compulsory, ...)

Modal Logics

- A family of logics
  - *Temporal*, handles propositions such as "It is always true that p", "Sometimes it is true that p", ...
  - *Deontic*, handles propositions such as "*p* is mandatory", "*p* is permitted", ...
  - *Epistemic*, handles propositions such as "I know that *p*", "I believe that *p*", ...
  - *Alethic*, handles propositions such as "*p* is necessary", "*p* is possible", ...
  - *Ethic*, handles propositions such as "*p* is good", "*p* is bad", ...

Modal Logics

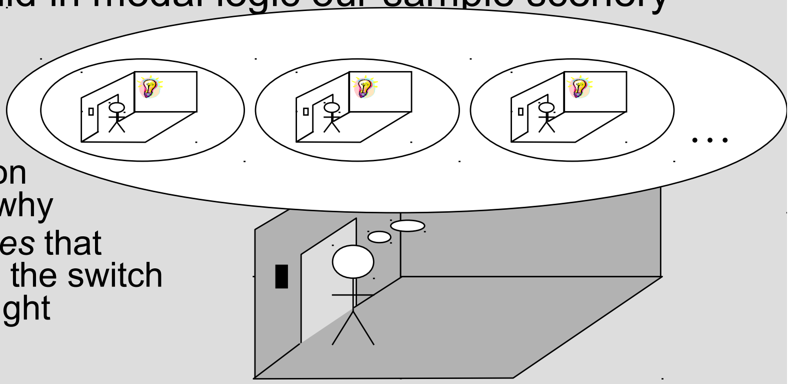
- Allows to model/formalize intelligence
  - To build intelligent agents that can operate in dynamic environments, like humans do



- Example:
  - Consider an agent moving in a dark room and, gropingly, discovering there is a switch on the wall
  - Understandably, the agent thinks the switch turns on the light...
  - Instead, after pressing it, a trapdoor opens beneath him and he falls down...
  - The agent escapes the trapdoor and, gropingly (he did not turn on the light), he reaches another room, again dark...
  - He finds another switch... but this time he thinks twice before turning it on!

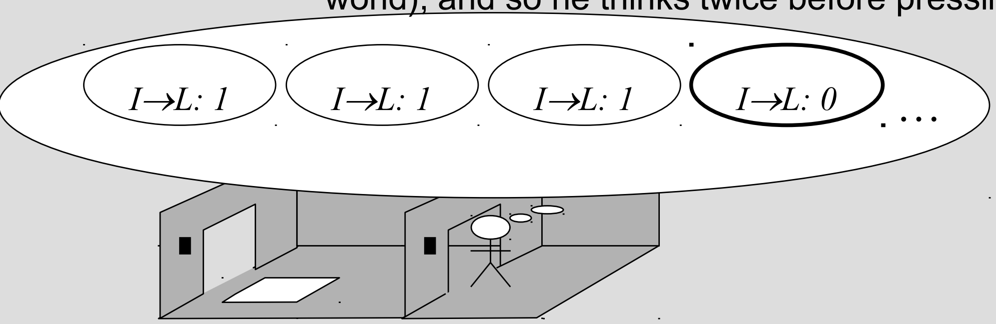
Modal Logics

- Agent's action was wrong (he shouldn't turn on the switch), not stupid
  - Objective: building *rational*, not *omniscient* software agents
    - At any time, makes the choice it believes is the best
  - Modal logics allow to build *rational agents*
    - Let us try and build in modal logic our sample scenery
      - In all rooms in which the agent was in the past, the switch on the wall turned on the light; this is why the agent *believes* that also in this case the switch will turn on the light



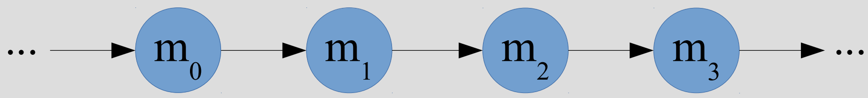
Modal Logics

- The rooms in which the agent remembers having been are just worlds accessible to him. In them, formula $I \rightarrow L$ holds, where I is fact "pushed button" and L "light is on"
- As a consequence, in the current room, i.e. the world in which the agent is moving, ($I \rightarrow L$) is *necessarily/probably* true, even if, as the agent will discover, in this world <u>$I \rightarrow L$ is false</u>
- **Warning!** The agent has no access to the world in which he moves: he falls in the trapdoor because he cannot see that, in it, $I \rightarrow L$ is false
- When the agent is in the new room, he recalls that, in the previous one, $I \rightarrow L$ was false (now, he has access to an additional possible world), and so he thinks twice before pressing the button



Temporal Modal Logic

- Allows representing knowledge with temporal references
- Extend classical logic with operators such as
  - ☐A   "A is always true"
  - ◊A   "A will be sooner or later true"
  - ○A   "A will be true in the next temporal moment"
- Interpreted in a structure consisting of a discrete set of temporal intervals (*worlds*)



Temporal Operators

- Meaning defined by determining when a formula is true in a world *w*
  - ☐A   is true in *w* if it is true at timepoint *w* itself and in all subsequent ones
  - ◊A   is true in *w* if it is true at timepoint *w* itself or there exist a timepoint w' following w in which A is true
  - ○A   is true in *w* if A is true at the timepoint immediately following *w*

- Example: proposition leave(train,rome)
  - "the train to Rome is leaving"
    - (a) $[\neg leave(train,rome)] \Rightarrow [\Box\ \neg leave(train,rome)]$
      - If the train to Rome is not leaving now, then it will never leave
    - (b) $[\neg leave(train,rome)] \Rightarrow [\bigcirc\ \neg leave(train,rome)]$
      - If the train to Rome is not leaving now, then it will leave in a moment
  - It might be appropriate to explicitly add time
    - leave(t,train,rome)
      - In which case the logic must include an arithmetic
      - (a') $\neg leave(0,train,rome) \Rightarrow \forall t\ (0 \leq t \Rightarrow \neg leave(t,train,rome))$
        - Equivalent to (a)
      - (b') $\neg leave(0,train,rome) \Rightarrow leave(1,train,rome)$
        - Equivalent to (b)

- Temporal aspects may also be represented by reifying propositions
  - treating them as objects to talk about

  using *higher level* predicates such as
  - holds(p,t)
    - "p holds at time t"
  - Example: Formulas (a) and (b) above become
    - (a'') $\neg holds(leave(train,rome),0) \Rightarrow \forall t\ (0 \leq t \Rightarrow \neg holds(leave(train,rome),t))$
    - (b'') $\neg holds(leave(train,rome),0) \Rightarrow holds(leave(train,rome),1)$

- One may even talk about the duration of an event, thanks to a predicate such as
  - happens(*p*,*t1*,*t2*)
    - “the duration of event *p* covers the temporal interval between *t1* and *t2*”
  - Example: This allows to express the fact that if the train travels from Rome to Turin in interval (t1,t2), then it arrives in Turin
    - happens(travel(train,rome,turin), t1, t2) $\Rightarrow$ holds(arrive(train,turin), t2)

Deontic Logic

- Allows to express obligation or permission
    - E.g., in juridic norms, one must be able to express
      - Making an action is *prohibited* or *permitted*
      - If someone makes a *prohibited* action he *must* be punished
  - Operators: given a proposition A
    - OA A is mandatory
    - PA A is permitted
      - O $\neg$A A is prohibited
  - Axioms describing operators features
    - PA $\Leftrightarrow$ $\neg$O $\neg$A
      - All and only non-prohibited things are permitted
    - OA $\Rightarrow$ PA
      - What is mandatory must be permitted

Reasoning about Knowledge

- Sometimes one needs to reason about knowledge and about reasoning itself
  - “If I know that the train to Turin leaves at time X I go to the station ten minutes before time X; if I have no idea about the time then I check the Internet”
  - “If it is not possible to prove that the defendant is guilty then one must conclude he is innocent”
- Need for modal logics that can work on a *meta-level*
  - Meta-knowledge
  - Meta-reasoning

Epistemic Modal Logics

- Knowledge $\neq$ Belief
- Operators
  - K denotes what an agent knows
    - KA the agent knows that proposition A is true
  - B denotes what an agent believes
    - BA the agent believes that proposition A is true
  - Examples
    - K $\exists x$ thief(x,C)
      - The agent knows that *someone* stole the picture
    - K $\forall x$ (thief(x,C) $\Rightarrow$ previous_offender(x))
    - $\exists x$ K thief(x,C)
      - The agent knows *who* stole the picture

Epistemic Modal Logics

- What is known is true ($KA \Rightarrow A$), but an agent may believe something false
- Possibility of representing situations with many agents and to use modal operators to represent what agents know or believe about each other
  - Example: agents α and β
    - $K(\alpha, A) \Leftrightarrow K_\alpha(A)$ agent α knows that A is true
    - $B_\alpha(A)$ agent α believes that A is true
    - $K_\alpha K_\beta A$ agent α knows that agent β knows A

Well-Formed Formulas

- Syntax in first-order modal languages
  - All wff in first-order predicate calculus are also wff in the modal language
  - If Φ is a closed formula in the modal language
    - Without free variables

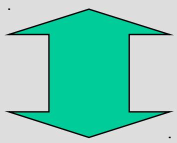
    and α is a ground term, then $K(\alpha,\Phi)$ is a wff in the modal language
  - If Φ and Ψ are wff, then all expressions that may be built from Φ and Ψ using the usual propositional connectives are wff, as well

Monotonic Reasoning

- Scheme based on certainty of deduction
  - Principle 1: In a monotonic reasoning system, the number of propositions proved true continuously increases
    - Knowledge is
      - Complete
        - All facts needed to solve a problem are present or may be derived)
      - Consistent
      - Updatable only by adding new facts that are consistent with those already asserted (monotonicity)

Non-monotonic Reasoning

- Often information may be incomplete, albeit temporarily, or conditions may change in time



- Need to formulate (even incorrect) hypotheses, to restart the reasoning process when it reaches a deadend
- *Non-monotonic* reasoning system
  - Keeps track of a set of hypothetical assumptions and continuously revises them based on new observed or deduced knowledge

Non-monotonic Reasoning

- Classical logic (FOPL)
  - Once a fact is asserted, its' forever true
  - Theorems never decrease with the increase of axioms
- Non-monotonic Reasoning
  - The set of currently true (or believed) facts does NOT increase monotonically
    - Can shrink or grow with reasoning
  - Adding a new fact might lead to an inconsistency
    - Need to remove one of the contradictory facts
  - Hypotheses based on default assumptions evolve as long as new information is acquired
    - Adding new information may change the correctness of the conclusion
    - From a formal viewpoint, adding new axioms to a theory T, not necessarily all theorems in T are preserved

Non-monotonic Reasoning

- Example
  - You are a student, it's 8am, you are in bed.
  - You slip out of your dreams and think: Today is Sunday. No classes today. I don't have to get up. You go back to sleep.
  - You wake up again. It's 9:30am now and it is slowly coming to your mind: Today is Tuesday. What an unpleasant surprise.

P1 = today-is-Tuesday	P2 = today-is-Sunday
P3 = have-class-at-10am	P4 = no-classes
P5 = have-to-get-up	P6 = can-stay-in-bed

Non-monotonic Reasoning

- 3 problems to be solved
  - How to extend the knowledge base so as to allow inferences based both on the absence and on the presence of knowledge
    - It is known that $\neg$P
    - It is not known if P
  - How to update correctly the knowledge base when a new fact is added or an old one is removed
  - How to use knowledge to try and solve conflicts that are generated when different non-monotonic, mutually inconsistent, inferences can be carried out

Non-monotonic Reasoning

- A formalism should:
  - 1. Be able to define the set of worlds tha may exist, given the facts that are certain
    - I.e., allow to define the set of models of any set of wffs
      - Interpretation of a set of wffs:
        a domain (set of objects) D with a function that maps
        - Each predicate to a relationship
        - Each *n*-ary function to an operator from D^n to D
        - Each constant to a member of D
      - Model of a set of wffs: an iterpretation that satisfies them
  - 2. provide a way to choose a model
  - 3. provide the basis for implementing reasoning
  - 4. provide a model whose conclusions correspond to our intuitions

Non-monotonic Reasoning

- Many formalisms proposed
  - Default logic (Reiter, 1987)
  - Non-monotonic logic (McDermott, Doyle, 1980)
  - Autoepistemic logic (Moore, 1985)
  - Circumscription (McCarthy, 1980)
  - Negation as Failure (Clark, 1978)
- All have to do with what we “know” at a given moment
  - Central question in the logic formalism has to do with the *extension* of operator $\exists$

Non-monotonic Reasoning

- Example: “All A’s are B’s”
    - A statement that many logics try to “weaken”, to reach more generality
  - Classical logic
    - $\forall x : x \in A \Rightarrow x \in B$
  - Default logic
    - An A is a B, unless exceptions
  - Non-monotonic modal logic
    - If B is conceivable and if A, then B
  - Autoepistemic logic
    - If a given A were not B, we should know
  - Circumscription
    - Any A which is not abnormal is B
  - Negation as Failure
    - If, given A, B cannot be proved, then B is assumed to be false

Non-monotonic Logic

- Extends the FOPL language with a modal operator M, meaning "is consistent"
  - Example: formula
    - $\forall x,y$: brothers(x,y) $\wedge$ M go_along_well(x,y) $\Rightarrow$ defends(x,y)
    - "For all x's and y's, if x and y are brothers and it is conceivable
      - Consistent with what one believes

      that x goes along well with y, then one concludes that x will defend y"
- Need to define the meaning "is consistent" for the theory to be at least semi-decidable
  - Normally the Prolog notion of Negation as Failure or some variant of it is used
    - To show that P is consistent:
      - Try and prove $\neg$P
      - If failure: assume $\neg$P to be false, declare P as consistent

Non-monotonic Reasoning

- Problem: possibility that many non-monotonic propositions, taken singularly, suggest extensions that are overall inconsistent
  - Example
    - $\forall x$: ignorant(x) $\wedge$ M studies(x) $\Rightarrow$ studies(x)
    - $\forall x$: studies(x) $\wedge$ M passes_test(x) $\Rightarrow$ passes_test(x)
    - $\forall x$: ignorant(x) $\Rightarrow$ $\neg$passes_test(x)
    - ignorant(pippo) $\Rightarrow$ studies(pippo)

      $\Downarrow$ $\Downarrow$
    - $\neg$passes_test(pippo) passes_test(pippo)

Non-monotonic Logics

- Defines the set of theorems that can be derived from a set A of wffs as the intersection of the sets of theorems resulting from all possible ways in which the wffs in A can be combined
- Assertions, albeit looking like rules, are wffs that can be manipulated using the traditional laws to combine logistic expressions
-
- $A \wedge MB \Rightarrow B$
-
- $\neg A \wedge MB \Rightarrow B$

$MB \Rightarrow B$

Default Logics (DL)

- Introduces a new class of rules

$$\frac{A : B}{C}$$

  - "If A is true (provable) and it is consistent to assume B, then conclude C"
    - A *requirement*
    - B *justification*
    - C *conclusion*

Default Logics (DL)

- Example
  - Sentence “Birds typically fly” can be expressed as

$$\frac{\text{bird}(x) : \text{fly}(x)}{\text{fly}(x)}$$

  - or

$$\frac{: \text{bird}(x) \Rightarrow \text{fly}(x)}{\text{bird}(x) \Rightarrow \text{fly}(x)}$$

    - “If it is conceivable that birds fly then I conclude that birds fly”
  - A belief becomes a theorem!

Default Logic (DL)

- Extension: a set of beliefs
  - The new inference rules are used as a basis to compute a set of plausible extensions of the knowledge base
  - Each extension ‘increments’ the knowledge base
    - Maximal consistent augmentation
  - Generated by applying inference rules without violating consistency
  - Logics admits as theorem an expression that is satisfied in some extension
  - If it’s needed to decide among extension, a decision is not made

Default Logic (DL)

- Expressions are inference rules rather than expressions in the language
  - Cannot be manipulated by other inference rules
    - Example: given two rules and no assertion about A
    - From

$$\frac{A \;:\; B}{B} \qquad\qquad \frac{\neg A : B}{B}$$

    - no conclusion can be drawn about B

Non-monotonic Reasoning

- Inheritance
  - Non-monotonic reasoning typically used in object-centered representation in order for instances of a class to inherit some attributes from a typical description of the class itself
  - Algorithm based on the fact that an object inherits an attribute value from all classes to which it belongs, *unless this leads to a contradiction*,
    - in which case a value coming from a more specific class has priority over that of a more general one

Non-monotonic Reasoning

- Use of *delineation* to deduce properties
  - Always requires to proceed from specific to general
  - With such an ordering defined on matching and retrieval processes, information present in lower levels protects the system from contradictory deductions based on higher levels
  - Example
    - Rule that considers inheritance of a default value for the height of a basketball player

$$\frac{\text{basketball\_player}(x)\ :\ \text{height}(x, 1.85)}{\text{height}(x, 1.85)}$$

    - In any case, a declared value blocks inheritance of a default value

Non-monotonic Reasoning

- When conflicts arise due to multiple inheritance, *delineation by default*, using this kind of rules, can be used

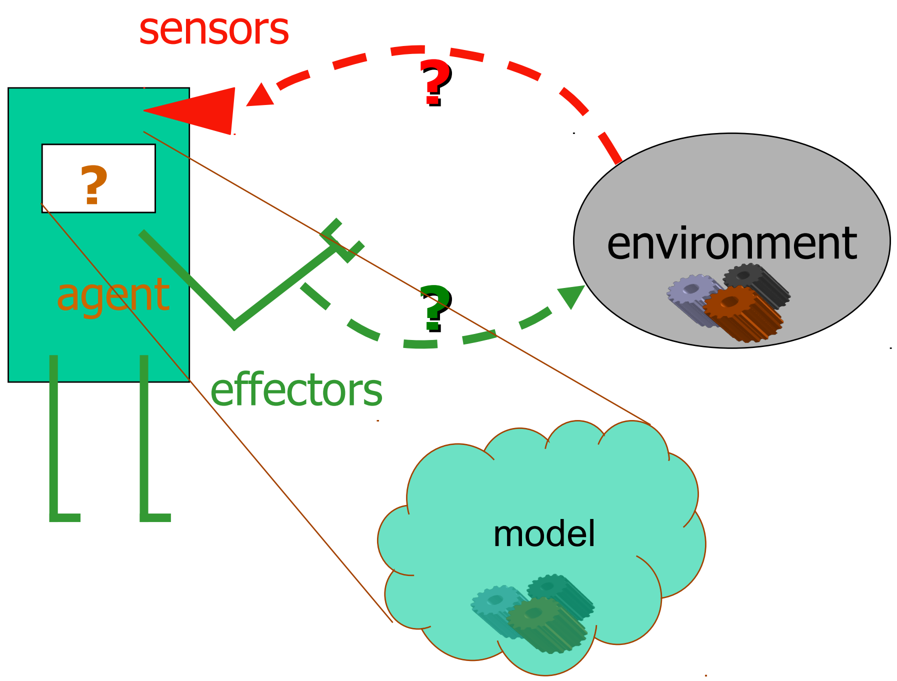
$$\frac{\text{elephant}(x) : \text{color}(x, \text{gray})}{\text{color}(x, \text{gray})}$$

$$\frac{\text{albino}(x) : \text{color}(x, \text{white})}{\text{color}(x, \text{white})}$$

$$\frac{\text{elephant}(x) : \neg\text{albino}(x) \wedge \text{color}(x, \text{gray})}{\text{color}(x, \text{gray})}$$

  - This rule can prevent application of default knowledge about elephants when more specific knowledge coming from class albino is available

Agent under Uncertain Conditions



Kinds of Uncertainty

- Uncertainty in prior knowledge
  - Example: in an expert system for medical diagnosis, some causes of the disease are not known and not represented in the knowledge base
- Uncertainty in action
  - Example: it is expected that the agent must turn on the light when entering a room, but it is necessary/ expected that there is power, that the switch works, that the bulb is not blown, etc.
- Uncertainty in perception
  - Example: sensors do not return the exact position, the environment is not enough well-lit, etc.

Fuzzy Theory

- A mathematical theory that encodes *qualitative* evaluations
  - Particularly suited for what people express in everyday language
  - Allows to deal with knowledge expressed qualitatively
- An extension of Boolean logics to values in the *continuous* range [0,1]
  - A *fuzzy* expression must not be true or false, but several *degrees of truth* are admitted

Fuzzy Logics

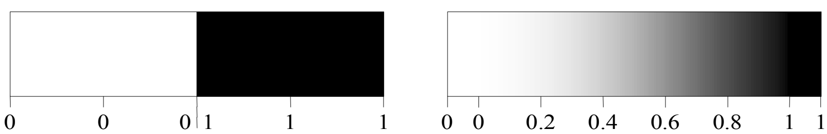
- Used to describe and operate with vague definitions
  - Example (controlling a cement plant)
    - If temperature is high, add few cement and much increase water
- Based on the idea that the elements of a set are defined through a *degree of membership*
  - Increased expressive power: many quantities can be represented in a fuzzy way
    - Examples:
      - The engine is very hot
      - John is very tall

Fuzzy Logics

- Typically used
  - Strictly speaking
    - It is a branch of fuzzy set theory, that deals with knowledge representation and inference
    - It deals with imprecise knowledge
  - Broadly speaking
    - Considered as a synonym of fuzzy set theory

Fuzzy Logics

- A set of mathematical principles for knowledge representation based on *degrees of membership* to a set
- Uses a continuum of logic values between 0 (completely false) and 1 (completely true)



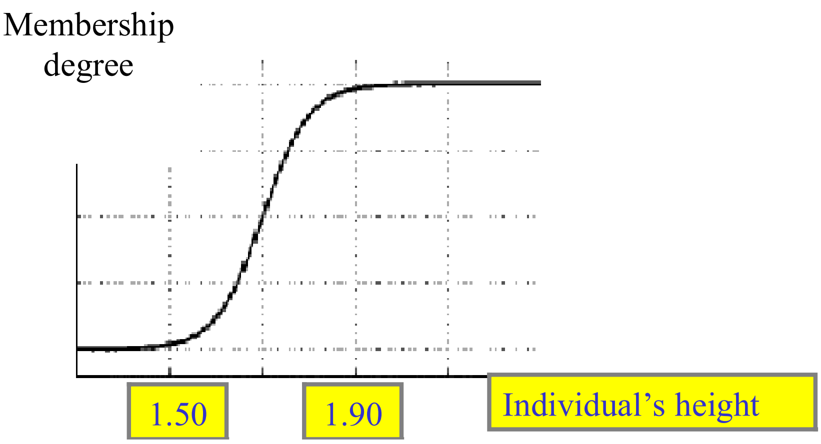
(*a*) Boolean Logic. (*b*) Multi-valued Logic.

Fuzzy Logics

- The degree of membership can be interpreted as a probability
  - Logic mechanisms of probabilistic type can be defined
    - Example
    - very_bad = [0, 0.2]; bad = [0.2, 0.5]; good = [0.5, 0.8]; very_good = [0.8, 1]
    - With different degrees of membership, e.g.:
      - 0 is very_bad with degree 1
      - 0.2 is bad with degree 0.5 and bad with degree 0.5
      - etc.
  - The *membership function* of a fuzzy set associates the features of each element (or *instance*, or *individual*) the value of the degree of membership to the set itself

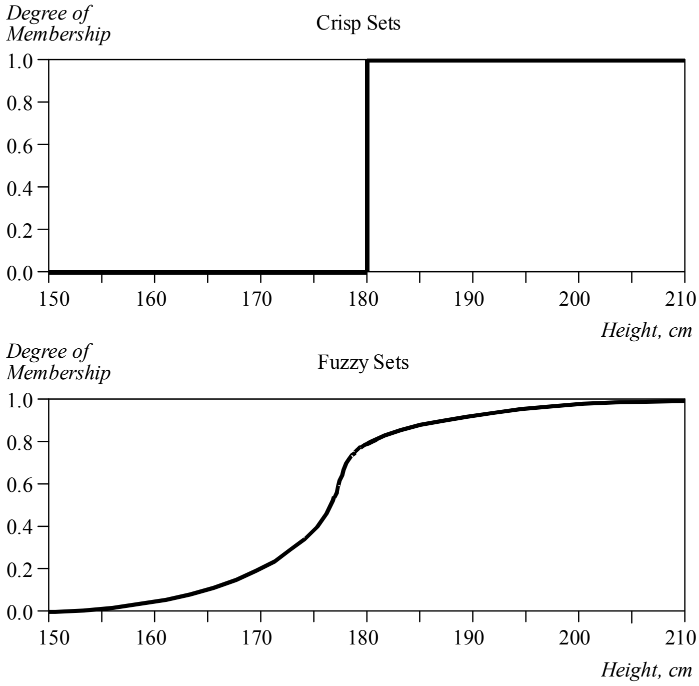
Fuzzy Logics

- Many kinds of membership functions are available
  - Can be chosen for pure mathematical convenience convenience or determined through experimental obsevations
  - Example: membership function for fuzzy category "quite tall"



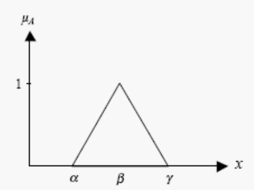
Fuzzy Logics

- Traditional (*Crisp*) sets vs Fuzzy sets
  - *x*-axis = universe of discourse
    - All possible values applicable to a given variable
  - *y*-axis = value of membership to the fuzzy set



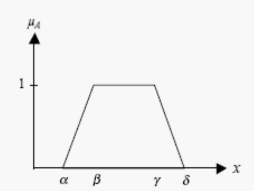
Fuzzy Theory

- Membership functions
  - Triangular



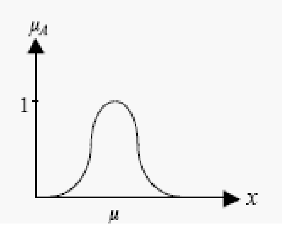
$$\mu_A(x,\alpha,\beta,\gamma)=\begin{cases}0 & \text{per } x<\alpha\\ \dfrac{x-\alpha}{\beta-\alpha} & \text{per } \alpha\le x\le\beta\\ \dfrac{\gamma-x}{\gamma-\beta} & \text{per } \beta<x\le\gamma\\ 0 & \text{per } x>\gamma\end{cases}$$

  - Trapezoidal



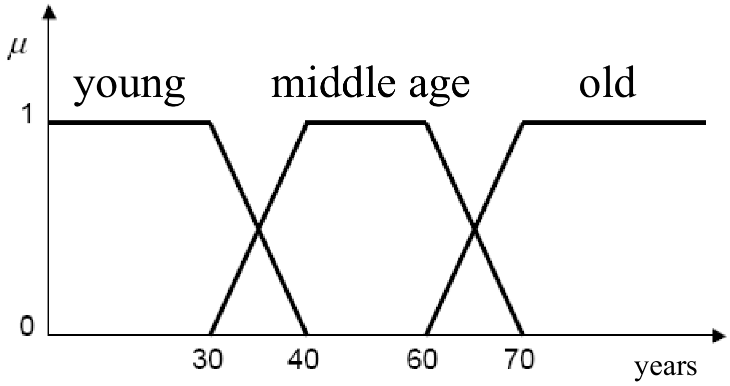
$$\mu_A(x,\alpha,\beta,\gamma,\delta)=\begin{cases}0 & \text{per } x<\alpha\\ \dfrac{x-\alpha}{\beta-\alpha} & \text{per } \alpha\le x\le\beta\\ 1 & \text{per } \beta<x\le\gamma\\ \dfrac{\delta-x}{\delta-\gamma} & \text{per } \gamma<x\le\delta\\ 0 & \text{per } x>\delta\end{cases}$$

  - Gaussian



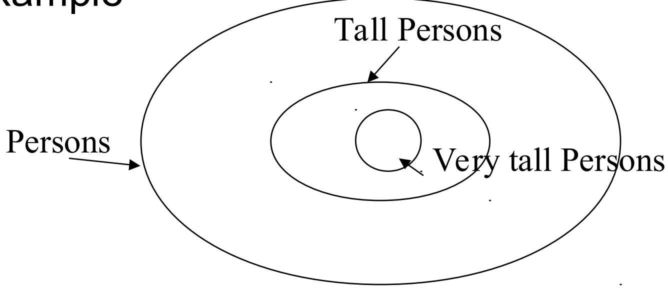
- Example: age of a person
  - A 30-years old person is young
  - How is a 31-years old person defined?

  - The fuzzy approach:



Fuzzy Logics

- Membership
  - In traditional sets, all elements in a set entirely belong to the superset
  - In fuzzy sets, each element may or may not belong both to the subset and to the superset
  - An element of a fuzzy set may have less degree of membership to the subset than to the superset
    - Example

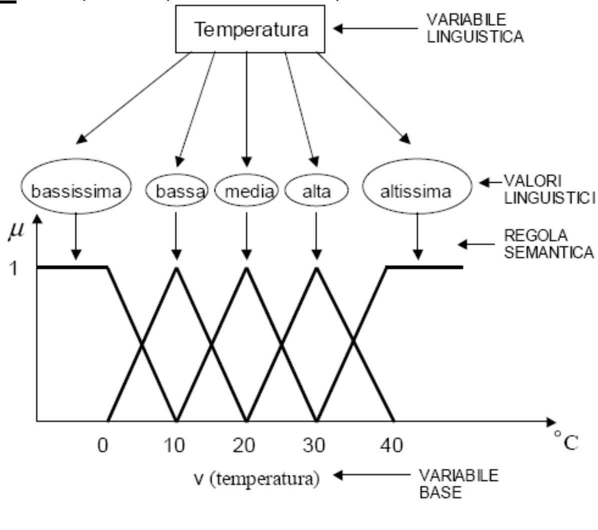


Fuzzy Theory

- Based on *linguistic variables*
- Linguistic variable
  - A fuzzy variable whose values are linguistic terms
    - E.g., the claim “John is tall” implies that linguistic variable John has linguistic value “tall”
  - In rule-based fuzzy systems, linguistic variables are used in *fuzzy rules*
    - E.g.:
      - IF wind is strong
      - THEN sailing is fine
    - or
      - IF speed is slow
      - THEN stopping_distance is short

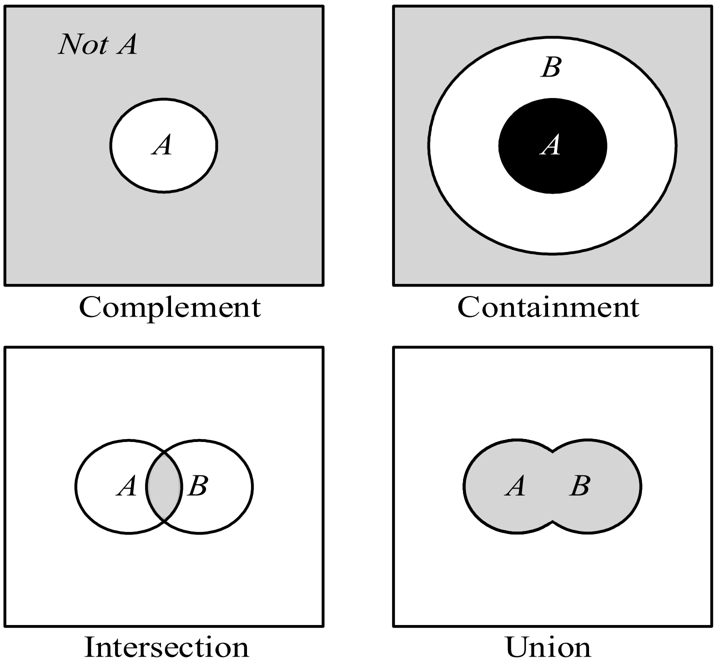
Fuzzy Theory

- Linguistic variables
  - Variable whose values are words or sentences in a natural or artificial language
    - Example: Variable *temperature* whose values are *very_low*, *low*, *medium*, etc.



Fuzzy Sets

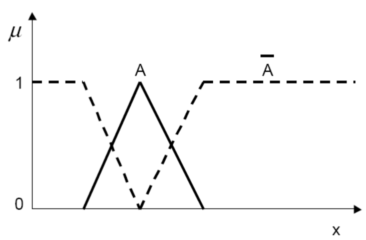
- Operations



Fuzzy Sets

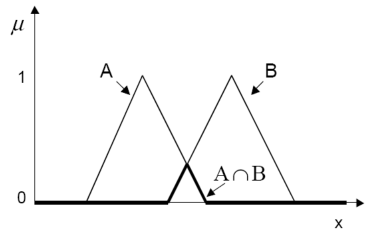
- Complement
  - Traditional sets: which elements do not belong to the set?
    - Opposite of the set
      - E.g.: the complement of the set of tall persons is the set of NOT tall persons
      - Removing tall persons from the universe of discourse, the complement is obtained
  - Fuzzy sets: by what degree an element does not belong to the set?
    - Complement ~A of a fuzzy set A obtained as follows:

$$\mu_{\sim A}(x) = 1 - \mu_A(x)$$



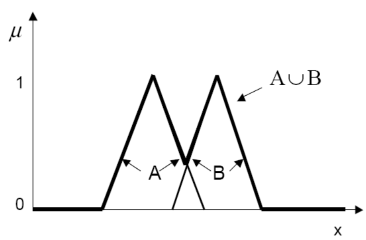
Fuzzy Sets

- Intersection
  - Traditional sets: Which elements belong to both sets?
    - Shared elements
  - Fuzzy sets: How much an element belongs to both sets?
    - An element may partially belong to the two sets with different degrees of membership
    - Defined as the lowest degree of belonging
    - Intersection between two fuzzy sets A and B on the universe of discourse X:
    - $\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)]$
      $= \mu_A(x) \cap \mu_B(x)$
      where $x \in X$



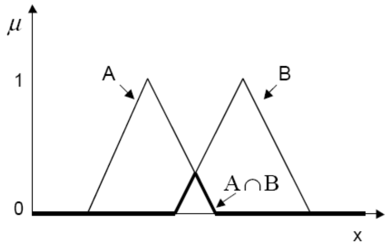
Fuzzy Sets

- Union
  - Traditional sets: which elements belong to either or both sets?
    - Elements belonging to at least one set
  - Fuzzy sets: how much an element belongs to either or both sets?
    - Reverse of intersection
    - Defined as the highest degree of belonging
    - Intersection between two fuzzy sets A and B on the universe of discourse X:
    - $\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)]$
      $= \mu_A(x) \cup \mu_B(x)$
      where $x \in X$

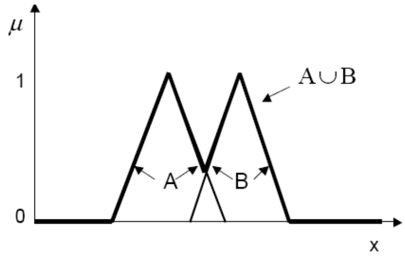


Fuzzy Sets

- Operators
  - AND (Intersection)
  - $\mu_{A \cap B}(x) = \min\left(\mu_A(x), \mu_B(x)\right) \quad \forall x \in X$

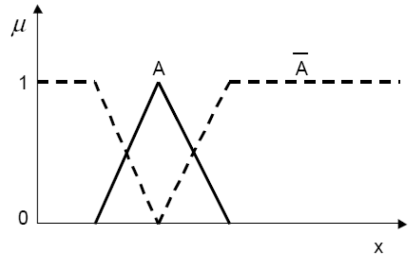


  - OR (Union)
  - $\mu_{A \cup B}(x) = \max\left(\mu_A(x), \mu_B(x)\right) \quad \forall x \in X$



  - NOT (Complement)

$\mu_{\overline{A}}(x) = 1 - \mu_A(x) \quad \forall x \in X$



Fuzzy Sets

- Properties
  - Equality
    - A fuzzy set is equal to another if
    - $\mu_A(x) = \mu_B(x)$, $\forall x \in X$
      - Example
      - A = 0.3/1 + 0.5/2 + 1/3
      - B = 0.3/1 + 0.5/2 + 1/3
      - Thus, A = B
  - Inclusion
    - A fuzzy set A, $A \subseteq X$, is inlcuded in a fuzzy set B, $B \subseteq X$, if
    - $\mu_A(x) \leq \mu_B(x)$, $\forall x \in X$
    - A is a subset of B
      - Example
      - Consider X = {1, 2, 3} and sets A and B
      - A = 0.3/1 + 0.5/2 + 1/3
      - B = 0.5/1 + 0.55/2 + 1/3
      - Then, $A \subseteq B$

Further Readings

- P. Smets, E.H. Mamdani, D. Dubois, H. Prade (Eds.) “Non–Standard Logics for Automated Reasoning” Academic Press, 1988.
- E. Allen Emerson “Temporal and Modal Logic” Report 1995, University of Texas, Austin
- G. Brewka, I. Niemela, M. Trusczynski “Non Monotonic Reasoning” 2007 Elsevier
- L. Zadeh “Fuzzy Logic and approximate reasoning” Synthese, 30: 407-428, 1975